DOCUMENT RESUME

ED 383 537 SE 055 807

AUTHOR da Ponte, Joao Pedro, Ed.; Matos, Joao Filipe, Ed.

TITLE Proceedings of the International Conference for the Psychology of Mathematics Education (PME) (18th, Lisbon, Portugal, July 29-August 3, 1994). Volumes I-IV.

INSTITUTION International Group for the Psychology of Mathematics Education.

REPORT NO ISBN-972-8161-00-X; ISBN-972-8161-01-8; ISBN-972-8161-02-6; ISBN-972-8161-03-4

PUB DATE Jul 94

NOTE 1,510p.

PUB TYPE Collected Works - Conference Proceedings (021)

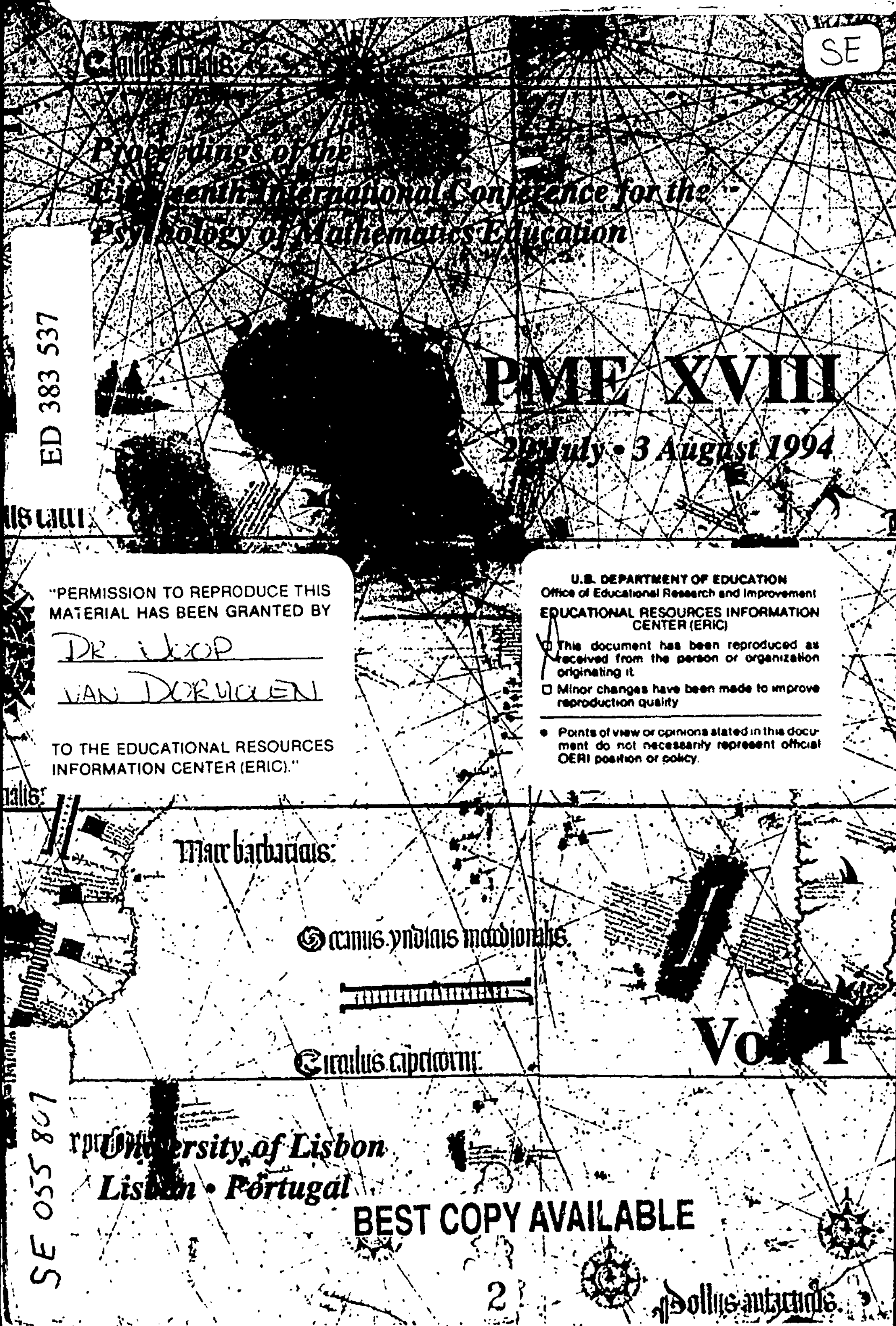
EDRS PRICE MF12/PC61 Plus Postage.

DESCRIPTORS Affective Behavior; *Algebra; Beliefs; Calculators; College Mathematics; Computers; Cultural Influences; Elementary Secondary Education; Epistemology; Functions (Mathematics); Geometry; Graphs; Higher Education; *History; Imagery; Language; *Learning Theories; Mathematical Models; Mathematics Instruction; Measurement; Metacognition; Number Concepts; Probability; Problem Solving; Proof (Mathematics); Rational Numbers; Ratios (Mathematics); Sex Differences; Socioeconomic Influences; Statistics; Student Evaluation; Teacher Education

IDENTIFIERS *Mathematics Education Research; *Psychology of Mathematics Education; Representations (Mathematics); Subject Content Knowledge

ABSTRACT

The Proceedings of PME-XVIII have been published in four separate volumes because of the large number of individual conference papers reported. Volume I contains brief reports for 11 Working Groups and 8 Discussion Groups, 55 "Short Oral Communications," 28 Posters, 5 Plenary Panel reports, and 4 Plenary Session reports. Volume II contains 50 Research Reports covering authors with last names starting with A-G. Volume III contains 52 Research Reports covering authors with last names starting with G-O. Volume IV contains 54 Research Reports covering authors with last names starting with P-Z. In summary, the four volumes contain 156 full-scale Research Reports, 4 full-scale Plenary Session Reports, and 57 briefer items. Conference subject content can be conveyed by a listing of the Plenary Panels and Plenary Session Reports. Plenary Panels: "The History of Mathematics and the Learning of Mathematics: Psychological Issues (Paul Ernest); "Relations Between History and Didactics of Mathematics" (Lucia Grugnetti); "The Case of Pre-Symbolic Algebra and the Operation of the Unknown" (Teresa Rojano); "What History of Mathematics Has to Offer to Psychology of Mathematical Thinking" (Anna Sfard); "Practical Uses of Mathematics in the Past: A Historical Approach to the Learning of Mathematics" (Eduardo Veloso). Plerary Session Reports: "The Historical Dimension of Mathematical Understanding--Objectifying the Subjective" (Hans Niels Jahnke); "A Functional Approach to the Introduction of Algebra: Some Pros and Cons" (Carolyn Kieran); "Researching from the Inside in Mathematics Education: Locating an I-You Relationship" (John Mason); "Mathematics Teachers' Professional Knowledge" (Joao Pedro da Ponte). (MKR)



Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

PME XVIII

29 July • 3 August 1994

Vol 1

University of Lisbon
Lisbon • Portugal

Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

PME XVIII

29 July — 3 August 1994

Volume I

University of Lisbon
Lisbon — Portugal

Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

Volume I

Published by the Program Committee of the 18th PME Conference,
Lisbon, Portugal

Editors: João Pedro da Ponte and João Filipe Matos

Cover: Eduardo Veloso

Depósito legal nº 78566 94

ISBN 972-8161-00-X

Printed by GRAFIS, Coop. de Artes Gráficas, CRL

Cover: The background is a reproduction of the *Cantino* chart. This Portuguese map was smuggled from Portugal in 1502, by the spy of the Duke of Ferrara, Alberto Cantino. It is the first world map of the modern times, and it takes into account the findings of the voyage of Vasco da Gama to India.
Cover overleaf: The astrolabe and the quadrant, the most used nautical instruments of the beginning of astronomical navigation, in the 15th century, are represented by two drawings included in the world map of Diogo Pereira, 1529.

Departamento de Educação
Faculdade de Ciências da Universidade de Lisboa
Edifício C1 — Campo Grande
1700 Lisbon — Portugal

PME XVIII PROCEEDINGS

Edited by João Pedro da Ponte and João Filipe Matos
Departamento de Educação
Faculdade de Ciências da Universidade de Lisboa
Lisboa — Portugal

PREFACE

The first meeting of the International Group for the Psychology of Mathematics Education (PME) took place in Karlsrure, Germany, in 1976. Thereafter different countries (Netherlands, Germany, U.K., U.S.A., France, Belgium, Israel, Australia, Canada, Hungary, Mexico, Italy, Japan) hosted the conference. In 1994 Portugal will host PME. The conference will take place at the University of Lisbon. The University is a public institution devoted to the creation, transmission and diffusion of culture and science, holding the heritage of

—the foundation of University in Portugal, in 1288,
—the creation of the Lisbon King's School of Surgery, in 1825,
—the creation of the Polytechnic School in 1837,
—the creation of the Higher Studies in Arts, in 1859,
—the institution of the University of Lisbon in 1911.

The Department of Education of the Faculty of Sciences has a strong record of commitment to research, teacher education and service in mathematics education. We are pleased to be the host site for PME XVIII.

The academic program of PME XVIII includes

— 157 research reports
— 55 short oral presentations
— 28 poster presentations
— 4 plenary addresses
— 1 plenary panel
— 11 working groups
— 7 discussion groups

The review process

The Program Committee received a total of 184 research report proposals that encompassed a wide variety of themes and approaches. Each proposal was submitted to three outside reviewers drawn from PME membership who were knowledgeable in the specific research area. In addition, one or more Program Committee members read each paper. Based on these reviews each paper was accepted or rejected. The short oral and poster presentations were also read by one or more Program Committee members and on the basis of this recommendation either accepted or rejected. If a reviewer submitted written comments, they were forwarded to the authors along with the Program Committee decisions.

List of PME XVIII reviewers

The Program Committee wishes to thank the following people for their help during the review process:

Paulo Abrantes (Portugal)
Josette Adda (France)
Conceição Almeida (Portugal)
Miriam Amit (Israel)
Michele Artigue (France)
Ferdinando Arzarello (Italy)
Bill Atweh (Australia)
Nicholas Balacheff (France)
Arthur Baroody (USA)
C. Batanero (Spain)
Luciana Bazzini (Italy)
Nadine Bednarz (Canada)
Alan Bell (United Kingdom)
Alan Bishop (Australia)
Ana Boavida (Portugal)
Paolo Boero (Italy)
George Booker (Australia)
Rosa Bottino (Italy)
Ada Boufi (Greece)
Chris Breen (South Africa)
Catherine Brown (USA)
Maria Bartolini Bussi (Italy)
Ana Paula Canavarro (Portugal)
Thomas Carpenter (USA)
David Carraher (Brazil)
Susana Carreira (Portugal)
Giampaolo Chiappini (Italy)
Douglas Clements (USA)
Paul Cobb (USA)
Jere Confrey (USA)
Thomas Cooney (USA)
Martin Cooper (Australia)
Anibal Cortes (France)
Kathelyn Crawford (Australia)
Robert Davis (USA)
Michael De Villiers (South Africa)
Bernardette Denys (France)
Willibald Dorfler (Austria)
Tommy Dreyfus (Israel)
Ed Dubinsky (USA)
Laurie Edwards (USA)
Ted Eisenberg (Israel)
Nerida Ellerton (Australia)
Paul Ernest (United Kingdom)
Gontran Ervynck (Belgium)
Ruhama Even (Israel)
Domingos Fernandes (Portugal)
Pier Luigi Ferrari (Italy)
Alex Friedlander (USA)
Fulvia Furinghetti (Italy)
Elisa Gallo (Italy)
Elisabeth Gallou-Dumiel (France)
Claude Gaulin (Canada)
Joaquim Gimenez (Spain)
Juan Godino (Spain)
Gerald Goldin (USA)
Brian Greer (Northern Ireland)
Jeef Gregg (USA)
Ulrich Grevsmuhl (Germany)
Henrique Manuel Guimarães (Portugal)
Angel Gutierrez (Spain)
Gila Hanna (Canada)
Guershon Harel (USA)
Kath Hart (United Kingdom)
Klaus Hasemann (Germany)
Lulu Healy (United Kingdom)

Rina Hershkowitz (Israel)
Sara Hershkowitz (Israel)
James Hiebert (USA)
Ichei Hirabayashi (Japan)
Fernando Hitt (Mexico)
Celia Hoyles (United Kingdom)
Adela Jaime (Spain)
Claude Janvier (Canada)
Barbara Jaworski (United Kingdom)
Rob Jong (Holand)
Rochelle Kaplan (USA)
James Kaput (USA)
Christine Keitel (Germany)
Carolyn Kieran (Canada)
David Kirshner (USA)
Konrad Krainer (Austria)
Chronis Kynigos (Greece)
Collete Laborde (France)
Susan Lamon (USA)
Marsha Landau (USA)
Leonor Cunha Leal (Portugal)
Miriam Leiva (USA)
Enrica Lemut (Italy)
Stephen Lerman (United Kingdom)
Frank Lester (USA)
Fou-Lai Lin (Taiwan)
Liora Linchevski (Israel)
Salvador Llinares (Spain)
Eric Love (United Kingdom)
Carolyn Maher (USA)
Nicolina A. Malara (Italy)
Maria Alessandra Mariotti (Italy)
Zvia Markovits (Israel)
Amy Martino (USA)
John Mason (United Kingdom)
José Manuel Matos (Portugal)
João Filipe Matos (Portugal)
Douglas Mcleod (USA)
Betsy McNeal (USA)
Hartwig Meissner (Germany)
Ana Mesquita (Portugal)
Stephen Monk (USA)
Cândida Moreira (Portugal)
Luisa Morgado (Portugal)
Carlo Morini (Italy)
Nitsa Movshkowitz-Hadar (Israel)
Hanlie Murray (South Africa)
Ricardo Nemirovsky (USA)
Dagmar Neuman (Sweden)
Richard Noss (United Kingdom)
Terezinha Nunes (United Kingdom)
Rafael Nunez (Switzerland)
Alwin Olivier (South Africa)
Jean Orton (United Kingdom)
John Pace (USA)
Michele Pellerey (Italy)
Barbara Pence (USA)
Marie Jeanne Perrin-Glorian (France)
Angela Pesci (Italy)
David Pimm (Canada)
João Pedro da Ponte (Portugal)
Norma Presmeg (USA)
Maria Reggiani (Italy)
Anne Marie Reynolds (USA)
Luis Rico (Spain)
Teresa Rojano (Mexico)
Thomas Romberg (USA).
Andy Rubin (USA)
Susan Russel (USA)
Victoria Sanchez (Spain)
Analucia Schlieman (Brazil)
Karen Schultz (USA)
Baruch Schwarz (Israel)
Falk Seeger (Germany)
Anna Sfard (Israel)
Michael Shaughnessy (USA)
Katsuhiko Shimizu (Japan)
Cristine Shiu (United Kingdom)
Edward Silver (USA)
Martin Simon (USA)
Judith Sowder (USA)
Larry Sowder (USA)
Kaye Stacey (Australia)
Heinz Steinbring (Germany)
Rudolf Straesser (Germany)
Leen Streefland (The Netherlands)
Alba Thompson (USA)
Cornelia Tierney (USA)
Dina Tirosh (Israel)
Gerard Vergnaud (France)
Shlomo Vinner (Israel)
Bruno Vitale (Switzerland)
Grayson Wheatley (USA)
Terry Wood (USA)
Erna Yackel (USA)
Michal Yerushalmy (Israel)
Orit Zaslavsky (Israel)
Nurit Zehavi (USA)

Acknowledgements

Departamento de Educação da Faculdade de Ciências da Universidade de Lisboa
Departamento de Física da Faculdade de Ciências da Universidade de Lisboa
Departamento de Geologia da Faculdade de Ciências da Universidade de Lisboa
Departamento de Matemática da Faculdade de Ciências da Universidade de Lisboa
Fundação Calouste Gulbenkian
Fundação da Faculdade de Ciências da Universidade de Lisboa
Fundação Luso Americana
Instituto de Inovação Educacional
Junta Nacional de Investigação Científica e Tecnológica
Projecto MINERVA - Departamento de Educação da FCUL
Câmara Municipal de Lisboa
Centro Nacional de Cultura
Comissão Nacional para as Comemorações dos Descobrimentos Portugueses
Gabinete de Incentivo a Congressos Organizados em Lisboa
Metropolitano de Lisboa
Automóvel Clube de Portugal
Beltrão Coelho
Caixa Geral de Depósitos
Casio
ICL
Investimentos, Comércio e Turismo de Portugal
Olivetti
Phillips
Prológica
TAP Air Portugal
TLP

INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Present Officers of PME

President	Carolyn Kieran (Canada)
Vice-President	Mariolina Bartolini-Bussi
Secretary	Martin Cooper (Australia)
Treasurer	Angel Gutiérrez (Spain)

Other Members of the International Committee

Michèle Artigue (France)
Jere Confrey (USA)
Bernardette Dennis (France)
Paul Ernest (UK)
Barbara Jaworski (UK)
Chronis Kynigos (Greece)
Stephen Lerman (UK)
Carolyn Mayer (USA)
Teresa Rojano (Mexico)
Keiichi Shigematsu (Japan)
Judith Sowder (USA)
Kaye Stacey (Australia)
Wacek Zawadowski (Poland)

Executive Secretary

Joop van Dormolen (The Netherlands)

PME XVIII Program Committee

Michèle Artigue (France)
Mariolina Bartolini-Bussi (Italy)
Domingos Fernandes (Portugal)
Angel Gutiérrez (Spain)
Barbara Jaworski (UK)
Carolyn Kieran (Canada)
João Filipe Matos (Portugal)
José Manuel Matos (Portugal)
Cândida Moreira (Portugal)
João Pedro da Ponte (Portugal, chair)

PME XVIII Local Organizing Committee

Paulo Abrantes
Susana Paula Carreira
Henrique Manuel Guimarães
João Filipe Matos
Madalena Santos
Ana Paula Canavarro
Maria José Delgado
Leonor Cunha Leal
João Pedro da Ponte

Conference Program Secretary

João Filipe Matos

HISTORY AND AIMS OF PME

At the Third International Congress of Mathematics education (ICME 3, Karlsruhe, 1976), Professor E. Fischbein of the Tel-Aviv University, Israel, instituted a study group bringing together people working in the area of the psychology of mathematics education. PME is affiliated with the International Commission for the Mathematics Instruction (ICMI). Its past presidents have been Prof. Efrain Fischbein, Prof. Richard Skemp of the University of Warwick, Dr. Gérard Vergnaud of the Centre Nacional de la Recherche Scientifique (C.N.R.S.) in Paris, Prof. Kevin F. Collis of the University of Tasmania, Prof. Perla Nesher of the University of Haifa, Dr. Nicolas Balacheff of the C.N.R.S. in Lyon, and Prof. Kath Hart of the University of Nottingham.

The major goals of PME are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof

Membership

Membership is open to people involved in active research consistent with the Group's aim, or professionally interested in the results of such research.

Membership is open in an annual basis and depends on payment of the subscription for the current year (January to December)

The subscription can be paid with the conference fee.

TABLE OF CONTENTS

Volume II

Research reports

Volume III

Research reports (cont.)

Volume IV

Research reports (cont.)

27

THEMATIC INDEX

Rational Numbers and Proportion

Social Construction of Mathematical Knowledge

Social Factors

Teacher Education

Theories of Learning

CONTRIBUTORS

Nadja Acioly Fax n° 26, rue du Pont, Auxerre 89000 FRANCE	Luciana Bazzini Fax n° 39 382 505602 <DIPMAT@IPVIANZ Dept. of Mathematics University Paiva Via Abbiategrasso 209 27100 Pavia ITALY
Jill Adler Fax n° 27 11 3393956 022BERYL@WITSVMA WITS.AC.ZA Education Department Witwatersrand University Private Bag 3 Powits 2050 SOUTH AFR	Richard A. Beare Fax n° 0203 523237 Dept of Science Education University of Warwick Coventry CV4 7AL UK
Janet Mary Ainley Fax n° 0203523237 serav@esv.warwick ac uk Department of Science Education University of Warwick Coventry CV4 7AL UK	Joanne Becker Fax n° 408 924 5080 becker@sjsumcs sjsu edu 3192 Malvasia CT Pleasanton California 94566 USA
Marta Elena Alvarez Valdemoros Fax n° 525 683 61 11 Cerro Huitzilac 146 - 301 D F Personal 03100 MEXICO	Nadine Bednarz Fax n° 514 987 4636 CIRADE@UGAM ER CA Cirade Universite du Quebec a Montreal C P 8888 Suc Centre Ville Montreal 43C 3P8 CANADA
Gilead Shmuel Amir Fax n° 061 275 3519 12 Meadfoot Ave - Prestwich Manchester M25 8AR UK	Alan Bell Fax n° 0602791813 rszab@unicorn nott ac uk Shell Centre University Park Nothingham NG2 5AR UK
Miriam Amit Fax n° 972 7 469 236 amit@nengus 38 Erez St. Omer 84965 ISRAEL	René Berthelot Fax n° 33 59 336569 8 Rue de l' Aubisque Maucor 64 160 FRANCE
Poulos Andreas Fax n° University of Salonica Department of Education Ostrovou 12-14 54453 Salonica GREECE	Janet Elizabeth Binns Fax n° 0908 653744 J E BINNS@open ac uk 116 Gurnards Avenue Fishermead Milton Keynes MK6 2EE UK
Ilana Arnon Fax n° 972 3 6422619 IBBY@ RAV.OPENU.AC IL Center of Educational Tecnology 16 Klausner St. P O Box 39513 Tel - Aviv 61394 ISRAEL	Hava Bloedy-Vinner Fax n° 20 Burla St Jerusalem 93714 ISRAEL
Michele Artigue Fax n° 1 44 27 56 08 artigue@ mathp7.Jussieu fr 15 Rue Mademoiselle Villebon Sur Yvette 91 140 FRANCE	Ana Maria Boavida Fax n° 01 2954461 Prac Ricardo Jorge N°3, 1°Esq. Pragal 2800 Almada PORTUGAL

Paolo Boero
Dipartamento Matematica Università
Via L. B. Alberti, 4-1
16132 Genova
ITALY
Fax nº
39 10 3538769

Marcelo Borba
Cx Postal 178
Departamento de Matematica
UNESP
13500 - 230 Rio Claro-SP
BRAZIL
Fax nº
?
UECB@BRFAPESP BITNET

Rosa Maria Bottino
Istituto Per la Matematica Applicata - C N R
Via de Marini, 6
16149 Genova
ITALY
Fax nº
39 10 6475660
BOTTINO@IMAGE CE CNR IT

Ada Boufi
University of Athens - Pedagogics
Department Elementary Education
33 Ippokratous Str Athens
10680
GREECE
Fax nº
30 1 7210478
aboufi@atlas.uoa ariadne_t gr

Gillian Marie Boulton-Lewis
School of Learning & Development
Faculty of Education, QUT
Locked Bag nº2, Red Mill, Old
4059
AUSTRALIA
Fax nº
07 8643987
g boulton_lewis@qut edu au

Chris Breen
School of Education
University of Cape Town
Rondebosh
7700
SOUTH AFRICA
Fax nº
021 650 3489
CB@EDUCATION UCT AC ZA

Anthony Mickage Brown
17 Burton Road - Withington
Manchester
M20 3GD
UK
Fax nº
061 445 4068

Alicia Bruno Castañeda
Departamento de Analisis Matematico
Universidad de la Laguna
La Laguna - Tenerife
38271
SPAIN
Fax nº
922 604023
ABRUNO@ULL ES

Ely Machado de Campos
R. Corte Real, 385 Apto 402
Porto Alegre
RS 90 610-080
BRAZIL
Fax nº
051 231 66 00

David Carraher
22 Railroad St # 407
Andover
MA 01810
USA
Fax nº
617 349 3535
DAVID_CARRAHER@TERC EDU

Jose Carrillo
Universidad de Huelva
Dep. Didactica de las Ciências
Avda Fuerzas Armadas, S/N
21007 Huelva
SPAIN
Fax nº
34 59 27 1000

Jaime Del Rio Castilho
Itam Rio Hondo Nº 1
Tizapan, San Angel
01000 Alvaro Obregon
D F Mexico
MEXICO
Fax nº
550 7637
JDRIO@GAUSS RHON ITA M

Olive Chapman
Depto Curriculum and Instruction
Ed T 702
2500 University Dr N W
Calgary Alberta
CANADA
Fax nº
403 282 8479

Christos Chasiotis
Department of Mathematics
University of Ioannina
Ioannina
G- 45110
GREECE
Fax nº
30 651 44112

Daniel Chazan
Michigan State University
116T Ericson Hall
East Lansing
MI 48824
USA
Fax nº
?
dchazan@msu.bitnet

Carles Romero i Chesa
Carretera Nova, 99, 3er 2ª
La Garriga
Barcelona
08530
SPAIN
Fax nº
3 581 1737

Marta Civil
Department of Mathematics
University of Arizona
Tucson AZ
85721
USA
Fax nº
602 621 8322
civil@math.arizona edu

David John Clarke
Math Teaching and Learning Centre
Australian Catholic University 17
Castlebar Road Oakleigh, Victoria
3166
AUSTRALIA
Fax nº
613 344 8739
DavidClarke@mac univ edu au

Douglas Clements
University of New York at Buffalo
593 Baldy Hall
Buffalo
NY 14260
USA
Fax nº

Lillie R F Crowley
Division of Psysical Sciences and Engineering Technologies, Lexington Community College Lexington KY 0235
40506
USA
Fax nº
606 257 4339
LCROWLEY@ UKCC.UKY EDU

Paul Cobb
Vanderbilt University
Peabodg College Box 330
Nashville
TN 37203
USA
Fax nº
615 322 8999
COBBP@VUCTRVAX bitnet

Robert B. Davis
501 South First Ave
Highland Park
NJ 2123
08904
USA
Fax nº
908 932 1318
rdavis@math.rutgers edu

Anne Cockburn
School of Education
University of East Anglia - Norwich
NR4 7TS
UK
Fax nº
0603 259388

Sandy Dawson
4172 Glenhavon Crescent
North Vancouver, BC
V7G 1B9
CANADA
Fax nº
1 604 291 3203
dawson@sfu ca

Claude Comiti
I.U.F M de GRENOBLE - 30
Av Marcellin Berthelot
38 100 Grenoble
FRANCE
Fax nº
33 76871947
Claude.Comiti@imag fr

Erik De Corte
Center For Instr Psych and Technology
Univ of Leuvem
Vesaliusstraat,2 Leuvem
B 3000
BELGIUM
Fax nº
32 16 286274

Jere Confrey
Corneil U - Dept of Education
Kennedy Hall, Ithaca
14853 - NY
USA
Fax nº
607 255 1255
JC56@cornell edu

Bernadette Denys
5/7 Rue de la Comete
75007 Paris
FRANCE
Fax nº
33 1 47 53 83 38
denys@mathp7 jussieu fr

Thomas Cooney
105 Aderhold Hall
University of Georgia
Athens, Georgia
30602
USA
Fax nº
706 542 4551
TCOONEY@ MOE COE VGA EDU

Helen Doerr
526 Engineering Theory Center Bldg
Cornell University
Ithaca, New York
14853
USA
Fax nº
607 254 8888
hdoerr@tc cornell edu

Moisés Coriat Benarroch
Dep de Didáctica de la Matematica
Facul de Ciéncias de la Education
Campus Universitario de Cartuja S N
18071 Granada
SPAIN
Fax nº
58 243949
MCORIAT@UGA ES

Brian Doig
ACER
P O Box 210
Camberwell
3124
AUSTRALIA
Fax nº
03 277 5500
DOIG@ACER EDU.AU

Anibal Cortes
Labpsydee - 46me St Jaques
75005 - Paris
FRANCE
Fax nº
40462993

Tommy Dreyfus
Center For Technological Education
P O BOX 305 HOLON
58102
ISRAEL
Fax nº
972 3 502 8967
NTOMMY@ WICCMAIL WEIZ MANN AC IL

Kathryn Crawford
511 Silva st
Tawarama
NSW - 2026
AUSTRALIA
Fax nº
61 2 692 4534
Kathyc@basse1 cs su oz au

Janet M Duffin
School of Mathematics
University of Hull
HU6 7RX
UK
Fax nº
0482 466218

Laurie Edwards
129 Vallier Hall
Crown College, UCSC
Santa Cruz
CA 95064
USA
Fax nº 408 459 4618
edwards@cats.ucsc.edu

Nerida Ellerton
Faculty of Education
Edith Cowan University
Pearson Street
6018
AUSTRALIA
Fax nº 61 9 387 7095
N.Ellerton@cowan.edu

Paul Ernest
University of Exeter
School of Education Exeter
Devon
EX1 2LV
UK
Fax nº 392 264 736
pErnest@cen.ex ac uk

Gontran Ervynck
K. UNIV LEUVEN
CAMPUS KORTRYK KULAK
KORTRYK
8500
BELGIUM
Fax nº 32 56 246999
Gortran Ervynck@felix.Kulak ac be

Isabel Escudero
Depto Didactica de las Ciencias Mat
Universidad de Sevilha
Av da Ciudad Jardin Nº 22
41005 Sevilha
SPAIN
Fax nº 4645861

Maria Candelaria Espinel Febles
Escuela Univ. del Profesorado de E G B
Univ. de la Laguna
C/Heraclio Sanchez.33.Tenerife
38204 Canarias
SPAIN
Fax nº 922 604023
MESPINEL@ULL ES

António Estepa Castro
C/O bispo Alonso Vera, nº4
23008 Jaen
SPAIN
Fax nº 53 212243

Jeff Evans
I Granville RD
London
N4 4EJ
UK
Fax nº 081 805 0702
jeff2@UK.AC.MIDDLESEX.VAXA

Estela Kaufman Faiguelernt
Rua Frei Leandro nº22 apt. 401
Jardim Botânico
Rio de Janeiro
RJ 22470 - 210
BRAZIL
Fax nº 021 5516446

Domingos Fernandes
Travessa Terras de Sant'Ana 15
1200 Lisboa
PORTUGAL
Fax nº

Joaquin Ferrer Sanchez
Dep de Estadistica e Investigaçion
Operativa - Campus Univers. de Almeria
La Canada
04071
SPAIN
Fax nº

Uri Fidelman
Depto of Education
Tecnology and Science
Haifa 32
ISRAEL
Fax nº

Keir Finlow-Bates
Southbank University S C.I S.M
Borough Road 103
London
SE1 0AA
UK
Fax nº 071 8157499
FINLOWK@UK.AC.SBU VAX

Efraim Fischbein
Tel Aviv University - Ramat Aviv
P O.B. 39040
69978 TELAVIV
ISRAEL
Fax nº 972 3 4609477

Misael Jose Fisico
3596 Davila St
Makati 1200
Metro Manila
FILIPINAS
Fax nº

Robin Foster
3 Tarnbrook Close - Crag Bank
Carnforth, Lancashire
LA5 9UL
UK
Fax nº

Fulvia Furinghetti
Dipartimento di Matematica
Via Alberti 4
16132 - Genova
ITALY
Fax nº 39 10 3538769

Aurora Gallardo
Nicolas San Juan nº1421
Col Del Valle
03100 Mexico DF
MEXICO
Fax nº 5 25 688 6111

Elisa Gallo
Dipartimento de Matematica
Università di Torino
Via Carlo Alberto,10
10123 Torino
ITALY
Fax n°
39 11 534497

F Ribeiro Gonçalves
Universidade do Algarve
Campus de Gambelas
8000 Faro
PORTUGAL
Fax n°

Rossella Garuti
7 - Fossoli di Carpi
Via del Melograno
Modena 41010
ITALY
Fax n°
39 59 650810

Simon Goodchild
College of St Mark and St John Derriford
Road Plymouth
Pl6 8BH
UK
Fax n°
44 752 761120

Peter Gates
School of Education
Nottingham University
University Park Nottingham
NG72RD
UK
Fax n°
44 602 791506
TEZPGA @ UK AC NOTTINGHAM UNICORN

Zahra Gooya
N° 4, 16th St NASR KISHA
Ave Tehran
14486
IRAN
Fax n°
Zangeneh@IREARN BITNET

Linda Gattuso
3417 Vendome
Montreal
H4A 3M6
CANADA
Fax n°
514 987 8477
GATTUSO LINDA@ UQAM CA

Edward Martin Gray
Department of Science Education
University of Warwick Coventry
CV4 7AL
UK
Fax n°
0203 523237

Joaquin Gimenez
General Mitre 216
08006
SPAIN
Fax n°
34 3 2176887

Susie Groves
Deakin University - Burwood
221 Burwood Highway
3125
AUSTRALIA
Fax n°
?
grovesac@deakin edu au

Gerald A. Goldin
Rutgers University Scro Bldg. RM 239
Busch Campus
Piscataway, NJ 1179
08855
USA
Fax n°
1 908 932 3477
gagoldin@dimacs.rutgers edu

Lucia Gruguetti
Universita Degli Studi di Parma
Dipartimento di Matematica
Via Massimo d Azeglio 85/a
I-43100 Parma
ITALY
Fax n°
39 521 902350

Colleen Goldstein
Maths Centre for Primary Teachers
P O Box 117
Auckland Park
2006
SOUTH AFRICA
Fax n°
011 726 1510

Angel Gutierrez
Dpto Didactica de La Matematica
Univ de Valencia
Apartado 22045
46071 Valencia
SPAIN
Fax n°
34 6 3864487
GUTIERREZ@ MAC UV ES

Bernardo Gomez Alfonso
Avda Hispanidad, n°8
Alzira Valencia
46600
SPAIN
Fax n°
346 3864487

Orit Hazzan
Dept of Science Education - Technion
Israel Inst of Tecnology
Haifa 32000
ISRAEL
Fax n°
972 4 325 445
TTR015O@TECHNION BITNET

Ines Mª Gomez Chacon
Visconde de Matamala 3
28028 Madrid
SPAIN
Fax n°
91 7259209

Josefa Hernandez Dominguez
C/ Malaquita 1 - 4° C
Santa Cruz de Tenerife
38005
SPAIN
Fax n°
922 603544

Rina Hershkowitz
Science Teaching Dept.
Weizmann Institute of Science Rehovot
76100
ISRAEL
Fax n°
972 8 344115
NTHERSHK@WEIZMANN.WEIZMANN.AC IL

Joel Hillel
Dept of Math. Stats Concordia - 7141
Sherbrooke ST.W
Montreal Quebec
H4B1R6
CANADA
Fax n°
jhillel@VAX2 CONCORDIA CA

Meira Hockman
P.O Box 926
Gallo Manor - Sandton
2052
SOUTH AFRICA
Fax n°
011 6401832

Mariela Orozco Hormaza
Traja 18 - 20, 4°, 3° E A
Barcelona - 08004
SPAIN
Fax n°
343 4023602

Ah-Chee Ida Mok
Department of Curriculum Studies
University of Hong Kong
Poakfulam Road
HONG KONG
Fax n°
852 8585649
iacmoK@hkucc bitnet

Bat - Sheva Ilany
27,8 Agnon st Ranna
43380
ISRAEL
Fax n°
972 3 5234117

Hans Niels Jahnke
Institut Fur Didaktik der Mathemank
Universitat Bielefeld
Post Fach100 131
48100 Bielfeld 1
GERMANY
Fax n°
49 521 1062991

Claude Janvier
367 Ave Victoria, Westmont Que
Canada
H3Y 2R8
CANADA
Fax n°
514 987 4637

Barbara Jaworski
University of Oxford Dept Of
Educational Studies,
15 Norham Gardens, Oxford
OX26PY
UK
Fax n°
0865 274027
JAWORSKI@UK AC OX

Kioko Kakihana
2011 - 167 Sasagu
Tsukuba Ibaraki
305
JAPAN
Fax n°
0298 55 9880
Kakihana@Kasei.ac jp

Maria Kaldrimidou
Early Child Houd Dep. of Education
University of Ioannina
Domboli 30 - Ioannina
45110
GREECE
Fax n°
30 651 40823

Lena Licon Khisty
216 So Maple Ave n°32
Oak Park IL
60302
USA
Fax n°
312 996 6400
U49827@UICVM

Carolyn Kieran
Dep Math - INEO
Universite du Quebec à Montreal
C P 8888, Succ Centre-Ville
H3C 3P8
CANADA
Fax n°
514 987 8477
R33770@ER UQAM CA

Eugeny Kopelman
Hebrew University High School
Ruppin Str 1
Jerusalem
95435
ISRAEL
Fax n°

Tadato Kotagiri
12 27 ban 1- chome Kubota
Okinawa City Okinawa
904
JAPAN
Fax n°
098 895 6099

Konrad Krainer
IFF-Univ Klagenfurt
Sterneckstrasse 15
9020 Klagenfurt
AUSTRIA
Fax n°
0463 2700 759
KRAINER@EDVZ UNI-KLAGENFURT ADA AT

Koichi Kumagai
Joetsu University of Education
Yamayashiki 1 Joetsu shi
Nigata Ken
943
JAPAN
Fax n°
0255 26 8408

Chronis Kynigos
Kleomenous 19
Athens 10675
GREECE
Fax n°
301 7249000
ckynigos@ATLAS UOA ARIADNE T GR

Maria Olivia Lago Marcos Fax nº 3943189	Facultad de Psicologia Campus de Somosaguas 28223 Madrid SPAIN
Shukkwan Susan Leung Fax nº 886 5 2264114 law@math nsysu edu tw	Dept. of Math & Science Educ National Chia Yi Teachers's College Minghsiung Chia yi, Taiwan R O C CHINA
Paul Eduard Laridon Fax nº 011 339 1054 036PEL@WITSVMA WITS AC ZA	Mathematics Dept P.O Wits 2050 SOUTH AFRICA
Liora Linchevski Fax nº 972 2 617 673	Hebrew University Mount Scoups School of Education 91 905 Jerusalem ISRAEL
Arturo Gonzalez Larios Fax nº 6 886111	Nicolás Sn Juan # 1242 Colonia del Valle D F 03100 MEXICO
Romulo Lins Fax nº 0195 340123	Av 24 A 1515 Matematica 13500 Rio Claro - SP BRAZIL
Christine Janet Lawrie Fax nº 067 729593 charvey@ metz.une.edu au	Dept Science Technology and Mathematics Education University of New England 2351 Armidale AUSTRALIA
Asuncion Lopez Fax n 003434023602	Pza España nº5 Barcelona 08014 SPAIN
Brenda Lee Fax nº ? BLEE@ALPHA I WFC EDU TW	Night School Luu Feng Instit of Technol & Commerce Minhsiung Chia Y I, Twaiwan, Rep of China CHINA
Daniele Louzon Fax nº BP1795JERUSALEM	Chavtai a Negbi 11 26 Guilo Jerusalem ISRAEL
Roza Leikin Fax nº 97 2 4 325 445 TTRAVIV@Technion Technion AC IL	Dereh Yad Labanim Haifa 32808 ISRAEL
Eric Love Fax nº 0908 652140 e love@open.ac uk	Centre For Mathematics Education Open University, Walton Hall Milton Keynes MK7 6AA UK
Stephen Lerman Fax nº 071 815 7499 LERMANS@UK AC SBU VAX	Centre for Math Edu Southbank University 103 Borough road - London SE1 0AA UK
Patricia Ann Lytle Fax nº 514 848 2831 R27141@ER UQAM	Dept of Mathematics & Statistics Concordia Univers. Loyola Campus 7141 Sherbrooke St West Montreal Quebec H4B 1R6 CANADA
Uri Leron Fax nº 972 4 325 445 ttr0128@technion.bitnet	Dep.of ScienceEducation Technion Israel Institute of Technology Haifa - 32000 ISRAEL
Mollie MacGregor Fax nº 03 344 8739 MOLLIE_MACGREGOR@MAC UNIMELB.EDU.AU	Dept of Sciences & Mathematics Educ University of Melbourne Parkvillle VIC 3052 AUSTRALIA
Richard Lesh Fax nº 609 734 1090 DLESH@ ROSEDALE ORG	7 Madison St Princepton NJ 08542 USA
Carolyn Maher Fax nº 908 932 1318 cmaher@ math rutgers edu	30 Hey Ward Hills Dr Holmdel, New Jersey 07733 USA

Nicolina A Malara
Dipartimento di Matematica Università
Via G. Campi 213/B
41100 - Modena
ITALY
Fax nº
0039 59 370513
IMOV X: GREM

José Manuel Matos
R. Cândido Guerreiro, 4, 1º Dto
1000 Lisboa
PORTUGAL
Fax nº
294 1005
jmm@mail.fct unl pt

Joana Mamona-Downs
University of Macedonia
Economics and Social Sciences
Egnatias 156 P O Box 1591-Thessaloniki
54006
GREECE
Fax nº
031 858497

Doug McLeod
San Diego State Univ
San Diego
CA - 92182 - 0315
USA
Fax nº
619 594 1581
DMcLEOD@Sciences SDSU EDU

Helen Mansfield
Faculty of Education
Curtin University of Techology
Box U 1987, GPO Perth WA
6001
AUSTRALIA
Fax nº
619 351 2547
H Mansfiel@educ curtin edu au

Luciano Meira
Federal University of Pernambuco
Mestrado em Psicologia Cognitiva CFCH
8º Andar
Recife 50670-901 PE
BRAZIL
Fax nº
55 81 2711843
LMEIRA@COGNIT UFPE BR

Helena Marchand
Fac de Psicologia e Educação
Universidade de Lisboa
Alameda da Universidade
1600 Lisboa
PORTUGAL
Fax nº
7933408

Maria do Carmo Mendonça
Av Piassanguaba Nº 237
São Paulo S P
04060 OCC
BRAZIL
Fax nº
011 275 8242

Maria Alessandra Mariotti
Dipartimento di Matematica
UNIV di PISA
Via Buonarroti, 2 - PISA 512
56100
ITALY
Fax nº
39 50 599524
MARIOTTI@DM UNIPI IT

Ana Mesquita
6, Rue d Anglevene
59800 Lille
FRANCE
Fax nº
33 20 105454
Ana Mesquita@univ_lille1 Fr

Zvia Markovits
14 - A Gilboa St.
Haifa 32716
ISRAEL
Fax nº
972 4 239423

John David Monaghan
22 Batcliffe Mount
Leeds
LS6 3PW
UK
Fax nº
602 791506
TEZJM@TEN 2 NOTT AC UK

Lyndon Charles Martin
Math Ed Research Centre
Oxford University
15 Norham Gardens Oxford
OX2 6PY
UK
Fax nº
865 274027
LMARTIN@VAX OX AC UK

Cândida Moreira
R Dr Henrique Miranda, 149
4300 Porto
PORTUGAL
Fax nº

Pedro Martinez López
C/Islas Cíes Comunidad Cervantes 4
Almeria
04720 Aguadulce
SPAIN
Fax nº

Candia Morgan
Department of Mathematics
Statistics Computing Instit of Education
Univ of London 20 Bedford Way
WC1H 0AL
UK
Fax nº
071 612 6686
temscrm@uk ac ioe

John Mason
Maths Faculty
Open University
Milton Keynes
MK7 6AA
UK
Fax nº
44 908 652140
J H.MASON@OPEN AC UK

Ceri Morgan
I Meadow View, Okoys Lane
Fernhill Heath, Worcester
WR3 8 RL
UK
Fax nº
0203 523237

Malca Mountwitten
26 Even Shmuel St
Jerusalem
97230
ISRAEL
Fax n°
02 294054

Ana Paula Mourão
Rua Càndido de Oliveira n°62, 4° Esq.
4700 Braga
PORTUGAL
Fax n°

Judith A. Mousley
Faculty of Education
Deakin University Aeelong
3217
AUSTRALIA
Fax n°
61 52 272014
judym@deakin edu au

Hanlie Murray
Faculty of Education
University of Stellenbosch
7600
SOUTH AFRICA
Fax n°
021 8813616

Nicole Nantais
Faculté d'education
Universite de Sherbrooke
2500, boul de l'Université - Quebec
J1K 2R1
CANADA
Fax n°
819 821 8048

Elena Nardi
2 Monard Terrace
Denmark Street - Oxford
OX 4 1QS
UK
Fax n°
NARDI@OX VAX.UK.AC.

Ricardo Nemirovsky
Terc - 2067 Massachusetts Av
Cambridge
MA 02140
USA
Fax n°
617 3493535
RICARDO_NEMIROVSKY@TERC.EDU

Dagmar Elisabet Neuman
Fagerstrand 131
S 165H Hasselby
SWEEDEN
Fax n°
468 739 8884

Marilyn Nickson
Research Evaluation Division
Univ. Cambridge Local Exam. Syndicate
1 Hills Road Cambridge
CB1 2EU
UK
Fax n°
223 60104
MTN1001@phx. cam ac uk

Richard Noss
Dept of Maths, Statistics and Computing
University of London Institute of Educ
20 Bedford Way
WC1H 0AL
UK
Fax n°
71 612 6686
rnoss@ioe ac.uk

Rafael Núñez
Institute of Cognitive Studies BLDG TA
Univ of California at Berkeley
California
CA 94720
USA
Fax n°
415 723 0758
nunez@csli.stanford edu

Nunokawa Kazuhiko
Department of Mathematics,
Joetsu University of Education
Yamayashiki, Joetsu, Nigata
943
JAPAN
Fax n°
0255 26 8408

Minoru Ohtani
Institute of Education
University of Tsukuba, 1- 1- 1
Tenno - Dai Tsukuba city Ibaraki
305
JAPAN
Fax n°
0298 53 6619

Isolina Oliveira
Rua Cidade Beira 46 - 9° D
1800 - Lisboa
PORTUGAL
Fax n°
351 1 690731

Alwyn Olivier
Faculty of Education
University of Stellenbosch
7600
SOUTH AFRICA
Fax n°
27 21 8084499
aio@akad.sun ac za

German Muñoz Ortega
Nicolas San Juan n° 1421
Col del Valle
D F 03100
MEXICO
Fax n°
525 688 6111
Ecordero@CINVESMX BITNET

Jean Elizabeth Orton
18 West End Lane - Horsforth Leeds
W Yorkshire
LS18 5JP
UK
Fax n°
0532 334683
EDU6J0@LUCS_03 NOVELL LEEDS AC Uk

Ralph Pantozzi
154 Hana RD
Edison
New Jersey
08817-2045
USA
Fax n°
PANTOZZI@GRANDALF RUTGERS.EDU

Victor Parsons Fax n° 081 316 9235	3 The Knoll Hayes, Bromley Kent, BR 2 7DD UK	Angela Pesci Fax n° 0382 505602	Dept of Mathematics University via Abbiategrasso 209 27100 - Pavia ITALY
Dorit Patkin Fax n° 972 03 6476620	12 Barazani St. Tel Aviv 69121 ISRAEL	George Philippou Fax n° 00357 2 366741 edphilip@JUPITERcca ucy cy	University of Cyprus Dept of Education Nicosia P O Box 537 CYPRUS
Tasos Anastasios Patronis Fax n°	Department of Mathematics University of Patras 26110 Patras GREECE	David Pimm Fax n° 44 908 652 140 d j pimm@open ac uk	Centre for Mathematics Education The Open University Walton Hall, Milton Keynes MK6 7AA UK
Erkki Pehkonen Fax n° 358 0 1918073	Dept. Teacher Education University of Helsinki PB 38 (Ratakatu 6a) 00014 Helsinki FINLAND	Susan Pirie Fax n° 865 274027 PIRIE@VAX OXFORD AC UK	Math Ed Research Centre 15 Norham Gardens-Oxford OX2 9PY UK
Barbara Pence Fax n° 408 924 5080	Dept of Math & Computer Science San Jose State Univ CA - 95192 USA	João Pedro Ponte Fax n° 7573624 EJP@SCOSYSV FC UL PT	Calçada do Galvão 21 1400 Lisboa PORTUGAL
Sylvie Penchaliah Fax n° 27 31 8202866 sylvie@pixie.udw ac.za	University of Durran Westville Dept. of Psychology of Education Private Bag X54001 Durban 4000 SOUTH AFRICA	Dave Pratt Fax n° 0203 523237 serba@esv warwick ac uk	Dept of Science Education University of Warwick Coventry CV4 7AL UK
Maria Isabel Pereira Fax n° 039 721955	Av. Fernando Namora 105 1º Esq 3000 Coimbra PORTUGAL	Norma Presmeg Fax n° 904 644 1880 npresmeg@garnet.acns fsu edu	Mathematics Education, Box 3032 The Florida State University Tallhassee FL 32306 USA
José Gonçalves Peres Monteiro Fax n°	Rua José dos Santos nº 20 Bairro da Luz, Guarda 6300 Guarda PORTUGAL	Luis Radford Fax n° 705 675 4816 LRADFORD@NICKEL LAURENTIAN CA	School of Education Laurentian University Sudbury, Ontario P3E 2C6 CANADA
Marie-Jeanne Perrin-Glorian Fax n° 1 44275608 glorian@ mathp7.jussieu fr	6, Rue Einstein 92160 - Antony FRANCE	Gloria Ramalho Fax n° 8860954	Av 5 de Outubro 54 2E 1000 Lisboa PORTUGAL

Edward Redden
STME
University of New England Armidale
2351
AUSTRALIA
Fax nº
067 729593
tredden@metz.UNE.edu.au

Janine Rogalski
38 Rue Bezout
75014
FRANCE
Fax nº
1 49 406754

Maria Reggiani
Dipartimento di Matematica Università
Via Abbiategrasso 209
27100 Pavia
ITALY
Fax nº
39 382 505602

Leo Rogers
Mathematics Education Digby Stuart College
Roehampton Institute London
SW15 5PH England
UK
Fax nº
081 392 3231

Joseph Relich
University of Western Sydney Nepean
P O.Box 10 - Kingswood
2747
AUSTRALIA
Fax nº
61 47 360 400
J RELICH@UWSEDU.AU

Teresa Rojano
Depto Matematica Educativa - Cinvestav
Nicolas San Juan 1421
Col. del Valle Mexico
D F 03100
MEXICO
Fax nº
52 5 688 6111

Anne Reynolds
The University of Oklahoma Educ/ilac
820 Van Vleet Oval
Norman, OK
73019
USA
Fax nº
405 325 1498
da1234@UOKMUSA BACKBONE.UOKNOR

Isabel Romero
Facultad de Ciencias de la Educacion
Dept de Didactica de la Matemática
Campus Cartuja S/N Granada
18071
SPAIN
Fax nº
243949
iromero@ugr.es

Jovana Ferreira Rezende
Rua Visconde de São Lourenço 195
Jardim Guanabara
Ilha do Governador Rio de Janeiro
RJ 21 940-310
BRAZIL
Fax nº
55 21 290 1095

Andre Rouchier
383, Ruede la Source
45160 Olivet
FRANCE
Fax nº

Maria Judith Sperb Ribeiro
Rua Riachuelo, 1324 Apt/1104
Porto Alegre
RS 90 010 - 273
BRAZIL
Fax nº
?

Luisa Ruiz Higueras
Avenida de Madrid, 70, 1º B.
23008 - Jaen
SPAIN
Fax nº
953 212343

Luis Rico
C Gonzalo Gallas, 37 - 4º G
Granada
18003
SPAIN
Fax nº
34 58 203561
LRICO@UGR.ES BITNET

Kenneth Ruthven
University of Cambridge
Department of Education
17 Trumpington Street - Cambridge
CB2 1OA
UK
Fax nº
44 223 332876
KR18@PHX CAM AC UK

Naomi Robinson
Dept of Science Teaching
Weizmann Institute of Science Reovot
76100
ISRAEL
Fax nº
972 - 8 - 344115

Adalira Saenz - Ludlow
2825 Covington St
West Lafayette
IN 47906
USA
Fax nº
317 494 7938
adalira@PYCCVM

Marc Rogalski
38 Rue Bezout Paris
75014 Paris
FRANCE
Fax nº
33 1 44272555
mro@ccr jussieu fr

Khoo Phon Sai
Depto Science and Mat Education
University Brunei Darussalam
Gadong BSB 3186
BRUNEI
Fax nº
427003

Ana Maria Ojeda Salazar
Nicolas San Juan 1421
Col.Del Valle
Mexico D.F. - 03100
MEXICO
Fax nº
52 5 688 61 11

Lucilia Bechara Sanchez
Av. Manuel dos Reis Araujo 812
São Paulo
S P - 04664-000
BRAZIL
Fax nº
55 11 62 7536

Manuel Santos
Dakota 379
Col Napoles
Mexico D.F. - 03810
MEXICO
Fax nº
525 543 0713
Lsantos@nvaxi red.cirv

Vânia Santos
R. Ministro Correia de Melo
99 Apt 103
Leblon - Rio Janeiro
RJ - CEP 22 430 -110
BRAZIL
Fax nº
55 21 2901095
VANIA@VMS1 NCE UFRJ BR

Analúcia Schliemann
TERC
2067 Massachussets Av
Cambridge
MA 02140
USA
Fax nº
617 349 3535
ANALÚCIA_SCHLIEMAN@TER ED.

Thomas Schroeder
Suny at Buffalo L.A I Dept
Graduate School of Education
At Buffalo - Buffalo
NY - 14260 - 1000
USA
Fax nº
716 645 3161
SCHROEDE@UBVMS CC BUFFALO EDU

Baruch Schwarz
School of Education
Mount Scopus
Hebrew University Jerusalem
ISRAEL
Fax nº
972 2 882 045
msschwar@pluto mscc huji ac il

Yasuhiro Sekiguchi
Dept of Education
University of Yamaguchi
1677- 1 Yoshida
Yamaguchi-shi Yamaguchi 753
JAPAN
Fax nº
0839 23 8612

Michelle Selinger
43 Colesbourne Drive
Downhead Park
Milton keynes
MK15 9AN
UK
Fax nº
0908 652218
M S SELINGER@OPEN AC UK

Fernando Sereno
Rua João B Lavanha 143 - 3º Dto
4100 - Porto
PORTUGAL
Fax nº
02 52 0772
fsereno@ civp1.nac.up.pt

Anna Sfard
The Science Teaching Centre
Hebrew University
Givat Ram Jerusalem
91904
ISRAEL
Fax nº
972 2 585708
SFARD@VMS.HUJI AC.IL

Gilli Shama
8 Histadrute St.
Hod-Hasharon
45270
ISRAEL
Fax nº

J Malcolm Shield
School of Maths. Science & Tech. Ed.
QUT Locked Bag nº2
Red Hill
4059
AUSTRALIA
Fax nº
61 7 8643985
M SHIELD@QUT.EDU AU

Elizabeth Bezerra Silva
Rua Dr Tavares de Macedo 541201
Higienopolis - Rio
RJ - 21051
BRAZIL
Fax nº

Luis Claudio da Silva
Trav José Bonifácio, 17 apt 404
Barro Vermelho
São Gonçalo Rio de Janeiro
RJ-24 416 - 430
BRAZIL
Fax nº

Leopoldo Zuniga Silva
Nicolas San Juan nº 1421
Col del Valle
D F 03100
MEXICO
Fax nº
525 6886111
MAT_COGN@CINVESMX BITNET

Pinder Singh
59 B Chevening RD
London
NW6 6DB
UK
Fax nº

Martin Manuel Socas Robayna
Escuela Universitaria del Professorado
C/ Heracio Sanchez Nº 33
Universidad de la Laguna - Tenerife
38271
SPAIN
Fax nº
922 603544
MSDCAS@ULL ES

Lai-Leng Soh
Trinity College Oxford
OX1 3BH
UK
Fax n°
865 274027
LLSOH@VAX.OX.AC UK

Judith Sowder
287 Hihill Way
El Cajon, California
92020
USA
Fax n°
1 619 5941581
JSOWDER@SCIENCES SDSU.EDU

Kaye Stacey
Dept.of Science & Mathematics Educat.
University of Melbourne
Parkville
VIC 3052
AUSTRALIA
Fax n°
61 3 344 3739
KAYE_STACEY@ MAC.UNIMELB.EDU AU

Ruth Stavy
School of Education
Tel-Aviv University
Tel-Aviv 69978
ISRAEL
Fax n°
972 36409477
ruth@ccsg tau ac il

Ruti Steinberg
Rabina St. 15/1
Tel Aviv 69395
ISRAEL
Fax n°
972 3 6990269
dms@math.tau ac.il

Belina Steren
Prof. E. Marti - Departamento Psicologia
Evolutiva Y Educacion
Facultad Psicologia C/Adolf Fiorensa S/N
08028 Barcelona
SPAIN
Fax n°
93 3346833
MARTI@FARMACIA UB ES

Leen Streefland
Tiberdreef 4 - Utrecht
Frewdemthal Institute
3561 GG
THE NETHRLANDS
Fax n°
31 30660430
LStreefland@fi.ruu nl

Jonathan Stupp
Dep of Economics, Hebrew
University of Jerusalem
MT Scopus
91905
ISRAEL
Fax n°
00972 2 816071

Rosamund Sutherland
Dept. of Math, Statistics Computing
Institute of Ed. Uni. of London
20 Bedford Way London
WC1 0AL
UK
Fax n°
071 612 6686
R.Sutherland@ uk ac ioe

Kevan Swinson
250 Moons Lane
Brookfield
Queensland
4069
AUSTRALIA
Fax n°
61 7 864 3985
K.SWINSON@QUT.EDU.AU

David Tall
Mathematics Education Research
Centre University of Warwick
Coventry
CV4 7AL
UK
Fax n°
0203 523237
serat@csv.warwick.ac.uk

Howard Tanner
119 Penlan Crescent Uplanos
Hendrefuelan Swansea
SA2 0RL Wales
UK
Fax n°
44 792 298499

Margaret Taplin
University of Tasmania at Launceston
Centre Education
P O Box 1214
7250
AUSTRALIA
Fax n°
61 03 243048
M Taplin@educ utas edu au

Anne Teppo
1611 Willow Way
Bozeman, MT
59715
USA
Fax n°
Teppo@Mathfs.Math Montana edu

Lucia Tinoco
Praça Radial Sul 25 / C02
Rio de Janeiro
RJ 22260 070
BRAZIL
Fax n°
55 21 2901095

Kathleen Truran
P O Box 157
Goodwood
South Australia
5034
AUSTRALIA
Fax n°
618 302 5101
Trurank@salisbury.unisa edu au

John Truran
P O. Box 157
Good Wood
South Australia
5034
AUSTRALIA
Fax n°
jtruran@arts.adelaide edu au

Pesia Tsamir
12 Tirza St Ramat-Gan
52364
ISRAEL
Fax n°
972 3 65625918

Marianna Tzekaki
Department of Eavey Childhood
Education Avistotle
University of Thessaloniki
54006
GREECE
Fax nº
030 31 20 7379

Behiye Ubuz
Shell Centre Mathematical Education
University of Nothingham
NG7 2RD
UK
Fax nº
44 602 791813
Rsxbu@ unicorn nott.ac uk

Palanisamy Veloo
Dept of Science and Math Edu.
Faculty of Edu
Univ. Brunei Darussalam
Brunei Darussalam
SOUTH AFRICA
Fax nº
673 2 427003

Eduardo Veloso
Av D. Rodrigo da Cunha 11-3ºD
1700 Lisboa
PORTUGAL
Fax nº

Gerard Vergnaud
Lapsydee
46 Rue Saint Jacques
75005 Paris
FRANÇA
Fax nº
33 1 40462993

Shlomo Vinner
Science Teaching Department
Hebrew University
Givat Ram Campus Jerusalem
ISRAEL
Fax nº
972 2 585708
VINNER@HUJI VMS BITNET

Tad Watanabe
Dept of Mathematics
Towson State University
Towson
MD 21204
USA
Fax nº
410 830 4149
Watanabe_t@ towsonvx bitnet

Jane Watson
School of Education
University of Tasmania
7001 Hobart
AUSTRALIA
Fax nº
61 02 202569
Jane Watson@educ utas edu au

Grayson Weatley
219 Carothers Hall
Florida State University Tallahassee
FL 32306 - 3032
USA
Fax nº
904 644 1880
gwheatle@garnet ECNS EDU

Robert Wright
Centre For Education
Southern Cross University
P.O Box 157 Lismore
NSW - 2480
AUSTRALIA
Fax nº
61 66 221833
BWRIGHT@ALSUID.SCU.EDU.AU

Tracey Wright
TERC
2067 Massachussetts Ave
Cambridge
02140 - Massachusetts
USA
Fax nº
617 349 3535
Tracey _Wright@terc edu

Erna Yackel
Department of Mathematics
Computer Science, and Statistic
Purdue University Calumet Hammond
Indiana 46323
USA
Fax nº
219 989 2750
YACKELEB@PUCAL BITNET

Michal Yerushalmy
Mt Carmel School Of Education
Haifa 31905
ISRAEL
Fax nº
972 4 240911
REDC410@HaifaUVM

Yudariah Mahammad Yusof
Mathematics Education Research Center
Science Education Department
University of Warwick, Coventry
CV4 7AL
UK
Fax nº
0203 523237

Vicki Zack
5822 Einstein Ave
Montreal-Quebec
H4W 2Y6
CANADA
Fax nº
514 486 5638
VICKI_Z@CED STS SYSTE

Orit Zaslavsky
Dep of Education in Technology of Science Techion
Haifa 3200
ISRAEL
Fax nº
972 4 325445
TTRORIT@TECHNION BITNET

Rina Zazkis
Faculty of Education - Simon Fraser University
Burnaby, British Columbia
V5A 1S6
CANADA
Fax nº
604 291 3203
RINA_ZAZKIS@sfu ca

WORKING GROUPS

PME XVIII Working Group

PSYCHOLOGY OF INSERVICE EDUCATION OF MATHEMATICS TEACHERS: A RESEARCH PERSPECTIVE

Group Organizers Sandy Dawson, Simon Fraser University, Canada
Terry Wood, Purdue University, USA
Barbara Jaworski, Oxford University, UK

This is the sixth year that the group has been studying the role of the teacher educator in providing inservice for mathematics teachers. Last year at PME XVII in Japan, the group did not meet. However, at PME in Italy, the work of the group focused on the propositions selected for a framework on inservice education (INSET). In Portugal, the time allocated to the working session will be spent preparing the draft of the text for a book on INSET from an international perspective.

Previously, the group had decided to develop a book that would present a sharing of experiences and practices on inservice mathematics teacher education from an international perspective. As a group, it has been decided that the text will contain cases, incidents or vignettes, in order to offer a sense of the nature and needs of INSET internationally. We have identified 4 propositions which will form the framework for the book

Since the meeting in Italy, group members have been preparing contributions for this text. In Portugal, these preparations will form the basis for the work of the group. During the first meeting, subgroups will be formed based on the 4 propositions. Each subgroup will then work during the time allocated for the working group sessions to prepare a draft of the book on INSET.

Group contact: Sandy Dawson, Simon Fraser University, Faculty of Education, Burnaby, British Columbia, V5A 1S6

Contributions : Terry Wood, Dept. Curriculum & Instruction, 1442 LAEB, Purdue University, West Lafayette, IN 47907 1442.

WORKING GROUP ON

CULTURAL ASPECTS IN THE LEARNING OF MATHEMATICS

The fact that mathematics curricula appear to be remarkably similar all over the world makes it possible to conduct international surveys such as those conducted by the International Association for the Evaluation of Educational Achievement (IEA).

At the same time, numerous recent studies question the nature of mathematics; the learning of mathematics is considered not only to be the learning of a body of knowledge but also a cultural process.

Various debates have arisen about the following points:

- The interactions between mathematical skills and culture-specific representations.
- The universal and non-universal aspects of cognitive development
- The nature of the effects of systems of signs (including language) on psychological functions: general or specific effects (Vygotski's theory questioned).

In the previous meetings at PME XVI and PME XVII, we identified three research directions which are consistent with our interests:

- Informal education and formal mathematical knowledge
- The effects of cultural environment, including language, on the mental representations of students and teachers
- Cognitive processes in learning mathematics, using a comparative approach for different cultures.

At PME XVIII in Lisboa, we intend to consider some definitions of the concept of culture and how they are expressed in our specific domain of the learning of mathematics.

One of them was offered by Geertz in *The Interpretation of Cultures* (1973).

Geertz considers the *culture concept:*

> *an historically transmitted pattern of meanings embodied in symbols, a system of inherited conceptions expressed in symbolic forms by means of which men communicate, perpetuate, and develop their knowledge about and attitudes toward life.*

Rotative panels will be devoted to the above-mentioned subjects about the current debates, taking into account the cultural context of the country in which PME is held. Then we should finalize the continuation of the work of the group for the next two years.

Coordinator : Bernadette Denys

RESEARCH ON THE PSYCHOLOGY OF MATHEMATICS TEACHER DEVELOPMENT

The Working Group *Research on the Psychology of Mathematics Teacher Development* was first convened as a Discussion Group at PME X in London in 1986, and continued in this format until the Working Group was formed in 1990. This year, at PME 18, we hope to build on the foundation of shared understandings that have developed over the last few years.

Aims of the Working Group

The Working Groups aims to:

- develop, communicate and examine paradigms and frameworks for research in the psychology of mathematics teacher development;
- collect, develop, discuss and critique tools and methodologies for conducting naturalistic and intervention research concerning the development of mathematics teachers' knowledge, beliefs, actions and reflections;
- implement collaborative research projects;
- foster and develop communication between participants;
- produce a joint publication on research frameworks and methodological issues.

Plans for Working Group Activities at PME in 1994

A striking feature of the Working Group for Research on the Psychology of Mathematics Teacher Development in Japan in 1993 was its cohesiveness. It is hoped that, through the activities of the Group in 1994, the dynamics of the Group can be continued and expanded.

Group members have expressed the need to have a deeper understand of a range of research methodologies which are particularly appropriate for research in the area of mathematics teacher development. Several approaches will be shared and critiqued within the Group, and will form the nucleus of the publication "Effective Methods for Achieving and Monitoring Change in Mathematics Teacher Development: An International Perspective." The focus of the publication would be on the collection, development, discussion and critiquing of paradigms and frameworks for research in the psychology of mathematics teacher development.

At PME in 1994, a summary of different strategies being used in different parts of the world for mathematics teacher development will be given, and research on the effectiveness of these will be summarised.

A start is to be made on the development of a reading/reference list which would be an ongoing activity for the Group, and possible collaboration on various research projects will be explored.

Nerida F. Ellerton, Convenor

WORKING GROUP ON SOCIAL PSYCHOLOGY OF MATHEMATICS EDUCATION

JEFF EVANS, Middlesex University, ENFIELD, Middx. EN3 4SF, UK
LEO ROGERS, Roehampton Institute, LONDON SW15 5PH, UK

The Working Group did not meet at PME XVIII in Japan. The programme for Portugal is planned to cover four 90 min. sessions as follows:

Session 1: Origins and Re-organisation of National Curricula (LR)

Superficial comparisons of mathematics curricula may be made in terms of statements of content, specifying in what year, if at all, a particular topic appears in a country's curriculum. Much more useful is a comparison such as the recently published IEA studies which begin to look at the contexts and motivations behind the "intended" curriculum in different countries. A study of the comparison between English and Spanish mathematics curriculum will be briefly presented, and some criteria for analysis proposed.

In this session, participants will be able to introduce themselves, and to report on the situation in their own countries.

Session 2: Children's Conceptions and Strategies in Designing (Tasos Patronis, Dept. of mathematics. Univ. of Patras, GREECE)

Our research group has tried to create a classroom activity that integrates physical, emotional and cognitive components of children's reality (d'Ambrosio, 1984, ICME-5) in the curriculum. Children of a rural primary school were asked to design and construct a model of a village in which a team of foreign children were supposed to spend their holidays. This "modelling activity" (in a broad sense) provides children with the opportunity to move from the real world to a "real model" by using mathematics and physics ideas at an implicit level.

Session 3: Alternative curricula for Democratic Citizenship (JE)

Many teachers of mathematics aspire to a curriculum which provides the basis for democracy, for "responsible citizenship", etc. What would such curricula look like: would there be any real maths? Or would it more resemble statistics? What problems might arise, e.g. in the "transfer" of school learning to applications in civic life, similar to the problems of applying "vocational" curricula at work.

Session 4: Looking ahead

Links with the Cultural Aspects working group.
Proposals for a name change; e.g. "Social Apects of Maths Ed.";
"Social and Political Aspects...";
"Social and Political Issues in..."
What might usefully be done between now and the next PME?

GEOMETRY WORKING GROUP

In 1993 the Geometry Working Group started a new period of activity, aimed to analyze the teaching and learning of Geometry on the light of results from recent research and of current trends in Mathematics Education. The first meeting of this period (Tsukuba, Japan, 1993) was devoted to an overview of the field, with presenters who talked about different components of the current problematic of Geometry Education (see the November 1993 PME Newsletter). The coming meetings, from this year, shall be devoted to monographic themes to be analyzed in depth from several points of view.

This year, the meeting of the Geometry Working Group shall be devoted to the theme of Geometry Problem Solving. Several approaches can be considered. For instance:

- From a classical point of view, Polya's classification of geometric problems in problems of construction and problems of proof, together with his recompilation of methods and heuristics for solving problems, form a framework that continues being useful for understanding the students' activity.

- However, current tendencies are also paying attention to the several types of processes that happen during the resolution of geometric problems, and to issues related to the curriculum and the position of problems in it.

- Traditional Geometry problem solving was based on constructions with ruler and compass and figures drawn on paper. However, the everyday more frequent use of computers in schools, and the development of specialized software rise new possibilities and new research questions that have to be analyzed.

And, hopefully, some other approaches shall arise during the meeting.

Angel Gutiérrez; Coordinator of the Geometry Working Group
Dpto. de Didáctica de la Matemática
Universidad de Valencia
Apartado 22045; 46071-Valencia (Spain)
Fax: 34 - 6 - 3864487
E-mail: gutierrez@mac.uv.es

The PME Working Group on Representations

Group organizers: **Claude Janvier, Université du Québec à Montréal**
Gérard Vergnaud, C. N. R. S., Paris

Preliminary statement

Representations are key theoretical constructs in the psychology of mathematics education. For the purposes of the working group, the meaning of this term is quite broad. It includes:

- External structured physical situations or sets of situations, that can be described mathematically or seen as embodying mathematical ideas. External physical representations range from pegboards to microworld.
- External structured symbolic systems. These can include linguistic systems, formal mathematical notations and constructs, or symbolic aspects of computer environment.
- Internal representations and systems of representation. These include individual representations of mathematical ideas (fractions, ratios, proportions, functions...) as well as broader theories of cognitive representation that range from image schemata to heuristic planning.

The group does not promote any particular theory of representations even though some of its participants may. It is essentially eclectic since it is open to any coherent "chunk" of principles or concepts, or theory in construction that can display a fair amount of "predictivity" in particular domains without being generalizable.

At Lisbon, come and check your theory against facts

As in the previous meetings, participants are invited to make presentations and discuss the issues they raise. A newsletter was sent to the regular participants with this intention in mind. Anyhow if you think you have some interesting things to report, get in touch with us.

Otherwise, you may simply want to check out your theoretical constructs against the supremacy of reality or you may be willing to see others becoming predictive on the basis of the constraints of a situation. Then we think we have what you desire: hard facts and the conditions prevailing during their production. We have, indeed, asked many people to bring in their luggage some experimental results that still need interpretations. Why not come and give yours?

Details and news

The Working Group was guided by Gerald A. Goldin from the 1989 conference in Paris . Participants were kept informed by a yearly newsletter. Over, the last years it has attracted roughly 50 participants at each conference. The idea of publishing a special issue of the Journal of Mathematical Behavior on the basis of the activities of the group will soon be a reality. Articles have been submitted and the referees at the time of the conference will have completed their work. More details to come.

Working Group: Teachers as Researchers in Mathematics Education

Co-ordinators: J. Mousley, V. Zack, C. Breen

This group has been meeting annually since 1988 and as a working group since 1990. It aims to discuss issues surrounding the theme of teachers becoming researchers in their own classrooms. Over the past few years, the group has also focused on teachers educators researching their own work and on collaborative, participant research methods.

People who attend the Teachers as Researchers Working Group for 1994 will review their own work, and that of other teachers, in attempting to understand and change classroom practices. Activities will include a small group discussion about how different "understandings" arise from the use of various research methodologies, as well as from different beliefs about mathematics, and about teaching and learning.

A variety of methods of collecting and using information about our own classrooms, in schools and in a variety of teacher education situations, will be discussed. Current and new members of the group will have the opportunity to give short presentations about any examples of formal or informal attempts to improve teaching which involves teachers in the research process, as well as to contribute to the group's activities throughout the coming year.

It is expected that the themes raised each year in our discussions, such as the positioning of teacher-researchers, reflective practice, the examination of individual and social pedagogical beliefs, support for collegiate research, and the facilitation of participatory research will be taken up again in Lisboa.

PME Working Group
Algebraic Processes and Structure

Coordinators: Teresa Rojano and Rosamund Sutherland

This Group was set up in PME XIV with the aim of characterising the shifts that appear to be involved in developing an algebraic mode of thinking and to investigate the role of symbolising in this development. The group is also concerned with the implications of their collective research for classroom practice.

The main issues discussed at PME XVII were: the necessity of distinguishing between the nature of mathematics and the nature of algebra; the danger of separating algebraic knowledge from symbolism when trying to characterise algebraic thinking: the importance of incorporating methods of solving problems as a main component to analyse the nature of algebra: the algebraic knowledge as such of school algebra, since there is always the possibility to include "othe algebras" like permutations of Boolean algebra; the role that computer environments might play when talking about an algebraic mode of thinking not necessarily linked to symbolic manipulation; the (semiotic) difficulties of passing from one type of representation to others, for example, geometric language: the shift from arithmetic to algebra and the question, 'if there is a hierarchy of growth in this shift, what role does natural language have in such growth?': the widespread tendency to relate the cognitive structure hierarchy to the sign system used to express the cognitive structure: the ways of using a determined sign system to express cognitive processes in algebraic thinking; the tendency in some psychological research works to see only the structures in the cognitive domain without incorporating the use of socially established codes.

The group are currently working on an edited book and draft chapters are being prepared before the meeting in Portugal. The work of the group at PME XVIII will centre around discussion and redrafting of these chapters, organised around four subgroups.

Research on Ratio and Proportion

During PME XVIII the working group on ratio and proportion will continue to explore the development of children's understanding of concepts of ratio and proportion. In addition, this year we will explore teachers' understanding of these concepts, and the necessary changes in teacher preparation and professional development programs to bring about an appropriate knowledge base for effectively teaching proportional reasoning.

The first session will be devoted to a discussion of the recent work of working group members in the area of children's learning of ratio and proportion concepts. New members will also be welcome to give brief presentations on current activities. This session will also give people a chance to introduce themselves and plan the remaining sessions in greater detail.

The second session will focus on teacher knowledge of proportionality and related concepts. One or more invited speakers will give brief presentations for discussion by the working group. We will discuss implications for the preparation of teachers.

Since concepts of ratio and proportion are part of the complex of concepts and skills Vergnaud refers to as the Multiplicative Conceptual Field, we will attempt to extend our discussion in the third session to connect ratio and proportion to related topics; rational numbers concepts and operations, quantitative reasoning, and so on.

The final session will be devoted to planning for future work by the Working Group, particularly for PME XIX in Brazil in 1995.

Anyone interested in participating and in brief presentations related to the session topics should attend the first session and/or talk to Judy Sowder.

Contact: Judy Sowder, San Diego State University, jsowder@sciences.sdsu.edu

ADVANCED MATHEMATICAL THINKING

Organisers: David Tall (U.K.), Gontran Ervynck (Belgium)

The group is concerned with all kinds of mathematical thinking of students beyond the age of 16, extending and developing theories of the psychology of Mathematics Education that cover development of mathematics over the full age range.

SESSION I: The Role of the Computer in learning advanced mathematical Concepts.

Initiators: John MONAGHAN (U.K.), Joe WIMBISH (U.S.A.), Thomas LINGEFJÄRD (Sweden).

Computing equipment is increasingly available in schools and universities. How does it affect the learning and understanding of advanced mathematics ? The computer can be used as a tool, introducing an opportunity for "experimental math"; doing the tedious calculations and displaying graphical representations it furthers understanding of the related concepts. But how does a computer help students to conceptualize and construct for themselves mathematics that are already completely formalized by others ? The use of symbolic manipulators is part of this topic as well. The relationship between the computer and the development of proofs will be discussed in the second session.

SESSION II: The Computer in Mathematical Proof.

Initiators: Gary DAVIS (Australia), Gila HANNA (Canada), Adrian SIMPSON (U.K.).

Recalling a statement of M. Atiyah (1984): "In mathematics there are several stages involved in a discovery, and formal proof is only the last. The early stage consists in the identification of significant facts, their arrangement into meaningful patterns and the extraction of some plausible law. Next is the process of testing this proposal against new facts, and only then does one consider a formal proof". We want a better understanding of the role of the computer in the stages preceding the construction of a formalized proof. E.g. we may raise the question if the views of Lakatos on the nature of proof have to be modified according to the role of computers ? This session may also be seen as a continuation of the work in the AMT Group at PME-17 (1993).

SESSION III:The *Psychology* of Advanced Mathematical Thinking, and Future plans.

Initiator: David TALL (U.K.)

The final session will focus attention on what psychological evidence may tell us about advanced mathematical thinking, the construction of formal concepts, relationships between visual and verbal/symbolic codings, cognitive aspects of proof. There will be opportunities to relate back to the earlier sessions.

The final half hour will be devoted to future plans.

PME Working Group

Classroom Research

The purpose of this group is to examine issues and techniques associated with classroom research and the impact of such research on educational conditions.

The focus of discussion at PME XVIII will be centered around the uses and limitations of innovative technology to enhance data collection and analysis in classroom research. Presentations by invited speakers will be made describing the following research studies. These presentations will be used as a springboard for active participant discussion of related issues.

Multiple, coordinated sources of data collection

A portable and inexpensive system will be described that uses audio and video taping coordinated with supportive software to investigate classroom learning. This system makes it possible to quickly access recorded data to facilitate post-classroom interviews.

Interactive multimedia

The use of and implications of using interactive multimedia in classroom research will be described. This technology consists of a CD-ROM disk with computer access, and involves the creation of QuickTime sequences of collected videos and the creation of a HyperCard interface. Reference will be made to the use of CD-ROM disks of children engaged in the active construction of mathematical ideas for teacher training and classroom research.

Qualitative data sorting, indexing, and theory building

The use of the computer application software NUDIST to search, index, and manage qualitative data will be discussed. Such software not only assists in structuring data but in developing educational theory. The potentials and problematics of theory building will be discussed with particular reference to how NUDIST is currently being used in a research project entitled "Quality Teaching in Mathematics Education."

Contact person: Anne Teppo, 1611 Willow Way, Bozeman, MT 59715, USA

DISCUSSION GROUPS

Cabri-geometry

Organizer: Nicolas Balacheff

PME 17 in Tsukuba gave an opportunity to start a discussion group specifically devoted to geometry in connection with computer environments, mainly (and not exclusively) based on the example of Cabri-géomètre.

Recent period actually shows a renewal of interest in geometry and the role devoted to geometry in the curricula of most of the countries around the World. Just to mention inside PME the Geometry Working Group, where active discussion has been taking place for many years.
One intriguing question stems from the irruption of computers in the classroom and at home and today several software exist that are used for geometry teaching and learning. Among them LOGO, the Geometry (Super)Supposer, Geometry Inventor, Geometer's Sketchpad and Cabri-géomètre, all share a more or less explicit relationship to the concept of "Discovery learning".

Questions to be addressed in the discussion group range as follows:

What is and what could be the **use of Cabri-type software** in classroom and outside classroom?
What could be the long term **changes in the geometry curricula** induced by the development of computer based teaching learning environments?
What are innovative uses of Cabri-type environments for exploring not purely geometry centred situations? (There is a growing interest in **modelisation using geometry**, e.g.in mechanics, optics, statics, etc.).

All the preceding issues should be examined according to different contexts
• the **traditional context of classroom**, where Cabri-type software can be used as an aid for proving geometrical facts. In this context, Cabri-géomètre is often used in conjunction with an overhead projector, as a demonstration program with great flexibility;
• the context of organisation of **problem-solving** sessions in a exploratory environment, where students work in small groups in front of and in interaction with a computer;
• the context of **mathematical research**: exploring new fields in geometry can lead to new results and insights for proving them. Cabri-géomètre and other computer based environments allow us a renewed attack on subjects of some complexity, which could not be investigated until now due to the lack of suitable tools;
• the context of production of **pedagogical documents**, made up of figures accompanied by text, to be used with or without Cabri-géomètre;
• the context of **research in didactique**: Cabri-géomètre can be used to analyse, study and compare the ideas and constructs of students in geometry, as produced in a standard pencil-and-paper situation or in the Cabri computer environment, or as the result of an interaction between both.

The PME 18 meeting of the discussion group will also give an opportunity to introduce a discussion about the new Cabri-geometry II which is to be released in August 94.

Discussion Group: **"Vygotskian Research and Mathematics Learning"**
Convenors: Kathy Crawford & Steve Lerman

The aim of the discussion group is to examine the contribution of Vygotsky and some of his compatriots and the implications of their theory of learning in a socio-cultural context for mathematics education. The assumptions underlying Vygotsky's position differ in several ways from the tacit philosophical and psychological position of the mathematics education community. In particular Vygotsky challenges the centrality of the individual in meaning-making and insists on a social ontogeny of consciousness. In addition his position transcends traditional Cartesian dualities such as self/other, mind/body, feeling/thinking and subject/object. His historical-cultural method of research differs significantly from predominant methodologies which typically focus on a part of the learning situation.

The discussion group will be collaborative and in keeping with the tradition of acknowledging the cultural histories of the people there. Participants should come prepared to make a contribution.

In the first session we will negotiate an agenda and then work together to interpret some aspects of Vygotsky's work. Some particular points of discussion might be:

The identification of points of difference between Vygotsky's theory and other research paradigms.
Practical application of the "historical cultural method"
Mediation
Activity
Dialectic thinking
Zone of proximal development
Concept (as a verb)

Discussion in the second session may centre around research questions that are raised by Vygotsky's work and issues of research design.

Mathematics classrooms as complex adaptive systems

A. J. (Sandy) Dawson, Simon Fraser University, Canada
Eric Love, Open University, UK
John Mason, Open University, UK

Current writings on the nature of mathematical learning directs attention towards cultural and social aspects of human life. This is exciting and challenging work for researchers in mathematics education. Equally provocative work is occurring in the areas of biology (e.g., Maturana and Varela), economics (e.g., Anderson, Arthur), complexity theory (Jen, Nocolis & Prigogine, Waldrop), chaos theory (e.g., Kauffman), computer science and AI (e.g., Langton et al), and the theory of coevolution and codetermination (e.g., Campbell & Dawson, Holland, Varela et al). The purpose of this discussion group would be to explore the possible connections amongst recent results in these various fields with a view to enriching and broadening the theoretical bases and research programs in mathematics education.

One of the results of recent research in those fields indicates that evolution is not just a result of random mutation and natural selection, but equally if not more importantly, it is the product of emergent behaviour and self-organization. Moreover, it is not as if a species or individual member of a species, evolves on its own; rather, species seem to co-evolve within a large system.

> Ecosystems, economies, societies--they all operate according to a kind of Darwinian principle of relativity everyone is constantly adapting to everyone else (Waldrop, p 259)

A area of interest to mathematics education researchers would be to view classrooms as societies, and the individuals within could be seen as coevolving together How might the use of this coevolutionary perspective enrich current classroom research by providing concepts and scaffolds from biological and coevolutionary research paradigms?

Complex adaptive theory suggests that complex behavior (such as is found in classrooms) need not have complex roots. Complex behavior can and is generated from simple roots Not only can very complex behavior arise from simple roots, but the development which occurs may take many different forms It is not possible to predict what the outcome will be from a simple set of roots Applied to the teaching and learning of mathematics, this result has profound implications for the outcomes teachers might expect from their lessons and the behavior they hope their students will display These implications would be explored in the discussion group

But recent results in the fields noted at the outset indicate that whatever behavior appears at a particular time is dependent upon events which have preceded it - yet the emergent behavior is *not* predictable from those events.

> . each emergence sets the stage and makes it easier for the emergence of the next level (Waldrop, p 297)

These results place teachers (of mathematics) in a difficult situation On the one hand, if students are to learn concept *y*, say, and if this might be facilitated by first learning concept *x*, then it would seem appropriate to teach *x* before *y* On the other hand, if students' learning is not predictable, then there is no guarantee that teaching *x* will make it easier (or even possible) for students to learn *y* And to further complicate the situation, the teaching of *x* may lead to some totally unexpected behavior Given these results, it would seem that classroom research which attempts to understand how teachers accomplish their goals, and how learners come to understand mathematics, needs at the very least to consider very carefully its methodology and the generalizability (if not validity) of its outcomes.

Life at the edge of chaos (Lewin) is a phrase increasingly common to discussions in the fields noted at the outset Those using this phase are attempting to describe a region located between the state where a system *dissolves into chaos*, where conditions are so fluid that no patterns are discernible, and a state where a system has become completely stable and in which it exhibits no fluidity at all The conjecture is that this region, beyond unchanging stability but just shy of random chaos--*the edge of chaos* - is where learning takes place What implications does the concept of *learning at the edge of chaos* have for the teaching, learning and researching in mathematics education?

Organizationally, the first discussion session would (1) provide a brief overview of recent developments in the fields noted (15 minutes), and (2) through a series of focused questions and by working in small groups explore the possible relevance of these developments for research in mathematics education (60 minutes), so as to (3) share the outcome of small group discussions with the goal of providing directions for further explorations (15 minutes) The second session, again working in a mixture of small and large groups, would take the results from session one and develop plans for action and communication to take place subsequent to the PME meeting

POST-STRUCTURALISM DISCUSSION GROUP

Organisers: Paul Ernest (UK) and Tony Brown (UK)

Post-structuralism is a loosely defined movement or set of perspectives that looks critically at structuralist theories. It critiques structuralism in the form of Jean Piaget's structural psychology, Claude Levi-Strauss's structural anthropology, and Roman Jacobsen's structural linguistics, for example. The critique argues that these and other structural theorists overstress static structure at the expense of personal agency, and ignore the contingent and historically shifting nature of the structures of human thought. A central figure in post-structuralist thought is the philosopher and historian of ideas Michel Foucault. More recently, a seminal post-structuralist critique and contribution to psychology is contained in Henriques et al (1984).

In the past few years, a growing number of researchers in the psychology of mathematics education have adopted or utilised post-structuralist perspectives. Perhaps the best known is Valerie Walkerdine who gave a plenary talk at PME 14, and is well known for her post-structuralist analysis of young children's language and reason (Walkerdine, 1988). Other recent papers have reflected or drawn upon a post-structuralist perspective, such as those of Jeff Evans (PME 17), Clive Kanes (PME 15) David Pimm (PME 14) and Paul Ernest (PME 15, 17) Thus post-structuralism is a part of PME thought, and needs to be discussed more fully.

Post-structuralism offers a number of important insights for the psychology of mathematics education. It stresses the import of social context, that of power and positioning in inter-personal relations, the central role of discourse, language and text, and the problematic and multiple nature of the learner or cognising subject. Currently these issues are topics of central empirical and theoretical interest in the psychology of mathematics education. Thus a discussion of the post-structuralist contribution is important and apposite.

It must also be recognised that post-structuralism is a controversial and problematic perspective. Apart from often being expressed in obscure language, and being as yet a nascent and incompletely worked out position, it also challenges many deeply held (and felt) assumptions widespread in the psychology of mathematics education. Thus it is appropriate to consider what post-structuralism might offer the psychology of mathematics education in undogmatic fashion in a discussion group. The group will proceed by short lead contributions followed by open chaired discussion on themes such as the following.

- **The nature and contribution of post-structuralism** Orientating over-view. Discursive practices. Power. Critiques of constructivist and cognitivist conceptions of the subject.
- **Critiques of post-structuralism.** The under emphasis on self, individual construction, mental structure. Constructivist critiques.
- **Post-structuralist research in the psychology of mathematics education.** Current work and perspectives. Discourse analysis. New potential sites: e.g semiotic analysis.

References

Henriques, J. *et al.* (1984) *Changing the Subject*, London: Methuen.

Walkerdine, V. (1988) *The Mastery of Reason*, London: Routledge.

Psychological Foundations for Systemic Reform in Schools

Richard Lesh

Psychological Foundations for Systemic Reform in Schools

In the USA, current curriculum reforms emphase systemic initiatives. But, how is systemic reform different from earlier initiatives based on mechanistic conceptions of mathematics, problem solving, learning, and instruction? This session will focus on ways that new curriculum reform efforts influence (and are influenced by) conceptions about what it means to understand and use mathematics in "real life" situations.

Current educational practice is based on a coherent set of ideas (about goals, knowledge, work, and technology) derived from "scientific management" principles from the industrial revolution. Consequently, schools are not likely to change unless these ideas are replaced with equally coherent "world views" in which knowledge is likened, not to a machine, but to a living organism. This discussion session will discuss why a similar shift in metaphors also must apply to the way students, teachers, and programs are assessed. The following themes will be addressed.

- Results from adult literacy examinations which emphasize current conceptions about basic mathematical literacy requirements for all citizens. (Saundra Young, Adult Literacy Program, Educational Testing Service)
- Principles for writing effective "performance assessment" activities which emphasize: (i) deeper and higher-order understandings of the most powerful constructs in elementary mathematics, and (ii) problem characteristics which are typical of situations in which mathematics is used in a technology-based society in an age of information. (Richard Lesh, Principal Scientist, Educational Testing Service)
- Characteristics of teachers' tools (e.g., response interpretation procedures, process observation forms) which are useful to identify conceptual strengths and weaknesses of individual students. (Bonnie Hole, Princeton Research Institute for Research on Science and Maathematics Learning)
- Examples of students' responses which illustrate why the preceding problems emphasize a broader range of mathematical abilities (and students) which were not recognized and rewarded by traditional tests, textbooks, and teaching. (Eamonn Kelly, Assoc. Professor, Rutgers University)
- Examples showing how performance assessment activities for students provide the basis for activities in which teachers are able to (simultaneously) learn and document what they are learning. (Miriam Amit, Math Curriculum Director, Israel).
- Principles for assessing the quality of emerging research methodologies to explore students' and teachers' thinking in complex situations. (Barbara Lovitts, Director, National Science Foundation Program for Research on Teaching & Learning).
- New models for program accountability using performance assessment activities. (Madeleine Long, Director, NSF Program on Systemic Initiatives).

VISUALIZATION IN TEACHING - LEARNING SITUATIONS

M. Alessandra Mariotti
Dip. Matematica
Università di Pisa - Italy

Angela Pesci
Dip. Matematica
Università di Pavia - Italy

Everybody agrees that images have a basic role in thinking, but so far, the mental processes involved have not yet been fully clarified.
Different and opposite positions can be found in the literature about the contribution of images in a variety of situations.
Of course exploring general processes and mechanism would be very interesting, but it is worth limiting our aim and considering the problem from the specific point of view of mathematical education, which is the domain we are interested in.
Thus, our discussion aims to focus on specific aspects in the particular school situation.
There are two main hypotheses about the role of imagery in education that we want to state explicitly, in order to be discussed and explored.

1. Thinking is spontaneously accompanied and supported by images, we usually call it 'visualization'.
The following can be considered a shared opinion among the psychologists "thinking makes use of representations, some of which are produced by imagery processes, and some by more abstract representational systems" (Denis, 1991, p. 104). Literature is very rich in this field, but "probably the greatest obstacle to imagery research is that the process it aims at specifying can be used in a wide variety of cognition situations" (Denis, 1991, p. 123).
On the other hand, it can be considered common opinion in the field of Math Education that "the cognitive manipulation of mathematical concepts is highly facilitated by the mental construction and availability of adequate image schemata" (Dörfler, 1991, p. 20).

2. It is possible to intentionally intervene on the process of 'visualization'.
For some respect, 'visualization' can be considered a subjective phenomenon, this idea agrees with everyday experience of a great variety in pupils' performances related to imagery.
This opinion may lead to neglect or even discard the possibility of direct interventions to promote pupils' imagery in mathematics.
On the contrary - together with many other researchers - we are convinced in the possibility of the positive influence of a purposeful teaching.
Among others the following questions seem meaningful to us:
- What kind of 'external images' should be used in school practice?
- Are there general criteria to select visual aids in math teaching?
- What is the relationship between external images provided by teachers and internal images which run in pupils' minds?
- When specific visual aids are introduced, how is it possible to check their influence on pupils' reasoning, both in problem-solving and concepts formation?
- How does teachers' opinion affect their attitude towards images in teaching practice?

Several research projects in this field have already started, we wish to promote a discussion and a cooperation among all interested people, contributing from different points of view.

References

Dörfler, W., (1991) Meaning, image schemata and protocols. *Proceedings PME XV*, Assisi, Vol. 1, 17-32
Denis, M. (1991) Imagery and thinking, in Cornoldi, C. & McDaniel, M.A. *Imagery and Cognition*, Springer-Verlag

Research on Learning Mathematics and Programming

Proposal for a Discussion Group at PME 18, Lisbon.

Richard Noss (University of London)
Doug Clements (University of New York at Buffalo)

The idea that learning to program a computer may have some beneficial effect on the learning of mathematics is now some thirty years old. The early eighties brought forward a flurry of research, largely catalysed by the appearance of *Logo* but this has largely subsided. Yet there is now a renewed interest in the question, not least because the notion of what it means to program is in a process of radical change The group will address a number of related questions:

- What does the research effort indicate so far?
- What issues have proved fruitful to research?
- Which methodologies have proved appropriate for the investigation of these issues?
- What are the implications for mathematics education?
- Where should research go in the future?

USING OPEN-ENDED PROBLEMS IN MATHEMATICS CLASS

Erkki Pehkonen, Dept Teacher Education, University of Helsinki (Finland)

The method of using open-ended problems in classroom for promoting mathematical discussion, the so called "open-approach" method, was developed in Japan in the 1970's (Shimada 1977). For example in the paper of Nohda (1991), one may find a nice description of the paradigm for the open-ended approach. This discussion group began last year in Japan (Pehkonen 1993), where the topic of discussions was the concept "open-ended problem" and its classroom usage.

In the group, we concluded that open-ended problems pertain to a larger class of open problems (i.e. problems with openness in the initial or goal situation). Furthermore, open problems contain e.g. problem posing, project work, and most real life problems. The presentations of the discussion group will be published in the International Review on Mathematical Education (in Germany known as ZDM).

In the sessions of this year, the discussion group will focus on the research results obtained around open-ended problems. There will be again some brief presentations (about 10–15 min) from different countries, in order to give some starting points for discussion. The main questions will be "What scientific knowledge do we have from open-ended problems?" and "Are there some under-presented fields on which we should focus our research?".

References

Nohda, N. 1991. Paradigm of the "open-approach" method in mathematics teaching: Focus on mathematical problem solving. International Reviews on Mathematical Education 23 (2), 32–37.

Pehkonen, E. 1993. Using open-ended problems in classroom. In: Proceedings of the seventeenth PME conference (eds. I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin). Vol. I, 104. University of Tsukuba, Tsukuba (Japan).

Shimada, S. (ed.) 1977. Open-end approach in arithmetic and mathematics – A new proposal toward teaching improvement. Tokyo: Mizuumishobo. [in Japanise]

SHORT ORAL COMMUNICATIONS

DILEMMAS AND PARADOXES: MATH TEACHERS' AWARENESSES OF TEACHING AND LEARNING MATHEMATICS IN MULTILINGUAL CLASSROOMS.

Jill Adler

Education Department, Witwatersrand University, South Africa.

Abstract: This paper reports on a preliminary analysis of the first stage of a longer research project on teacher awareness of teaching and learning mathematics in multilingual junior secondary classrooms in South Africa. The project is framed by a social theory of mind - that consciousness is formed through socially mediated activity (Vygotsky, 1978), and methodologically is concerned with a theoretically informed interpretation of empirical data gathered in a particular way. It is situated in South Africa where schooling contexts differ enormously and where multilingual classrooms are the norm. This presentation focusses on an initial set of in-depth interviews with six teachers in three different classroom contexts, where teachers talked about the context of their teaching, the tasks and challenges they face at the junior secondary level in general, and language related dimensions of their teaching in particular. A discourse analysis (Potter and Wetherall, 1987), and hence of presences and silences in the teachers' accounts, suggests that teachers in multilingual settings confront and produce language related dilemmas and paradoxes as they manage their teaching. These, in turn, have implications for educational access in the broader South African context.

Potter, J and M Wetherall (1987) Discourse and Social Psychology: Beyond Attitudes and Behaviour. London. Sage

Vygotsky, L (1978) Mind in Society. Harvard University Press. Cambridge.

USING PERFORMANCE ASSESSMENT ACTIVITIES TO PROMOTE INTEGRATION BETWEEN SUB-DISCIPLINES IN TEACHER EDUCATION

Miriam Amit
Ministry of Education, Israel

Currant reform in mathematics education calls for change in instruction and assessment. Instruction should de-emphasis drill and practice type activities and incense emphasis on open ended real-world problem. Testing and assessment should de-emphasis multiple choice activities and increase emphasis on performance assessment activities. In order for true educational change to occur, instruction and assessment must be linked. To address these changes, the Educational Testing Service (ETS) developed performance assessment materials for middle school mathematics based on research of Lesh and known as the PACKETS program activities. The program strongly reflect the NCTM standards and facilitates the delivery of interdisciplinary instruction, cooperative learning and application of mathematical thinking to real life situations. For each activity the student receives a newspaper article which provides a topical real-world starting point for a model eliciting project size problem. These problems are generally perceived as having the potential to contribute to the development of higher-order thinking, and assessment of performance and conceptual understanding. Teachers' guide include samples of student products of the project size problem, a "ways of thinking" sheet that analyze different approaches to solve the problem exhibited in the products, suggested interpretations using assessment tools and written reflection to the students. These materials were used in an inservice teacher training course for middle school mathematics teachers. The course was based on theory and practice in several "sub-disciplines" such as: instruction, testing and assessment, problem solving strategies, mathematical modelling, attitudes and beliefs, real-world and "pure" mathematics. Each of them was taught separately, with no interaction between the sub-disciplines. In order meet the changes in math education and to integrate between the above, performance assessment activities were used. At the beginning, the participants took the roll of students and solved a model eliciting problem, working in small groups and documenting their solutions. One activity, for example, was to create a model for adjusting given recipes to new situations under constrains that emerge from a newspaper article. The participating teachers used several solution paths and came out with different models, most of which met the requirements of the problem. The diversity of results was a surprise to the teachers who were used to "one problem, one path, one solution". Next, taking the roll of teachers again, participants analyzed samples of student products according to the "ways of Thinking" sheet provided in the PACKETS program, and expanded their analysis to peer and their own products. In performing the activity and and assessing the products,, participants had the opportunity to integrate between the different sub-disciplines taught in the course and even to re-consider some of their beliefs about problem solving. In the session the activity, the models and the integration process will be discussed.

THE DEVELOPMENT OF PROJECTIVE ABILITIES IN PRE-SCHOOL CHILDREN.

Poulos Andreas.

According to the Piagetian School the projective ideas in children's mind come after the topological ones and explicitly after the age of 7-8 years. This is mainly due to the fact that the child in this age is egocentric and he can't imagine the view from another person's position. There is some doubt about these conclusions and a reasoning on newer experiments of these positions. In this oral representation we will talk about two main research problems:

I: The role that the projective ideas play in the development of geometrical thinking on children's mind.

II: The effect of didactical intervention at earlier ages (5-6 years old) on these ideas.

Our research programme on acquisition of geometrical concepts is based on the follow:

1. The concepts of plane figures are learned with the use of solid bodies.
2. The perceptive perception is improved in the 3D space.
3. The micro-space of the table isn't sufficient for the development of perceptive perception.
4. The children in the play perform tasks that call for the view of objects from many different optical positions. This is a component of projective view of geometrical objects.

We study and control the evolution of children's behaviour in the process of our programme during the year's 1992-93 and 1993-94.

The first conclusions of our observations are:

a) The children have great deal of experience, so they can "read" two-dimensional complex figures. Also they can construct a block of objects looking at the photo or the pattern.

b) The differences between perspective and non-perspective drawings are not perceptible from the children without proper instruction.

We hope that the completion of our researching programme will throw light on the projective abilities of pre-school children.

References.

* Cox Mauren (1977) "Teaching perception ability to five-years-old", British Journal of Educational Psychology 47,312-321.
* Piaget Jean & Inhelder Barbel (1971) "The Child's Conception of Space", Routledge & Kegan Paul. London.
* Pillow Bradford & Flavell John (1986) "Young children's knowledge about visual perception, projective size and shape". Child Development, 57, 125-135.

INTEGRATING DERIVE IN SECONDARY LEVEL MATHEMATICAL TEACHING : THEORETICAL POTENTIALITIES AND THE REAL LIFE OF TEACHERS AND STUDENTS

Michèle Artigue, Maha Abboud, Jean Philippe Drouhard, Jean Baptiste Lagrange
Equipe DIDIREM, Université Denis Diderot Paris 7

The research presented here is an ongoing research whose aim is to study the impact of the use of C.A.S. softwares on the mathematical representations and practices of secondary school students.

It is carried out with students whose teachers can be considered as "experts" : they have been integrating computers to their maths teaching for many years and using DERIVE (the specific C.A.S. considered here) for more than one year. About 40 teachers are involved with students ranging from 14 to 18 years old.

The research combines external and internal methodologies :

- the external study is based on two questionnaires (one for teachers and one for students) aiming to collect detailed information about the way DERIVE has been used in each precise class during the whole academic year, what was expected and what really happened, as well as the opinions of both teachers and students about this software and its potentialities for maths teaching and learning.
- the internal study is based on qualitative observations of classroom sessions with DERIVE. Sessions to be observed are selected with a special subgroup of experts working together in a C.A.S. working group created three years ago by the Ministery of Education : they have previously experimented them and think that they are well adapted for testing some of the hypothesis currently found in the litterature about the potentialities of DERIVE or other C.A.S.

The research began in 1993. A first version of the student questionnaire was elaborated and tested with students belonging to the classes of the C.A.S. working group teachers mentioned above. Some observations were also carried out and allowed us to fix the internal methodology, specially the form taken by the a-priori analysis of situations to be observed and to get more precise criteria for selection.

In the short communication, firstly we shall present the problematic of this research, relating it to some others developed in this area, then we shall focus on the internal part of the research. By refering to some specific obervations, we shall show how the results obtained up to now lead us to reformulate most of the hypothesis initially made by the experts and currently found in the litterature and also to reflect on the conditions to be satisfied by mathematical situations for being "good DERIVE-session candidates".

CONTRIBUTION TO AN EPISTEMOLOGICAL AND EDUCATIONAL ANALYSIS OF TEACHERS' PERSONAL REPRESENTATIONS ABOUT PROBLEM SOLVING

Ana Maria Boavida, FCT- Universidade Nova de Lisboa

Several authors have claimed that the terms *problem* and *problem solving* have several meanings according to the teacher's perspectives about the nature of mathematics and its teaching and learning.

This presentation will report some components of a study which purpose was to understand how some Portuguese mathematics teachers interpret problem solving in the context of school mathematics in general and in the context of the students' development in particular.

The study assumed the plurality of ways of learning, the diversity of contexts in which knowledge is constructed, the unique character of the interaction of the individual with his or her environment, the complexity of success or failure in mathematics and the importance of a basic mathematical culture as a necessary condition for the construction of the Person in today's historical and social context.

This study had a two-fold objective: (a) to analyse and to understand teachers' *personal representations* of problem and problem solving in mathematics education; (b) to explore possible relationships between these *representations* and their *personal philosophies* about mathematics.

Methodologically, the study was organized around two complementary components. A theoretical component, which analysed contemporary trends in mathematical philosophy, and the meanings of the terms *problem* and *problem solving*. Another component included the analysis and interpretation of data collected through semi-structured interviews of teachers of mathematics (grades 7^{th} - 12^{th}).

The interpretation of these data suggests that the interviewed teachers have diferent meanings for problem and problem solving which affect the role and place each teacher gives to problem solving. The relationships between their *personal philosophies* about mathematics and the *personal representations* of problem solving are complex, simultaneously involving mathematical and non mathematical factors. It is also found that they claim predominantly *personal philosophies* of absolutist tendency for mathematics.

CONTEXTS AND STRUCTURES IN LEARNING TO NEGATIVE NUMBERS

Alicia Bruno y Antonio Martinón
Universidad de La Laguna

We expose the results of an experience carried out about the learning of negative numbers with 111 students (11-12 years old). The students worked with a curricular material in which negative numbers were presented through concrete situations in five different contexts: possesion and debt, temperature, above and below water level, time and road. The numbers appeared expressing stages (s) ("I owe 5 dollars", "temperature is 8 degrees below zero"), comparisons (c) ("I owe two more than you", "in Madrid there are two degrees less than in London") and variation (v) ("I lost 8 dollars", "temperature fell 8 degrees"). The addition and subtraction with negative numbers was studied through the problem solving with the following structures:

(1) the addition of two **states** gives the **total state**: $s_1 + s_2 = s_t$;

(2) the addition of the **initial state** and the **variation** gives the **final state**: $s_i + v = s_f$;

(3) the addition of two **variations** gives the **total variation**: $v_1 + v_2 = v_t$;

(4) the difference of the **final state** and the **initial state** gives the **variation**: $s_f - s_i = v$;

(5) the difference of two **state** gives the **comparison**: $s_1 - s_2 = c$.

This classification of the problems is similar to the one used by Vergnaud (1982).

We analyse the difficulty of the problems and the strategies used by the students (numerical and/or graphic) to solve them (Peled, 1991), according to the kind of the structure of the problem and the context. On the other hand, we study the kind of structures and contexts that the students choose with greater frequency to write problems (the solutions of which are fixed operations with negative numbers).

The problems allowing a best understanding of the operations with negative numbers were those of the context of the possesions and debt and those of the structures (1) and (2). Also these problems are chosen by the students to write problems in greater frequency. The most difficult problems were those of the context of the time and the structures (4) and (5), furthermore these structures were chosen occasionally to write problems.

REFERENCES

Peled, I. (1991). Levels of knowledge about signed numbers: effects of age and ability. **Proccedings of the XV International Conference of PME**. 145-152.

Vergnaud, G (1982). Cognitive tasks and operations of thought. In Carpenter, T, Moser, J. and Ronberg, T (eds). **Addition and Subtraction: A Cognitive Perspective**. LEA.Hilsdale, New Jersey.

A STUDY OF THE CHILD'S CONCEPTION OF FRACTION
- THE GENETIC DIMENSION -
USING THE CLINICAL PIAGET'S METHOD
AND SEARCHING METODOLOGICAL INFERENCES

Ely Machado de Campos

Centro Educacional La Salle de Ensino Superior

Canoas, RS. Brasil

The purpose of this investigation is to study the child's conception of fraction. The study deals with the elementary concrete partitions of three dimensional and two dimensional figures and the use of Piaget's methods of collecting and organizing · rimental data.

The principal objective is the chartring of the development of intellectual structures related with fraction, i. e., the study of the genetic dimension.

In order to grasp the context, this work more specifically includes two sections: a discussion of Piaget's investigation and the aplication of the clinical method to a experimental group with 35 children aged 3-10 years old.

In the current research, the subjects were given the task of bisecting, trisecting, etc ... a circular cardboard "cake" and other figures. In the clinical interview, which acompanied each task, subjects were questioned about their dominace of conservation.

The findings of this study were largely as Piaget indicated and they refer how important it is to explore the intellectual development theory about fraction where the viewpoints of teachers must include besides the what's to be taught, how children learn and when they learn (i. e., conditions to learn).

Bibliography:

1. COPELAND, D. *How Children Learn Mathematics* New York, Macmillan Publishing Co., Inc., 1974, 357 p.

2. DICKSON, L. BROWN, M. and GIBSON, O. *Children Learning Mathematics: A Teacher's Guide to Recent Research*, London, Cassell educational Ltd, 1988, 379 p.

3. FISCHER, H. *Didatique de la initiation mathématique à l'école primaire*, Génère, Bureau International d' Education, 1956. 180 p.

4. PIAGET, J. Essay sur quelques aspects du developpement de la notion de partie chez l'enfant. *Journal de Psychologie Normal et Patologique*, XVIII, 6.

5. PIAGET, J. INHELDER, B. et SZEMINSKA. A. *La Géometrie spontanée de l'enfant*, Paris, Presses Universitaires de France. 1948. 514 p.

FROM INTEGERS TO FRACTIONS

David Carraher*, Luciane Heim, & Analúcia Schliemann*
Universidade Federal de Pernambuco, Recife, Brazil

In learning about fractions students must rely upon what they know about integers, yet their understanding of whole numbers at times gets in the way. For example, research has shown that many students maintain the belief that multiplication "makes bigger" and division "makes smaller" (Bell, Fischbein, & Greer, 1984) well into their secondary school years. How can students reconcile former knowledge about natural numbers with knowledge to be acquired about fractions? Where must breaks with past knowledge and conventions be established?

An exploratory study was undertaken with 10 middle school children to investigate how they deal with the effect of multiplication and division by integers. This seemed a reasonable situation for clarifying how integer knowledge is related to fraction knowledge since successive multiplication and division by integers are equivalent to operating by a fraction. Children were first given single operation tasks to clarify their interpretations of multiplying and dividing quantities by integers Following this they were given tasks which involved successive division and multiplication upon a given quantity. [Both orders were given: division followed by multiplication and vice versa.]

Although all ten children accepted multiplication as n-folding a quantity, there was considerable variability about how to interpret division of a quantity by an integer. Four of the students were inclined to view "division by n" as "breaking the quantity into n pieces". Even when the interviewer drew attention to the written operation, "÷ 3", some students maintained that the amount of chocolate was the same as it was before the division. This suggest that the belief that "dividing by an integer makes smaller" may admit exceptions in certain circumstances.

There was considerable variation among students regarding the effects of successive integer multiplication and division upon a physical quantity. Some students treated division and multiplication (e.g. ÷ 4 followed by x 3) as producing a fractional part (e.g. 3/4) of the original amount, regardless of the order of the operations. However, students reaching this conclusion did not necessarily realize that a fraction could act as a multiplicative operator. Others claimed that the integer operations were not related to fractions. Overall, the interviews suggest that the links between integer multiplication and division, on the one hand, and fractions, on the other, are tenuous. That some students showed progress during the interview itself suggests that successive operations tasks may be useful for drawing out the interrelations between integers and fractions.

* At the Technological Education Research Centers (Cambridge, MA) during 1994.

USING A LOGO ENVIRONMENT TO CREATE A MATHEMATICS CLASSROOM COMMUNITY

Marta Civil
University of Arizona, Tucson, USA

In this paper I describe one aspect of our work with a fifth grade teacher, as we collaborate to change mathematics teaching and learning in her classroom. This research is part of a larger program of work that focuses on a socio-cultural approach to learning and rejects the deficit model for minority education (Moll, 1992). In our work, classroom teachers and university researchers collaborate to develop innovations in teaching by developing a classroom environment that sees the students and their families as resources for learning.

The work reported here is part of our effort to create a mathematics classroom community in which students engage in discussions on mathematically rich problems (Cobb, 1991; Lampert, 1988). Because of the characteristics of the school and the fact that these students have been acculturated to the school ways for five years, changing their perceptions about what it means to do mathematics in school was met with resistance. Immersing them in a completely new environment, namely the logo laboratory, gave us a way to break down some of the established behavior norms and roles in the classroom. In addition, because logo was new to every student in the class, it served as an equalizer through which students considered "less successful" were able to shine. After a few sessions with the whole class, a very diverse group of seven students expressed an interest in continuing working on logo. This paper will report on this micro-community of learning focusing on social and cognitive aspects.

At the social level, the patterns of behavior of the logo group in the lab were in sharp contrast with their behavior in the classroom. Students who were either aggressive or withdrawn in the classroom, became more open and friendlier towards their peers in the lab. They were curious about each other's work. Although students worked individually in the projects of their choice, there was constant spontaneous sharing of information. Ideas traveled throughout the lab. Some students became resources to their peers. In the paper I elaborate on the different patterns of interaction. The students' projects gave us a window into their world and into their understanding of some aspects of mathematics (e.g., decimals, angles, patterns). With varying degrees of success (as I will illustrate in the paper), we engaged in conversations around their work and their thinking. This helped us assess what was happening at the cognitive level. Issues of persistence, confidence, intellectual challenge, and frustration shaped each student's personal approach to his or her project. The projects gave them an opportunity to develop an area of expertise (Hoyles, 1985). The effects of this expertise were quite evident in their presentations to the whole class. Even the students whose thinking we had difficulty reaching, gave quite coherent and articulate presentations.

References

Cobb, P. (1991). Reconstructing Elementary School Mathematics. Focus on Learning Problems in Mathematics, 13 (2), 3-32.

Hoyles, C. (1985). Developing a context for LOGO in school mathematics. Journal of Mathematical Behavior, 4, 237-256.

Lampert, M. (1988). The teacher's role in reinventing the meaning of mathematical knowing in the classroom. In M. J. Behr, C. B. Lacampagne, & M. M. Wheeler (Eds.), Proceedings of the Tenth Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 433-480). De Kalb, IL: Psychology of Mathematics Education.

Moll, L. (1992) Bilingual classroom studies and community analysis: Some recent trends. Educational Researcher, 21 (2), 20-24.

THEORY AND PRACTICE: DO STUDENT TEACHERS ENCOURAGE YOUNG CHILDREN'S MATHEMATICAL KNOWLEDGE OF THE REAL WORLD?

Anne D. Cockburn
University of East Anglia, U.K.

Research suggests that many young children are unaware of the role of mathematics in adult lives (Desforges and Cockburn, 1987). Teachers have been encouraged to present their pupils with realistic, practical problems (e.g. Haylock, 1991; Kamii, 1989) and, indeed, such an approach forms part of the Mathematics National Curriculum in England and Wales. Despite this, many children have difficulty in applying their knowledge in unfamiliar contexts (OFSTED, 1993). Personal observation supports Desforges and Cockburn's 1987 finding that early years teachers tend to provide routine mathematics tasks rather than problems and investigations which require the application of basic skills.

In response to a questionnaire, early years student teachers indicated that they are strongly in favour (97%) of pupils engaging in practical mathematics in realistic contexts (N = 36). This short presentation will consider:

(i) in what ways students act upon these beliefs on teaching practice. (If they do not, why not?)
(ii) whether such an approach has any effect on pupils' understanding, progress and knowledge of mathematics within a real world context.

References

DESFORGES, C. and COCKBURN, A. (1987) Understanding the Mathematics Teacher. Lewes, Falmer Press.

HAYLOCK, D. (1991) Teaching Mathematics to Low Attainers. London, Paul Chapman Publishing.

KAMII, C. (1989) Young Children Continue to Re-invent Arithmetic. New York, Teachers College Press.

OFFICE FOR STANDARDS IN EDUCATION (OFSTED) (1993) Mathematics: Key Stages 1, 2 and 3. London, HMSO.

WORD PROBLEMS.
FROM NATURAL LANGUAGE TO MATHEMATICAL EQUATION: THE RESPECT OF HOMOGENEITY

CORTES Anibal
LabPSYDEE, 46 rue St. Jacques, 75005 PARIS.

Most students in putting into equation the following problem:

A shop-keeper selling smoking items buys from a wholesaler match-boxes and lighters by giving him 230F. If he buys 260 match-boxes and 30 lighters, the wholesaler gives him back 3F. If he buys 160 match-boxes and 40 lighters the whole saler gives him back 2F. What's the price of a match-box? and that of a lighter?

write directly the system of equations: $260x+30y=230-3$ and $160x+40y=230-2$. Now, to put in equation a word problem requires to pass through stages, in general implicit:

a) First, it is necessary to identify or to construct correspondences (in natural language) between two whole sets in our problem: *to a number* **n** *of mutch-boxes corresponds a sum of money Sa (only one)* **n -> Sa**. In the same way, for **m** cigarette-lighters: **m -> Sb** and **nUm -> Sa+Sb**.

b) Then, it is necessary to construct numerical functions starting from the correspondences previously constructed. There is a rupture here because these numerical functions express equivalences: **the homogeneity** of terms must be respected The functions in our problem are: *the total sum of money* ***St=Sa+Sb****; the sums of money payed in function of the number of objects:* ***Sa=xn*** *,* ***Sb=ym*** *and* ***St=xn+ym***. In general these functions are not written down.

c) The mental construction of these functions makes possible to connect the numerical data to the unknowns of the problem. Now, the numerical calculation of these unknowns requires, in our problem, to write a system of equations by substitution of datas in the function **St=xn+ym**. This process is in general implicit.

Analysis of one representative error: some students write *"x represents the match-boxes and y represents the lighters"*. The signification of the unknowns switches to the one of an object. Then, they write: $x+y=230$ and $260x+30y=230-3$. In the first equation the equal sign expresses only a correspondence: to a number of objects corresponds a sum of money. The homogeneity of terms is not respected: there are objects on one side and money on the other. The disregard of the homogeneity also appers when students give two different significations to one symbol. For example **x** can be aiso an *unit-price* in the second equation.

The numerical functions and the equations written down must always have the same units and the same signification. Therefore, we can identify an **operational invariant** or a principle: **the respect of the homogeneity** of terms in mathematical expressions and the homogeneity of the signification of symbols. This implicit principle guides the cognitive process involved in mathematical modelisations of conceptulisations of reality.

INTUITIVE NOTIONS ON SEQUECES IN PUPILS OF SECONDARY SCHOOL

Joaquín Ferrer Sánchez Departamento de Didáctica de la Matemática.Universidad de Almería. Almería.Spain.

ABSTRACT

This work is about the intuitions about the concept of sequence in pupils fourteen-fifteen years old.

A lest has bee passed to the children in a classroom in Secondary School,which consists in two items about sequence.In the first item one pupil is to explain what a sequence is to another pupil.The second item presents four examples,and the pupil needs to dicose whether one one of them is a sequence and justify his choice.The first example is about an infinite non-numerable set:a segment;the second example asks about of sand of a beach;the third one is the positions of a pendulum.The set of odd numbers is the last example.

Responses are classified following threecriterion: **oder,infinity** and **regularity**.Then,we analyte the consistence of the answers given to the different above mentioned examples.

References

Fischbein.E et al.1979.Educ. Stud. in Math,10(3),40
Moreno,Luis E. et al.1991.Educ. Stud. in Math.22:21
Sierpinska,A.1987.Educ. Stud. in Math.,18,371
Sierpinska,A.1990.For the Learn. of Math.,10(3),24

Exploring the source and effects of critical incidents in the growth of the professional knowledge of mathematics teachers

Peter Gates
The University of Nottingham

Abstract

The field in which I wish to report is that of the development of professional learning in mathematics teachers. Areas of current investigation outside of Mathematics Education include work on the structure of a teacher's knowledge (Shulman, Calderhead, Zeichner), the nature of knowledge about teaching (Schon, Calderhead, McIntyre and Brown, Olsen, van Manen) and the nature of professional development (Calderhead & Gates, Day, Eraut, Carr & Kemmis).

Shulman has written that one area which needs further examination is the effect subject knowledge has on a teacher's beliefs and classroom practice. Furthermore there is currently great interest in constructivist approaches to learning, not just for pupils learning mathematics, but for teachers developing their teaching. However there are some very recent signs that there is a shift away from viewing the individual as constructor of knowledge towards the social and cultural context of knowledge construction: 'social constructivism' and 'social interactionism'. This attempts to merge Vygotskian ideas of the role of language and situation, with a perspective that learning often does take place as a result of some interaction with another. However just as we are not just individual meaning makers, we are not just social negotiators. At the same time we are a product of our past, having psychological dispositions which may either liberate or constrain us.

It is within this context that I wish to report some work which looks closely at the act of developing new ideas about teaching which are significant enough to bring about a change in awareness which in its turn brings about a change in behaviour. The idea of exploring 'significant moments' (Mason, Jaworski, Gates, Davis), 'critical incidents' (Trip) or 'critical events' (Woods) is not a new one and might even be traced back to early works in Buddhism, the Vedics and other such philosophies. However it seems to be providing an access into the mind of the teacher 'in action'. One area which has received little attention has been the psychological disposition of teachers towards interpreting their experiences. Whilst much research from a constructivist tradition has identified the very moments of learning of young children (eg, Steffe, Cobb *et al*, von Glasersfeld) little has been done in the area of the growth of knowledge of the teacher - either novice, or expert - with a focus on the subject specialism.

References

Brown S. & McIntyre, D (1993) *Making sense of teaching* Buckingham, Open University Press
Calderhead, J. & Gates, P. (eds) (1993) *Conceptualising Reflection in Teacher Development*, London, Falmer Press.
Calderhead, J. (1991) 'The nature and growth of knowledge in student teaching' *Teaching and Teacher Education* 7, 5/6, pp 531 - 536.
Carr, W. & Kemmis, S. (1986) *Becoming Critical*, Lewes, Falmer Press
Ernest, P. (1989) 'The knowledge, beliefs and attitudes of the Mathematics teacher: a model', *Journal of Education for Teaching* 15, 1, pp 13 - 33.
von Glasersfeld, E. (1991) *Radical Constructivism in Mathematics Education* Dordrecht, Kluwer.
van Manen, M. (1977) 'Linking ways of knowing with ways of being practical', *Curriculum Inquiry* 6(3) p 205-228.
Mason, J., Davis, J., Jaworski, B., Gates, P., & Griffin, P. (1989) *Teaching Moments* unpublished manuscript, Centre for Mathematics Education, Open University.
Olson, J. (1992) *Understanding Teaching: Beyond Expertise*, Buckingham, Open University Press.
Schon, D. (1983) *The Reflective Practitioner* New York, Basic Books.
Shulman, L. (1987) 'Knowledge and teaching: foundations of the new reform', *Harvard Educational Review* 57, p 1-22.
Shulman, L. (1986) 'Those who understand', *Educational Researcher* 15, pp 4 - 14.
Tripp, D. (1993) *Critical incidents in teaching* London, Routledge.
Woods, P. (1993) *Critical events in teaching and learning*, London, Falmer Press.
Zeichner, K., Tabachnick, B. and Densmore, K. (1987) 'Individual, institutional and cultural influences on the development of teachers' craft knowledge' in Calderhead, J. (ed.) *Exploring Teacher Thinking*, Eastbourne, Cassell.

ROLE OF INTERVALS WHEN FUNCTIONS ARE INTRODUCED

Joaquin Giménez. Computer Engineering Department, Taragona

ABSTRACT

Some distractors were found in building intuitional knowledge of dependence in no linear graph situations. Conceptions of dependence and use of interval is analyzed.

INTRODUCTION AND OBJECTIVES

Our study try to analyze the students' conceptions (Artigue 1990) and images (Vinner 1983). (in a "in service teacher training" experience) of global interpretation, use of variation ideas of functions prior to the usual formal study (11-16 years old). The main questions were the following: (a) Could we characterize an intuitional knowledge (Kieren 1992) about dependence and variability prior the formal study? (b) What about the use and verbalisation of interval in using global interpretations of no linear graphs in comparison situations?

METHODOLOGY

120 scholars (11 to 16 years old) and 5 teachers in the Tarragona area participated in a cross longitudinal study. The set of questions included: (a) dependence, and corres-ponding linguistic framework, (a) interpretation of scaled function and comparison of graphs in real context situation, and (c) use of algebraic and table notation to represent a simple dependence.

MAIN RESULTS

(a) Functions are considered as graphs, and later as a relation, but the students need the formula at a starting point, for explicitation. (b) The observations revealed a basic geometrical view (Azcarate & Deulofeu 1994) of dependence and no explicit link with operative understanding. Global observations did not appear easily. (c)"Close contextual" activities as football game act as distractor to interpretations. (d) Operative discrete understanding act as a barrier for a global interpretation of graphs, if there is no related to a geometrical interpretation. (e) A lot of discrete viewing appears in functions' comparison tasks. Intervals are used for single functional interpretations of increasement, but it seemed not necessary to talk about variability even in comparison tasks. (f) Second part of the study (in progress) revealed that interpreting and using real situations in a figurative way, could be a powerful intuition to improve dependence understanding if comparison tasks are introduced.

REFERENCES

Artigue,M (1990) Epistemologie et didactique. *Recherches en didactique des mathématiques.* 10, (2.3),pp.241-286.

Azcárate & Deulofeu (1994) 14 to 16 yearsold pupils'problems in connection to function dependence and variation. In Malara & Rico (eds) *Proceedings of 1st Italian-Spanish Research Symposium in Mathematics Education*, 91-97.

Kieren (1992) "Building intuitional knowledge". ICMI lecture. Univ. Laval. Québec.

Vinner, S (1983) Concept definition, concept image, and the notion of function. *Int. Journal of Mathematics Education in Science and Technology*, 14 (3), 293-305.

TEACHER THINKING AND NARRATIVE ANALYSIS OF STUDENT TEACHER'S STORIES

Fernando RIBEIRO GONÇALVES
Helena RALHA SIMÕES
Carlos Marques SIMÕES

UNIVERSIDADE DO ALGARVE

ABSTRACT: Metacognition can provide an important conceptual tool for the study of teacher thinking (Clark & Peterson, 1986; Morine-Dershimer, 1991). In spite of his incipient nature, as a scientif domain, it helps eventually to understand the complexity of the several modes of knowing involved in teacher development, since his early beginnings (Eisner, 1985). But, if the modes of knowing and the events to which they refer are taken to be in a different domain from the practice of teaching and learning, then we must also take into account the distinction between theory and practice that permeates our reflexion.

Our effort in this paper is to construct a basis for working on these possibilities through the narrative analysis of **student teacher stories and histories** (Josselson & Lieblich, 1993). Such terms provide elements of metacognitive reflexion, and therefore we will try to reconstitue the emergence of the pedagogical content knowledge in teacher education (Gudsmundottir, 1991, Shulman, 1991).

References

CLARK, C. & PETERSON, P. (1986) Teacher's thought process. In W.C. Wittrock (ed.) Handbook of research on teaching. New York, MacMillan

EISNER, E. (ed.) (1985) Learning and teaching the ways of knowing - Eighty-fourth Yearbook of the National Society for the Study of Education. Chicago, Ill., University of Chicago Press

GUDMUNDSDOTTIR, S. (1991) Ways of seeing are ways of knowing. The pedagogical content knowledge of an expert English teacher. Journal of Curriculum Studies, 23, 5: 409-421

JOSSELSON, R. & LIEBLICH, A (ed.) The narrative study of lives. Newbury Park, Cal. , Sage

MORINE-DERSHIMER, G. (1991) Learning to think like a teacher. Teaching and Teacher Education, 7, 2: 159-168

SHULMAN, L.S. (1991) Ways of seeing, ways of knowing: ways of learning about teaching. Journal of Curriculum Studies, 23, 5: 393-396

"THE ATTITUDES IN PROBLEM SOLVING"

Josefa Hernández, Candelaria Espinel and Martín M. Socas
La Laguna University (Spain)

The influence of affective factors in education is beyond doubt, doing research aspects concerning with the pupil attitudes (McLeod, 1992) and teacher beliefs (Thomson, 1992) at the preent.

Some of the aims of our paper are analyzed:

. Differences between pupil attitudes toward Mathematics in general and towards problem solving.

. Differences by ages, sex, studies of their parents.

. Relation between the pupil attitudes and the qualification in mathematics.

. Relation between the attitudes and the style of the teacher.

Our research has been carried out with students (8-11 years) in Tenerife (Spain), in which 13 teachers of 7 schools and 355 children took part. Starting from some papers by Aiken (1974, 76) and by Gairin (1987), we made and validate two tests: one to measure their attitude towards mathematics and the other their attitude to problem solving. These are made up 25 items, referred to affective (10 reactives), comportamental (5), cognoscitive (5), implication (2), contextual (2) and beliefs (1) components. At the same time we make a questionaire for teachers to analyze their way of teaching.

Firstly we gave the test about attitudes towards mathematics to the students and them about the attitudes towards problem solving. This test is given again to the student after we has been carried out a instructional design about arithmetic problems. The assesment technique which we used was Likert's added clasifications and it was given to all students. As a consequence of these results we have found out that the attitudes of the children towards problem solving is slightly higher than towards mathematics, nevertheless these points do not change after carrying out the instructional design, which was assessed in a positive way by the student as well as by the teacher. Among the causes of these results we have found out the influence of teacher beliefs and conceptions, as well as the instructional design which is carried out in a short time does not make any significative changes in the student's attitudes.

References:
.Aiken, R. (1974) :"Two sclaes of Attitude toward Mathematics". J.R.M.E., 5.
.Gairín, J (1987): Las actitudes en educación. PPU. Barcelona.
.McLeod, D. (1992): Research on Affect in Mathematics Education: A Reconceptualization and
.Thompson, A. (1992): Teachers' Beliefs and Conceptions: A Synthesis of the Research. In Grouws, D. (Ed.):"Handbook of Research on Mathematics Teaching and Learning". (A Project of the National Council of Teachers of Mathematics. Macmillan Publishing Company. New York.

MEDIATING UNDERGRADUATE MATHEMATICS LEARNING THROUGH DIALOGUE AND CO-OPERATION.

M. HOCKMAN.
University of Witwatersrand.

A learning programme based on collaborative and constructivist ideas was developed and introduced into the Math 100 tutorial system at The University of Witwatersrand during the first semester of 1993.

The programme involves a cycle of learning, beginning with the lecture or class, moving through group work and pro-active tutorials, and ending with self study. The construction of the programme was influenced by the work of Vygotsky with respect to the Zone of Proximal Development, the constructivist views of Von Glasersfeld and Lochhead, and the existing precedents of group work and conflict lessons developed by the Shell Centre for Mathematical Education, University of Nottingham. The great success of pair-learning at religious Jewish seminaries also encouraged and influenced the development of the programme. It is hoped that the programme can be adapted to become part of the tutorial system of any maths course at a university.

The main concern of the project is the structuring of small-group and pro-active tutorials. The aim of both these tutorials is to encourage communication among the students, and between the students and the lecturers. Maths education at both school and university needs to move away from being teacher-centred and allow the students to take a more active part in their own learning.

The learning programme was piloted with 50 of the 350 students in the first year main stream mathematics course, and involved two lecturers. A questionnaire, relating to the general atmosphere during tutorials, was completed by the whole Math 100 class at the end of the semester. A positive result was obtained, with the pilot project students indicating an involvement and enjoyment of the tutorial system, not shared by the rest of the students. Despite these results, no significant change in academic performance was observed at this stage.

References

Lakatos, I. (1976). Proofs and Refutations. Cambridge University Press.
Lochhead, J. (Summer 1983). Thinking about Learning. The Journal Of Learning Skills.
Von Glasersfeld, E. (1989). Cognition, Construction of Knowledge and Teaching. Kluwer Academic Publishers, Neth.
Vygotsky, L. S. (1978). Mind in Society. Cambridge; MA: Harvard University Press.

SLUMS CHILDREN'S SCHOOL ARITHMETICAL KNOWLEDGE

Mariela Orozco Hormaza
Department of Psychology
Universidad del Valle
Cali, Colombia

Several authors have described successful performances of low socio economic level children to solve arithmetical tasks, provided they are presented in their "everyday context" (Saxe, 1991) or in their "context of transaction" (Carraher y Schliemann, 1985). I would like to describe children's successful procedures in solving a school arithmetic centered test. The test was applied individually to 375 "slums children" to identify their number comprehension before they enter first grade. Children's success in solving the test tasks suggest that it is necessary to modify assessment methods in order to be able to establish their school mathematical knowledge and ways of reasoning. The presentation will include: a description of the test, the way it was applied and the results obtained; and a reflexion on the test content and the methods used to apply it and to analyse the children's answers. Finally, I would like to propose that this type of assessment allows one to identify "children's mathematics" (Steefe (1990) and enables one to decide on "mathematics for children" as Steefe has suggested (Steefe, 1988, 1990).

CARRAHER, T.N., CARRAHER D.W. AND SCHLIEMANN, A.D. (1985) Mathematics in the streets and in schools. **British Journal of Developmental Psychology**, 3, pp. 21-29.

SAXE, G. (1990) **Culture and Cognitive Development: Studies in Mathematical Understanding.** Hillsdale, NJ: Lawrence Erlbaum.

STEFFE, L. P. (1988) Children Construction of number sequence and multiplying schemes. In J. Hiebert, M, Behr (Eds.). **Number Concepts and Operations in the Middle Grades.** Preston, Virginia: Erlbaum. National Council of Teachers of Mathematics.

STEEFE, L. P. (1990) Children's construction of meaning for arithmetical words: A curriculum problem. In D. Tirosh (Ed.). **Implicit and Explicit Knowledge. An educational Approach.** Norwood, NJ: Ablex.

Progression in the understanding of an algebraic rule

MOK, Ah-Chee Ida
Department of Curriculum Studies, University of Hong Kong

Most of the exercises in textbooks can be completed by recalling algebraic rules accurately. Thus, a correct piece of routine exercise may not guarantee that students have achieved progress in understanding. This study is an ongoing one and it aims to find a better picture of how students understand an algebraic rule and how their reasoning changes as their learning experience grows. Students from different levels in secondary school are chosen to do task-based interviews. The design of the tasks are based on the pattern of the distributive law. During the interviews, students have to distinguish the legitimate and the illegitimate situations, and they are asked to explain their answers.

In the trial study for designing the inteview tasks, the tasks were tried by three students with Hong Kong educational background and of age 12, 14 and 18. They had completed secondary year one, three and five in Hong Kong respectively. The responses of the students were found to be mainly unistructural according to the SOLO taxonomy. (Biggs & Collis, 1982) There was obvious difference in the reasoning strategies of students from different age group.

The student's responses are summarized in the table below:

Student age	Characteristics of responses
age 12	unistructural, mainly substitution, tautology, inconsistency, recall, wrong data
age 14	unistructural, mainly manipulations, goal-directed transformation, some substitution, tautology
age 18	induction & deduction, consideration of the nature of variable, unistructural, tautology, multistructural but closed too soon, relational during probing

References

Biggs, J.B. & Collis, K.F., 1982, Evaluating the Quality of Learning. The Solo Taxonomy (Structure of the Observed Learning Outcome), Academic Press.

Mok, A.C.I., 1993, "The understanding of the distributive law", poster presentation in PME XVII, Japan.

CONCEPTIONS OF FUNCTION: A TOOL FOR ANALYSIS OR A CONSTITUENT OF THE MATHEMATICAL KNOWLEDGE?

Maria Kaldrimidou, Dpt. of Early Chilhood Education, University of Ioannina, Greece.

The term conception (and/or misconception) appears more and more often in the litterature of the mathematics' education. The use and the meaning the notion has in the researches' reports, and especially to the researches concerning the concept of function, will be the first issue of this paper.

Usually, the term is used without an explicitly reference to his meaning. But, the context of the researches' reports led us to an interpretation of the term. So, "conceptions" are used in relation to:

1. the knoweldge of the subjects (students or teachers of mathematics), as the "analogon" (Vergnaud) construct of the concept in their minds (Breidenbach & al., Vinner)
2. the mathematical concept, to determine the different aspects the concept has in different topics (A. & J. Selden, Sierpinska).
3. the procedures used in mathematics, their epistemological value and other caracteristics of the mathematics such as generalisation, definition, iconic representation... (Artigue, Sierpinska, Dreyffus, Kaldrimidou & Ikonomou).

Moreover, and this will be the second issue of the paper, we will try to set up an unifying approach of the notion of conception and to present our considerations about:

1. the discrimination between the term conception and other similar terms such as ideas, beliefs, scheme of thought,...
2. the nature, the function and the structure of conceptions in the construction of mathematical knowledge, and consequently
3. the possibility of intervention, during the educational proces.

References

Artigue M., 1991, Epistemologie et Didactique, R.D.M., 10(2.3), pp. 241-285.

Breidenbach & al., 1992, Development of the process conception of function, E.S.M., 23, pp.247-285.

Dreyfus T., 1991, On the status of visual reasoning in mathematics and mathematics education, in Furinghetti F (Ed.), Proceedings of the 15th Int. Conf. P. M. E, Vol. I, (pp. 33-48), Assisi, Italy.

Kaldrimidou M. & Ikonomou A., 1992, Epistemological and metacognitive conceptions as factors involved in the learning of mathematics a study which focuses on graphic representations of functions, Papers WG7, ICME7, IDM / University of Bielefeld

Selden A&J, 1992, Research perspectives of conceptions of function, in G.Harel & E.Dubinsky (Eds.), The Concept of Function: Aspects of Epistemology and Pedagogy, MAA Notes, Vol 25, (pp. 1-16).

Sierpinska A., 1992, On understanding the notion of function G.Harel & E.Dubinsky (Eds.), The Concept of Function: Aspects of Epistemology and Pedagogy, MAA Notes, Vol. 25, (pp. 25-58).

Vergnaud G., 1985, Concepts et schemes dans une theorie operatoire de la representation, Psychologie Francaise, 30-3.4, pp 245-252

Vinner S., 1983, Concept definition, concept image and the notion of function, I.J.M.E.S.T., 14(3), pp.293-305.

THE COGNITIVE STRUCTURE AS A PROCESS OF DOING MATHEMATICS

Tadato Kotagiri
University of the Ryukyus

I have previously reported on the cognitive structure that is observed when elementary-school children successfully learn numbers and four arithmetic calculations according to the Suido Method.(See the Proceedings of the PME17, Ⅲ-240.) This structure has four phases, or four thinking modes, identified as the Real World, the World of Models, the World of Schemas, and the Mathematical World. All the empirical results show that every elementary-school child can achieve a satisfactory level of learning numerical concepts by going through the four cognitive worlds conducted by the Suido Method even if he/she is called a slow learner in mathematics learning because he/she can manipulate concrete objects and understand the result of his/her efforts.

We can begin here with the question, "Why can a child understand the result of manipulating concrete objects?" This question is not particular to the Suido Method, but is a general question because it is popular to use concrete objects, pictures of concrete objects, and everyday-life stories for teaching mathematics. Then, the question is, "Why can a child do and understand mathematics by using the concrete objects, the pictures, and the stories?"

I propose that there is a base-structure under the structure composed of the four cognitive worlds. Citing Freud's idea, this base-structure could be called Prelogical Knowledge, which is obtained un-, sub-, or pre-consciously. By a simple experiment, it is easy to prove that there exists Prelogical Knowledge, which is not intended to be memorized. From this point of view, the above statement of successful teaching should be rectified as follows: Every child who has stored enough Prelogical Knowledge in his/her mind can understand the mathematical relations explained in the thinking modes with concrete objects, pictures, and stories. The Prelogical Knowledge is different from the knowledge memorized through the Drill-and-Practice lessons because a child learns or memorizes something consciously in the Drill-and-Practice situation. The memory system of Prelogical Knowledge is different from computer memory because a child understands the meaning of information to be accumulated unconsciously even if he/she is just two or three years old.

We might now consider another question, "Why can a child understand the meaning of information to be stored unconsciously?" Since a child scarcely know the words to communicate with another person, child's way of understanding the meaning of words is different from adult's one. This question is generalized into the following, "Why can a young child, whose vocabulary is extremely limited, obtain concepts such as Large, Small, Heavy, Much, etc.?"

I propose that there are two conditions which must both be satisfied for a young child to understand the meanings of new words: One is Emotional Resonance, and the other is Contextual Understanding. It is easy to observe these conditions in a child's play, i.e., when the child is playing house or is enjoying a paper picture show. In other words, we can observe the two conditions when a child forgets himself/herself to do something. This suggests that an important principle of storing Prelogical Knowledge is to let a child do what he/she feels enjoyable, joyful, delightful, amusing, etc.

Constructing a culture of alternative pedagogy in a formal educational system: an analysis of two teachers' interventions.

Yanna Preen - Kynigou and Chronis Kynigos*
*University of Athens, School of Philosophy, Dept. of Philosophy, Education and Psychology

The context of this research is a logitudinal project, involving all teachers and children of a primary school using Logo programming to develop alternative pedagogy for mathematics. Episodes about the didactical intervention of two teachers with seven years' prior Logo teaching experience within the project are described, where, their developed strategies to influence the learning environment, on the one hand in order to discourage an unreflective use of the computational medium, and on the other, in order to help the children to focus on and become aware of the interesting and powerful mathematical ideas which they use in amongst their projects were challenged by the assumptions carried over from the wider educational paradigm. A qualitative analysis of all the teachers' verbalisations in the classroom was carried out supported by interview data based on a) video recordings of the teachers' activity and transcription of all their interventions throughout five teaching periods each and b) on semi - strucutred interviews regarding their beliefs about the didactical strategies they had formed

The preliminary results support the argument that the teachers have developed strategies to influence the learning environment, on the one hand in order to discourage an unreflective use of Logo, and on the other, in order to help the children to focus on and become aware of the interesting and powerful ideas which they use in amongst their projects However, their efforts are not facilitated by the assumptions carried over from the wider educational paradigm to the Logo work even though during the normal curriculum hours they did attempt to engage the children in classroom discussions or find time for them to work in small groups and the school explicitly supports such pedagogy The children come into the Logo classroom with the expectation that the teacher is there either to provide information or answers or to test whether the pupil can provide them The teachers have, in general gradually developed a meaningful way to communicate to the children why they are not readily giving them factual answers, why they will often throw the responsibility of a situation back to the children themselves, and that it is socially acceptable and legitimate for them to conjecture, theorize and make mistakes Even so this does not seem to be the case of simply explaining the rules of a new game and then playing it The teachers see important eduational value in consistently communicating these rules throughout the project's duration

References

Kynigos, C (1992) 'Insights into Pupils and Teachers' Activities in Pupil Controlled Problem Solving Situations A Longitudinally Developing Use for Programming by All in a Primary School' In Information Technology and Mathematics Problem Solving Research in Contexts of Practice NATO ASI Series 219-238 Berlin Springer Verlag
Hoyles, C (1992) Illuminations and Reflections - Teachers, Methodologies and Mathematics Proceedings of the sixteenth Conference for the Psycology of Mathematics Education New Hampshire

CARDINALITY UNDERSTANDING AND LEVELS OF ACQUISITION

Vicente Bermejo
M• Oliva Lago
Purificación Rodríguez

Universidad Complutense, Madrid, Spain

This work focused, on the one hand, on the process of cardinality acquisition and, on the other, on the relation between the cardinal concept and the counting procedure. To gain a better understanding of children's understanding of cardinality we presented four tasks: (a) conventional sequence (CS); (b) backward sequence (BS); (c) jump at the beginning of the standard sequence (JBSS); and (d) jump forward and backward in the standard sequence (JFBSS) to three groups of subjects (i.e., Group I: M: 3:9, range 3:5 - 4:0 years; Group II: M: 4:3, range 4:0 - 4:6 years; Group III: M: 4:10, range 4:6 - 5:0 years).

As for our first goal, in general, the results showed that the cardinality acquisition did not conform to an all-or-nothing process. On the contrary, the present data supplied further evidence to the six steps proposed by Bermejo and Lago (1990) in an earlier research regarding the cardinality understanding: (1) incomprehension of the situation and random responses; (2) repetition of the number word sequence used in counting; (3) counting the set of objects again; (4) giving the last number word of the counting sequence (i.e., the "how many" rule); (5) responding with the largest number word of the used sequence; and (6) a true cardinality answer.

Concerning our second goal, the results evidenced that did not seem to exist an essential relation between counting and cardinality. In effect, many children who counted accurately failed to answer to the cardinality question. Likewise, many children that didn't know how to correctly count were able to give precise cardinality responses. Thus, in accordance to previous research, a cultural or instrumental relation is suggested between counting and cardinality.

THE STABILITY OF ALTERNATIVE PROBABILITY CONCEPTIONS

P. E. Laridon and M. J. Glencross

Universities of the Witwatersrand and Transkei

As reported at PME 17 (Laridon and Glencross, 1993), ongoing research uses questionnaires adapted from Green (1982) to investigate the understanding of probability amongst grade 9 pupils in the Witwatersrand and Transkei areas of South Africa. This questionnaire has been applied to about 1 200 pupils.

A more detailed analysis is here presented of two series of items from Green's questionnaire. The first of these consists of questions related to expectations pupils have on drawing balls from bags; the second series deals with spinners. A novel sequential path approach to the statistical analysis of these series of items is used. Alternative conceptions are posited as being the basis on which pupils made choices amongst the alternatives presented with each item in the series. The group of pupils fitting into a particular conception as indicated by a choice is followed through the series. Some startling results emerge in terms of the actual final percentage of pupils who have used a particular conception consistently throughout the series. The outcomes of this analysis, questions statements often made about the stability and persistence of alternative conceptions. Serious doubts are also raised about the reliability of the usual statistical analyses of instruments consisting of multiple choice items.

In Green's questionnaire some of the multiple choice items are followed by an elicitation of reasons for the choice made. The cascade analysis of the items mentioned above is illuminated by an analysis of these free responses. The categories obtained are discussed in terms of possible underlying causes as found in the literature (Borovcnik and Bentz, 1991).

REFERENCES

BOROVCNIK, M & BENTZ, H J (1991) Empirical research in understanding probability (In KAPADIA, R & BOROVCNIK, M (1991) Chance encounters: probability in education. Kluwer, Dordrecht; pp: 73 - 105)

GREEN, D R (1982) Probability concepts in school pupils aged 11 - 16 years. (Unpublished Ph D Thesis, Loughborough University of Technology.)

LARIDON, P E & GLENCROSS, M J (1993) Intuitive probability concepts in South African Adolescents (In the Proceedings of PME17: I - 265)

THE BUILDING OF MEANINGS IN MATHEMATICS: AN APPROACH OF THE CONCEPT OF FRACTION.

Lopez Carretero, Asunción. I.M.E.B. (Investigation Area I.M.I.P.A.E). Council of Barcelona. Spain.
Gallego Lázaro, Carlos. Teaching Staff College. Ramon Llull University. Barcelona. Spain.

This work intends to be a contribution from a psychological perspective to the study of the mental functioning in the Mathematics field. It analyses the formation of concepts, and more specifically some aspects of the building of the fraction's notion.

The work places the problemetic in the frame of the mental representation in order to tackle the cognitive dynamics responsable for the building of meanings in Mathematics.

Thanks to the theoretical means of "the representational model" and "the operational context" (Moreno-Sastre, 1988), this work studies the interaction produced between the representational content -variables of the task- and the operations that the subject makes to relate these variables.

It has been noticed how different conceptual schemes together with forms of symbolic expression are generated and coordinated in the process of elaboration.

The building of meanings in Mathematics in the first stages could be considered as a progressive coordination of operations and representational contents. The dynamics of the formation of the schemes could be the responsable for the building of the mathematical knowledge.

The elaboration of a symbolic language able to express the mathematical relations is the result of a process of creation which does not only resultin the reorganisation of the conceptual relations at a more complex level, but it can also open impassable mental tracks up to know.

The aim of this communication is to relate the conclusions of this work to Vergnaud's concept of "the Conceptual Field", to the ideas around Brousseau's "Epistemologic Obstacle", and to the studies carried out by other autors such as (Post, Lesh, Berh) in the fraction's field.

Berh,M.J.,Lesh.R.,Post,Th.R.,Silves,E.A.(1983):Rational Number Concepts. In :*Acquisition of mathematics concepts and processes*.London, Ed. Academic Press Inc.
Brousseau,G. (1983):Les obstacles épistémologiques et les problèmes en mathématiques.*Recherches en didactique des mathématiques* 4.2 pp.165-198.
Lopez Carretero,A (1988): Aprender matemáticas,dogma o construcción colectiva. In Moreno,M (Comp.).
Moreno,M (1988): El dialogo de lo cientifico. In Moreno,M (comp):*Ciencia, Aprendizaje y Comunicación*. Barcelona, Ed. Laia.
Sastre,G (1988): Creación de estrategias comunicativas y las escrituras aritméticas. In Moreno,M (comp).
Vergnaud,G.: La théorie des camps conceptuels. *Recherches en didactique des mathématiques* 10/2.3 pp.133-169.

IMMIGRANT UNIVERSITY STUDENTS' EXPRESSION OF PROBABILISTIC CONCEPTS FROM THEIR LIFE EXPERIENCES

Louzon, D. and Zinn, B.

Rothberg School for Overseas Students, The Hebrew University of Jerusalem

The primary aim of our teaching is to enhance students' capabilities in dealing with probabilistic concepts, and secondly to support immigrant youth in their adaptation to a new country.

Students are invited to use theoretical models, for instance Venn diagrams or Bayes' rule, as a basis for illustrating a life situation of their choice. The process involves going from the model to the situation (the reverse of modeling). To successfully undertake this task, a student must understand the theoretical model and its structure, select an appropriate subject for this model, assign each part of the subject to its counterpart in the model, and to evaluate the fit between the subject and the model. Students must then pose mathematical questions about their representation. This process demands flexibility on the part of the student. Immigration is a stressful process, requiring flexibility in adapting to the host country. We make use of the fluid situation in which our immigrant students find themselves to instill new learning strategies for problem-solving.

A pilot study, using Venn diagrams as the theoretical model, was undertaken with French-speaking immigrant students. The pilot study gave an indication of students' increased interest, firstly in writing their own interpretations of the model and secondly, in active learning during class through generating their own exercises. The reverse modeling gave immigrant students a legitimate opportunity to express nostalgia for what they left behind as well as their reactions to the host country's culture during their process of adapting to the new country. These concerns were reflected in the students' choice of topics which ranged from Italian soccer teams, Israeli culture as viewed by Asterix comic strip, airport baggage control, and international politics. The degree of students' activity and the fusion of their affective and intellectual involvement led us to extend the study.

An experimental group, which did not differ from control groups on pre-tests of basic algebra and probability skills, was provided with many opportunities to write their own interpretations of models, during a semester course on Probability. The experimental group's achievements on the final exam were significantly higher than those of the control groups. Eliciting students' expressions of probabilistic concepts seems to be an avenue worth pursuing. We are trying to implement this teaching strategy with logic problems, which form part of psychometric tests for university entrance.

THE RESOLUTION OF TWO COMBINATORY TASKS BY MATHEMATICS TEACHERS

Helena Maria d'Orey Marchand
Faculdade de Psicologia e de Ciências da Educação-
Universidade de Lisboa

Formal operational thinking is the frailest of all stages of cognitive development studied by the Geneva School (Inhelder & Piaget, 1955). Research on formal thinking in adolescence shows that the average age of access to formal stage is substantially different from the one claimed by Piaget and Inhelder in 1955. Research, which was aimed to study adults' cognition, shows 1) that subjects that used formal thinking were few and 2) that a significant number of subjects never reached a fully formal level. Faced with these results, Piaget (1972) reformulated his first conception of formal thinking. In this reconceptualisation Piaget not only admitted that the age level of 11/12 years should be extended to 15/20 years but he also pointed out 1) the role of the environment, 2) the role of the capacities and 3) the role of professional specialisation in the construction of formal operational structures. In 1972, Piaget suggested that the cognition of adults be studied through tasks related to their profession or related with problems with which they were familiar.

What happens with adults with a high level of schooling (average = 17 years) and with a professional experience essentially in teaching? Do they control formal operations? Do mathematics teachers solve easier formal tasks than teachers of other scientific areas? In this type of population, do formal tasks with familiar content induce formal thinking more easily than formal tasks with academic content?

The aim of this communication is to report some research data gathered to answer these questions.

References

Inhelder,B. & Piaget, J. (1955), *De la logique de l'enfant à la logique de l'adolescent*. Paris: PUF.

Piaget,J. (1972), Intellectual evolution from adolescence to adulthood.*Human Development*, 15, 1-12.

OECD STUDIES OF EDUCATIONAL INNOVATIONS: THE CASE OF THE NCTM *STANDARDS*

Douglas B. McLeod
Center for Research in Mathematics and Science Education
San Diego State University (USA)

The National Council of Teachers of Mathematics (NCTM) has played a significant role in efforts to change school mathematics in the United States. NCTM has focused its efforts on the development and implementation of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The activities of NCTM provide an interesting opportunity for a case study of how a professional organization of mathematics teachers can provide direction for educational change in an entire country.

Romberg and Webb (1993) have presented the historical background for understanding NCTM's role in developing the *Standards*. In this case study we are analyzing the various factors that influenced NCTM's work. We are also gathering data on the extent to which the *Standards* have been implemented. Main sources of data include interviews with educational leaders and observations in classrooms.

Seven other case studies of educational change are being conducted in the USA (see Romberg & Webb, 1993) and a number of other researchers have begun to carry out similar studies of educational change in several other OECD countries (e.g., Keiichi Shigematsu in Japan). Researchers who are participating in similar studies are invited to identify themselves at the session.

References

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Romberg, T. A., & Webb, N. L. (1993). The role of the National Council of Teachers of Mathematics in the current reform movement in school mathematics in the United States of America. In *Science and Mathematics Education in the United States: Eight Innovations* (pp. 143-182). Paris: Organisation for Economic Co-operation and Development.

PROBLEMATIZATION IN THE TEACHING AND LEARNING PROCESS: STUDENTS AND TEACHER ATTITUDES

Maria do Carmo Domite Mendonça
University of Campinas

This paper relates the results of a study in a 6th grade class based in a problematization approach. This approach stresses active learning produced from information from students' social reality. Two aspects were emphasized: (a) the action, to motivate students to act within new situations of his or her reality in order to modify it; (b) the dialogue, as the teacher faces patiently the differences of points of view and, among the, to reveal enough similarities to establish communication concerning the goal aimed by the group.

Of major interest was the study of students' attitudes towards mathematics learning when the educator is looking from an externalist view of mathematics, as mathematics connects itself with other fields of study and as the teacher considers the socio-cultural context of the students.

The class had 36 students. They were regarded as the weakest 6th grade class of the school with a quite low achievement in mathematics and in other subjects. The average age of the students was 14 years old. They worked in groups of 4 or 5 during their math classes in a spontaneous arrangement. Two of the strategies used in the work were the following (concrete examples will be given at the presentation):

(1) SE—SPONTANEOUS: Make evident a situation in the school context. The teacher must be attentive to emphasize a situation that is significant for the students and, then, to carry on a discussion about it, in order to develop a problematization process.

(2) GTS—GENERATOR THEME: ask the students to choose a "generator theme" from their social reality and help them to observe and investigate it in order to unleash a problematization.

Experimental evidence shows that there was an increase in interest of most of the students. they were excited about mathematics classes. Some problematizations allowed the introduction of new mathematics topics. Others were used essentially to present some exercises to practice skills and techniques. Still others enabled a synthesis of what students had learned, providing them the opportunity to develop skill in communicating mathematical ideas. Finally, some problematizations helped to develop in students a greater sense of what mathematics is, how it has been created and why to study it.

ASSESSING STUDENTS' MATHEMATICAL ACTIVITY IN THE CONTEXT OF DESIGN PROJECTS

Judit N. Moschkovich
Institute for Research on Learning
Palo Alto, CA U.S.A.

Research in cognition and learning has pointed out the need for closing the gap between learning mathematics in and out of school (Carraher, Carraher, and Schliemann, 1985; D'Ambrosio, 1991; Lave, 1988; Saxe, 1991). This perspective redefines what mathematics is and extends mathematical activity to include more than using rote algorithms. Following this perspective, current curriculum guidelines and standards for mathematics call for engaging students in "real world" mathematics rather than mathematics in isolation of its applications. However, it is not clear how application projects will change students' activity or how these projects will affect assessment practices.

The Middle-school Mathematics through Applications Project (MMAP) is designing curriculum materials in line with these standards and investigating students' activity when using these materials in the classroom. In this alternative learning environment students explore mathematical concepts in the context of design projects. One of the 4-6 week units, Antarctica, puts middle school students in the role of designers who are creating a research station for a scientific expedition to Antarctica. This unit guides the students, working mostly in small groups, through the design and analysis process. Tools include ArchiTech, IRL-designed software that allows students to create floor plans and analyze information on their station's heating and building costs.

This paper presents the preliminary analysis of research undertaken in one MMAP classroom using the Antarctica project. The analysis focuses on several issues in assessment practices raised by the design project curriculum. The paper describes the central issues and tensions encountered in the design and investigation of assessment practices: How does an application based, long term project affect classroom assessment practices? How are the assessment needs different than in a traditional classroom? What activities do assessment practices focus on: the design process, the presentation of products, students' uses of mathematical tools, or students' mathematical arguments? On what other productive activities might assessment practices focus?

References

Carraher, T.N., Carraher, D.W., and Schliemann, A.D. (1985) Mathematics in the streets and in schools. British Journal of Developmental Psychology, 3, 21-29.

D'Ambrosio, U. (1991). Ethnomathematics and its place in the history and pedagogy of mathematics. In Mary Harris (Ed.), *Schools, mathematics and work.* Bristol, PA: The Falmer Press, pp. 15-25.

Lave, J. (1988). *Cognition in practice.* Cambridge: Cambridge University Press.

Saxe, G. (1991). *Culture and cognitive development.* New Jersey: Lawrence Erlbaum.

STUDENTS' PROBLEM SOLVING APPROACHES
AND THEIR SPATIAL AND ABSTRACT REASONING

Conceição Almeida
e
Ana Paula Mourão

Instituto de Educação
Universidade do Minho
Braga, Portugal

The investigation of students' problem solving abilities - nature, origine and development processe - has been one of the major concerns of our recent work.

The study reported here is part of this investigation. Its purpose is to investigate how different problem-solving approaches, used by 10th grade students to solve school mathematical problems and other problems (curriculum content free), are related to their spatial and abstract reasoning.

The results of the study are presented and discussed with reference to its purpose.

Some suggestions to the teaching and learning of school mathematics are made.

References:

Almeida, L. S. (1988). *O raciocínio diferencial dos jovens - avaliação, desenvolvimento e diferenciação.* Lisboa: Instituto Nacional de Investigação Científica

Grouws, D. A. (Ed) (1992). *Handbook of research on mathematics teaching and learning.* NCTM, New York: Macmillian Publishing Company.

Krutetskii, V. A. (1976). *The Psychology of mathematical abilities in schoolchildren.* Chicago: University of Chicago Press.

Tartre, L. A. (1990). Spatial orientation skill and mathematical problem solving, in *Journal For Research in Mathematics Education*, 21(3), 216-229.

USING INTERACTIVE MEDIA FOR CLASSROOM OBSERVATION[1]

Judith A. Mousley — Deakin University, Geelong, Australia

Peter Sullivan — Australian Catholic University, Christ Campus, Australia

The software presented is a multi-media package aimed at improving classroom observation and research skills. It is being developed for use in pre-service and in-service mathematics education, but has potential for use in other professional education contexts. Resources included within the program include films of pre- and post-lesson interviews with the teachers, videos of the lessons (and any parts thereof), transcripts and other lesson documentation, associated readings and bibliographic data bases, graphic representations, and other resource materials.

Four lessons, based on six components of quality teaching (see Mousley & Sullivan, this volume) were planned, taught and videotaped. They are now stored on CD-ROM disc to enable interactive analysis of the lessons. Our current research is examining how interactive media can be used to form comprehensive data bases of classrooms in action, and how these can be used in teacher preparation, in-service professional development and educational research.

Using on-and off-campus teacher education students, the project is using case study methods to determine different levels of interactions, (a) between users and the program, and (b) within small groups of uses, during the employment of a variety of pedagogical styles and student tasks. Teaching and learning range from a transmission mode, where the lecturer plots the learning path, to individual or group research projects of the students' own design using the program's resources.

This project has four aims. The first is to further understandings about the potential of multi-media technologies to promote re' ion, discussion and writing by groups of teachers and student teachers. The second is to increase knowledge about how technology can be used to stimulate more student-directed professional development and particularly group discussions and research projects (including teachers' research into their own practices). The third aim is to provide a detailed evaluation of some uses of interactive media to enhance classroom observation during the practicum as well as for researchers. Lastly, we aim to explore the potential for students on different campuses (or distance education students with access to appropriate networks) to contribute to a group analysis of mathematics classroom interaction and to a common data base.

Conference participants are encouraged to contribute to the further development of this project by adding to interactive documents that the software contains, including data bases of (a) questions which could be researched using the program's materials, (b) comments, suggestions and questions about the research project, and (c) short quotations and bibliographies appropriate for the styles of teaching demonstrated.

---

1 This research is being funded by the Australian Research Council, Australian Catholic University and Deakin University.

INTEGRATION OF INDIVIDUAL QUESTIONING IN THE ERROR ANALYSIS PROCESS.
(An example with the algorithm of division)

NANTAIS, Nicole
Université de Sherbrooke, Québec

Error analysis is a very good approach to identify errors that the children commit in applying arithmetical algorithms. Many authors (Ashlock, 1976, Cox, 1975, Radatz, 1979) have studied errors and tried to classify them as well as make hypothesis concerning their causes.

Errors can come from many different sources: computation error, understanding error, application of false rules, misunderstanding of numeration, and these kinds of errors can be corrected in different manners depending on their source. So, for teachers, it is important to know not only where the children have committed errors, but also what kinds of errors, why they were committed, and how to correct them effectively.

In teacher preparation courses, we have developed a global didactical approach in which we use error analysis for understanding arithmetical algorithms (Nantais, 1990, 1991a). This approach consists in six steps that allow future teachers not only to classify pupils' errors, but also to learn why the child does that and what to do to remediate a specific kind of error. We describe these steps and each of their aims below.

The first step consists of a **task analysis to identify difficulties** proper to each arithmetical algorithm which having its own rules based on numeration principles. The second step is the **classification of errors** of which we know the typology from the literature (Ashlock, 1976, Cox, 1975, Ginsburg, 1977, Radatz, 1979). The third step consists of **determining hypothetical causes.** The fourth step is the preparation of a **list of exercices** based on the difficulties identified in step 1 and the **classification of a group of pupils' answers in a grid** that permits to see, at a glance, where the difficulties lie for each child as well as for the group. Nevertheless, this step doesn't allow necessarily to know the understanding of the construction of the algorithm in the application of rules or of the understanding of the numerical principles underlying them. Then, the fifth step consists of the **design of individual questioning** developed in accordance with the specific problem identified in step 4; this questioning is based on the design of the mini-interview developed by Nantais (1991b). And the sixth and last step, the **development of a remediation program to aid individual pupils or a group of pupils** correct their errors as well as understand what they do when they apply arithmetical algorithms.

In this presentation, we will bring some results of an experimentation of this approach by future teachers and highlight the essential role of individual questioning of step 5 with fifth graders on their understanding of the division algorithm.

References

- Ashlock, R.B., (1976). Errors Patterns in Computation, a semi- programmed approach. Merrill Pub. Columbus, Ohio.
- Cox, L. S.(1975). Systematic errors in the four vertical algorithms in normal and handicaped populations, Journal for Research in Mathematics Education, november.1975
- Ginsburg, H. (1977). Children's arithmetic: the learning process. New York, Litton Educational Publishing.
- Nantais, Nicole, (1990). L'apprentissage des tables de multiplication versus la mémorisation. Instantanés mathématiques, APAME. Vol. XXIX, no. 3.
- Nantais, Nicole, (1991a). Une démarche d'analyse d'erreurs appliquée à l'algorithme de multiplication. Bulletin de l'AMQ, Vol. V, no. 3.
- Nantais, Nicole, (1991b). La MINI-ENTREVUE: un nouvel outil d'évaluation de la compréhension mathématique au primaire, Presses de l'Université de Montréal, Coll. Prix Jeanne-Grégoire.
- Radatz, H. (1979). Error analysis in mathematics education. Journal for Research in Mathematics Education, no. 10.

ABOUT SYMBIOSIS BETWEEN NOTION AND ALGORITHM IN INTEGRAL CALCULUS

Germán Muñoz Ortega & Francisco Cordero Osorio
Departamento de Matemática Educativa, CINVESTAV-IPN, México.

The phenomena that integral calculus describes are characterized by the relationship between magnitudes intervening and variations given among those magnitudes. When studying this phenomenon, particular problems are inferred (see group 1 in figure) which partially describe the behaviour of phenomenon. For instance, from movement two kind of questions can be posed: about "real causes" of movement or about "calculation" of the evolution of movement systems.

Ab → Ab → Ab.
Phenomenon Particular problem Mathematical Model Theories
Ap ← Ap ← Ap.
├── 1 ──┤ ├── 2 ──┤ ├── 3 ─┤
(Ab) Abstraction (Ap) Application.

Each particular problem requires the following components to be solved (see group 2,3 in figure):

* Processes of abstraction in which student's "notions" (with associated meanings) are present when he is faced to problems. These processes are essentially of two types: empirical abstraction and reflective abstraction (Piaget & García, 1992).
* Mathematical models are outer representations of de subject as an consequence of process of abstraction.
* Symbolic language in which the subject gives to symbols some rules and meanings (syntax and semantics).
* Theories appear when the structure of several mathematical models are studied and concepts, theorems and algorithms are part of it.

Based on the scheme above, we can state that:

* The present teaching of Calculus is focused on groups 2 or 3, that is, definitions of concepts are analysed and algorithms are practiced and then to go back to group 1 to study problems of applications In this sense, Dreyfus (1990, p.125) has reported that "students learn calculus procedures in a purely algorithmic level" and "teachers and students learn to say what integral and its geometrical representation is and they identify with difficulty a methodology which enables them to study continuous variations phenomena" (Cordero, 1993).
* Concepts and algorithms are the result of a set of problems or of a set of situation in the same sense as Vergnaud -conceptual field-(Vergnaud, 1990).
* Notions and algorithms cannot be isolated. On the contrary, a symbiosis exists between them.

Questions such as what mechanisms are present in this symbiosis? How to structure a conceptual field which enhances this symbiosis and the understanding of concepts and algorithms of Calculus?.

The specific problems we are studying are: a) problems "with initial conditions", b) prediction and accumulation notions, and c) Taylor's series and the foundamental theorem of calculus both used as a device of calculus, taking in account all the above elements for discussion.

REFERENCES

Vergnaud, G.(1990).La théorie des champs conceptuels. Vol. 10, No.13, pp.133-170. CNR et Université, Rene Descartes.

Piaget, J. & Garcia, R (1992). Psicogénesis e Historia de la Ciencia. Ed. Siglo XXI, México, 5a. ed.

Dreyfus, T (1992). Advanced Mathematical Thinking, pp. 113-134. Kluwer Academic Publisher Library.

DOES YOUR ANSWER MAKE SENSE?

THE ROLE OF TEACHER QUESTIONING IN STUDENT JUSTIFICATIONS

Ralph S. Pantozzi

Rutgers University

This paper describes a temporal sequence of questioning observed in interactions between a teacher and pairs of students during the course of a longitudinal study. Analysis of these interactions reveals processes that assist the teacher in constructing a working knowledge of children's thinking. While students develop solutions to a problem, the teacher asks them to explain aspects of their work.

Method

Data for this analysis came from seven videotapes of students working in pairs on a problem situation. Each videotape was accompanied by a complete transcript. Each session was examined for examples of interaction between students and teacher and the resulting effects upon the next interaction.

Results and Conclusions

My analysis of these videotapes reveals a temporal pattern of five levels of teacher questioning. This pattern of questions is important because it triggers the students to justify their solutions. During this sequence of questioning students begin to make sense of their own answers. They develop a self-motivated need to discover and justify multiple solutions.

References

Burns, M (1985) The role of questioning The Arithmetic Teacher 32(6), 14-17

Clarke, D & Sullivan, P (1990) Is a question the best answer? The Australian Mathematics Teacher 46(3), 30-33

Gall, M (1984). Synthesis of research on teacher's questioning Educational Leadership 42(3), 40-49

Klinzing, G, Klinzing-Eunch, G & Tisher, R P (1985) Higher cognitive behaviors in classroom discourse Congruency between teachers' questions and pupil's responses The Australian Journal of Education, 29(1), 63-75

Perry, M, Vanderstoep, S & Yu, S (1993) Asking questions in first grade mathematics classes Potential influences on mathematical thought Journal of Educational Psychology, 85(1) 31-40

Proudfit, L (1992) Questioning in the elementary mathematics classroom School Science and Mathematics, 92(3), 133-135

Sullivan, P & Clarke, D (1992) Problem solving with conventional mathematics content Responses of pupils to open mathematical tasks Mathematics Education Research Journal 4(1), 42-670

ON STUDENTS' CONCEPTIONS OF THE REAL CONTINUUM

Tasos Patronis and Joanna Mamona-Downs
University of Patras and University of Macedonia, Greece.

There is a fair amount of research on students' formation of the basic concepts of Analysis - especially of the concept of limit - but only a few works focus on the conceptions about the continuum of real numbers. This is a communication of some results of on-going research on the subject. We asked the following question to students of mathematics and to future primary school teachers:

" Given the two sets of real numbers $A = \{0.3, 0.33, 0.333, \ldots\}$ and $B = \{0.4, 0.34, 0.334, \ldots\}$, does there exist a real number x which is, at the same time, greater than all elements of the set A and less than all elements of the set B?"

Below we include some of the main conceptual formations and prejudices that we have identified in the responses of 68 students of Mathematics and 20 students of Education. Formations (A), (B), (C) and (D) are somewhat related to those identified by Mamona, and Mamona-Downs (1987, 1990), while (E) and (F) are new.

(A) Dependence on the symbolic representation (or form) of numbers.
Example : The number x in the question does not exist because " there is no digit between 3 and 4 , so it is impossible to form such a decimal number ".

(B) Each of the two given sets expresses an on-going process without end.
This leads to an diminishing interval which never degenerates to a single point between the sets A and B . (This conception was more frequent among future primary school teachers).

(C) "Infinite induction" used for the definitions of the sets A and B.
Each of the two given sets has a member of "infinite order", (expressed respectively as "0.333..." or as "0.333...4"), which is the "final result" of the process: these numbers consequently are not usually considered as equal to the real number x in the given question.

(D) Explicit Use of Limits:
Application of the usual rule to find the limit of a geometric series, sometimes together with an attempted "ε-δ" reasoning.

(E) If $a \le x$ for every member of an (infinite) set A, then we should also have $\sup A \le x$.
Example: "The number x in question does not exist; otherwise we should have $\sup A < x < \inf B$, which is not true since $\sup A = \inf B$".

(F) Arguing from R is dense in its ordering and is order -complete.
From the existence of an x_1 such that $0.3 < x_1 < 0.4$, an x_2 such that $0.33 < x_2 < 0.34$ and so on, some students infer the existence of x in the given question "obviously by passing to the limit".

References

1. Joanna Mamona (1987). "Students' Interpretations of Some Concepts of Mathematical Analysis" (Ph.D. dissertation).
2. J. Mamona-Downs, (1990). "Pupils' Interpretations of the Limit Concept; A Comparison Study between Greeks and English", in G. Booker, P. Cobb, & T.N. Mendicuti (Eds), Proceedings of the 14th Annual Conference of the PME International (pp. 69-76), Mexico.

PEER COLLABORATION - AN ESSENTIAL ELEMENT FOR THE SUCCESS OF THE PROBLEM - CENTERED APPROACH TO THE LEARNING OF MATHEMATICS

SYLVIE PENCHALIAH

(UNIVERSITY OF DURBAN-WESTVILLE, SOUTH AFRICA)

This report forms part of a research project by the author, on children's intuitive/spontaneous problem solving, with regard to multiplication and division word problems. The theoretical framework is based on Socio - Constructivism (basic premise - learning is a social activity as well as an individual constructive activity). The principles of Qualitative and Action Research Methodology were followed.

For a period of four months, work has been done with Junior Primary children, at a school that has implemented the problem-centered approach for approximately four years. Group sessions were conducted, twice a week, for ten weeks with small groups (i.e. 4 to 6 children). These children were given different types of multiplication and division word problems, which they worked out however they wished (eg. by modelling, drawing diagrams or mental/abstract methods). These steps were followed:

1. Their behaviours were observed in relation to:

1.1 their solution strategies
1.2 their social interactions

2. These observations were reflected upon and

3. subsequently modified in the succeeding group sessions.

One of the many interesting observations that was made, has to do with social interactions within these groups. Now fundamental to the Problem-Centered Approach based on Socio-Constructivism, children/individuals learn through their interactions and discussion with others. But many of these young children had great difficulty in listening to each other's ideas. In this paper/poster some ideas will be shared on how this problem was addressed by the author.

Feb94/SP/jp

A teaching of the absolute value in secondary school. St , of "institutionnalisation".

Marie-Jeanne Perrin-Glorian
Equipe DIDIREM, Université Paris 7

Previous researches (Perrin-Glorian 1993) showed how important is the articulation between the course of the teacher and the sense actually involved by the pupils during activity of problem solving. Even if students use with sense in probem solving some tool that we can identify as a mathematical concept, we may observe later a loss of sense, after the course when definitions and formalisms are given.
The present research intends to identify relevant variables for institutionnalisation (Brousseau 1987), on the side of pupils and on the side of teachers. In order to do it, we chose to make some cases studies by observing several classes on the same mathematical topic.
We chose the 10th grade (15-16 years old) because it is in France the first year of "lycée" (second cycle of general secondary school) and the last year before orientation of students to scientific, literary or economic section. We retained two mathematical topics that are new at this level : the absolute value (including the absolute value function) and homothety.

In our presentation, we broach two parts : on the one hand, the place of the absolute value in the French curriculum and specificities of the teaching of such a topic, on the other hand, the study of the non strictly mathematical part of the discourse of the teacher.

About 5 years ago, the absolute value was introduced in the 8th grade (13-14 years old) with regard to relative numbers. But it was considered as difficult and of no real use for this level. Now students meet the absolute value for the first time in the 10th grade as distance on the real line and as a particular function (the theme of functions is important at this level). But, since there is no more formalisation of the notion of limit in secondary school, it is not very useful for problem solving : it is a tool to have shorter formulations, to define functions with only one algebraic expression, instead of several ones, and it is a mean to make counterexamples : it gives the only function studied at this level that is continuous and not differentiable everywhere. The statute of this concept makes it difficult to introduce it as an implicit tool to solve a problem.

On the second point, we present the comparison of the discourse of the same teacher in two classes : the first one had a good level, the level of the second one was lower. We started from the hypothesis that the teacher fits to his students and we were looking for differences. This teacher has a somewhat innovative practise in the 2 classes : during one session, students are proposed problems involving the new concepts as implicit tools ; they are organized in small groups to solve them. During the next session, the teacher directs a synthesis, and, at this juncture, gives the lesson : definitions, explanations ... Moreover, this teacher gives much place to heuristics in his discourse.
We detected 2 main differences between the 2 classes :
- time is not managed in the same way : there are more digressions in the class of lower level. Moreover, in this class, they are caused by an error or an insuffisantly accurate answer of a student ; they give an opportunity to record previous lessons. In the other class, they give an opportunity to anticipate further lessons. Paradoxically, more difficult questions may be asked in the class of lower level.
- there is more heuristic discourse in the class of lower level, but it is more algorithmized.

References

BROUSSEAU G. (1987) Fondements et méthodes de la didactique. *Recherches en didactique des mathématiques* n° 7.2, p. 33-115. La pensée sauvage Grenoble.

PERRIN-GLORIAN M.J. (1990) Réflexions sur le rôle du maître dans les situations didactiques à partir du cas de l'enseignement à des élèves en difficulté *PME, Mexico, juillet 1990. Proceedings vol. II pages 209- 216.*English version available.

PERRIN-GLORIAN M.J. (1993) Questions didactiques soulevées à partir de l'enseignement dans des "classes faibles" *Recherches en didactique des mathématiques n°13.1.* p. 5-118 La Pensée sauvage, Grenoble.

KINDS OF ARGUMENTATION USED IN GEOMETRY

JOVANA REZENDE and LILIAN NASSER
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO – BRASIL

Secondary school students encounter, in general, many difficulties in geometric questions requiring justifications and proofs (Balacheff, 1987; Hanna, 1990) This study focuses on the kinds and levels of argumentation used and its main aim is to try to understand and alleviate students' difficulties on argumentation tasks

The first pilot study was carried out with 13 year-old-students starting the geometry systematic course. As they did not have sufficient previous experience with geometry, they were reasoning at a low van Hiele level (van Hiele, 1986) and could not answer the argumentation questions.

Another attempt was tried with freshman students at the Mathematics undergraduate course. According to their answers to several tasks, the following kinds of argumentation were identified:

- **Inconsistent reasoning:** the student cannot organize his reasoning to justify the assertion;
- **Empirical justification:** the student relies on the verification of some examples;
- **Graphic explanation:** the student justifies the assertion through a graphic or figure,
- **Reference to a higher authority:** the student accepts the truth of the assertion based on the authority of the textbook or the teacher;
- **Acceptable justification:** the student succeeds in justifying the assertion, expontaneously or after prompting;
- **Formal proof:** the student writes a proof to justify the assertion.

A deeper observation will be carried on with another sample of freshman undergraduate students at the Geometry course, in which two aspects will be investigated: if the kinds of argumentation suggested are validated and if it is possible to improve the kinds of argumentation used through suitable instruction.

References:

Balacheff, N. (1987): Aspects of proof in pupil's practice of school mathematics. In: Pimm, D. (Ed) (1987): Mathematics, Teachers and Children. The Open University.

Hanna, G. (1990): Some Pedagogical Aspects of Proof. In: Interchange, Vol. 21, nº 1, pp. 6-13.

Van Hiele, P.M. (1986): Structure and Insight. Orlando, FL: Academic Press.

Vygotsky, L.S.(1986): Thought and Language. Cambridge, MA: Harvard University Press

THE CONSTRUCTION OF THE CONCEPT OF FRACTIONAL NUMBER

Maria Judith Sperb Ribeiro

Universidade Federal do Rio Grande do Sul

Porto Alegre, RS, Brasil

The purpose of this work is to investigate the understanding of the concept of fraction by elementary school's students.

The application of a writen test and subsequent observations made in a fourth grade class, show that in general, the children have problems with fundamental concepts of fractions.

Interviews and other related activities realized with two students (10 years old) of this class, choosen at random, showed that the symbols were introduced before the students could understand their meaning. The students did the fractional number operations by just applying mechanically some rules that they didn't understand.

A suplementary work perfomed with this two children, using manipulative material, diagrams and real world situations, indicate that the difficulty was a consequence of insuficient exploration and discovery experiences and not a lack of mental hability.

From our experience we conclude that students in a ten-year-old mental level have intellectual conditions to construct the concepts of fraction if their teacher understands and respects their knowledge process. For the correctly understanding of fractions, we believe it is crucial the use of concrete material and that the students do some activities that allow them to develop strong mental images of fractions.

Bibliography:

1. BEHR, M. et alii. *Rational Number Concepts*. In LESH, R. and LANDAU, M. (eds.) Acquisition of Mathematical Concepts and Processes. New York: Academic Press, 1983.

2. DIENES, Z. Building up Mathematics. London: Hutchinson Educational, 1960.

3. LAGÔA, A. *Por que as crianças acham difícil entender frações*. Nova Escola. Ano VI, Nº 47, Abril, 1991.

4. WITHERSPOON, M. L. *Fractions: in search of meaning*. Arithmetic Teacher. Vol. 40, Nº 8, April, 1993.

The correspondence between rational numbers and the number line: a classroom experience in Secondary School.

Isabel ROMERO and Luis RICO. Dep. de Didáctica de la Matemática. Universidad de Granada. SPAIN.

Abstract

This work belongs to a wider study about the initial comprehension of the concept of real number by students of Secondary School. It has been carried out as an action research experience with a class of children fourteen-fifteen years old.

In order to explore the intuitions of the pupils about the correspondence of the rational numbers and the number line, two questions were made at different moments of the didactical process. In the first one, we asked wheter they thought that rational numbers filled the number line. The second question was posed as an exercise: pupils needed to asigne a number to some given points on a line in which the points 0 and 1 were also marked; in the end we asked whether they thought that there was always a number corresponding to any given point on the line, and which kind of number could it be.

The answers to the first question were Yes and No in a similar proportion; the reasons pupils gave were fuzzy and diverse, and in some ocassions served to justify opposite opinions. In the second question, pupils asigned finite decimals or fractions to the given points, answered that there was always a number corresponding to any point of the line, and that this number was an integer, a fraction or a decimal (finite).

This result ilustrates the difficulties of dealing with the correspondence between the geometrical continuum and the numerical continuum, which is a key point in the classical construction of the concept of real number.

References

-Arsac, G. (1987). *L'origine de la démonstration: Essai d'épistémologie didactique.* Recherches en Didactique des Matematiques, vol. 8, nº3. pags. 267-312.

-Gardiner, A. (1982). *Infinite Processes.* New York: Springer-Verlag.

-Janvier, C. (1987). *Problems of representations in the teaching and learning of Mathematics.* London: Lawrence Earlbaum Associated Publishers.

-Romero, I. (1993). *The introduction of real number in Secondary school.* Not published yet.

-Tall, D. (1981). *The notion of infinite measuring number and its relevance in the intuitin of infinity.* Educational Studies in Mathematics, vol.11, nº3. p.271-284.

AFFECTIVE VARIABLES AND MATHEMATICS ACHIEVEMENT: A STUDY OF A SAMPLE OF SECONDARY STUDENTS IN BRUNEI DARUSSALAM

Khoo Phon Sai and Palanisamy K. Veloo
Universiti Brunei Darussalam

The study explored the relationships among selected affective variables and mathematics achievement and the extent to which these affective variables accounted for observed variance in maths achievement. The subjects comprised 259 secondary students. Data was collected through the administration of an Assessment of Mathematical Abilities test (AMA) and a set of instruments consisting of Attributions in Maths (AIM) (Relich, 1986), Maths Belief Scales (MBS) (Kloosterman 1991), Maths Attitude Scale (MAS) and Maths Classroom Perception Inventory (MCPI).

The major findings of the study are generally consistent with those reported in the literature. For example, high maths achievement (as measured by AMA) is positively correlated with maths self-concept, enjoyment of maths and confidence in learning maths but negatively with maths anxiety (Reys, 1984; McLeod, 1992). Analysis of variance results showed that girls are more likely than boys to attribute success to effort, have greater self-concept and enjoyment of maths and more positive attitudes towards maths.

The method of systematic decomposition of variance based on a series of regression analyses showed that the unique contribution of each block of affective variables ranged from 0.3% to 4.7%. In contrast, variables such as gender, class type, age, previous exam grade and monthly maths test marks provided larger unique contributions ranging from 12.0% to 29.8%. The results of the study fail to confirm the hypothesis (Kloosterman, 1991) that beliefs about how maths is learned account for a considerable portion of the variance in mathematics achievement scores.

References

Kloosterman, P. (1991). Beliefs and achievement in seventh grade mathematics. *Focus on Learning Problems in Mathematics*, 13 (3), 3-15.

McLeod, D. B. (1992). Research on affect in mathematics education: a reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (p.575-596). New York: Macmillan.

Relich, J. D. et al (1986). The mediating role of attributions and self-efficacy variables for treatment effects on achievement outcomes. *Contemporary Educational Psychology*, 11, 195- 216.

Reys, Laurie H. (1984). Affective variables and mathematics education. *Elementary School Journal*, 84, 558-581.

USE OF ALTERNATIVE METHODS OF ASSESSMENT

Elizabeth Bezerra Silva and Vânia Maria Pereira dos Santos
Escola Municipal Alcide De Gasperi and Projeto Fundão-Matemática UFRJ
Universidade Federal do Rio de Janeiro (UFRJ) and Projeto Fundão-Matemática UFRJ

"Assessment practices send a powerful message to students about the mathematical thinking, experience and content that is valued" (Tzur, Brooks, Enderson, Morgan and Cooney, 1993, p. 151). Assessment methods should change if teachers want to have students valuing mathematics, knowing how to argue and question in-school as well as out-of school situations that deals with logical thinking, and willing to try out different solutions. Since 1993, a group of five school teachers together with five college students (math majors) and one university mathematics educator have been investigating the use of alternative assessment techniques in different grade levels. This paper discusses two experiences realized by one middle school teacher from this group. In the 1993 study, the 30 seventh grade students explored geometric topics (e.g., plane and spatial figures, angles and triangles) and the teacher tried out several assessment instruments. She used group work, open-ended questions, concept maps, group and individual contests, and self-assessment of their performance. The driven forces for her innovations in teaching and assessing were threefold. First, her own beliefs that should exist other ways to examine students' mathematical knowledge besides the traditional paper-pencil tests. Second, her long teaching practice giving evidence that good grades are not always equivalent to a meaningful mathematical understanding and bad grades do not necessarily means that students don't understand mathematics. And, lastly the discussions within our study group with the rich input of the college students offering their own point of view about assessment (Webb & Coxford, 1993).

The pilot experience in 1993 showed us the potential of (a) concept maps to diagnose students' conceptions and misconceptions, (b) open-ended questions to exhibit a variety of students' solutions and interpretations, (c) assessment in group to develop socialization skills and to teach students about other ways of thinking, and (d) self-assessment to develop students' awareness of their strengths and weaknesses in mathematics. The analysis of several instruments of assessment offered us with a richer picture of students' mathematical thinking and understanding. Therefore in the second study, that was initiated in February 1994 with a grade eight classroom we will pursue these aspects. Throughout the year this classroom will be observed and interviews will be conducted in order to gather further information about the advantages and difficulties of using alternative assessment techniques. In February 1994, while revisiting geometric topics explored in 1993, we have concentrated our analysis on the effects of the use of concept mapping and open-ended tasks and it was perplexing to perceive how students are achieving mathematical maturity about themselves and can solve problems in a variety of ways.

References

Tzur, R., Brooks, K., Enderson, M., Morgan, M, & Cooney, T. (1993). Teachers' use of alternate assessment methods. In J. R. Becker and B. J. Pence (Eds.), *Proceedings of the Fifteenth Annual Meeting of PME-NA, Vol. 1* (pp. 151-157).

Webb, N. L., Coxford, A. F. (Eds.). (1993). *Assessment in the mathematics classroom: 1993 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.

A GEOMETRIC-ALGEBRAIC APPROXIMATE TO THE TRIGONOMETRIC FUNCTIONS WITH THE USE OF SUPERCALCULATORS.

Leopoldo Zúñiga Silva
Tecnológico de Estudios Superiores de Ecatepec, México.

On the investigations made regarding the organization of a precalculus course, we have recoursed to the graphic representation, as the nearest to the student's intuition to agree with new concepts. It has the purpose of finding teaching options to make possible to give the students the knowledge, abilities and the necessary strategies for the study of calculus

This investigation was made in the Technological of Advanced Studies of Ecatepec, México with students in the first year of university level (between 17 and 22 years old) and it was made in the frame of the Didactic Engineering [Artige, M. 1990] as an investigation methodology.

We worked on the trigonometric functions theme attending the results obtained in the preliminary analysis of Engineering [Zúñiga, L. 1993].

The central part of the research was based in the study of the functions in relation to the mechanisms that operate the transference between the algebraic and geometric contexts.

Within the development of the experience we used the supercalculator (Texas Instruments TI-81) in order to examine to what point it gives information that can lead to the problem solution and how is it that the student interacts with it under the knowledge and abilities that was pretended to develop At the same time we are concerned of observing and analyzing the competence (as to be capable or not) of the students as the involved cognitive process'

On the other hand, the evaluation was constant, and it was made according to a resource called "Tasks", which consisted in problems designed to observe the advances and difficulties in the students throughout the research. We also applied a final exam on the theme. The type of problems worked were as the following examples.

Without the use of a supercalculator.- Determine amplitude, period, intersection points with the axis and draw the graphic of the function. $y = 3\,sen\,(x - \pi/6)$

With the use of a supercalculator.- How many solutions does the equation $sen(1/x)=0$ have? Argument your answer.

The results obtained in this investigation, shows a better competence on the students in the problems solution, as a comprehension of sufficient satisfactory concepts.

In the research (and after that) we also observed that the students appealed to the graphication in an spontaneous way with the purpose of fortifying its analytic procedures. We consider necessary in the didactic strategies manufacture and on the systematic observation of learning experiences, to study such student's conduct.

Also, the results show meaningful flowings; for example, to approach the proper calculus problems, the students subject to the study, will not have as an additional problem the graphication of functions, at least on which respects to the trigonometric functions Its possible that the abilities and knowledge acquired, the student reflect as use tools on their following courses.

REFERENCES

Artigue M., (1990) Ingénierie Didactique. *Recherches en Didactique des Mathématiques.* Vol. 9, No. 3, pp 281 - 308. La Pensée Sauvage: France.

Zúñiga L., (1993) Competencia, Cognición y Currícula en precálculo en un ambiente gráfico. un acercamiento cualitativo a las funciones trigonométricas en el marco de la ingeniería didáctica *Master Thesis.* CINVESTAV-IPN: México.

THE WORKER STUDENT AND HIS RELATIONSHIP WITH MATHEMATICS

Luiz Claudio da Silva and Vânia Maria Pereira dos Santos
Colégio Estadual Walter Orlandine and Projeto Fundão-Matemática UFRJ
Universidade Federal do Rio de Janeiro (UFRJ) and Projeto Fundão-Matemática UFRJ

There is considerable research documenting the gap between in and out-of school mathematics(e.g., Bishop & Abreu, 1991; Carraher, 1988; Lave, 1988) and there is still a need to investigate ways to actually bridge this gap with school teaching. This study is part of a project that explored the language and discourse of a group of 15 fourth grade students (with ages varying from 15 to 24) that came to an evening school after a full day of work (Silva, 1994). Data source included a dialogue, classroom observations and audio-taped interviews. The dialogue with the students was to identify their professional activity and mathematical concepts they used with no awareness of them. In this paper we talk about one student who affirmed in class that disliked mathematics. When describing his daily routine as an apprentice of baker he talked about a recipe for making French bread. He explained the effects of heat on the yeast, the different proportions he had to be aware of when doubling or halving the recipe, and ways he would divide the dough. The language he used to describe his task was full of mathematics and he wasn't aware of it. The two episodes (the oral and the written recipe) together with an audio-taped interview helped us to identify the richness of terms and meanings he had used. His interest for mathematics began to increase after these events with the researchers. This occurred because we could point out for him work situations that had a lot to do with school mathematics. In his discourse we could identify room to explore (a) differences and similarities between forms and size; (b) length, mass and volume conservation; and (c) fraction concepts (half and third of continuous and discrete sets, proper fractions and equivalence of fractions). His out-of-school experience with mathematics (as well as the other students' professional experience) offered a better context for learning mathematics meaningfully. And could easily raise more interest in students than the routine and out of context problems proposed by the teacher or textbook. But the teacher wasn't able to perceive the potential of her students' experience or to use the information provided by the research to make a bridge between their use of math in the work culture with the school mathematics. We strongly agree with Borba (1993) when he argues that an alternative for the classroom action would be for teachers to consider a closer relationship between tacit knowledge from culture and school knowledge.

References

Bishop, A., & Abreu, G. (1991). Children's use of outside-school knowledge to solve mathematics problems in-school. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education, Vol. 1* (pp. 128-135). Assisi, Italy.

Borba, M. C. (1993). Etnomatemática e a cultura da sala de aula. *Educação Matemática em Revista, 1*, 43-58.

Carraher, T. N. (1988). Street mathematics and school mathematics. In A. Borbás (Ed.), *Proceedings of the Twelfth International Conference for the Psychology of Mathematics Education, Vol. 1* (pp. 1-23). Veszprém, Hungary.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life.* New York: Cambridge University Press.

Silva, L. C. (1994). *Linguagem- Um obstáculo epistemológico na construção do conhecimento formal da matemática.* Unpublished master's thesis, GEPEM-Universidade Santa Ursula, Rio de Janeiro.

A VERBAL-ARITHMETIC PROBLEM SOLVING MODEL, THAT JUXTAPOSE TWO SELF-SUFFICIENT REPRESENTATIONAL SYSTEMS

Josefa Hernández and Martín M. Socas

La Laguna University (SPAIN)

The Polya' model (1957) to solve mathematical problems, inspiration source to a broad list of subsequent models (Schoenfeld, Mason-Burton-Stacey), and the model for competence based on representational systems proposed by Goldin (1985,87) within the framework of information-processing theory give us changes in the raising of arithmetic problem solving, since these are moved from a verbal-syntactic system to the formal notational system.

We are doing research on a model to solve verbal-arithmetic problems, adapted for Primary schoolchildren (8-11 years), which contemplate the Polya' stages and which facilitate the use of different representational systems.

This model consists of 6 stages, not necessarily lineal: reading; understanding; visual-geometric representation-performance and solution; formal representation-performance and solution; solutions and verification. Starting from reading, we are trying to understand it with the help of the use of drawings, which represent the situation as a whole together with the writing of the data and the unknown. The third stage tries the children to solve the problem in the visual-geometric system in a self-sufficient way, we are previously learning to use this representational system in its semantic and syntactic aspects. We emphasize that the visual-geometric solution or the diagram are not only considered to be a visual support, which is basically semantic in research, but also an autonomous way of solving problems. The fourth stage is referred to the problem formal solution by means of arithmetic operations. Both solution ways involve a verification in terms of equality or inequality (5th stage). The sixth stage consists of the verification of the solution within the framework of the problem wording. During the processes we are trying to control the planning system used by the children, that is to say, to develop a manager who control their own advance, as well as the attitudes which cause the performance of the model towards the problem solving and mathematics.

We are design a normal index card to implement this model, which can be extended to non arithmetic problem.

References:

-Goldin, G.A. (1985): "Thinking scientifically and Thinking Mathematically". In Silver (Ed): "Teaching and Learning Mathematical Problem Solving". Hillsdale N.J.: Lawrence Erlbaum Associates.
-Goldin, G.A. (1987): "Cognitive Representational Systems for Mathematical Problem Solving". In Janvier (Ed.) "Problems of representation in the teaching and learning of mathematics". Hillsdale N.J.: Lawrence Erlbaum Associates.
-Polya (1957): "How to solve it". New York.

Pre-algebra from a different perspective

L. Streefland
Freudenthal institute
Utrecht, The Netherlands

History

The term 'algebra' stems from 'Hisab *al djabr* we al moekabalah' by al Charezmi († 840) 'Algorithm' stems from the author's name. The title means *reorganising* or *regrouping* terms or parts, while changing their state. Those activities, that also might be characterised by *schematising* and *algebrising* concern the treatment of first and second degree equations in a verbal-numerical format Is not this algebra, or is it? Yes, it is: algebra without letters: Was pre-algebra anticipating al Charezmi's work?

François Viète (1540-1603) was the *first* mathematician who used letters for both known and unknown (values of) magnitudes in equations. It was a great step forward. The general treatment of equations had been achieved then. Is this algebra? Superfluous to ask!

Equations are only one part of algebra and their transition from an arithmetical environment to a formal algebraic one took a considerable amount of time. This implies a 'warning' If pre-algebra is defined as a stage of transition from arithmetic to algebra, then - as historical development shows - one needs to be careful about the duration thereof in mathematics education.

Pre-algebra

The historical learning process of mathematics shows both the foreshadowing of future events (anticipating) as well as looking back (viewing them retrospectively).

Examples of pre-algebra from the 'Maths in context project'[1] will be presented, together with some provisional results. They will illustrate both the beforementioned general principles, that is taking a retrospective view **aiming at** anticipation and their mutual connection in a course. It will turn out that the transition from primary school mathematics (arithmetic) to algebra can be anticipated rather early by intertwining [2] particular learning strands in primary school mathematics.

Conclusion

After all the examples will make clear, that mathematics education as a subject is a *design* science and that the educational profits of research, for instance, are not determined by the integrity of the research-design and the methods applied, but rather by the justice done to the subject matter in the background of the philosophical and theoretical framework one has This means that distortions need to be avoided. From the learners perspective formal mathematics is distorted mathematics due to its rigidity as a product. This is true for pre-algebra in particular.

notes

1) This is a co-project of the Freudenthal institute and the National Center for Research in Mathematical Sciences Education at the University of Wisconsin, Madison, USA.

2) Streefland, L. *The design of a mathematics course A theoretical reflection*, L. Streefland (ed), The legacy of Hans Freudenthal, special issue of Educational Studies in Mathematics, 25 (1/2) 1993, 109-135.

ACCELERATING THE DEVELOPMENT OF COGNITIVE AND METACOGNITIVE SKILLS: THE MATHEMATICAL THINKING SKILLS PROJECT

Howard Tanner and Sonia Jones
University College of Swansea, Wales, UK.

The ability to think formally and to generalise and explain underpins the secondary mathematics curriculum. Recent research studies using problem centered learning suggest that cognitive acceleration is possible in early adolescence. (Cobb et al 1992, Shayer & Adey 1992) There is also evidence to suggest that pupils' metacognitive development can be enhanced. (Tanner & Jones 1993)

This paper describes a project to develop and evaluate a thinking skills course. There are two strands to the course:

1. the development of a structured series of cognitive challenges to stimulate the progressive evolution of key skills in the areas of strategy, logic and communication;
2. the use and development of teaching techniques which encourage the maturation of the metacognitive skills of planning, monitoring and evaluation.

Underpinning both strands is a continual emphasis on the need to explain rather than describe, to hypothesise and test, and to justify and prove. Activities are structured to encourage the development of a small number of general strategic or cognitive tools. Each activity is targeted on at least one of the schema of formal operations.

350 children aged between 11 and 13 from twelve classes in six secondary schools in Wales followed the course and were compared with an equal number of matched control groups using pre-tests, post-tests and structured interviews. Assessment instruments were devised to assess pupils' levels of cognitive development, and their ability to use strategic and metacognitive skills. A Likert type attitude questionnaire was applied and subjected to factor analysis. Statistical data were supported by participant observations made and recorded during intervention lessons.

References:

Cobb, P., Wood, T., Yackel, E. & Perlwitz, M., (1992) 'A follow-up assessment of a second-grade problem-centered mathematics project', *Educational Studies in Mathematics*, **23**, 483-504.

Shayer, M. & Adey, P.S., (1992) 'Accelerating the development of formal thinking in middle and high school students II: Post-project effects on Science achievement', *Journal of Research in Science Education*, **29**,1,81-92.

Tanner H. & Jones S.A., (1993) 'Developing metacognition through peer and self assessment', in *Teaching and learning mathematics in context*, eds T.Breiteig et al, Ellis Horwood.

THE FUNCTION CONCEPT FOR 5TH TO 8TH GRADE STUDENTS

LUCIA A.A.TINOCO
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO - BRASIL

The function concept has been introduced at secon('y school in Brazil about fourty years ago, and its teaching is concentrated on the last three years in most of the schools. But the importance of this concept and the difficulty of its teaching have incentivated the research in this topic. Most of them reffer to the use of machines ir order to improve the students' aquisition of the concept.

The present works was developped by a group of the "Projeto Fundão" team, at 1992/93, and focused the four phases of the function concept construction, based on the classification proposed by Bergeron, J. and Herscovics, N (1982), particularly, the second and third phases: initial mathematization and abstraction which can be developped from grade 5th to 8th.

The work is grounded on the answers of secondary teachers and students to a questionary/test applied in order to know their main ideas about the function concept and its teaching. Analyzing the results of the questionaries and tests, we have observed that there is no comprehention of the function concept in the meaning reinforced by Ponte (1990): as an especific tool for explore variation problems.

In this sense, the group has concentrated its attention to these two tasks:
1) The explicitation of the main characteristics and involved ideas in the first three phases proposed by Bergeron and Herscovics (1982) (intuitive comprehention, initial mathematization and abstraction), to be developped in the first years of the secondary school.

Through a serious work at that phases it is possible to have a formal treatment of the functions in the last high school years, but now with the pertinent notation and nomenclature.

2) The experiments have been made in 6th, 7th and 8th grade classes of public schools in Rio de Janeiro by teachers and undergraduate students of the group.

Its results are beeing useful to:
- aprimorate the conclusions about the phases' characteristics and activities;
- recognize some other knowledges and tools necessary to improve the construction of the function concept by the students.

REFERENCES:

Bergeron, J.C. e Herscovics, N (1982) - Levels in the Understandig of the Function Concept Proceedings of the Workshop on Functions p.p. 1-11, Enschede, Holanda.
Ponte, J. P. (1990) - O Conceito de Função no Curriculo de Matemática, Educação Matemática, nº 15, pp. 3-9, M. Lisboa, Portugal.

CHILDREN'S UNDERSTANDING OF RANDOM GENERATORS

Kathleen Truran

University of South Australia

This paper discusses the premise that young children do not intuitively perceive similarities between the behaviour of different, but mathematically identical, random generators.This study is innovative because children are asked to compare the behaviour of one random generator with others. These experiences are not generally encountered as part of the classroom mathematics experience,and so far as the writer can ascertain, similar questions have not often previously been asked by researchers. Previous studies have in the main been limited to investigating the behaviour of one random generator, usually a coin, dice or spinner of the type of question presented by, for example, Green (1983) and Hoemann and Ross (1982)

The communication will summarise data collected using group tests and individual interviews from primary school children aged from 7-12 years from a range of primary schools in South Australia. It will explain children's perceptions of the behaviour of familiar and unfamiliar random generators in identical situations, for example, playing a game of Ludo. The child is asked doe example whether a game of L.:do would be equally fair if it was played using a six-sided die, six numbered balls in an urn, tickets numbered 1-6, or a spinner with 6 equal sections are used.

This research investigates the thinking processes and affective ideas children use when confronted with such situations. Examples of r.sults based on gender and culture will also be discussed.

Preliminary findings indicate that children predict different results depending on whichever random generator is used in a game. This raises the question are their responses developmental or experiential; or do children respond to a tactile stimulus? For example:

I. Why do you say the game of Ludo will not be fair if you played using the six-sided die and these tickets numbered 1-6?

R Because they are different. That (the die) is plastic and feels hard. These (the tickets) are paper and they are soft.

REFERENCES

Green, David R. (1983) "Shaking a Six" *Mathematics in School* 12 (5): 29-32

Hoemann, Harry W. and Ross Bruce M. (1982) "Children's Concepts of Chance and Probability" in Brainerd, Charles J. (ed) (1982) *Children's Logical and Mathematical Cognition. Progress in Cognitive Development Research* (pp.93 - 121) New York: Springer Verlag

CONTROL PROCESSES DURING THE SOLUTION OF GEOMETRIC CONSTRUCTION PROBLEMS IN COMPUTER

Marianna Tzekaki, Lecturer,
Department of Early Childhood Education, Aristotle University of Thessaloniki

The proposed oral communication refers to the operation of control process developed within the frames of the solution of geometric construction problems in computers. It presents data from the observation of students (12-13 years old) during the construction of geometric figures in a computerized environment, especially organized in order to offer control indications along the course of a construction.

Such indications, related to the faced geometric knowledge, allow the students to examine the validity of their decisions and their actions and to look for answers adjusted to the construction problems they face. It must be point out that the students, interpreting such external indications, tend to convert them into internal control criteria, according to their knowledge. This fact leads them to adjust or even to change the indications meaning, in order to correspond to their pre-existing structure. Thus, the paper presents the interpretation or change procedures of control indications, which are a priori homogeneous, but their reading varies per student and per faced geometric concept.

It is indicated that the functionality of such control processes depends on how an external indication tends to be assimilated in a proper internal criterion, that is in a criterion which, for each faced knowledge, gives to the pupil valid control means for his action or for his decision.

REFERENCES

1. BALACHEFF N., 1988, *Une étude de processus de preuve en mathlématique chez des élèves de Collège*, Thèse d'Etat, Université Joseph Fourier, Grenoble

2. BELLEMAIN F., CAPPONI B. 1992, Spécificité de l'organisation d'une sequence d'enseignement lors de l'utilisation de l'ordinateur, *Educational Studies in Mathematics*, vol 23, No. 1, pp. 59-98

3. BROUSSEAU G. (1986), *Théorisation des phénomènes d'enseignement des mathématiques*, Thèse d' état, Université de Bordeaux 1

4. CARREGA J.C. (1981), *Théorie des corps; la règle et le compas*, Paris, HERMAN

5. LABORDE C. (1989), L' enseignement de la géométrie en tant que terrain d'exploration de phénomènes didactique, *Recherches en Didactiques des Mathématiques*, vol.9.3 pp. 337-364

6. MARGOLINAS C, 1993, *De l'importance du vrai et du faux*, ed. La Pensée Sauvage, Grenoble

7. OSTA I., 1988, *L'ordinateur comme outil d'aide à l'enseignement. Une séquence didactique pour l'enseignement du repérage dans l'espace à l'aide des logiciels graphiques*, Thèse de l'Université Joseph Fourier, Grenoble

9. PIAGET J., 1975, *L'équilibration des structures cognitives*, coll. Etudes d'épistémologie génétique, ed. P.U.F., Paris

A PILOT STUDY ON STUDENTS' COGNITIVE DIFFICULTIES IN CALCULUS

Behiye Ubuz

University of Nottingham *
Shell Centre for Mathematical Education
Nottingham, NG7 2RD, UK

Middle East Technical University
Science Education
Ankara 06531, TR

This research reports the results of a pilot study which aimed to get feedback about students' cognitive difficulties in calculus. A diagnostic questionnaire was designed to observe them and piloted with 18 A-level second year students in England, 1993. A-level is a 2 year academic course for 16-18 year old students. The primary concern was to approach these difficulties by studying the students' errors and misconceptions. This research also examines the implication of these difficulties for constructing the design of the main study. The purpose of the main study is to investigate engineering students' cognitive difficulties in calculus in computer-based environment compared with non-computer-based environment.

A substantial amount of research in errors and misconceptions has been done at the precollege and college level in traditional environment (Orton, 1980; Selden, 1989; Amit and Vinner, 1990), but a smaller body of work exists at these levels in a computer-based environment (Tall, 1986).

The presentation will focus on the issues arising from the research to form the hypotheses of the main study. A brief explanation of findings correlated with literature will also be given.

References

Amit, M., & Vinner, S. (1990). Some misconceptions in calculus- anecdotes or the tip of an iceberg? Proceedings of PME 14, Mexico, 1, 3-10.

Orton, A. (1980). A cross-sectional study of the understanding of elementary calculus in adolescents and young adults. Unpublished Ph.D Thesis, University of Leeds

Selden, J., Mason, A., & Selden, A. (1989). Can average calculus students solve nonroutine problems? Journal of Mathematical Behavior, 8, 45-50.

Tall, D. (1986). Using the computer graphics to represent calculus concepts. Plenary lecture, Le IVème École d' Été de Didactique des Mathèmatiques, Orléans, Recueil des Textes et Comptes Rendus, 238-264.

* present address

AN ANALYSIS OF DIAGRAMS USED BY SECONDARY SCHOOL PUPILS IN SOLVING MATHEMATICAL PROBLEMS

Palanisamy K Veloo
Universiti Brunei Darussalam

Francis Lopez-Real
University of Hong Kong

Two hundred secondary school pupils were given a set of ten mathematical word-problems to solve. The pupils ranged from Form 1 to Form 4 and the mathematical content of all the problems did not go above the Form 1 level. The pupils were instructed to draw a diagram which they considered would be helpful for solving each problem and to incorporate any important information into the diagram. This presentation analyses the diagrams and discusses the results with respect to four classifications:
i) the degree to which the diagram was concrete or symbolic
ii) the degree to which the relational aspects of the problem were represented
iii) how much of the relevant data was present in the diagram space
iv) the extent to which the data was incorporated into the diagram.

The analysis showed a statistically significant difference in the *type* of diagram drawn by the four age-groups. The Form 3 & 4 pupils drew more diagrams at the Symbolic end of the continuum and less at the Concrete end compared with the Form 1 & 2 pupils. A similar result was found with respect to the Relational construct, again the Form 3 &4 pupils drawing more diagrams at the positive end of the continuum. As regards the correlation between a successful outcome for the solution of a problem and the type of diagram drawn, two highly significant factors emerged. These were: (i) the degree to which the *relational* aspects were represented, and (ii) the degree to which the data was *incorporated* into the diagram.

However, a very important 'negative' result also emerged from the study. Despite being given explicit instructions to draw a diagram, nevertheless *no* diagram was drawn in approximately 30% of all cases. The hypothesis that the diagram was omitted in cases where the pupils could easily solve the problem by other means was not borne out in the analysis. In fact the opposite is the case, the paired frequencies of 'Diagram Drawn' and 'Success' being significant at the 1% level.

The implications of these results are discussed and in particular it is argued that there is an urgent need to help children with a diagram-drawing strategy for problem-solving in an *explicit and positive* way rather than simply leaving it to chance and the spontaneous intuition of a few visually-oriented pupils.

References

Gonzalez, N.A. (1986) An exploratory study on the teaching of diagram-drawing as a problem-solving strategy in college algebra. Unpublished doctoral dissertation. Harvard University.

Lopez-Real, F & Veloo, P.K. (1993) Children's Use of Diagrams as a Problem-Solving Strategy. *Proceedings of PME XVII (Japan).*

The Construction and Re-presentation of Images in Mathematical Activity

Grayson H. Wheatley and Dawn Brown
Florida State University

Evidence is mounting that imagery plays a significant role in mathematical reasoning. When students are engaged in meaningful mathematics rather than rote computation, it is quite likely they will be using some form of imagery. Kosslyn posits three processes in the act of imaging: construction , re-presentation, and transformation of the image. In previous work we have discussed transformation of images and the role it plays in mathematics learning. Drawing upon the work of Brown, Kosslyn, Krutetski, Lakoff, Presmeg, von Forester, and Wheatley, this paper will lay out a theoretical position on the construction and re-presentation of images in relation to mathematical activity.

The construction of an image is the most fundamental of the three processes. If a student fails to initially construct an image there is nothing to re-present or transform. While engaged in mathematical activity, whether of a numeric or geometric nature, students construct images. For example, they may be shown a geometric figure briefly and asked to draw what they saw. When they make their drawing, they are operating from a constructed image. The nature and quality of the image will influence the drawing which results. If at a later time they are asked to draw what they saw, the students then must re-present the image. This act of re-presentation is complex and subtle. For example, Piaget has shown that the image constructed may undergo change over time with no intervening intervention. In many cases the re-presented image may have been modified or it might be a prototype which is then transformed based on the demands of the task. Furthermore, the nature of the re-presentation is greatly influenced by the intentions and goals of the individual at the time of re-presentation.

This presentation will provide elaborated descriptions of image construction and re-presentation in the act of doing mathematics as well as detailed examples of each. Examples of different re-presentations being evoked by different tasks will be illustrated. Based on our previous work, we believe activities which encourage the construction of images can greatly enhance mathematics learning. Further, some individuals are particularly successful in constructing and re-presenting images. Our programmatic research over the past five years suggests that students who naturally use images in their thinking easily make sense of novel mathematics tasks while students who score low on a test of mental rotations often do not.

THE INTERVIEWER'S ROLE IN THE EMERGENCE OF MEANING

Tracey Wright
TERC, Cambridge, MA

This presentation will be based on my experience interviewing elementary school children in the Students' Conceptions of the Mathematics of Change project. The project intends to contribute new foundations for the teaching and learning of the mathematics of change from elementary school to college. A major goal is to explore ways to foster continuity between elementary school mathematics and calculus. Some of the thematic focuses are qualitative integration and differentiation, the interpretation of numerical and graphical patterns to describe change, and the interplay between symbolic expressions and physical change over time or space.
Our elementary study is based on what we call Learning Situations, that is, four or five sessions during which individual students hold a conversation with the interviewer as they explore ideas and use tools to make sense of mathematical problems.

What is the goal of such a conversation? It is not to teach the child because we do not have a specific pre-defined way of understanding the problem situations that we want the child to adopt. It is not to evaluate the child because we do not use an apriori scheme to categorize children's responses. The primary goal of our interviews is to understand what becomes meaningful for the child as she deals with situations involving motion and graphing.

The focus of this presentation will be an analysis of the role of the interviewer in light of this ongoing challenge: How does meaning emerge for both the interviewer and the child in the context of an interview? How does the interviewer help the child to express what is significant to her? In what sense does the interviewer guide the child through the particular graphing phenomena we wish to explore? I will also focus on the interaction between the interviewer and the subject. How does what is meaningful get negotiated? Who is the teacher? Who is the learner? Is there an expert here?

After exploring these questions I will address a broader issue: How do we as researchers make sense of what the child does and says in light of our beliefs that the interviewer's words and actions must influence the child? I will give examples from my experience as an interviewer, looking at both the child and the interviewer as subjects.

Reference: Tierney, Nemirovsky, Wright and Ackerman (1993). *Body motion and children's understanding of graphs.* In PMENA 15th Proceedings (Pacific Grove CA).

POSTERS

MATHEMATICAL UNDERSTANDING AMONG SUGAR CANE FARMERS

Nadja Maria Acioly, C.N.R.S, FRANCE

Abstract

This study is aimed at analysing the cognitive functioning of illiterate subjects or subjects with little schooling in the mathematical domain. These subjects, while working as sugar-cane planters, have to deal with mathematical computations involving the determination of the area of different plots of land, and to develop, without any direct help from school instruction, an understanding of mathematical relations. This understanding, however, is developed through the use of non-precise schemes, which, depending on the measures chosen to be included in the computation, may sub estimate or overestimate the final results. Twenty-one sugar cane workers, occupying different levels in the hierarchy of the sugar cane plantation, from cane cutters to supervisors and administrators, took part in the study. Following an ethnographic study of the work in the plantation, data were collected in three phases. In phase I subjects were individually interviewed about how they would compute the area of plots of measures spontaneously given by themselves, followed by the computation of areas of plots with measures suggested by the interviewer. In this phase, the precise shapes of the plots (quadrilaterals or triangles) were not drawn. Phase II consisted of a more formal interview in which the subject was presented, in view of drawings on paper, with the same perimeter, but different surfaces. They had to: (a)compute the area of quadrilateral and triangles representing plots of land with all measures indicated; (b) compare the surfaces of figures representing plots of land; and (c) double the area of a square. Phase III consisted of field observations and interviews to clarify some points raised by the results of previous phases, and determine the most common types of plots that workers deal with in their everyday activities. Results of the first phase show the constant use of non-precise computational schemes with choice of elements that overestimate (in the case of cane cutters) or sub estimate (in the case of supervisors and administrators) the area to be worked. In the second, more formal phase, new strategies, aimed at correcting distortions increased by the irregular shape of plots were found. Finally, in the third phase, we found that workers at all levels seem to be aware of the distortions resulting from the use of a wrong formula and that these distortions can be compensated by other social mechanisms. The choice of measures in this context may be interpreted as a result of the particular social interactions in which computations take place, thus illustrating how the development of mathematical understanding is intertwined with the cultural context involved in the situation where problems are solved. Results are discussed in terms of methodological and conceptual issues relating culture and the development of mathematical knowledge.

AWARENESS OF LEARNING, REFLECTION AND TRANSFER IN SCHOOL MATHEMATICS

Alan Bell
Shell Centre for Mathematical Education
University of Nottingham

The aims of this project were to investigate the metacognitive skills and concepts possessed by students of secondary school age in some typical mathematical learning environments, to explore the feasibility of raising the levels of their awareness by appropriate interventions, and to study the effects of such enhancement on the students' mathematical attainments.

The outcomes of the project include, as well as these results detailed below; the Teachers' Handbook containing the set of suggested enhancement activities, trialied and including examples of students' work; the Evaluative Instruments, partially developed but needing further improvement; and the set of Case Studies of the seven classes during the main experimental year. *(Documents available at present from the Shell Centre are the Summary Report , the Teachers' Handbook , Evaluation).*

Excerpts from each of these aspects will be displayed.

NUMERICAL UNIVERSES

Alicia Bruno y Antonio Martinón (Universidad de La Laguna)

In the teaching of the natural numbers to primary school it is usually insisted that these numbers can be used to express .ardinals and not in the fact that can be used to express a measurement. We believe that, always that were be possible, activities would be carried out in which numbers for measure were used, because in this way the learning of the numbers would be more complete and more congruent with posterior numerical extensions.

Numerical Situation: A number is used to express different numerical situations ("I have got 2 sweets", "the length of the table is 2 meters,...).

Numerical Region: The situations can be grouped in numerical region according to a characteristic concept (cardinal, ordinal, ratio of cardinals,...). A numerical region is of the numerical set $\mathbb{A}$ if all the numbers of $\mathbb{A}$ can be used to express numerical situations of this region. For example, the following are regions of $\mathbb{Z}_+$: ordinal, cardinal, ratio of cardinal, measurement of scalar magnitudes of one direction, measurement of scalar magnitudes of two directions. However, the cardinal region is not of $\mathbb{R}$, because 0'3 does not express a cardinal.

Numerical Universe: The numerical universe of a set $\mathbb{A}$ is the set of the numerical regions of $\mathbb{A}$.

If we consider an extension of the numerical set $\mathbb{A}$ to the numerical set $\mathbb{B}$, the number of regions of $\mathbb{A}$ is greater than the number of regions of $\mathbb{B}$, that is to say, the numerical universe decreases. However, the numerical situations corresponding to a common region to both sets increase. For example, with the numbers of $\mathbb{Q}_+$ more lengths than with those of $\mathbb{Z}_+$ can be measured, but the cardinal and ordinal regions are not in the numerical universe of $\mathbb{Q}_+$ which are in universe of $\mathbb{Z}_+$. In the practice of teaching, when extending the number sets usually introduces pupils to an environment in which it seems that the interpretations of the numbers (their numerical universe) is increased, when in fact what increases are the numerical situations belonging to regions common to both sets. This causes that pupils associate each number set almost exclusively with a single region, the one on which it was insisted more. In the literature certain ideas are found that have some points in common with numerical universe: Kieren (1988), Greeno (1991) and Vergnaud (1990).

REFERENCES

Greeno, J. (1991). Number sense as situated knowing in a conceptual domain. **Journal for Research in Mathematics Education**, 22, 3, pp 170-218.

Kieren, T (1988). Personal knowledge of rational Numbers: Its Intuitive and Formal Delopment. In Hiebert, J. and Bher,M. (eds). **Number concepts and operations in the middle grades**. LEA. pp 162-181.

Vergnaud, G. (1990). La theorie des champs conceptuels. **Recherches en Didactique des Mathématiques**, 10, pp 133-170.

MATHEMATICAL EDUCATION AN PUPILS OF DEPRIVED CULTURAL BACKGROUNDS

Inés Mª Gómez Chacón
Instituto de Estudios Pedagógicos Somosaguas. Spain.

In this study we tray to evaluate students' ability in order to learn mathematics. This pupils belong to the Centre " Taller NORTE JOVEN" -placed in a suburban area of Madrid- and follow a Math program for students in a disadvantaged social-cultural context and all of them are out of school system and have a limited backgrounds in mathematics.

It describe how cultural background and affective factors could be important in the way they learn and they are taught.

The method I have followed has been elaborate mainly through:

- Autobiographies in about their experiences in clasroom' s teaching and learning mathematics.

- Strategy games. As games can replace some of the uninteresting routines of drill and practice with a self-motivating procedure, and they can offer to "piayers" a variety of problem solving experiences during which they are enabled to observe themselves as problem-solvers, and upon which they can later reflect, in order to appropriate mathematics as their own.

THE COLOR CARDS - LOGICAL REASONING TESTS (CC-LRT)

Christos Chasiotis

Department of Mathematics, University of Ioannina, Greece

The educational importance of logical reasoning, the unexpectedly low level of student performance in that area and the great diversity of the theoretical and methodological aproaches of the related educational and psychological research, justifies the search and development of suitable and convenient assessment and research instruments.

The Color Cards - Logical Reasoning Tests (CC-LRT), based on simple games with a collection of color cards, free of logical and mathematical terminology, can be utilised for the assessment of logical reasoning of students and teachers, in all school levels, and the investigation of the corresponting cognitive processes.

The variation of different variables of the situation, number of different colors and number of cards, logical form of propositions and type of logical tasks, leads to different forms, equivalent or not, of these tests. Some of these forms can be considered as variations or extensions of the well known Wason's *Four Card* or *Selection Task.*

Some forms of these tests, concerning deductive reasoning from a conditional hypothesis and testing a conditional hypothesis with complete or incomplete data, will be presented, followed by some results of alternative analyses of students responses, indicating that partial correct responses can be obtained by the application of incorrect rules, and leading to alternative hypotheses about the underlying cognitive processes.

References

Chasiotis, C. (1988). Etude didactique de situations utilisees dans l' enseignement du raisonnement logico-mathematique. *Communication at the I.C.M.E.-6, C.D.M.* Fas.1.

Johnson-Laird, P. N. & Byrne, R. M. J. (1991). *Deduction.* Laurence Erlbaum Ass. Pub.

O'Daffer, P. G. & Thomquist, A. (1993). Critical Thinking, Mathematical Reasoning, and Proof. In P. S. Wilson (Ed.) *Research Ideas for the Classroom: High School Mathematics.* N.C.T.M., Macmmilan Pub. Co. pp. 39-56.

Overton. W. F. (1990). *Reassoning, Necessity and Logic: Developmental Perspectives.* Laurence Erlbaum Ass. Pub.

Sutton, R. E., & Ennis, R. H. (1985). Logical operations in the classroom. In T. Husen, *The International Encyclopedia of Education.* Pergamon Press. pp. 3129- 3139.

Wason, P.C., & Johnson-Laird, P. D. (1972). *Psycology of reasoning: Structure and content.* Cambrige, MA: Harvard University Press.

CARNAVAL MATEMÁTICO
"FORMAL" MATHEMATICS IN AN "INFORMAL" SETTING

Marta Civil
University of Arizona, Tucson, USA

This poster describes a mathematics project for ethnic minority children (ages 11-13). The overall goal was to provide the participants with a mathematically rich and supportive environment to promote their interest and confidence in doing mathematics. The program used a combination of "formal" and "informal" time. During the formal time, the participants worked in small groups at four different stations on activities ranging over probability, geometry, patterns and functions, and measurement. Each station had a middle-school teacher whose role was to facilitate the children's work by encouraging mathematical conversations around the participants' ideas. During the informal time, the participants chose among an array of logic and strategy games and geometric and topological puzzles that were spread around the room. These "carnivals" took place in non-school settings: the Children's Museum and the Public Library.

This project set out to 1) broaden the participants' views about what mathematics is by engaging them in topics and methods that usually receive little to no attention in school; 2) promote group work in mathematics by presenting them with activities that called for cooperation with their peers; 3) enhance the participants' communication skills in mathematics by encouraging them to explain their thinking.
The participants expressed surprise that what they were doing was actually mathematics. The influence of their in-school mathematics experience was quite noticeable. Yet, they appeared to gain an appreciation for the different methods to go about a problem and for the variety in the tasks they worked on. The poster will elaborate on each of the above objectives as well as discuss implications for school teaching. This project provided a laboratory for in-school teaching innovation by giving the teachers a chance to focus on mathematics topics and instructional methods that are not commonplace in school curricula. The teachers have since then implemented many of the aspects of the project in their regular classroom teaching.

A research objective of this project was to study the participants' behavior during the informal time, in which they had more control over what to do and with whom. The conceptual framework guiding this work draws upon research on formal versus informal learning, especially on studies on mathematical performance in out-of-school versus in-school settings (Lave, 1988; Nunes, 1992; Schoenfeld, 1991). The sources of data consist of surveys, interviews, observations, the participants' folders, and videotaping. I looked at questions such as: What caught the participants' attention? How did they tackle the task? How persistent were they? Did everybody participate? The poster addresses these questions as well as implications for the development of mathematics programs in "informal" settings.

References
Lave, J. (1988). Cognition in practice: Mind, mathematics, and culture in everyday life. New York: Cambridge University Press.
Nunes, T. (1992). Ethnomathematics and everyday cognition. In D. A. Grouws (ed.), Handbook of research on mathematics teaching and learning (pp. 557-574). New York: Macmillan.
Schoenfeld, A. H. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. F. Voss, D. N. Perkins, & J. Segal (Eds.), Informal reasoning and instruction (pp. 311-343). Hillsdale, NJ: Lawrence Erlbaum.

Funding for Carnaval Matemático was provided by a grant under the Eisenhower Mathematics and Science Education Act.

TRIADS: The implications of a new theoretical structure

David Clarke
Australian Catholic University
Oakleigh, Victoria 3166, Australia

Carne Barnett
Far West Laboratory
San Francisco, Cal., USA

Sandra Frid
Curtin University
Perth, West Australia

ABSTRACT

Recent research in education, and mathematics education, in particular, has led to the identification of independent categorizing systems intended to mirror the structures found in such diverse fields as teacher professional development (Barnett, 1992); student writing in mathematics (Clarke, Stephens, & Waywood, in press); and student acquisition of calculus knowledge (Frid, 1992). There are particular characteristics of these categorizing systems which display a tantalising similarity:

- Contextual similarity - the common location of all three studies within educational environments;
- Structural similarity - the "three-valued" (triadic) structure of all three categorizing systems;
- Conceptual similarity - categories in each system resemble each other in the nature of their conceptual distinctions.

This degree of similarity suggests that each categorizing system is an independent manifestation of a more fundamental triadic system (TRIADS). This paper examines the characteristics of these triadic systems and makes comparison with other systems (or analytical frameworks) found in the research or theoretical literature, in an attempt to establish the significance of the degree of conceptual similarity found in the categorizing systems employed in mathematics education. It is proposed that the privileging of abstraction which has characterized contemporary models of learning is mistaken, and that cognitive sophistication be identified with personally contextualized knowledge rather than with formally abstracted knowledge. TRIADS is proposed as a robust structure having relevance in a variety of educational contexts. It is also proposed that conceptual similarities between the first two levels of TRIADS and Skemp's (1976) diadic structure for mathematical understanding support the addition of a third level, to be called *Contextual Understanding*.

References

Barnett, C. & Sather, S. (1992). Using case discussions to promote changes in beliefs among mathematics teachers. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Clarke, D.J., Waywood, A., & Stephens, M. (in press). Probing the structure of mathematical writing. Accepted for publication in *Educational Studies in Mathematics*.

Frid, S. (1992). Undergraduate calculus students' language use and sources of conviction. Unpublished doctoral dissertation, University of Alberta, Canada.

Skemp, R.R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, No. 77, December, 1976, 20 - 26.

Mathematics Curriculum Design - A case: 12-16 in Andalucía (Spain)

Rafael Carretero. Centro de Profesores. Aracena, Huelva (Spain)
Moisés Coriat. Dep. de Didáctica de la Matemática. Universidad de Granada. Spain
Pedro Nieto. Centro de Profesores. Sevilla (Spain)

Abstract

According to the 1990's Spanish *Ley de Ordenación General del Sistema Educativo* (LOGSE), Mathematics Curricula in Andalucia have been designed along various years and, for the 12-16 Step, they end with Advisory Documents. We give a visual sketch of an Advisory Document (Carretero, Coriat y Nieto [1993]):

Firstly, we present an overall view of Andalucía's **open** Mathematics Curriculum. The open character leads to state three levels in the teachers' planning and programming tasks: **Sequencing, Organizing and Class-Room Activities.**

Secondly, the attention is focused on the Organizing Level, and, within it, on the **Adapted Conceptual Structures** (ACSs) (Cockroft [1985], Rico [1992]), whose meaning and general framework are resumed.

The framework is then applied to an ACS exemple whose title is **Big and Small.**

Globally, Sequencing, Organizing and Class-Room Activities Levels are intended as tools allowing teachers to exchange their experiences or issues and to connect them with research in Mathematics Education.

References

Carretero, R., Coriat, M. y Nieto, P. 1993. *Etapa 12-16. Area de Matemáticas. Secuenciación, Organización y Actividades de Aula.*. Granada: Edición de los Autores. (En prensa, por la Junta de Andalucía.)

Carretero, R. et alii, 1989. *Educación Secundaria, 16-18. Diseños Curriculares de Matemáticas*. Junta de Andalucía: Sevilla.

Cockroft, W.H. y otros, 1985. *Las matemáticas sí cuentan. Informe Cockroft.* Madrid: Servicio de Publicaciones del MEC.

Decreto de enseñanzas de Andalucía. BOJA nº 56, de 20/6/92.

LOGSE. BOE nº 238 de 4/10/1990, pp. 28927-28942

Fernández, A. y Rico, L. (1992) *Prensa y Matemáticas*. Síntesis: Madrid. p. 60.

Freudenthal, H. 1983. *Didactical Phenomenology of Mathematical Structures*. Reidel Pub. Co.: Dordrecht.

Nieto, P., Ramos, E., Carretero, R., and Viera, A.M., 1989. *Educación Secundaria, 12-16. Diseños Curriculares de Matemáticas.* Junta de Andalucía: Sevilla.

REFLECTION ON PRACTICE: THE CONCEPT OF FUNCTION AS A CONTEXT

I. Escudero, M. García, S. Llinares y V. Sánchez
Departamento de Didáctica de las Ciencias (Matemáticas)
Facultad de Educación. Universidad de Sevilla, España.

There are lots issues about In-service Mathematics Teacher Education that worry both researchers and mathematics teacher. In this poster we focus on two interrelated aspects of mathematics teaching and professional development of the teacher:

(i) the analyses of practice and the design of tools that facilitate mathematics teachers' reflection process about their own practices, and

(ii) the design and analyses of teaching tasks that support and encourage mathematical reasoning and learners' perspectives to do mathematics.

The mathematics teachers' conceptions about mathematics, mathematics teaching and learning have on impact on (i) the analysis process on mathematical content and the way that teacher designs and posing teaching tasks, and (ii) their own professional development processes.

In this context (design of teaching and professional development of mathematics teacher), we think that there are some questions that must be addressed,

(i) What is the meaning of : 'doing mathematics', 'teaching mathematics', 'learning mathematics in the classroom', ...?

(ii) What is the meaning of mathematics content (both process and product aspects?.

If we think about the school mathematical content as a process, we are in favour of a perspective that hightlight process as problem solving, communication about mathematics (classroom discourse), mathematical reasoning (explore examples and no-examples to investigate a conjeture), etc. Consider the school mathematics as a process imply one different way to think about mathematics teaching situations.

In this poster we focus on several aspects of the mathematics teacher's job of teaching mathematics in one context of in-service teacher education. The analisis of relationships in the classroom between

- TASK-ACTIVITY
- MATHEMATICS TEACHER, AND
- LEARNERS,

into an institutional context are the content of the one pedagogical intervention in a inservice teacher education.

The concept of function is the mathematical content in the tasks analyzed. The relationship between the task and the activity should be the content of teacher's thinking process. Two tasks and their analysis are presented as examples of discusion in inservice mathematics teacher education.

This report forms part of the Project PS91-0102. DGICYT. Madrid.

GRAPHS AND CONCEPTUAL MAPS

María Candelaria Espinel Febles
University of La Laguna (Spain)*

From the point view of understanding, the explanatory power of a model depends upon its capacity to account for the observed phenomena using the smallest possible number of parameters. Today it seems to be widely recognised that the ability to compare and to classify constitutes one of the fundamental mechanisms which lie at the root of intelligent behaviour.

The conceptual map (Novak, 1990, 1988) is usually represented by a diagram which allows to show the conceptual structure of a discipline, or one part of it, following an order of greater to smaller level of generality. The objetive is to make use of it as a didactic tool for explaining their great help as an instrument for significant learning of concepts, this is, to memorise, recognise and finally to explain in the best way.

The technique for constructing this model is placing the most outstanding concepts or main explanatory power in the higher part of a net and they are going to add others of smaller nevel of generality. From this procedure, we have a net which vertices are the concepts and its edges are their relations between concepts. The mathematic model is a connected graph. If this graph is a tree the concepts are going to be organised as levels. If the graph has cycles it has interlace notions and that supposes a better understanding of the concepts. Graph theory is one of the most flourishing branches of combinatorial mathematics and constitutes an extraordinary mathematic tool to study the conceptual maps.

This research on study the conceptual nets introduces a few elements of the terminology of graph theory, concretely, X-trees, minimal spanning tree, Steiner trees and dendrograms. We take as examples the analysis conceptual maps in mathematics made by a group of teachers and student teachers.

REFERENCES

BARTHELEMY, J.P. - GUENOCHE, A. (1991) **Trees and Proximity Representations.** John Wiley & Sons.

NOVAK, J. (1990) Concept Mapping: A Usseful Tool for Science Education. **Journal of Research in Science Teaching.** 27, 937-949.

NOVAK, J. - GOWIN, D.B. (1988) **Aprendiendo a aprender.** Martínez Roca.

NONTRADITIONAL ACTIVITIES IN MATHEMATICS CLASSES

Misael Jose S. Fisico
Math Instructor, *Brent International School, Manila*
Editor, *Vita Mathematica* (A Series of Magazines for Elementary and High School Students)
Member, *Mathematics Society of the Philippines* (MSP)
Member, *Mathematics Teachers Association of the Philippines* (MTAP)
Trainer, *Philippine Delegates to the International Math Olympiad*

In mathematics education, the way a student learns about the subject matter is just as important as what he learns about it. Most math teachers, apparently, associate their learners to an empty can where knowledge is poured in. Thus, students learn in a very restricted manner.

To have a more conducive atmosphere for learning mathematics, a creative teacher who is also diligent and resourceful should always consider giving mathematics activities which do not only help in the learning processes but also captivate the attention, the curiosity, and the sense of wonder of the students.

Realizing that best lessons in life come from actual experiences, the author developed nontraditional mathematics activities where students experience learning in a more intimate and interesting manner.

Some of these activities* which the author would be sharing are the following.

a. *Cartooning the Lessons in Mathematics*

Cartooning caricatures mathematical ideas. It makes principles, theorems, numbers and symbols "come alive" through the art of exaggerating lessons (but making sure that basic principles are retained.) The humor or the essence of each doodle usually depends on the caption or the expression in each animation.

This activity which shows the aesthetic, philosophical and humorous aspects of mathematics is based on the assumption that learners scribble or make "nonsense" doodles whenever they feel like killing time or they are pressured to listen to uninteresting lectures.

b. *Essays in Mathematics Classes*

As a mathematics activity, expository writing helps learners understand mathematical concepts clearly and concisely. Asking students to write math-related compositions minimizes rote memorization on the processes in solutions which are usually given in class. Thus, an activity of this sort enhances a deeper understanding of problems, making the acquisition of skills in problem solving an easy task to achieve.

c. *Rhyming in Mathematics Classes*

Many topics and concepts in mathematics are perplexing to students. If these are not properly taught, misconceptions may arise. As an example, the *cancellation process* is a misnomer topic. When assessed, most students have a hard time explaining the difference between cancellation of expressions which sum up to zero and cancellation of rational expressions which give a quotient of one. Rhyming the lessons helps in eliminating math-misconceptions.

In addition, the use of rhymes makes learning ideas lighter and within the reach of the average student. It also helps students possess the following virtues - discipline, perseverance and the need to compromise. To relate rhyming words, say *prove, disapprove, move, mood,* and *groove* in such a way that the meter as well as the essence of each expression is correct involves a lot of patience and compromise with a students' belief without jeopardizing mathematical principles.

---

* *Samples of work of students will be distributed during the conference; Other activities such as newspaper clippings, translation activities, etc. will be delivered if time permits*

OPERATIONS WITH POLYNOMIALS-CONCRETE MATERIAL

ESTELA KAUFMAN FAINGUELERNT and FRANCA COHEN GOTTLIEB
UNIVERSIDADE SANTA ÚRSULA - Rio de Janeiro - Brasil

Purpose - Demystify the teaching of algebra introducing the concept of algebric expressions and operating with them by games with concrete material.

Strategy - Starting by the visualization of the areas of squares and rectangles, the pupils were able to:

- write algebric expressions of 1^{st} and 2^{nd} degrees
- write algebric polynomials of 1^{st} and 2^{nd} degrees
- operate with polynomials of 1^{st} and 2^{nd} degrees
- generalize to higher degrees without the use of the concrete material, descovering rules and writing them in corrent language.

Construction of the geometrical forms and writing the correspondent algebric expressions using the mesures of the sides.

Construction of the geometrical forms corresponding to algebric expressions previousely given.

Descover rules for notable products.

Accomplish factorations.

Focalization of reversibility and generalizations in every activity.

Concrete material used -

- 4 red squares with side of 8 cm
- 4 blue squares with side of 8 cm
- 20 red square with side of 2 cm
- 20 blue squares with side of 2 cm
- 12 red rectangles of 8 cm by 2 cm
- 12 blue rectangles of 8 cm by 2 cm

The experience - It was applied in three classes of the 7^{th} grade (age 12-13), total of 96 pupils. They worked in groups of four. Every group was accompanied by the teacher and the coordenator of Mathematics.

Result - The pupils who participated in the experience did not find any difficulty with equations and systems.
95% of these pupils were promoted.

Bibliography - Hilde Howden - Algebra Titles for the Ovrhead Projector-CCA. Cuisinaire Material

Jakubo e Lellis - Matemática na Medida Certa. Ed. Scipione.

Estela Kaufman Fainguelernt and others. Trabalhando com Geometria. Ed. Ática.

THE OSTENSIVE REDUCTION IN ALGEBRAIC MANIPULATION

Elisa GALLO
Department of Mathematics
University of Turin - Italy

Algebraic manipulation is what is used to transform a formal expression into another of the same algebraic object; the formal transformations of expressions make sense when they are inserted in a conceptually structured context, even if they operate on the ostensive plane of the expression.

The adaptation of the *models* activated by the pupil during the algebraic manipulation, in order to create a model ad hoc, is provoked by the action which links the subjective pole (what the pupil knows) to the objective pole (what the pupil produces), which we refer to as *control*.

The question at the basis of our research on algebraic manipulation is therefore: what is the dynamics of control during the solution of an algebraic exercise?

An algebra exercise may be taken as a problem or not: the latter is true when the pupil automatically applies a procedure without using controls. If, on the contrary, control is present in the solution, the exercise becomes a problem and control is developed at the level of perception and the perceptive reorganisation of formulas; at the level of the use of resources (models) and procedures (other models); at the level of syntax; at the level of meanings, namely global and local semantics.

The manipulation which takes place in terms of formal ostension must also be carried to the conceptual level, with the action of control: we are studying the phenomenon of the ostensive reduction of manipulation (present when the pupil does not manage to adjust his knowledge and the formal object with which he is working in the two processes of the instantiation of knowledge and of the application of semantics to knowledge) and its didactical overcoming.

References

Battù M., Gallo E., Testa C., 1993, Evaluation et contrôle dans le calcul littéral comme problème, *Proceedings of CIEAEM45*, Cagliari (in press).

Gallo E., Amoretti C., Testa C., 1990, Utilisation de modèles géométriques en situation de résolution de problèmes: contrôle descendant et ascendant, *Proceedings of CIEAEM41*, Bruxelles.

Gallo E., Battù M., Testa C., 1991, The control in problem resolution, *Proceedings of PMEXV*, Assisi, vol. I.

Gallo E., 1992, Control and solution of "algebraic problems", *Proceedings of Workshop on algebraic learning*, Torino (in press).

Gallo E., 1994, Algebraic manipulation as problem solving, *Proceedings of First Italian-Spanish Research Symposium in Mathematics Education*, Modena.

STUDENTS' GOALS IN LEARNING MATHEMATICS: A CRITICAL ANALYSIS OF YEAR 10 STUDENTS' ACTIVITY IN A MATHEMATICS CLASSROOM

Simon Goodchild

College of St. Mark and St. John, Plymouth U.K.

I am currently engaged in exploring the nature of students' activity in a mathematics classroom using a social-constructivist model of cognition as a basis for interpretation of events observed and students' accounts of their activity. The object of this presentation is to outline the model and describe how I am using it to explain students' activity.

Investigating cognition in everyday practice Lave (1988) offers an explanation of cognition as being "... constituted in dialectical relations among people acting, the contexts of their activity, and the activity itself" (p. 148). Lave develops her model of cognition from observations of individuals engaged in 'everyday' activities such as shopping in a supermarket or dieters preparing calorie controlled meals. This provides a starting point for explaining classroom activity but it is insufficient, as El'konin (1961) points out educational tasks differ from everyday tasks in that the goal of an educational task is a change in the acting subject rather than the object of the activity. This change in the acting subject is believed to be due to a process of interpretation or reflection (Goodchild 1992), Neisser describes 'the perceptual cycle' which illuminates how the reflective process may operate. Mathematical activity is clearly central to the practice of the class observed and this has been characterised as processes of metaphoric and metonymic transformations (Ernest 1993) Thus the model I wish to present synthesises the contributions of Lave, Neisser and Ernest.

The data interpreted by the above model arises from my attendance, throughout the year, at every mathematics lesson of one year 10 class where I tape record the teacher's remarks to the class and conversations I have with individual students regarding their interpretation of the activity. Using the model I believe it is possible to identify three different levels of goals towards which students work: their rationale (Mellin-Olsen 1987) for engaging in activity, their purpose in the set task and a dichotomy between production and interpretation.

REFERENCES

El'konin, D.B (1961) in Davydov, V.V. and Markova, A.K. (1983) A concept of educational activity for schoolchildren. Soviet Psychology 21(2).

Ernest, P. (1993) Mathematical activity and rhetoric: towards a social constructivist account, Proceedings of PME 17 Conference, Tsukuba, Japan, 2, 238-245

Goodchild, S. (1992) Active learning, reflection and interpretation. Mathematics Education Review 1.

Lave, J. (1988) Cognition in Practice. Cambridge, Cambridge University Press

Mellin-Olsen, S (1987) The Politics of Mathematics Education. Dordrecht: Reidel

Neisser, U. (1976) Cognition and Reality. San Francisco, W.H. Freeman

VERIFIER - A SELF LEARNING METHOD IN EARLY GEOMETRY & MATHEMATICS

by

H. Orbach, B. Ilany-Gigozinsky

BETH BERL TEACHER TRAINING COLLEGE
CENTER FOR THE INSTRUCTION
OF MATHEMATICS

The purpose of the "VERIFIER" teaching aid is to develop visual perception and discrimination of shapes as a basis for geometric thinking. Furthermore, it also aids in the identification of numbers and of groups, series, and basic facts concerning mathematical exercises.

"VERIFIER" is a series of four teaching aids with accompanying booklets for ages 3-4; 4-5; 5-6; 6-7 and so on. The exercises within each level are also presented in increasing levels of difficulty. Teachers can also create their own worksheets.

All the teaching aids are de ' ;ned to be self-correcting and allow the child to proceed at his own pace.

Each tool consists of a transparent box and a set of tiles. The child solves the exercises by selecting tiles and placing them in the appropriate places on the transparent bottom part of the box, turns it upside down and verifies his answers by comparing the pattern to the one in the booklet.

The "VERIFIER" and its accompanying booklets contain:

1. Shapes - the child learns to identify and distinguish between the basic shapes (circle, square, triangle, etc.) and discovers intuitively the principles of conservation of the characteristics of the various shapes during transformation within a plane.
2. Patterns - the child discovers the concept of mathematical patterns by identifying the missing shapes which complete the pattern.
3. Directions - the child develops his ability to perceive directionality on a plane and to use concepts such as above, below, right and left.
4. Coordinates - the child develops his ability to coordinate on a plane. He must take into consideration up to four criterions at once (shape, size color, full or empty).
5. Reading numbers and identifying groups - the child learns to count and recognize numbers, utilizing geometric groups. The emphasis is on separation and classification of shapes.
6. Exercises - include drills with addition, substraction, division, multiplication and equations.

EVERY CHILD CAN DO MATHEMATICS

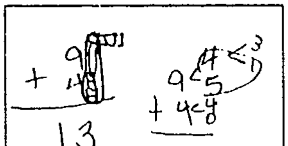
Tadato Kotagiri
University of the Ryukyus

What is Doing Mathematics? Creating mathematical ideas is probably one of the most important activities and building ideas into a whole system is hard work. In this process, symbols are used as, for example, concrete materials are used in planning and building a house. Therefore, operating symbols is a part of Doing Mathematics.

In the nature of things, creating mathematical ideas is pretty amusing and everyone can enjoy it. But actually many people have failed to enjoy it, although they think it would be amusing. The reason is that it is difficult to operate symbols in mathematics. This algebraic operation should be one of the main reasons why people have difficulties in doing mathematics.

MJ, a fifth-grade boy and slow learner in mathematics, could name the numerals up to a four-digit number and count concrete objects such as apples or a picture of apples. But he couldn't answer 6+7 correctly although he could do 9+4. He knew how to add one-digit numbers: that is, he could count his fingers one by one to find the answer. What I like to clarify here is that he didn't operate figures as mathematical symbols and he operated only numerals and fingers, i.e., the verbal and the concrete.

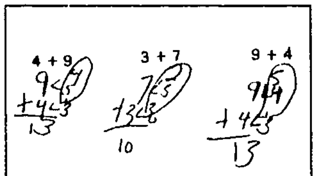
KB, a third-grade girl and also slow learner in mathematics, could answer 4+9 by using the concrete objects called the Tiles. But when she tried to operate figures in order to answer 4+9, she couldn't do the same thing in figures as she did in Tiles as shown in the Data-1.



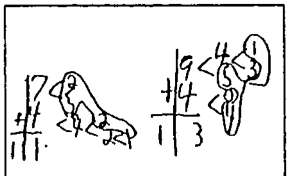
Data-1

It is unthinkable that a child, who is not good at operating mathematical symbols, is a good mathematics learner. Then, the issue is whether or not every child can operate mathematical symbols. My findings, which are based on the Suido Method, show that the answer is affirmative. (Refer to the Proceedings of the PME17, Ⅲ-240.)

The Data-2 is KB's and the Data-3 is MJ's. These data show that they became able to manipulate mathematical symbols as if they manipulated concrete objects. Also, it is obvious that they could continue to identify the quantitative meanings of operating figures. As the result, they can operate figures algebraically as if they paid no attention to the meanings of the mathematical symbols.



Data-2



Data-3

Note: The subjects in my research are Canadian and Japanese. The research results that I am interested in doesn't depend on the cultural difference between Canada and Japan. I am pleased to acknowledge the considerable assistance of Dr. T. Kieren, University of Alberta, and Dr. W. Szetela, University of British Columbia.

SOME PROBLEMS IDENTIFIED WITH MAYBERRY TEST ITEMS IN ASSESSING STUDENTS' VAN HIELE LEVELS.

Christine Lawrie
University of New England, Armidale, Australia.

In the early 80s Mayberry (1981) developed a diagnostic instrument to be used to assess the van Hiele levels of pre-service primary teachers. The test which was carried out in an interview situation, was designed to examine seven geometric concepts. There has been no reported attempt to (a) replicate this work in Australia; (b) consider the items in some alternative format; or (c) analyse the validity of the test questions.

To address these issues, a detailed testing and interview program of 60 first year primary-teacher trainees was undertaken at the University of New England. This paper considers one aspect of the findings of this study. When collating the results in the assessment of van Hiele levels, some of the students' reasoning was not consistent with expectation according to the Mayberry items. Interviews did not appear to clarify these inconsistencies. On analysis of the results by concept and by level, it was considered that certain aspects of the Mayberry items had the potential to lead to incorrect assessment of a student's level of understanding in geometry. In particular, four main features were found to account for major problems to the test validity. They were:

1. incorrect assignation of a level to certain items;
2. unequal treatment of concepts across levels;
3. uneven distribution of questions across levels; and
4. unbalanced distribution of question focus within levels.

Conclusion

This analysis not only gives us a clearer perspective about the Mayberry test and the results, but also provides further insight into the van Hiele Theory itself. In particular it provides further empirical evidence about what it means to work at a particular level. It reinforces the hierarchy of the van Hiele levels, and that the level at which a student is working can be discerned. However, it refutes the van Hiele notion that the levels are discrete, lending support to the findings of researchers including Gutierrez (1991) which suggest that there are some students whose answers clearly reflect two consecutive levels of reasoning simultaneously.

References

Gutierrez, A., Jaime, A., and Fortuny, J. M. (1991) An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education,* Vol. **22**, 3, 237-251.

Mayberry, J. W. (1981). *An investigation of the van Hiele levels of geometric thought in undergraduate in undergraduate teachers.* Doctoral Dissertation, University of Georgia, (University microfilms, no. 8123078).

Pegg, J. E. (1992). *Students' Understanding of Geometry: Theoretical Perspectives* Paper presented at the 15th Annual MERGA Conference, University of Western Sydney.

Colours Towards R^+

Moisés CORIAT BENARROCH · Departamento de Didáctica de la Matemática. Universidad de Granada. Granada, Spain
Pedro S. MARTINEZ LOPEZ, Inst. de Bachillerato "Bahía de Almería". Almería. Spain.
Julián BAENA RUIZ Inst. de Bachillerato "Montes Orientales". Iznalloz. Granada (Spain)

Abstract

Referring to real number first learning at Secondary School Level, a Phenomenology (in Freudenthal's sense), incorporating a full variety of examples and didactic situations, is lacking. To reach such a goal, we claim the need exists for tools to allow the design and study of didactic plans and obstacles (like those revealed by Sierpiñska [1987, 1990]) arising in the Class-room.

Colours towards $\mathbb{R}$+ suggests a complementary, manipulative approach to the real number learning and teaching at Secondary School Level; such an approach was first fully described by the same authors [1993].

We present an activity based upon colours (used by us in in-service teachers training courses), allowing (a) to ask significative questions related to a particular encapsulation process (Tall [1991]), exhaustion, and (b) to link infinite and limit intuitions, on the one hand, and real number first learning, ont the other. As far as human groups enjoy handling colours, the activity appears as culture-independent.

References

Coriat Benarroch, M., Martínez López, P.S., and Baena Ruiz, J., 1993. *Int. Jour. of Math. Educ.*, **24**(4), 501.

Fischbein, E. Tirosh, D., Stavy, R., and Oster, A., 1990. *For the Learn. of Math.*, **10**(1), 23.

Gardiner, A., 1982. *Infinite Processes. Background to Analysis*. New York: Springer-Verlag.

Sierpiñska, A., 1987. *Educ. Stud. in Math.*, **18**, 371.

Sierpiñska, A., 1990. *For the Learn. of Math.*, **10** (3), 24.

Tall, D. (ed.), 1991. *Advanced Mathematical Thinking*. Dordrecht, The Netherlands: Kluwer Academic. (Chapters 7, by Dubinsky, E., and 15, by Tall, D.)

ALGEBRAIC REPRESENTATIONS AND DISCOURSE IN A "TRADITIONAL" CLASSROOM

Luciano Meira
Federal University of Pernambuco

This poster looks at the ways in which the teaching and learning of algebra are intertwined with the representational and communicative processes going on in the classroom. The extensive body of research in algebra has demonstrated teacher's and students' difficulties regarding this subject. Many have argued against the current focus of algebra instruction on "formal procedures for transforming symbolic expressions and solving equations to find the hidden value of the variable." (Fey, 1990, p. 70) This general aim of traditional teaching generates other more specific difficulties such as (1) understanding the transformative character of algebraic representations; (2) building algebraic models of situations, (3) proving results for algebra problems; (4) understanding the function of algebra concepts and notations. Recently, several alternative approaches for algebra teaching have been suggested to help students overcoming these obstacles: (1) the "arithmetization" of algebra, in the sense of stressing the areas of continuity between algebraic and arithmetic problem solving (e.g., Smith, 1994; Falcão, 1992; Lins, 1991); and (2) the algebraic modeling of functions based on activity upon physical events and situations (e.g., Meira, 1991; Greeno, 1988).

Although we seem to know much about "algebraic thinking" (even if it is very hard to define), we lack a robust understanding of algebraic activity in the classroom context. In order to better understand the difficulties reported above and to place those difficulties in relation to classroom processes, this research analyzes the mathematical activity of teachers and students in the everyday practice of algebra teaching and learning in a traditional classroom The data for this presentation consists of videotapes of a teacher's and his students' algebraic activity in one eighth grade classroom, and videos of pairs of students from the same classroom as they participated in several problem solving sessions.

A preliminary microgenetic analysis of the data suggests that (1) there were many instances in the classroom where the students' procedural work with algebraic notation could be developed into more elaborated algebraic thinking; (2) there were not enough instances where the teacher was able to capitalize on emergent representational and interactional scenarios in order to develop an advanced sense of algebraic activity; and (3) a significant part of the students' algebraic activity during the problem solving sessions involved little or no direct engagement with algebraic notations. The poster will illustrate these claims through the presentation of protocols that show several categories of clashes between the teacher's and the students' understanding of algebra.

USING INTERACTIVE MEDIA FOR CLASSROOM OBSERVATION[1]

Judith A. Mousley
Deakin University
Geelong, Australia

Peter Sullivan
Australian Catholic University
Christ Campus, Australia

The software presented is a multi-media package aimed at improving classroom observation and research. It is being developed for use in pre-service and in-service mathematics education, but has potential for use in other professional education contexts. Resources included within the program include films of pre- and post-lesson interviews with the teachers, videos of the lessons (and any parts thereof), transcripts and other lesson documentation, associated readings and bibliographic data bases, graphic representations, and other resource materials.

Four lessons, based on six components of quality teaching (see Mousley & Sullivan, this volume) were planned, taught and videotaped. They are now stored on CD-ROM disc. Our current research is examining how interactive media can be used to form comprehensive resource bases on mathematics education, and how these can be used in teacher preparation, in-service professional development and educational research.

Using on-and off-campus teacher education students, the project is employing case study methods to determine different levels of interactions, (a) between users and the program, and (b) within small groups of users. Teaching and learning styles range from a transmission mode, where the lecturer plots the learning path, to individual or group inquiry-based projects of the students' own design using the program's resources.

This project has three aims. The first is to further understandings about the potential of multi-media technologies to promote reflection, discussion and writing by groups of teachers and student teachers. The second is to increase knowledge about how technology can be used to stimulate more student-directed professional development, including teachers' research into their own practices. The third aim is to provide a detailed evaluation of some uses of interactive media to enhance classroom observation during the practicum as well as for researchers.

Conference participants are encouraged to contribute to the further development of this project by adding to interactive documents that the software contains, including data bases of (a) questions which could be researched using the program's materials, (b) comments, suggestions and questions about the research project, and (c) short quotations and bibliographies appropriate for the styles of teaching demonstrated.

---

1 This research is being funded by the Australian Research Council, Australian Catholic University and Deakin University.

THE IMPORTANCE OF CONTEXT IN ASSESSING MATHEMATICAL LEARNING

Marilyn Nickson, Alison Green, Sarah Hughes
University of Cambridge Local Examinations Syndicate

The poster presentation will be provide details of a study of the effects of contextualisation in questions assessing mathematical learning. The effects of context in mathematics tests has been established (e.g. APU 1985, Carraher 1989) and some of these effects have been noted in connection with mathematics examination papers for 16 year olds (Key Stage 4) (SEAC 1992) Examinations at this level reflect the recommendations of the Cockroft Report (DES 1982) which advocated the testing of a larger proportion of mathematics in context, with a view to expanding the place of the uses and applications of mathematics within the taught curriculum. This has been accompanied by a move towards criterion-referenced testing which sets out to assess what the pupil can do and thus to reward positive achievement (DES 1982). Four levels of activity have been identified in the process of responding to a contextualised question and one of the concerns of the study is to explore whether it is possible to reward a pupils for achievement at each of these four levels. Examples of such questions and how they can be 'decomposed' into these levels will be illustrated together with the various combinations of components in contextualising questions (e g. numbers with diagrams or the written word with pictures). Major concerns of the study, therefore, are to establish the extent to which it is possible (a) to deduce any evidence of the mathematics a pupil knows at each level, (b) to reward a pupil's response at each level and (c) to determine the facilitating or inhibiting effects of particular contexts in relation to particular kinds of mathematics. A flowchart will be included in the poster to indicate the progression of the project and the questions being addressed.

The project is being undertaken by the Cognitive Psychometrics Section of the Research and Evaluation Division at the University of Cambridge Local Examinations Syndicate to investigate the effects of context in assessing the mathematical learning of children at Key Stage 2 (7-11 year olds) in the National Curriculum. The project is known as Cambridge Primary Assessment (CamPAs).

Bibliography

Assessment Performance Unit (APU) (1985) A Review of Monitoring in Mathematics 1978-82 Slough: NFER

Carraher T. (1989) Negotiating the results of mathematical computations International Journal of Educational Research 13(6) 637-646

Department of Education & Science (1982) Mathematics Counts London: HMSO

Schools Examinations Assessment Council (1992) Differentiation in GCSE Mathematics (A Report by the Inter-Group Research committee for the GCSE) SEAC Newcombe House 45 Notting Hill Gate London W11 3JB

A LOOK AT GEOMETRY IN NATURE AND AROUND US - ANOTHER WAY TO TRAIN UNDERGRADUATE PRESERVICE TEACHERS TO TEACH GEOMETRY

Dorit Patkin - State Teachers College Seminar Hakibbutzim

- Israel Educational Television, Tel-Aviv,62507

Ilana Levenberg - Municipally High-school, Haifa, Israel

Introduction- The activity described below is a part of a course in Euclidean geometry in the State Teachers College. It is known that geometry is a subject which needs visualization skill and students whose visualizasion skill is not developed enough have difficulties in studying geometry. In addition, students think that geometry is closed, dead, emotionless and boring.

In order to show the beauty of the geometry and to understand its implementation in our daily life the activity includes slides and pictures. The subjects which are presented through the slides are: from nature's architecture of diatoms to the bold shapes of city skylines, from naturally occuring polyhedra to artistic and functional adaptations created by humankind, from ratio and proportion conciderations of nature to the subtle use in proportion in our everyday life. By using these slides and pictures we bring together the wide expance of geometry and make this important subject much more meaningful to the students.

The aims- Using these slides and the activity around it makes the learning more meaningful by helping the students move from a concrete to a pictorial and finally to an abstract representation. This activity is a pictorial bridge from the concrete to the abstract. It engages the visual sense and brings to the students an awareness of geometry in functional as well as aesthetic setting. It is an inherent tidisciplinary approach relating geometry to a plethora of contexts such a. life science, social science ecology etc..

In such an activity every student can take part, and can develop his visual, drawing, logical and applied skills across the cognitive geometry levels of the students according to Van-Hiele's theory.

The activities- The activities can be done while introducing a topic, developing it or summarizing it. Several slides and pictures are introduce to the students and they are asked what they can see - for example a slide of a wheel of a car. The students are asked to determine the geometrical shape they are shown, to draw it or to explain how to draw it, to give other examles of wheels in other shapes and so on. Another activity is with a triangle garden. In the three corners there are roses. Questions are asked like - Where we have to put the sprinklers? - Can we do it in every triangle garden? Activities and questions such as these enhance the lessons and motivate the students to the awareness of geometry in nature and the world around us.

LOGO AND PROBLEM SOLVING STRATEGIES
AN EXPERIMENTAL STUDY WITH PRIMARY SCHOOL CHILDREN

Maria Isabel Pereira * and Luisa Morgado **

* Escola Superior de Educação de Leiria
** Faculdade de Psicologia e Ciências da Educação da Universidade de Coimbra

In an adequate environment LOGO programming can facilitate the development of cognitive skills, particularly in what concerns problem solving skills in mathematics

With the purpose of evaluating the effect of LOGO programming in problem solving strategies, we started an investigation with two groups of 7 and 9 years old children. Our main hypothesis is that LOGO facilitates the generalisation of the procedures that children use to solve problems.

This study had three different parts: pre-test, tasks and post-test The study was made in two primary schools in Leiria between January and June 1993 It involved sixty four children, half of them with seven years and the other half with nine years. Each age group was divided in a control and a experimental group Each of this four groups had sixteen children. All the children in the experimental groups had at least one year of experience with LOGO programming. In establishing these groups we tried to make them as homogeneous as possible in what concerns the psychogenetic development of the children involved, in particular the concepts of space, laterality and perspective coordination During the pre-test and the post-test, piagetian proofs were used to measure these concepts in the children involved. Between the two tests children were asked to perform two tasks. Those tasks involved the use of LOGO programming in experimental groups and cut-paper manipulatives in control groups.

The analysis of data collected from the tests made allows us to draw some conclusions about the strategies that LOGO programming children use to solve problems and to compare the generalisation skills of the children in experimental and control groups.

The Language LOGO and the construction of mental representation in the child, an experimental study

José Peres Monteiro
Escola Superior de Educação da Guarda

Luisa Morgado
Universidade de Coimbra

<u>Introduction</u>: We examine the question of knowing whether the language LOGO may become a favorable learning context for the users to develop their ability of mental representation or of building up concepts, when solving concrete problems.

<u>Hypothesis</u>: H_1- The practice of LOGO mainly benefits the development of the ability of mental representation or the building up of concepts of the individuals who have used this language of programming rather than the simple practice of using a piece of paper and pen.

H_2- The practice of LOGO helps the individuals to achieve a precise and rigorous verbal codification in the description of the behaviour adopted in the acomplishment of the problem solving tasks.

<u>Experience</u>: Twenty eight children aged eight to nine and attending the third year of primary school took part in this experimental research. There were two groups A (LOGO) and B (paper and pen). In the phases of pre-test and post-test all the students in both groups performed the same piagetian tests. They were used to evaluate concepts of length measurement, mental representation and space orientation. In the experimental phase the two groups performed the same type of tasks, where group A used LOGO and group B used only paper and pen.

<u>Results</u>: The results revealed that group A (LOGO) in the mental representation test (second post-test) obtained higher results than group B (paper and pen). No statistical differences were noticed.

The students of group A (LOGO) used a more precise and acurate language when describing the procedures subjected to a mental representation.

Mathematical notions and operative cognitive tools in occupational settings

Janine ROGALSKI
Senior Researcher Equipe Psychologie Cognitive Ergonomique
CNRS-Université Paris 8 2 rue de la liberté 93526 Saint-Denis Cedex 2

Supporting operators in complex occupational activities is a general trend in technical settings (such as process control). Mathematical notions are involved as tools, and tables of values and functional graphical representations often play a rôle in support systems. An important issue is concerned with the availability of such tools depending on the operators' vocational knowledge about the entities modelized in such mathematical ways and their knowledge about the mathematical notions themselves.

An empirical study was developed about the use of a graphical-based operative cognitive tool, devoted for helping operators on prognosis, anticipation for resources and actions in forest fire fighting. (We define as "operative cognitive tools" various types of artifacts -from graphical representations, abacus, tables of values, pocket calculators to expert systems- that take in charge part of the cognitive operations required by a task.)

The forest fire tool (FFT) integrates in the same graphical representation two (linear) models : one represents the relationship between the distance (y-axis) of the front fire by respect to time (x-axis), the other links the quantity of required means and the fire spread through a twofold valuation of the y-axis (distance and number of firetrucks). Available means depending on time may be represented on the same graphic. Comparing the curves of the expected and required means allows to evaluate time intervals were action is possible. Relating time intervals to distance intervals allows to evaluate (spatial) zones where action is possible. (Ground properties -read on topographical maps- are used for decision making.)

Operators are firemen Officers ; most of them get a scientific background (more than 2 years at universitary level). Nevertheless, using the FFT appears to present specific difficulties, some of them revealing ill-acquired notions about functions and graphs.

We will present global results about the main hypotheses : 1) specific training on the tool is required even if no new notions are involved and 2) specific experience in the domain (forest fire fighting) facilitates training and use of the tool, and qualitative data about the type of errors. Implications for required mathematical background and/or specific vocational mathematical training will be discussed.

References

Kaput J. (1987) Representation systems and mathematics, in C. Janvier (Ed) *Problems of representation in the teaching and learning of mathematics*, Laurence Erlbaum Associates.

Mason J. (1987) What do symbols represent, in C. Janvier (Ed) *Problems of representation in the teaching and learning of mathematics*, Laurence Erlbaum Associates.

Ochanine D. (1978) Le rôle des images opératives dans la régulation des activités de travail, *Psychologie et Education*, 3, 63-78

Rogalski J. (1984) Représentations graphiques dans l'enseignement : concepts et méthodes d'analyse appliqués au graphe des fonctions, in A. Giordan et J.-L. Martinand (Eds) *Signes et discours dans l'éducation et la vulgarisation scientifiques*, Paris : Université Paris VII, 379-388.

Rogalski J., Samurçay R. (1993) Représentations : outils cognitifs pour le contrôle d'environnements dynamiques, in A.Weill-Fassina, P.Rabardel et D.Dubois (Eds) *Représentations pour l'action*, Toulouse : Octarès, 183-207.

PROJECT EME: A EUROPEAN MATHEMATICS EDUCATION RESEARCH DEGREE

Leo Rogers, Mathematics Education, Roehampton Institute London SW15 5PH

1. Background.

a) An ERASMUS Network has been in operation for three years which now involves the universities of Thessaloniki[1], Bielefeld, Bologna, Grenoble, Institute of Education London, Ioanina, Parma, Regensburg, Roehampton Institute, and Southbank University London.

The subject of this programme is "Didactics of Mathematics" and involves the exchange of students on undergraduate programmes who are training to be teachers. There have been a number of meetings of the teaching staff involved, to discuss the details of their different programmes, and the project is continuing with opportunities for curriculum development and teaching staff exchanges.

b) Two TEMPUS projects have been submitted, one bringing the University of Sofia into the above network, and the other, a proposal by Roehampton Instittute involving, Warsaw, Wroclaw, Pavia and Utrecht in a scheme to develop a higher degree programme in Mathematics Education.

2. The Proposal.

As a result of this cooperation, and motivated by the report of the Committee for Mathematics Education published in the EMS Newsletter,[2] a proposal was later published[3] giving details of the work in progress, and the possible future development of European Graduate Level studies in Didactics of Mathematics.

The principal ideas are as follows:

a) Each university has developed its own characteristic programme in didactics of mathematics. The content and style of these programmes not only reflect local and national interests, but also the individual expertise of the tutors has its outlet in the topics included in the programme.

b) The idea of a European Programme in Didactics of Mathematics at Masters or Doctorate level is based on two principles:

(i) Modularisation of courses, and (ii) Credit Transfer.

For example, a student might enrol for a Masters or Doctorate degree in one country, but study for part of the time in another community country, thereby gaining credits which can be transferred back to the institution in the home country to count towards their final qualification.

The success of this scheme depends on the agreement to adopt the the principles of the European Credit Transfer System (ECTS), and to ensure that their programmes are constructed according to a modular scheme whereby all students are able to take advantage of self-contained and viable short courses.

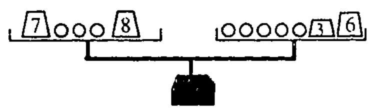
Central to this is the idea of a Unit of Study which may be spread over a period of one semester or one term, or organised as a period of "intensive study" of, say, two weeks. A number of European Universities are adopting this method of organising programmes, although the Units or Modules may need to be of different lengths, according to the aims of the courses.

Further information and a more comprehensive description of the programmes involved may be obtained from Leo Rogers at the address above.

1. Coordinator: Prof. Athanassios Gagatsis, Department of Mathematics, Aristotle University of Thessaloniki, 54006 Thessaloniki.
2. Dorfler,W.(1992) European Mathematical Society (EMS) Newsletter No.4, June 1992, (14-15)
3. Rogers,L. (1993) EMS Newsletter 7, March 1993

PUPILS' UNDERSTANDING OF BEGINNING ALGEBRA

Lai-Leng Soh
University of Oxford, UK



When Ben was asked to translate the above balance puzzle into an algebraic equation, he wrote

$$3x + 15 = 5x$$

Without completing this equation, he stopped and said,

"Can't do that ... ... unless you swap them round."

He then wrote

$$5x + 9 = 3x + 15$$

and solved the equation. This episode showed that Ben had the idea that an equation with one unknown could be solved only when there was more of the unknown on the left hand side of the equation than on the the right hand side.

This is an example of the ideas that pupils may form during the learning process, some of which may not be intended by teachers or authors of teaching materials. As such unintended ideas may be impediments to the learning of algebra at a later stage, it is essential to comprehend how pupils come to understand beginning algebra.

This study examines pupils' growth of understanding of beginning algebra in classes using individualised learning materials. Of particular interests are:

- What images do pupils construct?
- How do pupils go about constructing these images?
- How do these images compare with those intended by authors of teaching materials?

The study involves 5 case studies, each on a pair of pupils aged 11-13. The methods of data collection include video recording and personal observation in classrooms, interviews and post-tests.

The presentation displays some of the results obtained in this study.

USING A COMPUTER ENVIRONMENT IN THE CLASSRROM TO LEARN THE CONCEPT OF PROPORTION

Martí,E., Steren,B., and Garcia-Milà,M.
Universitat de Barcelona

The aim of the present study is to analyze the process of learning the concept of proportion displayed by 12-13 year-old student dyads when they use a computer environment in their mathematics class. We hypothesize that computers can be an effective mean to foster the process of learning difficult mathematics concepts such as proportion. When the students use a computer environment the process of learning becomes more interactive, motivating and self-regulated (Pea, 1987). The computerized didactic sequence we propose is the result of an interdisciplinary collaboration among cognitive psychologists, mathematics teachers, and software designers. The main principles that have guided the design of the didactic sequence are the following:

1. To integrate the didactic sequence in the curriculum of mathematics.

2. To present a set of similar problems based on the same physical phenomenon (light and shadows) but becoming progressively complex. This complexity is defined in terms of the students' rate of progress in the psychological construction of the concept of proportion (Vergnaud, 1983).

3. To help the students overcome their difficulties with a set of retroactive regulation (text and graphic feedback) according to the type of errors displayed.

4. To foster interaction between students as a way to optimize the process of learning using a computer environment.

In addition, two complementary analyses are undertaken. The first one consists of comparing the results obtained in the pretest and the posttest in both, the experimental (working with the computer environment) and the control (working with a noncomputerized version of the didactic sequence) groups. The second one is a microgenetic analysis of the working process displayed by a subset of dyads while they interact with the computer environment.

In the present poster, we describe and theoretically justify the didactic sequence involved in the computer environment, and we present the results of the first analysis (pre/posttest differences).

REFERENCES

PEA,R. (1987). Cognitive technologies for mathematics education. In A.H. Schoenfeld (Ed.), Cognitive science and mathematics education (pp. 89-122). Hillsdale, NJ:Erlbaum.

VERGNAUD,G. (1983). Multiplicative structures. In R. Lesh and M. Landau (Eds.), Acquisition of mathematics concepts and process (pp.127-174). New York: Academic Press.

PLENARY PANEL

THE HISTORY OF MATHEMATICS AND THE LEARNING OF MATHEMATICS: PSYCHOLOGICAL ISSUES

Paul Ernest (UK)

Introduction to the Plenary Panel of Lucia Grugnetti (italy), Teresa Rojano (Mexico), Anna Sfard (Israel) and Eduardo Veloso (Portugal)

Currently an historical reconceptualisation of the nature of mathematics is underway. Until recently the widely accepted view of mathematics was that of modernism which envisaged the building of timeless and indubitable structures of thought based on a logical masterplan, the Euclidean paradigm. However the failure of the prescriptive programs of the logicist, intuitionist and formalist schools to achieve absolutely certain foundations for mathematical knowledge is well documented (Ernest, 1991, Kline 1980, Tiles, 1991). In contrast, a new tradition in the philosophy of mathematics has been emerging which has been termed post-modernist (Tiles, 1991), maverick (Kitcher and Aspray, 1988) and quasi-empiricist (Lakatos, 1976). This perspective is primarily naturalistic, concerned to describe the nature of mathematics and the practices of mathematicians, both current and historical. It is and fallibilist in its epistemology as well as inter-disciplinary and historicist (Lakatos, 1976; Kitcher, 1984).

The reconceptualisation of the nature of mathematics as essentially historical, in philosophy, mathematics education and adjacent fields, such as Piaget's genetic epistemology, is a development of great significance. It has strong implications for epistemology, encompassing the nature and justification of all knowledge, for methodology, as well as for theories and research on learning. For example, much of the controversy around radical constructivism centres on the fact that this position adopts the view that all knowledge, including mathematics, is a fallible construction. Inducting learners into a body of knowledge that is the contingent product of historical, cultural and evolutionary forces is a different problem from teaching under the traditional view. It demands more humility from teachers and researchers, and the realisation that learner errors are not absolutely defined, but only relative to a conventional body of knowledge.

The admission of a historical perspective centre stage in the psychology of mathematics education thus is of deep and widespread significance. However the indications of its importance go beyond philosophy and methodology. There are also vital ways in which the inclusion of the history of mathematics is central to the *content* of research in psychology of mathematics education. This will be evident in the contributions of panel members. But I should also mention that there are important schools of thought beyond those represented on the panel in which history plays a central part, such as Dutch, French and German research in the didactics of mathematics. Thus, for example, the international research community in mathematics education has had its perceptions of the place of history in mathematics education, and in the psychology of mathematics education in particular, changed by the contribution of French researchers.

CONCEPTUAL DEVELOPMENT AND OBSTACLES

The parallel between the cultural development of mathematics and individual psychological development has been explicitly noted at least since the 19th century biologist Haeckel formulated his biogenetic law: 'ontogenesis recapitulates phylogenesis'. In the realm of mathematics this parallel was extensively elaborated by a number of authors such as by Branford in 1908 (Fauvel, 1991: 16). In psychology this insight was further developed by Vygotsky as a central feature of his psychological theory.

> Every function in the child's cultural development appears twice, on two levels. First, on the social and later on the psychological level; first between people as an interpsychological category, and then inside the child as an intrapsychological category. (Vygotsky, 1978: 128)

Now as Freudenthal said in 1983, "We know for sure that this [biogenetic] law is not true in a trivial way...The young learner recapitulates the learning process of [hu]mankind, although in a modified way." (Steiner, 1989: 27-28) Thus the history of mathematics cannot absolutely dictate the necessary order in the learning or development of mathematical concepts. But since mathematics concept formation appears to be a recursive operation, the historical stages in the elaboration and extension of concepts provides an illuminating if sometimes loose parallel of great richness. Consequently, the history of mathematics is widely used to provide a genetic epistemological analysis of mathematical concepts for psychological and didactical purposes.

During the course of its history science changes, and some of these changes are dramatic enough to be termed revolutions (Kuhn, 1970). Gillies (1992) claims that there are revolutions in mathematics too. There is a powerful analogy between Kuhn's theory of normal and revolutionary development, and Piaget's theory of assimilation and accommodation in cognitive growth, respectively. Both a scientific revolution and schema accommodation are triggered by a contradiction or cognitive conflict.

Bachelard (1951) anticipated Kuhn's ideas, and invented the concept of an 'epistemological obstacle' which prevents progress and development in the history of ideas. He writes of revolutions as 'ruptures' in thought when an old style of thinking which is acting as an epistemological obstacle is swept away. Thus in mathematics well as the learning of new knowledge, *unlearning* to overcome epistemological obstacles is also needed. For example the fact for natural numbers (n>1) that multiplication is a procedure that produces larger numbers ceases to be true for rational numbers or integers. Multiplication no longer always 'makes bigger'. Many examples like this, and deeper ones too (e.g. shifts in the definition of a function), are indicated in the history of mathematics. Thus analysis of the history of mathematics provides researchers with a tool for anticipating psychological obstacles in the learning of mathematics.

MATHEMATICAL PROCESSES

Problems and problem solving are central to both the history of mathematics and psychology of mathematics education. In both areas they stimulate knowledge growth. Historical problems, such as the Königsberg Bridge Problem, which stimulated Euler to create Topology (Wolff, 1963) also serves well to introduce students to network theory in today's classroom. But the parallel I wish to remark on is in the realm of mathematical processes and strategies. Pappus distinguished between analytic and synthetic problem solving methods. Two thousand years later the distinction was used by psychologists to distinguish different levels of cognitive processing (Bloom, 1956). More generally, methodologists of mathematics such as Descartes 1628) and Polya (1945) have offered systems of heuristics which have driven research in problem solving in psychology and mathematics education (Groner *et al.*, 1983). Problem solving research in mathematics education research still has much to learn from the history of mathematics, and in the era of computers that history continues to grow and evolve new methods.

ATTITUDES TO AND PERCEPTIONS OF MATHEMATICS

It is widely remarked in the mathematics education literature that student and teacher attitudes and perceptions of mathematics are important factors in learning (e.g. Ernest, 1991). Indeed, much of the research on girls or women and mathematics talks of the problems caused by the stereotypical perceptions of mathematics as a male domain (e.g. Walkerdine, 1988). Thus a further important area of impact of the history of mathematics is on student (and teacher) attitudes to and perceptions of mathematics. In his survey, Fauvel (1991: 4) suggests that the use of the history of mathematics in teaching has the following outcomes.

- Helps to increase motivation for learning
- Makes mathematics less frightening
- Pupils derive comfort from knowing they are not the only ones with problems
- Gives mathematics a human face
- Changes pupils' perceptions of mathematics

One of my own areas of research and interest has been in the area of the philosophy of mathematics education, concerning a historicised reconceptualisation of mathematics (social constructivism as a philosophy of mathematics), and its impact in theory and practice on teacher and student personal philosophies of mathematics (e.g. Ernest, 1991). Although there is insufficient research on the relationship between the use of the history and positive attitudes to and useful perceptions of mathematics (Stander, 1989), a number of experimental teaching programs reported at PME use a historical approach and have positive outcomes (e.g. some Italian experiments described by Bartolini-Bussi, 1991).

These are just some of the important relationships that exist between the history of mathematics and the psychology of learning mathematics. My colleagues will offer more detailed explorations into such relationships.

REFERENCES

Bachelard, G. (1951) *L'activité rationaliste de la physique contemporaine*, Paris.

Bartolini-Bussi, M. G. (1991) Social Interaction and Mathematical Knowledge, in *Proceedings of PME-15*, Assisi, Italy, 1, 1-16.

Bloom, B. S. ed. (1956) *Taxonomy of Educational Objectives 1, Cognitive Domain*, New York: David McKay.

Descartes, R. (1628) Rules for the direction of the mind, in *Philosophical Works*, Vol. 1, New York: Dover Press, 1955.

Ernest, P. (1991) *The Philosophy of Mathematics Education*, London: Falmer Press.

Fauvel, J. (1991) Using History in Mathematics Education, *For the Learning of Mathematics*, 11(2), 3-6 & 16.

Groner, R, Groner, M. and Bischof, W. F. (1983) *Methods of Heuristics*, Hillsdale, NJ: Erlbaum.

Gillies, D. A. Ed. (1992) *Revolutions in Mathematics*, Oxford: Clarendon Press.

Kitcher, P. (1984) *The Nature of Mathematical Knowledge*, New York: Oxford University Press.

Kitcher, P. and Aspray, W. (1988) An opinionated introduction, in Aspray, W. and Kitcher, P. Eds. (1988) *History and Philosophy of Modern Mathematics*, Minneapolis: University of Minnesota Press, 3-57.

Kline, M. (1980) *Mathematics the Loss of Certainty*, Oxford: Oxford University Press.

Kuhn, T. S. (1970) *The Structure of Scientific Revolutions*, Chicago: Chicago University Press (2nd ed.).

Lakatos, I. (1976) *Proofs and Refutations*, Cambridge: Cambridge University Press.

Polya (1945) Polya, G. (1945) *How to Solve it*, Princeton, New Jersey: Princeton University Press.

Stander, D. (1989) The Use of the History of Mathematics in Teaching, in Ernest, P. Ed. *Mathematics Teaching: The State of the Art, London*, Falmer Press, 1989: 241-246.

Steiner, H. G. (1989) in Bazzini, L. and Steiner, H. G., Eds, *Proceedings of the First German-Italian Bilateral Symposium*, Pavia, Italy: U. of Pavia, 1989, 25-35.

Tiles, M. (1991) *Mathematics and the Image of Reason*, London: Routledge.

Vygotsky, L. (1978) *Mind in Society*, Cambridge, Massachusetts: Harvard University Press.

Walkerdine, V. (1988) *The Mastery of Reason*, London: Routledge.

Wolff, P. Ed. (1963) *Breakthroughs in Mathematics*, New York: New American Library.

RELATIONS BETWEEN HISTORY AND DIDACTICS OF MATHEMATICS

Lucia Grugnetti, Dipartimento di Matematica, Università di Parma

History of mathematics in mathematics education: why?

Among all subjects involved in mathematics education, nowadays there is a growing debate concerning the role of the history of mathematics in mathematics education. On the other hand, at least for the time being, teaching of scientific subjects does not emphasize the historical point of view. Perhaps something will change: in the proposal of the programs for Italian high secondary schools, concerning *calculus* for example, it is written "It will be possible to follow the historical itinerary which, starting from Archimedes, arrives to Newton and Leibniz through the contributions of Cavalieri, Torricelli and other precursors of calculus." On the contrary, in the almost all Italian text-books for high secondary schools, the development of calculus is not the historical one; in fact it goes from the concept of function to the concept of limit (one of the most complex mathematical concepts) and then from the concept of derivative to the concept of integral. On the other hand, I believe that, with regard to mathematics, an historical approach is advantageous for the student because it allows them to think of mathematics as a continuous effort of reflection and of improvement by man, rather than a "definitive building" composed of irrefutable and unchangeable truths.

Concerning the teachers they must remain aware of the inherent relativity of knowledge, and that to provide students with an adequate view of the way science builds up knowledge is in the long run worth more than the acquisition of facts (von Glaserfeld, 1991) [4].

Risks and difficulties

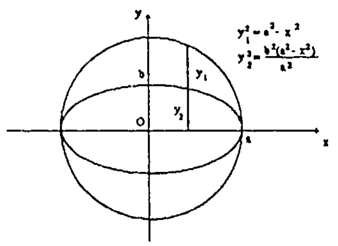
The crux of the matter is "how can we methodologically develop a teaching itinerary from a historical point of view?". In my opinion, the historical approach to mathematics involves the passage from a disciplinary to an interdisciplinary treatment in the broadest sense of the word. The passage from a disciplinary to an interdisciplinary approach is really fundamental if we want to try and bypass one of the several risks in introducing history of mathematics in mathematics education: the anachronism. As Weil (1978) [11] "There is a vast difference between recognizing Archimedes as a forerunner of integral and differential calculus, whose influence on the founders of the calculus can hardly be overstimated, and fancying to see in him, as has sometimes been done, an early practitioner of the calculus." Moreover, once we introduce at school a mathematician or, in general, a scientist, it is fundamental to analyse the political, social, economical context in which he lived. By this way it is possible to discover that facts and theories, studied in different disciplines, are concretely related. As Pepe (1990) [9] reminds us, the "meeting" between history and didactics of mathematics must be developed taking into account the negative influences that they can have one on the other. A possible negative influence of history on didactics is *the increasing of a notionistic ambit*, with interesting and curious references which are, in effect, not essential. On the contrary, the history of mathematics offers us several examples which "gain" by an interdisciplinary approach (L. Pepe, 1990) as, for example: the number systems of ancients; Galileo, the mathematization, the experimental method; Descartes and the analytical method. Fortunately we have not yet "standards" in the field of history of mathematics in the didactics of mathematics; in fact this field is nowadays open to discussions and suggestions. Among the different proposals there are the well known French works of some IREM [6] that are, in particular, concerned with a critical analysis of "originals" directly in classrooms. Another proposal comes from the USA where, for example, F. J. Swetz [10] says that a direct approach to historically enriching mathematics instruction and the learning of mathematics is to have students solve some of the problems that interested early mathematicians. Such problems offer case studies of many contemporary topics encountered by students in class. According to Swetz, that problems transport the reader back to the age when the problems were posed and illustrate the mathematical concerns of the period. According to me, the approach by problems can be easily utilised also by teachers that are not really concerned with history of mathematics, but there is the risk, if there is not a "contextualisation", that we offer to students a too much fragmentary idea of the history of mathematics. By this , I do not want to discourage the approch to mathematics and to history of mathematics by problems, but that we must be conscious of the inherent obstacles and misconceptions. In this perspective, the proposals of using originals, as IREM', are very interesting on condition that we pay a big attention to the inherent difficulties. Coming again to the Italian proposal of the mathematics program for high secondary school, whereas we want to follow the historical itinerary, we must be conscious, once again of needs and potentialities together with difficulties and obstacles. For example, concerning the teaching of calculus, the Cavalieri's theory of indivisibles could be an interesting point, on condition that since the originals by Cavalieri are very difficult, we use a sort of "didactical transposition" of that theory.

Some suggestions from Italian experiences

An interesting use, from a didactical point of view, of the Cavalieri's theory, can be found in a cultivate italian text-book (L. Lombardo Radice, L. Mancini Proia, 1979) [8] to find the area of the ellipse:

If we have the circle: $x^2 + y^2 = 1$ and the ellipse: $x^2/a^2 + y^2/b^2 = 1$

and if we consider the chords obtained by cutting the circle and the ellipse by stright lines which are parallel to axis y, we have: $y^2 = a^2 - x^2$; $y^2 = b^2 (a^2 - x^2) / a^2$



The ratio of the two chords is b^2 / a^2, so we obtain: $E : C = b : a$ from which $E = C\, b/a$. We know that $C = \pi a^2$; therefore $E = \pi\, ab$.

It is important to underline at least two important aspects of such an activity. The first is that it is inserted in a historical approach to calculus starting from the way in which Archimedes calculated the area between a parabola and one of its chords. This approach allows teacher to follow also with students the historical process in which the problem of the quadratures (integration) preceds the problem of the tangents (derivation). Moreover I think that, by this process, students can also appreciate later "modern" integration as need of generalisation. At the same time, if we try to follow, in a certain sense, the historical process of calculus and, as in the previous examples, the theory of indivisibles, we have to develop with students a critical analysis concerning that theory. So why the XVII century was the moment of the theory of indivisibles? Or why the calculation of areas and volumes being one of the main problems of that century? An implicit use of history in designing teaching is introduced by the research group of Modena (M. Bartolini Bussi). Early historico-epistemological studies of this research group resulted in the adoption of a particular context which could have emphasised the dialectical feature of mathematical experience as well as offered opportunities to develop meaningful historical studies in the classroom: it is the context of *mathematical machines* . For example, some elements of the study of conic sections are considered in the project *Mathematical Machines in High School*. The leading motives of this teaching experiment can be described under the following keywords: *geometry, history, machines*. [1]. The research group of Genova (P. Boero) considers, particularly, the following uses of the history of mathematics in the teaching of mathematics: 1) as a source of ideas, for the teacher, on "fields of experience" in which to construct mathematical skills and concepts, through suitable teaching itineraries and situations; when used this way, the pupil does not necessarely have to receive explicit information on the historic background used for the teaching project; 2) as a field of study for pupil, in order to work on particular mathematical objects (concepts and formalisms), to analyse their historical evolution and to translate from one formalism to another; 3) as an opportunity to set off developing mathematical discussions and demostrations, based on questions which have arisen in the course of the history of mathematics [2]. The research group of Genova (F. Furinghetti) uses the history of mathematics education to analyse mathematics teaching problems and educational phenomena. Moreover, according to this group, it is important to realize an integration of the history of mathematics in school mathematics [3]. From the research group of Cagliari (L. Grugnetti) comes an example of the history of mathematics also in view of an interdisciplinary teaching: the *Liber Abaci* by Leonardo Pisano (Fibonacci) as a source of problems (from XIIIth century) which concern different teachers and subjects: Italian and Latin (what kind of language is that of the Liber Abaci?), history (The development of the Middle Ages in Europe and Islam, geography (the West, the Middle East "Islam"), mathematics (the pupils' strategies for solving some questions from Liber Abaci; Leonardo Pisano's strategies: why he solved his problems by those strategies?) [5]. Interdisciplinarity is also the leading motivation for the interest of the research group of Rome (M. Menghini) in the history of mathematics. The purpose of interdisciplinarity is not to show how mathematics can be applied to other disciplines, nor to examine a certain period in different fields, but to show how different disciplines help all together to form a branch of knowledge. And this can be observed better "from a certain distance", when the concept is clearer; also because the sistematization process can be very long [7]. Also other Italian research groups are begining to work on the role of the history of mathematics in mathematics education. The different points of view allow a dynamic debate on this subject and point out a growing up of an important field of mathematics education.

Bibliography

[1] Bartolini Bussi M. & Pergola M., (to appear) History in the Mathematics Classroom: Examples from a Research Project for High School, Proc. of the Essen Symposium on History of Mathematics and Mathematics Learning.

[2] Boero P. Semantic fields suggested by history: their function in the acquisition of mathematical concepts, Proc. of the first Italian-German bilateral symposium on didactics of mathematics, Pavia 1988, ed. by Bazzini L., Steiner H. G., pp. 73-93.

[3] Furinghetti F., Insegnare matematica in una prospettiva storica, L'Educazione Matematica, Anno XIV, Ottobre 1993, pp. 123-134.

[4] Glaserfeld, E. von, A Constructivist's view of Learning and Teaching, International Confernce on Psychology of Learning and Mathematics Learning, 1991 (preprint in English), L'Educazione Matematica, Aprile 1992, pp. 27-39.

[5] Grugnetti L., The Role of the History of Mathematics in an Interdisciplinary Approach to Mathematics Teaching, Proc. of the first Italian-German bilateral symposium on didactics of mathematics, Pavia 1988, ed. by Bazzini L., Steiner H. G., pp. 9-23.

[6] IREM, Mathématiques: approche par des textes historique (tome 2), Univ.-Paris VII, Janvier 1990.

[7] Menghini M., Some remarks on the didactic use of the history of mathematics, Proc. of the first Italian-German bilateral symposium on didactics of mathematics, Pavia 1988, ed. by Bazzini L., Steiner H. G., pp. 51-58.

[8] Lombardo Radice, L. & Mancini Proia, L., Il metodo matematico, Principato, 1979.

[9] Pepe, L., Storia e Didattica della matematica, L'Educazione Matematica, suppl. al n; 2 Agosto 1990 (ottobre 1990), pp. 23-24.

[10]Swetz F. J., Using Problems from the History of Mathematics in Classroom, Mathematics Teacher, May 1989, pp. 370-377.

[11]Weil, A., History of Mathemathics: why and how, Proc. of the International Congress of Mathematicians, Helsinky 1978, Helsinky 1980, Vol. I, pp. 227-236.

THE CASE OF PRE-SYMBOLIC ALGEBRA AND THE OPERATION OF THE UNKNOWN

Teresa Rojano
Departamento de Matematica Educativa
Centro de Investigacion y Estudios Avanzados del I P N (Mexico)

The notion of **rupture** or **cut** in knowledge as in Bachelard (1970), which in turn gives rise to the concept of **epistemological obstacle**, is used as a central element linking the historical and educational domains. One characteristic of the case presented here consists of the formulation of conjectures on the **didactic obstacles of epistemological origin** (Brousseau, 1976) which are present in individuals in transition from arithmetical to algebraic thinking and the empirical exploration of such conjectures by means of clinical studies with subjects who are beginning to learn symbolic algebra. On the basis of the results obtained at an ontological level, the historical-epistemological analysis is once again taken up with the aim of seeking possible correspondence with the new didactic findings. This movement back and forth between history and didactics is what places this work in the field of mathematical education and not in that of the history or the epistemology of mathematics (Filloy, 1990).

Two moments of pre-symbolic algebra.

From the comparison between two moments in the symbolic evolution of algebra, before the appearance of *The Analytic Art* by F. Viète (ed. Witmer, 1983), the presence of cuts or **didactic obstacles** in the process of acquisition of algebraic language is conjectured. The moments in question are (a) that of the Practical Mathematics of the Italian Renaissance (XIII - XV centuries) represented by the *abbacus books* (Egmond, 1980), characterized by a dedication to the solution of practical problems expressed in vernacular form and using the methods of oriental mathematics. In these texts there is a diversity of strategies used to solve problems which, from the point of view of the symbolic algebra currently in use, would be identical problems; it is the specific numerical characteristics of each case which guides the strategy for problem solving and there is not an explicit operativity regarding unknown quantities (Filloy and Rojano, 1984a). The second moment (b) is that of the first text book of advanced algebra, the *De Numerics Datis* by Jordanus de Nemore, written in Latin in about 1225 (ed. Hughes, 1981). Concrete numbers never appear in the statement or the argument in this work. Most of the propositions correspond to the solution of systems of equations that can be reduced to a quadratic. The work with general numbers allows the cases solved previously to be recognized and thus, a general strategy of the book is to try to reduce the new cases to earlier propositions. In the argument, the quantities discussed are designated by letters and there is no syntactic operativity on them except juxtaposition to indicate addition (Puig, 1991).

As well as pertaining to well differentiated symbolic levels, the medieval works referred to also have common features: there is no systematic treatment in the operations carried out on those terms of the equation which involve unknowns. This disposition "not to operate with unknowns", which appears to be connected with the symbolic insufficiency of these works compared with the symbolic algebra inaugurated by Viète, leads to the formulation of conjectures as to the presence of didactic cuts in the process of transition from arithmetical thought to that of algebra. One of these cuts is located precisely at the moment in which, for the first time, students face the need to operate the unknown in the solution of linear equations with terms with "x" in both members.

A didactic cut of epistemological origin

Among the most eloquent manifestations of the rupture mentioned above are the typical spontaneous responses of students in the clinical interview to items of equation solving such as $2x+3=5x$. The children tend to assign an arbitrary value to the unknown on the right side and solve the equation $2x+3=\text{constant}$, which does not require operation of the term x. Or else, when they are asked to find the value of x in $x+5=x+x$, we get what we have called polisemy of x, that is the response: "this x (the one of the right) has a value of 5 and the other two (one on each side of the equation) can have any value (the same value for both)" (Filloy & Rojano, 1984b, 1989). The term **polisemy of x** is used when, within the same statement (algebraic), the same symbol is assigned meanings which pertain to different semantic fields. In one case this is the semantic field to which the notion of specific unknown pertains. In the other, it is the semantic field of general numbers. Examples of the eleven cognitive tendencies reported in this study are: conferring intermediate senses to more abstract mathematical texts, and focusing on readings made in language strata that will not allow solving the problem situation (Filloy, 1991 (b)).

Findings like those referred to previously, obtained at an ontoligical level, with subjects who are in transition from arithmetical thought to algebraic thought, correspond to moments of significant change in the historical development of algebraic language and lead to questions about how the mathematical signs systems used and their interpretation (as in the case of how pre-algebraic students read algebra's signs system) predetermine the modes of analysis and the strategies for solving problems modeled in these systems of signs. In turn, this opens up a new perspective for re-visiting the history of the symbolic systems within algebraic knowledge, taking it now as an element of theoretical analysis for cognitive sciences (Filloy and Puig, 1991).

Towards a new type of research program

An example of this new line of research is the analysis which Filloy (1991 (a)) carries out on proposition XIII of *The Book of Squares* by Leonardo Pisano (medieval mathematician, also known as Fibonacci). The proposition in question states that: "the square of any number can be expressed as the sum of successive uneven numbers, starting with the unit". In the analysis, the differences between the mathematical signs systems used by Leonardo to express his arguments and the mathematical signs system of current symbolic algebra play a central role and it is shown that this proposition, in the latter system of signs, can be proved in a simple and brief manner, unlike the long and complicated formulations of Fibonacci. This contrast is used to show the fact that the mathematical signs system used announce the strategies for solving problems and guide the sense of all the inferences made during the argument, which, in the case of Leonardo, can be related with what is observed in the development of thinking in children today, when they try to attain competence in the use of algebra at secondary school. That is to say, the new program of research proposed is to observe and describe in the work of pre-vietic algebrists, such as Fibonacci, the cognitive tendencies like those found in children who are beginning the study of algebra, in order to better formulate existing hypothesis and to develop new, plausible hypothesis about these tendencies.

Puig (1991) describes another piece of research of the same sort, on the mathematical signs system used by Jordanus de Nemore in *De Numeris Datis*. In this paper, the author attempts to describe *De Numeris Datis* as one of the strata of language which, looked at retrospectively from current symbolic algebra, are seen as less abstract. This can be related with the objective of throwing light on the uses of mathematical signs systems which will culminate in the competent use of the system of signs of symbolic algebra.

REFERENCES

Bachelard, G. (1970) *La Formation de l'esprit scientifique*, VRIN, Paris.

Brousseau, G. (1976) *Les Obstacles Epistemologiques et les Problems en Mathématiques*. C.I.E.A.E.M., pp 153-168.

Egmond, V.W. (1980) *Practical Mathematics in the Italian Renaissance. A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600*, Annali dell'Instituto e Museo di Storia della Scienza, fasciolo 1, Instituto e Museo di Storia della Scienza, Firenze.

Filloy, E. (1990) *PME Algebra Research: A Working Perspective* (in Booker, G.; Cobb, P.; Mendecuti, T. eds.), Proceedings of PME 14, Mexico, Vol. 1, Pt II, pp 1-33.

Filloy, E. (1991 (a)) *El Libro de los Cuadrados de Leonardo de Pisa*, (in Puig, L.; Rojano, T. eds.), Memorias del Tercer Simposio Internacional sobre Investigación en Educación Matemática, Historia de las Ideas Algebraicas, Valencia, España, pp 11-30.

Filloy, E. (1991 (b)) *Cognitive tendencies and abstraction processes in algebra learning* (in Furinghetti, F. ed), Proceedings of PME 15, Italy, Vol.2, pp 48-55.

Filloy, E. and Puig, L. (1991) *Historia de las Ideas Algebraicas (Presentation)* (in Puig, L.; Rojano, T. eds.), Memorias del Tercer Simposio Internacional sobre Investigación en Educación Matemática, Valencia, España.

Filloy, E. and Rojano, T. (1984 (a)) *La Aparición del Lenguaje Aritmético-Algebraico*, L'Educazione Matematica, Vol.5, No.3, pp 1-16.

Filloy, E. and Rojano, T. (1984 (b)) *From an Arithmetical to an Algebraic Thought. A clinical study with 12-13 year olds*, (in Moser, J ed.) Proceedings of PME-NA 4, Wisconsin University, pp 51-56.

Filloy, E. and Rojano, T. (1989) *Solving Equations: The Transition from Arithmetic to Algebra*, For the Learning of Mathetmatics, 9, 2, Montreal, pp 19-25.

Hughes, B., ed. (1981) *Jordanus de Nemore. De Numeris Datis*, University of California Press, Berkely

Puig, L. (1991) *el De Numeris Datis de Jordanus de Nemore: esbozo de un análisis*, (in Puig, L.; Rojano, T. eds.) Memorias del Tercer Simposio Internacional sobre Investigación en Educación Matemática, Valencia, España.

Sigler, L.E. (1987) *Leonardo Pisano Fibonacci: The Book of Squares*, Academic Press, Inc.

Witmer, T R., ed. (1983) *Françoise Viète. The Analytic Art*, The Kent State University Press, Kent, Ohio.

WHAT HISTORY OF MATHEMATICS HAS TO OFFER TO PSYCHOLOGY OF MATHEMATICAL THINKING

Anna Sfard
The Hebrew University of Jerusalem, Israel

A few years ago, in a conversation with a mathematician, I complained about my students' resistance to the seemingly simple idea of a complex number. The mathematician would not accept the claim that the subject might be inherently difficult. He hinted that it could be a problem of an incompetent teacher rather than of incapable students or a difficult subject matter. I insisted that there is a stumbling block the nature of which was not yet clear to me. I brought lots of evidence. Eventually, the mathematician seemed to be giving in. 'Yes, the problem might be somewhat more complex than I thought', he said. But then, after a thoughtful pause, he exclaimed: 'OK, it's difficult, but I assure you that I would make the students understand the idea in no time: I'd just write the axioms of a complex field and show that the concept is well-defined'.

I knew then, as I know now, that the mathematician I talked to was not an exception. Teachers are sometimes equally insensitive to profound difficulties students have with seemingly simple concepts. Several years and many hours of reading, researching, and thinking later, I began to realize the reasons of this insensitivity. It may be stated in many different ways, depending on the school of thought to which one belongs: one can say that the students and the teacher do not share one **paradigm**, or that, without realizing, they are engaged in different mathematical **practices**, or that they don't participate in the same mathematical **discourse**. In whatever words one puts the explanation, the message is always the same: the student and the teacher cannot communicate because, although they may be using identical expressions, they live in incompatible mathematical worlds, they are playing according to different rules, and their thoughts go in parallel rather than clash or coincide.

There is no better way to analyze this problem than by scrutinizing the historical development of mathematical knowledge. History is the best instrument for detecting invisible conceptual pitfalls. History makes it clear that the way toward mathematical ideas may be marked with more discontinuities and dangerous jumps than the teacher's are likely to realize. If the mathematician I talked to were aware of the turbulent biography of complex numbers, he would probably be more attentive to my classroom horror stories. Similarly, if the mathematics teacher of the French writer Stendhal bothered to look into some original mathematical texts, the latter would have been less likely to complain for the sake of posterity that his difficulties with negative numbers "simply didn't enter [the teacher's] head" (see Hefendehl-Hebeker, 1991). Indeed, it would be enough to have a brief glimpse at the emotion-laden statements of eighteenth and nineteenth century mathematicians to realize that there must have been some major disparity between the emerging ideas and established knowledge. Here is an example of such a revealing utterance:

> The talks about "impossible numbers, which, being multiplied together, produce unity.. [are] all jargon, at which common sense recoils; but, from its having been once adopted, like many other figments, it finds the most strenuous supporters among those who love to take things upon trust and hate the colour of a serious thought" (Frend in 1796).

The point I wish to make here is that historical disputes, however fierce, tend to be completely forgotten once mathematicians manage to reconcile themselves with the problematic notions. As follows from the conversation quoted above, for a modern mathematician nothing could appear simpler than complex numbers. Stewart and Tall (1983) comment, tongue in cheek: "Looking at the early history of complex numbers, the overall impression is of countless generations of mathematicians beating out their brains against a brick wall in search of what? A triviality" (p.5). In the classroom, however, problems are likely to appear at any of the junctions at which mathematicians themselves faltered and asked questions. Thus, history seems to have much to offer to the psychologist who studies the learning of mathematics.

To fully grasp the view I try to present, one has to understand far-reaching changes occurring right before our eyes in philosophers' conception of human knowledge. The Cartesian implicit assumption about the possibility of "God's eye view" of the world is gradually giving way to a non-objectivist position -- position which questions the claim that scientific theories "tell us what properties things have in themselves" (Putnam, 1987, p. 13). New winds in philosophy of mathematics may be viewed as after-shocks of this major conceptual upheaval. As a traditional stronghold of "absolute truth" and objectivity, mathematics may be the last domain of human intellectual activity to succumb, if only partially, to the doctrines of relativism and historicism (philosophy of science was revolutionized much earlier by Kuhn, Feyerabend, and their followers; in fact, it may well be that this is where all the trending away from the Cartesian paradigm started). Robust as it always was in its objectivist positions, even this most analytical of sciences could not, in the longer run, remain unaffected by the sweeping blow of the postmodern Zeitgeist. Since Lakatos declared it fallible and quasi-empirical, mathematics has been steadily proceeding toward a new identity. The image of mathematics as a "stark, atemporal, formal, universe of ideal knowledge" (Kaput, 1983) was abandoned for the sake of the idea that knowledge in general, and mathematical knowledge in particular, is a form of social practice. As such, it is inter-subjective rather than objective, it is judged by criteria of acceptability and usefulness rather than of "perfect correctness", and above all - it is in a constant flux and can never be regarded as a finished product. Thus, even though the term "practice" may have different meanings for different writers, one implication is common to all the possible interpretations: human thought and thinking in large, and mathematical thought and thinking in particular, cannot be fully understood in isolation from their history. According to Kuhn, understanding science and its development is a hermeneutical task, and hermeneutics means, among others, using a method of analysis which considers history as consequential (Berenstein, 1986). The same may be said in the context of mathematics.

Being deeply convinced of its importance, I have been using historical analysis in practically all my studies of concept formation. Whether I dealt with the learning of algebra or with the development of the notion of function, the historical perspective proved invaluable source of insights into the subtleties of learning process and into the nature of student's difficulties. This talk is too short to present any of these analyses. To illustrate my claims, let me therefore elaborate on the opening example. According to what was said above, my mathematician's insensitivity to student's difficulties with the concept of complex numbers could probably be successfully treated with a dose of historical insights and a hermeneutical exercise. If the mathematician looked into historical sources he might have realized that like in the case of mathematicians who lived a few hundreds of years ago, the student could be a captive of an inadequate system of epistemological and ontological beliefs. Today, like in the past, certain implicit meta-matematical assumptions may be the barrier which prevents the concept of a complex number from entering into the realm of mathematics. Indeed, the idea of extracting square root from a negative number is inadmissible to a person whose understanding of number is based on the metaphor of physical quantity, for whom numbers are mind-independent entities, and who does not consider the possibility of bringing such entities into existence by mere axiomatic stipulation. To realize all this it would be enough to read carefully and with a hermeneutical eye texts written by Leibniz, Euler, and many other mathematicians.

The task, of course, is far from easy. It is extremely difficult to put oneself into intellectual shoes of another generation of practitioners. This is probably why pupil's fundamental problems with the idea of complex number tend to be overlooked by the teacher. The latter's own implicit beliefs make him or her oblivious to the possibility of somebody having a different epistemological and ontological attitude. Those who stand at the top of a mountain might have forgotten the changing landscape they watched themselves while climbing the hill for the first time. One of the few works which have shown that the problem of an inadequate meta-mathematical framework does exist and obstructs learning of complex numbers is the study by Tirosh and Almog (1989). The authors attempt to probe learners' hidden conceptions led them to the conclusion that "the students are reluctant to accept complex numbers as numbers, and [they] incorrectly attribute to complex numbers the ordering relation which holds for real numbers". One finds here a striking similarity to the stance taken by past mathematicians who rejected the concept of complex number on the basis of the claim that "a square root of -1 is both greater than infinity and smaller than minus infinity".

In this brief talk I tried to make a case for historicism. This last term is used here in its "benign" non-Popperian sense, which reminds the researcher that "agent's epistemic principles and their standards of reflective acceptability ... vary historically" (Geuss, 1981). Mathematics, like any other kind of human knowledge, is inextricably entangled into its own past, and thus considering history when trying to understand the ways people conceive and construct mathematical universes is not an option -- it is a necessity.

Berenstein, R.J. (1986). Philosophical profiles, Polity Press, Cambridge

Geuss, R. (1981). The idea of a critical theory, Cambridge University Press, Cambridge.

Hefendehl-Hebeker, L. (1991). Negative numbers: obstacles in their evolution from intuitive to intellectual constructs For the learning of mathematics, 11(1), 26-32

Kaput, J (1979), Mathematics and learning. Roots of epistemological status., in Lochhead, J. and Clement, J. (eds.), ognitive Process Instruction, Franklin Institute Press.

Putnam, H. (1987), The many faces of realism, Open Court, LaSalle, Illinois

Tirosh, D. and Almog (1989), Conceptual adjustments in progressing from real to complex numbers. In G. Vergnaud, J. Rogalski, & M Artigue (Eds), Proceedings of the Thirteenth International Conference of PME (Vol.3, pp. 221-227). Paris: Laboratoire PSYDEE.

PRACTICAL USES OF MATHEMATICS IN THE PAST: A HISTORICAL APPROACH TO THE LEARNING OF MATHEMATICS.

Eduardo Veloso (Portugal)

If we accept that students attitudes and perceptions of mathematics are important factors in learning and if we believe that the integration of history in the teaching of mathematics has some effect on those attitudes and perceptions, two possible next steps would be i) to answer the questions raised by John Fauvel (1991) "why the value of using history is so hard to put across in the right quarters?" and "what is getting in the way of this simple beneficial improvement?" and ii) to discuss the present situation in what concerns *how* that integration is practiced, in order to appreciate what kind of effect it has or could have on those attitudes and perceptions. I will try to propose an answer to John Fauvel questions and to suggest and give examples of alternative ways of carrying that integration.

Firstly, the answer to John Fauvel questions could be based on the analysis of some "good reasons for using history" suggested in the same article and quoted by Paul Ernest in the introductory statement: "helps to increase motivation for learning", "makes mathematics less frightening", "gives mathematics a human face". In fact, many teachers today think that they can use *other* teaching strategies (for instance computers, manipulatives, challenging problems) to *increase motivation*, to *eliminate the fear of math* and even to teach *human* mathematics, without the additional burden of learning history of mathematics. And the *right quarters* simply don't understand why do we must integrate history in the teaching mathematics if our objectives for school mathematics are mainly the understanding of mathematical concepts and the acquisition of modern techniques. So, only when the point of view expressed in the Cockcroft Report that "the mathematics teacher has the task [...] of helping each pupil to develop so far as is possible his appreciation and enjoyment of mathematics itself and his realization of the role which it has played and will continue to play both in the development of science and technology and of our civilization..." (Cockcroft, 1982) will be openly accepted as a description of the *main* *objective* for school mathematics, we could be assured that a sound rationale for the integration of history in the teaching mathematics can be built.

Secondly, in what concerns the present practice of integration, we might say that the main stream is not satisfactory at all. The basis for this practice is the idea that the history of mathematics could be identified with the life and works of the great mathematicians. So, from the poorer ways of using history in the classroom — telling anecdotes about some mathematicians — to the ones that are most promising — activities using mathematical texts of the past — what is always implicit is the history of mathematics as a science. This is of course a valuable way to integrate the history of mathematics in education, and there are very good examples of this practice (see for instance (Fauvel 1990). But most mathematical texts written by mathematicians are difficult to be interpreted by the average student, and their use is more appropriate for classes of students with a good disposition to mathematics. So it is probable that this practice, if exclusive, would result, for the majority of students, in the perception of mathematics as a distant and mostly inaccessible domain, reserved for some gifted people. It is not in this direction that the integration of mathematics in its teaching is expected to change the perceptions of the students.

The history of mathematics is not just the history of the building of mathematics as a science, it includes the practical uses of mathematics, less or more sophisticated, during the slow development of our technological society. When the students develop activities concerning this practical uses of mathematics in the past, they may

• realize that people *like them* used elementary and intuitive mathematics with ingenuity, in order to make their work more productive and their lives better;

• see mathematics as a science "storicamente determinata e fortemente intrecciata con le altre manifestazione culturali delle varie epoche." (Bussi 1992) (free translation: "historically determined and strongly connected with the other cultural accomplishments of the different periods").

When possible, the examples of practical uses of mathematics gain to be taken from the local history (Fauvel 1993). This point could be illustrated with the experiences that are being carried, in Portugal, on the relations between mathematics and the techniques of navigation used by the Portuguese pilots during the voyages of the 15th and 16th centuries. This is a very rich domain, as suggested by the following examples (for more details, see (Veloso 1992)) .

The mathematics of astronomical navigation. The beginnings of astronomical navigation in the Atlantic occurred in the second half of the 15th century, when the Portuguese ships navigating near the coast of Africa were forced by strong headwinds to make long detours across the ocean to return to Portugal, without reference points on land. Navigators need to know their progression to the north, and they develop rudimentary ways to do it, resorting to the only references they had available, the position of the stars. Adapting old astrological instruments, like the quadrant, to new needs, they measured each day the altitude in degrees of the North Star and used a known correspondence of difference in degrees to distance navigated in latitude, in leagues (1° <> 16 1/3 leagues).
Pupils can recapitulate the reasonings of their ancestors when they study the simple mathematical principles of the quadrant, follow the method of Eratóstenes to measure the earth, and discover the correspondence between degrees and distances.

Interpretation of marine texts. The nautical handbooks of the 15th and 16th centuries were written by the pilots themselves, making successive copies of the most important rules used in the navigation, and introducing modifications requested by new conditions or findings. They are a rich source of texts, written in an interesting style and embodying almost always a mathematical component. There are rules for navigation with the Pole Star ("the regiment of the North"), with the Sun ("the regiment of the declination of the Sun") and with other stars (for instance,"the regiment of the southern cross"), rules to compute the corrections in distance when the course is deviated for any reason ("the regiment of the leagues"), rules to know the time during the night ("the regiment of the hours of the night"), and so on. Pupils will interpret these texts, discovering and exploring the mathematical underpinnings of the rules. When they read the nautical handbooks, written without the algebraic symbolism introduced later in the history of mathematics, they will appreciate the strength and utility of the mathematical language.

Building and graduating nautical instruments. After the quadrant, many other types of instruments of navigation were introduced, like the modified astrolabe, the cross shaft and the kamal. Several processes were developed for the correct graduation of these instruments. Pupils can understand the mathematical principles of the different instruments, work in small groups to build models of some of them, and even explore their uses in several ways.

REFERENCES

Bussi, Maria G. Bartolini and Quattrocchi, Pasquale (1992). Preface to the Catalog *Macchine Matematiche e Altri Oggeti.* Nucleo di Ricerca in Storia e Didattica della Matematica, Università degli Studi di Modena.

Cockcroft, W. H. (chair) (1982) *Mathematics Counts*, London, HMSO.

Fauvel, J. ed. (1990). *History in the Mathematics Classroom: the IREM Papers.* The Mathematical Association.

Fauvel, J. (1991) Using History in Mathematics Education, *For the Learning of Mathematics*, Volume 11, Number 2.

Fauvel, J. (1993). Utilização da História da Matemática Local na Educação do Jovem Matemático, *Educação e Matemática*, Number 27, Lisboa: Associação de Professores de Matemática.

Veloso, Eduardo (1992). *Portuguese Discoveries: a Source of Interesting Activities in the Mathematics Classroom.* Paper presented to the ICME-7 Satellite Meeting of the History and Pedagogy of Mathematics Study Group.

PLENARY SESSIONS

The Historical Dimension of Mathematical Understanding — Objectifying the Subjective

Hans Niels Jahnke
IDM, University of Bielefeld, Germany

1. The cognitive quality of culture — a plea against fast food mathematics

In the following, I shall ask for the role history and culture could play in the teaching and learning of mathematics. Besides L. S. Vygotsky, Jerome S. Bruner has doubtlessly been the psychologist who, in part against Piaget, has pointed out the significance of culture for cognitive development. Though the considerations I am going to quote date back nearly thirty years, they still appear worth considering. At the core of his early conception of cognitive development was the belief that *"the human, species-typical way in which we increase our powers comes through converting external bodies of knowledge embodied in the culture into generative rules for thinking about the world and about ourselves."* (Bruner [1972], 115) Though this may be a widely shared viewpoint, I have nowhere read a more impressive description of the resulting conceptual problem for (cognitive) psychology than with Bruner:*"It is always difficult for the psychologist to think of anything 'existing' in a culture or however one wishes to express the presence of traits and skills transcending any one individual's life or even the span of a generation. Psychology is in the main acultural and ahistorical in its theories: ... We are, alas, wedded to the idea that human reality exists within the limiting boundary of the human skin!"* (Bruner, Olver, & Greenfield [1967], 321) This has been written in 1967, and I am well aware that meanwhile a lot has been done to overcome the said deficit, particularly with regard to mathematics instruction and by scientists who feel obliged to the context of *"psychology of mathematics education"*. Nevertheless, I shall not be wrong in assuming that, on the whole, Bruner's statement is still true and that we find it as difficult as before to understand the cognitive quality of culture and history in a theoretically appropriate way. An indication for this is again found in Bruner when, today, he speaks of a *"cultural psychology"* (Bruner [1990]) as a desideratum and not as an existing discipline.

This unsatisfactory state of affairs within cognitive psychology is reflected in a corresponding pedagogical uncertainty about the role of culture and history in the teaching of formal sciences. Though, frequently, lip-services are payed to the cultural dimension, in reality only very few people take it serious. In fact, even if we accept the thesis that the formal sciences are founded in and by culture, it is not at all obvious that the cultural dimension should be made an explicite object of teaching. As far as I know, even

Bruner did not held the idea to assigne a greater relevance to the history of science in the teaching of formal sciences , and it must be stated that the movement of "New Math" to which Bruner adhered, at the time the above lines were written, was characterized by a profoundly unhistorical conception of mathematics and of mathematics instruction. Bruner's thesis about cognitive development is at first sight merely a proposition which states something about the evolution of human thinking. It is a cultural achievement that man generates ever more mediated and powerful systems of symbols. But it is doubtful whether making culture a topic will contribute anything towards making the conversion of cultural artefacts into generative rules more profound and more efficient. Somehow, it is as with logic. Thinking is logical, nevertheless it is doubtful whether a treatment of logic proper will contribute anything towards enhancing thinking.

There are two lines of argument for the significance of the history of mathematics for teaching which we owe to the 19th century. The first is known as "biogenetical law" and says that the cognitive development of the individual recapitulates the development of the human race on the whole. Hence, the conclusion is, teaching should also present the subject matter in the order of their historical genesis. Without doubt, this "law" may claim a certain plausibility. Mathematical reasoning in the individual develops from primitive counting skills to higher mathematical compentencies as it has been in history. In essence, however, this "law" seems to be too undifferentiated and, on the whole, wrong. Firstly, it provides, strictly spoken, no legitimation for treating history of mathematics in the classroom, but, at best, requires that the makers of curricula should know something about history. Secondly, every person grows up in a new and different culture, and if his mental development consists in transforming the artefacts of culture into his personal mental property, then the way of learning cannot recapitulate history.

The second argument in favor of history is much more specific and has exerted a strong influence on many mathematicians and mathematics teachers, at least in Germany. Mathematician Otto Töplitz formulated it most poignantly in the 1920ies. It starts from the observation that, for many learners, mathematics is a conglomerate of techniques and concepts whithout meaning. At some point in time, however, these contents have been invented as answers to concrete questions posed by human beings, and if we would go back to the times when they emerged, *"the dust of the times, the scratches of long use would drop from them, and they would rise before us as entities full of life."* (Toeplitz [1927], 90) Hence, history of mathematics could play an important, even decisive, role in teaching by giving recourse to the original, proper meaning of mathematical concepts. Without doubt, this thesis deserves a detailed discussion. I should like to confine myself to some short remarks.

It is certainly true that we obtain crucial insights into our thinking by going back to its beginnings. A comparison to teaching general history is useful. There, too, we invest much effort in teaching history to adolescents in the hope that they will grasp something about their identity als social and cultural beings. But the example of teaching general history shows also something else. In general history, there has been a significant and instructive change of paradigm. In the 19th century, educated European people held the view that it is most important to teach the ancient history of the Greeks and Romans, because this period constituted the origin of Western civilization. Knowing this history was identical with knowing our innermost identity. Today, this view is certainly no longer held. Of course, we continue teaching Greek and Roman history, but it has lost its prime importance. Modern history is deemed as significant as the ancient, and it is a matter of debate which part has contributed more towards forming our identity. Our present-day teaching of history has become more pluralistic. We learn something about our past in an exemplary way, but only few teachers will still conceive of history as of a process in which a "Western" identity was formed and developed ever higher.

To me, this has significant consequences for discussing Töplitz' thesis. On the one hand, in affirming it. Just as Greek history remains important, it is a significant experience to recur to the origin of a mathematical concept. On the other hand, in correcting it: I would deny that something like the proper meaning of a mathematical concept exists at all and as well that it can be found in its origins. Rather, it is an overwhelming experience that history introduces those who beome involved with it to a foreign world which may be interesting and fascinating, but which is considerably at odds with one's own notions about mathematics. I shall go into this in more detail with two examples in the main part of my lecture.

The tension between history and the notion of mathematics which is formed in today's classrooms has a simple and important consequence which is not always taken seriously enough by those who plead in favor of more history in mathematics teaching. History of mathematics is difficult! At all times, scientists have written for a close circle of colleagues and experts, and what they wrote was the fruit of long and intense reflections. There is no reason to assume that this should be easily accessible to adolescent students, even if subject matter is concerned which today seems to be quite elementary and which we can teach to a larger circle of students without great problems. Indeed, we have to state that while the idea to include history in one's teaching is considered attractive by many, it has not been very successful as yet. Where it is done, there are mostly entire groups composed of teachers and scientists devoting themselves to this task with much engagement (for efforts of this type in France and Italy see Commission Inter-IREM Épistémologie et Histoire des Mathématiques [1992] and Bazzini & Steiner [1989]). Of course, the achievements of individual excellent school and university teachers should not be neglected, too. However, as a rule, teachers consider history of mathematics as additional subject matter in face of too short a time and they do not feel prepared adequately. Somehow, history is felt to be alien to normal classroom work (see Jahnke [1991]).

In view of this situation, it seems to be crucial to arrive at an adequate understanding of the aims of integrating history into the curriculum. These aims should be realistic and regard the difficulties mentioned. Above all, we should develop an appropriate (neither too exaggerated nor too low) idea of what history can achieve. This idea will not be easy to elaborate and will require not only pragmatic work but profound reflection as well. Thus, I should like to pose my above question once more: does the explicit inclusion of cultural and historical references yield an indispensible cognitive support for learning mathematics, and if so, which is the cognitive quality of this contribution, and how can it be described? Or rather: does history provide only additional subject matter which has nothing to do with "mathematics proper" and which, at best, does no harm, but which has no intrinsic connection to the mathematics we teach?

Faced with questions of this type, it is often useful to take a step back and look for analogous situations in other fields. We know that every vital function of man is culturally mediated, however basic and close to his biological needs it may be. It would be easy to think that while this cultural moulding is a fact of life, becoming aware of it plays no role for exercising them. But this is simply not true. Even if we consider functions so basic as sexuality and nutrition, it is quite evident that their cultural dimension plays a quite important part in our everyday world. How strongly sexuality is culturally thematized, I need not dwell upon, nor on the extent to which this thematization influences our actual experience of sexuality. But even in regard to nutrition it is true that while one can eat without knowing anything about the culture of eating, it is not only gourmets who know that one cannot really relish food without knowing about manifold cultural aspects (eating habits, producing and refining foodstuffs, how and where the raw products are grown). This includes quite naturally a historical dimension as well, and quite a branch of experts is fully occupied with disseminating knowledge about these things in magazines, books, films and advertising. Of course, the cultural dimension of eating is differently perceived by different people, with some quite elaborately, with others only in rudiments. To formulate things polemically: eating without cultural perception is consuming "fast food", the latter meaning both the products and a certain attitude towards eating.

This example teaches us something else, too. The way, how the eating culture is conveyed to us, embraces also historical dimensions, though they they may be rather reduced. Thus, like any culture, the culture of eating should be differentiated into two aspects: into a <u>synchronous and a diachronous culture</u>. The crucial problem is then to understand how the synchronous and the diachronous culture are intertwined and how they interact.

What does this comparison teach us? To me, one insight seems to be most important: the cultural moulding and reflection of our basic needs is a <u>spontaneous, natural, human process</u>. It is not intentional. Nobody consciously decides to look for the cultural dimension of nutrition and sexuality in order to motivate people to maintain their own lives and their species, but vice versa, culture is an integral part of these functions. "Fast food", however, also shows: culture has natural enemies. Culture of eating is not simply there, but it has to be developed, enhanced and defended.

For mathematics teaching, this means: the value of the historical view is missed if we try to make it function for some other purpose. If we only intend to "motivate" our students and to keep their attention awake, the real importance of the historical view is missed. History should be a natural component of teaching and it cannot be justified by anything but itself. History of mathematics requires justification just as little as mathematics proper. Whoever gives mathematics lessons in which questions of how a concept came historically into being never arise, has already made a far-reaching decision about the cultural framing of his teaching. To say it polemically: he has opted for fast food mathematics.

In the subsequent parts of my lecture I should like to develop first some considerations on the relationship between mathematics and culture from two small case studies, and then, in the closing part, I shall present a model for the integration of history into teaching which aims at interlinking the synchronous and diachronous culture of teaching.

2. Cultural dimensions of mathematics — the case of quadratic equations

In order to study the significance of cultural and historical frames for understanding mathematics I shall discuss two cases: the history of quadratic equations and the introduction and justification of the transfinite ordinal numbers by Georg Cantor. Let us begin with quadratic equations. We shall confine ourselves to two texts (episodes). While they are presented here in their historical order, we shall see that, from the viewpoint of school teaching, it does not make sense to study them in this order. Indeed, the first text is difficult to understand, and its relation to quadratic equations is not evident. This highlights some of the difficulties historical texts frequently entail.

The text in question is theorem 6 from the IInd book of Euclid's Elements (in the following we say Euclid II, 6): *If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.* (*Heath [1956]*, 385)

Thus, the theorem says (see fig. 1): $AD \cdot DB + CB^2 = CD^2$.

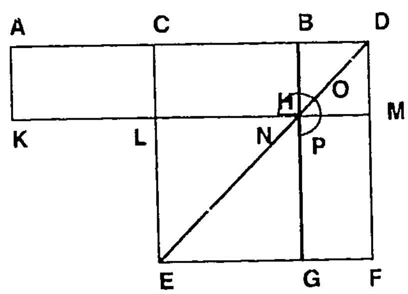


Figure 1

Obviously, the meaning of this and other theorems in Euclid II is not easy to decipher. The historians' standard view ist that this is algebra in geometrical disguise, the so-called "geometrical algebra". According to this, the theorem is a means to construct geometrically the solution of a quadratic equation of the form

$$x^2+ax=b^2.$$

This can be achieved in the following way. The idea is somehow concealed in the wording of the theorem, since the Greeks expressed it additively. If we reformulate the theorem slightly, the idea can be seen better:

$$AD \cdot BD = CD^2 - CB^2$$

Here, the difference between the two squares on the right side has a geometrical meaning. It is the angular area NOP situated around the point H (fig. 1). Such angular figures were designated by the Greeks with a proper term as a "gnomon". Since the gnomon at hand is the difference between two squares, it is suggestive that the Pythagorean theorem must somehow be applicable. Indeed, if we take

$$AB = a$$

as the given quantity and

$$BD = x$$

as the sought one, then we have

$$(a+x)x = x^2 + ax = \text{Gnomon}.$$

The gnomon, however, is a difference between two squares and shall be $=b^2$. Thus, the Pythagorean theorem can be applied to construct the solution of the equation:

$$\left(\frac{a}{2}+x\right)^2 - \left(\frac{a}{2}\right)^2 = b^2$$

$$\left(\frac{a}{2}+x\right)^2 = b^2 + \left(\frac{a}{2}\right)^2$$

This yields immediately a geometrical construction of a root by means of compass and ruler.

As has been said above, this interpretation has not remained unquestioned, as no source can be found with the Greeks in which this theorem is used in this way (see f.e. Unguru [1975/76] and van der Waerden [1975/76]. For a discussion cf. Artmann [1991]) One finds, however, that it has been used for geometrical problems which, today, we would describe algebraically by quadratic equations. Thus, for instance, in Euclid II, 11 the golden section is derived from our theorem Euclid II,6, and, algebraically, the golden section requires the solution of a quadratic equation. Likewise, Appolonius has characterized the conic sections by conditions which take recourse to the corresponding theorems in Euclid II. Nevertheless, it may rightfully be asked whether it is justified to speak simply of algebra here. Euclid II contains purely geoemtrical theorems which are used to solve purely geoemtrical problems. However this question may be answered, in any case, to me the conclusion seems irrefutable that Euclid II belongs into the history of algebra, because some subsequent authors referred explicitely to Euclid or used similar geometrical figures in order to justify the undisputably algebraic formulae for solving quadratic equations.

This is, for instance, the case with al-Khowarizmi, and with this we come to our second text. In the 9th century A.D., al-Khowarizmi (see Rosen [1831]) formulated a verbal rule to solve the quadratic equation of the type *"Roots and Sqares are equal to Numbers"* (e.g.: $x^2+10x=39$): *"...you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add to this thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine"* (l.c., 8). He justified this rule by the following geometrical diagram

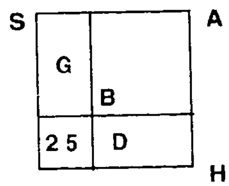


Figure 2

The square AB represents the x^2 sought, to which the two rectangles G und D of size $5x$ are made adjacent. In the lower left corner is then a square of size $5 \cdot 5 = 25$ "missing". If it is added, the large square SH has the area $25+39=64$ and, thus, the side length $8=5+x$. Therefore, a solution of the equation is $x=3$.

Obviously, there is a certain similarity between this figure and that in Euclid. Al-Khowarizmis's figure, however, is much simpler, and its meaning is much easier to decipher. It does not aim at a geometrical theorem which would have to be formulated statically, but rather for an arithmetical or geometrical, in any case algorithmic, determination of a certain quantitiy. Al-Khowarizmi's figure must be read dynamicall, and it is incomplete in the sense that it does not contain any equality at all. We must mentally add this equation. Al-Khowarizmi's figure has only an operative meaning, but this meaning is clear and unmistakable.

This purely cognitive description of the difference between Euclid and al-Khowarizmi, however, remains incomplete and unintelligible, if the respective cultural context is not considered. Indeed, there is evidence that al-Khowarizmi belonged to a group of Arabian mathematicians who followed an anti-Euclidean attitude and did mathematics not out of a theoretical, but out of a purely practical interest. Solomon Gandz has described this very impressively, and I should like to quote him at some length:*"At the university of Baghdad founded by al-Ma'mun (813-33), the so-called Bayt al-Hikma, 'the House of Wisdom', where al-Khowarizmi worked under the patronage of the Caliph, there and then flourished also an older contemporary of al-Khowarizmi by the name of al-Haijaj ibn Yusuf ibn Matar. This man was the foremost protagonist of the Greek school working for the reception of Greek science by the Arabs. All his life was devoted to the work on Arabic translations from the Greek. Already under Harun al-Rashid (786-809) al-Haijaj had brought out an Arabic translation of Euclid's Elements. When al-Mamun became*

Caliph, al-Haijaj tried to gain his favor by preparing a second improved edition of his Euclid translation. Later on (829-30) he translated the Almagest. Now al-Khowarizmi never mentions this colleague of his and never refers to his work. Euclid and his geometry, though available in a good translation by his colleague, is entirely ignored by him when he writes on geometry. On the contrary, in the preface to his Algebra al-Khowarizmi distinctly emphasizes his purpose of writing a popular treatise that , in contradiction to Greek theoretical mathematics, will serve the practical ends and needs of the people in their affairs of inheritance and legacies, in their law suits, in trade and commerce, in the surveying of lands and in the digging of canals. Hence, al-Khowarizmi appears to us not as a pupil of the Greeks but, quite to the contrary, as the antagonist of al-Haijaj and the Greek school, as the representative of the native popular sciences. At the Academy of Baghdad al-Khowarizmi represented rather the reaction against the introduction of Greek mathematics. His Algebra impresses us as a protest rather against the Euclid translation and against the whole trend of the reception of the Greek sciences."(Gandz [1932], 65/66)

In the history of medieval Arabic algebra, there were obviously two currents. One consciously adopted Greek traditions and founded the methods to solve quadratic equations by recurring directly to Euclid II. This current can be designated as a scientifc-academic one. The other current had its sources more in Indian mathematics and in Near and Middle East traditions oriented towards popular applications. Van der Waerden speaks of a tradition of popular mathematics in Egypt and in the Near East during the hellenistic and post-hellenistic epoch (van der Waerden [1985], 14); similar statments have been made by J. Hoyrup (see Scholz [1990], 43). Al-Khowarizmi was probably inspired by these sources. Thus, the history of algebra seems to be fed by different lines of tradition which had been influenced and shaped by various theoretical and practical interests.

If we take these facts for granted, we may ask whether such knowledge should be part of the mathematical teaching offered to our students. I should like to answer this clearly in the affirmative. Why? The core of my answer is that history serves to thematize problems which concern the students and for which there is no opportunity of discussion in the usual learning environment. This concerns a host of aspects among which I should like to single out two, which seem particularly important to me. The first is the contrast, already mentioned, between practical and theoretical mathematics. This contrast can first be connected with the question why al-Khowarizmi's solution of the quadratic equation is so easy to understand and why Euclid's theorems prove to be so inaccessible. We had already pointed out that al-Khowarizmi's diagram must be read dynamically, as a step-by-step calculation of a quantity which is represented as a geometrical variable. This type of mathematics is characterized by the immediately perceptible connection between the motive of action (one wants to know a concrete quantity), plan, and action. On the other side, there is the unwieldy Euclidean theorem. It does not immediately express its meaning and why we should bother at all with such a theorem. As mathematicians, we are familiar with this phenomenon. Many theorems of mathematics have no meaning out of themselves, but only as components of a theoretical structure. To establish and to develop such a structure, i.e. a theory, is not only a cognitive achievement, but requires also a certain social and cultural context within which dealing with such purely mental structures makes sense (and, as we know, leads often to substantial practical conclusions). In Greece, for instance, a book like Euclid's can only be understood if we take into consideration the cultural climate in which philosophies of mathematics like those developed by Platon and Aristotle could grow. Students, too, are confronted with the contrast between theoretical and practical mathematics from the very beginning of their mathematical career, but there is no place in teaching where the perceptions, attitudes, motives and sentiments linked with the different types of mathematics can be thematized and reflected upon. Thus, important distinctive features of mathematics are reduced to individual observations never publicly mentioned which frequently turn into bizarre sediments of an unreflected mathematical biography. Here, history would offer an opportunity to relate one's own experience to the historical lines of traditions and, thus, to experience subjective impressions as objectively founded.

My second point refers to the concept of variable. This is a fundamental concept of mathematics. It is fundamental for the mathematical mode of thinking and for the way how mathematical objetcs exist. However, it does not belong to its 'subject matter' and is not explicitely treated. Though being one of the most fundamental ideas of mathematics, the concept of a variable is more a notion of meta-mathematics than of mathematics itself. The simple idea to suppose something not yet known as known and to represent it by a symbol "x" required a long history before it was elaborated. In al-Khowarizmi, we evidently find variables in geometrical form, but already conceived of operatively as constituents of algorithmic procedures. Al-Khowarizmi did certainly not invent variables, but rather they had already existed for a long time within the practical popular tradition. In Greek mathematics, Pappus reported about the "analytical method" which embraces the very idea of assuming what is sought as known. It is very probable that this idea goes back to the tradition of practical mathematics. It is this very idea, Descartes mentions again at the beginning of his "geometry" (1637). The meaning of the concept of variable is obviously not confined to finding the correct algebraic equation for elementary mathematical problmes. In his dispute with Frege, Hilbert has pointed out that systems of axioms can be compared to systems of conditional equations. Every axiom is some kind of equation for the undefined terms occurring within it, which must in consequence be regarded as variables (cf. Steiner [1964], Steiner [1965] and Jahnke [1978], 147 - 151). In tis generalized sense, the use of variables is even not only constitutive for mathematics, but for any theoretical science. Every science contains basic concepts which are at first onyl partially interpreted and are progressively determined by new applications and new theorems in the course of theoretical progress.

Again, we have to note that current teaching has no proper place for such ideas though they are constitutive for an adequate understanding of mathematics. Consequently, we encounter only a small number of pupils and students who are somehow aware of the significance of the method of setting the "unknown" equal to "x". At best, they will see this as a "brilliant trick", but none will realize that the concept of variable is a key to large parts of the history and philosophy of science. Thus, there is again the danger that this important achievement becomes a "bizarre sediment" within an unreflected mathematical biography. Such ideas should and can be thematized independent of historical contexts, but I claim that the very individual character of a case like al-Khowarizmi's substantially contributes towards understanding the general meaning of a concept like that of variable. This insight is deepened by the contrast between our present purely symbolical algebra and its geometrical versions which can be found in history. This contrast provides the opportunity of imbedding the meaning of algebra in broader contexts and making it understandable.

These brief remarks will suffice to make clear how I think that history could be included into the topic of "quadratic equations". Obviously, not genetically as a means to teach the algebraic methods of solution. This should be done in one of the usual ways. In the subsequent period of exercises, however, one of al-Khowarizmi's texts could be read. His geometric derivation is mathematically fascinating and challenging for students. According to the classroom situation, the other, more difficult cases of quadratic equations can be treated as well. If al-Khowarizmi's book is then discussed as a whole, the class will be guided immediately to the cultural and historical context. With regard to his practical motives, al-Khowarizmi is very explicit. Among the interesting questions is the one which practical applications al-Khowarizmi offers for quadratic equations. The students will be bitterly disappointed and find not one unartificial example. Perhaps comparisons with the textbook in use will be possible. The cultural context would also provide a key for a brief glance at Euclid, but under normal circumstances this would be going too far. I am convinced that the students' material view on algebra will substantially profit from this excursion into history.

3. The cultural dimension of mathematics - Georg Cantor's introduction and justification of the transfinite ordinal numbers

As my second case, I should like to treat the introduction and justification of the transfinite ordinal numbers by Georg Cantor in 1883. Obviously, this case is far from the former with regard to content and historical period, yet there is a relationship. This is established by the generalized meaning of the concept of variable of which I have spoken above. Cantor introduced the transfinite ordinals as symbolic quantities having certain properties. This procedure was not as evident to him as it is for us today, and, hence, he gave quite a number of reasons which can only be understood when considering the cultural and philosophical context of his time. These I am going to analyze. As a result of this analysis, Cantor will prove to be a typical representative of a transitional period from the 19th century to modern mathematics, and getting familiar with his mode of reasoning offers us a unique access to the origins of modern mathematical thought.

To understand Cantor's situation, we must realize that there are two different components in Cantor's set theory. One is the set theoretical description of point aggregates of real numbers or of spaces of higher dimension. This was intuitively accessible to Cantor's colleagues, and as an informal way of speaking this type of set theory had existed a long time before Cantor. He was only the first to start a systematical study in this field and to attain remarkable and surprising results. His achievements in this area were received and used by his colleagues. Quite different is the situation with the theory of the transfinite cardinals and ordinals, which Hilbert later designated as *"one of the most audacious creations of the human mind"*. This part of set theory met with widespread lack of understanding and even sharp criticism among his contemporaries.

The emergence of set theory started in 1872, when Cantor published a paper on the uniqueness of Fourier series representations (in regard to Cantor's biography and the emergence of his set theoretical ideas cf. Dauben [1979] and Purkert & Ilgauds [1987]). For the theorem to be proved he needed the new concept of a "derived set" to characterize certain subsets of the reals. He defined the derivation P' of a subset P of the reals as the set of all limit points of P. In the same paper he gave also a new foundation for the real numbers, an achievemnet which had been sufficient to secure him a prominent place in the history of mathematics. In 1874, Cantor published another paper where he proved that the algebraic numbers are countable whereas the reals are not. In 1877, after many efforts, he was successful to prove the result, surprising even to himself, that the the set of points of the unit square has same the cardinality as the set of points of the unit intervall.

Though some of these results were a real sensation, they remained within the limits of the widely accepted set theoretical description of the continuum and its subsets. From 1879 on, however, Cantor began publishing a series of papers under the heading "Über lineare unendliche Punktmannichfaltigkeiten" ("On linear infinite point sets"), part 1 to 6, in which he accomplished the decisive step towards the theory of transfinite ordinals. This was in the 5th contribution published in 1883 (Cantor [1883]). In the introduction, Cantor wrote that his studies had progressed to a point *"where their continuation becomes dependent on an extension of the concept of the real whole number beyond the present limitations, and this extension falls into a direction where nobody has hitherto sought it."* (l.c., 73) He was well aware of the fact this his endeavour was *"in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers"* (l.c.). Therefore, he devoted most of this paper to historical and philosophical reflections on the infinite and his new transfinite ordinals. In its entire character, this paper went beyond what was then mathematically accepted use, and Felix Klein's amazing foresight was required to ensure that Cantor's work could be published at all in a mathematical periodical.

To understand Cantor's epistemological problem and the audacity of his step, we must go back a bit. Cantor had already introduced the transfinite ordinals in the second paper "On linear infinite point sets" of 1880 as symbols designating the order of a certain iterative procedure (Cantor [1880]). If P is a set of points, and if $P, P', \dots P^{(n)}, \dots$ are derivations of P

of finite order, we obtain the first derivation with infinite order and, thus, the first infinite ordinal according to $P^{(\omega)} = \bigcap_{n=1}^{\infty} P^{(n)}$. After $P^{(\omega)}$ follow then $P^{(\omega+1)}$, $P^{(\omega+2)}$. The derivation of order of $P^{(\omega)}$ is then defined by $P^{(\omega 2)} = \bigcap_{n=1}^{\infty} P^{(\omega+n)}$. This process can be continued over all polynomials in , up to expressions like $P^{\omega^{\omega}}, P^{\omega^{\omega^{\omega}}}$, etc. In the present context, the most important fact is that intuitive-geometric examples can be given for all these sets, as Cantor showed. Therefore, the transfinite operative symbols $\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots$have a clearly determined, intutive meaning, too. In order to illustrate the situatiuon, one could perhaps draw an analogy to fractions. Fractions like $\frac{3}{4}$ or $\frac{1}{7}$ can be understood as operative symbols producing parts from a whole (e.g. a cake). In this sense, fractions have a well-defined intuitive meaning. But it is something very different to detach fractions from this intuitive meaning and to conceive of them as of autonomous objects. for which an abstact calculus, the calculus of fractions, is established.

This is precisely the step Cantor does in the said 5th paper. The transfinite operative symbols - Cantor speaks of *"bestimmt definierte Unendlichkeitssymbole" ("definitely defined symbols of infinity")* (Cantor [1883], 74)- are detached from their intuitive context and conceived of as *"unendliche reale ganze Zahlen" ("infinite, real, whole numbers")* (l.c.). As we have seen, Cantor was well aware of the audacity of this step, and I am going to discuss some of his arguments for justifying it. My purpose is not to reconstruct Cantor's mathematico-philosophical views completely, but I shall select some of his arguments, classify them, and relate them to their cultural and philosophical background within 19th century Germany. I distinguish seven arguments:

1. Mathematics is a formal science.

Cantor distinguishes between two meanings of the reality of whole (finite or infinite) numbers, their intrasubjective or immanent reality and their transsubjective or transient reality (l.c., 90/91). The former consists in the relations of numbers to other *"components of our thinking"*, the latter says that numbers have to be considered also as expressions or images of processes in the exterior world with which the intellect is confronted. A concept's immanent reality, according to Cantor, presupposes that its relations to the other, previously formed concepts are well-defined and consistent. Consequently, pure mathematics must deal only with a concept's immanent reality and is not subject to a metaphysical control. In this sense, it is "free" mathematics. Against that "applied" mathematics (analytical mechanics and mathematical physics) must take into account the relationship towards external objects and is insofar metaphysical in its foundations and objectives.

The strategic function of this reasoning is clear. It provided a general legitimation against all those colleagues who frankly denied the existence of the actual infinite and therefore would reject the transfinite ordinal numbers. If Gauß, Abel, Jacobi, Dirichlet, Weierstraß, Hermite and Riemann had been subject to a metaphysical control, Cantor said, the modern theory of complex functions, which at first had no applications, could never have been developed. Cantor extended this general argument by another one specific to the mathematics of the 19th century.

2. The example of Ideal elements. The transfinite ordinals and the actual infinite, Cantor says, are creations just as legitimate and at the same time alien to the everyday mind as the complex numbers had been ages ago (l.c., 74). We may note that this argument has a certain tradition in mathematics. Leibniz and others had already justified their use of infinitesimally small numbers with the same remark. In the 19th century, this entire method of introducing new, intuitively uninterpretable concepts in a purely symbolical way had been elaborated into a systematic technique by various mathematicians. Thus, for example, infinitely distant points, straight lines and planes in projective geometry, one infinitely distant point in the complex plane and ideal algebraic numbers had been created and proved to be powerful mathematical instruments. Cantor pointed explicitely to these examples, and indeed a present-day mathematician who has grown up with these techniques has difficulties in understanding why Cantor was exposed to such a strong criticism. Thirty years later, David Hilbert used the very same argument to legitimize his formalist foundation of mathematics. Hilbert considered only finite combinatorial objects to be intuitively interpretable, and the rest of mathematics turned into a universe of "ideal elements". Cantor's proclamation of the liberty of pure mathematics thus was, under philosophical aspects, an offshoot into mathematical modernity which nevertheless had its distinctive genesis within the cultural and philosophical soil of the 19th century.

Cantor's roots in 19th century culture can be seen most impressively from the fact that he was not simply a formalist and was not content with a formal view on pure mathematics, either. For him, there was no doubt that immanant and transient reality will always be in accordance, and he saw the reason for that *"in the unity of the universe to which we belong as well"* (l.c., 91). Thinking and external reality are in the last instance tied together by an inner harmony even if this is not evident at any time and in all respects. In order to elaborate the material meaning of the transfinite ordinals, Cantor, thus, advanced quite a number of additional arguments which again have their roots in 19th century culture.

3. The Appeal to Internal Intuition ("Innere Anscahuung"). In various parts of his text, Cantor appeals to "internal intuition" ("innere Anschauung") as proof that the transfinite ordinals exist. In one place he says, that the concept of counting possesses an immeadiate objective reality in our internal intuition and that this proves the reality of ordinal numbers even when they are definitely infinite (l.c., 76). A little later, he states that the laws of these numbers can be derived with apodictic certainty from *"immediate internal intuition"* (l.c., 78). He even proclaims the theorem that any set can be well-ordered to be an universally valid law of thinking (l.c.). The entire scope of this appeal to internal intuition can be understood only within the cultural climate of 19th century Germany. The concept seems to appear first in Kant where it is the psychological correlate to pure intuition. It designates our ability to internally grasp spatial and temporal forms as the basis of geometry and arithmetics, independent of any external empirical perception. Later on, this concept was

extended and linked to that of "productive imagination", then designating the ability to imagine even objects which go beyond spatio-temporal intuition. Internal intuition thus meant the ability of imagining worlds which lie beyond our empirical everyday world. The concept also had an important pedagogical function, as all pedagogical authors from Herbart to Diesterweg agreed that the development of internal intuition in contrast to external perception is the main aim of mathematics instruction (c.f. Jahnke [1990], 34 - 62). During the entire 19th century, the concept of internal intuition played an important role in the mathematical literature, because ideal elements such as the infinitely distant points in projective geometry, while not being empirically perceptible, seemed accessible to internal intuition, and Cantor used it in precisely this way in the phrases mentioned above. He was firmly convinced that such an internal intuition existed and was apt to prove the existence of a mathematical object, and he could trust that this would convince his readers, too.

4. The Aesthetical Argument. The beauty of a theory was a much-used topos in the scientifc literature of the 19th century, and, consequently, Cantor also wrote that it was *"pure delight"* for him to see how the number concept splits into the two concepts of cardinal and ordinal number when one ascends to infinity (l.c., 90).

5. The Legitimacy of Abstract Speculation. Kronecker, Cantor's strong opponent, held the view that only the finite whole numbers really existed and that, in the last instance, all other analytical concepts should be conceived of as relationships between finite whole numbers. This conception, later designated as "early intuitionism", was directed against most analysts of the time, in particular, however, against Cantor. In a very poignant remark of his paper, Cantor supposed that Kronecker's intention was *"to draw limits to the free flight of the mathematical speculation, to prevent its falling into the chasm of the 'transcedent'."* (l.c., 82) This leads directly into one of the hottest and typically German disputes within 19th century science. While philosophy and science had been closely allied at the beginning of the century, such as in romantic natural science ("romantische Naturforschung"), an anti-philosophical turn occurred later which led to a predominance of positivistic views. Cantor, obviously, views himself as being on the side of speculative thought and opposed to positivism even in 1883. He believes to advance a strong argument when he reproaches Kronecker for being attached to *"utilitarianism"*. C. G. J. Jacobi had already instisted, against Fourier, that he *"did science for the glory of the human mind"*. Within the German academic climate, it was not opportune, even at the end of the century, and despite the positivism, to be considered an adherent of "utilitarianism".

6. Organismic Conception of Nature. Cantor does not state it openly, but he nevertheless suggests to the reader in connection with one of his remarks about Leibniz' and Spinoza's philosophy that transfinite numbers are a first step towards a mathematics which would be suitable for an organismic research into nature, whereas the mathematical analysis hitherto used sided with the mechanical explanation of nature. Again, we observe a cognitive motive of romantic natural science (l.c., 86).

7. Mathematical Generalization and Differentiation of Properties. Against those mathematicians and philosophers who saw in the existence of the actual-infinite a

contradiction to other concepts and principles of mathematics, Cantor argued that it is one of the essential features of mathematical generalization that the generalized objects do not necessarily have the same properties as those which had been assumed at the outset. For finite sets, cardinal and ordinal number (in German "Anzahl", we say for the sake of simplicity "ordinal number") are congruent in the sense that sets of the same cardinality also have the same ordinal number. This is no longer true for infinite sets. Sets of the same cardinality may very well have different ordinal numbers. This is so, because for infinte sets the ordinal number is dependent on the previously defined well ordering, whereas for finite sets the possible different orders lead to the same ordinal number. In the finite case, the commutative law of addition holds for ordinal numbers, but this is not true for infinite sets, since $1+\omega=\omega\neq\omega+1$. For finite sets, an ordinal number is either even or odd, in the infinite cases, it can be both, because of $\omega\cdot 2=1+\omega\cdot 2$. The situtaion is like that with complex numbers, where one cannot distinguish between positive and negative numbers either (l.c., 75, 77, 78). Teacher students should be able to see the analogies to problems learners encounter when switching from calculating with natural numbers to fractional or negative numbers. Among other things, with fractions the "law" that multiplication increases a number will not longer hold true.

If we try to sum up these arguments, we will not be able to percieve them as a closed edifice of reasoning. The very same internal intuition which made Cantor "see" his transfinite ordinals served him as a reason to consider the infinitely small quantities mathematically illegitimate. Nevertheless, there is, as we have seen, a perhaps surprising relation to some general trends in German cultural history of the 19th century. Cantor's entire arguments had already played a part eighty years before in the romantic movement and romantic natural science. Romanticism had had to cede to positivist views, but some lines of thinking which originated from romanticism remained vital within the German academic world. With Cantor they fused again into a surprising synthesis.

Again, we may ask whether our students can profit from such a text and from being able to comprehend it historically. Of course, today, one hundred years after Cantor, we will not justify the introduction of the transfinite ordinals in the same way. Nevertheless, the tension between theoretical concepts and their intuitive interpretation remains a problem which concerns every mathematician, I would even say, every scientist, and for which there is no definite answer. It is precisely this which makes the historical view indispensible. Cantor's text quoted contains a fascinating mixture of aspects which can be immediately linked to present-day learners' experience and other aspects which must be inferred from the contemporary context. But even the romanticist subcurrent in Cantor's thinking is not so alien to our present-day problems as may seem at first glance. The big topic ot romanticism of imagining worlds which lie beyond the everyday world has attained surprising actuality in the computer age with its artificial worlds. Thus, the historical view does not only lead us back into the past, but enables us to see our present-day problems more clearly.

4. The Twofold Cirlce

To close, I should like to sketch a conception for integrating historical dimensions into mathematics teaching. As I have said in the beginning, it is particularly important to relate the sychronous and the diachronous mathematical culture to each other. The synchronous culture contains the culture of dialogue and work in the classroom just as well as the role of mathematics in public life, in economy and technolgy and the image which is attached to it. The diachronous culture must be related to this life of mathematics, but it should not simply affirm the sychronous culture, but should rather widen our visions.

My principal thesis is that the concept of "hermeneutics" is suitable to describe the pedagogical interaction between synchronous and diachronous culture. At a first glance, this thesis has quite simple consequences. The mathematics teacher who wants to use history of mathematics in his teaching should understand something of the historian's work. I mean this in the sense that also a physics teacher should understand something about the phsicists's work, at least so much as to have an approximate idea how physical theories refer to reality, which is the role of experiments, how hyotheses are obtained and tested. If history of mathematics is not to deteriorate into a dead dogma, a mere addition to the dogmas of mathematics, then the teacher introducing history should know something about historical sources and the basically hypothetical character of large parts of our historical knowledge.

On the other hand, my thesis implies the claim that the historian's perspective represents an important element of an appropriate teaching culture. Seen under the aspect of method, history of mathematics, like any history, is essentially a hermeneutic effort. Theories and their creators are interpreted, and the interpreter is always aware of the hypothetical, even intuitive character of his interpretation. Interpretation itself takes place within a circular process of forming hypotheses and checking them against the text given. In the case of history of science, the objects of this process of interpretation, the scientific subjects (individuals and groups), are again hermeneuticians who interpret fields of objects. Of course, this view of scientific work will be adequate with varying precision in different times and different fields. But if we do not understand it too narrowly, this description can very well be advanced. Scientific interpretation, too, is now subject to the circular process of forming hypotheses, testing and revising them. He who is concerned with history, thus, has to do with a complex network of interpreters, problem fields, and interpretations (theories) which I have represented in a little diagram and which I should like to name the "twofold circle".

This diagramm consists of a primary circle in the right bottom representing the circular relation between a scientist (S), a theory (T) and a field of objects (O) and a large secondary circle representing the historian (S_1), a historical interpretation (T_1) and the primary circle as his field of objects.

My proposition that the teacher should know and understand something about the historian's perspective if he takes history of mathematics into the classroom, refers precisely

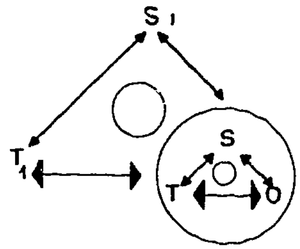


Figure 3

to the problem that he/she must be aware of this twofold circle and able to move within it. Only this will enable him and his students to acquire a certain freedom against the subject matter to form hypotheses and to be ready to think oneself into other persons who have lived in another time and another culture.

For me, this thinking oneself into another person and into a different world seems to be the core of an educational philosophy, providing a basis for historical contents in mathematics teaching. He who thinks himself into a scientist or into a group of people doing mathematics, has to do mathematics as well, he moves as a mental game in the primary circle. This compels him, for instance, to imagine the theoretical conditions the historical person is explicitely or implicitely supposing. These, however, he will know but incompletely, he will have to mobilize imagination to form hypotheses about them. The essential thing in this is that doing mathematics in the primary circle is guided, in its objectives, by other aspects which result from relations within the secondary circle.

Thinking ourselves into other persons leads us to reflect about our own relationship to mathematics. This reflection, in turn, is objectivated by the material (mostly texts) we are working upon. Certain aspects of the historical person will be easily accessible to us, others will reamin alien. Again: Why? As always in hermeneutics, our own self unavoidably comes into play, and it is not felt to be a disturbing factor, but a decisive prerequisite to insight.

A third element in this work is of especially pedagogical value. The roles of teachers and students have to change necessarily if hermeneutics is to be taken seriously. Surely, their roles will not become equal, but more symmetrical. At certain points the authority of the teacher will necessarily end.

Let me end with the statement that in all fields of life it will become more and more important in the future to consciously relate the diachronous and synchronous culture. The teaching of mathematics should participate in this process.

Bibliography

Artmann, B. (1991). Quadratische Probleme in Euklids 'Elementen' und ihre Behandlung im Mathematikunterricht. *Didaktik der Mathematik, 12*, 94-110.

Bazzini, L., & Steiner, H. G. (1989). *Proceedings of the first Italian-German bilateral symposium on didactics of mathematics*. Pavia.

Bruner, J. S. (1972). *The Relevance of Education*. London: Allen.

Bruner, J. S. (1990). *Acts of Meaning*. Cambridge, MA: Harvard University Press.

Bruner, J. S., Olver, R., & Greenfield, P. (1967). *Studies in Cognitive Growth*. New York: Wiley.

Cantor, G. (1880). *Ueber unendliche, lineare Punktmannichfaltigkeiten*. Mathematische Annalen (Vol. 17, pp. 355 - 358). Here quoted according to: G. Cantor, Über unendliche lineare Punktmannigfaltigkeiten. Arbeiten zur Mengenlehre aus den Jahren 1872 - 1884, ed. by G. Asser, Leipzig: Teubner Verlagsgesellschaft, 1984, 52 - 55

Cantor, G. (1883). *Ueber unendliche, lineare Punktmannichfaltigkeiten*. Mathematische Annalen (Vol. 21, pp. 545-591). Here quoted according to: G. Cantor, Über unendliche lineare Punktmannigfaltigkeiten. Arbeiten zur Mengenlehre aus den Jahren 1872 - 1884, ed. by G. Asser, Leipzig: Teubner Verlagsgesellschaft, 1984, 73 - 114

Commission Inter-IREM Épistémologie et Histoire des Mathématiques (1992). *Histoire de Problèmes - Historie des Mathématiques*. Lyon: IREM de Lyon

Dauben, J. W. (1979). *Georg Cantor*. Cambridge, MA: Harvard University Press

Gandz, S. (1932). *The Mishnat ha-Middot and the Geometry of Muhammed ibn Musa Al-Khowarizmi. Quellen und Studien zur Geschichte der Mathematik A2*. Berlin: Springer.

Heath, T. L. (1956). *The Thirteen Books of Euclid s Elements. Translated from the text of Heiberg. With introduction and commentary* (Reprint of the Second Edition). New York: Dover.

Jahnke, H. N. (1978). *Zum Verhältnis von Wissensentwicklung und Begründung in der Mathematik - Beweisen als didaktisches Problem*. Bielefeld: Materialien und Studien des IDM, vol. 10

Jahnke, H. N. (1990). *Mathematik und Bildung in der Humboldtschen Reform*. Göttingen: Vandenhoeck

Jahnke, H. N. (1991). Mathematik historisch verstehen-oder: Haben die alten Griechen quadratische Gleichungen gelöst? *mathematik lehren* (August-Heft), 6-12.

Purkert, W., & Ilgauds, H. J. (1987). *Georg Cantor 1845 - 1918*. Basel: Birkhäuser Verlag.

Rosen, F. (1831). *The Algebra of Mohammed ben Musa*. London

Scholz, E. (Ed.). (1990). *Geschichte der Algebra. Eine Einführung*. Mannheim: Bibliographisches Institut.

Steiner, H. G. (1964). Frege und die Grundlagen der Geometrie I. *Math.-phys. Semesterberichte*, *10*, 175-186 .

Steiner, H. G. (1965). Frege und die Grundlagen der Geometrie II. *Math.-phys. Semesterberichte*, *11*, 35-47 .

Toeplitz, O. (1927). Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, *36*, 88-100 .

Unguru, S. (1975/76). On the Need to Rewrite the History of Greek Mathematics. *Archive for History of Exact Sciences*, *15*, 67-114 .

Waerden, B. L. v. d. (1975/76). Defence of a "Shocking" Point of View. *Archive for History of Exact Sciences*, *15*, 199-210.

Waerden, B. L. v. d. (1985). *A History of Algebra: From al-Khwarizmi to Emmy Noether*. Berlin: Springer.

A Functional Approach To The Introduction Of Algebra — Some Pros And Cons

Carolyn Kieran
Département de Mathématiques, Université du Québec à Montréal, Canada

Why do we need to consider new approaches to the teaching of school algebra? In reporting the seventh- and eleventh-grade results of the fourth mathematics assessment of the National Assessment of Educational Progress (NAEP), Brown et al. (1988) concluded that: "Secondary school students generally seem to have some knowledge of basic algebraic and geometric concepts and skills. However, the results of this assessment indicate, as the results of past assessments have, that students often are not able to apply this knowledge in problem-solving situations, nor do they appear to understand many of the structures underlying these mathematical concepts and skills" (pp. 346-347). However, to cover their lack of understanding, it appears that students resort to memorizing rules and procedures that they eventually come to believe represents the essence of algebra. Brown et al. reported that a large majority of the students of the NAEP study felt that mathematics is rule based and about half of the students considered that learning mathematics is mostly memorizing.

Such findings are not limited to the United States. In the algebra research reports presented here at PME and at other international gatherings, time and again we have heard that students can learn to carry out the expression-simplification and equation-solving tasks we give them, but are unable to justify the operations they carry out. What is it in current algebra instruction that leads students to this impoverished state of cognition?

In this paper I will attempt to begin a discussion on an alternate approach to the teaching of algebra. The first part is a theoretical analysis of the development of algebra and of students' cognitive difficulties. The second part addresses questions related to the adoption of a functional approach to algebra from the very beginning. And the third part of the paper describes one researched attempt at organizing the teaching of algebra around the notion of function.

Theoretical Background

A. Process-Object Model

Anna Sfard, my co-researcher in the project to be described later in this paper, has analyzed the development of various mathematical concepts, definitions, and representations, from a combined historical and psychological perspective. This analysis, supported by research findings in the learning of school mathematics, has shown that abstract notions such as number and function can be conceived in two fundamentally different ways: structurally (as objects) or operationally (as processes). Sfard (1991) has presented the distinctions between the two approaches in the following way:

> There is a deep ontological gap between the structural and the operational approach. Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing--a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details. In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence only in a sequence of actions. Thus, whereas the structural conception is static and integrative, the operational is dynamic and detailed. (pp. 4-5)

Sfard has claimed that the operational conception is, for most people, the first step in the acquisition of new mathematical notions. The transition from a "process" conception to an "object" conception is accomplished neither quickly nor without great difficulty. After they are fully developed, both conceptions are said to play important roles in mathematical activity. The process-object duality of mathematical concepts, which is a pivotal part of the discussions in this paper, has also been elaborated in somewhat related ways by other researchers in the mathematics education research community, including, Douady (1985), Dubinsky (1991), Harel and Kaput (1991), and Gray and Tall (1991).

At the heart of these models is the notion that mathematical conceptions lie in the eyes of the beholder. To illustrate, I quote from a recent paper by Sfard and Linchevski (1994):

> When you look at an algebraic expression such as, say, 3(x + 5) + 1, what do you see? It depends. In certain situations you will probably say that this is a concise description of a computational process. 3(x + 5) + 1 will be seen as a sequence of instructions: Add 5 to the number at hand, multiply the result by 3 and add 1. In another setting you may feel differently: 3(x + 5) + 1 represents a certain number. It is the product of a computation rather than the computation itself. Even if this product cannot be specified at the moment due to the fact that the component number x is unknown, it is still a number and the whole expression should be expected to behave like one. If the context changes, 3(x + 5) + 1 may become yet another thing: a function -- a mapping which translates every number x into another. This time, the formula does not represent any fixed (even if unknown) value. Rather, it reflects a change. The things look still more complicated when a letter appears instead of one of the numerical coefficients, like in a(x + 5) + 1. The resulting expression may now be treated as an entire family of functions from R to R. Alternatively, one may claim that what hides behind the symbols is a function of two variables, from R2 to R. There is, of course, a much simpler way of looking at 3(x + 5) + 1: it may be taken at its face value, as a mere string of symbols which represents nothing. It is an algebraic object in itself. Although semantically empty, the expression may still be manipulated and combined with other expressions of

the same type, according to certain well-defined rules. (emphasis in the original version, p. 191)

B. The Development of Algebra

The above example is particularly appropriate because it sets the scene for the continuation of our discussion. In that example, we saw how an algebraic expression can be interpreted as a computational process, a fixed but unknown number, a function, a family of functions, and a string of symbols. The order of these interpretations is significant, because it reflects the historical development of algebra. As we next look briefly at this evolution, it is to be kept in mind that what was conceived in terms of process at one level became reconceptualized as an object in the next phase of development. (See Sfard (1991) and Sfard and Linchevski (1994) for more details.)

1. Operational algebra. During this period, which encompasses all of what we refer to as rhetorical algebra, algebra was characterized by the use of ordinary language descriptions for solving particular types of problems and, with a few exceptions, lacked the use of symbols or special signs to represent unknowns. Even the approaches of the Renaissance mathematicians of the 16th century were purely verbal and operational. Generalized computational processes on number continued to be the focus until Vieta's work of the late 1500s at which time algebraic objects were conceptualized. This is not meant to suggest that the presence of symbols can be equated with the development of a structural view. But it is extremely difficult to imagine conceptualizing a process as an object when one has only words to work with and does not have some symbol system.

2. Algebra of a fixed value (of an unknown). Algebra could enter a structural phase once a means was developed to present a computational process in a very concise and highly manipulable way. Symbolic algebraic expressions with an unknown represented by a letter were such a means. It was Vieta who, by the end of the sixteenth century, definitely set a basis for symbolic algebra. From now on, an expression such as $3(x + 5) + 1$ (using the symbolism of today) could represent not only a set of operations to be carried out on some unknown number, it could be viewed also as a number itself, albeit an unspecified one. The process-product distinction with respect to algebraic expressions, first mentioned by Davis (1975), captures the essence of the point to be made here.

Historically, this phase had to be attained before any future progress could be made. The process had to itself become an object; in fact, the algebraic expression came to be interpreted as having a double sense. Similar developments occurred earlier with respect to the operations of division and subtraction whereby objects such as 1/3 and -4 were created.

There is some evidence that algebra may have entered this structural phase with the work of Diophantus (3rd century A.D.). Diophantus, who used a combination of symbols and words and thus created so-called "syncopated" algebra, solved many different types of equations. Sfard and Linchevski (1994) have suggested that "while solving word problems, he constructed such expressions as 10 - x and 10 + x (in fact, he wrote equivalent strings of Greek letters) and manipulated them as if they were genuine numbers (e.g., he multiplied them obtaining 100 - x2); the fact that thirteen centuries after Diophantus mathematicians still preferred the awkward verbosity of rhetoric algebra bespeaks the inherent difficulty of his way of thinking" (p. 199). But this period can only be considered a first step in the development of structural algebra, for Diophantus did not develop any general methods; each of the 189 problems of his Arithmetica was solved in a different way. Kline (1972) has remarked that the work of Diophantus "reads like the procedural texts of the Egyptians and Babylonians, which tell us how to do things" (p. 144).

3. Functional algebra. This next phase of structural algebra was made possible by one ingenious contribution made by Vieta: his introduction of letters in two different roles--as unknowns and as givens. His use of letters in this double role was crucial to the development of functional algebra: it now became possible to express general, non-numerical solutions to problems, something that Diophantus had not imagined. In other words, the letter could now be viewed as a variable.

The use of symbolic expressions to "reify" computational processes brought a real breakthrough in algebra, and in fact in the whole of mathematics. Concise procedures were then developed for the solving of equations by mere manipulation of the algebraic objects. These processes were in stark contrast to the reversing computational processes applied earlier when algebraists were seeking to find the value of the unknown quantity.

Over the course of time, these parametrized expressions which were the objects of algebra came to be viewed in a functional way. Sfard and Linchevski (1994) have pointed out that "after the new invention [introduction of the symbolism] was transferred (mainly by Descartes and Fermat) to geometry to serve as an alternative to the standard graphic representations, and then applied in science (by Galileo, Newton, and Leibniz, among others) to represent natural phenomena, algebra was ultimately transformed from a science of constant quantities into a science of changing magnitudes" (p. 200).

Despite these advances, the new mathematical object of function still had a long way to go in its evolutionary path before it assumed the identity it has today. However, I will not dwell any longer on these historical unfoldings since they are of less relevance for my story which focuses on the introduction of high school algebra.

C. Reflections of History in Students' Learning of Algebra

The previous account of the historical development of algebra provides a perspective for interpreting research findings on student learning of algebra. Only a few samples of past research are presented here, in order to provide a flavor of how students' algebra learning difficulties can be interpreted as a series of cognitive adjustments to be made in moving from process to object conceptualizations. Bear in mind that, quite often, instruction has tended to follow the historical order. Thus, it is essential that we be cognizant of student difficulties in order to begin to evaluate the implications of altering our approach to teaching algebra.

1. Operational algebra. This perspective is shown in students' early work with algebra problem solving. Simple problems such as "Amy has 5 more marbles than Bill, and Bill has twice as many marbles as Ken; if Amy has 49 marbles, how many does Ken have?" are usually solved verbally and by reversing the operations specified in the story (49 - 5 and then 44 / 2). This operational approach, which reflects students' earlier work in arithmetic, is easily extended to cover the solving of simple equations such as $3x + 5 = 23$. With problems involving the algebra of a fixed value, however, this undoing approach fails.

2. Algebra of a fixed value (of an unknown). Much of the algebra research literature provides evidence of students' difficulties in handling tasks requiring a fixed-value-algebra interpretation. Collis (1974) has described one symptom of this difficulty as students' inability to accept Lack of Closure. Filloy and Rojano (1984) have characterized the phenomenon of being able to solve equations such as the one above, but not examples with an unknown on both sides, such as $3x + 5 = 2x + 12$; they have called this phenomenon the Didactic Cut. Bell, Malone, and Taylor (1987), in their report of the development of 14-year-old students' abilities at setting up equations to solve word problems, noted that the initial conceptual obstacle of how to express certain word-problem statements (e.g., "15 more than x") were overcome; however, the difficulties associated with treating an algebraic expression as an object (e.g., coping with "15 more than [x - 30]"), were less fully resolved. Even when students do appear to have acquired the beginnings of a fixed number interpretation (as evidenced by their ability to solve equations by performing the same operation on both sides), this knowledge is often found to be quite fragile when they are faced with explaining the underpinnings of their solving approaches for systems of equations (Sfard & Linchevski, 1994).

3. Functional algebra. Küchemann's (1981) report of the algebra segment of the CSMS study contained a few examples of tasks requiring a variable interpretation of the letter and of students' attempts at such tasks. For example, the question, "Which is larger, 2n or n + 2?", was successfully answered by only 10% of the 15-year-olds of the study. Several researchers (e.g., Clement, 1982; Mevarech & Yitschak, 1983) have also found that expressing functional relations by means of an equation poses difficulties to students. However, it is not only the function concept which is elusive to many (e.g., Leinhardt, Zaslavsky, & Stein, 1990); functional notation poses its own special barriers (Carpenter et al., 1981). The limits of student understanding of functional algebra are sometimes revealed with tasks involving parameters (e.g., Assessment of Performance Unit, 1980). In this regard, the potential of innovative computer programs, in conjunction with the necessary instructional support of a capable teacher, has been suggested by a few recent studies (e.g., Heid, to appear; Moschkovich, 1992; Schwarz, 1989).

In algebra curricula that have followed the sequence: operational algebra --> fixed-value algebra --> functional algebra, there has never been, to my knowledge, any serious effort to test whether the student difficulties that have been documented with respect to moving from an unknown-to-a-variable interpretation of the letter (Harper, 1987; Küchemann, 1981) are dependent on the nature of this instruction. In any case, even with those curricula that do not follow this order, students have been found to experience serious problems with functional algebra. Thus, the obstacles that the field of mathematics has experienced over the centuries in its lengthy evolution from computational to functional algebra are seen to be reflected daily in the microcosm of student learning.

Why a Functional Approach to Algebra From the Very Beginning?

The previous section devoted to the Theoretical Background was intended to make apparent that a functional approach to algebra is one in which the letter is viewed as a variable. As well, with representations that do not involve symbols, such as story contexts, the term functional approach is extended to include the expression of variable notions in a more rhetorical formulation.

1. Some Arguments Against

From the preceding historico-psychological analyses, reinforced by the available research literature, we see not only that functional algebra was late in developing historically, but also that students appear to have a great deal of difficulty with this advanced kind of algebra. It is widely accepted that the difference between thinking in terms of known and unknown quantities versus thinking in terms of variables and constants marked a demarcation line between two distinct kinds of algebra during the past centuries. Many researchers have emphasized how much of a challenge it is for students to switch from the former mode of thought to the latter. But what if students did not learn operational and fixed-value algebra before going on to functional algebra? What if they were to begin with functional algebra? The evidence from individual learning might then not correspond with historical development. Sfard and Linchevski (1994) have warned that "a direct jump, say, over the wide gap separating functional algebra from operationally interpreted algebra may end in broken bones" (p. 205); nevertheless, they have added that, due to computers, there is new hope for such a jump.

Functional approaches to algebra have been attempted in the past. During the New Math movement, function was presented as the thread tying together all of high school algebra. But that entire movement failed for reasons that have not been fully analyzed. Since the concept of function is extremely difficult for students, the odds are that the way it was then introduced may have had much to do with the failure.

In certain countries, Israel for example, the introduction to algebra starts right away with the use of letters as variables and not only as unknowns. Equation solutions, even the solving procedures, are described in set-theoretic terms. The official introduction of the concept of function comes later. At that point, solutions of equations and inequalities are supported with graphical representations. According to Sfard and Linchevski (1994), "this uncompromisingly structural way of dealing with the subject is certainly very attractive due to its mathematical elegance, consistency, and universality" (p. 213). However, the resulting student conceptualizations have been found to be considerably less structural than the instruction would suggest (Sfard, 1987, 1989) and, in fact, fraught with gaps and misconceptions (e.g., Dreyfus & Eisenberg, 1982; Markovits, Eylon, & Bruckheimer, 1986; Vinner & Dreyfus, 1989). This leads us to ask how one might conceive a structural approach to algebra involving functions that avoids some of the pitfalls of the past.

2. Some Arguments For

According to Schwartz and Yerushalmy (1992), "function" is the primitive algebraic object:

> We believe it is important for both students and teachers to have a reasonably simple coherent top-level view of the subject that they are learning and teaching. This belief, in turn, leads us to try and formulate the subject of algebra parsimoniously, building everything on the concept of function. There are several reasons for this choice. First, we believe that it is pedagogically workable. ... Second, the function is a mathematically powerful and pervasive idea. Third, the capacity of suitably crafted microcomputer software to present several different representations of functions simultaneously seems to us to be a pedagogic opportunity worth seizing. (p. 262)

The top-level view of algebra advocated by Schwartz and Yerushalmy is one for which it is possible to describe certain obvious cognitive advantages, such as possession of a general and concise conceptualization rather than one that is detailed and diffuse, versatility and adaptability of perspective that greatly increases the student's ability to cope with the task at hand, and so on.

The potential of beginning with a more powerful concept in the introduction to algebra has been researched by Davydov (1962) in the (former) Soviet Union with elementary school children. (See also Freudenthal's (1974) discussion of this work.) Even though Davydov's study was not oriented explicitly around functional algebra, but rather the teaching of part-whole relations as a means of introducing algebra, the positive findings of the study clearly indicate the feasibility of beginning with top-level views. The nature of Davydov's approach is summarized as follows:

> In the first phase of the study involving classes of 8-year-olds, strips of paper cut into parts, volumes of water, weights of grits, and so on were used to show children the meaning of whole and parts. From the beginning, the whole and parts were indicated by letters on drawings--numbers were never used. Later on, the wholes and parts were related in equations involving plus, minus, and equal signs. Children learned to make drawings corresponding to formulas such as $k = a - c - b - f$. After about three dozen lessons on the relation of whole and parts, the second phase on problem solving began. Texts such as, "There were a red and b blue pencils in a box, and together there were c

> pencils," were to be translated into a drawing, a scheme, and three formulas. Then the children had to invent texts to correspond to given drawings and, later on, to given formulas. Eventually numerical values were introduced. (Kieran, 1992, pp. 403-404)

According to Davydov, the use of literal data in problem situations compelled the children to fix their attention on the relations between magnitudes and on their variability. The report of this study does not state whether the children were, in fact, viewing these relations in terms of fixed-value algebra or of functional algebra; however, the point here is that they had not first gone through the phase of operational algebra.

Schwartz and Yerushalmy mention above the potential of the computer to work hand in hand with and, in fact, enhance a function-based approach to the teaching of algebra. Some examples of their approach are provided in Chazan, Yerushalmy, and Schwartz (1993). Even though they "expect that it will take the better part of the next decade to work out the full details of a curriculum in mathematical analysis for 12- to 18-year-olds" (Schwartz & Yerushalmy, 1992, p. 289), they do claim to have grounds for optimism: "In large measure, this optimism stems from the deep conviction that mathematics is at least as much a visual undertaking as it is a symbolic one; now, for the first time we have available a set of instruments that allow us to manipulate our mathematical constructs graphically" (p. 289).

Other technologically-supported projects that have recently been built around the concept of function, many of them via mathematical modeling, include the Computer-Intensive Algebra curriculum project (Fey & Heid, 1991), the Pittsburgh Urban Mathematics Project (Hadley, 1993), and the Empowering Teachers-Mathematical Inquiry Through Technology project (Carter, 1993). It should perhaps be mentioned that the way in which the computer is used in these projects is not of the early programmed instruction type; rather students use computers as tools to explore mathematical ideas and to solve mathematical problems.

The problem with several of these recent attempts at building algebra around the notion of function is the lack of a theoretical framework (except for those researchers who have adopted a process-object perspective) for both designing the functional approach and for testing it. Thus, we still know very little about whether such approaches are indeed a route to follow in future algebra teaching or not. I would like now to share with you one theory-based attempt at developing an approach to functional algebra and for researching its potential.

Our Project[1]

1. Overview

Our intention in this project was to develop an approach to teaching algebra that did not follow the historical sequence of operational to fixed value (of an unknown) to function. We aimed to introduce algebra from the start with a functional notion of variable. But as we have seen, on the face of it, there is really nothing so new or controversial about emphasizing the importance of variables and functions--even for beginning algebra students. The novelty of our project resides, we believe, in its theoretical dimensions; also, the actual route we take is, in some respects, quite original.

Looking at the matrix (Figure 1), we can follow the trail of history, as shown by the solid arrows. It begins in the top left cell with operational algebra, followed by the first phase of structural algebra--the algebra of a fixed value--which led in turn to the second phase of structural algebra--functional algebra. The lower left cell of the matrix does not contain part of the solid arrow because functional algebra does not seem to have developed directly from computational algebra.

Since the theoretical model underlying our work emphasizes the process-object duality of mathematical conceptions, we decided to design two alternative versions of a functional algebra approach (shown by the two dotted arrows in the matrix of Figure 1): (a) a process-oriented version which begins with an operational interpretation of functional algebra and heads toward a structural interpretation, and (b) an object-oriented version which does the opposite, that is, it begins with a structural interpretation of functional algebra and gradually adds an operational interpretation.

By separating our functional approach into these two theory-based parallel versions, we hoped to be able to observe critical features of student learning of functional algebra that can be identified with either process or object emphases, and to isolate cognitive difficulties associated with one version or the other. (The implicit assumption with respect to the theory is that the process-object model is applicable to the learning of functional algebra which according to the historical development of algebra had a strong structural component from the very beginning.) Our ultimate goal is to design an approach to functional algebra that integrates the essential aspects of both versions and that might lead to a refinement of the model with respect to its application to the learning of structural mathematics.

Before continuing, it is to be noticed that both versions explicitly bypass the algebra of a fixed value in which the letter is treated as an unknown. Our intention in this regard was to include in both versions examples involving an unknown by spreading them over the teaching units. Such examples of fixed-value algebra would be derived from the functional algebra (by providing some numerical value to the function). It is of significant research interest to examine the impact of such a reordering of traditional teaching approaches on student algebra learning.

In order to give a sense of the spirit of these two versions, I will illustrate each by showing some of the ways by which we intended to provide for the construction of meaning for (a) algebraic expressions and (b) the operations underlying symbolic manipulations. But, first, a few words describing methodological aspects of the project are in order.

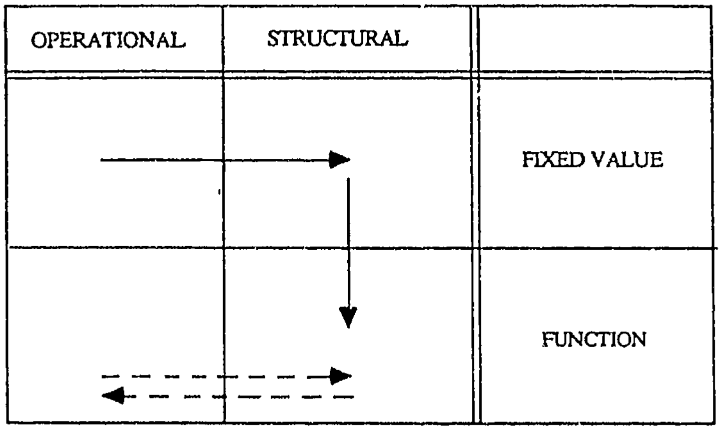


Figure 1: Theoretical matrix in which the path of the solid arrows represents the historical development of algebra, and the two dotted arrows indicate the process-to-object and object-to-process dimensions of our functional algebra.

2. General Description of the Experiment

Two teaching sequences with accompanying materials (teacher materials and student activity sheets) were developed--each one designed for thirty 45-minute lessons. The sequences were aimed at seventh or eighth grade students (12-14 years) who had not yet had any algebra before, but who had already learned to operate with integers. The object-oriented version began with introducing the students to Cartesian graphs in the context of story-based functional situations and went as far as the solving of equations and inequalities by graphing methods. The process-oriented version began with an emphasis on the computational processes involved in describing functional situations and ended with the solving of equations and inequalities, but by tabular rather than graphical representations. Both versions included a great deal of work on equivalent expressions. We had intended to cover algebraic methods of equation solving, as well as the use of graphical representations in the process-oriented class, but did not have enough time for these.

After pilot testing our materials with two groups of four students at another school, we embarked on the main part of the study in a relatively small, private, Montreal secondary school--in two of the Grade 7 classes. All students were pretested and many of them were also interviewed prior to the start of the classroom teaching sessions. It was planned that the students should work in pairs on all of the activities. In general, they chose their own partners, which led to various combinations of abilities being paired together. We found that these partnerships worked better than those few combinations that we ourselves engineered. Another pedagogical aspect of the study was an attempt to keep to a minimum the amount of time spent by the teacher doing frontal teaching.

After a brief introduction to new material, students went on to working at their activity sheets. It was only at the end of each unit (each unit being four to six days in length) that the students spent a rather larger portion of their time in whole-class discussions orchestrated by the teacher. In these discussions, there was an attempt to bring out and draw together the notions that had been tackled by the student pairs during the previous few days.

The teaching, which began at the end of January this year, was done by the two research assistants of the project and by a math teacher from the school; both principal researchers also circulated in the class to help out while the students were working on their activity sheets. All class and computer laboratory sessions were video-taped. While the students were working in pairs, the two cameras were focused on the same two pairs of students each day. After each week's lessons, the two pairs in each class were interviewed (two at a time) by the principal researchers. The aim of these interviews was to follow the students' conceptual development. We also attempted to identify those areas that they were finding either difficult or rather easy, as well as to probe into the kinds of meanings they were forging for the new mathematical symbols and transformations. Further interviews were conducted with these students on an individual basis immediately after the study ended, as well as three weeks later. Written tests were administered to the two classes of students midway through the study and also at the end. The same test that was written by the two seventh grade classes at the end of the study was given to all of the other students of the school from Grades 8 to 11.

3. Creating Meaning for Algebraic Expressions

Process-oriented version. In this version, the emphasis from the outset was on computational processes involving functional interpretations of the variable. One of the best ways in our opinion to encourage an operational perspective in learners is by means of their generating computer algorithms. The computer software that we used was CARAPACE (Boileau & Garançon, 1987). This software requires the entry of an algorithm before one can gain access to the associated table of values. The central idea of this approach was that the students develop algebraic symbols as means for programming the computer--as a way of encoding computational procedures.

An example illustrating the early emphasis on computational processes, which included activities with tables and computer algorithms, as well as a focus on verbal descriptions of a story situation (though the latter are not depicted here), is provided in Figure 2. This very explicit use of variables in an equally explicit computational setting within a computer environment is a rather new approach to the introduction of algebra. Other researchers have attempted to simulate this with paper and pencil, but student difficulties in carrying out their own computations each time they replace the input variable with another numerical value have tended to remove their attention from the critical aspect of functional change which is taking place.

The vocabulary, input and output variables, and computational procedures, was an integral part of this process-oriented teaching sequence. Other early activities included guessing rules from tables and translating verbal descriptions of computational recipes into expressions. In the CARAPACE environment, the transition from significant variable names to single letters was made without much fuss. Linking the input and output variables of computer programs with algebraic expressions was facilitated by the writing of one-line programs and the use of computer-generated tables of values that displayed the algebraic expression as the header of the output-variable column (see Figure 3). It is to be noted that, initially, students wrote only one operation per line in their CARAPACE programs, and that the one-line program illustrated in Figure 3 is the result of a few sessions devoted to the replacement of several one-operation lines by a single line containing the final "story-telling output variable."

You have been asked to take charge of preparing and painting a square-shaped notice board to advertise the school play. It has not yet been decided how long the side of this square notice board should be. The cost of painting the front of this notice board is $0.01 for each square centimeter.

a) Fill in the table below.

length of notice board (cm)	paint cost ($)
50	
100	
150	
200	

b) Write a CARAPACE program that would produce a table like yours.

(For the benefit of the reader, a sample of a student's program is provided.)

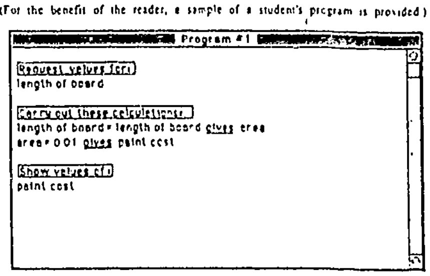
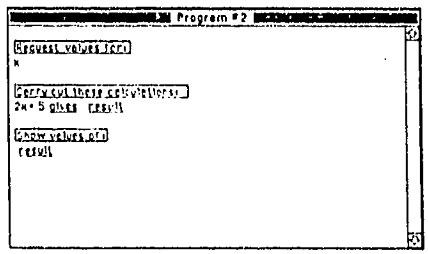


Figure 2



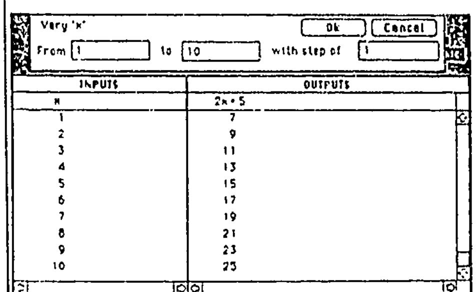


Figure 3

Object-oriented version. In this structural approach to functional algebra, the graphical representation was emphasized. Sfard (1991) has pointed out that "geometric ideas for which the unifying static graphical representations appear to be more natural than any other, can probably be conceived structurally [as objects] even before full awareness of the alternative [process] descriptions has been achieved" (p. 10). It should perhaps be mentioned that graphical representations can also be interpreted operationally in terms of computational processes, but the particular slant we used did not stress such aspects.

Because of the potential of computer technology, which not only makes available the intensive use of Cartesian graphs but also provides for novel and powerful explorations of the properties of functions, we were able to exploit the structural features of functional graphs in ways that would otherwise not have been feasible. The computer software used was Math Connections: Algebra II (Rosenberg, 1992), a learning environment that permitted the students to work with up to three graphs at a time--graphs that were color-coordinated with accompanying (if desired) tabular and algebraic representations. The central idea of this approach was that the students develop (a) algebraic symbols as means for describing the functions represented by the graphs and (b) algebraic manipulations as means for recording operations with the graphs of functions.

MAKING AND USING RULES

ACME Builders have put up a new apartment building with a new concept: every room in the building is exactly square. The rooms are built in several different sizes. ACME wants to put wall to wall carpeting in each room, as well as strips of edging along the perimeter of each room to hold the carpeting in place.

PART ONE:

(1) To put strips of edging in a square room whose side is 10 feet, what length of edging would have to be purchased? ____ feet

(2) If 66 feet of edging will fit around one of the square rooms, what is the length of one side of the room? ______ feet

(3) Fill in the following table which shows how much edging will be needed according to the length of a side of a square room

SIDE OF THE SQUARE ROOM (ft)	LENGTH OF THE EDGING (ft)
S	L
5	
6	
7	
8	
9	
10	
12 5	
15 2	
17 8	
20	

PART TWO.

(1) To carpet a square room which has sides of 9 feet, what area of carpet is needed? __________ square feet

(2) If the area of one of the square rooms is 121 square feet, what is the length of the side of the room? __________ feet

(3) Fill in the following table which shows how much carpeting will be needed according to the length of a side of a square room. You may want to use your calculator.

SIDE OF THE SQUARE ROOM (ft)	AREA OF CARPETING (sq. feet)
S	A
5	
6	
7	
8	
9	
10	
12 5	
15 2	
17 8	
20	

(4) Write the mathematical rule that you used to calculate area as a function of the length of side of the square

To calculate area: ______________________

(5) Plot the information from the table onto the same Cartesian plane as before

(6) Can the points that you plotted be joined? ______

Explain:

USING THE COMPUTER TO CHECK YOUR GRAPH

(1) Enter your two tables from the square room problem.

(2) Have the computer plot the two graphs by clicking on the f(x) button for each table. Notice that each graph will be the same color as its table.

(3) Compare what the computer drew with the graph that you made by hand.

(4) What length of side will make a square which has exactly the same area as its perimeter? __________

(5) When the side of the square is smaller than this, what can you say about the relationship between the area and perimeter?

EDGING OF SQUARE ROOM

C f(x) (x,y)

x	f(x)
6	24
7	28
8	32
9	36
10	40

AREA OF SQUARE ROOM

C f(x) (x,y)

x	f(x)
6	36
7	49
8	64
9	81
10	100

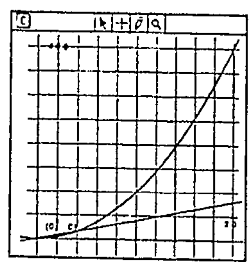


Figure 4

Global overviews of functional graphs were encouraged right from the beginning with story-based discrete and continuous graphs that did not have any underlying rule. For example, an age-height function provided the context for discussing how height varies with age, when height increases, when it increases more or less quickly, when it levels off, and so on. Then students were introduced to story-based graphs involving a rule, with activities like the one illustrated in Figure 4. Verbally-stated rules were translated into algebraic expressions which were subsequently graphed; the analysis of the graph was always a major focus. Soon students learned to associate the slope and y-intercept of linear graphs with the corresponding features of an algebraic expression and a suitable story context. Explorations in the computer graphing environment also permitted them to categorize expressions of different degrees into linear, parabolic, and others; and to begin an analysis of the parameters of quadratic functions vis-à-vis their graphical representations.

4. Creating Meaning for Algebraic Manipulations

Some of what we consider to be the most interesting and exciting aspects of the project occurred around the creation of an underlying support for algebraic transformations. Our aim for this part of the teaching sequence was the meaningful exploration of equivalence of algebraic expressions via computational and graphical work with the properties of associativity, distributivity, and commutativity, leading to general formulations of these properties.

For example, in the process-oriented version, we wanted to use computer algorithms and the underlying numerical relationships as a base for generalizing the properties, the latter being instantiated with, for example, partially-covered dot arrays. Many of the follow-up activities involved the generation of equivalent expressions and their testing by means of computer-produced tables of values (see Figure 5).

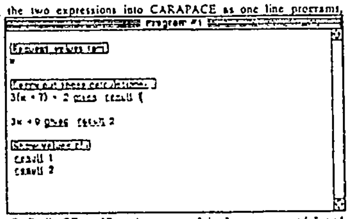
In the object oriented version, we decided to introduce algebraic manipulations as an equivalent of operations on functions, such as adding, multiplying by a number, and so on; these operations were first to be demonstrated with graphs. We began with adding the graphs of two linear functions. The expressions for these graphs would not be provided in advance. Thus, students would add vertical segments of the two linear graphs (for a given x-value they would add the two y-values), marking the sums by points, and join the points. They would then discover that the resulting graph is linear as well, and would then generate for it an expression for the sum. Let's say that it turned out to be $50 + 15x$. The students would then be asked to look again at the two original graphs and to generate their expressions ($15 + 10x$ and $35 + 5x$). They would thus realize that the expressions $(15 + 10x) + (35 + 5x)$ and $50 + 15x$ are equivalent. Equivalent expressions were defined earlier as "leading to the same graphs, in any domain," thus as "representing the same function." After working with several graph

In Mr. Swanson's seventh grade the students were given a list of expressions and were asked to find for each of them a short and simple equivalent expression. Below are some of the answers given by the students in his class. In each case you will be asked to check on the computer whether the students are correct.

EXAMPLE:

a) Mr. Swanson gave the expression $3(x + 7) + 2$. Mark thinks that $3x + 9$ is an equivalent expression. Is he right?

Typing the two expressions into CARAPACE as one line programs,



we ask CARAPACE to "Run the program" in "automatic mode" and if we ask the dialogue box to choose values from 1 to 10 with steps of 1 we get the table of values below

INPUTS	OUTPUTS	
x	$3(x + 7) + 2$	$3x + 9$
1	26	12
2	29	15
3	32	18
4	35	21
5	38	24
6	41	27
7	44	30
8	47	33
9	50	36
10	53	39

Since the two expressions do not have the same table of values they cannot be equivalent. So Mark's expression $3x + 9$ is not correct.

Someone else suggests that Mark may have forgotten to multiply the 7 in the brackets by 3 and thinks the correct expression would be $3x + 23$. The 9 in the above program is replaced by 23 and the tables are again produced using, for a change, values for x between 12 and 20. The table below appears

INPUTS	OUTPUTS	
x	$3(x + 7) + 2$	$3x + 23$
12	59	59
13	62	62
14	65	65
15	68	68
16	71	71
17	74	74
18	77	77
19	80	80
20	83	83

Mr. Swanson's class is convinced $3x + 23$ is the equivalent expression they were looking for. How do you think the student got this expression?

b) Next Mr. Swanson gave the expression $3x + 7 + 5(x - 2)$. Joel suggested that $8x - 3$ was a simpler equivalent expression.

Using the computer as in the example above, test to see if Joel's expression is equivalent to Mr. Swanson's. Is Joel's correct? _____
If it is not, find an expression that is.

Figure 5

additions of this type, the ground would have been prepared for discussions of the property of distributivity, for example, $3x + 4x = (3 + 4)x = 7x$.

The first session devoted to operations on functions gave an unexpected direction to the subsequent teaching, which was not quite what we had initially planned. In order to motivate the whole business of adding graphs, we placed the two starting graphs into a story context: "Two brothers have savings accounts in the bank. When they opened their accounts, they each had a lump sum to deposit; this amount was to be increased by regular deposits of their allowances. The two given graphs represent how their accounts have grown each month. If they had opened only a joint account from the beginning, what would the graph look like?" The students were given some help in figuring out how to go about adding together two graphs--by adding vertical segments from the horizontal axis to the corresponding point on each graph. They drew their sum-graph (by joining the resulting points) on the activity sheet containing the two given graphs (see Figure 6).

When the sum-graph was finished, they were to try to generate the expression for that function. As we were circulating around the class, observing how they were going about this task, we happened to hear one pair of students remark that they had figured out a quick way to add two functions: "Just add the y-intercepts and then add the slopes." When the research team got together at the end of the day to discuss all that had transpired during that class session, we decided to take advantage of this unexpected

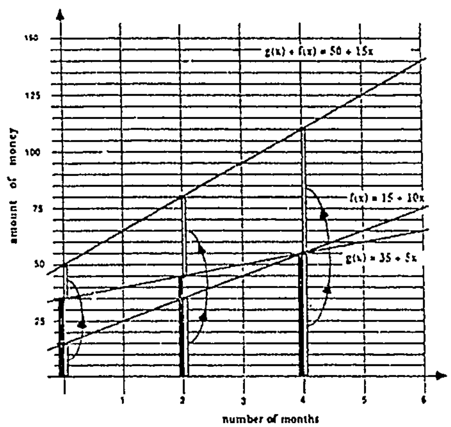


Figure 6

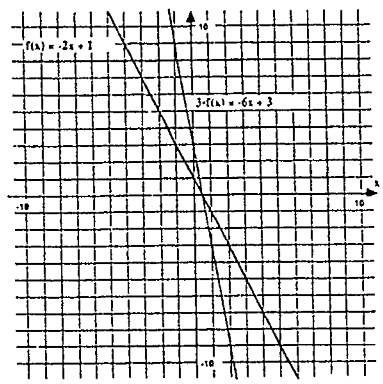


Figure 7

approach that had surfaced. We revised some of the subsequent activities in order to build on this formulation. This student-generated interpretation of the simplification of linear algebraic expressions was then tested by the class during the following days on all the major linear functions they were dealing with: those with graphs passing through the origin, those not passing through the origin, and constant functions. They also tried it out with more complicated sum combinations, for example, $(8x - 1) + (-2x + 5) + (3x - 2)$. Obviously, it was not immediately generalizable to full general versions of distributivity, associativity, and commutativity (all of which were involved in adding linear expressions), but this would not be a problem in the current study, which because of time constraints would not go any farther than the solving of linear equations and inequalities by means of graphical and tabular representations. What still remained to be done with this group before going on to the comparing of functions (equations and inequalities) was the multiplying of a function by a constant (see Figure 7). This too was approached by graphical operations, which, just as with the previous experience, provided the underlying support for further algebraic manipulations. They now had a coherent whole for operating on equations; they could meaningfully simplify any combination of linear functions into canonical form. Thus, equations such as $3x + 4 - 2 - 7x + 6(x + 5) = 2(3 - 4x) - 14$ could easily be solved graphically by first "adding both the slope and intercept terms on each side" to obtain the canonical form of each expression, and then graphing the two functions being compared by means of the slope and y-intercept terms of the two expressions.

Closing Remarks

There are many more stories that could be told at this point in our research, but most of them would be incomplete because we have not yet finished analyzing our data. You are probably curious to know which version seemed better. But there is no point in asking such a question because it was really not a competition. Until we have done a complete analysis of the data, it would be premature to say anything specific about the merits and the drawbacks of each of the two approaches. Thus, you must be kept in suspense for some time. All that might be mentioned at this point is that both versions showed certain strengths and weaknesses. Each one of them provided us with ideas on good opportunities for learning algebra in a meaningful way, and each brought its own caveats as to what shouldn't be done.

But this has not been a presentation about our research project. It has been an attempt to open up discussion on an alternate way of approaching the teaching of algebra to early high school students. Much of the past algebra research has tended to take the traditional curriculum as a given and to document the nature of the learning that accrues with such a curriculum; the kinds of learning obstacles associated with such instruction were noted. More recent research that has taken advantage of the potential of technology has pointed to ways in which we might rethink the teaching of algebra, but much of this work has lacked a theoretical framework by which we can not only assess what is occurring in these novel environments but also place it in a broader context.

In elaborating some of the pros and cons of introducing algebra right from the start with a functional interpretation of variable, I have attempted to situate such an approach within both a historical and a psychological perspective. The process-object model afforded the foundation for a theoretical matrix for generating two different versions of a functional approach to algebra. It is hoped that the few examples that were supplied and the description of the underlying theoretical framework were adequate to provide a basis for further dialogue and development in this area of mathematics learning.

Acknowledgments

[1]Part of this paper is based on a study, the first phase of which was recently carried out with the collaboration of the project's co-investigator, Anna Sfard. I want to thank Anna for her many contributions, especially those throughout the past year while she was here in Montreal on sabbatical. I also express my appreciation to our research assistants, Lesley Lee and Pat Lytle, and the audio-visual assistant, Michel Lalonde, for their dedicated work. We are grateful for the financial support of the Social Sciences and Humanities Research Council of Canada (grant # 410-93-0605), without whose help this study would not have been possible.

References

Assessment of Performance Unit. (1980). *Mathematical development, secondary survey, report no. 1*. London: HMSO.

Bell, A., Malone, J. A., & Taylor, P. C. (1987). *Algebra—an exploratory teaching experiment.* Nottingham, UK: Shell Centre for Mathematical Education.

Boileau, A., & Garançon, M. (1987). *CARAPACE* [Computer program]. Montréal, Canada: Université du Québec à Montréal, Département de Mathématiques.

Brown, C. A., Carpenter, T. P., Kouba, V. L., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Secondary school results for the fourth NAEP mathematics assessment: Algebra, geometry, mathematical methods, and attitudes. *Mathematics Teacher, 81*, 337-347.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Jr., Lindquist, M. M., & Reys, R. E. (1981). *Results from the second mathematics assessment of the National Assessment of Educational Progress.* Reston, VA: National Council of Teachers of Mathematics.

Carter, R. (1993, June). *Introducing algebra and the ET-MITT project.* Paper presented at the Working Conference on the Redesign of Introductory Algebra, Michigan State University, East Lansing, MI.

Chazan, D., Yerushalmy, M., & Schwartz, J. (1993, June). *Teaching algebra for understanding: Providing meaning for manipulations.* Paper presented at the Working Conference on the Redesign of Introductory Algebra, Michigan State University, East Lansing, MI.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*, 16-30.

Collis, K. F. (1974, June). *Cognitive development and mathematics learning.* Paper presented at the Psychology of Mathematics Workshop, Centre for Science Education, Chesea College, London.

Davis, R. B. (1975). Cognitive processes involved in solving simple arithmetic equations. *Journal of Children's Mathematical Behavior, 1* (3), 7-35.

Davydov, V. V. (1962). An experiment in introducing elements of algebra in elementary school. *Soviet Education, V* (1), 27-37.

Douady, R. (1985). The interplay between different settings: Tool-object dialectic in the extension of mathematical ability--examples from elementary school teaching. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 33-52). Utrecht, The Netherlands: State University of Utrecht, OW&OC.

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education, 13*, 360-380.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht, The Netherlands: Kluwer.

Fey, J., & Heid, M. K. (1991). *Computer-intensive algebra.* College Park: University of Maryland, Department of Mathematics.

Filloy, E., & Rojano, T. (1984). From an arithmetical to an algebraic thought. In J. M. Moser (Ed.), *Proceedings of the Sixth Annual Meeting of PME-NA* (pp. 51-56). Madison: University of Wisconsin.

Freudenthal, H. (1974). Soviet research on teaching algebra at the lower grades of the elementary school. *Educational Studies in Mathematics, 5*, 391-412.

Gray, E., & Tall, D. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 72-79). Genova, Italy: Dip. di matematica dell'Università di Genova.

Hadley, Wm. (1993, June). *The PUMP project.* Paper presented at the Working Conference on the Redesign of Introductory Algebra, Michigan State University, East Lansing, MI.

Harel, G., & Kaput, J. (1991). The role of conceptual entities and their symbols in building advanced mathematical concepts. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 82-94). Dordrecht, The Netherlands: Kluwer.

Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics, 18*, 75-90.

Heid, M. K. (to appear). Computer-Intensive Algebra: Reflections on a functions approach to beginning algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching*.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kline, M. (1972). *Mathematical thought from ancient to modern times*. New York: Oxford University Press.

Küchemann, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London: John Murray.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*, 1-64.

Markovits, Z., Eylon, B.-S., & Bruckheimer, M. (1986). Functions today and yesterday. For the *Learning of Mathematics, 6* (2), 18-24.

Mevarech, Z. R., & Yitschak, D. (1983). Students' misconceptions of the equivalence relationship. In R. Hershkowitz (Ed.), *Proceedings of the Seventh International Conference for the Psychology of Mathematics Education* (pp. 313-318). Rehovot, Israel: Weizmann Institute of Science.

Moschkovich, J. (1992). *Making sense of graphs and equations: An analysis of students' conceptions and language use*. Unpublished doctoral dissertation, University of California, Berkeley.

Rosenberg, J. (1992). *Math Connections: Algebra II* [Computer program]. Scotts Valley, CA: WINGS for Learning, Inc.

Schwartz, J., & Yerushalmy, M. (1992). Getting students to function in and with algebra. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemolgy and pedagogy* (MAA Notes, Vol. 25, pp. 261-289). Washington, DC: Mathematical Association of America.

Sfard, A. (1987). Two conceptions of mathematical notions: Operational and structural. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 162-169). Montréal, Canada: Université de Montréal.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 151-158). Paris: G. R. Didactique, CNRS.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification--the case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Schwarz, B. (1989). *The use of a microworld to improve the concept image of function: The Triple Representation Model curriculum*. Unpublished doctoral dissertation, Weizmann Institute of Science, Israel.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, 356-366.

Researching From the Inside in Mathematics Education — Locating an I-You Relationship[1]

John Mason
Centre for Mathematics Education, Open University, UK

INTRODUCTION

To express is to over stress

As I cast about for a suitable form for what I want to express, I find myself drawn deeper and deeper into the web of language, the philosophical disputes which have produced volumes of words over centuries. In preparation for this talk, I have been refreshing my reading of psychology, sociology, and philosophy, by skipping and wading through vast tracts of discourse. What came to mind repeatedly is the phrase 'about it and about', from FitzGerald's verse XXIX of Omar Khayyám's Rubáiyát.

> Myself when young did eagerly frequent
> Doctor and Saint, and heard great argument
> About it and about; but evermore
> Came out by the same door wherein I went.

Words generate more words in explanation, but often draw us away from the experiences from which they stem. I want to stay with my knowing. And so just as I have been seeking nuggets in the writings of, and about, others, so you will, I hope, find some nugget in what follows. Indeed, what follows is an expression, replete with concomitant over-stressings, of what I am tempted to call common epistemology, a description, elaboration, and precising of how we all come to know that, know how, and know to. But to make such claims requires me to ground my remarks in the discourse of philosophy, psychology, and sociology, as well as mathematics. And that there is certainly neither time nor space to attempt in detail.

I shall attempt to draw your attention away from cause-and-effect oriented direct approach that characterises so much of our collective research, and towards a more indirect action-based approach espoused by John Dewey (1933):

> Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time. Collateral learning ... may be and often is much more important than the lesson.

His sentiment is reflected in research such as Denvir & Brown (1986) and Bereiter (1991) in which children appeared to master components of a topic only indirectly connected with that topic, and not to learn some components which the researchers felt were core to the topic. I want instead to draw your attention inwards to the vital and essential role of the researcher's own sensitivities. I shall need to draw upon techniques for promoting effective shared meaning, what Maturana & Varela (1988) called consensual coordination (of consensual coordination of action), which produces language. I shall offer an opportunity to engage in actions of construal, spurred, as Sperber & Wilson (1986) propose, by the assumption that I believe that what I offer you is of relevance, and to engage in mutual co-determination (Varela et al. 1991). As my title suggests, one approach is through Buber's expression of the triad I-You-It but before developing that theme, a word or two about background motivation and beliefs.

---

1 An expanded version of this paper is available under the same title from CME, The Open University, Milton Keynes, MK7 6AA, U.K.

AIMS AND BELIEFS

I have long concluded that it is very hard to say anything new that has not been said more eloquently elsewhere. Plato and Greek Myths, Chinese, Indian, and Islamic philosophers, and more recently, Dewey, Piaget, Freudenthal, and Skemp, (to name but a few major theorists) have said it all beautifully. Nevertheless it is important to re-search, re-collect, re-connect, re-learn, re-integrate, and re-cast insights in the discourse of the times. I see working on education not in terms of an edifice of knowledge, adding new theorems to old, but rather as a journey of self discovery and development in which what others have learned has to b re-experienced by each traveller, re-learned re-integrated and re-expressed in each generation. Nor is effective teaching a matter of producing the correct causes which are then guaranteed to have a desired effect. Rather, each effective act of teaching is an action which involves the mutuality of three impulses (Bennett 1956, Mason, 1994). Each individual has to re-work the main themes for themselves as they become aware of their awarenesses in preparation for teaching. Researchers cannot hand results to teachers as recipes with guaranteed results.

My aim is

> to understand, to construe, ... for myself;
> to develop, socialise, and integrate my various selves;
> to attend to the process that I may help others do the same.

In short my aim is to set up actions whose working out will keep me alive to situations in which I find myself. As Gattegno (1990) put it,

> All I have to share is my awareness of my awareness

and Feyerabend (1991) offered:

> All you can do, if you really want to be truthful, is to tell a story.

So my aim is to extend my awareness, increase my sensitivities to others, and to do this by telling stories, and by evoking stories from others. My position is easily stated:

> I cannot change others, but I *can* work at changing myself.

I am not the only one to take this pragmatic view. Rice (1993) posed it as a question: "Should we, as Robinson (1989) suggests, abandon our attempts to change teachers, and concentrate on creating opportunities for them to change themselves?"

This runs counter to the systemic approach in which huge projects attempt to effect change in most or all teachers. I believe such approaches to be fundamentally flawed, precisely because they ignore the psychological process of searching, integrating, construing, and acting, which constitute the intra-human and inter-human processes of teaching and learning. I am interested in what is possible, more than in what is the norm, and in self-actualisation rather than changing others, so I align myself with Maslow (1971). I take this opportunity not to try to persuade you about something, but rather to use the energy of your attention to engage in the action of exposition, in which your presence (actual or virtual) as audience enables me to contact a world of experience in a fresh way, to talk *from* experience rather than talking *about* it, through relating directly to my topic rather than merely referring to it.

I - You - It.

Buber is well known for his contrast between I-It connection and I-You[2] relationship. The I-It connection is subject-object, owner-owned, user-used, whereas I-You is co-acknowledging, co-

---

2 Kaufmann (1980 p253) points out that Buber's original title was *Ich und Du*, and that the *Thou* appeared in the first English translation, made by a clergyman. Buber bemoans the inevitablility of transition of You into It, but Kaufmann points out that the underlying *malaise* reflects Buber's being deeply affected by separation from his

determining, co-evolving mutuality. I-It thinking characterised much of 19th and early 20th century attitudes to the environment: man controlling nature and the use of the engine as fundamental metaphor for thinking about nature and about people. Buber suggested that most human relationships are I-It in nature, rather than the humanistic I-You.

> There is no I as such but only the I of the basic word I-You and the I of the basic word I-It. (Buber 1923, p54)
>
> It borders on other Its; It is only by virtue of bordering on others. But where You is said there is no something. You has no borders. (Ibid p51)
>
> The world as experience belongs to the basic word I-It. The basic word I-You establishes a world of relationship. (Ibid p53)

Kang & Kilpatrick (1992) used Buber's characterisations to describe the 'fragility of knowledge' in which 'constant form cannot keep constant meaning' (p3) as it emerges from formless personal awareness (I-You) into form-full public articulation (I-It) in a journal, and is then further organised and edited into formulae, technique, and even mnemonic, as instruction. I-You knowledge is insightful, connected, and personally situated. I-It knowledge is behavioural action manifested in a material world. Unfortunately, instructional materials convert I-You expert awareness into I-It behavioural knowledge, which Chevellard (1985) called the *transposition didactique*.

Textbooks appear to be transmitters of knowledge, but in fact re-inforce mechanicality. The very act of simplification so that 'they will understand' is an act of reduction (Tahta 1991) from I-You awareness to I-It formality which each student has to reconstruct into I-You awareness if they are to re-locate complexity, to 'make it their own and achieve ownership' (a popular play on words in Mathematics Education in the UK in the 80's). Brousseau (1984) spoke in terms of a *contrat didactique* which has at its heart a didactic tension (Laborde, 1989):

> The more explicitly the teacher indicates the behaviour which would arise from understanding, the more likely students are to be able produce that behaviour without generating it from understanding (Mason 1989).

Of course to turn assertions like this into stimuli for informing teaching requires agreement on the meaning of terms such as *understanding*, *behaviour*, etc. which is not my focus here. Researching from the inside involves developing means to work at negotiation of meaning (giving brief-but-vivid accounts of incidents which illustrate the terms and the functioning of the tension so as to develop a shared vocabulary and the basis for sharing alternative actions) which not only assist in building an I-You relationship, but also offer teachers experience of processes which they can themselves use in their own teaching. Thus knowing-that can become knowing-to.

Constructivism draws attention to the necessity for each individual to construct an I-You relationship from I-It knowledge, through what Piaget called *assimilation* and *accommodation*, Skemp (1969) called *relational understanding*, Gattegno (1970) called *integration through subordination* (developed, for example, in Hewitt 1994), and Pirie & Kieren (1989, 199, 1994) extended into a picture of recursive unfolding and folding back. Whereas textbook authors often appear to be trying to replace or minimise the role of the teacher, thereby exhibiting distrust of teacher capabilities, all the authors quoted above suggest that the teacher is essential for exhibiting and displaying, in Vygotsky's words, 'higher-psychological functioning' (Vygotsky 1978, Wertsch 1985). Most pupils require the presence of exemplary awareness in order themselves to become aware, to be stimulated and

---

mother at the age of three, and suggests that this influenced the essentially longing quality of Buber's description of an I-You relationship.

prompted to seek an I-You relationship, to enter into and experience the complexity of knowing rather than being satisfied with simplified pre-digested I-It knowledge. This is the teacher's epistemological vigilance (Bousseau 1986), their 'sharing of awareness ' (Gattegno 1987).

Through I-You relationships, an educated person both listens and speaks, engaging in two way co-determination and co-evolution of interacting selves (Varela et al 1991). Neither acts upon or causes the other, but both develop in relationship. Buber saw human development as leading to the ability to make decisions in the moment, to responding sensitively and existentially, using principles and traditional practices as checks and reminders rather than as rules. This is non-trivial to achieve. It requires authenticity, the courage to *be* (as Tillich 1952 also stressed) rather than to *seem*. Like many educationalists before and since, Buber felt that learning had to be related to subsequent action. A similarly pragmaticist theme is present in researching from the inside, since the main test of validity of a distinction is that it informs future practice.

Peirce used the same words as Buber rather similarly some sixty years earlier:

> Though they cannot be expressed in terms of each other, yet they have a relation to each other, for THOU is an IT in which there is another I. I looks in, IT looks out, THOU looks *through*, out and in again. I outwells, IT inflows, THOU commingles. I is self-supported, IT bears on a staff, THOU leans on what it supports (Peirce 1861 reprinted in Moore 1982 p 45-46).

The developed and balanced triad I-You-It makes it possible to transcend cause-and-effect thinking by focusing on actions (Mason 1994).

RESEARCHING FROM THE INSIDE

Researching from the inside arose from interrogating my own experience of work on myself while addressing the question of how to support teachers in teaching mathematics. My conclusion was that they have to work on themselves, informed by research, and shared practices. I then realised that all educational research depends on blending research from the inside and from the outside.

There is a strong sense in which the most important effect of educational research is on the researcher themself (Mason 1994). What a researcher finds out most about is themselves, not just in mathematics education, but even in mathematics and science:

> We have found that where science has progressed the farthest, the mind has but regained from nature that which the mind has put into nature. We have found a strange footprint on the shores of the unknown. We have devised profound theories, one after another, to account for its origin. At last we have succeeded in reconstructing the creature that made the footprint. And Lo! It is our own. (Eddington, quoted in Kline 1980, p341)

No matter how hard we struggle to find out about the external world (and independently of whether our epistemological foundations even admit of such a construct), what we see and construe is in strong measure a mirror image of ourself. The theme of mirroring appears in fairy stories such as Snow White, and in philosophical treatises (Rorty 1979). Jackson (1992), in reporting on extensive classroom observations, illustrates mirroring beautifully through choosing to report incidents that acted as stimulus to enquire and interrogate his own perceptions and assumptions, rather than intrude on the teacher whose lessons he was observing.

At one end of the spectrum, you can literally research yourself, and at the other end of the spectrum, using personal experience (whether through analogy or more directly) you can at inform perceptions, observations, analysis and theorising.

Experience

The core of researching from the inside is attending to experience to get a taste of

> What it is like to ... be stuck on a problem, think you know how to ... but in fact have a buggy algorithm, face a new class of on the first day of term, etc.

so as to develop sensitivities to others and to be awake to possibilities. The practices of researching from the inside are derived from working on ways to learn from experience ourselves, so as to be assistance to each other.

Once attention is directed onto any word such as *experience*, it becomes woolly and problematic. Suddenly it evaporates into a myriad of unclear notions. I want to use it to refer both to that accumulation from conception to death which we call *life*, and also to moment-by-moment incidents (those fragments which can be re-called and re-entered). Often it is tempting to speak of *experience of* something (such as counting, factoring, reasoning, specialising or generalising, abstracting, using a computer, etc.), but this use begs the question of whether there is an 'it' that can be experienced, or whether life is a continual process of experiencing.

Philosophers and psychologists have worked hard to define *experience*,

> My experience is what I choose to attend to. (James 1890 p402)
> Experience is remoteness from You (Buber 1923 p60)

These sorts of definitions only make full sense when I-You connection is made with the total *weltenschauung* of the writer. I prefer a Wittgensteinian or Peircean-pragmaticist stance, indicating meaning through use in context, metaphor, and effect.

For example, in 1884, Abbott published the astoundingly popular Flatland. In it, he developed the notion that in any given world you only ever experience sections of objects of higher dimension. You cannot be aware of higher dimensionality except as you construct it virtually and analogically. Flatland's continuing popularity is partly due to the metaphoric content: what we see, what we attend to, is necessarily confined to the dimensions of what we are attuned to. Higher dimensions have to be imagined and pieced together. Experiencing a succession of sections does not ensure that you have an experience of that succession as a complete object, but working by analogy is one way to educate awareness about a higher-dimensional structure. The adage

> A succession of experiences does not add up to an experience of that succession.

has been attributed to Kant, and admirably summarises one of his central tenets in *Critique of Pure Reason* (1781). It also hints at the possibility of having experience of succession, or in other words, having a shift in the structure and focus of attention (Mason & Davis 1988) from participating in an action to awareness of that action.

It is quite remarkable that although it is a cliché that we learn from experience, one thing we don't seem to learn from experience is that we do not often learn from experience alone. Extra effort is often required. The child touching the hot stove may learn not to touch stoves, but the student who rushes in with the first idea that comes to them and then gets stuck does not seem to learn so readily, nor the teacher who answers their own questions because students don't, or the researcher who gets caught up in their own ideas while conducting an interview.

Reflection has been much touted as the way to learn from experience, but my experience *has* shown me that more is required more than mere 'thinking back'. The Discipline of Noticing (to be summarised shortly) was articulated in order to provide more systematic structure and methodical practices which are founded in psychological insight and which have proved effective in a wide variety of circumstances. In a different vein, Pearson & Smith (1985) offer a range of debriefing strategies to support practitioners in learning from experience, Boud et al (1985) offer a range of suggestions. Kemmis (1982) developed Stenhouse's notion (1975) of teacher as researcher into action research, but this has turned into a mechanical procedure in four stages which are the antithesis of what action research was designed to support, and Kolb & Fry (1975) offer a four stage model redolent of action research but omitting many of the elements which are crucial to effective research from the inside.

Using Experience

A vital part of participation in society is checking what others say against our own experience. Leonardo da Vinci put it as

> Avoid the teaching of speculators, whose judgements are not confirmed by experience (quoted in Zammattio, et al 1980 p133).

In mathematics it is possible to accept a result stated by another mathematician (although one ought to check it for oneself), but in education it is impossible to build upon a proposed 'result' without testing it in one's own experience and situation. If it checks out, sheds light, sharpens awareness, or extends the range of actions, it will be taken as valid for that individual, otherwise it will fade into the background.

Dewey (1933) turned his sense of the importance of experience as the grounding of all learning, all epistemology, into an educational principle:

> Since learning is something the pupil has to do himself and for himself, the initiative lies with the learner. The teacher is a guide and director; he steers the boat, but the energy that propels it must come from those who are learning. The more a teacher is aware of the past experiences of students, of their hopes, desires, chief interests, the better will he understand the forces at work that need to be directed and utilized for the formation of reflective habits. (p36)

My impression from the recent ICMI conference on *Research in Mathematics Education: its methods and its results* was a widespread acknowledgment that, to paraphrase Dewey,

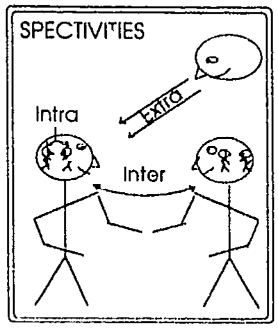
> Teaching is something the teacher has to do themself, the initiative lies with the teacher. The researcher can guide. But the energy that propels teaching comes from the teacher.

In other words, we cannot hand teachers facts or strategies with guaranteed effects (Artigue & Perrin-Glorian 1991). In order for research activity to affect anyone other than the researcher themself, it is necessary that an action take place, in which the practitioner's experience is spoken to directly, and acted upon by the practitioner. They have to reconstruct the awareness and check it in their own experience.

That is exactly what researching from the inside is about. The next sections contain a brief general description of my approach, followed by a particular case study. If you prefer to work from the particular to the general, you might begin with the case study.

Spection

One way to appreciate something is to see what it is not. To this end, it is useful to employ what Bohm (1983) called *rheomode*, building new words from old roots. In this case, *spection* (from the Latin *specare* meaning *to look*) is the root for such words as *inspect*· ' (related to observation and so at the heart of all research), ' ·*pect*, *spectacles*, and *speculation* (future looking). From *spection* we can produce words like



extraspection: looking from outside;

intraspection: one self observing another;

interspection: sharing and negotiating observations with another.

These three terms serve to distinguish aspects of research perspectives. Traditional research methodologies are based on *extraspection*. Studying what children, teachers, and others do, both inside and outside of schools, by observing them, analysing, making distinctions, and explaining and theorising is valuable work. I call it *extraspective* because it is observation from outside of the situation, inevitably separate and I-It, and perceived through different forms of attention from that of the participants. Only the most extreme forms of ethnography are anything other than extraspective.

By contrast, *intraspection* describes the development of an inner observer (Mason et al 1984, Schoenfeld 1985) who watches and witnesses, and when strengthened, can inform practice in the moment by awakening the functioning self to alternative possibilities. *Spection* is being awake in the moment, noticing and responding freshly and creatively in the instant, catching oneself before embarking on habitual behaviour. As noticing becomes less retrospective and increasingly spective, an altered structure of attention is created. Intraspection becomes more frequent and more vivid. This requires hard and lengthy work on different selves and on splitting attention.

Interspection describes interactions between colleagues who go beyond extraspection and begin to form a shared or taken-as-shared (Cobb 1991) world of experience. They develop labels for collections of similar incidents and actions to trigger a rich web of negotiated description and experience. One by-product of sharing descriptions is that strategies and gambits (Pimm 1987) emerge which can turn into future possibilities.

While reflection continues to be an ill-defined and overly used term, I use it to refer to *retrospective* re-entering of salient moments from the recent past, and attempting to give accounts of these in descriptions which do not embellish, judge or justify. Their purpose is to resonate similar experience in the listener through which they can enter the experience of the describer. To prepare for future actions, I am prospective by mentally imagining myself in a typical situation in which I wish to work differently, and projectively imagining myself responding in the way I wish.

It is essential not to get caught up in ones own little solipsistic world, and to make use of the rich source of insight and awareness provided by colleagues. It is necessary therefore to be *interspective* through describing salient moments to others by re-entering one's own moments and giving *accounts-of* them to others, and being triggered to re-enter moments by listening to other accounts.

None of these spections is to be confused with *introspection* which, although a useful word, is now tainted with the brush of 19th century Northern European psychologists such as Wundt (Gardener 1986 p102-106). Wundt and his colleagues promoted introspection, in which subjects (largely fellow experimenters, because they had to be trained as observers) attended to their own sensations and reported as objectively as possible. Behaviourism grew out of a logical-positivist reaction to excesses which introspection produced. Intraspection is based on accepting the need for training in inner observation, but not accepting any account as valid just because it is personal and idiosyncratic and hence unchallengeable. Instead, a rigorous discipline of reporting and validating is involved, avoiding the theorising of "I must have ...", and "I think that ...", and focusing on direct brief-but-vivid descriptions which resonate with other people's experiences.

Basic Processes of Noticing as a Discipline

> I never imagined that there could be so many ways to look mathematically at a box of raisins or our own families. It has made me open my eyes to the simple things around me.

This comment from a fifth-grade teacher (Muscella 1992) captures vividly a sudden awakening of awareness, a feeling in the moment that 'things will never be the same' which is characteristic of moments of sudden noticing.

Most of the time, students, teachers, educators, researchers, and administrators react to stimuli. Their actions are the working out or unfolding of decisions made hours, months or years earlier. In order to cope with the variety of situations that develop, it is essential to automate certain responses so that they become rehearsed reactions which can flow automatically without agonising all the time over making choices. But the trouble with habits is that they sink below the level of awareness, and then are not available for inspection:

> Habit forming can be habit forming

One aspect of inner research is to seek out such habits and make them available for re-questioning from time to time, to maintain a conjecturing 'as-if' stance towards the whole process of teaching and learning. Learning diaries and personal journals can serve to awaken an enquirer to issues which are of significant to them, simply by looking back over the incidents that were considered sufficiently significant to record, and locating patterns and commonalities in these.

Wheeler (1965) advocated description of classroom incidents as the currency of exchange between teachers, and this has been developed by many groups including the my own Centre. For extended descriptions, see OU (1988), OU (1992), and Mathematical Association (1991). The roots lie very deep. Plato advocated use of the particular in the way he set out his dialogues, while through the voice of Socrates, he commented on the difficulty that discussants have in dealing with abstraction without using the particular. Effective writers are those who are able to resonate reader experience through the clever use of brief-but-vivid evocations.

Study of teachers noticing in the moment, that is teacher decision-making, provides a study in transition from outer to inner research. Early studies (Peterson et al 1978a, 1978b, McNair 1978a, 1978b) focused on interviews and observations by objective researchers. More recently, teachers are finding that such decisions are available to self study, and that it is precisely in such moments that the taste of freedom, of being alive to the situation, is felt (Rosen 1991, Baird & Northfield 1992, Williams 1989, Chatley 1992, Billington 1992).

The Discipline of Noticing is elaborated in several sources (Mason, 1984, Davis & Mason 1989, Mason 1991, Davis 1992, Mason 1993). There are four aspects:

Systematic Reflection: retrospective re-entry through brief-but-vivid accounts of incidents, without judgement or explanation; once a corpus of accounts accumulates, it is possible to identify common threads.

Recognising and Labelling Choices: locating alternative strategies and gambits for use in particular situations in the future, either from reading, from sharing incidents, or from observing others; labelling typical incidents and corresponding gambits so as to trigger recollection of possible actions spectively.

Preparing and Noticing: entering recent experiences post-spectively, then imagining making an alternative choice pro-spectively. It takes considerable effort to move the moment of recognition from the retrospective to the spective.

Validating with Others: to guard against solipsism and self-delusion, continued interspective exchange of brief-but-vivid incidents and observations; refining task-exercises which permit colleagues to experience (directly or by analogy) something proposed as worthy of noticing.

Popper's fallibilism requires theories to be falsifiable. The Discipline of Noticing requires theories to be sufficiently resonant to inform future practice, but this may take work. Lack of resonance falsifies the theory only locally, at the given time and under the given conditions, since resonance may arise later, and attention may be focused elsewhere at that time. By maintaining a questioning attitude, seeking recognition of salient moments and resonance with the experience of others, you act against a tendency to become fixed in your new ways.

Attention and Awareness in Inner Research

Through systematic use of the discipline of noticing, and through interrogating my own experience (whether corresponding or analogic) and that of others I find that I am able to propose conjectures and focus attention when tackling a problem in mathematics education. These conjectures may sometimes be usefully investigated by measuring things, but often more efectively by working at locating experiences which contribute to enriching and deepening sensitivities so as to be awake to possibilities in the future. The test of validity is whether it generates convincing stories about the past, and whether it informs actions in the future.

My attention has been self-referentially or perhaps recursively drawn to the central importance of attention and awareness, and hence imagery and energies as research domains, as an explanatory component of a global theory of teaching and learning, and as the essence of methodology. Epistemological foundations of any research paradigm are based on how attention and awareness is employed, made explicit, and interrogated. Inner research makes explicit use of the study of attention, which in turn seems to me to be at the heart of learning, teaching, research, and administration.

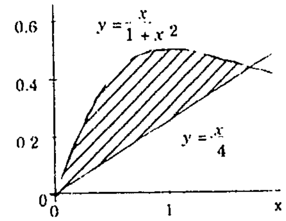
Incidents that remain salient in memory, even for a short time, can be taken as the phenomena under study, or as indicators of it. They can also be taken as a mirror in which to see oneself, to locate one's interests and concerns. Jackson (1992 p37-56) illustrates mirroring beautifully, through choosing to use things that stood out for him as stimulus to enquire and interrogate his own perceptions and assumptions rather than intrude on the teacher whose lessons he was observing.

> The universe is a mirror in which we can contemplate only what we have learned to know in ourselves. (Calvino 1986, p107)

which brings us back to mirors and the Eddington (quoted earlier). All research has an element of research of self within it.

CASE STUDY: WHAT DID YOU SEE?

My case study illustrates the beginnings of an investigation described in a manner which reflects the process of researching from the inside as closely as I can manage in text. I begin with an incident.

> A colleague and I were observing a lesson involving twenty-five 17-18 year olds doing A-Level (pre-university) mathematics. The teacher drew their attention to the following problem as a potential homework from an old exam paper:
>
> 
>
>
> Show that the shaded area is $\int_0^{\sqrt{3}} \frac{x\,dx}{1+x^2} - \frac{3}{8}$.
>
> She then asked the class "what do you see?". I distinctly heard a student say "root three", but the teacher appeared not to hear. She waited, and someone else offered "the equation of the curve". On getting no further reply within a few seconds, she drew attention to the subtraction in the 'show that' as a difference between two areas.
>
> It transpired in later conversation that she had indeed not heard the "root three", though she began from then on to refer to it as if she in fact had heard it. She expressed disappointment that they had not had more to say.

You might like to pause and consider what if anything strikes you about this incident.

What strikes me initially is the not-hearing. I felt a resonance between this incident and many others I have participated in. I recognise so clearly being caught up in the flow of what I am thinking that it is only later when I listen to a tape that I realise that I have not 'heard' what a student or colleague was saying to me. Instead I hear what I can hear, and run from there. It is only when I actuall hear an unexpected reply that I realise I had a very clear image in my head, and that my question was intended to point them to what I was seeing (Love & Mason 1992). It is a form of 'guess what is in my mind'. But I only recognise it as such when the response differs from what I am expecting *and* I am awake to that discrepancy. Then I can do more than react by asking a more focused question. Recognition offers a moment of choice:

> I can carry on and ask another question that is more focused, perhaps ending with a trail of more and more specific questions such as Holt (1964) captures so beautifully in his account of Ruth ("If I wouldn't tell her the answers, very well, she would just let me question her right up to them"); and which Bauersfeld called *funnelling*.
>
> I can acknowledge that I am caught in a guessing game and reveal (try to describe) what I am seeing;
>
> I can let go of my thoughts and concentrate on what the students are saying.

The latter strategy parallels another common strategy of asking students to construct their own question of a similar type which they consider to be hard, because students then reveal where their attention and concern is focused, and it is often not where the teacher expects it. Listening to what other students say they are seeing, and discovering that different students are seeing different features can be one way for students to realise that there are alternative ways of seeing which may be more effective.

I now have something to look out for (an unexpected response), and some strategies to try out when I notice an opportunity. I set myself to catch a similar moment in the future, and I do this by imagining myself in a class asking a question and being surprised by the response. Then I imagine myself choosing to tell them what is in my mind, and I also imagine myself taking time to listen to what they say, and giving time for others to make different reports. In order to help trigger my awareness of strategies to use, I produce a label for this sort of incident, for example, rather unimaginatively, *unexpected response.*

After working in this way for a while, I find perhaps that I am beginning to recognise I am about to ask a question pointed at what I am seeing, and am able to ask less pointed questions, to dwell more in the students' experience than in my own. Or perhaps I find that it does not help me, and I soon forget all about the incident, the label, and questioning as an issue. If I find my practice being informed, I seek out particularly sharp examples which I can turn into task-exercises to offer to others as the basis for discovering whether they too recognise what I think I am noticing, and whether it too informs their practice.

Discussion

That was an example of one phase of researching from the inside, working at extending sensitivity to notice opportunities to alter what I am attending to, to awaken my awareness in the moment. It might have been triggered by reading some research on questioning, from something a colleague said, or a moment of 'graced' noticing (Mason & Davis 1988).

What I have described here is a process of I-You connection; but it is presented in words, so you have to battle through an I-It reduction to re-construct an I-You of your own. The purpose of task-exercises is to provide specific highlighted recent experience to which to relate and through which to build connections.

Development

Having located one incident that struck me, suddenly it is everywhere. I begin to notice 'the phenomenon' in other situations (not even necessarily in the classroom). An action is beginning, parallel to the growth of early language (Brown 1973) in which meanings are over generalised and then re-specialised to common usage. All sorts of 'examples' come to mind, before an editing-selecting process refines the class, choosing some particularly central or paradigmatic examples (Lakoff 1987). Past and present experience are juxtaposed by a strength of intention to look out for such incidents. The very notion of 'example', 'such incidents', and 'this phenomenon' is an indication that abstraction is taking place. It is a delicate shift in the structure of my attention (Mason 1989). My experience with graduate students suggests that this is a significant and non-trivial component of any research.

As the investigation continues there will be a temptation to discriminate more finely, to produce more distinctions, to classify and taxonimise. Sometimes this can be helpful, but intricate classifications not based on direct experience or extant structures are usually very hard to remember. Dyads and triads and well chosen labels for metonymic triggering are much easier to reconstruct, and more likely to 'come to mind' than lists.

More Exemplars

The *unexpected-response* phenomenon is itself a particular case of a more general phenomenon of not-seeing turning into seeing, which I have labelled as *coming-to-see*, and which I recognise having worked on in a different guise as *shifts of attention* (Mason & Davis 1988). In the interests of conserving space, I limit myself to a very few examples from each of the domains mathematical content, mathematical themes, learning, teaching, and teacher education.

Mathematical Content

A student describes the surface of a ball as three dimensional. It "takes up space".

The perimeter of a region ignores the 'space inside'.

Learning

A student looks at $y = x/k$ and does not see this as the equation of a straight line with slope $1/k$ because it is not in the form $y = mx$.

A student can calculate how many handshakes will be needed in a group of strangers for specific sizes of group, but not how to 'do it in general'.

A student can read off points from a graph, but seems to have no sense of what the graph says about the situation being graphed.

This stimulated Janvier (1978) and led to insights and new materials (Swan 1985).

A student can talk about triangles and lines, but acts as if they are not aware of angles, vertices, and edges of the triangles

This observation forms the core of the van Hiele (Burger & Shaunessy1986) investigations leading to their classifications of stages in geometry.

Mr. Short's height is 6 paperclips. It is also 4 matchsticks. Mr. Tall's height is 6 matchsticks. What is Mr. Tall's height in paperclips?

Hart (1981) and many others used this and a variety of other contexts (based on ideas drawn from Karplus) to study children's use of addition strategies in what experts classify as ratio and proportion tasks (see for example Hiebert & Behr 1988).

For me these are generic examples of observations that most teachers will recognise. Sometimes it is not until their attention is drawn to them that teachers actually recognise them *as* phenomena, which in itself provides a generic example of the difference between *noticing* and *marking*. (You have noticed if you recognise what someone else remarks upon; you have marked if you can initiate a remark about it.)

What do I see that they have in common? In each case, the student appears to be unaware of some feature. They appear not to see what the teacher sees. This is not simply a feature of schooling, for it is thought that neonates are aware only of surfaces, and those rather fuzzy at first, and only later come to distinguish edges, colours, and textures (Kellman & Spelke 1983). Their perceptual apparatus has to learn to discriminate. So too, every act of learning can be seen as an act of new distinction-making, of coming to see what was previous undetected, undistinguished, unseen. It all depends on what you are attending to, and that is influenced by habits, practices, and integrated awarenesses.

Mathematical Themes

Mathematicians use the metaphor of not-seeing explicitly, as became clear when category theory introduced the forgetful functor in order to formalise the way mathematicians suppress extra information (MacLane 1986). For example:

treating fractions as ratios, or division operations, or rational numbers, according to the context and ignoring the other interpretations.

treating on polynomials as vectors which can be added and 'forgetting' that they can be multiplied or solved;

These are deliberate not-seeings, suppression of context when it does not suit. Suppression has positive and negative aspects, and there are clear resonances with psychological behaviour in which we suppress what we cannot cope with, and literally do not see it. This aspect deserves considerable more attention (see Tahta 1991, 1993).

Teaching

> When students are introduced to the modulus function ($y = |x|$), the lecturer sees this function as generic, illustrating how functions can be continuous at a point but not differentiable there, but the students see it is an exceptional case.

MacHale (1980) pointed out that most textbooks offer students a single counter-example to natural conjectures, such as that continuity implies differentiability. Consequently, students demonstrate Lakatosian facility in monster-barring, treating the single example as an isolated exception. Most students require assistance to build a rich web of examples and implications from a single example.

Teacher Education

Gates (1993) observed that when pre-service teachers look back over a lesson they have planned and carried out, they become aware that they 'just hadn't thought' of what would happen. For instance,

> One trainee had brought only one object for the whole class to feel or examine, but with the intention that everyone have hands-on experience. Asked about it later, he replied "I just hadn't thought of it."

It is not just a matter of not employing mental imagery to prospectively enter the class and see what would happen, for the novice teacher does not have the experience even to notice that there might be a difficulty.

Researching

As researchers we see what we are attuned to see, rather than 'what is there'. Researched events consist of the collection of memories and reconstructions that participants have of them. Goodman (1978) exploits his world-making metaphor and extends Hanson's (1958) observation that 'facts are theory-laden' to

> (facts) are as theory laden as we hope our theories are fact-laden ... facts are small theories, and theories are big facts (p96-97).

Donaldson (1978) among others showed how easy it is when conducting experiments with children to assume that they hear the question in the same way that you imagine it: "How many more white counters than counters?" may be heard as "how many more white then black?", and lead to puzzling results. When you ask a question, you assume that people respond to that question as if it matters to them and as if they hear what you are thinking.

When you think you see, when you dwell in what you see, you are less open to seeing more. This is consistent with Peirce and Dewey who stressed the role of doubt as essential for successful research: not temporary or interim doubt, but repeated and continuing.

Next Phase

The next phase of development is to isolate some task-exercises which highlight pertinent noticing. I will offer some examples. in the presentation. Their aim is to provide direct recent experience of what it is like to not-see, exemplifying the role of relating ideas to your own experience as directly as possible, through analogy. These tasks are then tried out with colleagues, modified and honed so as to focus attention on not-seeing and strategies for coming to see. They can then be used as the focus for crystallising past experience, for being sensitive to new situations with similar qualities, and for locating alternative strategies and gambits. The cycle of investigation continues.

Summary

I began with an incident, then drew out from that (lacking the possibilities of interaction) something worth noticing (being surprised by an unexpected response) which I have found fruitful. I illustrated it and suggested alternative choices that could be made. I then broadened my focus to situations in which people do not discriminate. I offered a sample range of situations (as substitute for collecting examples from you) to indicate the domain of applicability of the phenomenon. The examples in some sense constitute the domain, their generic or paradigmatic qualities being a matter of personal perception and collective negotiation. The test for whether what I have said and done is valid locally (here, today), is whether you feel you recognise the sort of situation I describe, and whether, at least for a short time, you find yourself thinking and perceiving in these terms.

I offered a variety of forms of analysis: some classifications, some parallels with other aspects of psychology, and some connections to specific mathematics education research. All of this was an attempt to locate how this framework helps me to see situations, and to structure my actions. Ideally, through sharing descriptions of relevant incidents, we would develop a range of strategies which various people have used in similar situations, we would report on attempts to employ these, and so develop a practice with a shared (or taken-as-shared) vocabulary of description.

Note the self reference of the case study in the process of validation with colleagues. Colleagues may not themselves be able to hear or see what it is I am pointing to, perhaps because their attention is fully taken up by other concerns, perhaps because my 'examples' are not in fact examples for them, perhaps because I am deluding myself to some extent. Hence the importance of conferences, where colleagues can share experience and negotiate the use of technical terms and frameworks in an extended community.

But is this research?

I know that this is a contentious issue. Let me put the case that what I have outlined *can* constitute a research practice. I have sketched how to ground the methodology in a consistent epistemology. I have indicated a domain of applicability of the methodology, not just the obvious and central aspects of psychology including attention, awareness, consciousness, and inner processes such as mental imagery and energies. Any issue concerning the experience of others can be informed and illuminated by careful interrogation and use of your own experience. I have illustrated examples of the sorts of questions it is best able to inform and the sorts of questions it tends to raise. I have discussed briefly the sort of validity and robustness of results that can be obtained (personal and collective insight and enhanced awareness).

COMMON EPISTEMOLOGY

I call it a *common epistemology* because it describes the processes which I have found in myself and which seem to accord with the experience of most of the people I have talked to in many different disciplines. It concerns how we commonly rather than theoretically come to know things in mathematics education.

Most researchers I know are unable to read everything published that pertains to their interests. So they are selective. They tend to read articles in a few particular journals regularly, and to cast about to others when connections arise. They tend not to read everything, and even the articles they read they attend to differently. Some authors are known to them and may receive special (or little) attention. Statements that conform to experience are passed over unless they pertain directly to current activity, in which case they are noted for future reference; statements that conflict with current thinking are either dismissed (with justifications to do with methodology or context or some other factor), or are noted and checked out at the next opportunity.

Even where a researcher tests a pedagogical activity, shows that 'it worked' and then publishes it, others must adapt and adjust the task for their conditions. They cannot simply import it and then build upon it. Indeed the failure of a sequence of major national projects intended to improve, change, or revamp mathematics teaching and learning have been singularly unsuccessful precisely where they have focused on behaviour and failed to work on awareness, have treated teachers and pupils as objects in I-It relation, and failed to attract the person within to work on whatever was being offered in I-You connection.

CONCLUSION

I began by observing that to express is to over stress. Once catapulted into language, it is very hard to preserve the complexity of teaching and learning mathematics because of the inevitable distinctions and classifications through which we simplify the world of our experience. As soon as emphasis is placed on separation, on I-It, as for example in researching solely from the outside, there is a de-emphasis on I-You relationship; as soon as emphasis is placed on relationship (which in itself is hard to express or manifest in linear text[3]), there is a corresponding de-emphasis on separation. Gilligan (1982 p48) added significance to this tension by suggesting that separation and relationship form one dimension along which women's and men's ways of knowing differ. Just as you think you are noticing something useful, someone else sees that your seeing is partial and biased!

I am grateful to my colleague Liz Binns for her frequent comments during the drafting of this paper, and to my many colleagues whose insights and incidents have contributed to my seeing.

BIBLIOGRAPHY

Abbott, E. 1884, *A. Square*, Seeley & Co., London, republished 1952 as *Flatland: a romance in many dimensions*, Dover, New York.

Artigue, M. & Perrin-Glorian, M. 1991, Didactic Engineering, Research and Development Tool: some theortical problems linked to this duality, *For the Learning of Mathematics*, 11 (1) p13-18.

Baird, J. & Northfield, F. *Learning from the Peel Experience*, Monash University, Melbourne.

Bereiter, C. 1991, Implications for Thinking about Rules, *Educational Researcher* 20 (3), p10-16.

Billington, E. 1992, *Beyond mere Experience: the role of reflection in the learning of mathematics*, M.Phil thesis, Open University, Milton Keynes.

Bohm, D. 1980, *Wholeness and the Implicate Order*, Routledge & Kegan Paul, London.

Boud, D. 1985, Keogh, R. & Walker, D. 1985, Promoting Reflection in Learning: a model, in Boud, Keogh, & Walker,(Eds), *Reflection: turning experience into learning*, Kogan Page, London.

Brousseau, G. 1984, The Crucial Role of the Didactical Contract in the Analysis and Construction of Situations in Teaching and Learning mathematics, in *Theory of Mathematics Education*, Paper 54, Institut fur Didaktik der Mathematik der Universitat Bielefeld (ed, H. Steiner), p110-119.

Brousseau, G. 1986, Fondements et méthodes de la Didactique des Mathematiques, *Recherches en Didactique des Mathématiques*, 7 (2) p33-115.

Brown, R. 1973, *A First Language*, MIT, Cambridge.

Buber, M. 1923, *Ich und Du*, English translation: Kaufman, 1970, *I and Thou*, T. Clark, Edinburgh.

---

3 Perhaps the repetition in this paper is one effect of trying to maintain a semblance of the multitude of interconnections which flood my mind as I write.

Burger W. & Shaunessy J. 1986 Characterizing the Van Hiele Levels of Development in Geometry, *Journal for Research in Mathematics Education*, p31-48.

Calvino, I. 1986, *Mr. Palomar*, Picador, London.

Chatley, J. 1992, *The Use of Experiential Learning Techniques in Mathematics Education: a personal evaluation*, M.Phil thesis, Open University, Milton Keynes.

Chevellard, Y. 1985, *La Transposition Didactique*, Le Pensée Sauvage, Grenoble.

Cobb, P. 1991, Some Thoughts About Individual learning, Group development, and Social Interaction, *Proceedings of PME XV*, (Furinghetti, ed), Assisi, Vol I p 231-238.

Davis, J 1992, The Role of the Participant Observer in the Discipline of Noticing, in Seeger, F. & Steinbring, H. *The Dialogue Between Theory and Practice in Mathematics education: overcoming the broadcast metaphor*, Materialen und Studien Band 38, Institute für Didaktik der Mathematik der Universität Bielefeld, p167-176.

Davis, J. & Mason, J. 1989, Notes on a Radical Constructivist Epistemethodology Applied to Didactic Situations, *Journal of Structural Learning*, 10, p157-176.

Denvir, B. & Brown, M. 1986, Understanding Number Concepts in Low Attaining 7-9 year-olds, *Educational Studies in Mathematics*, 17 (1) p15-36.

Dewey, J. 1933, *How We Think*, Heath, Boston.

Donaldson, M. 1978, *Children's Minds*, Norton, New York.

Feyerabend, P. 1991, *Three Dialogues on Knowledge*, Blackwell, Oxford.

Gardner, H. 1985, *The Mind's New Science: a history of the cognitive revolution*, Basic, New York.

Gates, P. 1993, A Focus on Learning To Teach, in Jaworski, B. & Watson, A. (Eds) *Mentoring in Mathematics Teaching*, Mathematical Association & Falmer Press London, p13-28.

Gattegno, C. 1970, *What We Owe Children, The Subordination of Teaching To Learning*, Routledge & Kegan Paul, London.

Gattegno, C. 1987, *The Science of Education: part I theoretical considerations*, New York: Educational Solutions.

Gilligan, C. 1982, *In A Different Voice: psychological theory and women's development*, Harvard University Press, Cambridge.

Goodman, N. 1978, *Ways of World Making*, Harvester press, Hassocks.

Hanson, N. 1958, *Patterns of Discovery*, Cambridge University Press.

Hart, K. 1981, Ratio and Proportion in Hart, K. Kerslake, D. Brown, M. Ruddock, G. Küchemann, D. and McCartney, M. (Eds), *Children's Understanding of Mathematics 11-16*, John Murray, London.

Hiebert, J. & Behr, M. (Eds) 1988, *Number Concepts and Operations*, Laurence Erlbaum & NCTM, Reston.

Hewitt, D. 1994, *Economiy and Learning Mathematics*, unpublished PhD dissertation, Open University, Milton Keynes.

Holt, John 1964 *How Children Fail*, Penguin, Harmondsworth.

Jackson , P. 1992, *Untaught Lessons*, Teacher's College Press, New York.

Janvier, C. 1978, The Interpretaion of Complex Cartesian Graphs Representing Situations: Studies and Teaching Experiments, Shell Centre, Nottingham.

James, W. 1890, (reprint 1950)*Principles of Psychology* Vol 1 Dover, New York.

Kang, W. & Kilpatrick, J . 1992, Didactic transposition in mathematics textbooks, *For The Learning of Mathematics*, 12 (1) p2-7.

Kant, I. 1781, Smith, N. (trans) 1929,*Critique of Pure Reason*, MacMillan, London.

Kellman, P. & Spelke, E. 1983 Perception of Partly Occluded Objects in Infancy, *Cognitive Psychology*, 15 p521.

Kemmis, S. (Ed) *The Action Research Reader*, Deakin University Press, Geelong.

Kline, M. 1980, *Mathematics: the loss of certainty*< Oxford University Press, New York.

Kolb, D. & Fry, R. 1975, Towards a Theory of Experiential Learning, in Cooper, C. (Ed) *Theories of Group Processes*, Wiley, London p33-58.

Laborde, C. 1989, Audacity and Reason: French research in mathematics education, *For the Learning of Mathematics*, 9 (3) p31-36.

Lakoff, G. 1987, *Wom.-:, Fire, and Dangerous Things: what categories reveal about the human mind*, Chicago University Press, Chicago.

Love, E. & Mason, J. 1992, *Teaching Mathematics: Action and Awareness*, Open University, Milton Keynes.

MacHale, D. 1980, The Predictability of Counter-Examples, *American Mathematical Monthly*, Nov. p752.

MacLane S. 1986, *Mathematics: Form and Function*, Springer-Verlag, New York.

Maslow, A. 1971, *The Farther Reaches of Human Nature*, Viking Press, New York.

Mason J , Burton L., & Stacey K. 1984, *Thinking Mathematically*, Addison Wesley, London.

Mason, J. 1984, Towards One Possible Discipline of Mathematics Education, presentation to ICME 5 Adelaide, in *Theory of Mathematics Education*, Inst. fur Didaktik der Math. der ,U. Bielefeld, paper 54, p42-55.

Mason, J. 1989, Mathematical Abstraction Seen as a Delicate Shift of Attention, *For the Learning of Mathematics* 9 (2) p2-8.

Mason, J. 1991, Epistemological Foundations for Frameworks Which Stimulate Noticing, *Proceedings of PME-NA* 13 Vol 2, p36-42, Oct .

Mason, J. 1993, *Noticing*, Sunrise Research Laboratory, Melbourne.

Mason, J. 1994, Researching From the Inside in Mathematics Education, Background papers to the ICMI Conference on Research in Mathematics Education, Washington, May.

Mason J. & Davis J. 1988, Cognitive and Metacognitive Shifts, PMEXII, (ed Borbás) vol 2 p487-494.

Maturana, H. & Varela, F., 1988. *The Tree of Knowledge: The Biological Roots of Human Understanding*, Shambala, Boston.

Mathematical Association, 1991, *Developing Your Teaching*, Stanley Thornes, Cheltenham.

Mcnair, K., 1978a, Capturing Inflight Decisions: Thoughts While Teaching, *Educational Research Quarterly*, 3 (4) p26-42.

Mcnair, K., 1978b, Thought and Action, a frozen section, *Educational Research Quarterly*, 3 (4) p16-25.

Moore, E. (Ed) 1982 *Writings of C.S. Peirce: a chronological edition*, Indiana University Press, Bloomington.

Muscella, D. 1992, Reflective Practice: a goal for staff development, *Hands On*, TERC 15 (2) p1.

OU 1988, *ME234: Using Mathematical Thinking*, Open University Course, Open University, Milton Keynes.

OU 1992, *EM236: Learning and Teaching Mathematics*, Open University Course, Open University, Milton Keynes.

Peterson, Penelope and Clark, Christopher, 1978a, Teacher's Reports of Their Cognitive Processes During Teaching, *American Educational Research Journal*, 15 (4) p555-565.

Peterson, P., Marx, R., Clark, C., 1978b, Teacher Planning, Teacher Behaviour, and Student Achievement, *American Educational Research Journal*, 15 (3) p417-432.

Pearson, M. & Smith, D. 1985, Debriefing in Experienced-based Learning, in Boud, D. Keogh, R. & Walker, D. (Eds), *Reflection: turning experience into learning*, Kogan Page, London.

Pimm, D. 1987, *Speaking Mathematically*, Routledge & Kegan Paul, London.

Pirie, S. & Kieren, T. 1989, A Recursive Theory of Mathematical Understanding, *For the Learning of Mathematics*, 9 (4) p7-11.

Pirie, S. & Kieren, T. 1991, Folding Back: dynamics in the growth of mathematical understanding, in Furinghetti, F. (ed), *Proceedings of PME XV*, Assisi.

Pirie, S. & Kieren, T. 1994, Beyond Metaphor: formalising in mathematical understanding within constructivist environments, *For the Learning of Mathematics*, 14 (1) p39-43.

Popper, K. 1972, *Objective Knowledge: an evolutionary approach*, Oxford University Press, Oxford.

Rice, M. 1993, Professional Development for Teachers: Into the 1990's, in Mousley, J. & Rice, M. (Eds) *Mathematics: of primary importance*, Proceedings of MAV 1993, Mathematical Association of Victoria, Melbourne, p189-195

Robinson, I. 1989, The Empowerment Paradigm for the Professional Development of Teachers of Mathematics, in Ellerton, N. & Clements, M. (eds) *School Mathematics: the challenge to change*, Deakin University, Geelong p 269-283.

Rorty, R. 1979, *Philosophy and the Mirror of Nature*, Princeton University Press, Princeton.

Rosen, J. 1991, Moment-by-moment Decisions a teacher makes in the Classroom: a new approach to teaching style, *SET research information for teachers*, 2, item 3, ACER, Melbourne.

Schoenfeld, A. 1985, *Mathematical Problem Solving*, Academic Press, New York.

Swan, M. 1985, *The Language of Functions and Graphs*, Shell Centre, Nottingham

Skemp, R. 1969, *The Psychology of Mathematics*, Penguin, Harmondsworth.

Sperber, D. & Wilson, D. 1986, *Relevance: Communication and Cognition*. Oxford: Blackwell.

Stenhouse, L. 1975, *An Introduction to Curriculum Research and Development*, Heinemann, London.

Tahta, D. 1991, Understanding and Desire, in Pimm, D. & Love, E. (Eds) *Teaching and Learning School Mathematics*, Hodder & Stoughton, London , p221-246.

Tahta, D. 1993, Victoire sur les Maths, *For the Learning of Mathematics* , 13 (1) p47-48.

Tillich, P. 1952, *Courage To Be*, Collins, London.

Varela, F., Thompson, E. & Rosch, E. 1991, *The Embodied Mind: cognitive science and human experience*, MIT Press, Cambridge.

Vygotsky, L. 1978, *Mind in Society: The Development of the Higher Psychological Processes*, Harvard University Press, London.

Wertsch, James (1985) *Culture, Communication and Cognition: Vygotskian Perspectives*, (Cambridge, Cambridge Univ. Press).

Wheeler, D. 1965, Teaching for Understanding, Mathematics Teaching, 33, p47-50.

Williams, H. 1989, Tuning-in to Young Children: an exploration of contexts for learning mathematics, unpublished M.Phil thesis, Open University, Milton Keynes.

Yates, J. 1978, *Four Mathematical Classroom: an enquiry into teaching method*, Report for the Faculty of Mathematical Studies, University of Southampton.

Zammattio, C., Marinoni, A. & Brizio, A. 1980, *Leonardo The Scientist*, McGraw-Hill, New York.

Mathematics teachers' professional knowledge*

João Pedro da Ponte
Universidade de Lisboa, Portugal

This paper addresses the study of teachers' knowledge, beliefs, conceptions and practices, presenting some illustrations from the area of problem solving. In mathematics education, the teacher has attracted much less work than the student. This may be due, in part, to the different knowledge base of interest in each case. Regarding students, we are concerned with their learning of mathematics. The nature of mathematical knowledge is itself problematic, yet that does not seem to rise too many difficulties for our work. Regarding teachers, it is much less clear what is the specific knowledge (necessary for teaching mathematics) that we should be looking at. Is it knowledge of mathematics content? of mathematics pedagogy? of students' cognitive processes? some mixture of several of these?

In recent years, the teacher emerged as a key figure from whom depends much of the success of current reform efforts in mathematics education. At our research group, the study of teaching became a major area of interest. Problem solving appeared as an interesting focus for inquiry since (a) it is strongly valued by the new curriculum orientations, (b) there are many different views about it, among both teachers and mathematics educators (Schoenfeld, 1992), and (c) it deals with processes that we all agree to be at the heart of the mathematical activity.

In the first part of the paper I will briefly review work done on teachers' professional knowledge and related concepts within and outside PME. Then, I will present cases taken from empirical research and discuss a few concepts used in our investigations. And in the final part I will contrast some general frameworks to study mathematics teachers' professional knowledge and draw some perspectives for future work.

* This work was done within the project "O Saber dos Professores" ("Teachers' Knowledge), supported by a grant from JNICT—Junta Nacional de Investigação Científica e Tecnológica, Portugal, under the contract PCSH/379/92/CED. The project team also includes Henrique M. Guimarães, Ana Paula Canavarro, Leonor Cunha Leal and Paulo Abrantes. The author thanks José Manuel Matos and Maria dos Anjos Caseiro who made helpful comments in a previous version of the manuscript.

Teachers and research on teaching

The place of the teacher in mathematics education research. Until quite recently, teachers received just a very devaluated view both in curriculum development projects (Elbaz, 1983; Howson, Keitel and Kilpatrick, 1981) and in psychologically oriented studies in mathematics education such as those reported in PME (Hoyles, 1992).

> Research on teaching, like classroom research, has generally viewed teachers in a fragmented way, in terms of isolated characteristics, and from a negative stance... Such approaches reinforce the view of the teacher as an instrument; she is a cog in the educational machine, and one which often seems to fall below the quality-controlled standards of the whole, at that... Part of the problem of such research undoubtedly lies on viewing the teacher and her work in isolation from the substance of what she teaches, that which gives much of its meaning and direction to her work. But the main failing of these approaches is that they view teachers as passive, dependent and often unsuccessful participants in the educational enterprise. (Elbaz, 1983, p. 9-10)

This devaluated view is quite natural within the dominant *individualistic* tradition of educational research, "focusing on learners, their intelligence, their abilities, and their thinking" (Bauersfeld, 1994, p. 133). However, it is not acceptable to an *interactionist* view in which the individual and the society are seen as inseparable units, having a mutually interdependent relationship, as "teacher and students interactively constitute the culture of the classroom" (p. 139).

The interactionist perspective, requiring the consideration of both psychological and sociological theories, is receiving increasing acceptance. Many mathematics educators now concur that a comprehensive understanding of the main issues bearing in the processes of teaching and learning mathematics cannot be studied just by looking at the learner. It is necessary to take into account also other factors, such as the social context of learning (Balacheff, 1990) and the nature of the knowledge being learned — also a social construct (Berger and Luckmann, 1976). The teacher is certainly one of the most important elements of the learning context and a key person in the definition of what is knowledge.

Not surprisingly, the individualistic tradition of educational research, focusing on the learner, basically sees the teacher as a complicating variable. However, a growing awareness of the limitations of this individualistic stream of educational research — for example, making it impossible to guarantee reproducibility conditions in teaching experiments (Artigue, 1992) — and the current pressures for reform in mathematics education all led to a greater awareness of the complexities of the mathematics classroom and of the important role of the teacher.

Recent work in PME. In PME, many studies have addressed (in one way or another) *mathematics teachers' beliefs*. This research mostly derives from the pioneer work of Thompson (1982) and Cooney (1985), standing on the assumption that what teachers do in the classroom mostly depends on their beliefs about mathematics and mathematics teaching. Recently, the view that the classroom environment and social, educational, and personal constraints also shapes teachers' beliefs is favoring a more dialectical perspective of the relationship between beliefs and practices (Hoyles, 1992; Thompson, 1992).

Another line of research in PME has to do with teachers' knowledge of subject matter and teaching strategies, reflecting the influence of Shulman' s (1986) ideas about the key role of *subject matter pedagogy*. These studies investigate teachers' knowledge of mathematical concepts and how to teach them (Even and Markovits, 1991; Llinares and Sánchez,1991; Sánchez and Llinares, 1992). Their assumption is that teachers who do not know well their subject cannot do a good job in teaching it, which is certainly a relevant point. However, this research has a strong focus on declarative aspects of knowledge and may be leaving out of the scene the most important issues regarding teachers' instructional activity, clearly stressed in Shulman's (1992) more recent writings.

Some research about teachers mostly stand on general frameworks of *constructivism* and *activity theory*, usually either presenting these ideas as content for teachers to learn or using them to inform the approach followed in teacher education initiatives (Adler, 1992; Crawford, 1992; Hart and Najee-ullah, 1992; Rice, 1992). Such general frameworks may provide useful starting points, but still need to be developed into more specific concepts and models related to teachers' actual professional roles.

The NCTM *Standards* (1989, 1991) key concepts of mathematical power, discourse, problem solving, etc. are the basic frame for *didactic oriented studies* (Davenport, 1992; Dougherty, 1992; Santos and Kroll, 1992). Such studies are clearly grounded in teachers' subject matter and pedagogical knowledge but seem to require a more elaborated view about the specific ways how such knowledge develops and works in practice.

Finally, recent work has been done in the perspective of teachers as *reflective practitioners*, or, going one step further, of teachers as *researchers* (Chapman, 1993; Jaworski, 1992; Lerman and Scott-Hodgetts, 1991; Mousley, 1992). Teachers, in this perspective, are seen as playing an important role in the definition of the purposes and goals of their work as well as on the means to attain them and, therefore, participate in the production of knowledge about teaching (Zeichner, 1993). Most of these ideas spring from the influential work of Schön (1983), discussing different kinds of reflection, notably reflection-on-action and reflection-in-action.

Work not much represented in PME. However, there are other important lines of work on teaching which, perhaps surprisingly, have not been strongly represented in PME:

a) The study of teachers' thinking within a *cognitive psychology approach*, dealing with teachers' interactive thoughts and decisions, both in planning an in conducting classroom activities. Most of this research has been done contrasting "expert" and "novice" teachers, looking at their squemata or knowledge structures (Berliner et al., 1988; Leinhardt, 1988).

b) Whereas that program of study presupposed that actions are guided by knowledge structures existing in the individual mind, more recently *situated cognition* began defining expertise from the perspective of knowledge use in practice, assuming that the acquisition and use of expert knowledge is essentially bound to particular contexts (Lampert and Clark, 1990).

c) With a similar concern but with a different origin, the study of teachers' practical knowledge has been carried from an *interpretative-phenomenological perspective*. For example, Elbaz (1983) views teachers' knowledge as a complex, practically-oriented set of understandings used to shape and direct the work of teaching. The content of this knowledge is revealed in the responses that teachers give to the situations that they live in their professional activity. Also in this perspective, Clandinin (1986) describes personal practical knowledge as being experiential, value-laden, positive and oriented towards practice. In her view, such knowledge is acquired through trial and error, is subject to change, and implies a dialectical relationship between theory and practice.

In mathematics education research, the teacher has been mostly viewed as a *deficient professional* — a person with deep misconceptions, lack of mathematical knowledge, and inappropriate and inconsistent beliefs, contradictory with current re orm efforts. This brief review suggests that research on mathematics teaching may consider alternative perspectives. Outside mathematics education, other lines of study have considered important aspects of teachers' activity and knowledge. Without denying the difficulties that teachers face, both in conceptual and practical grounds, these perspectives may yield new and more intersting ways of looking at them.

Teachers conceptions and practices regarding mathematical problem solving

In Portugal, recommendations to radical reform in mathematics teaching have been strongly supported by teacher education institutions and the association of mathematics teachers and were partially adopted by the Ministry of Education. In just a few years, positions that were minority became part of the official discourse. Problem solving is at the heart of the new curriculum orientations that also emphasize aspects as enhancing students' attitudes and values, applications of mathematics, use of calculators, active methods, group work, history of mathematics, and new assessment methods.

I will consider several examples of teachers' professional knowledge and practice as it relates to problem solving. The ideas and data presented in this paper were developed using an interpretative qualitative methodology, based in case studies of teachers. Of special concern were their personal and professional experiences, including their conceptions, motivations, and areas of difficulty[1]. And I will take these examples as starting points to discuss some related theoretical concepts.

Carolina[2]. Carolina has 5 years of experience teaching 7th-11th grade students. She graduated two years ago from the Faculty of Sciences of Lisbon, where she had some difficulty in completing all her mathematics requirements. She felt that in the mathematics methods course there was a stress in just a few ideas (as problem solving) that she did not view as particularly relevant for mathematics teaching. During the internship, under the influence of the supervising teacher, she began viewing more favorably the new orientations for mathematics education. Stimulated by her colleagues, she has participated in meetings of the association of teachers of mathematics, co-leading some sessions on the use of graphic calculators.

Problem solving, as a personal activity, does not attract much of her enthusiasm. As she indicated: "I am not very good in solving problems." Her attempts to introduce specific problem solving activities in her classroom have not been very encouraging. She feels a particular difficulty in the discussion of the solutions:

> Last year... With one of my classes I did some problems... I think that the big interest is the discussion that is generated... And my discussions are terrible! [Each] problem has a discussion, [contrasting] the several ways they did them, and it is a disaster... Those classes go very poorly... And it is something from which I run away, not because I do not find it interesting, but because [I do not feel good].

Carolina also indicates some trouble in finding good materials to support problem solving classes. She considers that her classes follow basically a "traditional" model, with moments for exposition and periods for solving exercises. She does not feel comfortable in unforeseen situations and tries to prevent those from arising. She has trouble in changing her approach if the situation requires so. She likes to present mathematical material in the form of games and is very concerned (and apparently successful) in establishing a good relationship with her students. Notwithstanding all her difficulties and frustrations she has a prevailing feeling of satisfaction — not because of the intensity of the mathematical activity she is able to promote but because her good relationship with students.

A distinction among knowledge, beliefs and conceptions may be helpful in interpreting this case. I take *knowledge* to refer to a wide network of concepts, images, and intelligent abilities possessed by human beings. *Beliefs* are the incontrovertible personal "truths" held by everyone, deriving from experience or from fantasy, having a strong affective and evaluative component (Pajares, 1992). *Conceptions* are the underlying organizing frames of concepts, having essentially a cognitive nature. Both beliefs and conceptions are part of knowledge.

Beliefs are just a part relatively less elaborated of knowledge, kept from confronts with empirical reality. Belief systems do not require social consensus regarding their validity or appropriateness. Personal beliefs do not even require internal consistency. This implies that beliefs are quite disputable, more inflexible, and less dynamic than other aspects of knowledge (Pajares, 1992). Beliefs play a major role in domains of knowledge where verification is difficult or impossible. Although we cannot live and act without beliefs, one of the most important goals of education is to discuss and promote our awareness of them.

Conceptions, as underlying organizing frames, conditionate the way we tackle tasks, very often in forms that others find far from appropriate. The interest in the study of conceptions stands on the assumption that, as a conceptual substratum, they play an essential role in thinking and acting. Instead of referring to specific concepts, they constitute a way of seeing the world and organizing thought. However, they cannot be reduced to the most immediate observable aspects of behavior and they do not reveal themselves easily — both to others and to ourselves.

Carolina moved from rather definite beliefs about the non-appropriateness of current curriculum orientations to a somehow middle ground position: these ideas seem all right but they are very difficult to put into practice. She does not know how to conduct several aspects of a problem solving activity (how to get materials? how to lead a discussion?) and so keeps following what she recognizes as a traditional approach. She is perfectly aware that her teaching practice does not conform to the current curriculum orientations which she does not know how to carry out. Her beliefs about the importance of the personal relationships between teacher and student constitute a very strong core around which she constructed her professional role. Her conceptions regarding the educational usefulness of games (that include a game metaphor to describe mathematics learning) provide her an important framework to organize her teaching.

Carolina is a rather insecure teacher — she expects that experience will help her to become more confident. She has many problems left to solve in her relationship with mathematics — an issue that she wants to avoid. But the essential point is the remarkable distance between what she considers didactically desirable and what she does in practice, and this seems to stand mostly on difficulties in the practical knowledge regarding instructional activities and classroom management.

Isaura[3]. Isaura is a mathematics teacher in a middle school (5th and 6th grade) with 18 years of experience. She is a very responsible person, enthusiastic about teaching, who likes to speak about her work. At the age of 15 she already intended to become a mathematics teacher. However, a new mathematics class she had at 10th and 11th grade was a rather negative experience. She disliked the teacher who, she said, "made me feel anxious about everything". And Isaura chose to study agricultural engineering. But even before completing this degree she was already teaching in a secondary school.

Isaura is now fully certified to teach. She has been a mathematics head teacher in several schools, is a regular active participant in the national meetings of the association of teachers of mathematics and has been involved in the MINERVA project dealing with the introduction of computers in schools.

To this teacher, problem solving is the essence of mathematics and should underlie all mathematics teaching. She supports the new curriculum orientations and says that the good teacher "puts in the classroom all the innovations... New technologies, group work, materials". And she indeed puts a lot of effort in selecting and preparing learning situations. In her classes, she uses investigations, games, puzzles, and activities related to students' out of school interests. In practice, problem solving is a means to introduce or apply different concepts and is only used when it fits within the curriculum:

> [Problems need to fit] not just the sequence of topics but the curriculum... [I find] it difficult to elaborate problems suitable for students and, finally, how to evaluate the students on this domain... From a theoretical point of view much is said about problem solving but from a practical point of view [not much]... How are we supposed to take problems to the classroom? How do we integrate them in the sequence of topics?

> In terms of observation, I circulate among [students], I observe the difficulties but I cannot see the support that each student requires... And that makes me very confused... I begin thinking: Are they stuck? What kind of help shall I give to let them arrive at a given conclusion? Will this yield a great mess, each of us saying a different thing?... Sometimes the problems are not very well chosen.

Isaura has difficulty in finding good problems and in integrating problem solving activities within the sequence of topics. She prepares many materials but she is still uncertain about which are the really suitable ones. She also indicates difficulties in conducting her classes, especially at the level of interaction and discourse. Group work is frequent but the working climate is marked by many distractions and interruptions. Also, there is a noticeable pressure to rush up things to draw quickly conclusions without giving all the students the chance to think thoroughly the proposed tasks. Isaura's concern in "losing no time" is quite apparent in most of her activity.

There is a sharp discrepancy between Isaura's stated beliefs about mathematics teaching and her actual practices. However, when we look closer on the possible reasons why practice finds problems in matching those beliefs we also find issues that have to do with knowledge. We may understand Isaura a little better if we view our knowledge as broadly structured in different *worlds of experience*, each with its own "meaning-structure" leading us to operate with a particular "cognitive style" (Schutz and Luckmann, 1973).

There are several dimensions of these worlds of experience, all mutually inter-related. One is its coherence and self-sufficiency. The very intensive and practical nature of teaching demands this restricted world to be taken as a whole and complete. Therefore, the world of teaching, although overlapping the world of everyday life, has a distinctive coherence and distinctiveness (Elbaz, 1983).

Each world of experience is also characterized by a particular tension of consciousness and a form of spontaneity. The tension of consciousness bears on our level of interest and attentiveness. Teachers need to be aware of many simultaneous phenomena, which makes teaching a very stressing activity. The form of spontaneity of a given world of experience is different from that of other worlds. This explains why, to a "similar event" (to an outsider's perspective), one person may respond very differently in distinct worlds of experience. Teachers' forms of spontaneity in teaching may significantly differ regarding those of everyday life. They may even vary noticeably according to the specific professional context and the students they are working with.

This structure of our experience in different worlds makes quite comprehensible that a teacher as Isaura expresses (sincerely) some beliefs about the teaching of mathematics in an interview setting and acts in a rather different way in a classroom plenty of "difficult" students. There is no inconsistency but just a gap that can be bridged by research that looks more closely at teachers' actions in the classroom and their contextual and conceptual basis.

Our knowledge within a given world of experience may have different degrees of density and consistency. It conveys complex and detailed information relative to the domains with which we have to work with frequently. As our knowledge works in a satisfying way we suspend doubts about it. Only when our maxims fail in delivering what they promised in the world of experience in which they are supposed to work they may become seriously problematic (Berger and Luckmann, 1976).

In this framework, all our knowledge, including our beliefs and conceptions, has social roots in our activity and is shaped by our experience. Beliefs and conceptions cannot be viewed determining practice, since it is the nature of the social institutions in which we move — including schools — that mostly shapes them. In the long run, however, these conceptions are mostly framed by experience within social contexts. However, the interactionist perspective does not indicate the absolute domain of the social. There is a margin for the individual which may be widened by conscious reflection. Specific actions are framed by existing conceptions acting in a given world of experience which enable to make sense of situations and choose among alternatives.

Isaura has attended several workshops and discussions on problem solving and reads regularly the professional literature. However, she indicates that her personal relation with problem solving is not a very easy one:

> I do not always have persistence... For some [problems] I do not have much patience... I like to know how it is done... I go to the solutions...
>
> Many of the existing difficulties begin with the teachers and their lack of preparation in this field. In most of the cases, the teachers while they were students were not used to solving problems but just exercises... On the other hand, the word 'problem' has for us a very strong connotation, associated with the idea that was transmitted that who could not solve problems was stupid.

The activity of this teacher seems to be marked by some anxiety, which is probably a specific character trait. This results in some difficulty in managing a stable and productive working relationship with the students. As with Carolina, there are also some problems left to solve in her relation with mathematics. However, Isaura's beliefs supporting the new curriculum orientations seem to have a positive impact in her classes, enriching her the students' learning experiences.

Júlia[4]. Júlia is a teacher of mathematics with 10 years of experience, who graduated from the Faculty of Sciences of Lisbon. She teaches 7th-11th grade students in a secondary school. She is a communicative and active person, who usually manages to accomplish what she plans and enjoys chatting and laughing. Júlia attended a couple of national meetings of the association of teachers of mathematics and was involved for several years in the national project MINERVA. She enjoys artistic activities, something she would now gladly consider for a profession. In teaching, she highly regards the possibility of designing and conducting what she views as "creative" classes.

Her classes have an enjoyable working climate, with most of the students involved in the proposed activities. Mathematical situations with a problematic character are the starting point for the activities that she carries with students. Most of these situations involve just mathematical notions and do not refer to real life contexts. She stresses interrelating graphical and analytical approaches, establishing relationships among concepts, generalizing and formalizing. All of this is carried encouraging communication among students and teacher and students.

During the interviews, she made in two different moments the following statements about problem solving:

> Look, I do not solve many problems myself... Perhaps because I am not a person that... That gets much interested about problem solving. It is not because I feel that as not important, you know?... But I think that it is important to propose problems to students. There, I miss some opportunities. As I do not solve many problems myself, I only know those most trivial ones and those that are on the books.
>
> I think it is important that [mathematics teaching] is done this way: putting the student in the position of the mathematician, of the mathematician that discovered, because he is also a person... [The student] also thinks and is able to discover the same as the [mathematician]. Or else, always in a [perspective of proposing] a problem that we have to solve... [I say], 'let us see how we can solve this' and here it comes the mathematics.

Júlia seems to have more than one conception of mathematics problem. On one hand, she regards a problem as a self-contained situation, referring to a mathematical or non-mathematical context, but not much related to the mathematics curriculum — something that she indicates to be important but that does not play a significant role in her teaching. On the other hand, by problem she means a strictly mathematical question that requires a non immediate response, implies to relate several concepts and yields the discovery of new mathematics knowledge — a perspective that is consistent with her views and practices of mathematics teaching. She seems to refer to very different kinds of "problems", but the use of the same term to express distinct ideas generates what can be regarded as an inconsistency in her responses. Júlia is just referring to different worlds of experience: one concerns general talk about mathematics and mathematics education and another refers to actual classroom activity.

Situations of practice have unique characteristics of complexity, specificity, instability, disorder, and undetermination. Schön (1983) considers *knowing-in-action* as a kind of knowledge that is built into and reveals itself in action. In his view, knowing-in-action has an intuitive and tacit nature, is marked by spontaneity, and is learned through action and reflection in situations of practice. With a similar intent other authors speak of personal professional knowledge (Connelly and Clandinin, 1986) or teachers' craft knowledge (Grimmett and Mackinnon, 1992; Brown and McIntyre, 1993).

Educational theory is unable to direct practice taking into account all the myriad features underlying practical situations. If one is concerned with practice and with knowledge that evolves out of contextualized activity and informs intelligent action (such as that of the teacher in the classroom), we need to focus in a different kind of knowledge, which I will call professional knowledge.

A professional activity is characterized by the accumulation of practical experience in a given domain. It is not adequate to judge the professional knowledge of a practitioner by the standards of academic (scientific or philosophical) knowledge. Judgment, and on the case of teachers, judgment on the spot, plays an essential role in professional activity. This judgment may profit from academic knowledge but requires the use of other resources. It needs an intuitive apprehension of the situations, an ability to articulate thinking and action, a sense of personal relationships, and self-confidence. That is, *professional knowledge* is essentially knowing in action, based both on theoretical knowledge and on experience and reflection on experience.

We may differentiate among academic, professional and common knowledge based on the ways the basic underlying beliefs are articulated with specific patterns of thinking (based on logical reasoning and experience). Experiential aspects are pervasive in more elaborated professional knowledge. Rational arguments predominate in academic knowledge. Academics and professionals (when they act in their quite circumscribed special domains) have a strong explicit or implicit concern for consistency and systematicity. Common people (and academics and professionals when they act outside their activity domains) have other priorities and do not worry too much about such matters.

If we want to regognize professional practice on its own right, we need to take it as the starting point for research and not just the place for application of theory. The concern becomes not just in applying theory to practice to improve it, but instead, to work with professionals to better understand practice and its constraints — to make it stronger. Or, as Connelly and Clandinin (1986, p. 294) put it, "to enhance its ongoing practicality".

Somehow disturbing is Júlia's little personal involvement in the practice of mathematics investigations and problem solving, suggesting a perplexing dissonance between what she considers a valuable learning experience for her students and what she values in her personal life. However, this teacher has the ability to make the mathematical content appear quite naturally in a problematic way. Her practices seem to fit what Stanic and Kilpatrick (1989) described as problem solving as context to develop mathematical ideas. One may wonder if her students get a sharp notion of what are mathematical problems, what are problem solving strategies and how is mathematics applied in real world situations. But Júlia certainly provides an example of great mathematics teaching, with plenty of opportunities for students to reason and communicate mathematically. Although not quite adjusted to all requirements of the new curriculum orientations, she gives us a superb case of professional knowledge.

The structure and development of teachers' professional knowledge

Structure. Elbaz (1983) suggests that teachers' professional knowledge is structured at three levels: images, practical principles, and practical rules. A *rule of practice* is a concise statement of what to do in a frequent practical situation. It may apply to rather specific or fairly general situations, but it always refers to their concrete aspects. Rules of practice concern means — the purposes of the action are taken for granted — and are quite idiosyncratic. They are formulated for the purpose of eliminating the need for unnecessary deliberative thought.

For Elbaz, a *practical principle* is a less explicit statement indicating a purpose. They are more expressive of the personal dimension of professional practical knowledge. Principles may stand on theory, may develop out of experience, or both. Although they do not guarantee similar courses of action in similar situations, they still enable us to say that practice is principled. It is worth noting that such practical principles and rules of practice seem rather similar to actions and operations as assumed in activity theory (Crawford, 1992).

Images of how teaching should be are the less explicit and the most general level of teachers' practical knowledge. They are broad and metaphorical statements that express in a clear way some purpose and result from the combination of feelings, values, needs and beliefs. Images organize the teachers' knowledge in different areas. They capture some essential aspects of their perceptions of themselves, of their teaching, of their situation in the classroom, and of their subject matter. Summarizing her conclusions of a case study of a teacher, Elbaz (1983) writes: "Sarah's practical knowledge is structured around a small number of images that reflect the entire body of her knowledge and serve to hold together the principles and rules she uses in bringing her knowledge to bear on practice" (p. 144-5). The idea of image is also present in other authors. For example, Clandinin (1986) views images as "the coalescence of diverse experiences and their expression in diverse practices" (p. 180) and Berliner et al. (1988) seem to regard images as reference frames of how things should go in a classroom.

Leinhardt (1988) uses two other constructs to study teachers' professional knowledge: script and agenda. A *script* refers to a specific lesson segment and concerns the goals and structured actions enacted to teach a particular topic. For each lesson the teacher has an *agenda*, which includes goals and actions, tests, and an overall strategy. Typically, in each lesson, the teacher sets an agenda and uses several scripts. A fundamental objective of teaching is to go on using a script, or, according to the intended agenda, to move to the next script. Modest adjustments are made when necessary in response to the needs of the students.

Leinhardt contends that teachers construct models, but not of individual students' knowledge. She found that teachers essentially diagnose what went well and what went wrong in their teaching and not the specific mental representations of their students. Therefore, the core of these models refers to the ways of teaching the subject matter and provides an orientation for the future — how each topic will be taught next time. This is a line of thinking that brings us close to Shulman's (1986) ideas about the centrality of subject matter pedagogical knowledge.

There are several differences between these authors. Elbaz is concerned with the general structure of the teacher's professional knowledge while Leinhardt, Berliner and their colleagues focus on the knowledge structures mostly bearing in the development of the lesson. Also, Elbaz regards teachers as having just a few organizing images while Berliner seems to extend this term to include practical principles, suggesting that teachers (especially experienced teachers) hold a great variety of images regarding classroom work. And thirdly, Elbaz strongly takes into account the personal and contextual factors bearing on teachers' knowledge while Leinhardt and Berliner seem to be essentially concerned with classroom performance.

Adopting Elbaz's terminology, we can say that Júlia seems to base all her teaching in a strong image of the working mathematician, as someone who has successes and failures in striving to relate concepts and solve problems. Her self-image as a creative and autonomous professional is also crucial and has a strong impact in her classroom practice. Her practical principles and rules of practice do not seem to be in conflict with her fundamental professional images.

Isaura also has a strong and clear (rather ambitious) image of what is an innovative teacher. However, when she comes to concrete action this image is overshadowed by concerns about the curriculum, the classroom dynamics, and her view of the students' capacities. She may face a problem of competing images. Alternatively, we may conjecture that she does not dispose of appropriate practical principles and rules of practice to enact her espoused teaching images. There is little doubt that if we just studied her stated beliefs and conceptions we would gather a very different picture of how she is as a teacher and what may be her most significant professional development needs.

Finally, Carolina distinguishes what may be all right in principle from what she feels confident in doing. She has a self-image of not being very well succeeded in mathematics nor in conducting classroom discussions. She organizes her professional practice around activities in which she feels secure and constructs her professional role through her personal relationship with the students.

Development. Experience is certainly one major factor contributing to the development of teachers' knowledge. As Berliner et al. (1998) indicate, experience changes the way we see the world around us, creating insensivity towards ordinary things and prompting us to notice atypical aspects. This is extremely helpful, specially in dealing with complex situations, since we do not need to look at usual features and can concentrate in a few selected issues.

Preservice teachers do not have a professional experience. But we may argue that their personal experience in elementary, secondary, and higher education yields the essential frame in which they organize their teaching images and tentative practical principles and rules of practice (Crawford, 1992). So, let us see in what ways may experience contribute to the development of teachers' professional knowledge.

Referring to their empirical research, Berliner et al. (1988) indicate that the responses of many of the experienced teachers reveal that they have rich images or prototypes for students and classroom events. These teachers also accumulate a great quantity of information about students, so that, in some sense, they seem to know their students even before they meet them. They also use routines in more areas of instruction more frequently and with more success. Supporting the view that professional knowledge is somehow distinct from common everyday knowledge, these authors say that experienced teachers show evidence of more reasoned thinking, referring to concrete evidence in their explanations and justifications of their actions.

Experienced teachers constantly monitor students and the class activity. If things are going as expected, there is no need to give too much attention to details and it is possible to follow the intended agendas and scripts. A similar view is proposed by Brown and McIntyre (1993), who claim that teachers are mostly concerned in maintaining some "desirable normal state of pupil activity" enabling progress towards the intended goals. But, if something unforeseen arises, the teacher needs to act in a different, deliberative mode, close to Schön's (1983) idea of reflection-in-action. Both through positive and negative instances, in an intuitive way, we subsume our experience in practical principles but we can also do it at a more conscious level, through deliberate and systematic reflection.

The influence of the context is a major concern of recent approaches to cognition. In these three teachers we note the influence of context through preservice and inservice opportunities. However this influence seems to be mediated by (a) their attitude towards the profession, (b) their personal relationships with mathematics, shaped by experience as elementary, secondary and university students, and (c) the way they personally relate to students, shaped by all their former experience of interpersonal relationships.

Júlia derived most of her images of the mathematics classroom from her preservice and inservice experiences, participation in projects, and other professional activities. Isaura, also took an interest in innovative approaches to mathematics education mostly from her intensive contacts with a professional association. Carolina, quite differently, did not accept the ideas proposed in her preservice education, showing how the effect of these professional learning contexts highly depends on a personal readiness factor. During her internship, under the influence of an enthusiastic and supportive supervisor and within a group context, she changed notably her general attitudes towards the current curriculum orientations and accepted to give a try to some of the new ideas. However, regarding the ways she presently organizes her classes, one may ask how strongl' .e changed her essential conceptions. All these personal processes take many years, show.ıng that teachers' professional development has to be studied in a very different time scale of that usually adopted in teacher education programs.

Conclusion

This paper presented some theoretical concepts which may prove to be useful in studying teachers' professional knowledge. It also showed some evidence suggesting that a different view of the teacher's knowledge and professional activity may be fruitful in studying mathematics teaching.

Recent research has most emphasized teachers' conceptions and beliefs. We may also study teachers' images (a clearly less evaluative concept), practical principles and rules of practice (more directly related to teachers' actions), and how these relate to teachers' agendas and scripts in specific lessons. We should be looking for the internal integration of the different levels of the structure of teachers' knowledge as well as for their ability to guide actual practice in a variety of contexts.

The difficulty in integrating problem solving into the mathematics curriculum is a feature common to all these teachers. Júlia presents her students with situations with a problem solving flavor, but does not value specific problem solving activities or students' learning of problem solving strategies. Isaura proposes many problem solving tasks, but does not explore them to their full potential. She is aware that sometimes things do not go very well but has trouble in understanding the specific nature of the difficulties. Carolina agrees that problem solving activities would be desirable but simply does not feel comfortable in doing them and chooses to work in other directions.

Practice, an inherently complex and unpredictable realm, has its own specific characteristics that need to be valued on its own right. Teachers work within many constraints (of which we need a better understanding) but still create quite sensible solutions for their practical situations. Innovative curriculum orientations, such as problem solving, need to be studied more closely from the point of view of practitioners.

There is a need to keep discussing general models and concepts of teachers' professional knowledge as well as carrying specific studies on the external influences and on the internal development processes. Such research may provide important guidance for the development of new professional development programs and promote a better account of the role of the teacher in curriculum development initiatives. To be successful, research in this field needs to include a strong participation of teachers where they are granted the role of active partners speaking on their own voice (Jaworski, 1992). This collaborative process may turn out to be a most valuable key for a better understanding and improvement of mathematics education.

Notes

[1] Although specific methodological aspects are not discussed here, important issues arise in this kind of research, such as the relative role of observation and interviewing and the relation of the researcher with participating teachers and students.

[2] This case is reported in detail in Ponte et al. (1993). In this and the following cases, I kept the pseudonyms used in the original research reports.

[3] This case is discussed in Delgado (1993).

[4] This case is taken from Canavarro (1993) and Ponte and Canavarro (1993).

References

Adler, J. (1992). Action Research and the Theory-Practice Dialetic: Insights From a Small Post-Graduate Project Inspired by Activity Theory. *Proceedings of PME XVI*, Durham, USA, pp. 41-48.

Artigue, M. (1992). Didactic Engineering. In R. Douady & A. Mercier (Eds.), *Research in Didactique of Mathematics: Selected Papers* (pp. 41-66). Grenoble: La Pensée Sauvage.

Balacheff, N. (1990). Beyond a Psychological Approach: The Psychology of Mathematics Education. *For The Learning of Mathematics, 10*(3), 2-8

Bauersfeld, H. (1994). Theoretical Perspectives on Interaction in the Mathematics Classroom. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Wilkelmann (Ed.), *Didactics of Mathematics as a Scientific Discipline* (pp. 133-146). Dordrecht: Kluwer.

Berger, P. L., & Luckmann, T. (1973). *A Construção Social da Realidade* (Floriano de Souza Fernandes translator). Petrópolis: Vozes.

Berliner, D. C., Stein, P., Sabers, D., Clarridge, P. B., Cushing, K., & Pinnegar, S. (1988). Implications of Research on Pedagogical Expertise and Experience for Mathematics Teaching. In D. Grouws, T. Cooney, & D. Jones (Eds.), *Effective Mathematics Teaching* (pp. 67-95). Reston, VA: NCTM e Laurence Erlbaum.

Brown, S., & McIntyre, D. (1993). *Making Sense of Teaching*. Buckingham: Open University.

Canavarro, A. P. (1993). *Concepções e Práticas de Professores de Matemática: Três Estudos de Caso*. Masters thesis, Lisbon University.

Chapman, O. (1993). Facilitating In-Service Mathematics Teacher Self-Development. *Proceedings of PME XV* (pp. I/228-235), Tsukuba, Japan..

Clandinin, D. J. (1986). *Classroom Practice: Teacher Images in Action*. London: The Falmer Press.

Connely, F. M., & Clandinin, D. J. (1986). On Narrative Method, Personal Philosophy, and Narrative Unities in the Story of Teaching. *Journal of Research in Science Teaching*, *23*(4), 293-310.

Cooney, T. J. (1985). a Biginning Teachers' View of Problem Solving. *Journal For Research in Mathematics Education*, *16*, 324-336.

Crawford, K. (1992). Applying Theory in Teacher Education: Changing Practice in Mathematics Education. *Proceedings of PME XV* (pp. II/161-168), Durham, USA..

Davenport, L., & Narode, R. (1992). School Math to Inquiry Math: Moving from Here to There. *Proceedings of PME XVI* (pp. II/169-176), Durham, USA.

Delgado, M. J. (1993). *Os Professores de Matemática e a Resolução de Problemas: Três Estudos de Caso*. Masters thesis, Lisbon University.

Dougherty, B. (1992). Project DELTA: Teacher Change in Secondary Classrooms. *Proceedings of PME XVI* (pp. II/201-208), Durham, USA.

Elbaz, F. (1983). *Teacher Thinking: A Study of Practical Knowledge*. London: Croom Helm.

Even, R., & Markovits, Z. (1991). Teachers' Pedagogical Knowledge: The Case of Functions. *Proceedings of PME XV* (pp. II/40-47), Assisi, Italy.

Grimmett, P. P., & Nackinnon, A. M. (1992). Craft Knowledge and the Education of Teachers. *Review of Research in Education*, *18*, 385-456.

Hart, L. C., & Najee-ullah, D. H. (1992). Pictures in an Exhibition: Snapshots of a Teacher in the Process of Change. *Proceedings of PME XVI* (p. II/257-264), Durham, USA.

Howson, A. G., Keitel, C., & Kilpatrick, J. (1981). *Curriculum Development in Mathematics*. Cambridge: Cambridge University Press.

Hoyles, C. (1992). Illuminations and Reflections: Teachers, Methodologies and Mathematics. *Proceedings of PME XVI* (pp. III/263-286), Durham, USA.

Jaworski, B. (1992). The Emancipatory Nature of Reflective Mathematics Teaching. *Proceedings of PME XVI* (pp II/289-296), Durham, USA.

Lampert, M., & Clark, C. M. (1990). Expert Knowledge and Expert Thinking in Teaching: A Response to Floden and Klinzing. *Educational Researcher*, *19*(4), 21-23 and 42.

Leinhardt, G. (1988). Expertise in Instructional Designs: A Example From Fractions. In D. Grouws, T. Cooney, & D. Jones (Ed.), *Effective Mathematics Teaching* (pp. 47-66). Reston, VA: NCTM e Laurence Erlbaum.

Lerman, S., & Scott-Hodgetts, R. (1991). 'Critical Incidents' in Classroom Learning: Their Role in Developing Reflective Practice. *Proceedings of PME XV* (pp. II/293-300), Assisi, Italy.

Llinares, S., & Sánchez, M. V. (1991). The Knowledge About Unity in Fraction Tasks of Prospective Elementary Teachers. *Proceedings of PME XV* (pp. II/334-341) Assisi, Italy.

Mousley, J. (1992). Teachers as Researchers: Dialetics of Action and Reflection. *Proceedings of PME XVI* (pp. II/334-341), Durham, USA.

NCTM (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

NCTM (1991). *Professional Standards for Teaching Mathematics.* Reston, VA: NCTM.

Pajares, M. F. (1992). Teachers' Beliefs and Educational Research: Cleaning Up a Messy Construct. *Review of Educational Research, 62*(3), 307-332.

Ponte, J. P., & Canavarro, A. P. (1994). A Resolução de Problemas nas Concepções e Práticas de Professores. In D. Fernandes, A. Borralho, G. Amaro (Ed.), *Resolução de Problemas: Processos Cognitivos, Concepções de Professores e Desenvolvimento Curricular* (pp. 197-211). Lisboa: IIE.

Ponte, J. P., Guimarães, H., Leal, L. C., Canavarro, A. P., & Silva, A. (1993). *Viver a Inovação, Viver a Escola.* Projecto DIF e APM.

Rice, M. (1992). Teacher Change: A Constructivist Approach to Professional Development. *Proceedings of PME XVI* (pp. III/250-257), Durham, USA.

Sánchez, V., & Llinares, S. (1992). Prospective Elementary Teachers Pedagogical Content Knowledge About Equivalet Fractions. *Proceedings of PME XVI* (pp. III/274-275), Durham, USA.

Santos, V., & Kroll, D. L. (1992). Empowering Prospective Elementary Teachers Through Social Interaction, Reflection and Communication. *Proceedings of PME XVI* (pp. III/282-289), Durham, USA.

Schoenfeld, A. H. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and sense making in Mathematics. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 334-370). New York: Macmillan.

Schön, D. A. (1983). *The Reflective Practioner: How Professionals Think in Action.* Aldershot Hants: Avebury.

Schutz, A. and Luckmann, T. (1973). *The Structures of the Life-world* (R. M. Zaner & H. T. Engelhardt Jr, trans). Evanston: Nortwestern University Press.

Shulman, L. S. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1992). Renewing the Pedagogy of Teacher Education: The Impact of Subject-Specific Conceptions of Teaching. *Las Didácticas Especificas en la Formacion del Professorado* (pp. 53-69), Santiago de Compostela: Tórculo Edicións.

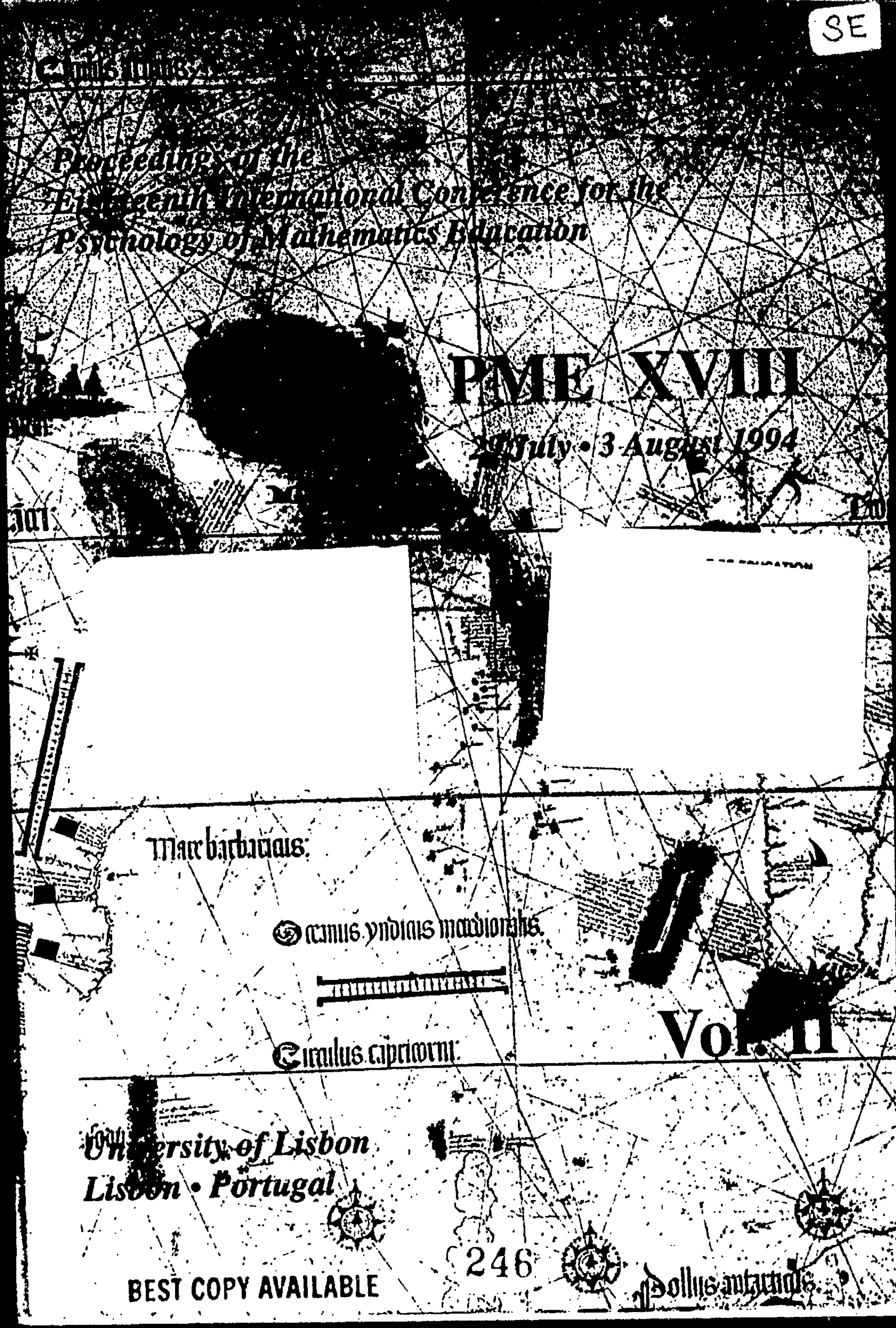
Stanic, G. M. A., & Kilpatrick, J. (1989). Historical perspectives on Problem Solving in the Mathematics Curriculum. In R. I. Charles, & E. A. Silver (Ed.), *The Teaching and Assessing of Mathematical Problem Solving* (pp. 1-22). Reston, VA: NCTM e Lawrence Erlbaum.

Thompson, A. G. (1992). Teachers' Beliefs and Conceptions: A Synthesis of the Research. In *Handbook of Research in Mathematics Teaching and Learning* (pp. 127-146). New York: Macmillan.

Thompson, A. G. (1992). Teachers' Beliefs and Conceptions: A Synthesis of the Research. In D. A. Grouws (Ed.) *Handbook of Research in Mathematics Teaching and Learning* (pp. 127-146). New York: Macmillan.

Zeichner, K. (1993). *A Formação Reflexiva de Professores: Ideias e Práticas* (A. J. Carmona Teixeira, M. J. Carvalho, and M. Nóvoa translators). Lisboa: Educa.

SE



Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

PME XVIII

29 July • 3 August 1994

Vol. II

University of Lisbon
Lisbon • Portugal

Proceedings of the Eighteenth International Conference
for the
Psychology of Mathematics Education

PME XVIII

29 July — 3 August 1994

Volume II

University of Lisbon
Lisbon — Portugal

Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

Volume II

Published by the Program Committee of the 18th PME Conference, Lisbon, Portugal

Editors: João Pedro da Ponte and João Filipe Matos

Cover: Eduardo Veloso

Depósito legal nº 78567 94

ISBN 972-8161-01-8

Printed by GRAFIS, Coop. de Artes Gráficas, CRL

Cover: The background is a reproduction of the *Cantino* chart. This Portuguese map was smuggled from Portugal in 1502, by the spy of the Duke of Ferrara, Alberto Cantino. It is the first world map of the modern times, and it takes into account the findings of the voyage of Vasco da Gama to India.
Cover overleaf: The astrolabe and the quadrant, the most used nautical instruments of the beginning of astronomical navigation, in the 15th century, are represented by two drawings included in the world map of Diogo Pereira, 1529.

Departamento de Educação
Faculdade de Ciências da Universidade de Lisboa
Edifício C1 — Campo Grande
1700 Lisbon — Portugal

Contents of Volume II

BUILDING ON CHILDREN'S INTUITIONS ABOUT LINE GRAPHS

Janet Ainley
Mathematics Education Research Centre
University of Warwick
Coventry, UK.

This paper describes one aspect of the work of the Primary Laptop Project, which aims to study the effects of continuous and immediate access to high levels of computer provision on children's learning of mathematics. Children who are competent in the use of spreadsheets to record data are able to produce graphs quickly and easily, and so it may seem that more traditional skills of constructing graphs become redundant. However, evidence presented in this paper suggests that children as young as 8 - 9 years are able to assimilate these skills, and produce and use reasonably accurate hand-drawn graphs as a result of their computer experience without any explicit teaching.

Background

The Primary Laptop Project has the long-term aim of studying the effects of continuous and immediate access to high levels of computer provision on children's learning of mathematics. Two classes within the school - currently Y4 (8/9 year olds) and Y5 (9/10 year olds) - are equipped with one portable computer for every two or three children. They have access to these machines throughout the school day, and take them home overnight and at weekends. The software available includes *LogoWriter* and *ClarisWorks*, a commercial package containing wordprocessor, database, spreadsheet and graphics facilities.

Two researchers are working alongside the class teachers, focusing on work in mathematics and science. The researchers operate as a teacher/researcher pair, each responsible for teaching one class, and observing lessons in the other. Field notes taken in the classroom (directly onto a portable computer) are supplemented by examples of children's work, recorded interviews with children and teachers, discussions between researchers and class teachers, and field notes taken by the teachers themselves.

Activities in the classroom are planned jointly by the class teachers and researchers, following the thematic approach normally used in the school, but building into this opportunities to exploit the potential offered by the children's high levels of computer competence.

Initial evidence of children's intuitions

As part of a series of activities linked to the theme of *Growing and Shrinking*, the Y4 class had been researching their own body measurements, and how these had changed since they were babies. They had experience of the use of a spreadsheet and its graphing facilities through entering their own body measurements in a table, and drawing bar charts to compare heights, foot sizes etc. In an attempt to extend the children's experience of graphs, an activity using line graphs of children's growth was introduced.

I introduced the activity by showing some data about the height of a baby (in fact my own daughter Lilian) between birth and age three. From this data I produced a line graph as shown in Figure 1, and invited the children to comment on what they saw.

Immediately Emily said: *It says two and a half there* (i.e. on the axis) *but that wasn't in the data.* I asked if they could work out how tall Lilian was when she was two and a half. Several ideas were offered.
Joanne: *Would it be half way along that part of the line?*
Emily: *You could look at the numbers and work out half way between them.*
A number of other children did not say anything, but were moving their fingers in the air, apparently trying to read off the height on the vertical axis.

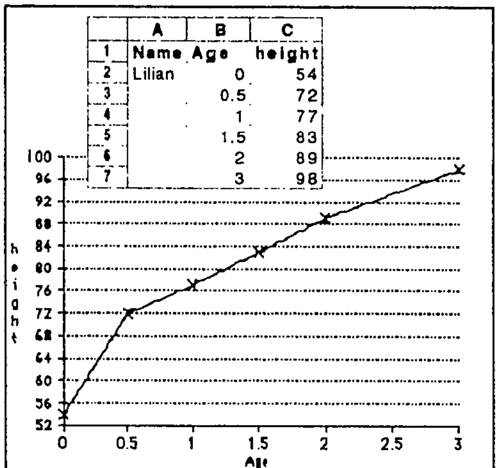


Figure 1: Line graph drawn from spreadsheet data.

After some discussion, the next stage of the activity was introduced. The children were presented with tables showing the heights of four imaginary children at various ages. The task was to produce a line graph from one set of data, and to find out from it how tall that child was at age 3 and age 13.

The task of entering their chosen data into the spreadsheet, and producing a line graph from it presented few problems to the children, and most groups were able to make a good attempt at interpolating the values for the heights at age 3 and age 13. They used a variety of methods for doing this: pointing with fingers, using the mouse pointer, using graphics lines from the drawing tools. In addition, conversations with the children while they were working revealed their understanding of the meanings of the graphs they produced.

Holly and Verity quickly produced a line graph. Verity: (Pointing with the mouse to 11) *That's about 11 there and* (pointing again) *about 3 there.* Holly: (reading across the graph)*So it's about 80 centimetres at age three Is that right?* They then went over to get a ruler to see what 80 cm looked like. They had a quick discussion about whether this could be the height of a three year old. In the end they decided their answer was all right.

Emma and Phillipa showed their graph to the class teacher because it didn't look as they had expected. She asked them what the chart told them; they said that Danny had shrunk as he got older. The teacher asked: *Is that likely?* Phillipa thought that they had probably put in the wrong data. They went away to check, and found that Phillipa had put in 191 instead of 119 for the height at age 7. They quickly changed this entry, and were satisfied with their new graph.

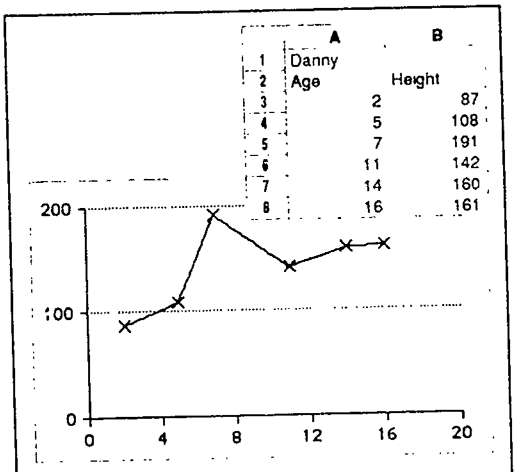


Figure 2: Emma and Phillipa's surprising graph.

Both teacher and researchers were excited about the events of this lesson because of the intuitive ease with which the children seemed to handle line graphs. The children's facility with the graphs was remarkable for a number of reasons.

- This was the children's first introduction to line graphs in school (though they would have been exposed to their use in the media.) Line graphs are generally considered to be relatively difficult, and would not normally be introduced to children as young as these.
- By default, the software draws graphs showing only the main horizontal grid lines (as can be seen from the figures above), so very little support is available for interpolation.
- The software chooses a scale for the graph which best fits the frame available, and the scale then alters as the size of the frame was changed. This meant that the children had to deal with a variety of scales, some of which were not ones which we would have chosen as being easy to read.
- The children had had no explicit instruction in how to interpolate points on the graph, or how to read scales on the axes.

Intuitions explored

In order to explore further the strength of the children's intuitive understanding of the work they had done with line graphs, individual structured interviews were conducted with about a third of the class, taking a cross-section of abilities. In the interview, the children were given new sets of data for the growth of two more imaginary children. They were asked to produce a line graph on the computer from one set of data, and to read from the graph the person's height at 3 and at 13, as in the previous lesson. If they did this confidently, the computer was removed, and they were then asked if they could draw a similar graph from the other set of data by hand, on a sheet of centimetre-squared paper. If this task seemed beyond them, they were offered a sheet with ready drawn axes on which to produce the graph. We posed this task not because we particularly valued the ability to draw graphs by hand, but as a way of exploring the children's understanding of the structure of the graphs they had been using.

All of the children interviewed were able to repeat the original task of entering data into the spreadsheet and using the software to produce a line graph. All but two of the children interviewed were prepared to attempt drawing a graph by hand. These varied in their accuracy, but most showed a reasonable choice of scales on the axes, often copying the scales used by the computer. The children found more difficulty in plotting points efficiently. The general strategy was to start drawing a line, often with a ruler, from the bottom left corner, and hope to arrive at the right place for the first point. We did give a strong hint at this point about marking the point first. Once provided with prepared axes, all but one of the children (a boy with very low self-confidence who would not attempt the task) produced a graph from which they could interpolate satisfactorily.

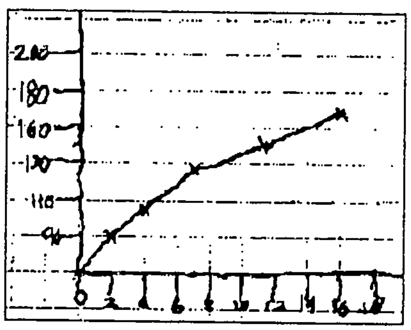


Figure 3: Sam's graph

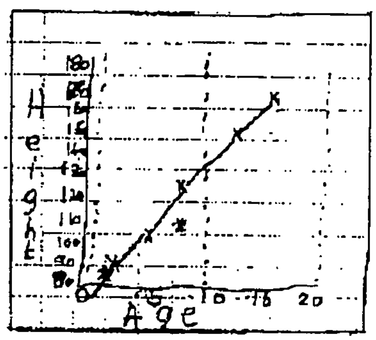


Figure 4: Emily's graph

Initial conjectures

From the evidence described above, we conjectured that the computer played a significant role in enabling children to gain access to work with line graphs, and furthermore in allowing children to build on their intuitive understanding to construct for themselves the skills required to draw such graphs by hand. It seemed possible that being able to produce graphs without needing to worry about the problems of scaling axes and plotting points, freed children to focus attention on using the graph in a meaningful way. Also, experiencing a number of examples of similar graphs enabled them to assimilate some features of the use of scale, which they were then able to use to produce their own graphs. One feature of the software seemed to be potentially important here: if the size of the frame within which the graph is drawn is changed (something which the children would naturally do to get the graph looking as they wanted it) the scale is altered to fit the new frame. We had a sense that this might be powerful in implicitly drawing children's attention to significant features of the graph which did not change under these conditions.

These initial results are in clear contrast to the experiences of other researchers, for examples those of Padilla, McKenzie and Shaw (1986), who used pencil and paper tests to investigate success rates in a number of different skills relating to line graphs. They open their paper with the statement that *as many experienced teachers are well aware, creating graphs and interpreting data from them are skills not easily acquired by most students.* With students in grades 7 - 12, (i.e. 3 to 8 years older than the pupils in the Primary Laptop Project) they found that, although 84% were successful in reading and plotting points, only 57% were able to interpolate and extrapolate, and only 32% were able to scale axes successfully. Swatton and Taylor (1994), reporting on tests of graphing skills within the Assessment of Performance Unit studies, observed similar levels of competence with children at age eleven: 78% success at reading points, but only 35% success for interpolation.

Although there appears to be considerable difference in the results we had obtained and those reported by these two papers, it is worth pointing out two factors which we recognise as having considerable significance. The children in our project class were carrying out this work within the context of a project they had been closely involved with for some weeks. The data they were working with was, although artificial in the sense that it referred to imaginary children, real and meaningful to them. This would not be the case for pupils in either of the studies referred to above. Secondly, the line graphs the children produced were ones in which the appearance of the graph matched the phenomenon which was being graphed; the graph goes up as the child grows up. Kerslake (1981) suggests that graphs of this type are the easiest for children to interpret, and it is not clear whether Padilla's or Swatton's test items contained only graphs of this kind.

A comparative study

In an attempt to replicate our results and to explore the role of the computer in this activity, we undertook a comparative study with a group of 9/10 year old children in another of the project classes. Our aim was to introduce the idea of line graphs in two ways, one making use of the computer, and one relying on more traditional resources. Our conjecture was that children who had used the computer would be better able to produce their own graphs by hand, and to interpolate from them. We gave the whole class a pre-test which presented a table of data and a hand-drawn line graph of a child's growth, and asked them to recognise specific points, and to interpolate. The results from this pre-test were used to establish a baseline of skills, and to divide the class into two groups of matched pairs.

Our aim was to provide parallel experiences for each group without explicitly teaching either group how to draw graphs. One group (the **computer** group) was given a series of activities, following closely the work done by the Y4 children. That is, they were given tables of data for various imaginary children, and asked to produce graphs from this on the computer and read off information which required them to interpolate. The second group (the **paper** group) worked with the same data, but had it presented to them on duplicated sheets containing the table of data and a hand-drawn graph. They were asked to discuss and compare the graphs, and to read off information in the same way as the computer group. The hand-drawn graphs used the same scales as those on the computer, in order to keep the experiences of the two groups as similar as possible.

After working on these activities for about 45 minutes, the computers and the duplicated sheets were put away, and both groups of children were given a new set of data and asked to produce a line graph by hand. The results at this stage were generally disappointing, and we felt that the groups had not had sufficient time to get to grips with the work, particularly as they had been introduced to the topic in rather an artificial way, without it being part of an on-going project. We wanted to focus the children's attention on the task of drawing their own graphs, but without explicitly teaching the skills of constructing axes and plotting points. In order to do this, we returned the children's initial attempts at drawing graphs without comment, and asked them to compare and discuss them in small groups. We then lead general discussions in which the computer group compared their attempts to graphs produced on the screen, and the paper group looked back at examples of hand-drawn graphs. After this both groups were given a post-test. For this they were given squared paper, and a table of data about a child's growth. They were asked to produce a graph by hand and interpolate to read off information about both the height of the child at various ages, and her age at a particular height.

In analysing the post test, one result was immediately clear: there was no difference between the computer group and the paper group in their ability to produce hand-drawn line graphs. After an initial response of considerable disappointment that our conjecture about the effect of the computer had not been supported, we began to consider the results more carefully. We believe that they point to exciting possibilities for building on children's intuitive responses to line graphs of this kind.

A large number of the children (more than half) were able to complete the post-test successfully: that is, they could construct sensible axes using an appropriate scale, plot the points they had been given using these, and interpolate from their graph to read off information with reasonable accuracy. (We accepted as accurate any response which fell within the correct square on the grid.)

To analyse the performance of the children more precisely, we scored them separately for their ability to perform each of these parts of the task. In order to get a graphical representation which reflected this analysis, we used a binary scoring system, awarding one point for the ability to interpolate, 2 points for plotting points correctly and 4 points for constructing sensible axes. Those children who completed the post-test correctly scored 7. Figure 5 shows the results for both groups presented in this way, and makes a number of features immediately clear. For example, no child scored 6: in other words, there were no children who could plot points and construct sensible axes but who could not then interpolate successfully from their graphs. In fact, apart from the 3 children who failed to score, only one child failed to interpolate from the graph she had drawn. Two children made a reasonable attempt at interpolation even when their graphs were not 'sensible' ones: for example see James' graph in Figure 6. A significant group of the children (5 out of 25), were more successful at constructing suitable axes than at plotting points accurately

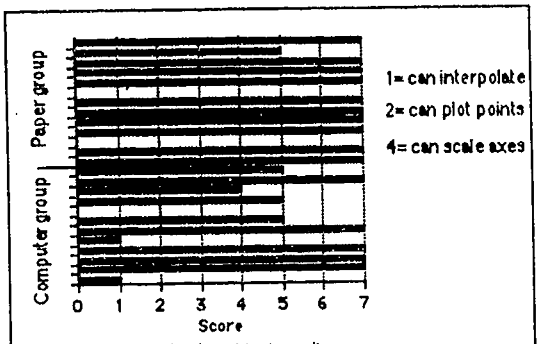


Figure 5: Analysis of post-test results

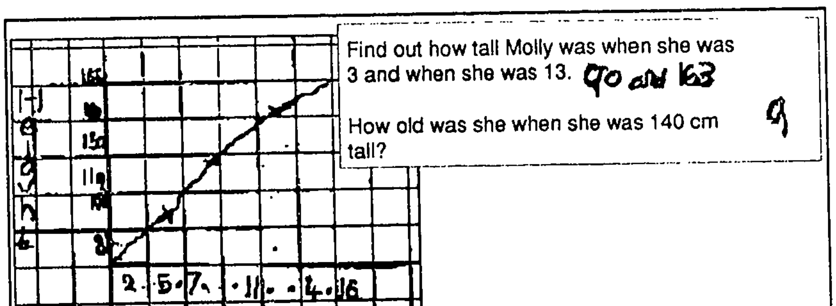


Figure 6: James' graph and attempts to interpolate

Conclusions

Our results suggest that these relatively young children were able to respond intuitively to these graphs; interpolating points not actually marked on the graph, and reading information off axes with differing scales did not seem to be difficult for most of them. Furthermore, in the process of working with graphs, many of the children seemed to assimilate skills which enabled them to draw simple line graphs by hand.

On reflection, we conjecture that the children were able to interpolate, handle scales, plot points and construct sensibly scaled axes because we *did not* attempt to teach them these skills. In contrast to the way in which line graphs are traditionally introduced, we did not deconstruct the task into constituent skills, but presented the children with complete images of the graphs in contexts which focused not on the representations themselves, but on meaningful tasks using the graphs. We see this as an example of what Hewitt (1994) refers to as *functionalisation*, the process by which skills reach a level at which we are able to function with them automatically, when they are encountered in contexts in levels subordinated to other tasks.

The presentation of a complete image was the common feature of the approaches taken with both the paper and computer groups within the comparative study, and this seemed to be more significant than the differences between working with or without the computer. More traditional approaches to introducing line graphs would necessarily begin by teaching construction skills: constructing suitably scaled axes and plotting points. If attention is focused on these, it is difficult for children to keep in mind the context and purpose for which the graph is being drawn. Indeed, the skills of constructing graphs are often taught in isolation from any meaningful context, and so appear to children to be an end in themselves.

We now believe that the presentation of complete images of line graphs in purposeful contexts offers exciting opportunities for building on young children's intuitive ideas to introduce difficult but powerful skills such as interpolation and manipulating scales. Such images could be, of course, provided by using printed materials, but these resources are not under the children's control, and will not generally be drawn from what we have come to call *hot* data (Ainley & Pratt, 1993). This is data which has been collected or researched for a clear purpose, as part of some wider exploration, rather than as an end in itself. Using computers allows children to have control: to select the data which is appropriate for their work, and to produce graphical images of that data quickly and easily. Our experience suggests that given this opportunity, young children's ability to work with line graphs is far greater than is generally recognised.

Ainley, J. & Pratt, D. (1993) Portable Mathematics and Integrated Technology, Proceedings of Technology in Mathematics Teaching Conference, University of Birmingham (UK)

Hewitt, D. (1994) The Principle of Economy in the Learning and Teaching of Mathematics, unpublished PhD thesis, Open University.

Kerslake, D. (1981) Graphs in Hart, K.M. (Ed) Children's Understanding of Mathematics 11-16, Murray.

Padilla, M.J., McKenzie, D.L. & Shaw, E.L. (1986) An Examination of the Line Graphing Ability of Students in Grades Seven Through Twelve, School Science and Mathematics, Vol. 86.

Swatton, P. & Taylor, R.M. (1994) Pupil Performance in Graphical Tasks and its Relationship to the Ability to Handle Variables, British Educational Research Journal, Vol. 20.

Cognition and competence in mental calculation

Gómez Alfonso, Bernardo
Depto. de didáctica de la matemática.
Universidad de Valencia-Spain

Abstract

The present article deals with the performance of Spanish students in a Teacher Training College when solving problems of mental computation with natural and decimal numbers before and after receiving instruction based on formal methods (i.e. methods taken from the written tradition found in arithmetic textbooks). The study seeks to establish the cognitive processes underlying lack of competence and to establish a typology of the different errors of the students.

Resumen

Este artículo trata sobre el desempeño de estudiantes españoles para futuros profesores al resolver ejercicios de cálculo mental con números naturales y decimales, antes y después de recibir enseñanza de los métodos formales (entiéndase que son aquellos que recoge la tradición escrita en los libros de Aritmética). El estudio se orientó hacia la determinación de los procesos cognitivos involucrados en la falta de competencia, y en particular en la tipología de los errores que cometieron .

Introduction

Recently there has been a growing interest in the roots of students systematic errors. Matz (1980), in an article about errors in the solution of algebra problems, suggests that they stem from adaptations, which though reasonable do not always work, of previously acquired knowledge to new situations and that some of these errors in algebra are rooted in a lack of mastery and understanding of arithmetic procedures.

On the other hand it has been empirically established that one of the problems in mental computation is the generalized lack of competence and more specifically in a diminished competence when problems involving decimal numbers, rather than integers, are to be solved. It has been claimed that this is because the concepts are not well developed or because learners lack valid strategies (Reys, Trafton, Reys and Zawolesky, 1984, *in* Reys, 1985, p.25).

It has also been argued that mental computation is an excellent method for bringing out and checking students' conceptions of arithmetic, which normally remain concealed when standard pen and paper calculations are performed.

An experimental study based on these ideas has been carried out to determine how far the lack of strategies, insufficiently developed concepts, lack of mastery and understanding of arithmetic procedures and faulty adaptations of previously acquired procedures are the cause of the generalized lack of competence in students who have already demonstrated adequate capacity in arithmetic and algebra at school.

Methodology

The study was carried out throughout the academic year 1992-93 with Spanish students training to be future teachers in their normal venue of study the E.U.d F.P. of the University of Valencia.

The students were administered two tests: one prior to instruction in formal methods of mental calculation and another after the period of instruction. Both tests were divided into two one-hour sessions, one session of each test consisted of subtraction problems and the other of multiplication problems. The problems were selected after a number of pilot experiments and were characterised by the following features: they were not excessively difficult, it was possible to use alternative strategies to solve them, they allowed for comparison between data gathered for natural numbers and decimals, the number of problems in each test was such that students were able to solve them in a one hour session.

Pre-instruction Problems

547-189	243-75	1300-875	461-166	265-199	13-8,75	2,23-1,58
$3,1^2$	37x0,25	28x35	47x99	41x42	3,4x0,15	64x25

Post-instruction Problems

737-289	634-75.	1400-675.	481-186	245-197	14-7,75.	2,32-1,67	54,7-18,9	24,3-7,5	46,1-16,6
$4,1^2$	47x0,25	26x35	37x98	31x42	2,4x0,15	64x75	2,8x0,45	4,9x5,1	19x18

Students were required to solve the problems in their head and although there was no time limit for each item, the time assigned to each session was one hour, which proved to be sufficient. They were instructed to write only the result of their computation so that they had to compete the operation before writing anything. On the same sheet, they then had to explain in their own words how they had obtained the answer. Subsequently the answers and the students' explanations were analyzed and a typology of student errors was established, which was refined by means of individual interviews.

The instruction in methods of mental computation was developed following an experimental program which had been previously devised by the author (Gomez, b 1994) and based on a broad study of the history of mental calculation in the school curriculum.

Unlike the program which is currently popular in school textbooks and which is analytic and case dependant, the experimental program is a global and simultaneous synthesis of formal methods for the four operations based on levels of generalization and common guiding principles.

The students were introduced to this synthesis. The basis of the synthesis was discussed in class to bring out the basic facts of the number system and the properties and concepts of the four operations and to promote flexibility rather than mere training or pointless discussion of which is the best method.

Results and conclusions

Before receiving instruction in formal methods, most of the students were restricted to column methods and the most common errors were those that seemed to stem from carelessness or lapses of memory, however some errors were also observed that appeared to be symptomatic or evidence of more specific problems.

After instruction students showed more flexibility and there was a greater number of error types in particular for symptomatic errors . In the subtraction problems 15 different types of error were identified, of which 7 could be put down to carelessness or lapse of memory and 8 seemed symptomatic; in the multiplication problems, of the 27 types of error 10 seemed to be due carelessness and 17 symptomatic.

The high number of error types may be explained by the way error types have been distinguished, since we have considered as belonging to different types both errors that appear in different methods and errors that appear in the same method but which seem to require a different explanation.

This wide range of error types can, however, be arranged in terms of three categories: rule extrapolation, generalized method, or centring on the modified element instead of on the modification.

1. Examples of Rule extrapolation:

$3\bullet 1^2$ = "96•1. I multiplied 3•1x3•1 -> 3•1

```
                    3•1
                   ----
                    3 1
                   9 3 .
                   -----
                   9 6•1."
```

This error seems to be an extrapolation of the rule for placing the decimal point from the column based algorithms for adding and subtracting. Here it seems that the student applies the same rule to multiplication so that he brings the decimal point down respecting its initial position.

The mistake of misplacing the decimal point by counting final zeroes in a decimal number:

2•4x0•15= "0•036; 2•4x0•15=1•2x0•30=0•6x0•60=0•036"

seems to be an extrapolation of the rule which counts decimal places of the data in order to place the decimal point in the product. In this case the student seems to count the zeroes to the right of the final significant decimal place, as though he considered 0•60 to be different from 0•6.

The error of eliminating or recovering improperly the decimal point after moving it in both factors

3•4x0•15=10•5 "I multiplied 34 by 1•5 and got 6•0+4•5 =10•5"
(In this example there is the additional error of not recognising the place value of 3 in 34.)

seems to be an extrapolation of the rule used in the division algorithm to eliminate the decimal point in the divisor by moving the decimal point simultaneously in the dividend and the divisor.

2. Examples of errors that seem stem from generalizing methods from one operation to another.

The error of decomposing a number and multiplying its first digit by the other number and subsequently multiplying the result by the second digit:

28x35= "I multiplied 5x28 and then multiplied the result by 3".

appears to be a generalization of the method of addition by decomposition combined with a failure to recognise the place value of the 3.

The error of invalid "adding and taking away" which involves multiplying, not the data given, but the results of adding a number to one of the data and taking away the same number from the other datum.

64x75=4830. "I take 5 away from 75 and add 5 to 64 so I get 69x70 (I multiply by an easier number and I get the same result)

seems to be a generalization of the compensation approach to adding, whereby a quantity is added to one datum and taken away from the other.

The mistake of "front-ending" or place grouping the elements of a multiplication without recognising the intermediate products.

Completed digits (i.e. including their place order zero)

28x35=640 "I multiplied 20x30=600 and then 8x5=40 and I added 600+40 = 640".

appears to be a generalization the method for adding whereby digits with different place values are added separately: units with units, tens with tens.

Isolated digits

$3{\bullet}1^2$="9•1. I squared 3 and then 1 (I multiplied 3•1x3•1)"

This variation seems to be evidence of the same theorem implicit in the previous example, but here the student is not aware of the initial decimal form 0• (zero point) before the 1 of 3•1 when this is decomposed into 3+0•1.

3. Examples of errors that seem to result from centring on the element modified rather than on the effect of the modification.

The error of increasing the multiplier by x and subtracting the product of the multiplier and x instead of subtracting the product of x and the multiplicand:

28x35= "994. I multiply 35x30 and I subtract 56"

seems to be evidence of centring on the modified element. It seems that here the student does the following: 28x35=(30-2)x35=30x35 -2x28.

The error of subtracting x from the multiplier and adding the product of the multiplier and x instead of adding the product of x and the multiplicand:

41x32=1294. "41 times 30 = 1230 and I add 64".

seems again to indicate centring on the modified element. Here it appears that the student multiplies each of the elements of the decomposition (30+2) by each of the original numbers: 30x41 and 2x32.

The error of miscounting the zeroes after multiplying numbers ending in zero:

47x99="423; 47x100=470-47=423".

appears to indicate centring on a hundred when multiplying by a hundred and not on the effect that it produces.

Implications

The problems identified in student errors seem to originate not so much in the existence of badly developed concepts as in rigid rule applications, generalizations, extrapolations and centring, that could be the consequence of teaching which lays too much emphasis on the creation of automatism at the expense of and of promoting reflection on procedures and on the effect that modifications to data have on the results.

To avoid these undesirable consequences it seems reasonable to propose a change in teaching so as to improve students' conceptions of arithmetic procedures as a meaningful, non-automatic actions on numbers.

References.

Butlen, Denis & Pezard Monique (1991). "Calcul mental, calcul rapide". *Grand N.*, 47, 35-59.

Gómez Alfonso, B. (1994). "Los métodos de cálculo mental en el contexto educativo y los procesos cognitivos involucrados en los errores que cometen los estudiantes al aplicarlos". (Doctoral dissertation, University of Valencia-Spain)

Matz, Marylin. (1982). "Towards a process model for high school algebra errors". En *Intelligent Tutoring Sistems*. Ed. D. Sleeman & J. S. Brown, (pp. 25-49). London, Academic Press.

Reys, B. B. (1985). "Identification and characterization of mental computation algorithms used by seventh and eighth grade students on visually and orally presented mental computation exercises" (Doctoral dissertation, University of Missouri-Columbia). *Dissertation Abstracts International*, *46*, 3279A.

VARIOUS REPRESENTATIONS OF THE FRACTION THROUGH A CASE STUDY

Marta Elena Valdemoros Alvarez
Research and Advanced Studies Center of the IPN, Mexico

In the development of a research centered on the construction of the "language of fractions" and its relationship with the involved concepts, we applied an exploratory questionnaire in the fourth grade of an elementary school. We also carried out various qualitative observations of classes and designed the study of some cases (through individual interviews). The present report is only referred to FABIOLA CASE, characterized by the effective use of pictorial representations connected with the sum of fractions and the generation of adequate representations tied to the manipulation of concrete materials (in contrast with her own algorithmic handle).

In general, Sastre (1984) identified some primal connections between the verbal language, the arithmetical signs and pictorial representations that resulted in conjugated arithmetic teaching and learning. The cognitive activities develop by any individual, in the mentioned symbolic fields, are inmersed in multiple difficulties. To carry out the passage of one symbolic plane to another, the introductory approach most accessible could be the *pictorial representations* which also could fulfill the role of precursor of the arithmetic writing.

Lesh (1987) recognized distinct types of representation systems tied to the mathematic learning; between others, the representations derived of manipulative models, the drawings or diagrams and the written symbols.

Laborde (1990) has emphasized the internal complexity of the mathematic language, jointly has ratified the constructive character of their symbolic processes and their tight links with the thought.

Opportunely (Valdemoros, 1993a, 1993b), we had recognized which activities were included in our exploratory questionnaire of the "language of fractions". In Valdemoros, 1993b, we centered on results derived of simple tasks referred to the fraction recognition. The necessity to deepen toward the additive situations contemplated in the exploratory questionnaire, made us realize some case studies (Valdemoros, 1993a); here we will limit ourselves to present only one of them, relying on the circumstance that this case accentuates the role of semantic enrichment fulfilled by the pictorial representations, in the sum of fractions context.

FABIOLA CASE

The results of the fifteen addition problems incorporated in the exploratory questionnaire, permitted us to identify some very noticiable tendencies, in the escolar group selected for the research. The most relevant and frequent phenomenon, in such questionnaire, defined the profile of various cases. A very generalized difficulty among the children was related to the recognition of the *notion of unit* implicated in each task; nevertheless, in the contexts which included the pictorial representations, we detected that there were a considerable number of pupils who successfully solved such obstacles and expressed an adequate result, with the help of drawings (Fabiola was the student that evidenced such situation with greater clarity).

Through the observation of some classes developed by the teacher in charge of the group, we noted that what was manifested by the last mentioned children was not found to be induced by the common teaching (because the instruction processes tended to be centered on the algorithmic procedures and included a very restrictive use of concrete treatments).

From those antecedents we described the case of the last children like *a concrete domain of the additive situations referred to the fractions, favored and expressed by the use of pictorial representations.* Not being made propitiated by the teaching, *we conceived that such concrete domain was generated by a generalization and ampliation of the "constructive mechanisms" that Kieren (1988) attributed to the fraction (partitioning, quantitative equivalence and formation of a divisible unit); this is the hypothesis of the case.*

Fabiola registered many algorithmic inconsistencies in various word addition problems of the exploratory questionnaire. Nevertheless, she accomplished adequate solutions each time that the addition problem included pictorial representations.

Although we did not ask for any drawing in the subtraction problems, Fabiola included some pictorial representations spontaneously. Taking into account the aforementioned aspects, we designed ten tasks that were developed within three hours of interview and were taped. Next we exposed these tasks (Table 1).

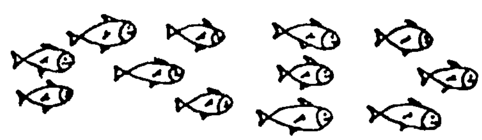
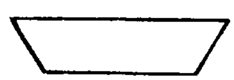
Task 1	Invent a problem that contains: 2/8 + 2/4.
Task 2	Invent a problem that contains: 1/2 - 1/4.
Task 3	Choose a figure in which you can represent: 2/8 + 2/4.
Task 4	How will you represent 1/2 - 1/4 with pencil and paper?
Task 5	Demand of confrontation between the Tasks 1 and 3.
Task 6	Requirement of contrast between the Tasks 2 and 4.
Task 7	Demand of reconstruction of some expressed solutions, in certain problems of the exploratory questionnaire.
Task 8	In this collection represent: 1/2 + 2/6. 
Task 9	Use the blocks* to represent in this figure: 2/8 + 2/4. 
Task 10	Use the blocks* to represent in this figure: 1 - 2/5. 

Table 1

* Blocks designed by Zullie (1975).

SOME RESULTS AND THEIR INTERPRETATIONS

Briefly, we centered our attention on the tasks where the most important passages of the activities were concentrated (we realized a detailed presentation of the case in Valdemoros, 1992).

Task 1:

Invent a problem that contains: 2/8 + 2/4.

Fabiola wrote *"Benito has 2/8 of apples and 2/4 of plums = 2/8 + 2/4 = 2/12"*. Static text where, in spite of the situation of measurement chosen, was absent the unit of measure. Likewise, both fractions were referred to distinct classes of objects. Such inconsistencies in the global elaboration of the text generated a lack of an interrogative statement (because of the indicated components disabled the semantic closure of the text, through a question). On an algorithmic level Fabiola added the denominators as though they were natural numbers.

Task 2:

Invent a problem that contains: 1/2 - 1/4.

In the first attempt, FABIOLA wrote *"Carlos has 1/2 kilo of chocolates and 1/4 of candy"* without getting to finish this text (the operation was associated to distinct classes of objects and the first fraction was related to a unit of measure). In the second attempt, it emerged as a definitive text *"Carlos has 1/2 kilo of chocolates and gave 1/4 to his brother. How much chocolate does he have left? 1/2 - 1/4 = 1/4"*. The final rectification indicates that the subtraction has permitted FABIOLA to refer the operation to a unique class of objects, for what she has identified satisfactorially a unit of measure. Presumably, she could manage in an adequate manner the equivalence between fractions required for the solution of the problem because both fraction were very familiar to Fabiola.

The solutions presented by FABIOLA, in the context of word problems, revealed certain instability (this characteristic was notably accentuated in the exploraty questionnaire).

Alternatively, FABIOLA selected distinct continuous whole and subdivided them (according to what we demonstrated in Figure 1). The motives of the variations of one to another of the options were able to be established with great effort, because she was very concentrated on the particular designs and did not notice the great affluence of her own thoughts. After a very extense dialogue and a very advanced development of the task, we were able to find that she tried to obtain three parts in the initial figures because *"they are two quarters and one quarter (that signifies two eighths)...they are three... three quarters"*. In other words, recognizing three quarters as a result She was centered on the numerator and intented to indicate three parts in the first square represented and in the triangle that then was substituted by the final square. (See figure 1).

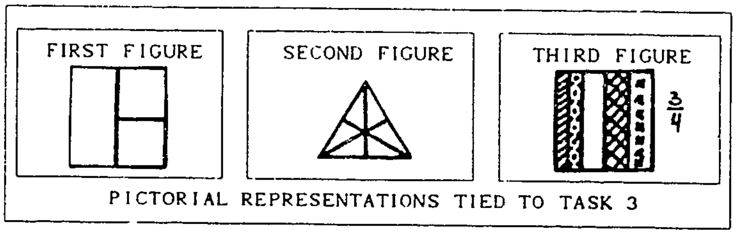


FIGURE 1

As soon as she could graphically place the complement of three quarters (the part not marked of the last figure) FABIOLA developed the final adequate representation. Later she marked with her hand, the presence of *"one eighth, two eighths... one quarter and one quarter"* in the last square; she wrote *"3/4"* as a result of it and colored in *"Egyptian style"* each one of these parts (with four different colors). Such shaded areas ratified what was stated by Kieren (1985), with respect to the generative role of the unitary fractions (meaning the ones that have numerator 1). Many children of this group realized to similar shades in other contexts.

<u>Task</u> <u>4</u>:

How would you represent 1/2 - 1/4 with pencil and paper?

In the exploratory questionnaire we omitted the tasks of this nature, considering the marked difficulties that brought about the pictorial representation of subtraction. Here, we formulated with great extent the requirement, giving more liberty to FABIOLA for the interpretation of the consignment of work. In the first instance she wrote this text: *"Juan bought 1/2 sheet of craft paper, but only used 1/4. How much paper does he have left?"*

To orient her answers toward the pictorial representations and their intrinsic difficulties, we suggested she make a drawing. FABIOLA designed what we exposed in Figure 2. To give meaning to this pictorial representation, she invented the next text: *"Juan bought half of a chocolate and only ate half of what he had, meaning one quarter of chocolate".*

This spontaneous resource added to her desing a dynamic model, in a way not deliberate (drawings preserved sufficient ambiguity in spite of having been organized as a sequence).

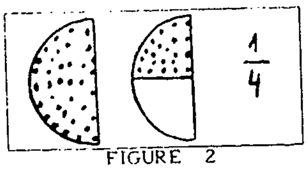


FIGURE 2

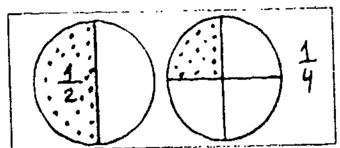


FIGURE 3

We asked Fabiola if she had something to add to the pictorial representation (in Figure 2) or if it was established for another person foreign to the interview could understand it. As a way of answering, FABIOLA completed the drawing, as we show it in Figure 3. With such representation, she identified with great clarity the unit of reference.

Task 5:

Demand of confrontation between Tasks 1 and 3

In this field, having to corroborate that the same addition of fractions she has assigned different results (in the Tasks 1 and 3), FABIOLA retreated inward in a strong conflict. FABIOLA ratified both solutions and, trying to resolve the contradiction recently recognized, she stated that *"both fractions resultant of the addition"* were equivalent, to justify such false statement. The girl was supported exclusively in verbal formulations. We required her to support her statement via creating a design.

Once again, in association with the pictorial representation FABIOLA proved the unexistence of the equivalence between recognized fractions in those tasks, as a result of the addition.

This circunstance revealed to us the strength given by the pupil to the algorithm that prevailed over the intuitive sureness brought about from an adequate use of the pictorial representations (even though she expressed a non conventional algorithm and it was false, in this task).

We suppose that a teaching based on those empiric evidences, will propitiate in FABIOLA the generation of important meanings and proving tools.

The instruction received by FABIOLA dropped such possibilities and in this spontaneous dynamic confrontation, the intuitive support tended to be subordinated to the algorithm in a context not appropiate for such hegemony.

<u>Tasks 9 and 10</u>:

Use the blocks to represent in this figure: 2/8 + 2/4.

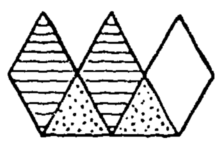


FIGURE 4

Use the blocks to represent in this figure: 1 - 2/5.

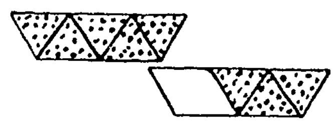


FIGURE 5

With respect to the manipulation of the concrete materials and to the *"active representations"* involved, the Figures 4 and 5 showed the respective final distributions of the colored blocks, mended to both figures, by FABIOLA. Without showing evidence of difficulties FABIOLA went on comparing all existing blocks probing them over the worksheet and decided upon such solutions. With this she demonstrated the ability to generalize the domain manifested in the pictorial systems, from such of another planes of concrete representation very close to the action.

CONCLUSIONS

About the use of distinct ways of representation articulated in the integrated field of arithmetic language, FABIOLA could generalize a recognition markedly intuitive of links and operations between fractions, from the pictorial domain towards other symbolic concrete representation systems.

So, such instruments appeared to conform the most fluid and accesible processes, in this particular case.

Globally, these mentioned resources showed to propitiate the identification of divisible units, the adequate partitioning that the diverse tasks required, and the recognition of quantitative equivalen relations. Such mechanisms were seen as seriously limited when the beliefs sustained by FABIOLA, with reference to the algorithm, generated contradictions in which resolution FABIOLA fell back on these false convictions, in detriment of her own empiric evidences. The direction of that problematic and the maximum unfolding of the mentioned intuitive resources (in the direction suggested by Sastre, 1984), requires being aboard through a efficient teaching.

REFERENCES

Kieren, T. E. (1985). Graphical Algorithms in Partitioning Tasks. *The Journal of Mathematical Behavior*, 4, 25-36.

Kieren, T. E. (1988). Personal Knowledge of Rational Numbers: Its Intuitive and Formal Development. In: J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades*, 2 (162-181). Reston: National Council of Teachers of Mathematics.

Laborde, C. (1990). Language and Mathematics. In: P. Nesher & J. Kilpatrick (Eds.) *Mathematics and Cognition* (53-69). New York: Cambridge University Press.

Lesh, R., Post, T. & Behr, M. (1987). Representations and Translations among Representations in Mathematics Learning and Problem Solving. In: C. Janvier (Ed.) *Problems of Representation in the Teaching and Learning of Mathematics* (33-40). Hillsdale: Lawrence Erlbaum.

Sastre, G. (1984). *Aprendizaje de los Signos Aritméticos y su Generalización*. Barcelona: Instituto Municipal de Investigación en Psicología Aplicada a la Educación.

Valdemoros, M. (1992). *Las Fracciones en Cuarto Grado: Estudio de Casos. Primera Parte*. Mexico: CINVESTAV, Matemática Educativa.

Valdemoros, M. (1993). *La Construcción del Lenguaje de las Fracciones y de los Conceptos Involucrados en él*. Doctoral dissertation. Mexico: CINVESTAV, Matemática Educativa.

Valdemoros, M. (1993). The Language of Fractions as an Active Vehicle for Concepts. *Proceedings of the Fifteenth Annual Meeting of PME-NA*, I, 233-239.

Zullie, M. E. (1975). *Fractions with Pattern Blocks*. Palo Alto: Creative Publications.

THE INFLUENCE OF CHILDREN'S CULTURE ON
THEIR PROBABILISTIC THINKING

Gilead Amir, Julian Williams
University of Manchester

Abstract

This research examines the influence of children's culture on their probabilistic thinking. In the first stage 38 11-12 year old children were interviewed. The interviews included discussion of their concepts of 'chance' and 'luck', their beliefs and attributions, their relevant experiences, and their probabilistic thinking. Interpretations of the concepts 'chance' and 'luck' were varied, often not involving randomness. Several distinct types of reasoning were identified. In some cases religion and superstitious beliefs appear to influence their inclination to use probabilistic thinking. Certain heuristics and approaches were common: the 'outcome approach', 'representativeness', 'availability', the 'equiprobability bias'. Some children did not understand coins and dice as random devices. In the second stage, a questionnaire will be administered in an attempt to further validate these results.

1. Literature and theoretical framework

The object of this investigation is to research the probabilistic thinking of 11 year old children in their first year of secondary school, before any special instruction has begun, and how this is affected by their 'culture'. We define the latter as their language, their experience and their beliefs, and explore these in so far as they relate to probability and chance.

Piaget and Inhelder (1975; original in French, 1951) analysed children's thinking about probability into the usual stages (pre-operational, concrete operational, etc.), culminating in a formal understanding of probability through combinatorics. This structure to some extent still holds good, and Green (1982), in a survey of 3000 secondary pupils aged 11-16 methodologically inspired by the CSMS studies (Hart, 1981), showed how their development followed a hierarchy which was consistent with this. But Green also concluded that most English pupils finish secondary school without achieving the level of formal operations. Green found that reasoning ability was the main significant factor associated with students level of understanding, explaining 44% of the variance, whilst age group explained only 14%.

Another focus of criticism on much of Piaget's research, see e.g. Donaldson (1978), has been targeted on the gap in communication which may develop between researcher and child. Borovcnik and Bentz (1991, p.101) cast doubt on some interpretations of answers to commonly used probability questions: "Seemingly false answers can be traced to such breakdowns in communication rather than missing concepts." For this reason we adopted an open, flexible interview format in trying to deduce children's patterns of reasoning and thinking.

With Green's conclusion that most English 16 year old do not reach Piaget's 'formal' stage it is not surprising that Kahnemann, Slovic and Tversky (1982) show how even adults reason in situations of uncertainty using intuitive 'heuristics', which are rules of thumb which seem to be developed to guide our behaviour in daily living. They are often used erroneously to make mistaken judgments, which they refer to as biases, in probabilistic situations. 'Representativeness', which involves expecting a sample to be representative of its parent population, and 'availability', which involves estimating odds according to memories of similar past experiences, are just two which are relevant to children. Lecoutre (1992) added a further important 'equiprobability bias' in which random events tend to be assumed to be equiprobable by nature.

Konold (1989, Konold et al, 1993) suggests that in additional to normative, formal reasoning and to reasoning with heuristics, people also reason in some situations (e.g. when predicting the weather) according to an 'outcome approach': they are inclined to view probability inappropriately as 'operative', i.e. as attempting to predict the outcome of an event. This is accompanied by a desire to seek causal explanations. In other words - people might be making mistakes not in their probabilistic judgments, but in applying to a probabilistic situation another type of judgment. In our research we will be checking to what extent do children apply the 'outcome approach'. We will also examine to what extent are different situations seen by the child as related to probability.

Fischbein (1975) showed that some intuitions (and biases) in young children's thinking are important in helping (and hindering) their pre-formal probabilistic thinking. These intuitions are a product of personal experience (Fischbein, 1987) . A teaching programme can bring significant improvement (Fischbein & Gazit, 1984). The tendency to look for causal explanations is noticed, and there is a suggestion that this might actually increase with age given the existing way of teaching.

We hypothesise that all these heuristics, intuitions, inclinations and biases might be affected by cultural factors. Influences of culture on cognition have been shown to be important in general, (Holland and Quinn, 1987; Buscaglia, 1987). Saxe (1991) has shown how children's experiences of sweet selling on the streets of Racife might have influenced their mathematical thinking. Fischbein, Nello and Marino (1991) indicate that cultural influences on probabilistic reasoning might be important. Phillips and Wright (1977) speculated that differences between UK and Hong Kong Chinese students' inclination to take a probabilistic view of events might be due to the relative 'fate-orientation' of Chinese culture.

It seems likely that 'availability' will be circumscribed by the frequency, strength or recency of relevant experiences in culturally determined experiences and practices like gambling or board games. We also expect that strong beliefs or attitudes to fate might influence children's inclination to view an event as 'random'. Children strongly causal in their view of the world might be expected to adopt the 'outcome approach' to random events. Indeed, we wonder whether children understand the term 'chance' in an appropriate way.

Our theoretical standpoint is therefore to view attitudes, beliefs, experiences and use of language as the means by which culture might influence children's probabilistic thinking, (in particular their attribution of events to chance, use of heuristics, level of understanding, and use of an outcome approach).

Our aim here is to find and describe examples of this. In the interviews certain types of thinking, and certain stereotypes of students did indeed emerge.

2. Method

The **first stage** of the research included interviews. These were conducted for two main purposes:
a) Building communication: clarifying main concepts used; trying out and modifying items from previous research and new items; trying out interview techniques.
b) Trying an holistic approach: looking for patterns linking the different fields of the child's probabilistic thinking, and between these and the child's culture. These patterns could lead to hypotheses, with the intention of further validation in the second stage of research.

38 pupils were interviewed, all in their first year of high school (11-12 years old), from two inner-city schools in Manchester. 33 were born in England. Of the parents - 20 children had two parents born in England, 13 had two parents born elsewhere, and 5 had one of each. Of families with one or two parents not born in England, 5 were born in Pakistan, 4 in Jamaica, 4 in Africa, 3 in the Far East (2 other). Each interview lasted for about 45 minutes.

The interviews were not uniform, due to the need to develop suitable questions and interview techniques, and due to the flexibility needed when an interesting aspect of the child's thinking emerged, requiring clarification. The main topics discussed were:

- Language related to probability, mainly - the concepts of 'chance' and 'luck'.
- Beliefs and superstitions, and the system of attribution.
- Background and experience related to chance.
- Specific questions in probability.

Although some quantitative analysis was done, the main emphasis remained qualitative, due to the small numbers and non representative sample. The perspective was basically a case-study perspective, trying to reach cautious generalisations as hypotheses for further research.

3. Results

3.1 Language

The pupils' understanding of what is meant by 'chance', 'things happening by chance' or 'luck' was very varied. Some of the main interpretations of **chance** were

The **possibility**, or **opportunity** of something happening, that is to 'have a chance'. For example:

Chess is analysed as a game of chance, because "you've got to try, and you've got a chance of winning and a chance of not winning." (INTERVIEW NO 1)

Interviewer: Could you tell me what do we mean when we speak of "things happening by chance" in our lives?
N.: Like - chance to become something.
I.: What do you mean by that?
N.: When you leave school and go to college or university, to become something. (INTERVIEW NO 24)

Something to do with **uncertainty**:

I.: What would be something that does NOT happen by chance?
R.: When you know what's going to happen.
Later weather is discussed.
R.: I'd agree that was chance, cause you wouldn't know. (INTERVIEW NO 31)

Something that **just happens, without planning or intention**:

A road accident? By chance I suppose... unless the driver is trying to kill the person, or the person that gets run over is trying to kill himself. It's usually by accident if you stepped into the road in the wrong time; or two cars skid - it's usually by chance. (INTERVIEW NO 22)

Some of the main interpretations of **luck**:

Something good or bad that happened, not necessarily random:

The discussion is about a computer game called 'lemmings':
I've got a sister, she's younger than me. She's not very lucky. You've got to use a mouse and she can't use a mouse properly. So she's always killing these lemming. (INTERVIEW NO 21)

Example of luck: I got 39 out of 39 in a maths test. (INTERVIEW NO 6)

An **unexplained**, perhaps **supernatural** factor.

About dice: My sister is very lucky. She gets 6 eight times in a row. I sometimes get a 6 straight away, sometimes I wait for ages... (INTERVIEW NO 2)

In many cases children used several interpretations of these concepts during the interview.

3.2 Beliefs

Beliefs appear to be the element of culture with the most influence on probabilistic thinking. The pupils' beliefs were explored in several areas: their religion; their view of God's role in the world; their superstitions; their attributions.

Some pupils thought God controls everything that happens in the world. Others thought God's control was partial - God controls the things He chooses to control. Others do not believe in God, or think He does not control anything in the world.

Several of the pupils believed in superstitions, such as walking under a ladder, breaking a mirror, lucky and unlucky numbers, black cats, etc. There were also beliefs directly connected to coins and dice, like, for instance, that when throwing a coin 'tails' is luckier.

After clarifying what the child meant by the terms 'chance' and 'luck', it was attempted to understand to what areas of life does he think these terms apply, i.e. where does he use them as attributions? This is important as an independent goal, of seeing how chance is mapped in the children's general view of the world. It is also important to see if random devices commonly used by researchers and teachers (dice, coins, counters, etc.) are seen in the same way by the children.

When discussing the broader issue, several contexts were used, such as road accidents, the weather and success in sports, and their relationship to chance was discussed. The variation in views was quite broad. For example road accidents were sometimes seen as known in advance or even planned by God; they were mostly seen as controllable by people's behaviour; some saw chance as playing a role.

Common devices used in probability teaching and research are dice, coins, counters, etc. It is often taken for granted that children see these devices as random. But, quite a number of children thought, in different degrees of certainty, that their results depend on **how you throw**, or handle, these different devices. This was especially so with coins. Some associated this with cheating; others - with experience; others could not explain why, but suggested some people are luckier than others. Specifically - quite a number of children prefer 'tails' when tossing a coin (9 of the 12 asked, as compared to 1 that preferred heads and 2 that did not prefer either), some quoting the sentence 'tails, tails, never fails'. Most of these had no rational explanation for this, with exceptions such as explaining that 'heads' sticks out, and so it is heavier, causing the 'tails' result to be the more frequent.

A few pupils used God as their major attribution, others tended to use causal attributions, and a few used chance a lot. As it will be shown later, these seemed to be types of reasoning modes, used along a lot of their responses. But many pupils, perhaps the majority, could not be described as revealing a consistent type of reasoning, but rather used a mixture of types.

3.3 Experience

Pupils' experiences seemed to be generally quite similar, regarding probability. All pupils play board games with dice and computer games, few mentioned card games. There were differences in the amount these games were played, but these didn't seem to be an important factor in explaining differences in probabilistic thinking. Perhaps experience that you do not reflect upon or discuss with someone does not influence concept building as much as could be assumed. For example, a pupil describing himself as his football team's 'coin flicker' seemed to have a lot of experience in flicking coins, but this did not diminish his belief that he could manipulate the coins' results (even though when trying to show his abilities in the interview - he succeeded about half the time... but he was still proud of himself!). Even so - experience might influence the strength of an '**availability**' heuristic, discussed in the next section.

3.4 Probabilistic thinking

Many of the interviews revealed application of **heuristics**. These sometimes caused the children to see common random devices in a non normative way. Many children remembered from their experience with board games waiting a long time for a '6' on the die, often needed to begin a game. This made them conclude that it is harder to get a '6' than other numbers (17 pupils). This could be thought of as an example of the '**availability**' heuristic (Kahnemann et al,1982). Sometimes this view did not emerge immediately, but only after some probing.

Another common heuristic used, mainly in the context of coins, can be classed as '**representativeness**' (14 pupils). For example, some expected that when you toss two coins you will mostly get one 'heads' and one 'tails' (rather than two 'heads' or two 'tails').

Some children used equiprobability - but did so automatically, reflecting the '**equiprobability bias**'. For example - when asked about a raffle in a class comprising 16 girls and 13 boys, 10 out of 23 interviewees thought a girl was just as likely to win as a boy. Part of their explanations seemed to indicate a view that chance is 'naturally' equiprobable.

The '**outcome approach**' was common in the interviews. In fact - a majority of the children revealed this approach to some extent. For example - 16 out of 21 interviewees answered that if after a prediction of 70% chance of rain it did not, in fact, rain - then the probabilistic prediction was wrong.

3.5 Possible links between culture and probability

A crucial question in this study is - does culture play a significant role in the building of a child's probabilistic thinking? Although, due to the nature of the interviews, this is not easy to generalise definitely, we claim that such a role exists. Examples of influences are:
6 children with a high level of **superstitiousness**, leading to views of dice and coins as not

equiprobable, without any explanation, and to crude, unrefined probabilistic thinking.
A child with a strong influence of **religion** (see discussion below).
3 children with a strong tendency to **causality and determinism**, in areas normatively seen as involving randomness.
A child whose world of chance was dominated by **tricks and suspicion**.
A child with a view of chance that was extremely **'equiprobable'**, as in Lecoutre (1992).

This is not to say that these views are the main influence on the child's probabilistic thinking. As shown in previous research (Green, 1982), **ability** is by far the major factor effecting probabilistic thinking. But beliefs do seem to be an additional factor, leading to several **types of reasoning**. The types that have been presented are probably not all existent types, and most children probably reflect not a 'pure' type of thinking, but some kind of individual combination. But this might be the **link** between culture and probabilistic thinking.

We will now present one detailed example of this type of possible link between a belief system and probabilistic thinking. In interview no 2 **religion** seems to have a strong impact on the boy's feelings toward chance. G. was born in England, and so were his parents. G's father is head of a Nazarene theological college, serving in the past as a missionary in Africa, so G. spent ages 2 to 5 in Africa. The result of these influences appears to be a mixture of rational and irrational elements in his interpretation of events:

When I was in Africa a poisonous snake almost killed my sister. By chance I came outside so I could tell someone. But it was not luck, it was God. He saw that my sister didn't die. That wasn't chance... Fruit machine, that's chance. I don't think God would like that..
I don't gamble... I'm against it. You just get ripped off. Most places are fixed.

So some things may look like chance, but are actually acts of God. Others - have a touch of evil in them. This could explain G.'s answers about probabilities, revealing a mixture of fairly correct knowledge with irrational ideas. Dice are seen on one hand, quite rationally, as unpredictable.

You can't say what you will get. I sometimes get a 6 straight away, sometimes I wait for ages.

But on the other hand :

My sister is very lucky. She gets 6 eight times in a row... With two dice: there is more chance to get different numbers. But my mum is good in getting the same number.

So G. seems to hold the irrational belief that certain people (the 'good' people?) have better chances than others when throwing dice.
In a question involving combinatorics G. reveals quite advanced combinatorics reasoning, but still gives an unexplained preference:

I.: What number would you bet on when summing results of two dice, a 3 or a 6?
G.: I'd bet on a 6, because there are three possibilities to get it (3,3 ; 4,2 ; 5,1) and 3 has only one chance (2,1).
I.: And if you'd have to choose between 3 or 11?

G.: It'll be difficult because both have one possibility. But I prefer 5,6.

So, to sum up, G. reflects a duality of rational with irrational ideas that seem to be influenced by religious belief. The hypothesis that probabilistic reasoning is influenced by religious belief will be one of those investigated in the second stage (questionnaires) of the research.

4. Conclusions

1. We have found that certain intuitions, approaches, biases and heuristics noted in the literature take a strong and common form in 11 year old thinking, e.g. the 'outcome approach', 'representativeness', 'availability', 'equiprobability'. Coins and dice may not be understood by some children as random devices.

2. We have instances where individuals' reasoning appear to be related to their beliefs about the world and events. A few stereotypes of religious, superstitious, causal and suspicious thinkers emerged.

Further research might add to the list, test the reliability of these stereotypes and discover their frequency in the population. We might also correlate the strength of these beliefs with the different measures of probabilistic thinking, approaches and heuristic reasoning.

References

Borovcnik, M. & Bentz, H. J. (1991). Empirical Research in Understanding Probability. In R. Kapadia & M. Borovcnik (eds) (1991). *Chance Encounters: Probability in Education.* Dordrecht: Kluwer.

Buscaglia, J. (1987). The Development of Thinking Skills Within the Cultural Identity of Latin America. In D. N. Perkins, J. Lochheads & J. Bishop (eds), *Thinking: The Second International Conference.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Donaldson, M. (1978). *Children's Minds.* Glasgow: Fontana Press.

Fischbein, E. (1975). *The Intuitive Sources of Probabilistic Thinking in Children.* Dordrecht : Reidel.

Fischbein, E. (1987). *Intuition in Science and Mathematics: An Educational Approach.* Dordrecht: Reidel.

Fischbein, E. & Gazit, A. (1984) Does the teaching of probability improve probabilistic intuitions? *Educational Studies in Mathematics, 15,* 1-24.

Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Estimating Odds and the Concept of Probability. *Educational Studies in Mathematics, 22,* 523 - 549.

Green, D. R. (1982). A Survey of Probability Concepts in 3000 Pupils Aged 11 - 16. In D. R. Grey, Holmes, V. Barnett & G. M. Constable (eds), *Proceedings of the First International Conference on Teaching Statistics.* Sheffield.

Hart, K. M., (1981). *Children's Understanding of Mathematics: 11-16.* London : John Murray.

Holland, D. & Quinn, N. (1987). *Cultural models in language and thought.* Cambridge: Cambridge University Press.

Kahnemann, D., Slovic, P. & Tversky, A. (eds) (1982). *Judgment under Uncertainty : Heuristics and Biases.* Cambridge : Cambridge University Press.

Konold, C. (1989). Informal Conceptions of Probability. *Cognition and Instruction,* 6(1), 59-98.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J. & Lipson, A. (1993). Inconsistencies in Students' Reasoning About Probability. *Journal for Research in Mathematics Education, 24(5),* 392-414.

Lave, J. (1988). *Cognition in Practice : Mind, mathematics and culture in everyday life.* Cambridge: Cambridge University Press.

Lecoutre, M. (1992). Cognitive Models and Problem Spaces in "Purely Random" Situations. *Educational Studies in Mathematics* 23, 557-568.

Phillips, L. D., & Wright, G. N. (1977). Cultural Differences in Viewing Uncertainty and Assessing Probabilities. In H. Jungermann & G. de Zeeuw (eds.), *Decision Making and Change in Human Affairs,* 507-519 Dordrecht: Reidel.

Piaget, J., & Inhelder, B. (1951). *La genese de l'idee de hasard chez l'enfant.* Paris: Presses Universitaires de France.

Piaget, J., & Inhelder, B. (1975). *The Origin of the Idea Of Chance in Children.* London: Routledge & Kegan Paul.

Saxe, G.B.(1991). *Culture and Cognitive Development : Studies in Mathematical Understanding* New Jersey : Lawrence Erlbaum Associates.

BEST COPY AVAILABLE

ACTIONS WHICH CAN BE PERFORMED IN THE LEARNER'S IMAGINATION: THE CASE OF MULTIPLICATION OF A FRACTION BY AN INTEGER

Ilana Arnon, Ed Dubinsky, Perla Nesher

Abstract

Two learning theories, both based on the work of Piaget, are brought together to contribute to curriculum development, research, and teaching mathematics to children of 7-12 years of age. The theory of Learning Systems (Nesher, 1989) provides the curriculum developer with a structure of learning units, using physical objects to teach mathematics. The question "How does the learner become independent of the physical objects?" remains without an explicit answer in this theory. We suggest that the scheme of learning: Action - Process - Concept (Dubinsky, 1991) provides an answer to this question. The analysis is presented via an example: fractions in grade-four.

I. Theory and the research question

Piaget (Piaget & Inhelder, 1958; Piaget, 1974; 1975; 1976) taught that at the age of concrete operations, children learn and construct abstract ideas while reflecting on actions they perform with their own hands on physical objects.

The theory of Learning Systems (**Nesher** 1989), based on Piaget's assumptions described above, provides the curriculum developer with a structure of mathematics learning units which .e physical objects. A learning unit consists of a Knowledge Domain, an Exemplification Domain, and a series of activities defined in the theory.

The Knowledge Domain consists of a collection of related mathematical concepts which one aims to teach. For example: fractions in grade four. The mathematical concepts discussed here are of four kinds: new objects (such as fractions); qualities of these objects (such as being greater than or smaller than 1); relations between the objects (such as one fraction is greater, smaller, or equal to another); and arithmetic operations on these objects (such as binary operations, e.g. addition of fractions, unary operations, e.g. the expansion of a fraction, or any n-ary operation, e.g. the average of n fractions).

The Exemplification Domain consists of the collection of physical objects which we want to use when teaching the Knowledge Domain. The collection should be carefully chosen according to two main criteria:

a) The objects should be familiar to the children in advance.

b) The collection should be as analogous as possible to the Knowledge Domain: the mathematical objects should be represented by the physical objects, on which one can check qualities and relations analogous to the mathematical qualities and relations of the Knowledge Domain. Furthermore, there exist physical actions isomorphic to the arithmetic operations of the Knowledge Domain and easy to perform in class, which represent the arithmetic operations, and are taught, at least at the beginning, instead of numeric algorithms. Such a physical action becomes the definition of the arithmetic operation it represents.

Example: a learning unit on fractions for grade four was planned according to this theory, the physical objects being ready made circle cuts representing a variety of fractions (1/2 ,..., 1/6, 1/8,...1/10, 1/12, 1/16, 1/20), n cuts for each 1/n fractior The Knowledge Domain and Exemplification Domain of this learning unit will be presented in detail in the presentation.

The Activities include three stages:

a) Activities within the Exemplification Domain, emphasizing those qualities of the objects which are relevant to the Knowledge Domain.

b) Activities which combine the Exemplification Domain and the Knowledge Domain. The mathematical language and symbols are introduced at this stage with their meaning in the Exemplification Domain: they are defined and given meaning to in the terms of the Exemplification Domain. Here is an example of a typical stage B activity: the child is faced with the arithmetic problem 1/5 * 3 =. In order to solve it, he or she takes three 1/5 circle cuts (three "times"[1]), arranges them in a consecutive array, which, according to the rules of correspondence between the Exemplification Domain and Knowledge Domain, stand for the fraction 3/5. Therefore, the pupil offers 3/5 as the answer of the multiplication problem.

c) Activities within the Knowledge Domain: this is the stage where a child acts without touching the physical objects of the Exemplification Domain. They might even not be present any more.

A research question: how does the shift from stage B to stage C activities happen? What brings about the learner's independence of the physical objects of the Exemplification Domain?

---

1) Another meaning, frequently attached to multiplication of a fraction by an integer - the meaning "part of a quantity" - is not dealt with in this research report.

Dubinsky (1991) suggests that every mathematical object is constructed by a learner as a result of an action on objects (concepts) previously constructed. The learning of a new concept starts with the "Interiorization" of the action. Interiorization is one of the five different reflective abstractions defined in this theory: Interiorization, Coordination, Reversal, Encapsulation and Generalization.

Interiorization is revealed by two major characteristics:

a) The learner's ability to perform the action in his or her imagination.
b) The learner's awareness of his or her action, including many of its details. A high degree of awareness is expressed by the learner's ability to describe his or her action verbally, and to feel free to invent examples, problems and shortcuts of his or her own.

When the interiorization of an action is completed the learner's knowledge reaches a stage called "Process" in this theory. The next stage in the development of a concept is the stage in which the learner is already able to operate new actions on the process. The process (the interiorized action) has become a concept. This stage is brought about by the reflective abstraction called by Dubinsky "Encapsulation".

II. The contribution of the Action - Process - Object theory to the theory of Learning Systems

When joining the two theories, we add a new criterion to the planning of the Exemplification Domain in the Learning System theory. If in the original planning the new mathematical objects of the Knowledge Domain were represented by physical objects, according to Dubinsky's theory, new mathematical objects have to be constructed by the learner via actions on previously constructed objects. According to Piaget, for children of ages 7-12 these objects should be physical. So, instead of starting the learning with a collection of physical objects that satisfies the above criteria, we start with an **action**: the action of producing such objects. It is the action of producing a physical object that stands for each individual mathematical object of the Knowledge Domain (for example: a fraction), and not the physical object itself. The learner will start working with ready made physical objects only after there exist enough indications to the fact that the action has been interiorized to a degree where the child knows, and can explain verbally, how to produce each object.

Example: in the unity for grade four, special tools were suggested and produced, which enable the children to draw, with satisfactory precision, circles divided into n equal parts, and shade k such parts. This action was taught as representing a fraction k/n (greater than, smaller than, or equal to 1). Other learning units, constructed for the learning of further mathematical concepts related to fractions, consisting of Exemplification Domains using collections of different physical objects, were structured according to this principle: the learning always starts with an action of producing a physical object. This action is carefully chosen so that it serves as a definition of the new mathematical object (idea) that is the main topic of the learning unit.

III. The contribution of the Action - Process - Concept theory to the research question.

An hypothesis was made: if a mathematical idea is taught via an action on physical objects, the learner will stop using these objects once he has interiorized the action to a degree that he can perform it in his imagination, no longer needing the actual touch of the physical objects. The question raised within the Learning System theory, of how the learner moves from using the Exemplification Domain in order to solve problems of the Knowledge Domain to acting exclusively within the Knowledge Domain, is answered in the following way: at the beginning, learners only **seem** to work exclusively within the Knowledge Domain. In reality, they use an imaginary Exemplification Domain: when presented with an arithmetic problem of the Knowledge Domain, they imagine the necessary physical objects, act upon them in their mind's eye, produce an imaginary physical result, and hand in an arithmetic solution (in the mathematical language, analogous to the Exemplification Domain terminology).

Do children really work that way? That is the question our research set out to find.

IV. Research Method

Two grade four classes of schools of similar children population learned fractions using two different sets of materials: booklets and sets of physical objects.

One class used materials planned according to the principles of Learning Systems: the set of physical objects consisted of ready made circle cuts representing the fractions 1/2 ,..., 1/6, 1/8,...1/10, 1/12, 1/16, 1/20, n cuts for each 1/n fraction, each circle cut carrying its appropriate fraction in print. This class is called Class A.

The second class used materials planned according to the combination of the Learning Systems ideas and the Action - Process - Concept scheme: the set of physical objects consisted of the same set of circle cuts, except that here most of them did not carry printed fractions (an important activity was to find out and write the appropriate fraction or fractions). In addition, the set included circle cuts for fractions with numerator greater than 1 (such as 3/4, 4/5). The last addition to the set of physical objects, which, in our opinion, makes the most significant difference between the two Exemplification Domains, was the tool with which to partition a circle into n equal parts. This tool enabled the learning unit to start with the action of drawing a representation of a fraction in a circle, long before the ready made circle cuts were first introduced in class. The class using this Exemplification Domain is called Class B.

Another difference between these two classes lies in the use of numeric algorithms: in Class B actions on the physical objects were the only methods introduced in class, either to solve problems, or as reference in discussions. In Class A, at the beginning of grade five, the teacher found it necessary to teach an additional method of solving problems of multiplication of a fraction by an integer: the numeric algorithm $a/b * c = (a*c)/b$.

At the beginning of grade five (after the summer vacation, and after the learning of the numeric algorithm took place in class A), the pupils of these classes were interviewed with the purpose of exposing the degree of interiorization of some of the actions they learnt in these learning units. We will describe some of the findings concerning the arithmetic operation "Multiplication of a Fraction by an Integer" .

The Multiplication Part of the Interview and The Prompt

No physical objects were present during the interview. The last question of the interview was either 1/5 * 3 or 3 * 1/5. The children were asked to explain their answers. If the pupils did not solve the problem, or if they did not refer in their explanations to the physical objects used in class, the interviewer gave what we will call The

Prompt: she suggested that the pupils try to recall and refer to the physical objects which were used in class.

V. Findings

Here are some illuminating findings of the multiplication part of the interview:

a) Many children of both classes used some numeric algorithm in solving the problem. The source of such algorithms for children who had not been taught any is a matter of further research.

b) Some children used **wrong** numeric algorithms. The prompt enabled the interviewer to find the origin of their wrong algorithm: it was always the interiorization of a wrong action on physical objects.

c) Children who did not correctly solve the problem before the prompt, but after it offered a correct answer followed by an explanation relying on the physical objects, were found in both classes: five of them in class B, but only one in class A.

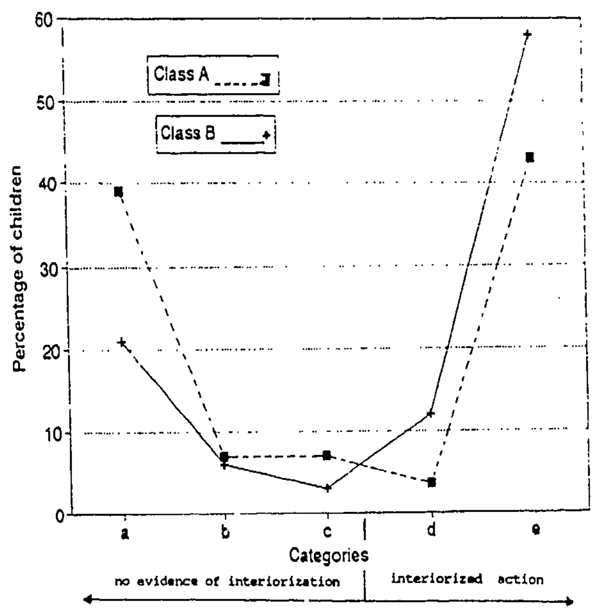
d) One girl offered a numeric algorithm, but spontaneously explained that she did not solve that way, but used an action on imaginary circle cuts.

Comparison of the two classes

The children's reactions were analyzed according to their success in solving the problem, their success to do so with the help of the prompt after having failed without it, and their ability to relate their explanations to the actions on the physical objects. Following this analysis, they were grouped into five categories described in the table:

	Category	Number and % of children class A	Class B
wrong answer or none	a) no influence of the prompt	11 (39%)	7 (21%)
wrong answer or none	b) the prompt helps to expose source of misconception	2 (7%)	2 (6%)
wrong answer or none	c) after the prompt: improvement	2 (7%)	1 (3%)
	d) no good answer before the prompt, satisfying answer and explanation after	1 (3.5%)	4 (12%)
	e) satisfying answer and explanation without the prompt	12 (43%)	19 (58%)
	total	28	33

All the children of the first three categories did not solve the multiplication problem correctly. Yet the categories differ from each other by the kind of influence the prompt had on the children. In the first category (a) the prompt had no influence on them. In the second category (b) after the prompt the children gave a physical object oriented explanation. This explanation helped the researcher to understand the source of their wrong answers. Category (c) is the first category with children on whom the prompt had a real positive influence: after the prompt they have improved their answer of the multiplication problem, although not to a completely correct answer. Category (d) is the category in which the influence of prompt is the strongest: these children did not correctly solve the problem before the prompt, but after it they offered a correct answer followed by an explanation relying on the physical objects. In category (e) the children solved the problem correctly and offered a satisfactory explanation without a prompt. The difference between the two classes is shown in the following figure:



In this figure we can see that the section point between those children who interiorized the action of multiplying a fraction by an integer and those who did not is between categories (c) and (d). The line of class B goes below that of class A to the left of this point, and above the line of class A to the right of this point: in class B, where the children worked according to the combined theories, the percentage of children who achieved some degree of interiorization of the action is higher than in class A. In terms of the research question we say that in class B, where the teaching put more emphasis on learning via actions which can be performed in the learner's imagination, more children interiorized the multiplication of a fraction by an integer to a degree where they became independent of the concrete objects (stage C activities in the theory of Learning Systems).

VI. Conclusions

We believe that the evidence collected in this research indicates that the solution suggested by Dubinsky's scheme of learning to the question raised within Nesher's framework is effective: teaching fractions to elementary school children using physical objects according to the Learning Systems framework and the Action - Process - Concept scheme seems to facilitate the child's prospect of becoming independent of the physical objects, without the need to teach and memorize numeric algorithms. Additional learning units on fractions have been developed and are being tried out in the light of this paradigm.

BIBLIOGRAPHY

[1] Dubinsky, Ed: 1991, Reflective Abstraction in Advanced Mathematical Thinking, in D. Tall (ed.), ADVANCED MATHEMATICAL THINKING, Clair Academic Publishers, Dordrecht/Boston/London, pp. 95-123.

[2] Nesher, Perla: 1989, Microworlds in Mathematical Education: A Pedagogical Realism, in L.B. Resnick (ed.), KNOWING, LEARNING & INSTRUCTION, Hilldale N.J., pp. 187-216.

[3] Piaget, J. and B. Inhelder: 1958, THE GROWTH OF LOGICAL THINKING FROM CHILDHOOD TO ADOLESCENCE, (A. Parson and S. Milgram, trans.), Basic Books, Inc., Publishers, New York.

[4] Piaget, J.: 1974, Introduction, in B. Inhelder, H. Sinclair & M. Bovet, LEARNING AND THE DEVELOPMENT OF COGNITION, (S. Wedgwood, trans.), Routland & Kegan Paul Ltd, London.

[5] Piaget, J., Piaget's Theory. In P.B. Neubauer (Ed.), THE PROCESS OF CHILD DEVELOPMENT (pp. 164-212). New York: Jason Aronson, 1975.

[6] Piaget, J.: 1976 (Original published 1974), THE GRASP OF CONSCIOUSNESS (S. Wedgwood, trans.), Harward University Press, Cambridge.

THE PROCESS OF NAMING IN ALGEBRAIC PROBLEM SOLVING

Arzarello F.(*) - Bazzini L.(**) - Chiappini G.(***)

(*) Dipartimento di Matemativa Università di Torino - Italy

(**) Dipartimento di Matematica Università di Pavia - Italy

(***) Istituto per la Matematica Applicata di Genova , C. N. R. - Italy

Summary. *This study is framed in a wider analysis of the nature of algebraic language and of its related cognitive processes, which has been carrying out by the authors. These analysis point out the difference between "sense" and "denotation" of a symbolic expression as a key element of algebraic thinking in problem solving: this distinction and the mutual interaction of the two poles are investigated more deeply. Particularly, the report focuses on the relationship between sense and denotation during the process of naming, that is the process of assigning names to the elements of the problem. An analysis of the process and of the difficulties met by students is given through examples; evidence is given about the crucial role of the naming process in determining the success of the solution.*

Introduction.

In a previous work (Arzarello-Bazzini-Chiappini, 1993) we have analysed the nature of algebraic language distinguishing between sense and denotation in an algebraic expression and we have identified the activation of frames as the driving force of algebraic thinking in problem solving. In this report we analyse more deeply the relationship between sense and denotation in the knowledge domains where algebraic representation is used. Particularly we focus on the so called process of naming, which makes possible incorporating and clarifying the sense of a problem into the system of algebraic representation.

This process is typical of algebra and constitutes a breaking away with arithmetic. Thus it is a novelty for students, as well as it represented a novelty in the historical development of the discipline. At this regard let's remind the work by Viéte, who emphasised the crucial role played by the process of predication in analytical methods ("Logistica numerosa est quae per numeros, Speciosa quae per species seu rerum formas exhibetur, ut pote per Alphabetica elementa").

In contemporary literature on algebra learning (Chalouh&Herscovics, 1988; Chevallard, 1989; Kieran, 1989) there is evidence that much work towards solution is done, once a good process of naming has been produced by the subject.

It is equally clear that many troubles are rooted in the students' incapability to construct suitable formulas, which should incorporate the meanings of the objects involved and their mutual relationships. Such difficulties have different reasons, as recent research has pointed out: for example the epistemological difference between arithmetic and algebra (Chevallard, 1989), obstacles due to the gap between procedural and relational aspects in formulas reading (Arzarello, 1991; Linchevski&Sfard, 1991; Sfard 1991), didactic cuts between levels of complexity in using formulas (Filloy&Rojano, 1989).

For many students there is an objective difficulty in coaching their stream of thinking in order to condense the most relevant aspects of a given problem into a formula.

The process of naming: a theoretical analysis

The process of naming consists in assigning names to the elements of a problem and is aimed at constructing an algebraic expression able to make explicit the sense of the problem. It implies the activation of a frame and of the dynamic relationship between sense and denotation of the algebraic expression within the frame (see below and Arzarello-Bazzini-Chiappini , 1993).

In the naming process the role of constructing and interpreting letters (variables or parameters) is crucial. Letters can be used to give a name to the extra linguistic elements (mathematical or not) which are involved in the problem and to emphasise the relationships among the elements within an algebraic expression. The choice of names to designate objects is strictly linked with the control of the variables introduced. In this process, one main difficulty, especially for novices, derives from the impossibility of sustaining the stream of thought using natural language: in fact, it is very unusual that algebraic formulas are a linear stenography of what is expressed by means of natural language.

Let us consider as an example the following problem.

Problem: *By means of a good choice of names to designate two subsequent odd numbers, show that their sum is a multiple of 4.*

Experiments carried out with students attending junior secondary school (11-14 year old) and University courses have revelled similar typologies of errors:

* $x + y$
* $2h+1+2k+1$ or $2h+1+2k+1+2$ instead of $2h+1+2h+3$;
* moreover, a high percentage of students made only arithmetical checks.

The example shows that some students, who can express the relationships among the elements of the problem using the natural language or the arithmetic code, are unable to express them suitably by means of the algebraic code. More specifically, they are unable to use the algebraic code as a mediator between the identified goals of the problem and the qualitative and quantitative relationships among its elements. In fact, they are unable to express the algebraic sense of the problem in adequate algebraic terms. By "algebraic sense of a problem" we mean the clarification of the relationships among the elements of a problem that the problem exhibits when is interpreted according to the rules of the algebraic language.

A good mastery of the algebraic code consists in the capability of incorporating the sense of the problem into an algebraic expression.

The process of naming is highly influenced by connotative aspects, when the subject stresses some specific properties, because of their adequacy with the aims of the problem. Connotation is thus highly connected with anticipatory and heuristic aspects; connotative features are important insofar as they feature the starting point of the thinking process towards the aims.

From the very beginning, good solvers have usually a glimpse of a possible path, and put it implicitly in their first trials of naming; usually they are able to incorporate the relationships among the elements and prefigure transformations apt to reach the solution. Not so for novices, who proceed more randomly while choosing and naming variables: their naming is weaker and often superficial, in any case not linked with anticipatory aspects but rather influenced by rigid stereotypes.

Sometimes the process of naming is impoverished or blocked when the subject constructs or interprets symbolic expressions in a rigid way, without understanding the complex and flexible relationship between sense and denotation. As a consequence, this subject usually does not grasp the potential of the algebraic code, i.e. the possibility of incorporating different properties within the names. The name becomes a rigid designator and may be an obstacle to algebraic reasoning since it inhibits flexibility, which is a basic support of the functions underlying the use of algebraic language.

In order to better understand such difficulties, we need a more detailed analysis of the meaning of algebraic expressions.

In our previous work (Arzarello-Bazzini-Chiappini, 1993, 1994) we have referred to Frege's semiotic triangle to distinguish between sense and denotation of an algebraic expression. More precisely, we call "denotation" of a symbolic expression in algebra the number set that is represented by the expression. It is determined by the symbolic expression and by the universe in which the expression is considered (for example the equation $x^2+1=0$ denotes the empty set when it is considered in **R** and the set {+i, -i} when considered in C).

On the contrary, the sense consists in the way the denoted set is given through a symbolic expression. In Mathematics, as well as in natural language, it is not difficult to find expressions whose senses are different but have the same denotation: for ex., both "the least prime number" and "the least even number" denote the same number, even if their sense is different.

Moreover, the same expression may have different senses.

A first one is given by the way the algebraic expression incorporates in its signs a computational rule, by which the denoted set can be obtained: we call it "algebraic sense of an expression". For ex. the expressions 4x+2 and 2(2x+1) mean different computational rules, and thus they have different algebraic senses, but denote the same number set.

But in a symbolic expression it is also relevant its reference to a specific knowledge domain: we call it "contextualized sense". In this frame the contextualized sense of an expression can be seen as a mapping between the culture of the knowledge domain under consideration and the syntactical structure of the expression, where its algebraic sense and its denotation are incorporated . A given expression may have different contextualized senses, without that its algebraic sense changes: for example $2(x+1)$ may be interpreted as the double of a number or as the area of a rectangle whose sides measure 2 and $x+1$.

Typically, in algebraic problem solving, the same expression can be used to solve different problems: its algebraic sense is the same but its contextualized sense may change very deeply. That's why the process of naming is crucial: in fact it is successful insofar it allows to grasp the algebraic sense of a problem by the algebraic sense of an expression, which reflects the meaning of the problem by means of a suitable contextualized sense. From this point of view, our use of such notions as sense and denotation is near to Vygotsky's distinction between "reference" and "meaning" (1962). In fact, Vygotsky's theory of concept development shows how the child's grasp of the "meaning" or "mode of presenting" of words changes from purely perceptual criteria to purely conceptual or symbolic features. This distinction is critical for understanding how the levels of generality expressed by words contribute to the levels of generality expressed by a sentence. Similarly, we can speak of levels of generality of symbols and expressions in the language of algebra, namely when naming is used in algebraic expressions in order to generate a contextualized sense suitable to grasp the sense of the problem.

The process of naming: an experimental study

This study has been designed to investigate the role of naming in the solution process. We have conjectured that the dynamics made active by students while solving problems highly depend on the process of naming and on the connotative aspects which characterise such process.

A sample of 70 undergraduate students in Mathematics at the University of Turin and a sample of 67 students (age range: 16-18) attending a scientifically oriented high school (Liceo Scientifico) in Pavia have been engaged for the study.

All students have been required to record their reasoning while solving problems as faithfully as they could. They have been informed that their findings would not be assessed according to the common standards at school.

Here is an example of a problem, which has been administered to the students individually:

Show that if you add to a number of four figures the number obtained reversing the order of figur : (ex. 1235 + 5321) you get a multiple of eleven.

Strategies have been classified according to the types of naming and the activated frame as shown below. We have adopted the notion of frame, as introduced in Artificial intelligence: i. e. a structure of data that is stored in memory and is able to represent generic concepts or stereotyped situations.

Types of naming:

N1 The position value of the figures is pointed out by means of the polynomial representation ($abcd = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$)

N2 The position value of the figures is not pointed out (the number is represented as abcd).

N3 Others

Types of frame:

F1 Multiple of 11; **F2** Criteria of divisibility by 11;

F3 Divisibility of polynomials; **F4** Arithmetical checking

If both the process of naming and the activation of frames are taken into account, four major strategies are identified:

S1 (N1 ; F1) or (N1; F4+F1) ; **S2** (N1; F3) ; **S3** (N2; F2) or (N2; F4+F2) ; **S4** (N3; F4)

Here we report some typical protocols of these strategies:

<u>TYPE S1</u>

$abcd = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$

$dcba = d \cdot 10^3 + c \cdot 10^2 + b \cdot 10 + a$

$abcd + dcba = (a+d)\, 1000 + (b+c)\, 100 + (c+b)\, 10 + (d+a)\, 1$

$(a+d)(1000+1) + (b+c)(100+10) =$

$(a+d)1001 + (b+c)110$

---> I have found that (a+d) is multiplied by 1001, which is a multiple of 11 and also 110.

---> I think I have found the solution .

TYPE S2

$a = A \cdot 10^3 + B \cdot 10^2 + C \cdot 10 + D$

$b = D \cdot 10^3 + C \cdot 10^2 + B \cdot 10 + A$ *I try to write the number in this form, otherwise if I write a=ABCD, b=DCBA I cannot add*

$a+b = (A+D)\, 10^3 + (B+C)\, 10^2 + (C+B)\, 10 + (A+D)$

$(A+D)\, 10^3 + (B+C)\, 10^2 + (C+B)\, 10 + (A+D)$ | $10+1$

$-(A+D)\, 10^3 - (A+D)\, 10^2$ | ---

--- | $(A+D)100 + [(B+C)-(A+D)]10 + (A+D)$

$[(B+C) - (A+D)]\, 10^2 + (C+B)\, 10$ |

$- [(B+C) - (A+D)]\, 10^2 - [(C+B) - (A+D)]\, 10$

$(A+D)\, 10 + (A+D)$

$- (A+D)\, 10 - (A+D)$

// *the remainder is zero: so a+b is a multiple of 11*

TYPE S3

[The student checks the property with numbers]

1235 *I try to see if there is any simmetry*

5769 *I try with numbers >5*

I have simmetry only if the sum of the figures in each number is <10

a b c d

d c b a

....

(a+d) (b+c) (c+b) (a+d) if (a+b) and (b+c) <10

(a+d) + (c+b) = (b+c) + (a+d)

If I have 4 figures in the result the sum of the first and of the third figure equals that of the second and of the fourth figure.

The strategy classified as Type S4 is exclusively based on arithmetical checks, which, taken by themselves, do not stimulate a more general reasoning.
The following table compares the type of strategy adopted and the percentage of success.

	University Students		High School Students	
	Chosen	Success	Chosen	Success
S1	33%	96%	73%	100%
S2	8%	83%	0%	
S3	32%	0%	18%	0%
S4	18%	0%	9%	0%

Discussion

The strategy S1 is the most common in both samples, even if the percentage of high school students overcomes that of university students. We observe here a good naming which allows the solution quite easily. Also the strategy S2 is successful : it shows a good naming and a good mastery of anticipatory thinking. It was only adopted by a small percentage of university students and it is absent in the sample of high school students.

The strategy S3 and S4 were totally unsuccessful for both samples.

In the example reported above (type S3) the student has the flexibility necessary to remind possible carries, but the bad naming (which does not make decimal powers explicit) and the anticipation of some possible symmetry inhibits her from guessing a possible solution.

The results we got show very clearly the crucial role of naming in the solution of the problem. As conjectured, a good mastery of this process is linked to anticipatory aspects and allows the subject to orient the solution process in view of the aim of the problem. On the contrary, rigid designators can hinder or block the process. In our case one main block was constituted by the number written as a sequence of letters without any connection with their decimal value. In no case this led to success.

It is interesting to observe that the two frames F1, F2 are somewhat similar and correlate sensibly with the type of naming: in fact N1, N2 facilitate respectively the frames F1, F2.

The former (namely strategy S1) allows students to enter into the problem in a very transparent way. The latter (namely strategy S3) drives students towards the use of such automatic rules as criteria of divisibility by 11. In this case the semantic control is less easy. In fact the reasons why one must add the odd and even figures to check if a number is divisible by 11 are very obscure for most students.

The results we obtained in the above mentioned problem are confirmed by the analysis of solution processes in other problems. Rigid designators show the student's incapability to use algebraic expressions as flexible thinking tools; as such, the algebraic code loses its representative function. Hence students do not see the different senses of formulas and use them in a purely instrumental way.

Conclusions

The experiments we have carried out at pre university and university level have revealed that many students still have difficulties in the use of algebraic code, even when facing easy problems. On the ground of these persisting difficulties, we have come to the

conclusion that learning how to use the algebraic code correctly and meaningfully could only be the result of a long cognitive apprenticeship during which the student learns to separate the representative function of the algebraic code from the purely instrumental function. Then he learns how to incorporate senses in algebraic terms and expressions, senses which have to be adequate with the problem situation.
Of course, we should not forget the risk that the teacher and the learner interpret algebraic expressions in different ways. On the one side, the teacher combines sense and denotation according to the problem situation and is able to use the representative function of algebraic language as a semiotic mediator. On the other side, the beginning learner is not able to master suitably the representative function of algebraic language, that remains at a very instrumental level. In short, he does not control the sense of expressions and uses names as rigid designators. In this phase the risk of misunderstanding between teacher and learner is very high and can easily induce misunderstandings in learning.

Taking this issue into account should be the basic ground for designing activities oriented towards a meaningful learning and use of the algebraic code.

References

Arzarello F., 1991, Procedural and relational aspects of algebraic thinking, *Proc. of PME XV*, Assisi, Vol 1, 80 - 87.

Arzarello F., Bazzini L.,Chiappini G.P., 1993, Cognitive processes in algebraic thinking: towards a theoretical framework, *Proc. of PME XVII*, Vol I, 138-145, Tsukuba, Japan.

Arzarello F., Bazzini L.,Chiappini G.P., 1994, *L'algebra come strumento di pensiero. Analisi teorica e considerazioni didattiche*, Progetto strategico TID del CNR, Quaderno 6, NRD, Pavia.

Chalouh L., Herscovics N., 1988, Teaching algebraic expressions in a meaningful way, In Coxford, A. F., *The ideas of Algebra, K- 12*, 1988 Yearbook , NCTM, Reston,Va.

Chevallard Y.,1989, Arithmetique, Algebre, Modelisation, *Publications de l'IREM* d'Aix-Marseille.

Filloy E. and Rojano T., 1989, Solving equations, the Transition from Arithmetic to Algebra, *For the Learning of Mathematics, Vol* 9, 19 - 25.

Frege G., 1892, Über Sinn und Bedeutung, *Zeitsch. für Philosophie und Philosophische Kritik*, 22 - 50.

Kieran C., 1989b, The Early Learning of Algebra, A Structural Perspective, in Wagner,Kieran, (eds.), *Research Issues in the Learning and Teaching of Algebra*, LEA, National Council of Teachers of Mathematics.

Sfard A. ,1991, On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics* , Vol. 22, 1 - 36.

Vygotsky L.S., 1962, Thought and language, MIT Press, Chicago

An Investigation of Different Approaches to Using a Graphical Spreadsheet

Richard Beare

Department of Science Education, University of Warwick, Coventry CV4 7AL, U.K.

Abstract

An experiment is described involving students using a graphical spreadsheet in three different ways, labelled 'prescriptive', 'constructivist', and 'open-ended'. The clear superiority of the constructivist approach is described and theoretical reasons for this are discussed. Specific difficulties that students have with using graphical spreadsheets are briefly described and discussed. Recommendations are made regarding the optimum educational use of graphical spreadsheets.

Introduction

Spreadsheets with advanced graphical capabilities have been around for several years now and their educational potential in a range of subject areas has been widely recognised, (e.g. Beare, Hewitson and Stephen, 1993, Healy and Sutherland, 1991). Apart from their graphing capabilities spreadsheets are characterised by the dynamic represention of algebraic relationships and the ability to replicate formulae in order to represent iterative processes. Amongst the perceived pedagogic benefits of spreadsheets as learning tools is their ability to support a variety of learning styles including, especially, open-ended investigative learning, in which the student has a large measure of control and ownership of the learning process, (e.g. Beare, 1992).

The most important criterion for using software for aiding learning is that it actually *aids learning* in some way, and with an investment of time, money and computer resources that makes it all worthwhile. We therefore need to know: 'In what pedagogic situations can spreadsheets enhance specific types of learning?' and 'How can spreadsheets best be used in these situations?' This paper describes an experiment to measure the benefit of using one particular graphical spreadsheet as a simulation, and to compare three quite different ways of using it. However many of the conclusions are relevant for spreadsheets in other contexts and also for other types of software with graphical output.

Description of the Experiment

A spreadsheet was designed to compare two different types of growth pattern for biological species, known as r-selection and K-selection. It was used with 56 first year teacher training undergraduates on an 'Ecosphere' course whose main subjects of study were either Biology or Geography. None of the students had studied the topic previously and it therefore made a good choice for the experiment since no students had prior knowledge which would have influenced the experimental results. Before working with the spreadsheet the students had a one hour lecture from their usual lecturer on r- and K-selection. They had been randomly assigned to three different seminar groups, and were given different sets of instructions for using the spreadsheet (loosely) labelled 'prescriptive', 'constructivist' and 'open ended'. The students had to give written answers to the questions on their worksheets and these were analysed together with their answers on a pre-test and an identical post-test, to learn as much as possible about how they used the spreadsheet and how their cognitive understanding was developing.

It is necessary to outline a few of the ideas involved. See any Ecology textbook for details (e.g. Krebs 1985). *r-selecting* species are 'opportunist' and rapidly increase in numbers in favourable conditions, e.g. aphids (like greenfly) in a greenhouse. Typically they are small in size, live a short time, reproduce rapidly devoting a lot of energy and material to reproduction, and are poor competitors for resources such as food and space. *K-selecting* species reproduce relatively slowly, but out-compete other species, producing a stable equilibrium population. They are sometimes called 'equilibrium species', typically are larger in size, live longer and devote less energy and material to reproduction. Oak trees and elephants might be good examples.

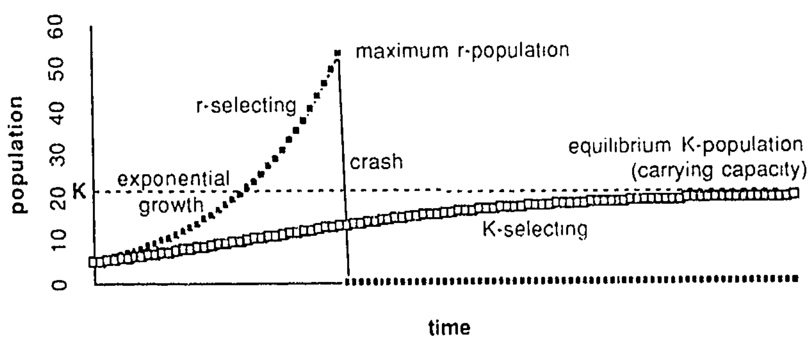


Figure 1: idealised growth curves for r and K-selecting species

Idealised growth curves for the two types of species are shown in figure 1. The growth curve for an r-selecting species is exponential at first, finally crashing when circumstances change (e.g. temperature suddenly drops). The growth part of the curve is described by the differential equation $\frac{dN}{dt} = rN$ where r is the growth rate constant and N is the population density. The idealised growth curve for a K-selecting species is given by the equation $\frac{dN}{dt} = rN\left(\frac{K-N}{K}\right)$ where K is a positive constant called the 'carrying capacity'. Initially the growth is almost exponential but it soon slows down as individuals compete for resources such as food or space and a final equilibrium population density equal to K results.

The spreadsheet calculated idealised growth curves for an r-species and a K-species. These were labelled 'A' and 'B' and and plotted on the same axes. Students could alter the values of r for both species, the population at which the r-species crashed, and the value of K for the K-species. The spreadsheet was written using the *Warwick Spreadsheet System* (based on *Microsoft Excel*) (Beare,1993) and this enabled the students to superimpose different growth curves quickly and easily.

During the lecture the students had been introduced to the biological differences between the two types of species and their different growth strategies and growth curves, but without any mathematical explanation or analysis. At this stage 'r' and 'K' for them were simply descriptive labels for two different types of species. Tall (1989) has suggested a useful term for analysing the conceptual frameworks or cognitive structures involved. He says: *We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes........ As the concept image develops it need not be coherent at all times........* In the present case elements of both biological and mathematical understanding would be needed to form a complete concept image of r and K-selecting species. The spreadsheet was intended to help develop the more quantitative aspects and to help make links between different conceptual components, biological and graphical, (but without any knowledge of the underlying differential equations used).

K-selection

(1) Imagine a population of bluebells in a deciduous wood. When first observed they have an average density of 1 plant per square metre. Assuming that the bluebells are K-selecting and the wood has a carrying capacity K of 12 bluebell plants per square metre sketch the kind of population growth curve that you might expect using the axes below. Label your curve 'A'.

(3). Suppose that the carrying capacity K is limited by the available space. Sketch a second curve to show the bluebells would grow in numbers if half the available space was already occupied by daffodils. Label this curve 'B'.

Table 2: Extracts from the identical pre-test and post-test.

To make the students *apply* the ideas involved, rather than just recall information, the questions on the pre-test and identical post-test related to two specific (idealised) situations that were different to those the students 'investigated' using the spreadsheet. For the r-selecting and K-selecting species respectively, these were aphids in a greenhouse and bluebells in a deciduous wood, rather than dandelions on a recently cleared plot of land and elephants in the bush. This prevented students transferring what they remembered from the spreadsheet to the post-test answers without thinking about the conceptual ideas involved. Two typical questions on the test are shown in table 2.

The students worked in pairs at the computer so that they would have to clarify their ideas by communicating with each other. Students could alter parameters such as r and K on the spreadsheet and observe the resulting changes on the growth curves and they could also superimpose different curves. Each of the three groups of 18 to 20 students was given a worksheet for use with the spreadsheet with a different set of instructions so that although they all investigated the same aspects of the growth curves they had to take different approaches. The specific wordings at the top were:

Prescriptive: Go through the numbered instructions and questions in order. Record any answers required in the spaces provided.

Constructivist: Before carrying out each investigation I would like you to try and predict the outcome of it and give reasons for your prediction. Record these things in the spaces provided. Afterwards record your findings. If they differ in any way from your predictions try and modify your explanation to account for the differences.

Open-ended: Before carrying out each investigation think about how you are going to carry it out. In some cases the approach might be quite obvious; in other cases it may require some thought. After carrying out the investigation give a short explanation in the spaces provided of how you went about the investigation and what you expected to find, what you actually found out and what your conclusions were.

Having to record observations on worksheets which were subsequently collected in and analysed was important. It directed students' attention to specific aspects of the situation being modelled and ensured they actually observed them. Analysis of worksheet answers in conjunction with the pre- and post-test answers showed what students actually did with the spreadsheet and gave useful insights into individual students' cognitive structures and processes as will be made clear below.

Discussion of Prescriptive, Constructivist and Open-ended Results

The prescriptive group were told what values to try, and asked to answer specific questions, such as *'What final equilibrium density does the elephant population reach?'* The constructivist group and the open-ended group were given identical questions to investigate, e.g. *'Investigate the significance of the carrying capacity K for the growth curve [for elephants]'*. Unlike the prescriptive group they had to decide for themselves what values to try and what to look for. The constructivist group had to think about what they expected to happen first and then make a prediction. They then recorded their observations using the headings *prediction, reasons, observations, modified explanation if any differences*. The open-ended group on the other hand did not have to make specific ***reasoned*** predictions. Their observations were recorded using the headings *how we went about the investigations and what we expected to find, what we observed, our conclusions.*

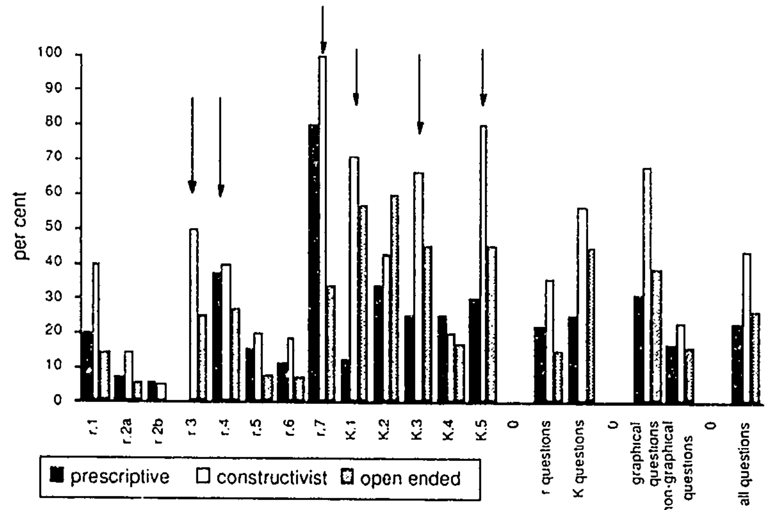


Figure 2: Percentages of students increasing their scores on initially wrong questions. Graphical questions are arrowed. Half right answers count 0.5.

	Prescriptive Group	Constructivist Group	Open-ended Group
Graphical questions	31 (41)	68 (34)	39 (50)
Non-graphical questions	17 (69)	23 (81)	16 (87)

Table 3: Overall percentage increases for initially incorrect graphical and non-graphical questions within the three groups (sample sizes in brackets).

Of the total of 13 pre and post-test questions, 6 (3 r and 3 K) tested *graphical* aspects of understanding by asking the students to sketch growth curves, and 7 tested *non-graphical* aspects (e.g., the environmental or biological factors that might influence K for bluebells in a wood). Answers to individual questions for each student were graded 1 (for correct), 0 (for incorrect) and 0.5 (for partly correct). A spreadsheet was used to calculate the percentages of students in each group getting each individual question correct ***having previously got it incorrect.*** In this way, the effectiveness of the spreadsheet exercise in improving understanding could be compared question by question for each group separately, taking into account only those students who did not initially know the correct answer. In this way inherent differences between the 3 groups were largely eliminated. See figure 2 and table 3.

Overall the spreadsheet exercise was quite successful in promoting understanding of growth curves. On *graphical questions* the constructivist group improved about twice as much (68%) as the other two groups, with the open-ended group (39%) doing a little better than the prescriptive group (31%). On *non-graphical questions* the increases were less, as one would expect, but with the constructivist group still faring best, although by a smaller margin. The clearest message emanating from this is the superiority of the ***constructivist approach*** to using the spreadsheet as compared with the prescriptive and open-ended approaches. The key element in this is making a ***reasoned prediction*** before each investigation. In order to do this they had to ***reflect on their own understanding***.

This is illustrated in figure 3. The broken lines indicate the traditional pedagogical methods of observing, scientific experimention and using analogies in teaching understanding of phenomena in the real world. The thin solid lines refer to using a graphical spreadsheet as a simulation to teaching scientific understanding. Assuming that students are able to interpret the spreadsheet-generated graphs correctly, the end result of this process is *knowledge* of how the real world behaves in specific instances (e.g. a bluebell population in a particular habitat) rather than *understanding*.

For students to *understand* this behaviour in terms of more fundamental knowledge and understanding it is necessary for them to *reflect on their own understanding*. This is indicated by the thick solid lines in figure 3. The constructivist group were forced to do this through the *predict with reasons - observe - explain any differencies* sequence. The open-ended group were also asked to say what they thought would happen, but without basing their predictions on any reasoning. They could simply guess without any thought being given as to *why* the graphs would change in a particular way and given the natural human tendency to take the easiest path this is exactly what they did in almost all cases as their written worksheet answers showed. The prescriptive group were able to simply observe without any thinking at all.

Graphical and other specific difficulties experienced by some students

Six specific difficulties experienced by some students were discovered from analysis of their worksheet answers and those on the pre-test and post-test but space only permits a brief mention of these. Details will be found in Beare (submitted). These difficulties can all interfere with the effectiveness of a graphical spreadsheet as a learning tool and this has been indicated by the question marks in figure 3. The first

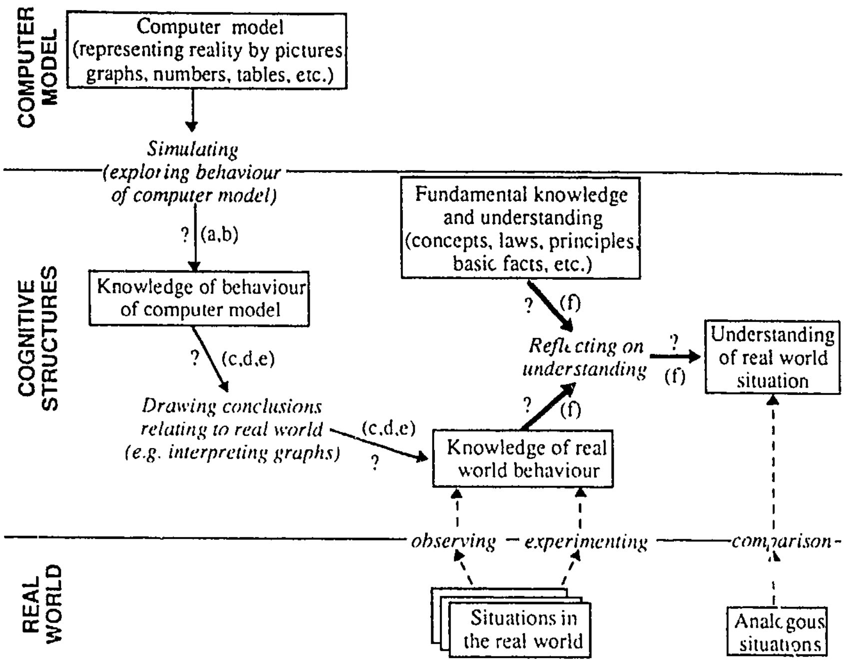


Figure 3: Cognitive Processes Involved in Using Simulations as Learning Tools (letters in parentheses refer to students' difficulties mentioned in the text)

two difficulties are graphical in nature and can prevent students observing adequately the behaviour of the computer model: *(a) difficulties with observing, recalling and interpreting the shapes of graphs* and *(b) difficulties in recognising the quantitative features of graphs*. Much of the research on children using graphs has either concentrated on the mechanics of graph-plotting and reading values rather than the interpretation of graphs as a means of communication, or it has introduced distracting extra problems of interpretation, (e.g. Kerslake, 1981 and Bell et al., 1985). The next three difficulties are ones students have developing higher-order more generalised abstract concepts from specific situations and these can prevent students drawing conclusions about the behaviour of real world phenomena from the behaviour of a computer model: *(c) taking graphs at face value rather than interpreting them in terms of the underlying processes they are intended to represent, (d) difficulties in generalising graphs from specific numerical examples and (e) difficulties with deriving generalised concepts from specific situations.* Finally: *(f) a lack of adequate basic understanding of the underlying scientific ideas* prevented some students reflecting adequately on the processes being modelled by the computer so that their understanding could develop.

Conclusions

Figure 3 makes it clear that a number of separate processes are involved in the effective use of a simulation based on a computer model. The experiment described in this paper pinpoints some of the ways in which these processes may fail to occur and shows the steps needed to ensure that they do in fact happen so that effective learning takes place. First of all the ***constructivist approach*** with a *predict with reasons - observe - explain any differences* sequence helps force students to reflect on their understanding of how the how the real world behaves in terms of more fundamental knowledge and understanding. Secondly specific steps need to be taken to ensure that the specific difficulties mentioned previously are circumvented: *(i) help and practice with interpreting the shapes of graphs correctly, (ii) help and practice with the quantitative interpretation of graphs, (iii) help and direction in making the links between the understanding of a phenomenon and its graphical representation,* (iv) *help with identifying generalised patterns from specific examples, and (v) a pedagogic approach that links use of a computer simulation to understanding of what it represents (either beforehand afterwards or concurrently).*

References

BEARE, R.A. (1992) Software Tools in Science Classrooms. *Journal of Computer Assisted Learning. 8, 221-230.*

BEARE, R.A. (1993) *Warwick Spreadsheet System,* (software) (Amo Publishing, Mill House, Mill Lane, Carshalton, Surrey SM5 2WZ. U.K.)

BEARE, R.A. (submitted) An Investigation to Evaluate the Educational Effectiveness of a Graphical Spreadsheet and Different Approaches to Using It. (copies available from Mathematics Education Research Centre, University of Warwick, Coventry CV4 7AL, U.K.).

BEARE, R.A., HEWITSON, J.F. and STEPHEN, M.J. (1993) *Warwick Spreadsheet System: First Physics and Astronomy Pack, First Biology Pack, First Chemistry Pack* (Amo Publishing, Mill House, Mill Lane, Carshalton, Surrey SM5 2WZ, U.K.)

BELL, A., BREKKE, G. and SWAN, M. (1985) Diagnostic Teaching 4: Graphical Interpretation and 5: Graphical Interpretation: Teaching Styles and their Effects. Shell Centre for Mathematics Education, University of Nottingham.

HEALY, L. and SUTHERLAND, R. (1991) *Exploring Mathematics with Spreadsheets.* (Blackwell)

KERSLAKE, D. (1981) Graphs. In HART, K.M. (ed), *Children's Understanding of Mathematics: 11 - 16,* pages 120-136. London, John Murray.

KREBS, C.J. (1985) *Ecology: the Experimental Analysis of Distribution and Abundance, (3rd ed.).* New York: Harper and Row.

TALL, D. (1989) Concept Images, Generic Organizers, Computers and Curriculum Change. *For the Learning of Mathematics,* 8, 3 (Montreal)

DEVELOPING A COMMUNITY OF RISK-TAKERS[1]

Terry Peluso, Cambell Union High School District, San José, CA USA

Barbara J. Pence

Joanne Rossi Becker

San José State University, San José, CA USA

This paper provides a small scale evaluation of a high school mathematics teacher staff development project which aims to support teachers as their schools implement de tracking and put all ninth grade students in algebra I. Two forms of data are reported here: results from three administrations of an Instructional Practices Scale (n = 27) and results from a two hour focus discussion group with five teachers from the first year of the project. Both forms of data are used to identify those aspects of the program which have had the most impact on teachers' beliefs and classroom practices. Areas in need of further development to support teachers as they implement innovations in curriculum and instruction are discussed.

Background

This study is part of an ongoing series of investigations by Becker and Pence (1990a, 1990b; unpublished manuscript) about inservice education for secondary teachers. Although there is a large body of literature in the areas of learning and teaching mathematics, little is known in the field of teacher education in mathematics. In the last edition of the Handbook of Research on Teaching, the chapter on mathematics education (Romberg and Carpenter, 1986) hardly mentions research on teacher education. As Grouws (1988) pointed out, there is little information available about the overall design features of inservice education programs which maximize changes in teacher beliefs and classroom behavior. Grouws has called for studies which focus on the impact of various features of inservice education on classroom practice. In previous work, Becker and Pence have found that although changes in teacher beliefs about mathematics and its teaching may be necessary for changes in teacher behavior to occur, such changes may not be sufficient. This paper extends that previous work and includes a new methodological approach to better ascertain the impact of the program on participants.

Program Description

The project discussed here, "Building Bridges to Mathematics for All," is an extensive staff development project with four main components: an intensive summer institute of 13 days; five days of followup sessions during the academic year; classroom coaching; and materials purchase for participant teachers. The project is part of a collaboration with Equity 2000, a College Board initiative which aims to increase the attendance and graduation rates of students traditionally underrepresented in college in the US: students of color and low income students. In a study of students' college-attending rates, Pelavin and Kane (1990) found that, among students who had

[1] This research was supported by a grant from the California Postsecondary Education Commission under the Eisenhower program for mathematics and science education. The views expressed here represent those of the authors alone.

completed geometry in high school, college attending rates of Hispanics and African-Americans were very close to those of European-Americans. Therefore one of the strategies Equity 2000 is pursuing in cooperating schools (16 schools) is to have all ninth graders (11,000) enroll in at least algebra 1 by 1994.

As the schools implement this bold initiative, it is clear that staff development for teachers is just one component of the support needed. Figure 1 shows the other components of the Equity 2000 program. In this paper we are concerned with the staff development done with the high school teachers; this is outlined in Figure 2.

The goals of the staff development include the following:

• Teachers will learn new ideas about how students learn mathematics. They will change their instruction to a constructivist, student-centered one using a variety of instructional modes and they will learn to serve a diverse student population.

• Teachers will learn how their expectations of student achievement can make a critical difference on student achievement and what their role is in achieving educational equity for all students.

• Teachers will learn new content and learn to extend content across topics and strands as needed for implementation of integrated courses 1 and 2.

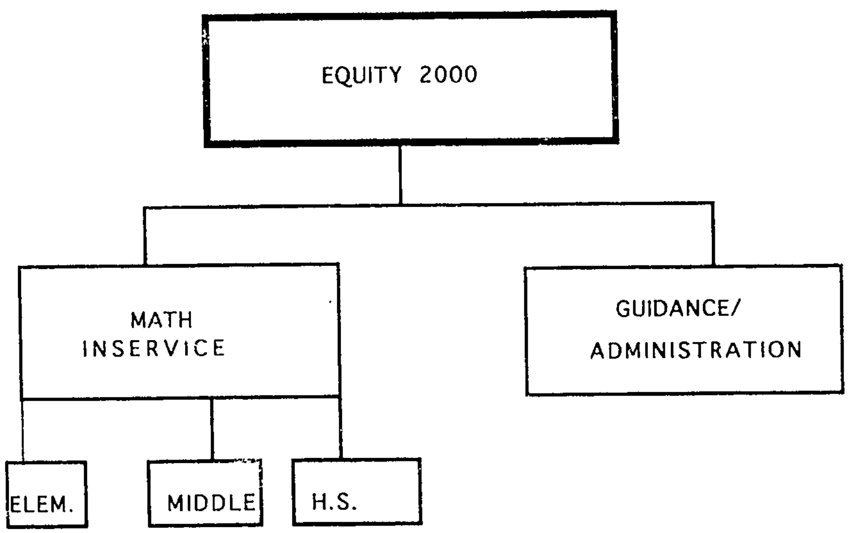


FIGURE 1

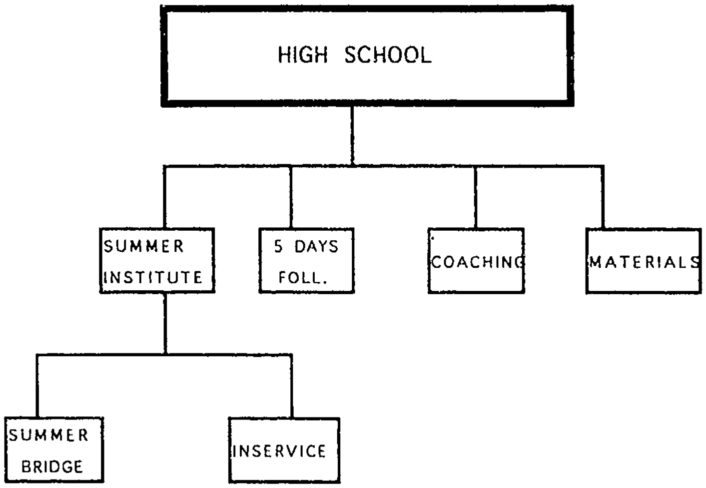


FIGURE 2

The summer institute of the high school program involved the teachers in two types of activities: traditional workshops, and visits to summer bridge classes for incoming ninth graders in which they could observe exemplary teachers working with the type of students they themselves would face the following Fall. The observations focussed on a number of topics, including: use of sheltered instruction with limited English proficient students; use of manipulatives to develop new concepts; and use of technology. Topics for the workshops included alternative approaches to teaching algebra, sheltered instruction, alternative assessment, use of technology (especially graphing calculators), and equity. Each teacher received a graphing calculator, and over the three years of the project all participating schools will receive a class set of graphing calculators with an overhead one included. In addition, the teachers get to select manipulative materials and software which are needed at their school to implement changes in curriculum and instruction in algebra and succeeding courses.

Methodology

The Instructional Practices Scale (Becker & Pence, 1990a) was administered to all teachers in the project at three points during the year: the first day of the summer institute, when the focus was on practices used the previous year; at the end of the summer, when the focus was projecting how much one would use each practice the upcoming academic year; and in May, when teachers reflected on how much they had used each practice the preceeding year. A total of 27 forms are available for all three administrations. This scale is a Likert scale with 21 items derived from recommendations in the National Council of Teachers of Mathematics Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). Items include positive and negative ones focusing on problem solving, use of alternative instructional modes, use of technology, assessment, and connections. The largest subscale is instructional modes which provides a contrast between a classroom which has students actively involved and a more traditional teacher-centered one (See Table 1 in the Results section below). An individual score on each administration was made for each teacher; in addition, each item was averaged across teachers for each administration of the test.

The individual scores were used to select five teachers to participate in a focus discussion group conducted by the first author, someone outside the project. Teachers were selected who had increased scores from the first to second administration of the test, but then went down considerably in the posttest.

Questions for the focus group include the following:

- Has it (the project) been harder to implement that you thought it would be?
- How adequate has planning time been?
- Describe any resistance from students. Staff
- What other obstacles have gotten in the way?
- Can you implement (the innovations) with the level of training provided by this project?
- What changes would you make in the inservice given?
- What do you believe your influence has been on others?
- What currently is the difference between what you believe and what you feel you can do?

This discussion was audio-taped and analyzed by the first author.

Results

Only whole-group results on the Instructional Practices Scale will be discussed here. Of the 21 items, four showed a substantial increase from pretest to posttest: less memorization of facts; more use of cooperative learning; less text-based curriculum; and more interrelating strands in mathematics. The first three showed a progression of increase from pretest to intermediate test to posttest. For the rest of the items, the pretest and posttest differed slightly, but in all instances, the intermediate score was higher than both the pretest and posttest. In fact, the averages for the whole test were: pretest 3.12; intermediate test 3.61; and posttest 3.1. From these data it seems that there were a small number of areas in which the teachers self-reported major change. More interesting, perhaps, is that they intended to use the recommended practices more that they found they actually did when they reflected back on the academic year. (See Table 1 below).

From the focus discussion group two types of results are discussed here: what positive impact teachers felt the project had on them, and areas of need for further development.

All of the participants feel the project to have been a success and its intervention a critical factor, and perhaps the critical factor, in affecting their own attitudes toward change as well as their willingness to work to implement new curricular design, methodology and materials. Comments made included the feeling that most participating teachers would still be doing the same thing had it not been for the project, which gave a necessary push, provided support, opened doors, met needs, pulled widely separate elements together, provided a multi-district effort and created a community of risk-takers.

From the point of view of the participants, the project served several purposes. It presented the opportunity to explore different ways to present algebra, to look into various systems and assessment procedures and to see demonstrations and other teachers in action. It also offered the opportunity to observe group work strategies, to learn new methodology and to be introduced to the uses of calculators and other technology in the classroom. But most importantly, in the opinions of the participants, it offered a platform for the exploration and evolution of perspectives and points of view, as well as a safe and encouraging atmosphere for the risk-taking necessary to the implementation of new curriculum and methodology. It further provided, for these participants, a resource and support structure for change, as well as a forum for active association with other participants as "risk partners."

There was a strong feeling that implementation was much harder than anyone thought it would be and that, perhaps, the level of staff development offered by the project is not sufficient, in and of itself, for full implementation to take place. The experience of the participants is that implementation too often finds the innovator in the role of a salesperson to students, parents and colleagues. The project, they feel, gave good preparation for this role with respect to parents and students, but more in-service in areas such as group dynamics is necessary for implementation. They stress that the main goal of the project, in their eyes, was to change point-of-view, not implementation. The project gave the opportunity and the resources to sample and experiment and to overcome the fear of risk.

All see a need for more planning time during the academic year, as well as the opportunity for more networking and sharing time. The more experience with technology the better, and joint in-service and articulation across grade levels were considered important. More staff development in writing and group work would be important as they are major points of resistance on the part of students. A point all of the participants kept coming back to, again and again, was the need for staff development in the skills of facilitating change. Resistance from colleagues and staff was underestimated by the participants, and they felt unprepared to deal with it.

They feel their influence on others has been to change their philosophy of teaching mathematics a little, to raise expectations for all students and to affect teachers' attitudes through discussion, and the use of materials and technology they were able to bring back to their school sites. The project, they feel, puts practitioners in a position to begin to affect student outcomes and to positively influence the attitudes of their colleagues toward change.

The participants believe that additional staff development in the areas of group management, classroom management and study skills is needed but they are strong in their belief that their students are truly learning more math than they did three years ago. They see their participation in the project as the main factor in making that happen.

Finally, the group thought that assessment must be addressed in greater depth including assessment skills and options. There seemed to be a common feeling that new methods of assessment were not well understood. How do you use these new methods, how many do you use, how are they used together and for what purposes do you use which methods are only some of the problems expressed by the participants in what seemed to the interviewer to represent almost an assessment anxiety.

ITEM	PRETEST	INTERMED.	POSTTEST
T presentation	1.7	2.15	1.7
Active involvement	3.88	4.35	3.9
Memorization of facts	2.61	2.9	3.1*
Questioning of Ss	3.95	4.1	3.8
Cooperative learning	3.35	3.78	3.8*
Text-based	2.28	2.65	2.8*
Skill work	2.27	2.65	2.3
Manipulatives	3.66	3.35	2.8
Discussions	3.38	3.85	3.4
Whys/hows of math	3.34	3.79	3.1
Use of calc computer	3.87	4.23	3.8
Problem solving as means of instruction	3.69	4.25	3.5
Prob solving to introduce concepts	3.16	3.93	3
Ss comm orally	3.35	3.95	3.6
Ss comm in writing	3.02	3.75	2.8
Tests to grade	2.74	2.92	2.5
Use of projects	2.36	3.42	2.6
Assessment integral part of instruction	3.44	3.79	3.3
Interrelating strands	3.34	4.18	3.8*
Integrated approach	3	3.65	3.3
Connections	3	3.93	3.3
TOTALS	3.12	3.61	3.1

TABLE 1

Discussion

These two types of data provide us with multiple ways to examine the project. Based on the results from the Instructional Practices Scale it seems that the project did encourage:

teacher movement away from having students memorize facts;

teacher movement towards more use of cooperative learning;

less emphasis on text-based curriculum;

and interrelating several mathematical strands.

In addition, except for use of manipulatives, teachers' rating of all other items rose from pretest to intermediate test. However, they then fell again from intermediate to posttest, in half of the cases to a lower level than on the pretests. It seems that, at the end of the intensive summer institute, teachers had plans for extensive implementation changes in their instructional practices. However, they were unable to carry out these ambitious plans. The focus discussion group provides some insight into why that may be the case.

From the frame of reference of the five focus group participants, the project created a community of risk-takers who support and act as resources for each other. They speak of a gradual evolution of point-of-view as a participant is exposed to new information and strategies in the presence of encouragement, resources and support. These aspects of the program, however, did not fully prepare the teachers to overcome resistance to change which they encountered in their individual school communities. These obstacles included students, other teachers, parents, and administrators.

These findings point out some of the weaknesses and areas in need of further development. Over the 3 years of the project we will work with more teachers at each site, which should extend this community of risk takers and provide additional support to the initial group of teachers. More work is obviously needed to inform parents and administrators about the mathematics reform movement in general and Equity 2000 specifically. The classroom coaching was not fully implemented during the first year of the project but it holds promise (Madsen-Nason & Lappan, 1987) to support teachers as they try to implement new curriculum, instructional strategies and materials.

The multiple strategies used in this project: traditional workshops; visits to exemplary classes; followup sessions during the academic year; and provision of resource materials such as calculators, manipulatives and software, provided for some change in teacher attitudes and willingness to try new instructional techniques. Viewing these techniques in action in the summer bridge classes and having time to network and plan with other teachers were identified as the most useful aspects of the total program. However, these alone did not provided enough support for the teachers to implement intended changes in their classrooms as they met obstacles at their school sites.

Once back in the schools, these teachers became a small component in a larger system. In order to support these agents of change at the site level, efforts need to be extended beyond a core group of teachers. How can we not only develop a community of risk takers, but also support them so they can truly implement change? This question remains for further investigation.

References

Becker, J.R. and Pence, B (unpublished evaluation manuscript). Linkages between teachers' views of mathematics teaching/learning and classroom practice.

Becker, J.R. and Pence, B (1990a). Teachers' l 'efs and classroom practices. Paper presented at the Fourteenth International PME Conference fexico.

Becker, J R. and Pence, B. (1990b) Linkages between teacher education and classroom practice In Dossey, J.A., Dossey, A E. and Parmantie, M. (eds.) Preservice teacher education The papers of action group 6 from ICME 6, Budapest, Hungary. Normal, IL: Illinois State University, 201-207.

Grouws, D. (May, 1988) Overview of the role of research on mathematics teaching and learning in improving research in classroom instruction and teacher education. Paper presented at the First Wisconsin Symposium on Teaching and Learning Mathematics, Madison, Wisconsin

Madsen-Nason, A and Lappan, G. (1987) The middle grades mathematics project Coaching as a strategy for changing teaching practice In Bergeron, J.C , Herscovics, N. and Kieran, C (eds.) Proceedings of the Eleventh International Conference of the Psychology of Mathematics Education. Montreal, 100-106

National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematics. Reston, VA author

Pelavin S.H. and Kane, M. (1990) Changing the odds. Factors increasing access to college NY College Entrance Examination Board.

Romberg, T. and Carpenter, T. (1986). Research on teaching and learning mathematics Two disciplines of scientific inquiry. In Wittrock, M. (ed.) Handbook of research on teaching NY. Macmillan Publishing Co.

THE EMERGENCE AND DEVELOPMENT OF ALGEBRA IN A PROBLEM SOLVING CONTEXT: A PROBLEM ANALYSIS

Nadine Bednarz, Bernadette Dufour-Janvier

CIRADE, Université du Québec à Montréal

ABSTRACT

The difficulties met by students when they solve problems in algebra led us to reflect on one hand on the nature of the problems given to students at various grade levels, and on the other hand, on procedures used to solve them. In many studies on algebra problem solving, the analysis of problems has often been reduced to the explicitation of the equation structure. Our study of "algebraic" problems as well as our first experimental work with different problems lead us to question this symbolic approach. Indeed, this approach does not adequately convey the involvement required from the solver facing these problems. On the basis of "relational calculus" (Vergnaud, 1982) involved in the representation and the resolution of such problems, the research team elaborated a reference framework and an experimentation to be conducted with students at different grade levels (12 to 16 years). The results obtained single out the relative complexity of the different tasks faced by students, as well as the continuities and discontinuities from one grade level to the next.

RÉSUMÉ

Les difficultés que rencontrent les élèves dans la résolution de problèmes en algèbre nécessitent une réflexion d'une part sur la nature des problèmes utilisés à différents niveaux scolaires et d'autre part sur les procédures utilisées par les élèves pour les résoudre.L'analyse des problèmes en algèbre a souvent été réduite, dans les recherches conduites dans ce domaine, à l'explicitation de leur structure sous-jacente en termes d'équation.Notre étude des problèmes"algébriques" et nos premières expérimentations portant sur différents types de problèmes nous conduisent à questionner cette approche symbolique qui rend mal compte de l'engagement qu'exigent ces problèmes du point de vue du solutionneur.Un cadre d'analyse développé par l'équipe sur la base des calculs relationnels(Vergnaud,82) qu'impliquent la représentation et résolution de tels problèmes a été mis à l'épreuve lors d'une expérimentation conduite au près d'élèves de différents niveaux scolaires(12 à 16 ans).L'analyse des résultats permet de faire ressortir la complexité des différentes tâches pour les étudiants,et les continuités,discontinuités d'un niveau scolaire à l'autre.

Resolution of "algebraic" problems represents one of the important goals of the high school program, where students often encounter numerous difficulties. Among these, mathematization was studied in Clement (1982); Lochead (1988); Kaput (1983) and others, mostly in its phase of statement translation into the equation form. These studies agree with Kucheman's results (1981) by showing that in that specific translation procedure, most of the students treat letters like concrete objects instead of number of objects. Their results, with those of Bell, Malone, Taylor (1988) and Mayer (1982) point out the obstacles met along this process of symbolization. How then should we introduce students to algebraic problem solving in order to make them better at mathematizing?

We believe that, in order to put forward teaching model principles that would lead to improvement, much deeper reflexions on the nature of the problems given to students are needed. We must establish criterias which would allow us to analyze the relative complexity of the problems. Much more has to be known on students reasoning and solving strategies at different stages of their learning of algebra: just when they start studying algebra, after their first year, after they are introduced with problems involving several unknowns.

Indeed, any didactic setting necessarily relies upon the knowledge of the relative difficulty of situations usually encountered in algebra, the repertory of available procedures to handle them, and the obstacles encountered. This knowledge can help in the choice of pertinent situations and interventions in that domain. These reflections question more particularly the type of transitions expected from the student in the resolution of "algebraic" problems all along their learning of algebra (Bednarz and al., 1992; Janvier, Bednarz, 1993). They are part of a wider research program which aims at clarifying the conditions for the construction and evolution of algebraic reasoning in situations that allowed or allow it to emerge.

OBJECTIVES

At first, we worked at establishing criterias from which a grid of analysis of problems emerged. This tool allowed us to characterize the different types of problems generally met in algebra textbooks and help us to shed some light on the difference between what we would call an arithmetic problem in opposition to an algebraic problem. This grid is also a good one to analyze the relative complexity of problems, and consequently we could wish to use it to produce a repertory of problems covering a range from easy to difficult depending on the selected criterias. Above all, this part of the research displays the continuities and "leaps" in relation with the solving of these different "algebraic" problems, particularly in the transition from arithmetic to algebra, or within algebra itself.

These first findings mostly related to the analysis of problems had to be confirmed and completed by informations concerning students' responses when solving these problems. Another aspect of the research was to investigate on the different ways students, at different grade levels, represent themselves problems and resolve them spontaneously, in relation with their underlying experience in arithmetic and their knowledge of algebra.Apart from the objective just mentioned, we wished to single out solving procedures developed by students within a class of problems in the evolution form one grade to the next.

THEORETICAL FRAMEWORK

Studies conducted on an analysis of problems in algebra and on their classification remain somewhat limited. These studies rejoin the results of more global pieces of research on problem solving with regard to the impact of the formulation of the problem and the students' knowledge structures (Greeno, 1985; Mayer, 1982). More specifically, Mayer's study (1982) in this domain uses relatively simple problems to provide a first categorization of these "algebraic" problems in terms of the different types of propositions present in the problem statement (assignment propositions, relational propositions). However, this linguistic analysis has its limits. Indeed, it does not account adequately for the relative complexity of different types of problems found in the same class (referring to the same types of propositions) and it is hard to extend the analysis to more complex algebraic problems. Other studies with a symbolic approach of analysis of problems and their complexity in algebra were also conducted. Thus, numerous studies have laid emphasis upon the so called stage of translating the problem statement into symbolic language (Hinsley, 1977; Clement, 1982; Lochead, 1988; Fisher, 1988; Malle, 1990. .). The stress put on the equation will lead some researchers to talk about a didactic cut when passing from solving problems which underlying equation is of the type $ax + b = c$ (Filloy, Rojano, 1984) to problems that can be solved by equations of the type ax + b = cx + d. This didactic cut would justify a certain graduation in the choice of possible problems in the introduction to algebra, and an important conceptual "leap" from the problems referring to the ax + b = c pattern to these types of problems (ax + b = cx + d).

Our analysis of the textbooks shows us that this "equation approach" also guides the distribution of the problems from one level to another. This gradation will guide the choice of problems at a given academic level, and most importantly the choice of the first problems in which solving with one variable is expected. Our study of "algebraic" problems, as well as our first experimental work with different problems (Bednarz and al. 1992; Janvier, Bednarz, 1993) leads us to question this approach.

Indeed, we can produce problems with similar equation patterns that will be answered very differently by the students (as shown later), and on the other hand, problems having different equation patterns will be answered relatively similarly. Therefore, an eventual gradation of the problems in terms of the underlying equation will not reflect adequately the involvement required from the student.

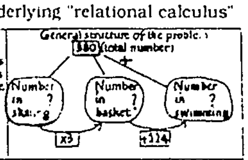
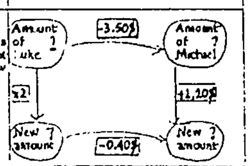
This first part of our research done in the perspective of our past work on problem solving with complex additive arithmetic problems, has led us to elaborate classification of different "algebraic" problems, in terms of what Vergnaud (1982) has called relational calculus (calculs relationnels), involved in the representation and resolution of problem situations. Our analysis of the problem's structure, realized with regards to the nature of the relations between the quantities of the problem, known and unknown quantities and the linkage of these relations[1] unables us to lay emphasis on the eventual difficulty of the problem for the student and his possible involvement in terms of resolution.

REFERENCE FRAMEWORK

The reference framework developed by the team was based on a systematic analysis of different types of problems found in arithmetic and algebra sections of textbooks at various grade levels. The analysis has identified three major classes of problems (see table 1) based on the nature of the quantities involved and the relationships between them.

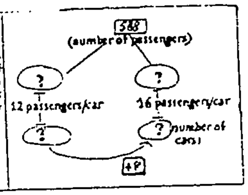
Table 1

Classes of problems in algebra: Underlying "relational calculus"

Problems of "inequitable partition" Example: "380 students are registered in the three sports activities offered during the season. Basketball has 3 times as many students as skating, and swimming has 114 more students than basketball. How many students are registered in each of the activities?" (Problem from our experimentation)	
Problems involving a magnitude transformation Example:[2] "Luc has 3,50$ less than Michael. Luke doubles his amount of money, whereas Michael increases his amount by 1,10$. Now, Luke has 0,40$ less than Michael. How much did Luke and Michael each have to begin with?" (This problem was suggested to us by François Conne)	

1 The set of criteria developed also involves other elements (formulation, context...).

2 A more complex version of this problem was given to us by François Conne during the meeting with our research group.

Problems involving a relationship between non-homogeneous magnitudes and a rate Example: "588 passengers must travel from one city to another. Two trains are available. One train disposes only of 12 passengers cars while another train disposes of 16 passengers cars. Supposing that the train with 16 passengers cars will have 8 cars more than the other train, how many cars must be attached to the locomotives of each train? (Problem from our experimentation)	

The general structure of the three main types of problems represented in table 1 lays emphasis on the quantities, knowns and unknowns, their relation to one another and the type of relation involved. These relations are given more or less explicitly in the statement of the problem and they have to be reconstructed by the student (with the help of the known quantities or other mathematical or contextual knowledge acquired before the resolution of the problem). For example, the general structure of the first problem in table 1 involves two types of relations: an additive **comparison** (114 more) between two unknown quantities, a multiplicative **comparison** (3 times more) also between two unknowns. The two relations join together and continue in the same direction: the number of students playing basketball obtained from the number of students skating, and the number of students swimming from the number of basketball players. A different type of link between the quantities is also explicit in that problem, the **whole quantity related to its parts** (which should be given equal consideration). In problem one, the whole quantity is known (number of participants in the combined activities).[3]

With the schemas used (see table 1) to illustrate the general structure of problems, the criterias we use to characterize the problems, as well as their anticipated solving difficulties are made more evident.

-For example, we can see that no easy bridging can be made between the known data: this addresses an important characteristic of the problems generally encountered in algebra, in relation to those proposed in arithmetic (see Bednarz and al., 1992; Janvier, Bednarz, 1993).

The arithmetic procedures generally organize themselves in function of the processing of the known quantities, by trying to create links between them in order to be able to operate (see Bednarz and al., 1992).

- In the preceding problem, the known quantities are the relations, or the whole. This is the only known state in this problem which constitutes a possible opening for the student to "bridge" from one given known quantity to produce a new one.

-Therefore with this technique of analysis, problem 2 (cf. table 1) will appear more complex to resolve than the problem 1 in so far as here the only known quantities are formed from relations or transformations (no state is provided in the problem).

3 For the analysis of the other problems see Bednarz and al. (1992).

-Moreover, problem 1 requires the simultaneous processing of two types of relations (x and +) by the student, this composition appears complex (as we will see later on) especially when one approaches it arithmetically.

This a priori analysis allowed us to underscore criterias which might influence the complexity of the problems: the nature of the relationships involved between the data, their linking (continuous or not, direct or not), the number of relationships involved, the formulation of the relations. A bank of whole-parts problems was set up based on these criterias and using this kind of diagram (see table 3 for some examples of problems).

DATA SOURCE

The grid of analysis was put to the test in an experiment involving students from different grade levels (first year high school, 12-13 years old) before any introductory course in algebra; second and third years, (after an introductory course in algebra). Up to this level, the current curriculum restricts students to single variable solution. Students in the fourth year of high school, after the introduction of solution with several variables were also included (see table 2). At this level, students were selected from a regular course, as well as an optional course leading to studies in science. Three to four groups of students per level from different schools had to solve the proposed problems and explain their solving strategies. Certain "arithmetic" problems were used as a "barometer" in the test and allowed us to weigh the relative strength of each of the groups.

Table 2
Data source

Sec. 1 (12-13 years old) before any introduction to algebra	Sec. II (13-14 years old) after an introductory course in algebra (1 variable)	Sec. III (14-15 years) after two years of algebra (1 variable)	Sec. IV (15-16 years) after an introduction to several variables Regular	Option
4 groups 132 students	4 groups 132 students	3 groups 105 students	5 groups 129 students	4 groups 119 students

We will restrict the discussion that follows to one clear class of problems through which the introduction to algebra is generally made (table 1, class 1). It is for this whole-parts class that a first bank of problems was set up.

RESULTS

1. Experimental results (see table 3 for partial results in some problems) explain, more or less the great difficulty students have with these problems. Their inability to create links between the known

quantities prevents them from getting "arithmetically" into the problem (in grade I, before any introduction in algebra).

2. The results of this experimentation confirm the influence of the criterias of complexity identified a priori in analyzing the problems:

2.1 In the whole-part structure, we observe the influence of the nature of the relationships between the quantities. Homogeneous (two relations of the same type as in problems a and b) appears relatively difficult. When solved arithmetically the homogoneous addivite problem appears more complex. When we treat them algebraically, the additive versus multiplicative relationships appear at the same level of difficulty. Non-homogeneous composition of the two relationships as in problem d are difficult problems. Thus, the problem d appears more complex than the others in terms of an algebraic treatment (see II, III, IV eg) given the composition of multiplication and addition involved. The algebraic treatment implies here resorting to parentheses, and to a genuine composition, which is not the case for problem c.

2.2 The influence of the sequencing of relationships is evident: direct linking as in problem g (involving a composition of two multiplicative relationships) versus non-linear (where no composition is explicitly involved as in problems e and f) convergent as in problem e (where the different quantities can not be expressed directly in terms of one unknown) and divergent as in problem f (where the different quantities can be easily expressed in terms of the first one). Problem e appears very difficult to solve at all grade levels, as predicted, due to the non-obvious choices it imposes on the unknown or unknowns, and the complex operations involved.

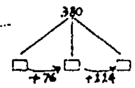
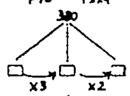
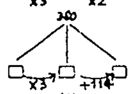
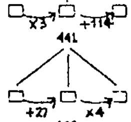
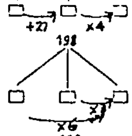
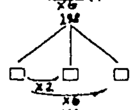
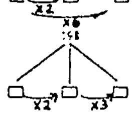
2.3 Other influences (not mentioned in this table) were confirmed as well during the introduction to algebra (sec. II), the number of relationships involved, making the data more or less complex to organize; the formulation of the relationship , confirming the conclusions of other studies, Clement (1982) and Lochead (1988)]

3. These results (table 3) show, moreover, the fundamental differences between the "relational calculus" on which an arithmetic mode of reasoning is founded (results of first year high school students before the introduction of algebra) and the "relational calculus" on which an algebraic mode of reasoning is based (results at the other grade levels). Thus, for example, problem c appears complex to solve arithmetically yet not algebraically. This problem involves the simultaneous managing of two types of relationships (x et +) whose arithmetic composition turns out to be quite complex. The student must see that the same quantity (number of students skating) is repeated and is found in the givens for basketball (repeated 3 times) and for swimming (once again repeated 3 times).

Similarly, the analysis of problems a and b shows us that these problems are differentiated arithmetically and not algebraically.

Table 3

Performance at different grade levels in some problems

The problems	Problem structure	Sec I	II	III	IV Reg	IV Opt
a) 380 students are registered in sports activities for the season?... Basketball has 76 more students than skating and swimming has 114 more than basketball. How many students are there in each of the activities?		5%	22	63	24	69
b) 380 students. Basketball has 3 times as many students as skating and swimming has twice as many as basketball. How many students are there in each of the activities?		15%	21	62	23	83
c) 380 students. Basketball has 3 times as many students as skating and swimming has 114 more students than basketball. How many students are there in each of the activities?		3%	33	72	54	82
d) 441 students. Basketball has 27 more students than skating and swimming has 4 times as many students as basketball. How many students are there in each of the activities?		12%	17	45	14	86
e) Three children are playing marbles. Together they have 198 marbles. Pierre has 6 times as many marbles as Denis and 3 times as many as Georges. How many marbles does each child have?		6%	10	21	9	32
f) Three children. Pierre has 6 times as many marbles as Denis and Georges has 2 times as many as Denis. How many marbles does each child have?		32%	46	87	50	82
g) Three children. Georges has 2 times as many marbles as Denis and Pierre has 3 times as many as Georges. How many marbles does each child have?		20%	20	51	42	96

4. Certain distinctions appear interesting within one level, and between different academic levels in so far as they bring about significant differences in the manner in which the students solve the situations. Thus, certain problems find more difficulty in raising up the students arithmetic involvement (for example the problems a, c and e). These problems are may be interesting to use to motivate a transition to algebra.

   Some other problems suscitate more in algebra, one variable treatment as in (problem f), or a multiple variable treatment (as in problem e where x is defined as the number of beads Denis possesses, and y as Georges's beads, Pierre therefore possesses 6x but also 3y). Thus, a more in depth analysis of the results (implicative graphs) allowed us to find different classes of problems and different ways to engage in the problem.

CONCLUSION

The preceding results, give us a framework that unables us to assign relative difficulties to problems generally met in algebra. A deeper analysis of the results by means of implicative analyses, had been realized at each level involved in the set of problems. This analysis confirmed what was previously stated and unables us to go further in the analysis of the relative complexity of each of the problems.

Such an analysis of the problems was realized for the case of arithmetics for the conceptual field of additive and multiplicative structures (Vergnaud, 1982) but was never achieved for algebra. It should shed some light on the relevant situations and their progressive organization within a learning of algebra stressing problem solving processes. More particularly, it allows us to question teaching approaches to algebra problem solving.

REFERENCES

Bednarz, N., Janvier, B., Mary, C., Lepage, A. (1992). L'algèbre comme outil de résolution de problèmes: une réflexion sur les changements nécessaires dans le passage d'un mode de traitement arithmétique à un mode de traitement algébrique. *Actres du colloque portant sur l'émergence de l'algèbre, avril.* Cahiers du CIRADE, pp. 17-31.

Bell, A., Malone, J., Taylor, P. (1988). *Algebra: an exploratory teaching experiment.* Shell Centre, University of Nottingham. SMEC, Curtin University, Perth.

Clement, J. (1982). Algebra word problems solutions: thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 14*, pp. 16-30.

Filloy, E., Rojano, T. (1984). From an arithmetic to an algebraic though'. *In* J.M. Noser (Ed.): *Proceedings Six Annual Meeting PME-NA*, Madison, University of Wisconsin.

Fisher, K. (1988). The students and professors problem revisited. *Journal for Research in Mathematics Education*, pp. 260-262.

Greeno, J. (1985). *Competence for solving and understanding problems.* Pittsburgh University. Reports Research.

Hinsley, D.A., Hayes, J.R., Simon, H.A. (1977). From words to equations: meaning and representation in algebra word problems. *In* M.A. Just and P. Carpenter (Eds): *Comprehension and cognition*, Erlbaum, Hillsdale, N.J., pp. 89-106.

Janvier, B., Bednarz, N. (1993). The arithmetic-algebra transition in problem solving: continuities and discontinuities. *Proceedings of the 15th annual Meeting PME-NA*, Asilomar, October 17-20.

Kaput, J. (1983). Errors in translations to algebraic equations: roots and implications. *Focus on learning problems in mathematics*, 5, pp. 63-78.

Kuchermann, D.E. (1981). Algebra. In K.M. Hart (Ed.): *Children's understanding of Mathematics: 11-16*, London: Murray.

Lochead, J., Mastre, J. (1988). *From words to algebra: mending misconceptions. the ideas of algebra*, k-12, NCTM Yearbook.

Malle, G. (1990). Semantic problems in elementary algebra. *Proceedings of BISME-2*, Bratislava, pp. 37-57.

Mayer, R.E. (1982). Memory for algebra story problems. *Journal of Educational Psychology, 74(1)*, pp. 199-216.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. *In* Carpenter and Moser (Eds): *Addition and subtraction. A cognitive perspective.* Erbaum, Hillsdale, N.J.

COMMON SPATIAL REPRESENTATIONS, AND THEIR EFFECTS UPON TEACHING AND LEARNING OF SPACE AND GEOMETRY

Berthelot R. et Salin M.H.
LaDiST, University of Bordeaux 1, France
Members of the GR Didactique du CNRS
Maîtres de conférences à l'IUFM d'Aquitaine

Abstract

The authors support, with empirical observations, curriculum analysis and didactic engineering, the theoretical hypothesis put forward by Brousseau (1983) and Galvez (1985) about micro spatial, meso spatial and macro spatial common representations.

They show some major links between many of the students' errors observed during the elementary learning of geometry, and the micro spatial representation, and they support these links by alternative didactic engineering.

They also support that it is necessary and possible to help students to improve their meso spatial and macro spatial representation during their compulsory school attendance.

Résumé

Les auteurs appuient, en la précisant par des observations empiriques et des travaux d'ingénierie didactique, l'hypothese théorique de Brousseau (1983) et Galvez (1985) selon laquelle les connaissances spontanées des éleves sont structurées selon au moins trois représentations : microspatiale, mésospatiale et macrospatiale. Leurs travaux permettent de relier a la représentation microspatiale de nombreuses erreurs constatées tant dans le déroulement de l'initiation a la géométrie que dans les évaluations de cet enseignement. Ils montrent aussi l'importance et la possibilité de faire évoluer les représentations méso- et macro-spatiales des éleves dans la scolarité obligatoire.

I. Presentation

Specific spatial knowledge learning is missing in the French compulsory school attendance curriculum. Geometry acts as a substitute for it. In the introduction to geometry, teachers and pupils can only rely on common space knowledge. So, if this common knowledge is inadequate in some of its aspects, it can have serious consequences on difficulties encountered by teachers and pupils in our classes.

MG Pêcheux (1990) assesses the results of main works led during the last twenty years to clarify the spatial knowledge development in childhood. She notices that the concept of representation gathers many varied mechanisms, and that it is very difficult to organise them in their effects upon the development of spatial abilities from baby to adult level. She shows that the question must be studied from the point of view of children's spatial practices, and she points to the problem of classifying these practices. Then, she concludes on the incapacity of psychology to classify the spatial relations in order to take in account the variety of the student's knowledge and their potential multiple processes in spatial behaviour.

The didactic processes developed about mathematical education enable us to exploit the tools developed by the theory of situations in school practices.

Relying on analysis of different restrictions set by the environment on judicious information capture and treatment in common situations, Brousseau (1983) and Galvez (1985) propose a classifying process of these situations by which they get three main situations classes, by using the space scale variable.

The space scale has already been studied by many researchers, and particularly the difference between small and large scale spaces. Although Acredolo (1981) has already pointed to the difficulty to isolate one from the other in the experimental tasks proposed by researchers.

Brousseau (1983) and Galvez (1985) developed a theoretical analysis of the usual practices, based for one part on the specific restrictions set though the space scale to people's environment interactions, and for the other, by their frequency and importance in our society. So, these authors showed the pertinence to consider not only small and large scale space interactions, but also the interactions with a third kind of space scale that he named *meso space*. They classify common spatial interactions with three values of the space scale variable, and they support the hypothesis that a specific spatial representation at least corresponds to each of these classes. Brousseau (1983) named, as a spatial representation, a group of specific conceptions (each conception corresponding to a geometrical concept) linked by their frequent common implementation in the same kind of situation..

We have extended this researche and our works have specified the respective roles of the space interaction restrictions from one part, and the scale space on the other. We got mainly pupils' observations, engineering elaboration and implementing in classrooms, and former works analysis.

II. The three main common space representations linked with the space scale hypothesis

In order to present these representations, we'll choose one of the geometrical notions we studied.

1) The micro space, or space of common interactions with manipulated small objects.

This space is familiar to each one from birth and most of the problems met don't need any geometrical conceptualisation.

The central notion of this kind of interaction is the notion of object. This space is mainly composed of objects. One object can be distinguished from another if you can separate them by a space(ing), that you can eleminate in one instant, without any effort. In this environment, the notion of distance can hardly be distinguished from spacing ; so it has few links with the geometrical notion.

The micro spatial conception of length is linked to the objects recognition, or cutting, and it is nearer to the geometrical notion.

2) the meso space, or space of domestic moves and interactions

It gathers situations of interactions linked with moves or position choices and modifying them within a field of domestic space. This part of space can be perceived by a global vision, obtained from successive but very brief perceptions. The moves cost more in the meso space than in the micro space. Thence the necessity of a intellectual representation of space which enables people to master their moves.

The central notions are here those of locates, routes, and objects. The new objects are fixed or semi-fixed ones. No space is necessary between two objects (walls...). Between the biggest objects there are several possible routes, the possibility to bring other objects (furniture...) ; so there is a distance concept, linked with length, height, depth, and their measures.

3) The macro spaces (urban, rural, maritime...)

It corresponds to areas the scale of which is so large that you can only get information though successive local visual glances separated by your moves on the earth surface.

In a macro space, you cannot get a simultaneous global view of the area with which you are interacting. The objects stay, you move. To direct your moves, you must build a global space representation to link your local views and recover the continuity of the travelled space.

A conceptualisation is needed to build an image of the whole which is out of direct perception access.

A macro space is mainly a space composed of locates and travels. The specific objects are fixed ones and fulfil a landmark function. distances are measured in km or travelling time.

4) Our contribution

We studied the theoretical features of the common spatial representations that we can associate to the main spatial and geometric notions.

The main result is that there is no reason for people to build a homogenous space representation, only by individual or social interactions with common situations. Only professional situations require such a representation which enables to master meso space or macro space tasks and concepts with their geometric properties.

The geometric concept(s) of angle(s) particularly requires such a homogenous representation, and may be a good theme to observe the pupils' spatial representation level of homogeneity.

III. Pupils' observations illustrating the space scale effects in the building of different conceptions of a geometric notion

The rectangle seems to be one of the best known by pupils among geometric notions with angles which are explicitly taught. We tried to find signs of different conceptions.

1) micro space

According the 1992 ministry evaluation of CE2-6ème[1], 90,6 % know how to draw a 7cm sided square. So we infer that about the same proportion know how to draw a little 4cmx7cm rectangle, for example.

The capacity to recognise rectangles has been acquired quite sooner as long as the Wermus (1976)

[1] CE2 means 9 year old pupils and 6ème means 12 year old ones. The sample was 2000 representative pupils

conditions are fulfilled (not too small width, proportion between length and width, and so on...).

2) meso space

On the contrary, pupils' abilities seems to be quite different when the space scale grows as the following samples show :

a) We marked on the ground two places distant of 1m (the virtual line was not parallel to the walls). We announced that the (rectangular) table (1mx2m) would be turned upside down. We have asked two good level pupils to mark on the ground where the two last corners of a table would be when the others were placed on the marks.

The pupils measured the length and width, and reported the length, with particular attention to parallelism but no attention to right angles.

When we turned the table upside down to evaluate the marking, the pupils noticed that their marks were far from the corners, but they were not able to identify the origin of their error.

b) On a circular lawn, we stuck two stumps 7 m away. We successively asked to four groups of two pupils to stick the other two stumps so that the four should mark the four corners of a 9m long rectangle. In order to determine the position of the stumps, each group checked the right angle either bodily, or with a square. But, the last stump stuck, three of the groups made adjustments only by the sides length and failed. The other group, who succeeded, controlled the direction and the length, even the diagonals lengths, but they still were not persuaded they had succeeded : they said "Some quadrilaterals may have four right angles and the opposite sides of the same length, but they are not rectangles !"

3) macro space restrictions

We also made an empirical study upon two samples of about 50 pupils (11 year old) to determine the respective influence of the scale (between micro and meso space) in one part and of the restrictions in space interactions (near to macro spaces conditions). We found that these limitations succeeded for about 50% of the samples, in blocking the micro spatial knowledge of the rectangle.

4) Conclusion

At the end of elementary school, all pupils can act nearly as if they had a micro spatial conception of the rectangle.

As soon as the spatial interactions lose they micro spatial characteristics, a great number of pupils are no more able to spontaneously call on the learned notions of right angles. They have not yet developed meso spatial or macro spatial conceptions of the rectangle. But some of them, may be 50% have.

We made other observations in order to show some different conceptions of spatial or geometric notions which should be linked with the space scale. This work is very complex, and the main difficulty we came across is the question of the invariability of the question in pupils' minds, when the space scale varies a great deal. Actually, it is exceptional to spontaneously consider the same question in such different environments. A question which is meaningful in an environment, is meaningless when you make the space scale vary.

IV. The common space representations influence on geometry teaching and learning

The hypothesis of different spatial representations enables us to :

- renew the analysis of pupils' errors in geometry. Most of the problems are set in the paper sheet space environment, which presents a lot of micro spatial characteristics. Consequently, we can expect to find signs of a micro spatial treatment mode in pupils' behaviour.

- find out the conditions which favour a geometric conceptualisation, to be able to produce or to simulate them in the teaching situations.

1) The micro spatial representation as an analysis tool of the pupils' behaviour in geometry.

a) Solid or void spaces

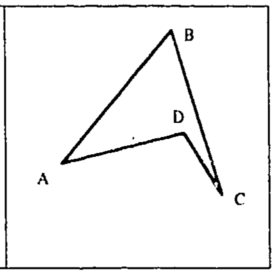
We give an example of a new pupils' test results interpretation with a test from the ministry of education evaluation in the first class after elementary school.

The pupils are asked to measure with their graduated ruler, in one hand the length of the [BC] segment which is traced, and in the other hand the distance from point A to point C.

En utilisant la règle graduée, mesure en centimètres :

La longueur du segment [BC] : $\boxed{BC = \text{..........cm}}$

La distance du point A au point C : $\boxed{AC = \text{..........cm}}$



The difficulties of the directions and measuring the two segments measuring difficulty are quite the sam· nd very well got over, since 93.2 % of the pupils succeed in the first question. But only 54.4 succeed the second question. Unfortunately, all the errors are not identified but it is said that 23.6% badly position their rulers (1.8% for the first one), and 10% measured and gave AB+BC or AD+DC instead of AC. The non response percentage is very low.

One might think that those pupils don't know the meaning of the word "distance". But in the 1991 tests, the same question (measure the distance between two points of a traced segment), the success score had been 91%.

Consequently, we can assign this difference to the influence of the heterogeneous micro spatial representation (in the solid-void meaning) applied to the paper sheet space.

The obstacle to a punctual conception of geometrical figures showed by different researchers (Artigue and Robinet 1982) (Grenier 1988) is a very clear consequence of the figures taught as objects, obviously micro space objects.

Bautier (1987) showed the heterogeneity of 15 year (and more) old pupils' spatial representations from the solid and void point of view, at the time of conic perspective learning..

b) The angle teaching didactic obstacle

We point here to a string of well known errors or blocks, attested up to the angle definition in some mathematical books (Berdonneau 1980), linked with the attribution to angles of sides traced length properties.

We showed that these errors can easily be explained if you see how the teaching presents angles as (micro spatial) objects from traced sides : "this is an angle..."

We developed in the COREM, with 10 and 11 year old pupils, a very well controlled alternative process of teaching angles, which saves from presenting angles as objects, and we found that we could save teachers and pupils from the usual obstacle.

c) The over-figures and under-figures

many researches showed the pupils difficulty (13 year old and after) to consider the possibility to include the figure they traced in an over-figure to be traced or to consider under figures, by mental removal of some of their elements.

The analysis of the traced figures' teaching, during the elementary schooling, shows that these figures are always assimilated to concrete objects. The micro spatial representation is implicitly solicited in this environment, and becomes an obstacle to a geometrical treatment. So are, for example the segment continuation difficulties (Grenier 1988) or the treatment of non connect figures.

2) Curricula problems

We examined the balance between the use of the showing geometrical teaching method (until 14 year old) and the restriction of the interactions with a sensible space to micro spatial interactions with the paper sheet space.

This method persistence and reappearance as a disguised showing method ("what can you tell about ?") is explained : it allows the teaching process to expel the effective space problems, and seems to take into account the pupil's activity.

The consequence is that the deductive geometric teaching will be result into an exclusion expulsion of sensible space out of the classroom practices, and not by its mastering.

The flat triangle problem (4cm, 9cm, 5cm) studied by Arsac (1989) shows the logic of this exclusion : the teacher can't accept the real triangles obtained by pupils, which puts in contradiction the showing method of teaching and the mathematical requirement of the three points alignment. So the real space must be excluded to let deductive geometry exist.

3) Engineering problems

So, if you limit yourself to micro spatial interactions, it is impossible to organise teaching processes which help pupils construct good space geometric models by effective interactions with space.

Let us take the example of tracing a superimposed figure to another. To make pupils become aware of the geometrical properties, the space interactions must not be micro spatial interactions. which allow pupils to compare continuously the traced object and its model, but meso or macro spatial interactions ; we found these conditions in the carpenter's work when he must prepare on the ground a beam so that it can adapt to a particular place in the framework, or in the work of the glazier who must cut a parallelogram shaped glass pane in its workshop.

Following Galvez (1985), we showed (1992 and 1993) that some space interactions conditions which are specific to macro spaces can be introduced in the teaching situations inside the classrooms. These situations allow pupils to interact effectively with a sensible space and construct some learning which were blocked in the usual teaching situations.

By these methods, we built and studied in detail two teaching learning processes :

- The first aims at some macro spatial learning that the school should make each pupil succeed : to be able to relate a real space and a plan of it. We worked with a class of 9 year old pupils. One year later, those pupils' abilities were better , with a significant difference, than 11 year old pupils' of a similar class in the same school. Weil Fassima and Rachedi (1993) noted the difficulties of adults in similar tests.

- The second aims at an elementary geometric learning : it is a teaching process of sector angles, for 10 and 11 year old pupils. We showed that the usual obstacle - the sides length- that we link with the micro spatial representation, can be avoided by this kind of process, where an angle is not an image of a concrete object, but firstly a means to get and treat pertinent information about some kind of object. So we proved that the difficulties met by teachers and pupils about sides length is an effect of didactic choices, and not a specific pupils' problem about angles.

V. Conclusion

We hope we showed you that the hypotheses of Brousseau (1983) and Galvez (1985) are confirmed by a lot of corroborating and socially important facts.

The previous examples, and many others, enable us to support the hypothesis that an important proportion of 16 year old French pupils leave the compulsory school attendance only provided with common and heterogeneous space representations,

- which are inadequate in relation with people's needs in the today's society, particularly concerned by space graphics,
- which have not been improved by the teaching of geometry,
- which have been an often still are obstacles to a geometrical learning.

Otherwise, we showed that it is possible to facilitate some important space learning by choosing some didactic variables in order to simulate, in a school limited space, some specific macro spatial conditions of interactions.

References

ACREDOLO L.P. (1981) : Small and large-scale spatial concepts in infancy and childhood. *Spatial representation and behavior across the life span.* Liben, Patterson, Newcombe, Academic Press, 63-82

ARSAC G. (1989) : la construction du concept de figure chez les élèves de 12 ans. *Actes PME* 1989.

ARTIGUE M., ROBINET J. (1982) : Conceptions du cercle chez les enfants de l'école élémentaire. *Recherches en didactique des mathematiques 3.1*, 5-64

BAUTIER T. (1987) : Etude didactique de l'introduction de la perspective conique. *Le dessin technique.* Rabardel P. , Weil Fassima A. ,Hermès Editeur

BERDONNEAU C. (1981) : Quelques remarques sur l'introduction à la géométrie démontrée à travers les manuels en usage dans l'enseignement post-élémentaire en France au vingtième siècle, Thèse Université Paris VII

BERTHELOT R. et SALIN M.H. (1992), *L'enseignement de l'espace et de la géométrie dans la scolarité obligatoire*, Thèse de didactique des mathématiques, LADIST, Université de Bordeaux I

BERTHELOT R. et SALIN M.H. (1993) : Conditions didactiques de l'apprentissage des plans et cartes dans l'enseignement élémentaire, *Espaces graphiques et Graphismes d'Espaces*, La Pensée Sauvage.

BROUSSEAU G. (1983) : Etudes de questions d'enseignement. Un exemple : la géométrie", *Séminaire de didactique des mathématiques et de l'informatique*, LSD IMAG, Université J. Fourier, Grenoble (1982-1983)

GALVEZ G. (1985) : *El aprendizage de la orientacion en el espacio urbano: Una proposicion para la ensenanza de la geometria en la escuela primaria*, Tesis, Centro de Investigacion del IPN Mexico.

GRENIER D. (1988) : *Construction et étude du fonctionnement d'un processus d'enseignement de la symétrie orthogonale en 6eme*, Thèse Université J. Fourier Grenoble I.

MINISTERE DE l'EDUCATION ET DE LA CULTURE (1992), *Evaluation CE2-6eme. résultats nationaux*, Direction de l'évaluation et de la prospective.

PECHEUX M.G. (1990) : *Le développement des rapports des enfants a l'espace.* Nathan université

WEILL-FASSINA A. ET RACHEDI Y (1993) Mise en relation d'un espace réel et de sa figuration sur un plan par des adultes de bas niveau de formation, *Espaces graphiques et graphismes d'espace*, La pensée sauvage

WERMUS H. (1976) : Essais de représentation de certaines activités cognitives à l'aide des prédicats avec composantes contextuelles, *Archives de psychologie vol. XIV, n°171 Geneve*, 205-221

ONE'S COMPANY, TWO'S A CROWD - PUPILS' DIFFICULTIES WITH MORE THAN ONE VARIABLE

J. E. Binns

Open University, Milton Keynes, U.K.

I give some accounts of and for difficulties faced by sixteen- and seventeen-year-old pupils in meeting situations where they have to deal with more than one variable. I describe one approach to overcoming these difficulties.

Pupils' difficulties with early algebra are well-documented if not well understood. In particular there is a lot of research evidence of pupils' discomfort with the presence of an "unknown". The CSMS research, for example (Küchemann 1981), identifies "letter evaluation" as a common strategy for dealing with an unknown or variable. The researchers found, in other words, that faced with a letter in place of a number, pupils would often invent a replacement number to use rather than deal with the letter as a variable. It may be that the ubiquity of this reaction is related to the historical emphasis on solving equations in the British early algebra curriculum.

My interest is in older pupils and the difficulties they face with more advanced algebra. In particular I have been studying situations where pupils are faced with problems involving two or more variables which play quite different roles. Some examples of these situations are

using "standard" forms such as $y=mx+c$, $ax^2+bx+c=0$, where the roles of the x and y variables are familiar, but a, b, c and m are replacements for what have, up till now, been numbers

considering functions such as $k(k-1)x^2+2(k+3)x+2$ in which the roles of the two variables could be seen as equivalent but are more likely to be seen as very different because of the familiarity of one (x) and the relative unfamiliarity of the other

using the notion of a variable point, (x, y) or (a, b) rather than a single variable, or of a variable line or curve.

I will offer you some short accounts of pupils' working on these types of problems, together with some reflections on and accountings for these events. My aim in doing this is to recall for you the similar experiences that you may have had and to encourage you to use your recollection of such events to enter into my reflections.

Each of the following accounts is of an event in the same classroom in a local school. I have been working with this class and their teacher twice a week since September 1993. They are studying the Pure Mathematics component of an 'A'-level[1] course in mathematics and are mainly aged 16 and 17 years. Peter is their teacher and any other names used in the accounts are of members of the class.

> Peter spends today's lesson going through the test which the class did last Thursday.
>
> Question 5 has caused difficulties for everybody (nobody has got any marks for it) and reads
>
> "A point P, co-ordinates (a, b) is equidistant from the x-axis and the point $(3, 2)$. Find a relationship connecting a and b."
>
> Peter says that he cannot understand why it was difficult and asks the class why they have found it difficult. David H replies that it is complicated and that "you have to remember all the formulas".
>
> Peter asks them to write down the two distances involved i.e. the distance from (a, b) to $(3, 2)$ and the distance from (a, b) to the x-axis. We both walk around the class giving assistance as they do this.
>
> I speak to James, Ewan, Mai and Simon C, in particular to James. He has drawn a diagram showing the position of $(3, 2)$ and the axes but has not marked on (a, b). I ask him where he is going to put (a, b) and he says that he does not know. I say that it is a general point and can go anywhere. He is trying to place it so that it is equidistant from $(3, 2)$ and the x-axis. I assure him that this is not necessary and suggest a general area in which he should place it. Meanwhile Simon has placed <u>his</u> (a, b) on the x-axis. I suggest that he put it somewhere else and he says "But you said it could go anywhere!" When they have got

[1]'A'-level is the main academic qualification taken at 18 plus in England and Wales. Most Universities accept it as an entrance qualification.

> suitable diagrams I ask them to find the distance between the two points. Nobody mentions the formula for finding the length of a line joining two given points so I ask them, by pointing at the diagram, about the horizontal and vertical distances from the axes of the two points (3, 2) and (a, b). James knows that the horizontal distance of (3, 2) from the y-axis is 3. When I ask him the horizontal distance from the y-axis to (a, b) (by pointing) he says "It looks about 1". I say "I don't want 'it looks about ..', I want an exact distance". He says, "Okay, exactly 1 then!"
>
> After this I point out that the distance is a and they go on to write down the length of the line joining the two points without much prompting from me. Simon C seems more confident than James.

Peter is an enthusiastic and well-informed teacher of six years' experience and he had had time since marking the test papers to consider why it was that the question had proved so difficult for everybody in the class. Yet he did not anticipate the nature of the problem that some pupils were having. Speaking to me the following day he said that dealing with (a, b) as a general point was so familiar to him that could not see where the difficulties lay for the pupils.

I interpret James' initial behaviour as being similar to that of the pupil who wants to know the value of the letter. He interprets the action of placing (a, b) on the diagram as equivalent to choosing values for a and b and so he is cautious about where to place it. In particular he tries to place it so that the condition expressed in the question is true. Because he sees the placement as assigning values to a and b, when I ask him about the distance from (a, b) to the y-axis he gives me a numerical answer. The more experienced mathematician can cope with this feature of geometry, that a representation which is necessarily particular can be treated as general. She can distinguish those properties of the point which would be true of any point from those which are consequences of its particular position.

One interpretation of Simon's behaviour is that he had sufficient confidence in my statement that (a, b) could go anywhere to place it somewhere which was convenient for the subsequent working. What is clear is that he failed to distinguish between general and particular properties of (a, b).

My second extract is from an earlier lesson in which we had been working on the equations of straight lines defined by various properties. Prior to the events described in this extract I had asked them to write down the equation of a straight line with gradient 3 and intercept with the y-axis at (0, 1). Next they had to write down the equation of a straight line with gradient 3 and intercept at (0, -3), and then a general equation for a straight line with gradient 3. After that I showed them how to find the equation of a straight line with gradient 3 and going through the point (2, 8). This process involved substituting the values 2 and 8 for x and y in the equation $y=3x+c$. They did one more similar question themselves and then I split them into pairs and asked one of each pair to explain to the other in general terms the process by which they had found the equation of a straight line given the gradient and a particular point on the line.

> Quite early in the lesson I ask them to find the equation of a straight line with gradient m and going through the point (x_1, y_1). Although we have done two particular examples of this process and they have described the process to each other, they find this very difficult. In particular there is a lot of confusion about the roles of x and y and x_1 and y_1. For example, having used the equation $y_1=mx_1+c$ to find an expression for c, Gary substitutes this expression for c into the equation $y_1=mx_1+c$ and gets $0=0$. Sonny gets the equation basically correct but cannot decide which of the xs and ys should have subscripts. There is again some difficulty when I ask them to check their formula with a numerical example. Some do not realise that they must leave x *and* y and substitute the given numbers for m, x_1 and y_1.

The roles of x and y in this problem are familiar to the pupils, though their ability to deal with them is not necessarily robust. I shall refer here to the role of x and y as that of *variable*. Each of them takes no specific value, either known or unknown. Their importance is not in the values they take but in their relationship to each other. For these pupils the form $y=mx+c$ has become a representation of a straight line, and they can substitute numbers for m and c and interpret the meaning of the equation. However, the introduction of another type of literal symbol changes the level of difficulty of the problem for them. Firstly the role of m is slightly changed. It is no longer to be replaced by a number immediately and before any manipulation is required. It is no longer simply a placeholder. It is an *unknown-to-be-given*[2] but the stage at which it will be given has been delayed. Secondly the point through

[2]The person usually credited with first using letters in this way is Vieta (see, for example, van der Waerden (1985) p85). He did this in the context of writing a general method for solving certain types of equation and used consonants for unknown-to-be-given quantities to distinguish them from unknown-to-be-found quantities which were represented by vowels.

which the line must pass is now represented by letters rather than numbers. These are also unknowns-to-be-given and must be worked with as letters in the first instance.

Gary fails to distinguish between the roles of variable and unknown-to-be-given. The letters x and y never enter into his calculations, x_1 and y_1 playing both roles. Sonny knows that there are two types of literal symbol involved (he has made no attempt to simplify an equation which involves two appearances of x and y) but their similarity confuses him and he ends with no clear picture of their roles.

The events described in my third extract took place two weeks after those of the second extract.

> After some work on the remainder and factor theorems and on division of polynomials, Peter asks the class to factorise
>
> (1) x^3-1
>
> (2) x^3-8
>
> (3) x^3-a^3
>
> (4) x^3+a^3
>
> We both walk around the class looking at students' work. David and Gary are the first to finish (1) and (2). In question (1) they have identified $(x-1)$ as a factor by seeing that $f(1)$ is zero and have then used long division to find the other factor. In question (2) they have used the same method. When I reach them they are beginning (3) and remark, "If we had done this one first we needn't have bothered with the first two".
>
> Some time later I go to the table where Sonny is working. He is also about to begin (3). He asks for my help, saying "I don't know what a is". I tell him that he cannot find out what a is. Somebody else at the table attracts my attention for a moment and when I return to Sonny he is keying something in on his calculator. I tell him that it is no use trying to use his calculator since he does not know the value of a . He seems stuck so I say that (3) is really a generalisation of (1) and (2) and explain that, in (2) the 8 can be seen as 2^3, so that it conforms to the pattern expressed in (3). I ask him to find the linear factor of $x^3 -27 =$

x^3-3^3, which he does without difficulty. Then I ask him for the linear factor of x^3-16^3. He pauses a long time over this, then reaches for his calculator to work out 16^3. After another long pause he says "x-16?" I go on to show him how to do (3), relating it to these special cases.

After another few minutes Simon asks for help. He has started (3) but has put $(x-a^3)$ as the linear factor. Ewan and I explain to him why this is wrong. My argument is along the lines that $f(a^3)$ is not zero. Ewan says "It's $f(a)$ because it's like the others".

In this lesson, which was focussed on methods of factorising polynomials and solving polynomial equations, Peter wanted to take the opportunity to introduce the pupils to the factorisations of a^3-b^3 and a^3+b^3. All the polynomials they had factorised so far were in x and had numerical coefficients. The three conversations I report seem to me to represent different stages in the development of an appreciation of the connection between (1) to (3). Sonny's original statement ("I don't know what a is") betrays that the unfamiliarity of this situation had thrown him back into the state of wanting to evaluate the letter. Notice that he was not concerned that he did not know what x was. The role of x as a variable (in the sense I used earlier) was well-established. By the end of my conversation with Sonny he still was not comfortable with the presence of a. His hesitation and use of the calculator in factorising x^3-16^3 indicate that the commonality of form of (1) to (3) was not apparent to him.

Ewan, on the other hand, had seen the connection between the first three expressions and was using his experience of the first two to solve the third. In fact he preferred his explanation "It's $f(a)$ because it's like the others" to mine ($f(a^3)$ is not zero). The "pattern" which he had seen was then more salient to him than the method which he used in performing the first two factorisations.

David and Gary had seen not only that (3) conformed to the same pattern as (1) and (2) but also that (1) and (2) were special cases of (3), that they were therefore included in it. This aspect of the role of the unknown-to-be-given was not emphasised by the way in which the task was set. The special cases were set up to lead to the more general case rather than to flow from it.

This kind of "top-down" generalisation, which starts with the general and moves to the particular is labelled by Krutetskii (1976) as "on-the-spot generalisation" and identified by him as a feature of the work of very able pupils. Davydov (1990) suggests that the top-down approach should be used in mathematics teaching with pupils of all abilities and ages and that the alternative "inductive" or "special to general" approach is most commonly used in schools now (that is in Soviet schools of the nineteen sixties) and has failed. The teaching strategies he suggests include the use of letters to stand for quantities which will not for the moment be enumerated, for example the length of a stick.

The approach with which I have begun experimenting in this class falls, I think, somewhere between the two. I have tried to use a technique which I have labelled the "generic example". By this I mean that I have approached a generalisable procedure by tackling, in front of the class, a particular example, but using it to point to the general. During the exposition I would repeatedly stress that my choice of numbers has been (at least in some sense) arbitrary and that other numbers could have been used in their place. I would leave numerical expressions unevaluated until the last possible moment so that the structure of the solution would remain visible. I would refer to a number as, for example "the x-coordinate of the first point", rather than using its numerical value. This then leads into a general expression of the procedure.

I have also begun to leave parts of this process to the pupils, at the same time making explicit to them what I am aiming to do and why.

It is my hope that being party to this process will assist pupils in

> clarifying the distinction between variable and unknown-to-be-given - they will have seen a set of numbers being replaced by the letters representing the unknown-to-be-given
>
> appreciating what is meant by "the letter could stand for anything" and overcoming the need to evaluate
>
> understanding which aspects of a "specific but standing for general" diagram are specific and which are general
>
> speeding up or removing the need for "inductive generalisation" of the kind envisaged by Peter in my third extract

References

Davydov V. Types of Generalization in Instruction. Translated by Teller J,. Soviet Studies in Mathematics Education, ed. Kilpatrick J. Reston, Virginia: NCTM, 1990.

Krutetskii V A. The Psychology of Mathematical Abilities in Schoolchildren. Translated by Teller J,. Survey of Recent East European Mathematical Literature, ed. Wirszup I. Chicago: University of Chicago Press, 1976.

Küchemann, D E. "Algebra." In Children's Understanding of Mathematics :11-16, ed. Hart K. London: John Murray, 1981.

van der Waerden B L. A History of Algebra. Berlin: Springer-Verlag, 1985.

THE ANALGEBRAIC MODE OF THINKING - THE CASE OF PARAMETER

Hava Bloedy-Vinner

Hebrew University, Jerusalem, Israel

Abstract: Algebraic language is analysed for the case of parameter. The term analgebraic is defined. A conceptual framework is suggested to explain students' difficulty to distinguish between the role of parameters and the role of unknowns and variables. The difficulty and its explanation are illustrated with data from students' written responses and interviews.

1. Introduction

There are many studies about students' difficulties with algebraic langauage. This paper presents a part of a study in which I suggest a conceptual framework for dealing with phenomena related to these difficulties, some of which are well known and treated in literature, and some less known or less treated.

The understanding that learning algebra includes the acquisition of algebraic language lies behind all related research. In my study I attempt to systematically analyze the structure of algebraic language, compare it to the structure of natural language, and learn about the influence of the latter (for better or for worse) on the acquisition of algebraic language.

It turns out that many students do not understand algebraic language correctly, and as a result, their thinking and performance are badly affected. I would like to call this language defficiency analgebraic (and use the word algebraic here, to describe understanding of algebraic language). The meaning of these terms vary from one context to another. In this paper I will deal with the analgebraic mode of thinking in the context of parameter.

2. Letters can be used in different ways

Although letters in algebraic expressions always denote numbers, and the meaning of an expression is always a number (or a process which results in a number), there is another level of meaning that depends on the context and application of the expression. Kuchemann (1981), and Usiskin (1988) suggest the following usage and meanings for letters:

(1) Generalized numbers (pattern generalizers), used in rules, patterns, and formulae, e.g. a+b=b+a or A=LW (area of rectangle).

(2) Specific unknowns, used when solving equations.

(3) Variables (function arguments), used when the referents of letters vary, as in "what happens to the value of $1/x$ as x gets larger and larger?" (Usiskin 1988), or in "which is larger, $2n$ or $n+2$?" (Kuchemann 1981).

Sfard & Linchevski (to appear) use the terms algebra of fixed value for usage of letters as specific unknowns, and functional algebra for usage of letters as variables.

In this paper, I would like to distinguish between using letters as parameters, and using letters as unknowns or variables. Like the above mentioned distinctions, this distinction too depends on the context and application of the algebraic expressions.

3. Parameter in high school algebra

In high school algebra parameter is encountered explicitly, when learning about parameter as such, as well as implicitly, when learning about families of equations (as in analytic geometry), families of functions, and in some problem solving.

Here are some examples of problems and statements that involve parameters:

(a) In the following equation x is an unknown and m is a parameter: $m(x-5)=m+2x$. For what value of the parameter m will the equation have no solution?

(b) The graph of $(x-m)^2+(y-n)^2=R^2$ is a circle.

(c) Find an equation for the line through (2,5) with slope 3.

(d) Write the coordinates of the extremum of the graph of the function $y=ax^2+bx+c$.

(e) X is the price of one pencil. Find the price of one eraser if I paid \$K for 10 pencils and 7 erasers.

The knowledge of which letters are meant to be parameters and which are unknowns or variables is not built into the expression or the equation itself. This knowledge is sometimes gained from an explicit declaration about which letters are used as parameters, as in example (a), sometimes from common knowledge about types of equations or functions as in examples (b)-(d), and sometimes by the formulation of the question as in example (e).

Moreover, the meaning of a letter as a parameter or as an unknown or variable, might change throughout the process of solving a problem as in example (c). Solving this problem starts with writing an equation $y=ax+b$, where common knowledge determines that x and y are variables whereas a and b are parameters. The process continues by substituting the constant 3 for a, and solving an equation with unknown b, where constants are substituted for x and y. The process terminates by substituting the constants found for a and b, and by letting x and y be variables in $y=3x-1$.

4. The algebraic and the analgebraic modes of thinking in the context of parameter

Understanding algebraic language related to parameter means understanding from the context, which letters are used as parameters, and understanding the role of parameters as opposed to the role of unknowns or variables. The different roles are explained by the fact that an equation or a function with parameters stands for a family of equations or functions, where specific instances may be created by substituting numbers for the parameters, while the other letters still assume the roles of unknowns or variables.

In understanding this difference of roles, there are some implicit quantifiers involved. (The existence of implicit quantifiers, in a different mathematical context, have already been mentioned by Esty (1992).)

Let me specify the quantifier structures corresponding to equations with parameters. I will use example (a) from the above (this is an equation with a parameter and an unknown. A similar treatment can be given to equations with parameters and variables):

(a1) "<u>for all</u> m, there <u>exists</u> E(x), so that E(x) is the equation $m(x-5)=m+2x$"

This means that in E(x) here, m (having been quantified) is now a specific constant, and x is an unknown.

Note that the existential quantifier in this statement is of second order, namely, quantifying an equation and not a number. As will presently be shown, this quantifier may be followed by different statements about the equation.

The difference between the roles of parameters and the roles of other letters is related to a <u>dynamics of possible substitutions</u>: first substitute for the parameter, get an equation, then substitute for the unknowns or variables to check if the equality holds. This dynamics may be expressed by adding (a2) to (a1) from

above:
(a2) "... and for all x, E(x) is either true or false"
(meaning that each x may or may not be a solution of E(x).)
Note that the unknown is quantified in (a2) after having quantified the parameters and the equation in (a1).
This order of quantifiers expresses the dynamics of substitutions, which is characteristic of equations with parameters.
In specific parameter tasks, one is asked to find values of the parameters for which the equations have special properties. To continue with example (a), (a1) should now be followed by (a3):
(a3) "there exists m, so that E(x) has no solution"
where the latter part may be expressed by (a4):
(a4) "there does not exist x, so that E(x) holds"
Here again, as in (a2), the quantification of x comes only after the existential quantification of the equation, performed in (a1).

If we look at natural language, we might find structures similar to unknowns ("someone called") or to generalized numbers ("kids like ice cream"). In natural language, there do not exist, though, structures similar to usage of variables and parameters, nor structures with (implicit) quantifiers which are as complex as described here.

The analgebraic mode of thinking in the context of parameter means failing to understand these implicit quantifier structures. As will be shown, the analgebraic students, will wrongly (implicitly) quantify unknowns and variables before quantifying the equations.
The failure to distinguish between roles of parameters and other letters is almost not mentioned in literature. Goldenberg, Louis, & O'Keefe (1992) mention this failure in the context of graphing software. Sfard and Linchevski (to appear) bring this failure as illustration to Sfard's theory of reification.

5. Method

As mentioned before, the purpose of this study is to examine to what extent students are algebraic or analgebraic in the context of parameter. For this purpose I compiled a questionnaire, part of which is presented in figure 1.

Figure 1: Five questions administered in the study

1. In his homework assignment Gadi was asked to explain why a line has infinitely many equations. Gadi answered: A line equation is $Ax+By=C$, and there are infinitely many possibilities for x and y, therefore there are infinitely many equations. What do you think of his answer?

2. Is the line equation $y=b$ a specific case of the line equation $y=ax+b$? <u>Explain your answer</u>!

3. Students were asked when the following equation was not quadratic: $mx^2+(m+5)x+m-1=0$ (m is a parameter). Avi answered: when $m=0$. Michael answered: When $x=0$. What do you think of each of the answers? (right, wrong, <u>explain your answer</u>!)

4. A number A is given. Can we always find a number B so that $(A-3)(B-2)=1$? Check the correct answer:
   a. Always. b. Only when $B\neq 2$. c. Only when $A\neq 3$.
   d. Only when $B\neq 2$ and $A\neq 3$.

5. Can you write the equation of a circle with center (4,6) and radius 5 which goes through (1,2)? Check the correct answer:
   a. Yes, the equation of the circle is: $(x-m)^2+(y-n)^2= R^2$.
   b. Yes, the equation of the circle is: $(1-4)^2+(2-6)^2=25$.
   c. Yes, the equation of the circle is: $(x-4)^2+(y-6)^2=25$.
   d. No, there is superfluous data.

For each of these questions, a correct answer requires an (at least implicit) understanding of the order of quantification as described above, while a wrong answer reflects a wrong quantification order. Task analyses will be given in the next section.

The questionaire was administered to Israeli students who had taken 3-5 unit matriculation exams in mathematics (a unit is one weekly hour during 3 years of high school). They answered the questions after having restudied the related material in a course, at the end of which, they repeated the matriculation exam. This course is a part of a one year university program for students who do not meet university admission requirements. By a rough estimate, more then half of high school graduates are on their mathematics level or below. Results will be given for two groups: Group A of 48 students repeating 4 or 5 unit exams, and preparing to study science at the university, and group B of 34 students repeating the 3 unit exam, and preparing to study humanities or social sciences. Some students in group B were interviewed after answering the questionaire.

6. Results

For all questions, answeres were classified into 3 categories:

Category 1: The algebraic answers.

Category 2: The analgebraic answers.

Category 3: Answers that could not be classified according to the algebraic/analgebraic criterion, and will not be analysed here.

The distribution of the answers is given in table 1.

Let me now, for each question, give its task analysis, illustrate the categories, and analyse some of the answers:

Question 1: The correct analysis is: "for all A, B, C, there exists an equation E(x,y)." The analgebraic students think that: "for all x, y, there exists an equation." The order of quantifiers is implicitly reflected in answers in categories 1 and 2. All answers that related correctly to the roles of x, y and of A, B, C, like "One equation with infinitely many solutions" or "His answer should be that x and y have infinitely many combinations that when multiplied by constants A and B they give result C," were included in category 1, eventhough they do not point out Gadi's wrong implication. The answer "Gadi is wrong, it depends on A, B, C" does point this out. All answers that justified Gadi's claim, were included in category 2, e.g.: "Gadi's answer is correct", "true! each number we substitute for x, y gives a different result for the equation (C)", "His answer is correct, there are infinitely many equations because on the line there are infinitely many points." Apparently, some students think that different x, y substitutions make different equations because they make different C's, and some think that each substitution is in itself an equation. This was also supported by the interviews, which for lack of space cannot be quoted here.

Question 2: The correct analysis is: "for all a, b, there exists an equation, and when a=0, that equation is y=b." The analgebraic students think: "for all x, there exists an equation, and when x=0, that equation is y=b." This is implicitly reflected in the answers "yes, when x=0" and "yes, when a=0 or x=0" which are in category 2. The answer "yes, when a=0" is in category 1. Note, though, that in the interviews, some of those students, readily accepted x=0 as an additional answer. This suggests an analgebraic mode of thinking which produced a seemingly algebraic response only because it was incomplete. The high incidence of unclassifiable answers in this question was due to the fact that students did not specify the substitution in their explanations: "yes, when $ax=0$" or "yes, $y=b$ is a specific line and $y=ax+b$ is a general line".

Question 3: The correct analysis is: "<u>for all</u> m, there <u>exists</u> an equation, and there <u>exists</u> m, so that the equation is not quadratic, and this m is $m=0$". Analgebraic students wrongly think: "<u>for all</u> x, there <u>exists</u> an equation, and there <u>exists</u> x, so that the equation is not quadratic, and this x is $x=0$." Category 1 includes all answers claiming that Avi is right and Michael is wrong. Some of the answers added explanations like "...because then [when $x=0$] it's not an equation" or "... with parameter you are concerned with m, not with x". All answers justifying Michael were included in category 2, e.g.: "Both are right because x^2 becomes zero." or "... Michael is right because then [when $x=0$] $m-1=0$, and it is not quadratic". When interviewing these students it was found that they (like those who wrote x^2 becomes zero) considered "$m-1=0$" not quadratic because it did not have x^2.

Table 1: Distribution of categories 1 (algebraic), 2 (analgebraic) and 3 (unclassifiable), for groups A (n=48), B (n=34).

question	group	category 1	category 2	category 3	no answer
1	A	26 (54%)	14 (29%)	8 (17%)	0
	B	5 (15%)	22 (65%)	1 (3%)	6 (17%)
2	A	19 (40%)	22 (46%)	7 (14%)	0
	B	7 (21%)	14 (41%)	11 (32%)	2 (6%)
3	A	27 (56%)	18 (38%)	3 (6%)	0
	B	3 (9%)	29 (85%)	1 (6%)	0
4	A	8 (16%)	39 (81%)	1 (3%)	0
	B	1 (3%)	31 (91%)	2 (6%)	0
5	A	33 (69%)	13 (27%)	2 (4%)	0
	B	18 (53%)	12 (35%)	2 (6%)	2 (6%)

Question 4: The role of A as parameter and the role of B as unknown are declared implicitly by the formulation of the question. The correct analysis is: "<u>for all</u> A, there <u>exists</u> an equation E(B), and when $A\neq3$, E(B) has a solution for B" (answer c). Answers b and d wrongly state: "<u>for all</u> A, B, there <u>exists</u> an equation E, and when $B\neq2$ (and $A\neq3$) that equation has a solution." Category 1 includes answer c, and category 2 includes answers b and d (answer a is

category 3). In the interviews the analgebraic students explained that when B=2 we get 0=1 and therefore the equation has no solution. They did not understand that by substituting B=2, one does not get an equation, but a specific substitution inside an equation, and therefore the correct conclusion should be: "B=2 is not a solution", and not: "the equation has no solution."

Question 5: The correct analysis: "for all m, n, R, there exists an equation". The analgebraic students wrongly think: "for all a, b, R, x, y, there exists an equation", namely, they substitute for all 5 letters to obtain the circle equation (being tempted to do so by the extra given point (1,2).) Thus, category 1 includes answer c, and category 2 includes answer b (no one checked answer a. Answer d is category 3).

7. Conclusion

Most of the questions had a high rate of analgebraic answers, and even a routine question like 5, was answered analgebraically by many students. These results can be explained by the fact that the notion of parameter includes implicit but unavoidable quantifier structures which are even more complex then advanced notions like ϵ,δ definitions in calculus, which teachers try to avoid.

As was to be expected, group A usually performed better than group B. Question 2 is an exception. This may be due to the higher technical skill of group A which led them to specify the substitutions a=0, x=0, thus performing skillfully but analgebraically, not paying attention to the meaning.

References

Esty, Warren W. (1992). Language concepts of mathematics. Focus on Learning Problems in Mathematics, 14(4), 31-54.

Goldenberg, P., Lewis, P., and O'Keefe, J. (1992). Dynamic representation and the development of a process understanding of function. In Dubinsky, E., and Harel, G. (Eds.) The concept of function: Aspects of epistemology and pedagogy, (pp. 235-260), MAA Notes.

Kuchemann, D. E. (1981). Algebra. In Hart, K. (Ed.), Childrens understanding of mathematics: 11-16, (pp. 102-119), Murray: The CSMS Mathematics Team.

Sfard, A. and Linchevski, L. (to appear in ESM). The gains and pitfalls of reification - The case of algebra.

Usiskin, Zalman (1988). Coceptions of school algebra and uses of variables. In Coxford, Arthur F. and Shulte, Albert P. (Eds.), The ideas of algebra, K-12, (1988 Yearbook, pp. 8-19), Reston VA: NCTM.

APPROACHING RATIONAL GEOMETRY: FROM PHYSICAL RELATIONSHIPS TO CONDITIONAL STATEMENTS

Paolo Boero
Department of Mathematics, Genoa University

Rossella Garuti
Institute for Applied Mathematics, CNR, Genoa

Reflections on some historical and epistemological aspects of the statements of theorems in geometry suggested a teaching experiment with students in grade VII, concerning the production of geometry statements and the comparison between the statements produced and the statements contained in the text-books. An analysis of the students' papers proves that through such activities, in an adequate educational context, they are able to approach geometry statements constructively.

1. Introduction

Research concerning the didactical and cognitive problems about approaching theorems and proofs has expanded especially on the proofs (see References); particularly, few works concerning the construction and analysis of the statements of rational geometry have been published. This report concerns an exploratory study on the students' early approach to the statements of theorems in geometry through the production of statements during suitable tasks, and the comparison between the statements produced and the statements in the text-books. Our hypothesis is as follows: in an adequate educational context students from 11-12 years of age should be able to perform these activities and constructively approach "rational geometry" [*].

Our research developed from a reflection on some historical and epistemological aspects of the approach to geometry statements (synthetically mentioned in § 2): from this we derived the guidelines of a teaching experiment concerning the approach to geometry statements in grade VII[**] (see § 3).

The aspects considered in §2 also suggested some keys suitable to analyze students' papers (see § 4).

2. Historical and epistemological aspects concerning geometry statements

The statements of theorems in "rational geometry" are **general** (each statement expresses the properties of a class of figures - all rectangular triangles, all circles..., not of a particular figure), **abstract** (the statements concern geometrical figures and not concrete objects), **conditional** (statements are logically articulated in the form: *"in the hypothesis that ...it is true that ..."*).We may also observe that (also depending on their content) statements may be expressed in a **procedural** way (especially when they concern how to perform a geometric construction or to work out a measure: see Heron's Theorem) or in a **relational** way (as many theorems in Euclid's "Elements").

From a historical point of view, Euclid's "Elements" have been the first systematic organization of a hypothetical-deductive kind, based on the "evident" properties (axioms) of the elementary geometrical entities, of geometrical knowledge expressed in the form of general, abstract and conditional statements. Historians consider it a controversial problem how in the Greek culture the complex transition was achieved from the particular methods used for particular concrete problems, to the

[*] we shall use the expression "rational geometry" as in the Italian syllabi for secondary education (*"geometria razionale"*), to signify an axiomatic-deductive theoretical organization of the geometrical knowledge, apart from the particular system of axioms chosen.

[**] in Italy, it is usually in grade VII that comprehensive school students get in touch with the first theorems of geometry (generally the theorem of Pythagoras, sometimes the theorem of Thales, more seldom the theorems of Euclid).

"Thales statements" as general properties of abstract geometrical figures, and from these to the "Euclidean statements" as statements of a conditional kind which are true since they can be proved (in them the hypotheses are conditions under which a thesis can be proved). Indeed the oldest geometrical propositions of the times of Thales probably were statements on the truth (based on evidence) of some general properties of abstract geometrical figures not requiring any proof based on other more elementary properties (Szabo, 1961). Surely at the time of Aristotle the conditional (as well as abstract and general) nature of the geometry statements was known (Heath, 1956).

Serres (1992) assumes that the investigation concerning peculiar contexts (such as that of sun shadows), in a suitable cultural environment, might have suggested the transition to abstract geometrical shapes. Szabo (1961) relates about different hypotheses interpreting the transition from Thales to Euclid, and points out their possibile complementarity. He refers to ideas of Kolmogorov (concerning the development of mathematical proof in a socio-cultural context which was very advanced for the time), of van der Waerden (concerning the need - felt by the Greeks - to build general statements in order to overcome the limitations and contradictions existing between different practical methods derived from the Egyptians or the Babylonians), of von Fritz (who analyses the relationships between the development of Aristotelian logic and the structure of geometry statements and mathematical proof). And to its own ideas (concerning the implications of the development of the early Greek philosophical thinking - especially the Eleatic School, on the development of geometrical thinking towards abstraction, generalization and logical deduction). Arsac (1961) discusses Szabo's position, thus showing possible interactions between internal motivations of the synchronic development of the structure of geometrical statements and mathematical proof (depending on crucial problems, such as that of incommensurability between the side and the diagonal of the square), and external cultural development (especially concerning philosophy, starting from the Eleatic School).

Historical and epistemological research may offer interesting suggestions to plan didactical activities aimed at making the transition to statements of "rational geometry" accessible to students, starting from their working experience in particular and concrete geometrical situations. This is in our view an open didactical problem, not so far exhaustively and globally investigated in the research on mathematical education. Particularly, in the few experimental works to be found in literature on the costruction of geometrical statements by younger students, we can see that they are only concerned with the transition from particular cases to general properties (recently, by also exploring geometrical situations realized through suitable softwares, such as Cabri (Laborde & Laborde, 1992) or Geometric Supposer (Yerushalmy & Chazan, 1990)), or from concrete to abstract situations (Gerdes, 1988), while the problem of the construction of "conditional" statements is virtually overlooked.

3. Planning the teaching experiment

Based on the analysis briefly explained in the previous paragraph, we have identified a preliminary didactic context and two subsequent tasks which seemed adequate to effectively and rapidly achieve a direct constructive approach of the students to the aspects of the statements of the theorems of rational geometry that we deem essential.

The educational context has been the study of the phenomenon of the shadows of the sun (see Garuti

& Boero,1992 for the motivations and the first part of the didactic itinerary); in this context the students of the two classes we will consider in this report had gradually come to solve problems concerning the determination of the heights of objects (inaccessible by direct measuring), by using the proportionality existing between the heights of the objects projecting the shadows, and the lengths of the projected shadows.

To introduce the didactic activities forming the object of this report, two ancient versions [1] of the "Thales and the height of the pyramids" anecdote were told, then compared and discussed.

In this context, the notion of "triangle similarity" was introduced under the double aspect of "preservation of the ratios between the sides" and "preservation of the angles".

The **first task** proposed was: *"Imagine you are Thales, and write what he might have written down in his testament in order to explain his findings to posterity. If you can, write a general statement"*

From the point of view of motivation, this task could be expected to stimulate a number of students to reconstruct the meaning of the work done. It was also assumed that the task would stimulate some students to produce statements with the typical attributes of the statements of rational geometry, to be used in the following comparison and discussion activities. Indeed the students had initially faced the problem of the height of a lamp-post and, subsequently that of a tower; the Thales anecdote was concerned with the height of a pyramid ... It was therefore assumed that some students would feel uneasy (in writing down their testament) about the many different examples of "concrete" problems, and that the progress to generality and abstraction of the statements relating to geometrical figures might be for them an adequate and practicable answer to the need to overcome their embarassment.

It was also assumed that in the transition to generality and abstraction of the statements relating to the geometrical figures, some students might feel the need to underline the parallelism of straight lines as a "hypothesis". In fact, in facing the "concrete" problems met in the "field of experience" of sun shadows many students had become convinced that the proportionality between the heights of the objects and the lengths of the shadows projected depended on the parallelism of the sun rays, and therefore might consider this parallelism as the condition of the ratio invariance.

At this point, we deemed impossible a spontaneous evolution of all the students towards statements of rational geometry (due to a lack of reference models). This was the motivation for the **second task**, proposed a few days after the "testament" had been produced, which reads as follows: *"Try to establish which of the following four statements resembles yours, and explain why"*.

In this task, the students are offered four different formulations of Thales's statement (see Annex) taken from high school text-books.The statements have been selected according to different criteria: an analysis of the texts produced by the students, so that each one of them may identify with an "official" statement; statement content; significance in relation to the strategies adopted by the students during the

---

[1] *"Thales measures the heights of the pyramids by measuring their shadows, after having observed the time when our shadow is the same as our height ..."* [Diogenes Laertius] .
"I especially admire you, Thales of Miletus, because, by placing your stick at the end of the shadow of a pyramid, you formed two triangles with the sun rays, and proved that the height of the pyramid is to the length of the stick as the shadow of the pyramid is to the shadow of the stick" [Plutarch].

work on proportional reasoning; diffusion in the text-books. Drawings were modified in comparison with those usually found in books so that the students may perceive analogies with the shadows phenomenon without however representing this phenomenon explicitly.

As to the objectives of the second task, we can remark that the request to recognize analogies and differences between a student's own statement and one of the four proposed, calls upon the distinctive attributes of a geometry statement (see § 2) and can therefore represent an important step in the process of becoming aware of them.

As for the **methods of observation**, in planning the teaching experiment we considered that the two classes (which had been working two years with the same Maths and Science teacher) were by then used to writing down their reasoning extensively and accurately. It could therefore be assumed that the verbal records produced during the two tasks would supply enough material for our study.

4. Analysis of the students' behaviour (see TABLE I, annexed)

4.1. Statement production

The texts produced by the students can be analyzed according to different keys, linked to the analysis carried out in the second paragraph.

I) Particular-General

The text types identified are listed below:

a) some students stick to a particular case (Pc), for instance the case of the equinox [*] or to another numerical example;

b) others generalize (G) by stating general properties of the sun rays and/or generally describe the process adopted for which no examples are given;

c) others still have mixed behaviours (Pc/G) because they go from a particular case to a general statement or else state the general aspects first, and then apply them to a particular example.

The G or Pc/G texts are prevailing perhaps because, beyond the explicit request in this sense, the production of a testament shifts the attention from the geometrical representation of a particular phenomenon to the construction of a text concerning general properties.

However, we can see that it is not easy to make a distinction between the "particular" as an example to express a general property, and the "particular" as a failure to have a more general view of the property in question (see later on examples concerning the "testaments" of students [1] and [27].

II) Concrete-abstract

The behaviours are of a different kind:

a) some students stick to the phenomenon they have observed, and the elements they describe are concrete objects (rays, sticks, pyramids, ...) (C);

b)others explicitly state the properties of abstract geometric entities(parallel straight lines,triangles..)(A)

c) finally, some students begin by mentioning concrete elements, but later in the discussion they gradually go on to talk of geometrical entities (C&A).

[*] they had experienced that in the equinox day at noon (at our latitude) the length of a pole is equal to the length of the projected shadow

According to this key we can see how the majority of the students are still tied to the physical phenomenon observed. This can be explained on the one hand, by the relevance of the geometrization work performed in the previous year, and on the other hand, by the fact that the transition to abstraction possibly requires a further didactical mediation to go from sun rays to parallel straight lines, from the cross section "drawing of shadows/objects/rays" to similar triangles.

III) Conditional nature of the statements

About the mechanism determining the transition from "parallelism of the sun rays as a cause of proportionality" to "parallelism of the straight lines as a condition of proportionality" it can be said, in all four cases ([1],[2],[3],[18]) where this transition spontaneously occurs (IF), that this might imply some degree of awareness that the straight lines (unlike the sun rays) may not be parallel, and the conviction that parallelism is the property of the rays granting proportionality between objects and their shadows.

IV) Procedural-Relational

The students' behaviours are essentially of three kinds:

a) Procedural (P), when the students describe only the measuring process of an object of height unkown by using its shadow;

b) Relational (R) when the students detect the parallelism as a cause (or a condition) for the ratio invariance and relate the elements of the phenomenon (parallel rays, same angular height,...) or of the geometrical figure without however caring to describe the process to work out the measure;

c) Procedural-Relational (P&R): when the students describe both the process to reckon the height unknown and the "causes" (or "conditions") for the ratio invariance.

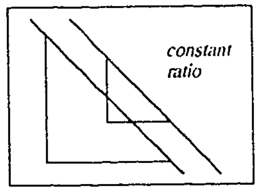
According to this key, we can see that the students are almost equally distributed in the three groups: the procedural statements probably reflect a will to make explicit (in a more or less general way) the resolutive methods concerning the problem of inaccessible height constructed during the previous year's activities; while the relational statements probably correspond to an idea of mathematics developed by a number of students of this age, due to their school experience about mathematics (as a "study of the properties of mathematical entities"). We can see how the distinction between the two types of statements does not seem to be cognitively relevant here (both types of statements are proposed by students of any level).

Let us now analyze some of the papers to see how the aspects we have so far observed have mixed in reality; examples such as these strengthen our conviction that, in the particular situation of statement production we are considering, the attributes "general" and "relational" or "abstract" do not indicate any superior cognitive performance.

a) [27]: he only considers the equinox case and concrete objects (Pc-C-P):
« I have considered a pyramid, I have measured its shadow and knowing that it was the equinox day, I have found the height of the pyramid. The ratio existing between the pole and the shadow is one »

b) [24]: his text is characterized by a high degree of generalization both in the description of the procedure and in the drawing illustrating it . In this drawing we may detect an approach to abstraction (G-C&A-P&R).

« I, Thales leave the following statements to my descendants:
- the sun rays are parallel
- an object and another taken at the same time and exposed for the same length of time in the sun have an element in common: if we measure their shadows at the same time and divide the larger by the smaller, we find a ratio equal to that of the heights.
- if I know the ratio of the shadows, I can find, through it, the height of an object through the height of another.»

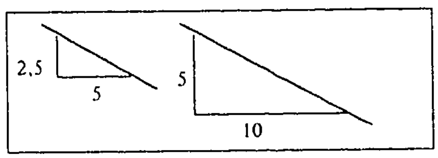


c) [33]: he builds the text starting from the case of the equinox and goes on to the general case by linking the ratio invariance with the parallelism of the sun rays. He therefore completes the picture by giving a numeric example where he makes the procedure explicit. In this general statement, he states a general property of the "parallel rays". This text is unique for its complexity and the dialectic existing among the various aspects (Pc&G-C-P&R).

« I, Thales, [....] have discovered [...] : given that the sun rays are parallel, I have worked out the height of the pyramid [...] I have used another object, i.e. a short pole [...] In terms of proportions the stick was as long as its shadow, and the pyramid was as high as its shadow and so [...] I then saw that even if the shadows were not exactly as long as the objects [...] Therefore, if the stick is contained in its shadow a certain number of times, then another object is also contained in its shadow the same number of times ... [numerical ex.]. *Since the rays are parallel, if a shadow is contained in another a certain number of times, hen a stick is also contained in another the same number of times ».*

d) [1]: his text opens with the equinox example, then goes on to a different numeric example. This student does not express any relation or generalization on these two cases. Afterwards, he "jumps" to a purely geometrical statement. In this text, the various aspects do not flow together at an explicit level, and there is a sudden progress to the "theorem" (Pc&G-C&A-IF-P).

«Dear descendants, I have made a wonderful discovery: I know how to work out the height of an object by measuring its shadow. [...............]
Statement: if three angles of two triangles are equal, I can work out the size of another side unknown, since the ratio between the two is the same.»



4.2. Comparison between produced and official statements: an outline of results.

During the recognition process the students should interpret systems of different signs: text, figure and symbols. They try to detect the statement analogue to their own by recognizing analogies in the drawing (DA) or the text (TA), or (also depending on the content of their texts) through a more articulated process of interpretation; in this second case, some students (Row), stimulated by "official" statements, revise their texts making them more general and abstract as well as closer to an official one; others (Ro) transform one of the official statements to bring it closer to their own statement, which therefore becomes a consequence (or a realization) of the official statement.

5. Discussion

As is shown in the examples and also in the table, many texts produced during the first and second task are characterized by remarkable variety and complexity, and by the dialectical presence of the different aspects: particular/general, concrete/abstract, procedural/rational; this allows the students, from time to time, to rest on one and/or the other aspect according to what they wish to communicate.

The official statements proposed in the second task are not interpreted by a number of students as formal texts, but seem to "speak" to them about shadows, sun rays and objects projecting shadows, and to also evoke the conquest by the students of the proportionality model. In the official statements they thus tend to recognize their own geometrical work experience.

The comparison between produced statements and official statements seems suited to fulfill the double function of maintaining the relation between the official statements and the students' geometric experience and introduce them to the cultural system of rational geometry.

With these considerations in mind, the teaching experiment suggests a certain optimism as regards the feasibility of the objective to get grade VII students constructively involved in approaching "rational geometry", obviously in an adequate educational context.

The teaching experiment also suggests some questions, concerning the teacher's didactical choices, we would like to point out very concisely: the opportunity or not of forcing (by modifying the first task) the production of relational statements; or especially the production of conditional statements; the most appropriate moments to introduce classroom discussions (which might be quite useful to foster the classification process of statements and the detection of the typical characteristics of the statements in rational geometry); the ways to accomplish the transition to proof once the "conditional" nature of the statements has been recognized.

A further question concerns the usage of the history of mathematics in mathematics education, apart from the usage that may be made in planning teaching experiments by deriving some suggestions from the hypotheses of historians (as we did). We think it might be useful to clarify the role (that we deem important both on the affective and cognitive ground) explicit references to personalities and intellectual activities so far back in time might have for students.

References

Arsac,G.: 1987, 'L'origine de la démonstration:...', *Recherches en didactique des mathématiques*

Balacheff, N.:1987,'Processus de preuve et situation de validation', *Educational Studies in Math.*

Balacheff,N.: 1988, *Une étude des processus de preuve en mathématiques*, thèse d'état, Grenoble

Boero, P., Chiappini, G.P. and Garuti, Rossella: 1993, *Towards proportional modelling: an exploratory study about context sensitivity* (preliminary report), DI.MA., Univ. Genova.

Duval, R.: 1991, 'Structure du raisonnement deductif et apprentissage de la démonstration'. *Educational Studies in Mathematics*

Garuti, R. and Boero,P.: 1992, 'A sequence of proportionality problems: an exploratory study', *Proceedings of P.M.E.-XVI*, Durhan

Gerdes, P.: 1988, 'On Culture, Geometrical Thinking and Mathematics Education', *Educational Studies in Mathematics*

Hanna,G. & Winchester, I.(Eds.): 1990, *Creativity,Thought and Mathematical Proof*, OISE, Toronto

Hanna,G. & Jahnke, N. (Eds.):1993, *Aspects of Proof*, special issue of *Educational Studies in Math.*

Heath, T.: 1956, *The Thirteen Books of Euclid's Elements*, Dover, New York

Laborde, C. & Laborde, J.M.: 1992, 'Problem Solving in Geometry:...', in J.P.Ponte et Al. (Eds.). *Mathematical Problem Solving and New Information Technologies*, Springer-Verlag, Berlin

Mesquita,A.L.:1989,'Sur une situation d'eveil à la déduction en géométrie',*Educational Studies in Mathematics*

Piaget, J. et Inhelder, B.: *De la logique de l'enfant à la logique de l'adolescent*, P.U.F., Paris

Serres, M.:1993, *Les origines de la géométrie*, Flammarion, Paris

Szabo, A.: 1961,'The transformation of mathematics into deductive science and the beginning of its foundation on definitions and axioms',I/II, *Scripta mathematica*, vol. XXVII

Yerushalmy, M. & Chazan, D.:1990, 'Overcoming Visual Obstacles with the Aid of Supposer', *Educational Studies in Mathematics*

TABLE I : this table summarizes the main characteristics of students' behaviours

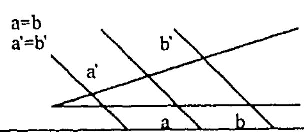
Stud.	I	II	III	IV	Rc
1	Pc&G	C&A	IF	P	TA
2	G	A	IF	P	Ro
3	Pc&G	C	IF	P	TA
4	*	*	*	*	*
5	Pc&G	C		R	DA
6	*	*	*	*	*
7	G	C		P	TA
8	G	C&A		R	Ro
9	G	C		R	Ro
10	Pc&G	C		P&R	Ru
11	G	C		P&R	Ru
12	Pc&G	C		P&R	Ru
13	*	*	*	*	*
14	Pc&G	C&A		R	Ro
15	G	C		R	*
16	G	C		P&R	DA
17	*	*	*	*	*
18	G	C	IF	P&R	DA

Stud.	I	II	III	IV	Rc
19	G	C&A		P&R	Row
20	G	C		R	Ro
21	Pc&G	C		R	Ro
22	G	C		R	TA
23	G	A		P	TA
24	G	C&A		P&R	*
25	Pc&G	C		P&R	Row
26	Pc&G	C		P	TA
27	Pc	C		P	DA
28	Pc&G	C&A		R	Row
29	G	C		P	*
30	Pc	C		P	Ro
31	Pc&G	C		P&R	DA
32	G	C&A		R	Ro
33	Pc&G	C		P&R	Ro
34	Pc&G	C		R	TA
35	Pc&G	C&A		R	Row
36	Pc&G	C		R	Ro
37	Pc	C&A		P	Ro

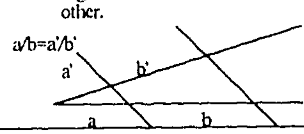
Pc=Particular; G=General; C=Concrete; A=Abstract; P=Procedural; R=Relational; IF=Conditional; Rc= Recognition; TA=Text Analogy; DA= Drawing Analogy; Ro= Revision official text; Row=Revision own text; *=absent.

ANNEX : official texts of the Theorem of Thales

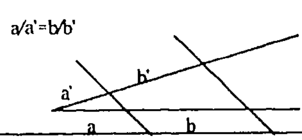
1. If a sheaf of parallel straight lines cuts two transversals, then equal segments on one of these transversals correspond to equal segments also on the other.



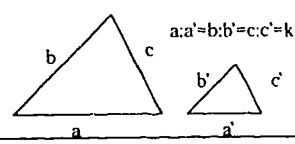
2. If a sheaf of parallel straight lines cuts two transversals, the ratio between the segments that are on one of these is equal to the ratio between the corresponding segments that are on the other.



3. If a sheaf of parallel straight lines cuts two transversals, the ratio between corresponding segments on the two transversals is constant.



4. Two triangles that have equal angles have corresponding sides in proportion.



A MODEL FOR STUDENTS' UNDERSTANDING IN A MULTI- REPRESENTATIONAL ENVIRONMENT

Marcelo C. Borba

Graduate Program of Mathematics Education/Mathematics Department
UNESP-Rio Claro, São Paulo, Brazil

Abstract

This paper presents a model to describe students' understanding of transformations of functions in a multiple representation computer environment. A case study is presented of a student who generates rules about stretches of a function and finds justification for the rules through noticing discrepancies and support among different representations. The teacher has an important role in this model since the "why question", which can be asked by either the teacher or the student, is crucial in the model developed. The model is seen as being potentially extended to other topics of the curriculum.

INTRODUCTION

The relevance of the study of transformations of functions has been associated with its importance in the modeling process (Confrey & Smith 1991). The use of multiple representations in mathematics education has also been discussed intensively in the last few years (Wenzelburger, 1992, Eisenberg & Dreyfus (1991), Rubin, (1990), Arcavi & Nachmias 1989, Kaput, 1987). In particular the use of multiple representations in the learning of transformations of functions has been attracting attention (Confrey, 1991). Many have argued that multiple representations would help students to connect realms of meaning which might have otherwise remained separate. Moreover, with the availability of computers, the use of multiple representations in the classroom has become feasible. However, few examples of how students have actually worked in such environments exist. In this paper I will present a case study of a student working with transformations of functions and a model of how students may understand functions in multi-representational environments. The research involved investigation of students' understanding of transfonnations of functions, although I believe that the model developed to describe students' understanding can be applied to students' understanding of concepts other than functions.

THE RESEARCH

Traditionally the study of transformations of functions has been dominated by algebra. This study uses visualization of graphs and direct actions on graphical software (through the mouse) as the starting point of the study of transformations. Then the impact of these graphical transformations on values in the table window and on algebraic equations is explored.

Function Probe (FP) (Confrey, 1989) is a multi-representational software that allows a graph to be transformed into another through direct actions on the graph using translation, stretch or reflection icons. FP allows both the reflection line and anchor line for the stretch to be moved. Therefore, a graph can be reflected on or stretched from any line parallel to the y or x axis. Points can be sampled from a graph allowing a transformation in a continuous graph to be studied as actions on discrete points. These samples can be sent to the table where they will be stored in "x" and "y" columns. Any pair of columns can be sent back to the graph window as points and columns of numbers can be altered in the table window. In this study, three students were interviewed in a teaching experiment fashion (Cobb & Steffe, 1983) over approximately eight two-hour sessions. Students were introduced to the features of the software through three tutorials and then given tasks involving absolute value (AV), quadratic, and step functions. In the first part of the experiment, the focus was entirely visual. Using the transformations icons, subjects attempted to transform graph A into graph B. In the second part, a numeric component was added as students were asked to predict what would happen to the coordinate values of a given point of a graph when the graph was transformed. Finally, students were asked to explore the relationships between actions in the graph and coefficients of algebraic expressions. The interviewer was the author of this paper.

RELEVANT RESULTS FOR THIS PAPER

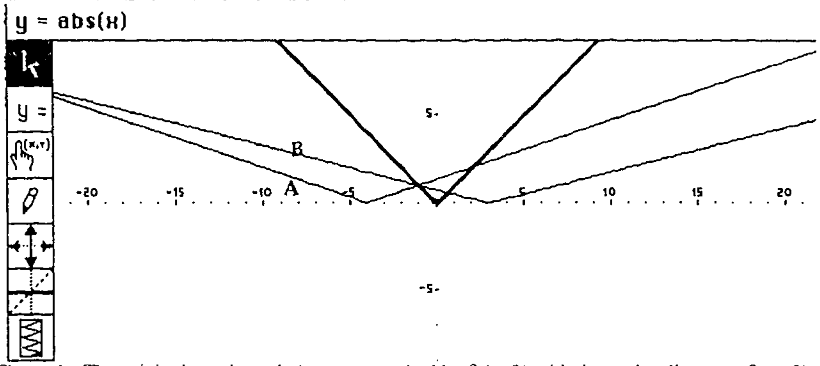


Figure 1 - The original graph y=abs(x) was stretched by 3 (s=3) with the anchor line at x=2 (a=2) resulting in graph A. The original graph was also stretched (s=4, a=-1) resulting in graph B. On the left hand side from the top down six icons of the graph window of FP can be seen: the selection icon, the equation icon, the cartesian point icon, the sketch pad, the translation icon, the reflection icon and the stretch icon.

Borba & Confrey (1992), Confrey & Smith (1991) have discussed the potential and the relevance of using multiple representations, visualization and discrete points in the learning of transformations of functions. Borba & Confrey (1992) reported an example of how Doug, a 16 year old high school student in Ithaca, New York, coordinated tables and graphs and actions in this graph to formulate a law of what would happen to a given point of y=|x| when the function is horizontally stretched. Doug had already worked through the first visual part of the teaching experiment and was working through the second part, trying to incorporate numerical value to transformations. In other words, he was trying to discover what would happen to a given ordered pair of y=|x| when a graph is transformed. In this second part, Doug has also already gone through translations and reflections and stretches, and was dealing with stretches with anchor lines which were not at the y-axis. Under these conditions when a horizontal stretch (HT) was performed the line of invariance would not be at the y-axis. Doug formulated, in regular language, a law of what would happen to this function when stretched, as shown in figure 1.

In a Lakatosian (Lakatos, 1976) fashion, Doug worked through a sequence of examples in the computer and found that a horizontal stretch by "s" with the anchor line at "a" could be described by the following law x'=sx-a(s-1), where x' is the transformed coordinate, "s" the stretch factor and "a" represents the position of the anchor line at the line x=a. It should be noted that Doug was working at this point only with covariational equations (Borba, 1993). Co-variational equations are equations such as x'=4x-9 and y'=y which represent algebraically a transformation of the function (in this case a HS by 4 with the anchor line a=3). Co-variational equations are heavily influenced by reasoning in a table environment and are embedded in a view of function as covariation instead of the input-output view (Rizzuti, 1991; Confrey, 1991).

Doug reached the conclusion described above after analyzing several transformations made in the graph window and analyzing these transformations within a table environment, looking at sets of points which were sent from the graph window. Not yet described by Borba & Confrey (1992) is the effect of a "why question" (Henderson, 1992) that was then posed by the interviewer. Doug was asked why his rule held true, leading him to explore the problem further.

This question took Doug on a very different track. His first attempt to solve the problem was a visual one. Doug had generated the rule essentially through an analysis of discrete points in the table. Now he turned to the graph window to look for support for his rule. In front of him he had now the graph window shown in figure 2 which was left from a previous investigation, since he was now focusing on the analysis of the results which were stored in the table window.

Apparently looking at the part of the computer screen shown in figure 2, Doug came up with an initial explanation:

> "what I am thinking right now . . is . . if I have my anchor line at a given point . . .say positive . . one . .x=+1, . then . . each point that is already . .to the left or negative side of it . . .is gonna continue moving left . . . .and each point that is already on the right . . .or positive right is gonna keep moving right or becoming . . more positive . . . a greater positive value . hum . . and . . . each stretch . . is . . .as I increase the value of the stretch . . . it moves even further . . . right . . if it is already on the right . . . or even left if it's already on the left . .that's where I am right now . . ." [4:12]

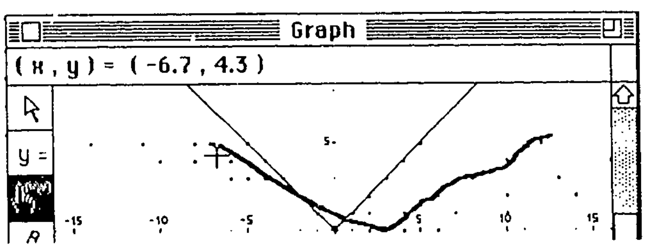


Figure 2 - Doug's sketch for HS (s=2, a=-2). The dots are the results of HSes he made from the original sampled set of points of y=|x|. The stretches on the points were made after his prediction (the highlighted sketch).

Doug seems to be explaining the rule he has found in visual terms inspired by the activities he had developed in the graph window. He was trying to explain the way a given point moved to the left or right when a graph is stretched. This train of thought did not lead him to fully answer the "why question", but on the other hand it helped him to better understand stretching, enabling him to make a wider variety of predictions.

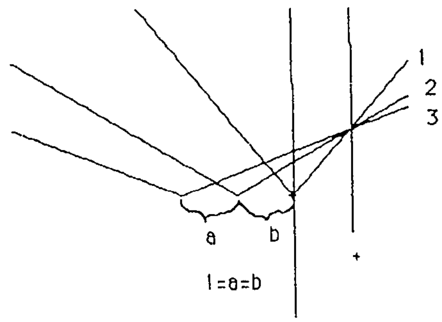


Figure 3 - a reproduction of Doug's sketch. The vertical line on the right hand side is his representation of the anchor line.

As Doug realized that the track he was following was not being very productive, he turned away from the computer and drew a sketch similar to the one shown on figure 3. This sketch shows what aspect of stretching he is emphasizing at this point. Doug is trying to look at a family of sketches with different stretch factors on a given function (y=|x|) with a fixed anchor line (x=1).

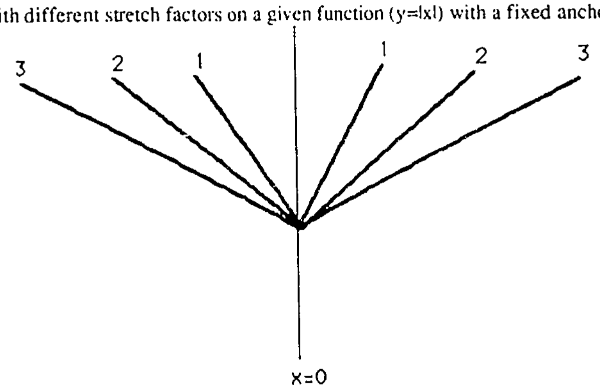


Figure 4 - Reproduction of the sketch made by Doug upon the interviewer's suggestion.

Although Doug could reassure himself of some features of the rule he had generated for stretches and anchor lines, he seemed to have stumbled in his reasoning. The interviewer suggested that he draw another set of stretched graphs with the anchor line at the y-axis so that he could contrast the sketch on figure 3 and the one on figure 4. The interviewer's suggestion had the effect of leading Doug to compare the sketches reproduced on figure 3 and figure 4 but also to look at the computer screen shown on figure 2. Doug compared these figures and, with the help of the interviewer who pointed out what he had in mind when he suggested that he draw figure 4, he arrived at the following explanation:

> I. OK, right. But what I'm trying to tell you is, when you look at this one here without moving the anchor line [figure 4], all you do is multiply by 2, OK? In a stretch by 2.
>
> D: Right.
>
> I: In this one that you move [the anchor line] by 1 [figure 4], what you do is you multiply by 2 and you subtract 1. Then when you made a stretch by 3
>
> D: It's a . . huh! It's a . . it's actually . . I can think of it in 2 ways. It's actually pretty simple I wasn't looking at all the different . . the fact that I add the same number or subtract the same number to each one, which is a translation after a stretch. It's, so that . . yeh, wow. I didn't even realize that. *I was thinking too much about how each value changed Not . . oh, my goodness. I didn't realize that. So that.. . it is a translation, I don't know exactly . well . . . And then the translation is just, um . . increases for each time the stretch increases.*
>
> I: When I do a translation?
>
> D: The trans . . . the . . each one is a translation. Even this [figure 4] is a translation of 0. This is a translation of -1. . . *I can . . the fact that the values are inverse in that if my anchor line is a =1, it's a subtraction, I can get from the fact that, if my, um . . x=0, um . . that even though my y, uh, my x, no . . my y-axis is always gonna be the same line, if I look at it as though my anchor line were the, um . . the y-axis, then this would be +1, even though its actual coordinate is -1. You see, this is -1, but the y-axis in relationship to it is +1. And the opposite is true here, that even though this is +1, um . . the relationship of the y-axis to this line is -1. That explains why it's an inverse value, at least to some degree.* [4;18-19]

It seems that Doug arrived at a new finding in the problem he was working with. He was able to construct an explanation for the first minus sign in the formula x'=sx-(s-1) he had found before. He saw the inversion of the sign - a movement to the right of the y-axis of the anchor line would mean a minus sign - as a way of explaining the relationship between the position of the anchor line and what happened to a given point when a given stretch has been finished.

DISCUSSION AND THE MODEL FOR STUDENTS' UNDERSTANDING

This case study indicates that understanding mathematics in a multi-representational environment may be something quite different than it has been in other environments. Doug had first worked only in the graph window of FP learning how to transform a function through icons driven by the computer mouse. Next, he generated rules, including the one describing stretches with anchor lines different than the y-axis which resulted from the observation of several sets of points he had previously stored in the table window. This rule was a result of an attempt to explain in the table environment regularities he had visualized in the graph window. However, the explanation in the table environment had "taken a life of its own", since he was working only on the numeric realm without being concerned with the visual part any longer. In a certain sense, it was this emphasis on one single aspect that allowed him to generate his powerful rule to link a stretched graph - with the anchor line not coinciding with the y-axis - to the original one. On the other hand, once the why question was asked, he had to look for an explanation in the visual realm. He combined both the paper and pencil and the computer media, using a suggestion by the interviewer, to understand further the law he had generated before and obtained an explanation to the crucial minus sign in his oral explanation of the rule he had generated. Moreover he could associate the translation of the anchor line - thinking of the anchor line as having a default position at the y-axis - with the stretch made on the graph.

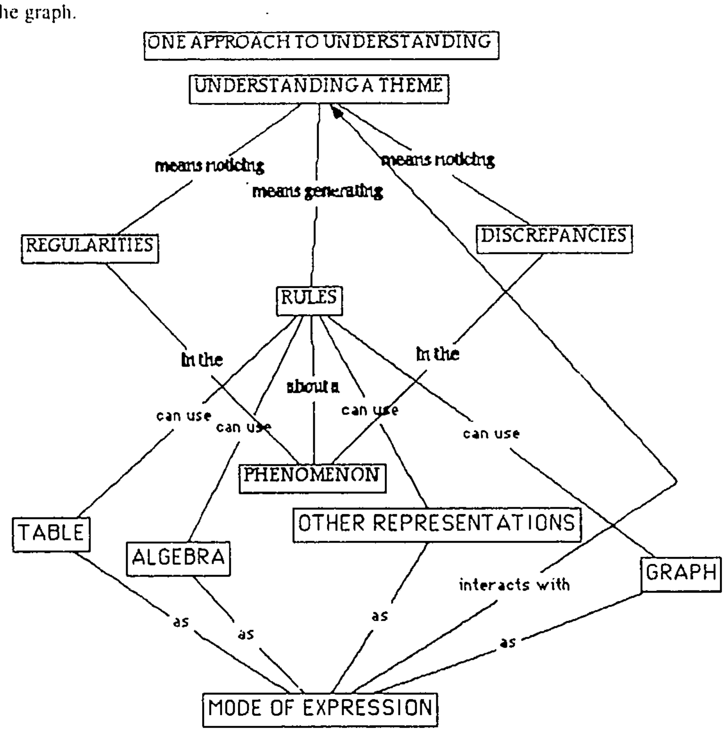


Figure 5 - Concept map showing part of the model for understanding in a multiple representation environment.

It is important to note the dialectical aspect of this "game" among representations. As he looked at the graph window and the table window, he was able to synthesize them to generate his powerful rule. What was once powerful was not as powerful anymore when the why question was asked and it became clear that rule had been generated with its basis on just one representation. At this point, he had to look again at both the graph and table representation to explain the rule he found.

As a result of the analysis of this excerpt and others from Doug and several others from other subjects which were interviewed, a model for students understanding in a multiple representation environment was synthesized in figures 5 & 6. Figure 5 emphasizes the notion that in trying to understand a theme, which could be a contextual problem (Confrey 1991) or transformations of functions in the "computer context" (Borba, 1993), the student would generate rules by either noticing regularities in an observed phenomenon or discrepancies between what he observed in that phenomenon and his previous experience.

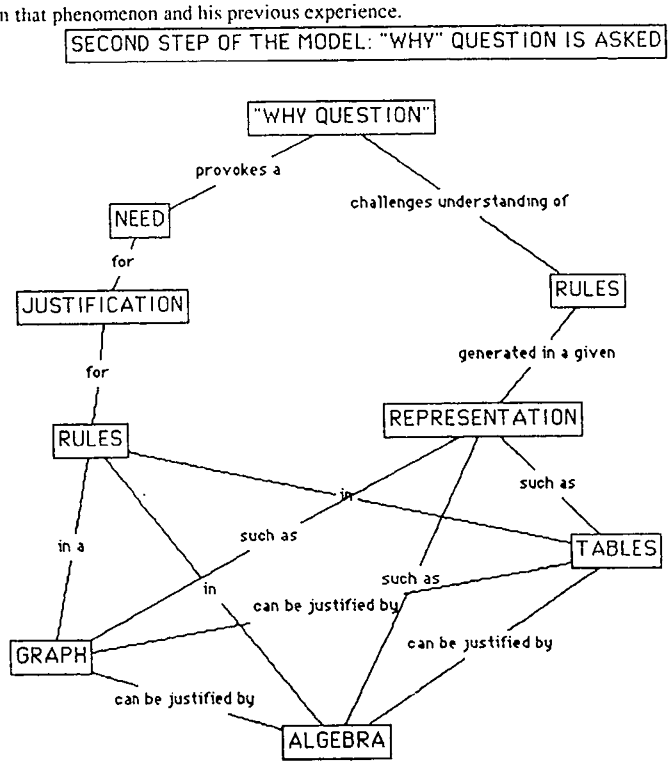


Figure 6 - The second step emphasizes the role of a representation in supporting (or not) a regularity noticed in another representation.

In Borba (1993) there is a collection of instances in which the "why question" is asked either by the interviewer or by the student him/herself. This why question in a multiple representation environment is usually motivated by discrepancies among patterns found in different representations or by regularities found in one representation but lacking any kind of justification that can give any kind of assurance to the knower about the result found. The justification provoked by the why question can serve as a means of reassurance. If we consider the idea of Henderson (1992) of proof as being a "convincing argument", it might be the case that the model presented on figures 5 & 6 is also a draft of what mathematics proof might become if multiple representations become pervasive in mathematics and mathematics education.

Finally, it should be mentioned that Confrey (1991) has argued that multiple representations bring diversity to the classroom. The diversity observed by Confrey (1991) might be extended if one considers the model below, especially the role of the why question. As the reader might have noted, the answer given by Doug to the why question could be unsatisfactory in other circumstances (different characters involved or the same character involved in different setting and different time). That means that a new why question can arise and new connections with other representations and knowledge about other phenomena may arise. As a consequence, the process described by the model can be seen as endless, allowing teachers to allow different students to answer a different amount and different types of why questions, asked by a teacher or by a student himself. It also seems that, the model could be used to describe students' understanding of other topics of the curriculum, although more research will have to be developed to assure this point.

BIBLIOGRAPHY

Arcavi ,A. and Nachmias, R. (1989) - Familiar concepts with a new representation. Proceedings of Psychology of Mathematics Education. Vol. I. July 9-13, Paris, France, pp. 77-84.

Borba, M. C. (1993) - Students' understanding of transformations of functions using multi-representational software - Tese de Doutoramento, Cornell University, Ithaca, Nova Iorque, EUA.

Borba, M. C. and Confrey, J. (1992) - Transformations of functions using multi-representational software: visualization and discrete points. A paper presented at the Annual Meeting of the Psychology of Mathematics Education Group. New Hampshire, USA, pp 149, vol. 3.

Cobb, P. and Steffe, L. (1983) - The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14 (2), pp. 83-94.

Confrey, J. (1991) - Workshop: "Using computers, contextual problems, and multiple representations in mathematics instruction." In press: Shirley Malcom, Linda Roberts & Karen Sheingold, *This Year in school science 1991*. Washington, DC, U.S.A: American Association for the Advancement of Science, 1992).

Confrey, J. and Smith, E. (1991) - A Framework for functions: Prototypes, multiple representations, and transformations. Proceedings of the Thirteenth Annual Meeting of the Psychology of Mathematics Education, Vol. I. October 16-19, Blacksburg, VA, USA, pp. 57-63.

Eisenberg, T. and Dreyfus, T. (1991) - On visualizing functions transformations. Technical Report, The Weizmann Institute of Science, Rehovot, Israel.

Kaput, J. J. (1987) - Representation systems and mathematics. In Claude Janvier (Ed.), Problems of representation in the teaching and learning of mathematics. Hillsdale, NJ, USA: Erlbraum Associates.

Henderson, D. (1992). Proof as a convincing argument that answers: Why? A paper presented at the Seventh International Conference of Mathematics Education, August 17-21, Quebec, Canada.

Rizzuti, J. (1991) - Students' conceptualizations of functions: Effects of a pedagogical approach involving multiple representations. Unpublished dissertation, Cornell University, Ithaca, NY, USA.

Rubin, A. (1990) - Concreteness in the abstract: Dimensions of multiple representations in data analysis. In Barbara Browen (Ed.), Design for learning: Research based design for learning. Cupertino,CA, USA: Apple Computer, Inc.

Wenzelburger, E. (1992) - The learning of thigonometric functions in a graphical comuter environment Proceedings of Psychology of Mathematics Education. Vol. III, New Hampshire, USA, pp. 106-113.

TEACHING MATHEMATICS AND USING COMPUTERS: LINKS BETWEEN TEACHERS' BELIEFS IN TWO DIFFERENT DOMAINS

ROSA MARIA BOTTINO (*) AND FULVIA FURINGHETTI (**)

(*) Istituto per la Matematica Applicata del Consiglio Nazionale delle Ricerche
Via De Marini 6 - 16149 Genova - Italy
(**) Dipartimento di Matematica dell'Università di Genova
Via L.B. Alberti 4 - 16132 Genova - Italy

ABSTRACT

The general framework of this research is the problem of the curricular changes determined by the introduction of computers in school. We investigate, through a case study methodology, how in-service mathematics teachers are reacting to this int· ·duction. In the paper we briefly outline the methodology of our work and the context in which ›et. Then we identify a number of issues we consider significant to investigate links between teachers' beliefs in mathematics and the use of computers. Findings resulting from the analysis of the case studies are presented according to these issues. They allow to enlighten the links between beliefs in maths teaching and the use of computers.

INTRODUCTION

The problem of curriculum changes and the consequent renewal in teaching emerges periodically in mathematics instruction of secondary school. (Wojciechowska, 1989) points out two major changes which took place in the secondary school mathematics curriculum in this century. The first one, referring to the *Meran Syllabus* (1905), had its principal advocate in Felix Klein who identified the development of pupil's geometrical imagination and functional thinking as being the basic objectives of mathematics teaching. The second major change, dating back to the early 60's, was supported by a French group of mathematicians known as Bourbaki: it ascribed a central role in mathematics to the abstract notion of structure, based on the set theory.

Today the great challenge offered to maths education by computers is leading to a third curriculum change, which is still in progress. Unlike the previous ones, this change does not come from within of the mathematical community as a consequence of certain cultural developments of the discipline, but it is the consequence of the great changes in the social and economical reality provoked by the impact of new information technologies.

Studies of the consequences of this impact in mathematical instruction show that the influence of computers in mathematics teaching is a complex phenomenon that has been approached with very different orientations depending on the context in which curriculum changes are carried out as well as on the cultural (mathematical and technological) background of mathematics educators. The updated edition of the ICMI studies on the influence of computers and informatics on mathematics and its teaching (Cornu and Ralston, 1992) offers a perspective of the different orientations observed in research and in school practice as regards the relationship between informatics and mathematics. We can sum up these different orientations under the following labels:

- computers as an aid to teaching and learning mathematics;
- introduction of informatics elements in the mathematics curriculum;
- mathematical basis of computer science.

In this paper we consider aspects of the development of the first two orientations observed at secondary school level; in particular, we focus on the reactions of teachers facing curricular innovations involving the use of computers. In this regard, while in the past one could mainly observe 'extreme' positions (absolute refusal or enthusiastic acceptance), it is now possible to detect a variety of *nuances* of attitudes between the extreme poles. An analogous pattern was followed in research: pioneering studies tended to single out the causes of teachers' attitudes in extreme factors (think, for example, of 'computer anxiety' or of 'computer appeal'), research has subsequently passed from this somewhat oversimplified point of view to a more articulated one by considering a wider range of factors in analysing of teachers' behaviour. In this paper we focus on some of these factors and, in particular, we attribute a crucial role to teachers' beliefs, in line with the recent trend of research (see, for example, Hoyles, 1990; Moreira and Noss, 1993; Ponte, 1990). Teachers' beliefs influence the transformation of the 'official' proposals of curriculum innovations into their 'real' implementation in classrooms. Different beliefs can be considered as a sign of different awareness of the curriculum changes proposed, both at the level of contents and methodology.

We observe that, in our particular case, beliefs have to be analysed in two different domains: the domain of mathematics and its teaching and that of informatics and its teaching. According to the terminology adopted in the IFIP "Guidelines for good practice on informatics education in secondary school" (Taylor, Aiken and van Weert, 1991, p.2) henceforth the term informatics will indicate «the science concerned with information processing» that is to say the theoretical aspects as well as the different applications of related technology, which in our case means the use of computers for educational purposes.

We believe that the relevance of our work is twofold: on the one hand, it may suggests elements to understand the reasons of the gap between proposals of curricular innovations and their actual implementations in classroom, and, on the other, it may offer researchers in educational computing elements for producing effective 'tools' for the teaching of mathematics.

CONTEXT AND METHODOLOGY OF THE RESEARCH

The present situation of Italian secondary school (age 14-18) offers a suitable framework for our research, since, at this school level, new mathematics curricula have been proposed by the Ministry of Education and are now under experimentation in a great number of schools.

These curricula, in addition to updating the teaching of classical topics (algebra and geometry), provide for the introduction of new ones (informatics, statistics and probability, logic) and suggest innovations in methodology.

In this paper we deal with the innovations induced in mathematics teaching by the introduction of informatics, which, as previous researches have pointed out, see for example (Bottino and Furinghetti, 1990), is perceived as being one of the most impressive innovations both for its cultural weight and for the pressures which teachers are placed under by students and by the world outside the school. It is not by chance that informatics is the only subject for which a widespread national initiative of training (addressed to mathematics teachers of upper secondary school) has been carried out. This training programme is one of the first initiatives of national training ever realised in Italy, and it has been a powerful stimulus for teachers in the direction of experimenting informatics and the use of computers in their classrooms. An analysis of the training project carried out and of its results can be found in (Bottino and Furinghetti, 1991).

In this context the objective of our research is to study the different beliefs on the use of computers in maths teaching formed by teachers required to deal with a project of curriculum reform contemplating the introduction of elements of informatics.

To pursue the stated objective we used a case-study methodology; this methodology seems to be appropriate, given the exploratory character of the study and the complexity of the relationships between teachers' orientations as regard informatics, the use of computers and mathematics.

We have chosen the case studies on the basis of the results of a questionnaire, which was aimed at investigating the above mentioned orientations, distributed to 120 mathematics teachers. The considered sample was significant with regard to types of schooling, social context and teachers' characteristics. According to the data collected, we singled out five teachers (the five case studies) which we considered representative of the emerged orientations.

Once selected the five case studies, we performed a deeper investigation of these cases through interviews (transcribed *verbatim*) and the analysis of the instructional materials used by the teachers during classroom work and of the tests employed by them in assessing students' learning.

ANALYSIS OF THE CASE STUDIES

In analysing the considered case studies we focus on the issues reported hereunder which we consider significant for the investigation of the links between teachers' beliefs in mathematics and the use of computers:

1) Peculiar features of mathematics teaching.
2) Teacher's inner philosophy of maths.
3) Ways in which informatics is introduced in relation to maths.
4) Motivations for the introduction or the refusal of informatics in maths teaching.
5) Ways of looking at informatics.
6) Opinions about new ways of working induced by the use of computers (e.g. team work).
7) Motivations at using (or not using) software packages.
8) Software packages used.

Issues 1) and 2) provide a framework of teachers' conceptions of maths and its teaching. Firstly, teachers were asked a number of questions concerning their behaviour in teaching (i.e. contents, time dedicated to the teaching of each content, degree of depth, types of problems faced, methodology of work, assessment, etc.). Answers encompass many factors which are difficult to synthesise into a unique orientation. Nevertheless, for each teacher, we found some feature which seem to characterise his/her teaching orientation. These features are related to the methodology of work in maths and/or to the way of transmitting maths knowledge. Then teachers were presented with statements that briefly outline different positions as regard philosophy of mathematics (platonism, formalism and constructivism). The teachers had to reflect on these statements and to state whether they recognise their way of looking at mathematics in one of these positions. It seems to us that this information, aimed at a first (rough) explanation of the teachers' inner philosophies of maths, may have relevance in studying the elements affecting teachers beliefs, their decisions on curricula and their methodology of teaching, as discussed in the specific studies (Jurdak, 1991; Lerman,1983).

Issues 3) and 4) are related to the way in which a proposal of curriculum reform which includes informatics is perceived by maths teachers. In this context it is important to stress how links between the teaching of maths and informatics were perceived. In particular, our interest is in investigating whether the links pointed out by teachers are conceptual (i.e. related to common objectives, methodologies, skills, etc.) or only instrumental (i.e. related to the opportunities offered by computers as tools for problem solving or as means to motivate students or to update teaching, etc.). Obviously the way in which links are perceived is connected to the way in which teachers look in general at informatics (issue 5), for example, if prevalence is given the idea of informatics as a technological subject or as a discipline with a strong theoretical foundation. The investigation on issues 3) and 4) enables us to point out also some awkwardness teachers feel about their traditional maths teaching and whether or not they perceive informatics and the use of computers as means to overcome these difficulties.

Issue 6 considers teachers' beliefs with regard to the new ways of classroom work induced by the use of computers (e.g. team work, 'experimental' strategies, autonomous work, etc.). We feel that the analysis of this aspect is important in order to ascertain the effective acceptance of the proposed curriculum change by the teachers, in other words, to establish whether the introduction of computers induces only superficial changes or more substantial ones.

Issues 7) and 8) point out the links between the different choices made by teachers with regard to the use of software tools (e.g. didactical or applicative packages) and the needs they feel in maths teaching. Differences concern both the type of chosen packages and the ways in which they are used. We remark that, as it will be pointed out from the reported findings, teachers usually use the software packages which are enclosed with the adopted maths textbook or use some applicative packages, such as Lotus spreadsheet, for which they had received some preparation in the training courses.

In table 1 we briefly outline findings resulting from the five case studies on the lines of to the issues discussed above.

Issues analysed	Case Study 1	Case Study 2	Case Study 3	Case Study 4	Case Study 5
1) Peculiar features of mathematics teaching	Stress on logic-deductive method	Pragmatic choice of objectives (priority given to those actually realizable)	Reliance on repetitive work as a means for maths learning	Prescriptive transmission of knowledge	Stress on 'aesthetic' aspects of maths without considering students' problems of learning
2) Inner philosophy of maths	Formalist	Constructivist	Formalist	The teacher is uncertain about her choice	Platonist
3) Ways in which informatics is introduced in relation with maths	Informatics teaching integrated with that of maths; stress on common objectives	Informatics teaching integrated with that of maths; prevalence of maths objectives	Informatics as a service-subject for maths; attention to informatics technical details	Passive use of computers with focus on 'appealing' aspects	Informatics is not introduced
4) Motivations for the introduction or the refusal of informatics in maths teaching	Need of a mediator of maths concepts	Need of reification in maths teaching	Need of proving the usefulness of maths in real life through the solution of problems	Wish to update teaching	Conflict between maths and informatics nature: maths certainty vs informatics uncertainty, existence vs computational aspects...
5) Ways of looking at informatics	Discipline with aspects (rules, methods, ...) 'close' to maths ones. The use of computers is only one aspect of informatics	Informatics as a discipline consisting of both theoretical and practical aspects	Informatics as technology which offers tools to problem solving (speeding up of computations, fostering of 'trial and error' strategies, ...)	Informatics as technology with no significant links with maths	Informatics as technology with no links with maths; computers outputs are not reliable since depend on the tool
6) Opinions about new ways of working induced by the use of computers (e.g. team work)	Interest on socialization of knowledge; positive acceptance of team work	Positive acceptance of team work; interest in the use of computers for checking maths processes	Team work is difficult to be managed in classroom; little space is given to individual initiatives	Refusal of team work; prescriptive attitude in teaching; no autonomy left to students	Conflict between ways of working in maths and informatics: individual vs team work; deduction vs 'trial and error' methods...
7) Motivations for using (or not) software packages	Little use of software tools: focus on formal methods rather than intuitive ones (e.g. visualization)	Use of software tools as a source of intuition; usefulness of visualization in maths learning	Software tools can be useful to develop maths exercises but no to introduce concepts	No use for software packages: they take too much time in relation to the opportunities offered;	Maths has to live for itself without the intervention of external elements
8) Software packages used	Lotus; a didactic package for geometric transformations and graphical representation of functions	Lotus; Microcalc; a didactic package for geometric transformations	A didactic package for the study of functions	A program for the visualization of fractals	

Table 1: Issues analysed and findings resulting from the case studies

Case Study 1

The teacher's attitude in mathematics teaching (stress on deduction processes and on theoretical aspects) guides her approach in the use of computers: she focuses on the basic informatics ideas rather than on technological aspects and concentrates on concepts (such as variable, syntax, algorithm) considered common to the two disciplines. The teacher sees computers as mediators of maths concepts since they offer motivations for the use of a formal language, they foster the adoption of a structured approach in problem solving and they make concrete the use of abstract concepts. For these reasons she lays greater emphasis upon the aspects related to programming than upon those closer to the use of software tools. These tools are considered useful in some case, especially at the beginning, but the teacher does not ascribe a substantial role to them in her teaching. This attitude points out her little confidence in an intuitive approach to learning and is linked with her orientation towards a formal view of mathematics. We remark that, during the interview, the teacher expressed her interest at using Cabri, since this didactical software is considered more congenial to her way of teaching geometry.

Case Study 2

This teacher reveals a pragmatic view in teaching which is present also when she introduces informatics elements and stresses those aspects she considers more relevant in preparing students for their future work. Computers are used in maths teaching as a motivating tool and as a means to facilitate, to strengthen or to build maths ideas and concepts. The use of software tools is to be considered in this context. They are seen as a means to introduce some maths notions more quickly or to reify concepts; this is done without forcing the *per se* introduction of the tools, producing, for example, *ad hoc* exercises, but using them as needed by the teaching situation. Teacher's attitude towards teaching can be put in relation to her inclination towards constructivism in looking at maths and is transferred also to the way in which she approaches informatics teaching. As a matter of fact she seems to regard informatics as a subject that allows the discovery of properties and relationships through personal inquiry.

Case Study 3

The teacher looks at maths as a system of formal rules, informatics is seen mainly as a technological discipline, service-subject to maths, that can be useful to solve problems and to motivate students. Its insertion in the program does not substantially change the way of approaching and of introducing maths concepts. Links with maths teaching concern more function (use) than common methodologies and methods. The teacher defines an advantage of the use of computers in maths teaching the stepping up of computation procedures and the development of trial and error strategies in problem solving. He is also interested in providing students with views on the practical utility of computers and their impact on social life. The way how software packages are used should be considered according to this general orientation: they can be useful to develop maths exercises but not to introduce or visualise concepts.

Case Study 4

This teacher looks at informatics as a technological subject which can be useful to attract students' interest and to 'surprise' them (see, for example, the visualisation of fractals), but she does not ascribe real educational potentialities to the use of computers in maths teaching a part from a surface updating of contents. She seems to have accepted the curriculum change only to comply with the suggestion of the new maths programs but maintains a substantial lack of confidence in the opportunities offered by computers, as evidenced also by her refusal of the new ways of work induced by the use of these tools. In general she has a prescriptive attitude in maths teaching that is transferred also to the use of computers which is carried out mostly in a passive way.

Case Study 5

This teacher has chosen not to use computers in his maths course; we have decided to include him in our study since we think that it may be relevant, for the purpose of our research, to analyse also the reasons for which, in certain cases, informatics is waived. The teacher looks at informatics as a technological subject and any relation with maths is denied, even at the level of service-subject. This opinion is connected with the teacher's conception of maths and of its teaching: he considers deduction as the central method in maths and working by trial and error the method peculiar to the computer world. These two methods are not seen as complementary but antithetical and the latter one is rejected. According to the teacher, the interest in maths is in 'existence' aspects and not in the computational ones. He is further not interested in problem solving as an activity to be performed in his teaching. From a pedagogical point of view, the teacher does not recognise a role of stimulus to computers when doing maths and dislikes team work since in his opinion mathematical learning needs individual work.

CONCLUSIONS

We believe our work may give new insight into the studies of maths teachers' beliefs since it stresses the links between the existing beliefs and those induced by changes in curricula. The case we consider concerns the introduction of computers in maths secondary school curriculum. We observed two levels of looking at the role of computers in maths teaching: a 'surface level' that regards computers as tools for better presenting topics and a 'deep level' where computers are seen as means for constructing knowledge. The analysis we have performed by means of the case studies points out that teachers' beliefs on the role of computers are mainly a projection of their beliefs on maths teaching. If teaching maths is interpreted as a transmission of knowledge, without a real participation of students, the use of computers appears of little relevance (or even more as a disturbance) since teachers are not interested in its 'constructive' facilities. On the other hand, teachers who are interested in constructing knowledge (and, as a consequence, in making concrete abstract ideas, in concepts reification, in knowledge mediators, etc.) find in computers answers to their needs.

The beliefs on the nature of mathematics are less influential in the acceptance or refusal of computers but play a role in the choice of the type of software tools used, see, for example, the relation between the interest on deduction processes expressed by case study 1, her interest on Cabri and the little relevance ascribed to other kinds of software tools.

From the previous considerations, it follows that in planning maths teachers training for the use of computers, it is necessary to carefully discuss the aims of maths teaching, presenting not only characteristics and opportunities offered by computers (and of software tools) but also didactical itineraries in which their use is functionally linked to maths objectives. This integrated approach is a necessary premise for effecting 'real' curricular changes with the introduction of computers.

REFERENCES

Bottino, R. M. and Furinghetti, F.: 1990, 'Computer science in basic education: curricular issues and school practice prospects', in McDougall, A. & Dowling, C. (editors) *Computer in education. Proceedings of IFIP TC3 Fifth World Conference on Computers in Education - WCCE 90*, North Holland, Amsterdam, 25-30.

Bottino, R. M. and Furinghetti, F.: 1991a, 'Computer science and mathematics education in upper secondary school (age 14-16)', in Waerbecq, A. (editor) *Proceedings of CIEAEM 41*, 333-341.

Bottino, R. M. and Furinghetti, F.: 1991b, 'A large-scale teachers training programme in informatics: chances and limits', in Winterburn, R. (editor) *Aspects of educational technology, Vol XXIV. Realizing human potential*, Kogan Page, London, 140-145.

Cornu, B. and Ralston, A.: 1992, *The influence of computes and informatics on mathematics and its teaching*, Science and technology education, 44, Unesco, Paris.

Hoyles, C.: 1993, 'Mathematics teaching and mathematics teachers: a meta-case study', *For the learning of mathematics*, 12, n. 3, 32-44.

Jurdak, M.: 1991, 'Teachers' conceptions of math education and the foundations of mathematics', in Furinghetti, F. (editor) *Proceedings of the PME XV*, 1991, II, 221-228.

Lerman, S.: 1983, 'Problem-solving or knowledge-centered: the influence of philosophy on mathematics teaching', *International journal of mathematical education in science and technology*, 14, 59-66.

Moreira, C. and Noss, R.: 1993, 'Catalysing teacher attitude change with computers', in Hirabayashi, I., Nohda, N., Shigematsu, K. & Lin, F-L. (editors) *Proceedings of PME conference XVII*, I, 268-275

Ponte, J.: 1990, 'Teachers' perceived roles of the computer in mathematics education', in Booker, G., Cobb, P. & De Mendicuti, T. N. (editors) *Proceedings of the PME XIV*, 1990, III, 183-190.

Wojciechowska, A.: 1989, 'Curriculum reform in mathematics: beyond the impossible revolution?', *Journal of curriculum studies*, 21, 151-159.

Taylor, H. G., Aiken, R. M. and van Weert, T. J.: 1991, 'Guidelines for good practice. Informatics education in secondary schools', *Document IFIP Group 3.1*.

A CASE STUDY OF A TEACHER'S CHANGE IN TEACHING MATHEMATICS

Ada Boufi

University of Athens, Greece

One of our purposes as we worked with a first-grade teacher, was to develop, in her classroom, a form of practice compatible with socioconstructivist assumptions about students' learning of mathematics. Supporting her efforts to cope with the constraints of a compulsory national curriculum, which has a traditional character, gave us the opportunity to study her process of learning In this paper, we present our first attempt to analyze the process by which she accommodated our purpose and changed her practice in the course of her interactions with the students and us.

Recent reform recommendations in mathematics education (NCTM, 1989, 1991) are linked with many research studies related to teachers' education (e.g., Ball, 1990; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb, Yackel, & Wood, 1991; Cooney, 1985; Simon, 1991; Thompson, 1984). These studies allow us, in a certain extent, to become aware of the complexities involved in innovative approaches of mathematics teaching. The crucial role of the teacher in any reform movement has led most of these studies to analyses of different aspects of teachers' knowledge, beliefs, and practices. However, research focused on the processes by which teachers reorganize their pedagogical beliefs and related practices is still incomplete. A few studies related to this issue do not account for teachers' learning in the context of their classrooms. Although the contextual nature of teaching has long been recognized, only in a series of studies by Cobb, Wood, and Yackel, classrooms are viewed as learning environments for teachers, and teachers' learning is tied to their classroom interactions (Cobb, Wood, & Yackel, 1990; Cobb, Yackel, & Wood, 1991; Wood, Cobb, & Yackel, 1990; Wood, Cobb, & Yackel, 1991). These studies provide an impetus for our work presented in this paper. Our purpose is to analyze the process by which a first-grade teacher reorganized her traditional pedagogical beliefs and practices, and thus, learned as she was trying to teach in a way that encourages students' autonomous construction of mathematical knowledge. Our analysis takes into account the teacher's intentions as well as the meanings she interactively constituted with her students. In addition, we take into account the fact that our teacher was obliged to cope with the constraints of a compulsory national curriculum, traditional in character.

THEORETICAL PERSPECTIVE/METHODOLOGY

Our theoretical perspective for studying our teacher's learning derives from socioconstructivism. From this perspective, teachers' learning can be viewed as both an individual and an interactive activity. More specifically, teachers' learning can be characterized as a process of self-reflection and mutual adaptation by which they reorganize their pedagogical beliefs in order to give meaning to their personal experiences in the process of interacting with others. In this sense, we believe that our teacher's interactions with her students in the classroom, with us, as well as with other members of the school community provided opportunities for the transformation of her traditional beliefs and practices.

The above characterization of teachers' learning is analogous to the way we view students' learning. Therefore, in the analysis of our data, we used theoretical constructs and methods that have been developed for the analysis of students' learning in classrooms (Voigt, 1985). In particular, as we analyzed the discourse in the episodes we selected, our focus was on the nature of classroom social interactions. This allowed us to infer the patterns of interaction, and thus, the social norms negotiated in the classroom. The evolving nature of these patterns was taken as an indication of our teacher's learning. In turn, based on the reflexive relationship between norms and beliefs, we were in a position to characterize the teacher's implicit beliefs about her own role, her students' roles, the nature of mathematical activity, and the role of the mathematics textbook. In this way, our teacher's learning was analyzed from a psychological and a sociological perspective (Cobb, Yackel, & Wood, 1993).

PRESENTATION AND ANALYSIS OF DATA

Our teacher was induced in the project after having participated in a seminar that we organized the year before. Some information about this seminar can be found in Boufi and Kafousi (1993).

In the course of an interview prior to the seminar, we collected some data about our teacher's beliefs. As a teacher of mathematics, she believed that she was the only person in the classroom responsible for students' learning. In accounting for the reasons of her students' difficulties, she felt strongly that, "They are only due to bad teaching, which results from the teacher who doesn't prepare himself/herself adequately." Her students' role in learning mathematics was viewed as passive. As she tried to characterize "the successful student" in mathematics, she mentioned that, "He/she is the student who can focus his/her attention and solve problems step-by-step by applying what he/she already knows." Mathematics, for her, was a subject which, compared to the others, could be characterized as, "The lesson with the very well-defined answers and fixed solutions...the lesson in which words, like 'maybe' and 'possibly' do not have a place and are not allowed." Finally, textbooks were described as having a mechanistic approach to mathematics, and as she said, "I am dissatisfied with them, but I am obliged to use them." Observations in her classroom, before the interview, confirmed that the above beliefs were compatible with her practice.

Starting our collaboration with her, we were audiotaping and, occasionally, videotaping most of her lessons. Also, twice a week, we were observing her lessons, and every week, we were meeting with her in order to discuss students' progress and plan the following week's activities.

The three episodes that we present and analyze below were selected because they illustrate some crucial points in our teacher's development and because each of them reflects her practice for a significant period of time, thus, allowing us to infer regularities in her classroom social interactions.

1st Episode:

This episode occurred in November. Students were working on a family of word problems. The whole class discussion that follows concerns the second problem.

Teacher: So, at the second stop, the bus arrived with 10 people. There, 3 people got off the bus. Could you tell me how many people are in the bus now? Hands, thinking and explanation. What do you say Katerina?

Katerina: In the bus, I think there were 5 people because 3 people got off...and there were 8 altogether [raises 8 fingers simultaneously] and as 3 people got off [folds down 3 fingers simultaneously], now, there are 5.

Savas: Ma'am, I disagree.

Teacher: Why do you disagree?

Savas: There are not 5, there are...

Teacher: No. Why do you disagree with Katerina?

Savas: Because...you said that there are 10 people and 3 people got off. If 2 more people had gotten off the bus, there would again be...5 more people on the bus...so, but there are not 5.

Teacher: The people were 10. Tell me, how many people are in the bus?

Teacher: You said that there were 10 people. You, Katerina, how many people did you say that there were on the bus? What did you say Kosmas? Speak louder.

Kosmas: 8 people were in the bus when it started. Now, there are 10 people.

Teacher: There were 10 people and then 3 people got off. What do you say Danae?

Danae: There were 10 altogether and 3 got off [shows 10 fingers and folds down 3 fingers]. Now, there are 1,2,3,4,5,6,7 [counts on her fingers].

Teacher: Okay. That's very nice.

[Students agree, and the teacher moves on to the next problem.]

In this episode, the teacher expects that finger counting can be the only means to an answer. By not asking for different ways of finding the answer, she and her students take for granted that finger counting is the only acceptable strategy that they can use. As the textbook is full of pictures with discrete objects and the teachers' guide encourages the use of manipulatives, we might take this expectation as reasonable. However, she is in the process of establishing, in her classroom, norms that facilitate students' communication among themselves. By asking students to give reasons for their disagreement, she, in effect, encourages and expects students to listen and understand others' solutions. Another point that we could make is that she was reluctant to let students resolve their differences. When she does not make sense of students' explanations, she always decides to funnel them to the answer that she has in her mind through direct questioning.

2nd Episode:

The following episode occurred just after the Christmas break. Students had worked in pairs on the following subtraction sentences: 20 - 8 = ___, 20 - 3 = ___, 16 - 3 = ___, and 18 - 3 = ___. Segments of the whole class discussion that followed the small group work are presented below:

Teacher: Let's see what you have done. Who wants to tell us the first? Will Chrissoula, who is raising her hand, tell us?

Chrissoula: We put together with Nikos 20 [both children hold up their two hands], and I took out 8 [Chrissoula closes eight fingers]...and there remained 1,2,3,4...12.

Teacher: Do you agree that 12 remain?

Students: Yes.

Teacher: I would like to hear many ways in order for me to agree. We listened to one way, but don't forget that if you wanted to find it, and you did not have a partner in order to take fingers, what would you do? What would Antonis and Yannis like to say?

Antonis: Ma'am, we found it by using the counting frame.

[The teacher is letting students present their solutions. Most of the students are using their and their partner's fingers. However, she only repeats solutions that are not based on primitive finger counting.]

Marianna: I had the 20 in my mind, and then I was taking off, one by one, from my hands.

Costas: That reminds me of something...that Marianna has done in the past.

Marianna: [She counts down while keeping record on her fingers.] 20,19,18,17,16,14.

Teacher: [Intervenes] 15.

Marianna: 15,14,13 and 12 remain.

Yannis: I did not understand.

Teacher: Yannis says he did not understand. Can you repeat it, Marianna?

Danae: You go down and you confuse us.

Marianna: I take the 20,19,18,17,16,15,14,13,and 12. The 10 are in my mind.

Student: Another ten?

Marianna: 2 and 10 in my mind makes 12.

Teacher: That's a wonderful way. Do you want her to explain this once more?

Student: But always when we do math, she always finds this way either with another one or by herself.

Teacher: But this way is comfortable for her.

Student: It's like someone else found it and she repeats it.

Teacher: Someone else found it?

Kosmas: She always says the same way.

Teacher: We are going to think, and we will find many ways; that does not mean that her way is not wonderful.

Student: But it confuses.

Teacher: Let is confuse us a little.

Student: Every day she says this way.

Teacher: Your way is very nice, Marianna. Shall I repeat is one more? I will raise my fingers so that you can see them well. Well, Marianna said as I want to take off 8, I will go back from 20, 8 times. And she goes down 20,19,18,17,16,15,14,13. How many remained?

Students: 12.

The teacher seems, now, to be aware that counting fingers in primitive ways has become an institutionalized practice in her classroom. Her effort to make students use different methods is apparent in this episode. The fact that the textbook does not any longer emphasize counting makes her even more reluctant to accept finger counting. On the other hand, her increased understanding of students' competencies informs her interactions with them and helps her to abandon her romantic attitude that students can use any method they like. She now expects that they can use more sophisticated methods. The norm that students must explain their answers seems to be well established. Also, now when a student presents a sophisticated solution, she usually repeats it. This action does not only create learning opportunities for the rest of the class, but is also used as a means to encourage solutions that she values.

3rd Episode:

In April, students are discussing the problem: 4 x ___ = 16, after they had found different ways for 2 x 8 = ___.

Teacher: Let's hear Danae.

Danae: I said, 4,...5,6,7,8,9,10,11,12,13,14,15,16[looks at her fingers]. So it must be 12.

Antonis: That's wrong...2 times 12 is 13,14,15,16,17,...you see, it's more than 16.

Teacher: Did you understand Danae?

[Danae seems confused.]

Teacher: Can you repeat, Antonis, what you said?

Antonis: 12 plus 12 is...

Teacher: Antonis, can you draw a figure? [Antonis makes 4 circles and places 12 in each of them. Then, he tries to find out with Danae how much is 12 plus 12.]

Teacher: What do you think Danae?

Danae: I am wrong. I thought that it was addition.

Teacher: Kosmas, how did you do it?

Kosmas: I took 4 times 2 and it was 2 plus 2, 4 plus 2, 6, plus 2, 8. [He raises 4 fingers to keep track of his counting.] Then, I tried 4 and it was right. [He repeats the same strategy.]

Teacher: [Repeats his solution, and, at the same time, she circles the first 2 circles and places an 8 next to it and does the same thing as she moves to the next 2 circles.]

The teacher is now more willing to let students resolve their disagreements. Her goal when she intervenes is to facilitate their communication. Another aspect of her practice is the use of symbolic representations to model her students' thinking. As we can see, students are also expected to use symbolic representations for their explanations.

CONCLUSIONS

The above analysis shows some of the changes in our teacher's beliefs and practices. Her classroom experiences have persuaded her that students can share a part of the responsibility for their learning. As a consequence, her views about her own role as a teacher have also changed. Mathematics is not anymore the subject with the fixed answers. As she works with her students, she experiences a way of doing mathematics much more different than the one she was used to. The textbook ceased to be the source of her teaching. In our meetings, she was increasingly involved in the development of instructional activities.

It is apparent that the changes in her beliefs were reflexively related to the changes in her practice. However, we need more systematic analyses of the teachers' practice as they try to be innovative if we want to develop models that characterize their learning. Then, teacher education programs will be easier to design.

REFERENCES

Ball, D.L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*, 449-466.

Boufi, A. & Kafousi S. (1993). Learning opportunities in an in-service teacher development program. *Proceedings of the Seventeenth International Conference of the Psychology of Mathematics Education* (vol. 2, pp. 207 - 214). Tsukuba, Japan: Psychology of Mathematics Education.

Carpenter, T.P., Fennema, E., Peterson, P.L., Chiang, C.P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*, 499 - 531.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R.B. Davis, C.A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics.* (Journal for Research in Mathematics Education, Monograph No. 4, pp. 125 - 146). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Yackel, E., & Wood, T. (1991). Curriculum and teacher development: Psychological and anthropological perspectives. In E. Fennema, T.P. Carpenter, & S.J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 92 - 131). Albany, N.Y.: SUNY University Press.

Cobb, P., Yackel, E., Wood, T. (1993). Theoretical orientation. In T. Wood, P. Cobb, E. Yackel, & D. Dillon (Eds.), *Rethinking elementary school mathematics: Insights and issues.* (Journal for Research in Mathematics Education, Monograph No. 6, pp. 21 - 32). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16*, 324 - 336.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

Simon, M. (1991). Initial development of prospective elementary teachers' conceptions of mathematics pedagogy. *Proceedings of the Fifteenth International Conference of the Psychology of Mathematics Education* (vol. 3, pp. 270 - 277). Assisi, Italy: Psychology of Mathematics Education.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*, 105 - 127.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathematiques, 6*, 69 - 118.

Wood, T., Cobb, P., & Yackel, E. (1990). The contextual nature of teaching: Mathematics and reading instruction in one second-grade classroom. *Elementary School Journal, 90*, (5), 497 - 513.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal, 28* (3), 578 - 616.

AN ANALYSIS OF YOUNG CHILDREN'S STRATEGIES AND USE OF DEVICES FOR LENGTH MEASUREMENT

Gillian M. Boulton-Lewis, Lynn Wilss & Sue Mutch

Queensland University of Technology, Brisbane, Australia.

This is a study of the strategies and length measuring devices used by 70 childrer in years 1,2 and 3 in two schools in Brisbane. Children were given matericls to use and asked to explain their procedures. The teachers were interv ewed to determine the procedures that they were introducing. The overall developmental sequence was a peak in use of arbitrary devices in year 2, a high interest in use of standard devices in all years with a significant increase in year 3, and a significant increase in understanding in the length measuring process over the three years. These results are in conflict with the normal recommended curriculum sequence and are considered from the perspective of the demand they might make on capacity to process information.

The intention of the research was to present children with a range of measurement tasks in order to determine the strategies and devices that they would choose to use to measure length. An analysis of possible strategis and devices was made in advance from an information processing perspective and from earlier research in the area and compared with the strategies and devices that the teachers were introducing. The hypothesis was that some of the representations and strategies that teachers currently accept as good practice may not actually be the most appropriate or effective.

THEORIES OF COGNITIVE DEVELOPMENT, MEASUREMENT AND USE OF MEASURING DEVICES

Length measurement requires knowledge of number, of units of length and of application of both of these in the length measuring process (Boulton-Lewis, 1987). Aspects of the length measuring process can be analysed on the basis of theories of cognitive development such as those of Halford (1992; 1993) or Case (1992) and approximate ages of achievement can be proposed, assuming that children possess requisite prior knowledge.

Halford's structure mapping theory of cognitive development (1988; 1992; 1993) accounts for cognitive development in terms of the structural complexity of tasks, together with the factors of capacity and knowledge. He has defined four levels of structure mapping the processing demands of which have been demonstrated empirically and by task analysis.

In terms of Halford's theory the ability to recognize and name the attribute of length in a nominal sense is a task at the element mapping level and should begin to be possible from 1 to 2 years of age even if an incorrect term such as big is used. Comparison of two lengths and the ability to use a standard measure without understanding the reasons for construction of such a measure or the relationship of units of length is an ability at the relational mapping level. This should allow the child to recognize equality or inequality of length and to order objects by pair by pair comparison. All these abilities should begin from 2 to 5 years. From about 4 or 5 years, the child should begin to be capable of understanding aspects of length at the system mapping level which require cognition of three elements and the relation between them. This should allow recognition of the invariance of the length of an object despite distracting perceptual factors, such as conservation of length as described and tested by Piaget, Inhelder & Szeminska (1960), transitive reasoning about length in situations that require both standard and non standard measures, and procedures for measuring length that require consideration of comparative length and number of sub units.

Case (1985, 1992) hypothesized four stages, rather than levels, of cognitive development (sensorimotor, interrelational, dimensional, and vectorial). In terms of Case's (1992) theory cognitive development takes place recursively. That is earlier separate operations are integrated to produce higher order operations. An element mapping task would be one at the last interrelational substage from about 31/2 to 5 years where, for example, numbers can be used to tag a set of objects and presumably words can be used to tag objects according to the attribute of length. From about 5 to 10 years of age children move through the dimensional substages where they can cognize third order relations. From about 5 to 7 years of age (unidimensional substage) children can understand the relation between enumeration and quantity, which is a relation in a single dimension (Case & Griffin, 1990; Case & Sowder, 1990), hence presumably they should be able to understand single dimension length relations such as longer than. At the bidimensional thought substage, from about 7 to 9 years children can co-ordinate two units of operations which allow them to understand part whole number relations and hence presumably relations between the length of two objects each compared with a third object. The final stage of integrated bidimensional thought allows children from about 9 years to construct and compare entities that result from numerical operations and hence this understanding should be able to be used in length measuring situations which require comparison of lengths of objects in terms of numbers and lengths of units.

In research with children aged from 3 to 7 years Boulton-Lewis (1987) predicted sequences of length measuring knowledge on the basis of a logical analysis from measurement theory, from a literature search, and from an information processing analysis of the tasks on the basis of Halford's (1982) and Case's (1982) theories. Empirical research found three clearly separated groups of tasks which included at the first level variables such knowledge of equality and inequality of length, ordering of lengths by pair by pair comparison and related number knowledge; at the second level recognition of length invariance, correct response to a transitivity task and then conservation of length on a Piagetian task; and at the third level, in order, were measurement with a ruler, a length measurement strategy when arbitrary units were involved and then the ability to reason transitively.

Predicting conservatively from Halford's and Case's theories one would expect children to begin to cognize and name the attribute of length in some way from 1 to 3 1/2 years, begin to compare lengths from 4 to 5 years, and begin to conserve length and reason transitively from 5 to 7 years onwards and hence be able to understand the length measuring process.

Resnick (1987) discussed differences between learning in and out of school and provided an example of a real world situation in which money is used. In a real buying situation children would use the coins they have to make up an amount rather than calculate in an abstract way what they would need. Similarly it is proposed that children will want to measure length in the way they see it done out of school.

On the basis of the theories of cognitive development described above, and from what we know about learning in and out of school the conventional wisdom about the kind of sequences usually recommended in curriculum documents for teaching length measuring to young children is questioned here. The Queensland curriculum guides recommend starting with arbitrary units then progressing to the standard units of metre and centimetre respectively by the end of year three. It would seem that this sequence would defeat the purpose of what it is intended to achieve. In the third group of variables identified by Boulton-Lewis (1987) children could succeed with measuring using a ruler before they could devise a measurement strategy using arbitrary units and before they could reason transitively. In order to realise that arbitrary measures are not reliable a child must reconcile the varying lengths and numbers of arbitrary units and reason transitively. On the other hand to use a standard device such as a 30cm or metre ruler to make pair by pair direct comparisons of lengths of objects is a less demanding task. It also has the advantage that it appears to be a real world meaningful activity.

RESEARCH IN CHILDREN'S LENGTH MEASURING STRATEGIES

Research concerned with young children's knowledge and strategies for length measuring addresses two main issues; the sequences and stages children go through (Boulton-Lewis, 1987; Hiebert, 1981), and children's understanding of identified measurement concepts and problems arising in the practical application of measurement strategies (Carpenter, Corbitt, Kepner, Lindquist,& Reys, 1981; Hiebert, 1984; Miller, 1989).

Hiebert's (1981) posited that children move initially from no understanding of the measurement situation, to employing an appropriate measurement strategy but achieving an approximate solution, to using finally an appropriate strategy with an accurate solution.

Carpenter et al. (1980), Hiebert (1984), and Miller (1989) believe that children need to have an understanding of basic measurement concepts if they are not to make mistakes. Problems that can arise in the use of a ruler include, incorrect alignment with the object to be measured, focussing on the numbers rather than the standard unit of measurement, starting at a point other than zero and leaving gaps (Hiebert, 1984; Kouba et al. 1988).

Haylock and Cockburn (1984) maintain that comparisons can be made directly without using measuring devices i.e. visual peception. For indirect comparions they believe that children should begin measuring with non-standard (arbitrary) devices to gain an awareness of the need for a standard unit. However this belief does not take into account the information processing demand that the use of arbitrary units would make, neither does it allow children to believe that they are really measuring as they see in the real world.

This research was to determine what strategies and measurement devices children would choose to use in a range of measurement situations and then to consider the results in terms of the theories described above and the procedures that the teachers were using. The study was undertaken in two schools at early and late in the school year, with a sample of children from years 1 to 3.

METHOD

Sample

This consisted of 70 children, 23 in year 1, 25 in year 2 and 22 in year 3, in two suburban schools in Brisbane in low to medium and high socioeconomic areas.

Design

The teachers were asked to identify in advance their objectives for length measurement, and the representations and strategies that they intended to use. Each child was interviewed in term 2 and term 4, with an interval of 16 weeks, to assess use of concrete materials and the processes or strategies they chose to use. All interviews with the children were videotaped, viewed, transcribed and coded by the authors.

Materials

Materials were selected on the basis of what teachers said they intended to use and included unifix cubes, unmarked sticks, half and full matches, string and scissors, children's body parts, metre rulers and 30 centimetre rulers.

Procedure

The child and the interviewer discussed the materials. Any unfamiliar materials were explained in case the child wanted to use them. Three tasks were presented as follows.

Task 1. The child was shown 3 pieces of string, 2 pieces were 27 centimetres in length, the third piece was 24 centimetres long (cf. Boulton-Lewis, Neill & Halford, 1987).

The child was asked 'Do you think any of these pieces of string are the same length? Why? 'Could you measure them using any of the materials to find out if you are right?

Task 2. The child was shown a skipping rope and told 'This is my favourite skipping rope and I want to buy another one just like it, so I need to know how long it is.' The child was then asked 'How could you find out how long it is? Would you measure it?'

Task 3. This task posed problems of comparison between two rows of matches, with one of the rows being constant for each comparison and where the child could not move the matches to use direct comparison as a strategy. The child was shown four cards with matches glued onto them (cf Boulton-Lewis, Neill & Halford, 1987). Card 1 (C1) had five full matches glued in a straight line and was paired with cards 2 (C2), 3 (C3) and 4 (4C) respectively as shown in Figure 2.

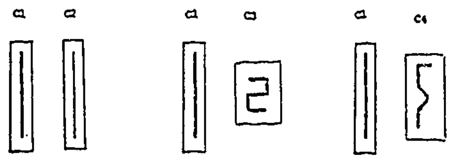


FIGURE 2 Cards for length as a function of size and number of match stick units.

The child was asked 'Do you think these lines of matches are the same length, or are they different? Why? How could you find out?'

RESULTS

Teacher expectations and strategies

All teachers indicated that they followed closely the objectives, materials and activities recommended in the state curriculum as described above.

Categorisation of responses

Strategies were derived from a literature review and examination of our data. The responses that the children gave were categorised first by successful and unsuccessful use of a strategy. These were as follows, SVP/UVP - successful/unsuccessful use of visual perception, SAD/UAD - successful/ unsuccessful use of arbitrary device, SSD/USD - successful/unsuccessful use of standard device, SSDL - successful use of standard device with language error or lack of measurement language, SSDA/USDA - successful/unsuccessful use of standard device in an arbitrary way, NSDC - nonconventional use of standard device but correct answer, MUC - mixed use of standard and arbitrary units, correct answer, MUIC - mixed use of standard and arbitrary units, incorrect answer, NO - no response at all, no measurement attempted, probably lacking the concept of length measurement. The categories were then collapsed to specific strategy use rather than correct and incorrect strategy use, that is VP, AD, SD, SDL, NSDC, SDA, MU and NO.

Analyses

A MANOVA for time by year was computed for SVP, UVP, SAD, UAD, SSD, USD, SDL, SDA, USDA, NSDC, MUC, MUIC and NO. The overall time by year effect was significant (Pillai's trace = 0.6, F = 1.8, DF = 2,67, p = .015). The univariate analyses for time by year showed that differences for UAD (unsuccessful use of arbitrary device) and NO (no response, no measurement attempted) were significant, F = 2.9 (p = .06) and F = 10.6 (p = .00) respectively. Tukey post hoc analyses were used to confirm the significant differences between groups in these categories at the 0.05 level. These differences are shown in Figures 3a and 3b. There was a significant increase in the un-successful use of arbitrary devices late in year 1, a decrease to late year 2, an increase early in year 3 and then a decline. There was a sharp decline in the use of the NO strategy late in year 1, followed by a marginal increase early in year 2 with this decreasing to no use of this strategy at all in late year 3.

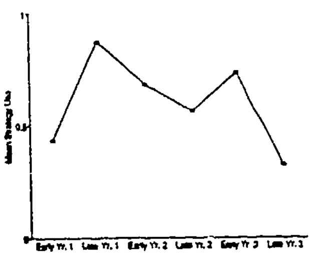


FIGURE 3a Unsuccessful use of arbitrary device early and late in each year

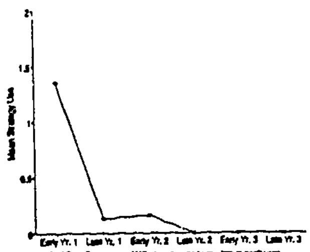


FIGURE 3b Frequency of NO response early and late in each year

Oneway analysis of variance for VP, AD, SD, SDL, NSDC, SDA, MU and NO showed significant differences between years for SD, SDA and NO which were confirmed at the 0.05 level by Tukey post hoc analysis. These differences were for SD between year 3 and both years 1 and 2 (Fig. 4a), for SDA between years 1 and 3 (Fig. 4b) and for NO between years 1 and both years 2 and 3 (Fig. 4c). The overall developmental trends shown in these figures indicate that the use of arbitrary devices increased in year 2, probably due to teaching effects, that there was high interest in the use of a standard device even in year 1, that this increased over the three years and significantly in year 3, that standard devices were used in an arbitrary fashion from year 1 and this kind of use declined significantly by year 3, and that there was a significant increase in understanding in the length measuring process from year 1 to years 2 and 3 as indicated by the decline in NO responses.

An examination of strategy use by task over the three years showed that the most frequently chosen were for task 1 (SD, SDL), task 2 (SD, SDL and AD equally), task 3a (AD, SD), task 3b (AD, VP), task 3c (AD, VP). This analysis of strategy use also showed that children change strategy to suit their perception of the difficulty of the task as proposed by Siegler and Robinson (1982) for number and that there was a steady preference for use of a standard device, where possible, despite the fact that teachers were emphasising the use of arbitrary devices until year 3.

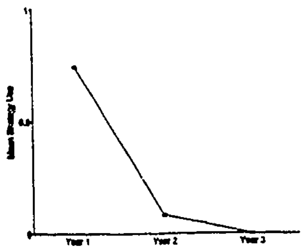


FIGURE 4c: NO response by year

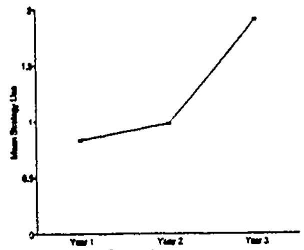


FIGURE 4a: Use of standard device by year

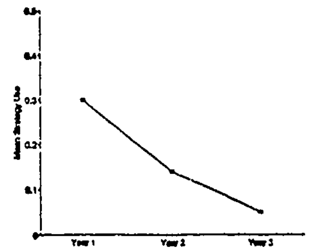


FIGURE 4b: Use of standard device in arbitrary way by year

Oneway analysis of variance of strategy use by school showed no significant differences between them.

DISCUSSION

The results confirm some of the predictions made from the theories of cognitive development. Generally there was a significant increase in understanding of the length measuring process over the three years from about 5 to 8 and a half years. They also confirm that children will choose to use a standard measuring device, if possible, even if they do not understand it fully or use it accurately.

CONCLUSION

The research indicates that a careful examination should be made of the sequences for length measuring recommended in curriculum guidelines. We would suggest that children be encouraged to measure directly and indirectly with standard measures (and arbitrary units if they so desire) from the first year of school. To avoid the kinds of errors of usage associated with standard measures described in the literature and in this research, teachers could discuss explicitly with children how people use standard measures, that is that they ignore the gap at the end of a ruler, do not leave spaces between measures and so on.

The results also suggest that if children prefer to use standard measures then measurement activities would have more interest and meaning for children if they were encouraged to use them and if the activities appeared to be for some real life purpose.

ACKNOWLEDGEMENT

This research was funded by a grant from the Australian Research Council.

REFERENCES

Boulton-Lewis, G.M., Neill, H. & Halford, G.S. (1987). Research into educational potential of aboriginal Australian children in south-east Queensland. Report to the Australian Institute of Aboriginal Studies, February.

Boulton-Lewis, G. M. (1987). Recent cognitive theories applied to sequential length measuring knowledge in young children. *British Journal of Educational Psychology, 57*, 330-342.
Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. R. (1981). Measurement. In *Results from the Second Mathematics Assessment of the National Assessment of Educational Progress.* Reston, Virginia: NCTM.
Case, R. (1985). *Intellectual Development: Birth To Adulthood.* Orlando, FL: Academic Press.
Case, R. (1992). *The Mind's Staircase.* Hillsdale, NJ: Erlbaum.
Case, R. & Griffin, S. (1990) Child cognitive development: The role of central conceptual structures in the development of scientific and social thought. In C. A. Hauert (Ed.) *Developmental Psychology: Cognitive, Perceptual-motor, and Neuropsychological Perspectives.* North Holland, Elsevier.
Case, R. & Sowder, J.T. (1990). The development of computational estimation: A neo-Piagetian analysis. *Cognition and Instruction, 7*(12), 79-104.
Chi, M. T. H. & Ceci, S. J. (1988). Content knowledge: its role, representation and restructuring in memory development. *Advances in Child Development and Behavior, 20*, 91-142.
Halford, G.S. (1988). A structure-mapping approach to cognitive development. In A. Demetriou (Ed.), *The Neo-Piagetian Theories of Cognitive Development: Toward an Integration.* Amsterdam, Netherlands: North-Holland.
Halford, G.S. (1992). Analogical reasoning and conceptual complexity in cognitive development. *Human Development, 35*, 193-217
Halford, G.S. (1993). *Children's Understanding: The Development of Mental Models.* Hillsdale, N.J.: Erlbaum.
Haylock, D. & Cockburn, A. (1989). *Understanding Early Years Mathematics.* London: Paul Chapman.
Hiebert, J. (1981). Cognitive development and learning linear. measurement. *Journal for Research in Mathematics Education. May*, 197-210.
Hiebert, J. (1984). Why do some children have trouble learning measurement concepts? *Arithmetic Teacher. March*, 19-24.
Kouba, V. L., Brown, C. A., Thomas, P., Lindquist, M. M., Silver, E., & Swafford, J. (1988). Results of the Fourth N.A.E.P. Assessment of Mathematics: Measurement, Geometry, Data Interpretation, Attitudes and Other Topics. *Arithmetic Teacher.* May, 10-16.
Miller, K. (1984). Child as the measurer of all things: measurement procedures and the development of quantitative concepts. In C. Sophian (Ed.) *Origins of Cognitive skills.* New Jersey: Erlbaum.
Miller, K. (1989). Measurement as a tool for thought: The role of measurement procedures in children's understanding of quantitative invariance. *Developmental Psychology.* 590-597.
Piaget, J., Inhelder, B. & Szeminska, A. (1960). *The Child's Conception of Geometry.* London: Routledge and Kegan Paul.
Queensland Department of Education. (1989). Years 1 to 10 Mathematics Sourcebook: Activities for teaching mathematics in Year 1.
Queensland Department of Education. (1990). Years 1 to 10 Mathematics Sourcebook: Activities for teaching mathematics in Year 2.
Queensland Department of Education. (1991). Years 1 to 10 Mathematics Sourcebook: Activities for teaching mathematics in Year 3. 102-107.
Resnick, L. B. (1987). Learning in school and out. Educational researcher, 16(9), 13-20.
Resnick, L. B., Bill, V. & Lesgold, S. (1992). Developing thinking abilities in arithmetic class. In A. Demetriou, M. Shayer & A. Efklides (Eds.) Neo-Piagetian Theories of Cognitive Development: Implications and Applications for Education. London: Routledge and Kegan Paul.
Siegler, R.S.and Robinson, M. (1982). The development of numerical understanding. Advances in child development and behavior. 16, 241-312.

AN INVESTIGATION INTO THE LONGER TERM EFFECTS OF A PRE-SERVICE MATHEMATICS METHOD COURSE.

Chris Breen and Wendy Millroy

School of Education, Univers ty of Cape Town, South Africa.

Abstract

One of the dangers of pre-service mathematics teacher education courses is that the presenters of such courses can become so involved in their own construction of the imperatives of mathematics teaching that they fail to pay sufficient attention to the real contexts in which their students will be teaching. This potential split becomes even more problematic when the context is one of contestation of a society in transition. The focus of this paper is on one particular method course for pre-service mathematics teachers at a South African university. The research reported here was undertaken in an attempt to determine the perceived effectiveness of the course by former students.

Introduction

The research reported in this paper takes place against a context of the present contested transformation of South African society. It focuses on the evolvement of a Mathematics Method pre-service course for secondary school mathematics teachers at the University of Cape Town during a period where education was often at the forefront of the struggle for change (Breen 1988). In summary, the course prepares an average of 45 students each year to teach in schools across the racially and regionally divided 19 different education departments. Lectures occur over a period of 20 weeks in a single year and are held on one day over a period of 4 ½ hours. The initial phase of the content section of the programme, between 1983 and 1986, included an emphasis on open-ended investigations and an exploration of Gattegno's description of geometry as 'an awareness of imagery'. The section of the course which tried to contextualise mathematics education into society focused on readings on current debates on the non-Eurocentric roots of mathematics, ethnomathematics, and gender issues in mathematics education.

The programme was thoroughly revised in 1987 in the light of the ineffectiveness of the programme in getting students to recognise their own need to integrate the theory of the course into their own practice (see Breen 1990). The political context of the period seemed to encourage students to take up simplistic polarised positions. In an attempt to challenge students to confront their own preconceptions about themselves, society and the teaching of mathematics, an experiential programme which focused on cognitive and affective conflict was introduced. The programme was based on a reflection model which is described in more detail in Breen (1992). The model drew from the work of Gattegno's statement that 'only awareness is educable' (Gattegno 1970) and tried to integrate Vygotskyan concepts of the teacher (in this case teacher educator) as mediator and the value of the social interaction in learning (Vygotsky 1978), and a Rogerian sense of valuing the whole person (Rogers 1983). The course became an experiential course where students were put in situations which forced them to mix socially, and to participate in activities which highlighted uncomfortable aspects of the teaching reality (Breen 1990, 1991, 1992). Throughout the course students were required to reflect on their assumptions and actions, and a crucial element of the programme was the requirement that students keep a reflective journal on their own growth as learners, mathematicians and teachers. This journal served as the major source of evaluation for the year.

Ratio'...ie for study

By the time the programme had run a few years, it had become extremely popular and students were describing it as one of the most important courses in their university career. While this was gratifying for the lecturer it did not mesh with the continuing reported injunction from teachers in schools to student teachers that they should forget all the idealistic nonsense that they learnt at university and deliver what was required in schools. A further concern was the extendibility of many of the ideas of the course to the realities of teaching mathematics to large classes of 75 pupils in severely underresourced schools. In a context of increasing turmoil in schools and an urgent need to reform the process of schooling in the country, it became apparent that the evaluation of the course by students finishing the course left a number of unanswered questions. Accordingly, it was decided to conduct some research which attempted to access the field experiences of practising teachers who had completed the course.

Method of research.

Dr. Wendy Millroy was appointed as a senior researcher during 1992 to conduct the research. Her first task was to develop a data base of past students and their current addresses. At the same time as this long process was in operation, Dr. Millroy conducted a series of six short interviews of past students teaching in the Cape Town region. The data obtained from these interviews was used to develop an interview script In analysing the data from these initial interviews it became apparent that teachers enjoyed the opportunity to talk to the researcher, particularly about the method course The considerable time taken for this section of the interview meant that far less time vas available to focus on their present teaching situation and their effectiveness at introducing changes into their schools. Accordingly the order of the interviews was changed to emphasis their current practice and a new script was used for a further set of nine long interviews with practising teachers. Transcripts were made of each interview.

These interview transcripts were then analysed and searched for possibilities for the construction of a questionnaire which could be sent to all past HDE students. The questionnaire generally consisted of statements which required a response based on a five point Likert scale and was then pilot studied with a group of ten students from the current method class of 1992. Their comments and criticisms were noted and the final questionnaire drafted. The revised questionnaire consisted of 93 questions which were subdivided into four sections which focused respectively on their present teaching practice; their experience with introducing change in their schools; their recollections of the Maths Method course; and their present needs for further professional development. The questionnaires were sent to 320 latest known addresses of students who had completed the course since 1983. 102 completed questionnaires were returned and subsequently analysed.

Results and implications.

Present Teaching

The picture painted by the results of the questionnaire are both reassuring and generally predictable. There is a sense of enormous perceived optimism in their responses. For example, 87% felt that they were free to use their own ideas about teaching in their classrooms,; 85% disagreed with the statement that "I do not encourage too many questions in class because it can be too time-consuming"; and 85% replied that they were open to new ideas and to trying new things in the classroom. In contrast to this sense of confidence and freedom, their responses to a question which asks them to identify their main concerns during their first years of teaching coincide with many of the frequently documented issues raised by first year teachers. Teachers in this sample list problems of discipline in class and the difficulties associated with achieving this; getting through the syllabus in time and the negative effect this has on creative teaching; and the importance of ensuring that students get good examination results.

Experiences with Change.

There is an interesting spread of perceived power in controlling change. 58% disagreed with the statement "I don't feel that I have the power to change anything at school (compared to 24% who agreed), 75% agreed that the teacher needs the support of a progressive principal to successfully make changes at school, while 63% agreed that pupils have the power to implement changes at their schools.

As regards their own involvement in trying to introduce change in their schools, 53% agreed that they can think of several areas where they have been able to make significant changes (28 % disagreed); 47% agreed that the changes they have been able to make have been positive and long-lasting; 45% disagreed that they had met lots of resistance when trying to change things at their school (35% agreed); and 50% agreed that change has to be introduced slowly at schools because teachers have a natural resistance to change (30% disagreed).

Influence of the Course.

The course appears to have had a far reaching influence on those who have responded. The results of the questions which focused on their recollections of the HDE course are generally encouraging. 84% agreed or strongly agreed that the course had taught them to reflect upon themselves as teachers; 83% agreed that their experience in the course had encouraged them to treat learners with respect; 77% agreed that they had tried to develop an atmosphere of freedom and enjoyment in their maths classes as a result of the Maths Method course; and 74% agreed that their experiences in the Maths Method course had changed their ideas of what it means to be a mathematics teacher. 65% agreed that their experiences in the Maths Method course had a great influence on their approach to teaching and the same number agreed that their experience in the course had encouraged them to work co-operatively with others.

Practical Implications of the course.

Responses in this section are less clear but some of the unmet needs begin to become evident and generally centre around their first year needs and their difficulties in ensuring that pupils deliver good results. For example, 50% felt the course equipped them with useful strategies to survive in the classroom, and 36% agreed that the course had taught them survival skills which helped them through their first years of teaching compared to 47% who disagreed. 35% agreed that they had been able to use a lot of the activities with their pupils at school. 62% agreed that time should be devoted to the special problems associated with teaching Matric maths. 39% felt there should have been more emphasis on maths content while 46% disagreed that the lecturer should have been far more prescriptive in covering specific subject topics and giving teaching ideas.

Significance of the Course.

48% agreed with the statement that "the Maths Method course was the most important tertiary experience I had". In responding to the statement, "The most significant aspect of the course for me was....", the most frequent responses were:

i) the variety of interesting, new and different approaches to teaching that were experienced
ii) the fact that the students got to know each other and to work co-operatively at group activities and interactive games
iii) the course promoted self-reflection and critical self-evaluation both as a person and as a teaching professional.
iv) the idea of the teacher as a learner and the possibility of the teacher being able to put him/herself in the learner's shoes.

The activities which were most frequently mentioned were: the role playing of Mr. Smith, the silent lesson; video recording and playback micro lessons; and circle dancing

While these results convey a picture of the continuing impact of the course and of a general optimistic attitude towards classroom innovation, the data gathered in the long interviews present a much richer source of understanding different dimensions of teacher realities. In an attempt to explore these different realities, two case studies taken from these long interviews will be described.

Case Studies

Paul had already taught for several years in one of the sectors of education which have been disadvantaged by apartheid before he came on the HDE course. He had continued to be very active politically during the year of the course, and had challenged the lecturer on several occasions during the year because he felt that the introduction of a humanistic element through the playing of games masked the political struggle and asked people to accept or tolerate each other at a time of national contestation. There were several heated debates between lecturer and student during the year, both in written form in the journal and in personal conversations.

In reflecting on the course to the researcher, he regretted that he had not participated earlier in group activities. At the end of the course at one stage he found that his non-participation had led to his being marginalised at a time when he needed support. In addition, during the year he had seen the importance of engaging with people with different beliefs and experiences rather than totally reject them for their positions. On returning to teaching, now as a head of department, he had been able to apply this new sensitivity and tolerance to a conflict he had had with a new staff member and had found the issue more easily resolved than previously would have been the case. Colleagues had also given him the feedback that it was the first time they had seen him smiling and he believed that he was able to achieve more in implementing changes in the school because he was more open to negotiation and understanding the other person's point of view than he had been before the course.

His strongest criticism of the course was that it created a safe and supportive environment for students which in no way resembled the reality that they would encounter in the school situation. He recalled an incident where students were criticising each other and the lecturer had intervened to control the situation as having been a missed opportunity for students to experience the reality of teaching. The successful separation of people through apartheid was also of concern to him and he felt that it was essential for students to teach or observe the different possibilities on the spectrum of teaching situations.

In talking about the extent to which he was able to incorporate ideas from the course into his own teaching, he raised the interesting point about students trying to teach using the lecturer's method. If it didn't work first time they would reject it. He stressed the importance of teachers owning the methods they use. "This is my method now. I am going to do it my way. If the situation is different from in the HDE class, I'm going to change it to suit my circumstances".

Dennis tells a very different story. He left the HDE course with a great deal of enthusiasm for what he called 'the discovery principle', a belief in the merits of group work and the need to use apparatus to teach mathematics. His meeting with the reality of teaching was harsh.

In the first place he was confronted by the absence of teaching aids in his school. He had chalk and duster and could not do anything about purchasing anything for himself as he did not receive a salary for the first three months. His enthusiasm was greeted with scepticism by his new colleagues who gave him two months before he'd be teaching like them because they said his fancy theories wouldn't work. The classes he was assigned were another problem. He taught 5 std 6 classes each day and the average class size was 70. This meant at times that 4 students shared a desk built for 2. He also found that students were severely disadvantaged through previous deficiencies in their knowledge of mathematical foundations.

His story bears testimony to his strength of character. He carried on and tried to apply his ideas of the 'discovery principle' and was pleased to find that the students enjoyed them. His class motto was 'Take Risk'. He told the class that at the end of the day that they'll see there will be a discovery. Students got excited when they arrived at a group answer. Introducing this method took time and he organised Saturday classes so that they could keep up with the other classes. He said that if students did not come to these extra classes he would have to stop doing extra things in class. They came and other classes started to ask for his help. He conformed a lot more near the June exams so that his class would not be disadvantaged.

The principal heard of the enthusiasm Dennis's teaching was generating and decided to sit in on a lesson. At the time the Principal was registered for a Masters degree at UCT and was open to progressive teaching ideas. He became extremely excited at what he saw and asked Dennis to share and develop some of his ideas. Dennis enjoyed this opportunity and started thinking about different possibilities. In the first place he said that it was no good teaching the same standard each year - it was important that teachers should follow the class up each year so that there was continuity. The principal liked this and agreed that Dennis should take a Std 7 class in the next year. He also asked Dennis to take over one of the Standard 9 classes to improve the results of the class. Dennis suggested a form of team teaching to avoid being seen as a thereat to the present teacher. In this method teachers would develop their own area of specialisation and support the others in that particular field. They should also examine the syllabus and make recommendations to other teachers as to where the important emphasis for each year should be. The principal was excited at this initiative. At the end of the year he rewarded this enthusiasm and dedication by promoting Dennis to Head of Department. Everything was going like a fairy tale......
The honeymoon ended abruptly at the beginning of the next year. The other teachers were not happy that this youngster has been promoted over the heads of those who had greater

experience just because he was better qualified and had new ideas. They agitated in the local community and got Dennis and 4 other teachers suspended from the school. He was out of teaching for three weeks without pay. In the meantime the principal faced the wrath of the teachers. The Department came to the principal's aid and he was promoted to circuit inspector of another district.

Dennis's class becomes angry that he was not there to teach them and they started a counter movement to get him back through the PTSA (Parent Teacher Student Association). It succeeded partially and he was reinstated but since his mentor had left the school his position was downgraded to the rank of assistant teacher and his standard 7 class was taken away from him so that he returned to teaching the 6's.

One would think that he would be devastated by this turn of events, but he returned to teaching at the school and his 7's joined him for the extra Saturday classes. But more problems hit him and soon after his interview for this project he was suspended again. Not surprisingly at the top of his list of needs was a desire to get together with like-minded teachers to form a support group. He wants UCT to set up a school in the township so that other teachers can be convinced of the viability of these new methods and that he has a place to go to get support.

Implications of the research

Before drawing implications from this research, it is important to highlight some of its limitations. In the first place the divisive nature of South African society and the education system in particular means that the students are teaching in a wide variety of contexts. Added to this is the dynamic nature of the course which undergoes annual revision. The statistics obtained from the questionnaire have been present for the group as a whole and show some interesting trends, but it is the differences between categories which would be of greater value. Unfortunately grouping according to education department is likely to be simplistic. However an attempt will be made to re-analyse the data using different categories including those of the two different stages of the course.

However, it is apparent that there are several important considerations arising from the research which need to be addressed. In the first place there is the issue of the concerns of first year teachers and the degree to which they are going to struggle to cope. To what extent should the pre-service course attempt to give them a tool kit of skills to survive the first year of teaching? Dennis's experience underlines the urgent need for there to be a support structure for first year teachers in particular. Yet Coombe (1992) describes how she attempted to set up such a support group for the 1991 Method class at UCT. Despite initial student enthusiasm for the idea, in the end it was the teachers from schools which had well established internal support structures for their teachers who found the time to attend meetings. In analysing her experience Coombe suggests that it is impossible to prepare students sufficiently to understand and cope with the different contexts in which they are likely to teach.

Paul's experience of the course and the way in which he was able to apply aspects of it to his teaching situation suggest that this type of experiential course would be more valuable for teachers who have gone through an induction period in teaching. This familiarity with the classroom context and a knowledge of their own capacity to cope with the daily stresses allows these teachers the possibility of introducing changes into their practice that are not available to a beginner teacher who has still to develop a style of teaching. The difficulty in South Africa is that no ethos of in-service continuous education exists and teachers can generally secure a life long position without doing any further study after their initial training. Addressing this issue and creating opportunities for accredited in-service upgrading of teaching skills would seem to be a crucial aim for the education department created by the new government of the country.

A final implication can be drawn from Dennis's case study where he has tried to implement a different model of teaching. The danger of presenting a radically different methodology is that it runs the danger of separating teachers from their existing knowledge base. A more sensible approach would be to get students to take a fresh look at their assumptions about teaching and get them to experience some of the strengths and weaknesses of different methodologies, so that they are in a position to choose for themselves what they would like to employ in any particular situation. The need for balance is particularly important in a country in transition where it is tempting to set up polarities which lead to the destructive creation of opposites (Breen 1993) which actually disempower teachers (Ellsworth 1989).

References

Breen,C (1988) Policing Exams : The Impact on Teachers and Students in Western Cape Schools, 1985. Perspectives in Education, 10,1,3-11.

Breen,C (1990) Reflecting on energy activities: an attempt at liberating pre-service mathematics teachers at a South African university. Proceedings of the First International Conference on the Political Dimensions of Mathematics Education: Action and Critique, University of London, April 1990.

Breen,C (1991) Concerning Mr Smith and his (very brief?) reign of terror. Pythagoras, 25, 31-37.

Breen,C (1992) Teacher Education: Confronting Preconceptions. Perspectives in Education, 13, 1, 33 - 44.

Breen,C (1993) Holding the Tension of the Opposites, For the Learning of Mathematics, 13,1, 6-10.

Coombe,J (1992) Researching First Year Teaching. Unpublished mimeograph, University of Cape Town.

Ellsworth,E (1989) Why doesn't this feel empowering? Working through the repressive myths of critical pedagogy. Harvard Educational Review, 59, 3, 247-324.

Gattegno,C (1970) What We Owe Children. Routledge and Kegan Paul : London.

Rogers,C (1983) Freedom to Learn in the Eighties. Charles Merrill: Ohio.

Vygotsky,L (1978) Mind in Society. Harvard University Press: Cambridge, Mass.

MATHEMATICS LIVING IN A POST-MODERN WORLD

Tony Brown

Manchester Metropolitan University

Abstract

In this paper it is suggested that mathematical phenomena generally have their roots in man's practical concerns and the underlying social dimension can be seen as an essential feature of the educational process in learning mathematics. By seeing mathematics as ideas accessed through their cultural presentation, an analogy is drawn with post-modernist views within which there is a rejection of over-arching schematic accounts of the world in which we live. It is shown how such an analogy can lead to a view of learning mathematics where it is seen primarily as being concerned with capturing, in language, the 'experience' of mathematical activity. By attending to a reconciliation between the nature of mathematical activity as viewed in a holistic sense and as viewed through particular formalised statements made within it we can re-introduce the human into mathematical ideas.

Introduction

Platonic views of mathematics (see Ernest, 1991) which see it as having an independent existence offer little help to those concerned with the teaching of mathematics. Mathematics, as described in this paper, is seen as being socially derived and its manifestations are always subject to the unique perspective of an individual human's gaze. As such mathematics cannot ever be surveyed neutrally in a holistic way; rather it is always mediated by some form of access. Embedded as it is in the social fabric, mathematics cannot be disentangled from its communicative dimension as it only ever arises in human activity, and then in the form of a language developed through historical social processes. For someone to engage in mathematics, an initiation into the socially conventional forms which embed it, is necessary. To do mathematics is to share the symbols of a particular community. Walkerdine (1988) and Lave (1988) who soften the distinction between individual and society are followed in describing a socially derived psychology of which the individual is part. Drawing on post modern accounts of culture this paper seeks to outline an alternative approach to viewing and reporting on mathematical activity.

The roots of mathematics

The prestige mathematics enjoys because of its manifestation in every day situations is not necessarily easily reconcilable with the prestige it derives from its status in the academic world. The 'queen of the sciences', in all of its supposed purity, is often seen as quite separate from its mundane existence as a discipline supporting human endeavours. For many mathematicians, applied mathematics is not quite mathematics: a certain loss of its purity seems to go hand in hand with using it in the real world. This is, however, a curious state of affairs given the origins of mathematics. The subject comprises, by and large, ideas passed down by our ancestors which, more often than not, emerged in respect of practical considerations. Most of these ideas can be seen as linguistic framings of phenomena perceived by the human gaze. For example, circles, right angles, or number systems did not preexist man; all of these have been created by humans. Names have been assigned as a consequence of repeated use. School syllabi today are full of those areas of mathematics imagined to be useful in some way, or at least testable. Indeed, it might be suggested that the rationale underlying most of school mathematics is essentially economic and utilitarian. Similarly, vast areas of mathematics have not been pursued by the population at large since any potential application is seen as being too obscure, extinct or non-existent.

The world as seen by humans has necessarily been tainted by the conventional linguistic framings employed within a given culture. Such framings have become embedded in our collective reality such that we now have difficulty in seeing them as anything other than natural. The things of the world assert themselves in their very familiarity in our ways of looking. For an Australian aborigine, however, in a world lacking the rectilinearity we experience in western cultures, or for a fly with its own form of optical equipment, we can see that the structures imposed on the world by western humans are not unique. As a culture we have introduced a structuring of the world as a means of orientation - a classification which allows us to highlight recognisable and repeatable features. Introducing the use of conventional notation can be seen as an initiation into social practices of this community rather than an uncovering and recognition of natural phenomena (see Walkerdine, 1988). As a society we identify and name certain features of our collective reality to provide markers. This is true of mathematics although the 'truths' of mathematics have become so embedded in our social fabric that we have forgotten they are constructs.

Owning mathematics

In Saussure's (1966) seminal work in linguistics, carried out at the turn of the century, an important distinction is drawn between *langue*, the system and structure of language, and *parole* ('speaking') the manifestation of language in everyday speech and writing. I suggest a similar distinction can be drawn between 'mathematics' existing in a Platonic sense and 'mathematical activity', its manifestation in everyday human usage in school, jobs of work etc. Drawing such a line enables us to focus on the language associated with the performance of mathematics by people. This permits an examination of mathematics as a social practice rather than as a subject with independent existence. Brown (1993) discusses this in greater depth.

By assuming that mathematical knowledge is socially derived there is a need to attend to the assumptions implicit in descriptions offered of mathematical phenomena. The language of mathematics reflects the society generating it. Further, in focusing on individual acts of *parole* certain perspectives are implicated, emanating from variously perceived *langues*, thus removing any possibility of neutrality in the observing. Mathematical learning is not so much directed towards a body of knowledge but rather is a process of transforming knowing. The time dependency implicit here results in a difficulty in locating and fixing the knowledge at the heart of the learning process. The notion of a mathematical equivalent to *langue* becomes problematic since it can only ever be accessed through individual acts of *parole* governed by particular perspectives. This presents difficulties for those whose job entails the sharing of perspectives.

There is a story about a man who throws a stick for a dog but the dog fails to see where it goes. The man points in the direction of the stick but the dog only looks at his pointing hand. For pupils in classrooms there is a frequent conflict between attending to the teacher's understanding and attending to the object of that understanding. Do you pay attention to the pointing hand or to the thing being pointed at? Such concerns about the location of knowledge have been discussed by Brousseau and Otte (1991). This problem might be characterised as a 'double-bind' (Mellin-Olsen, 1991) of the sort that might arise in a teacher's delivery -

-Look at what I see - but it's what you see that's important.

Brousseau and Otte (1991) outline the problems endemic in what they call the 'didactical contract', where a teacher finds him or her self forced into giving the child knowledge rather than allowing the child to re-construct the knowledge for themselves. The child, in paying attention to what the teacher wants, making this the focus of their learning task, is drawn away from the 'thing' itself. Brousseau and Otte resolve this potential dichotomy by rejecting notions of knowledge seen as a found object preferring to assert its essential 'fragility', something that can not be held as fixed. Learning is not about the reproduction of the teacher's knowledge in the mind of the child but rather can be seen as a transforming of both positions. Knowledge is not a fully constituted object being confronted by a fully constituted student, rather, both change through a time-dependent process. Learning is not just about adding to knowledge, rather knowledge, or at least our state of knowing, can be transformed in many ways; one subtracts from it as well as adds to it, forgetting as well as remembering, one re-organises so that known 'things' get new meanings - and knowing is not just about things. It seems to me the 'fragile' learner described by Brousseau and Otte has much in common with the human subject described variously within modern philosophical traditions such as post-structuralism and hermeneutic phenomenology. Walkerdine's (1982) post-structuralist psycho-semiotic approach, for example, casts the individual learner of mathematics in a rather different role to that supposed in cognitive psychology. She replaces

> 'Piaget's abstract epistemic subject.. with the notion of human subjects as created through the incorporation, through the medium of signs, of children into the social practices which make up our everyday life (p. 129)'

In her later work, Walkerdine (1988) provides many examples of how this is manifested in the classroom. These traditions have been further described in relation to the teaching and learning of mathematics by Brown (1993).

Post-modern mathematics

The boundaries of these modern traditions are not clearly defined but the tradition of post-modernism might be seen as embracing post-structuralism, perhaps with a certain degree of discomfort. Post-modernism offers some possibilities in describing the sort of world 'fragile' people might inhabit. Rooted in the work of Lyotard and Baudrillard (for an introduction see Easthope et al (1992)), post-modernism essentially attends to the medium of culture. Jencks (1989) suggests, '(t)he challenge...is to choose and combine traditions selectively, to *eclect* ...those aspects from past and present which appear relevant for the job at hand'. Within post-modernism there is always something outside of theory since a theory requires someone to expound it with a particular interest and from a certain perspective. According to Lyotard (1979) there is 'no universal meta language' - individuals can only express themselves within a particular story. To import Saussure's terminology, there is no universally agreed *langue*, but rather we live in a world governed by acts of *parole*.

Mathematics is often presented as such a meta-language, as an overarching *langue*, and this often results the cultural packaging, within which it is transmitted, being disregarded. A quote from Adair (1992, p.5) discussing art captures the issue:

> The work of art becomes a 'work of culture' only when it has been externalised, extrapolated from, thought about, talked about, read about, communicated and shared. Culture in this sense is applied art, as one refers to applied mathematics.

Mathematics too is highly dependent on how it is presented. As teachers and learners we are concerned with initiation into the culture of mathematics, and in particular with the culture of school mathematics - examinations, for example, demand a particular style of mathematics. The media through which we view mathematics condition what we see and the stories we tell about it. Such media include the published teaching scheme being used, the teaching philosophy of the teacher, the technical facilities available, the curriculum being followed, all of which change through time. A view of school mathematics, which places more emphasis on the cultural packaging, is offered in the work of the 'realist' school (e.g. Broekman et al, 1994). Zawadowski (1988) has further discussed school mathematics in relation to post-modernism.

The activity of mathematics is often characterised as being oriented around the certainties it contains. However, stories describing 'mathematical' experience are less reliable and \ need to find other approaches enabling orientation. Post-modernist accounts, through their emphasis on individual perspectives, might be seen as moving us away from seeing a universal 'correct' view of mathematics anchored around the 'facts' of mathematics. Rather it can be seen as a process of making sense of snippets of experience. Dave Wilson (in conversation) has suggested that 'a post-modernist view might be to perceive mathematics as just being a set of facts, not linked together by a characterisable story - in the same way that an architect might design a building using a 'bricolage' (Levi-Strauss, 1970) of styles, assemblies - working mathematically might be about using a series of 'facts', put together to create an answer'. The next section makes tentative moves towards outlining possible approaches to coping with the complexity of an interpretable mathematical field.

Orienting mathematics

Mason (1989a) and Brown (1991) have described the outcomes of mathematical activity in terms of organising and making sense of the experience of doing mathematics. Such a view presupposes a selection from the different aspects of a given mathematical activity and a softening of approaches to mathematics that see the subject as being about exercises comprising strings of questions with singular correct responses. Dockar-Drysdale's (1990) work with emotionally deprived children perhaps offers a useful analogy here. She describes some children who have difficulty in making sense of their everyday experience. This is characterised by a breakdown in their ability to capture and store personal incidents resulting in a feeling of disorientation. By teaching the children to characterise particular events in their lives she helps them to develop nodal points as a means of orientation which enable the child to 'store a realised experience in such a way that this can be preserved and, if need be communicated' (p.102). For these children structure does not emerge naturally, but is constructed, perhaps making use of components identified by the broader community.

The way in which children embed mathematical terminology in personal and family experience is also explored by Tahta (1993) and Walkerdine (1988). Brown (in press) describes an approach used by teachers on a masters course, which echoes Dockar-Drysdale's work. For these teachers, engaged in a task of developing their own professional awareness towards re-organising their practice, there was a need to heighten their awareness of how they captured their own professional experience. By telling anecdotes about specific issues in their practice they became more able to give an account of their practice in general, grounded in particular events. This mechanism facilitated increased control over change in their practice.

The sociality implicit in mathematical learning necessarily gives rise to an interpretive dimension. Certain styles of school mathematics activity, for example, investigation work, are perhaps more likely to result in the student exercising their interpretive skills. Other styles, more focused on responses to specific questions, might require a re-orientation in the teacher's role towards more actively encouraging the student to articulate the nature of their understanding in respect of the questions (Brown,1990a). In emphasising the interpretive aspects of mathematics the student needs to fit mathematical statements to their experience and there is a need to decide on their appropriateness in particular situations. Brown (1991) has followed Ricoeur in showing how explanation and understanding can be seen as a complementary duality under the umbrella of interpretation. In reflecting on a mathematical investigation or on some piece of research, for example, I may offer some comment, which for the time being, encapsulates my understanding. For example, Brown (1991, p. 477) offers some statements by children attempting to characterise some symmetrical drawings:

'"If it goes up there, the other side goes up the other way."
'It's the same both sides, like it's cut in half."
"That line up the middle is like the line of symmetry."
"It's straight up the middle in that it divides."'

However, as work proceeds refinements can be made. Nevertheless, in order to engage with a mathematical task it is necessary for me to make occasional tentative statements towards providing markers which help me in finding my bearings. In this way I describe, in a personal way, the mathematics in terms of my perceived relationship with it but necessarily in doing this I use the language of the society of which I am part (see Brown, 1994). Brown (1990b) offers a fuller account of children framing their experience of a mathematical investigation in language. In such work there is necessarily an interaction between engagement in mathematical activity and making statements in respect of it. As Elliot (1987, p.160) puts it, in discussing the hermeneutics of Gadamer: 'Interpretation constitutes a moment within the process of understanding'. Seen in this way mathematics is less about finding correct answers to specific questions than about selecting, framing and focusing on the key questions. This sort of approach has been discussed in more depth in the work of Mason (e.g. 1988, 1989b).

Conclusion

By rejecting the notion of an agreed and shareable mathematical *langue* in favour of seeing mathematics as being manifested and shared through individual acts of *parole* it becomes possible to assess mathematics as a social practice. In meeting mathematics in educational settings and in the world of work a social dimension is endemic in the mathematics itself which cannot be extracted -nor is it desirable that it be extracted. Mathematics in the social world of work cannot be about 'body of knowledge' type mathematics since the interpretive dimension cannot be removed. Communication is not about passing around ready made objects, nor is it about re-constructing knowledge, rather it is about operating on knowing. Knowing cannot be seen separate to action since the social practices which host specific actions are imbued with the society's preferred ways of doing things. Indeed the individual cannot be seen as separate to his or her society - since the society *speaks* itself through the actions of its individual subjects. For example, Walkerdine's (1988) psycho-semiotic account of developing mathematical understanding offers a framework which takes us away from more individualist notions of psychology rooted in cognition.The dynamic qualities of knowing in a social setting have rules of operation quite separate to those that might be supposed for individuals dealing with body-of-knowledge type visions of the various disciplines. This is discussed further by Elliot (1987). Lave (1988, pp.76-93), for example, suggests softening the distinctions between psychology and anthropology towards bringing cognition and culture into a more complementary relation. By placing too much emphasis on the culturally bound manifestations of mathematical 'facts', as confronted by individuals, we are diverted from attending to mathematics as a social practice and the skills associated with socially located mathematical activity.

References

Adair, G.:1992, *The post-modernist always rings twice*, Fourth Estate, London.

Broekman, H & Streefland, L.:1994, 'A realist approach to percentages', *Mathematics Teaching*, 145.

Brousseau, G & Otte, M.:1991, 'The fragility of knowledge', in Bishop, A et al, *Mathematical Knowledge:its growth through teaching*, Kluwer, Dordrecht.

Brown, T.:1990a, 'Signs of power', *Mathematics Teaching*, 132.

Brown, T.:1990b, 'Active learning within investigational tasks', in *Mathematics Teaching,*

133.
Brown, T.: 1991, 'Hermeneutics and mathematical activity', *Educational Studies in Mathematics*, Kluwer, Dordrecht, 22, 475-80.
Brown, T.:1993, 'Creating and knowing mathematics through language and experience', *Educational Studies in Mathematics*, Kluwer Academic Publishers, Dordrecht
Brown, T.:1994, 'Describing the mathematics you are part of: a post-structuralist account of mathematical learning' in Ernest, P (Ed).: *Mathematics, education and philosophy: an international perspective*, Falmer Press.
Brown, T.:in press, 'Constructing the assertive teacher, *Research in Education.*, Manchester University Press.
Dockar-Drysdale, B.:1990, *The provision of primary experience*, Free association, London.
Easthope, A & McGowan, K.:1992, *A critical and cultural studies reader*, Open University, Buckingham.
Elliot, J.:1987, 'Educational theory, practical philosophy and action research', *British Journal of Educational Studies*, June No.2.
Ernest, P.:1991, *The philosophy of mathematics education*, Falmer Press, Basingstoke.
Jencks, C.: 1989, *What is post-modernism?*, Academy/St Martins, London, p.7.
Lave, J.:1988, *Cognition in practice*, Cambridge University Press.
Levi-Strauss, C.:1970, *The raw and the cooked*, Penguin, pp.1-32.
Lyotard, J.: 1979, *The post-modern condition*, Manchester University Press, chap. 10.
Mason, J.: 1988, 'Fragments: the implications for teachers, learners and media users/researchers of personal construal and fragmentary recollection of aural and visual messages', *Instructional Science*, Kluwer, Dordrecht, 17: 195-218
Mason, J.:1989a, 'Teaching (pupils to make sense) and assessing (the sense they make)', in Ernest, P (Ed), *Mathematics teaching: the state of the art*, Falmer Press, Basingstoke.
Mason, J.:1989b, 'Mathematical abstraction as the result of a delicate shift of attention', *For the Learning of Mathematics*, 9,2,
Mellin-Olsen, S.:1991, 'The double bind as a didactical trap', in Bishop, A et al, *Mathematical Knowledge:its growth through teaching*, Kluwer, Dordrecht.
Saussure, F.:1966, *A course in general linguistics*, McGraw Hill, New York.
Tahta, D.:1993, 'Victoire sur les maths', *For the Learning of Mathematics*, FLM, White Rock, Canada, 13,1.
Walkerdine, V.: 1982, 'From context to text: a psychosemiotic approach to abstract thought', in Beveridge, M, *Children thinking through language*, Routledge.
Walkerdine, V.: 1988, *The mastery of reason*, Routledge, London.
Zawadowski, W.:1988 'The post modernism and post-modern mathematics in school', Department of Mathematics Education, University of Warsaw.

THE RELATIONSHIP BETWEEN THE TEACHER'S CONCEPTIONS OF MATHEMATICS AND OF MATHEMATICS TEACHING. A MODEL USING CATEGORIES AND DESCRIPTORS FOR THEIR ANALYSIS.

José Carrillo & Luis C. Contreras
University of Huelva (Spain)

Abstract

This research report outlines a framework for the analysis of the teacher's conceptions of mathematics and of mathematics teaching. It derives from a theoretical model, tested against data obtained from a case studies, which also make explicit the relations found between such conceptions. It greatest interest lies in the contribution of a categories system and descriptors that make a more detailed characterization of the teacher's beliefs of mathematics and its teaching easier.

Introduction

The identification of Teacher's conceptions[1] of Mathematics and Mathematics Teaching (hereafter CM and CMT, respectively) occupies an important place in Mathematics Education Research (for a detailed account of this importance the reader is referred to Thompson, 1984 and 1992).

Such conceptions act as a filter and as a decodifying element upon all information from whatever other research area. Thus, a specific CM and/or CMT could determine the interpretation of and decisions about students' conceptions, mistakes and epistemological obstacles, it would orient the selection of content and the related didactic situations, and it would allow or would serve as a basis for the negotiation frame (implicit or explicit) of a specific didactic contract (Brousseau, 1989). Hence, we consider that such conceptions constitute a central thread of teacher's professional development.

The implicit nature of these conceptions makes them difficult to identify and whatever system that seeks to make them explicit (verbally explicit) is conditioned and controlled by distinct intrinsic or extrinsic factors (Llinares and Sánchez, 1990). Furthermore, the interactive nature (theoretical-practical) of professional knowledge can lead the teacher to express ideas which he or she considers desirable but which have not yet been assimilated into their practice. This, together with the scant awareness of their own conceptions, accounts for the inconsistencies between the teacher's stated awareness of these and what can be inferred from the observation of teaching practices (Thompson, 1984).

We are amongst those who believe that an important relationship exists between conceptions (CM

[1] We use the term conception following Thompson (1992)

and CMT) and problem-solving modes (Hart, 1991), in the sense that the latter can provide data about the former and can also be used as a tool for their eventual modification. This conviction characterizes the way we will continue this research, in which the information about verbally expressed conceptions will be seen to be enriched by the comparison of what can be inferred from teacher observation and the data that derive from the processes used in problem solving. Moreover, a training strategy based on a individualized design (depending on personal conceptions) about problem solving, in which monitoring plays a main role (Hart, 1991), will be as useful in challenging the aforementioned conceptions as in promoting a possible alternative to them.

Methodology

The researc.ı presented here consists of a case study (6 mathematics teachers who give class to 14-18 years students). The data have been collated from the answers to a questionnaire and from an individual open-ended interview, both related to their beliefs of mathematics and mathematics teaching.

In order to organize all the information, obtained from questionnaires and interviews, it is necessary to have a theoretical model, that can be used as a basis for the categorization of the items of information; but, likewise, this model must be submitted to a continuous revision, so that it can reflect the reality shown in the sample (see table 1). It is clear that the classification, obtained by the final categorization in the manner explained below, can not be extrapolated, but it reflects the tensions that exist between the theoretical frame and the observations, and, what is more important, it offers a practical classification of the individuals studied.

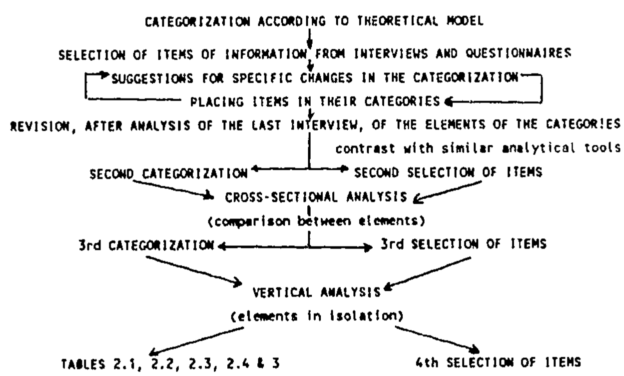


Table 1 (Revision process)

For the identification of the CMT we started from a model with 4 didactic tendencies[2]: **traditional, technological, spontaneous and investigative** (Porlán, 1992), in which we established 6 categories in order to get descriptors that could help us to state precisely our data in the most exhaustive manner. These categories are: **methodology, subject significance, learning conception, student's role, teacher's role and assessment**. It constitutes a total of 35 descriptors for each tendency, many of which coincide with those used in previous studies (Porlán, 1989,92; Kuhs and Ball, 1986; Ernest, 1989,91; Fennema and Franke, 1992). These descriptors, in a cross-sectionall reading, establish differences among the diverse tendencies and, in some cases, they have an accumulative nature when reading from left to right, also giving an idea of the possible development followed by a teacher from the traditional tendency to the investigative one.

Some descriptors acquire their full meaning when they are considered in context with the other descriptors of a specific tendency (vertical reading), and in some cases they are observed to have a complementary nature. In the same way, a descriptor belonging to 2 tendencies is differenciated by the said vertical reading (see tables 2.1 to 2.4).

CATEG.\DID.TEND.	TRADITIONAL	TECHNOLOGICAL	SPONTANEOUS	INVESTIGATIVE
METHODOLOGY	Repetitive practice	Reproduct. practice	Experimentation	Problem solving
	Conceptual objectives:contents	Final practical objectives	Flexible and orientative object.	Flexible and revisable object.
	Lecture style (textbook)	Short research simulations (technical means)	Random discovery, models manipulation	Planned research
	Official prescript. rigid program (discr. st. units)	Sequential, structured y closed program	Random program, negotiated contents	Organized conceptual maps
SUBJECT SIGNIFICANCE	Conceptual emphasis	Aplicability (process-product)	Emphasis on proced. and attitudes	Proced concepts and attitudes
	School Math.= formal Math.	School M.= adaptat. of formal M. to real life	School M.=M. that derives from real life	Sch. M.= synthesis of formal M. and everyday M.
	Informative	Informative and utilitarian	Form.(attitudes and rational values)	Formative (learner training)

Table 2.1

[2] We use the term tendency because we consider, in agreement with Porlán (1992), that, in practice, it's difficult to find teachers who can be identified with a concrete didactic model

The attempt to synthesize in a table all underlying ideas in the descriptors has forced us to compress them in a few words -sometimes only one-, attaining in practicability what is lost in richness.

The two preceeding ideas (complementation and compression) become more explicit in the category of "Teacher's role", in which the first and third descriptors answer the question "What does he/she do?", the second "How does he/she do what is expressed by the first descriptor?", and the fourth accounts for the action expressed by the third descriptor.

CATEG.\DID.TEND.	TRADITIONAL	TECHNOLOGICAL	SPONTANEOUS	INVESTIGATIVE
STUDENT'S ROLE	They don't take part in syllabus/ lesson design	Doesn't take part in syllabus/ lesson design	Takes part in syllabus/ lesson design	Takes part in syllabus/ lesson design
	The only responsable of T-L transferences. Submission	The main responsable of T-L transfer. (motiv. through context)	The key to T-L transfer. is the motivation through the activity	The key to T-L transf. is the process(motivation through meanings)
	Listens and copies	Reproduces and imitates	Does things	Researches
	Is attendive	Is attendiive	Plays	Reflects
	Accepts	Believes	Engages in dialogue	Questions
TEACHER'S ROLE	Transmits verbally	Transmits through technol. processes	Persuades	Provokes
	Dictates	Explains	Promotes	Guides
	Reproduces	Organizes	Analyzes react. and answers to their proposals	Researches the activity
	Content specialist	Content and didact. design technician	Humanist, special. in group dynamics	Interact. exper.of cont. and methods

Table 2.2

The theoretical model corresponding to the CM (table 3) has its source in the work of Ernest (1989,91) (of which we have taken the names of the tendencies) and in the contributions of similar studies (Lerman, 1983; Skemp, 1978). After the revision process, already described, the three tendencies (**instrumentalist, platonic and problem solving**) are organized into 21 descriptors grouped in three categories: **type of knowledge, aims (of mathematical knowledge) and means of development (of mathematics).**

CATEG.\DID.TEND.	TRADITIONAL	TECHNOLOGICAL	SPONTANEOUS	INVESTIGATIVE
ASSESSMENT	Cumulative (end product)	Cumulative (process dependent on prod.)	Formative (process)	Format.-cumulative (proc. and prod.)
	Quantitative	Quantative	Qualitative	Qual.-Quantative
	Doesn't make criteria explicit	Make criteria explicit	Variable criteria set by consensus	Explicit and negotiable criter.
	Memory	Achiening of objetives	Degree of involvement	Meanings and degree of involv.
	Mech. application	Mech. application	Meaningful application	Meaningful and relevant applicat.
	Rigid minimun objectives	Rigid minimun objectives	Negotiable minimun objectives	Revis.min. object. (dep. on stud., proc., subject and school context)
	No individual differentiation	No individual differentiation	Implicit individual differentiation	Explicit individ. differentiation
	Subjective	Taxonomical (observ. behavior)	Undefined	Holistic
	Recuperating means global repetition, isolated from normal development	Recuperating means specific repet., isolated from normal development	Motivation is recuperated through each activity	Personalised recuperat.,complex & inserted into the normal devel.
	Exam preparation fixes learning	Exam preparation fixes learning	The exam distorts the framework of relations and activities	Exam as student's creative act.; one learns during its execution
	Initial information based on previous contents	Init.identification of errors in order to immed.elim. them before starting	Initial diagnostic analysis of student's interests	Init.diagn. anal. which gives inf. for program elab. and execution
	Grading through product-based tests	Grading through tests of the objectives	Grading through revision of student's tasks and participation	Grading takes account of various sources(ex. book, obs.,exams,GW,...)

Table 2.3

CATEG.\DID.TEND.	TRADITIONAL	TECHNOLOGICAL	SPONTANEOUS	INVESTIGATIVE
LEARNING CONCEPTION	Accumulative rote learning	Sequential rote learning	Meaningful random learning	Mean. relevant learning (semmap)
	Deductive processes	Simulated inductive proc.and ded. proc.	Inductive processes	Induction-deduction
	By osmosis	By assimilation	By spontaneous construction	By directed construction
	Individual work	Individual work	Group work and discussions	Diversity of grouping and discussions
	Subject logic	Maths. logic	Students' interests	Logic and interests of stud. and subject
(aptitude)	Predetermined	Predetermined	Transformable	Transformable
(attitude)	Predetermined	Partly transformable	Transformable	Transformable

Table 2.4

CATEGORIES/CONCEPCTION	INSTRUMENTALIST	PLATONIST	PROBLEM SOLVING
TYPE OF KNOWLEDGE	Set of unquestionable truths	Logically structured knowledge	Revis. knowl.; relative truths dep. on context
	Set of unrelated rules and tools	Meaningfully structured knowledge. Core: concepts and rational values	Core: concept. struct., math. proced. and general strategies
	Set of utilitarian knowledge	Objective, absolute, universal, value-free and abstract	Knowledge which implies values
AIMS	External	Internal	Intelectual development
MEANS OF DEVELOPMENT	Math. as a process of creating and using algorithms. Pragmat. view	Math. is not created, but discovered. Dogmatic view	Knowledge problem-driven; potential for creativity. Dynamic view
	Knowledge built on the basis of looking for cause and effect relat. Determinist view	Knowledge built on the basis of previous results and problems arising from other sciences	Knowledge built through social interaction. Anthropological view
	Emphasis on empirical argumentation	Logical reasoning	Ind.-ded.reas.(conject., proofs and refutations)

Table 3

Discussion

The categories and descriptors of the previous tables have made it possible to locate each teacher in a didactic tendency more in accordance with they have verbally expressed, and in the model closest to their . athematics conception, it remaining clear that the objective is not to classify these teachers, but to obtain a starting point for an eventual modification of such conceptions through reflection.

In the results showed in table 4 we can observe a high degree of consistency between CM and CMT (according to Lerman, 1983). A pure problem-solving model corresponds to a pure investigative tendency; a pure instrumentalist model corresponds to a pure technological tendency; the three cases in which platonist and instrumentalist tendencies appear are natureized by a strong traditional tendency; and the last case shows the relation that exists between a mode of understanding what is problem-solving and the spontaneous tendency.

Teacher's code\conception	CM	CMT
RC	problem solving	investigative
JR	instrumentalist	technological
CG	platonist with substantial instrumentalist features	between traditional and technological
AP	platonist with some instrumentalist traits	traditional with some technological characters
PG	instrumentalist with platonist features	traditional with some technological characters
LD	problem solving with platonist features	spontaneous with substantial investig. traits, some techn. & very few traditional

Table 4

We have not looked for consistency in the relations between CM and CMT, although, as has been seen before, we were aware of it. Our aim is to provide an analytical tool that can be used to achieve a better characterization as much of the didactic tendencies as of the mathematics conception models.

On the other hand, this research reflects the predominance of the traditional and technological tendencies in the CMT, confirming, among others, Dougherty's work (1990). This predominance could be explained by the teachers' tendency to remake, above all during the first period of their professional life, the models which they have received (Ball, 1988).

References

BALL, D.L.(1988). Unlearning to teach mathematics. **For the Learning of Maths.**, 8(1), 40-48.

BROUSSEAU, G. (1989). Utilidad e intereses de la Didáctica para un profesor. **Suma**, 4, 5-12.

DOUGHERTY, B.J. (1990). Influences of teacher cognitive/con· 'ual levels on problem-solving instruction.**Proceedings of the 14th PME Conference**, 119-126.

ERNEST, P. (1989). Beliefs Influence in Mathematics Teaching. **Mathematics Education and Society.** Document Series 35. Unesco, 99-101.

ERNEST, P. (1991). **The philosophy of mathematics education.** London: The Falmer Press.

FENNEMA, E. & FRANKE, M.L. (1992). Teacher's knowledge and its impact. En GROUWS, D.A. (Ed.) **Handbook of Research in Mathematics Teaching and Learning.** New York: McMillan.

GARCIA, F. (1988). **Un modelo didáctico basado en la investigación.** Unpublished work document. Teacher' Training Center. Sevilla.

HART, L.C. (1991). Monitoring change in metacognition. **Proceeding of the 15th PME Conference**, Vol II, 141-148.

KUHS, T.M. & BALL, D.L.(1986). **Approaches to teaching mathematics: Mapping the domains of knowledge, skills and dispositions.** East Lansing: Michigan State University.

LERMAN, S.(1983). Problem solving or knowledge centered: The influence of phylosophy on mathematics teaching. **International Journal of Mathematics Education in Science and Technology**, 14(1), 59-66.

LLINARES, S. & SÁNCHEZ, M.V. (1990). El conocimiento profesional del profesor y la enseñanza de las matemáticas. In Llinares, S & Sánchez, M.V. (Eds.) **Teoría y Práctica en Educación Matemática.** Sevilla: Alfar.

PORLÁN, R. (1989). **Teoría del Conocimiento, Teoría de la Enseñanza y Desarrollo Profesional.** Unpublished Ph. Thesis. Department of Sciences Didactics. Sevilla University.

PORLÁN, R. (1992). Teoría y práctica del currículum. El currículum en la acción, en AA.VV. **Curso de actualización científico-didáctica**. Madrid: MEC.

SKEMP, R. (1978). Relational understanding and instrumental understanding. **Arithmetic Teacher**, 26(3), 9-15.

THOMPSON, A.G. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. **Educational Studies in Mathematics**, 15, 105-127.

THOMPSON, A.G. (1992). Teacher's Beliefs and Conceptions: a Syinthesis of the Research. En Grouws, D.A.(Ed.) **Handbook on Mathematics Teaching and Learning.** New York: McMillan.

ON UNDERSTANDING: SOME REMARKS ABOUT A CALCULUS OPTIMIZATION PROBLEM.

Jaime J. del Río Castillo

Departamento de Matemáticas
Instituto Tecnológico Autónomo de México

The analysis of an optimization problem may give some insight into the discussion of the hermeneutical nature of concept construction. The aim of this study is to include ideas and frameworks proper of philosophy and center a discussion around the possibility of mathematical concepts being objects that constitute a reality to the extent that they are essential constitutive parts of schematas of thought and understanding The question of understanding in general and understanding advanced mathematics is addressed in light of the hermeneutical circle.

Introduction

Reflective thinking happens to be provocative vis a vis situations where we can perceive some uneasiness that occasionally would lead us to single out a problem The conceptual identification of a problem carries the recognition of a goal which in turn may provide possibilities for a prototype to emerge Prototypes seen as mental representations (Presmeg 1992) that actualize a category could be hepful in an aim to develop strategies in advanced mathematics problem setting Acknowledgment of a problem with its consequent series of goals stated as objectives that need to be reached by means of a dynamic process brings us to the mere act of understanding The process of going from initial primitive knowing to inventing goes beyond acquisition or progressive linear development Optimizing a system has been very often an objective and always a preoccupation The topic of optimization becomes central in common courses of differential calculus in several variables for sophomore business and economy majors Getting an optimal solution is quite often a plausible objective When a situation is recongized as susceptible of being optimized it is because there is necessarily a stated objective That could serve us in a classroom situation as a very basic motivation to pursue constructivist environments where concepts like function, continuous variation and derivative could be inserted as a constitutive part of some of the schemes characteristic of calculus

The construction of mathematical concepts as well as learning mathematics are complex phenomena in which discontinuities, jumps and abrupt qualitative changes play a central role in the study of modes of understanding advanced mathematics A significant contribution in the study of discontinuities in science from a historical perspective is given by a generous reading of Kuhn (1971) whose intention is to show the way to a more open, flexible, and historically oriented understanding of scientific inquiry as a rational activity Kuhn is suggesting that we need to transform both our understanding of scientific inquiry and our concept of rationality Piaget and Garcia (1987) give an extensive account of similar discontinuities in mathematics Freudenthal (1983) deal with concepts as the main support for cognitive structures Formal rigor and logical consistency by themselves are not sufficient conditions for efficient teaching practices but technical byproducts of a very elaborate dynamic process

All this needs to be closely connected with the development of mathematics as a fundamental scientific tool and as an outstanding paradigm for learning and understanding What is now required is to understand understanding itself in the context of mathematics and to do this in a manner that permits us to make sense of the claim that understanding belongs to the sphere of meaning Dewey (1978) points out "All knowledge, all science, thus aims to grasp the meaning of objects and events, and this process always consists in taking them out of their apparent brute isolation as events, and finding them, which in turn, accounts for, explains, interprets them, i e renders them significant " p 272

Research Framework

A viable theoretical framework for my study is provided by Gadamer (1975) and Sierpinska (1992) Three interrelated levels of knowledge have been established

i) beliefs, credences which are knowable and communicable in a discoursive form,

ii) schemes of thought, things that are learned by practice and imitation in the course of our socialization and education which are mostly unconscious and

iii) technical knowledge whose validity is affirmed by logical coherence and consistency which calls in turn for rational or scientific justification

In this sense, the mathematical technical knowledge that can be achieved is strongly influenced and conditioned by beliefs and schemes of thought Gadamer includes, in a prominent way, beliefs and preconceptions in an interpretative model that lighten the understanding process In this context, understanding of mathematics in general and understanding calculus in particular has to do with construction of concepts referred necessarily to a series of complex imbrications that eventually lead to a explanatory stage

Imagery and visualization play a significant role in the development of concepts and strategies in advanced mathematics not just as symbolic and diagramatic devices but also as a mode that allows possibility conditions to manage concepts such as continuous variation

It is possible to regard imagery as a form of enunciation (Foucault 1987) which it does not have as necesary and sufficient conditions to exist the presence of a well defined propositional structure

The theoretical position adopted in this paper stresses that strategy in calculus problem solving situations is closely related with possibilities of arising prototypes or certain forms of metaphores in the constituency of visual enunciations

The Study of an Optimization Problem.

Many of the optimization problems provided in standard textbooks have an important geometrical component usually contempted or seen subyugated to the algorithms, rarely considered as the kernel or locus of the germinal ideas that allows a more insightful use of optimization methods and envisage new prespectives in concept developing

The following is the report of classroom experience It was carried out in three different class groups of sophomore bussines and economy majors What is reported is based on my observations and getting notes from the students

I asked them to discuss in small groups of three or four students the problem of bending a rectangular plate (fig 1) of given dimensions l and a, in such a way that once the transversal sections were covered to shape a container (fig 2) would have a maximun volume

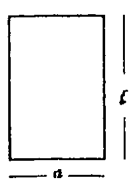


Figure 1

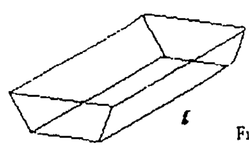


Figure 2

This is a problem that often appears in standard calculus textbooks

Through this experience three stages can be distinguished that involve multiple and varied actions of folding back to inner less formal understanding in order to use that understanding as a springboard for the construction of more sophisticated outer level understanding

Stage 1

Some paper models were made and most of the students, with certain ease, reached the point of determining that folding the plate was paramount to find where and how it must be bent, i e at what distance x from the sides and with which angle α with respect to the horizontal (fig 3) In addition it was stated that the larger the area of the trapezoidal transverse section, the larger the volume of the container and the length l of the plate was irrelevant for the purpose of optimizing

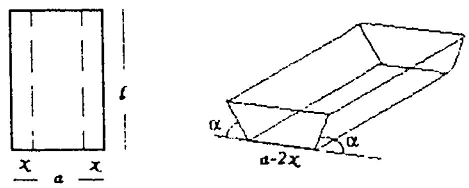


Figure 3

With the above, it was concluded that it was in fact a situation where there were two significant quantities x and l that could vary continuously Furthermore, the meningful range of values that x can have are is within the interval of [0, a/2] and the pertinent range of values of the angle α is within the interval of [0,90°]

Stage 2

With some difficulty, and without a generalized agreement, some students said that the plate shoud have been bent in three equal parts Some arguments were presented along the line that the rectangle with a fixed and given perimeter that encompasses the larger area was a square The areas of triangles, a squares and a rectangles with identical perimeters were compared At this point, there was the educated guess that it was plausible but still conflicting that to bend in $x = a/3$ could be sensible Two daring proposals were made to bend the plate forming a transvesal square section or to bend having an angle of 45° as shown in figure 4

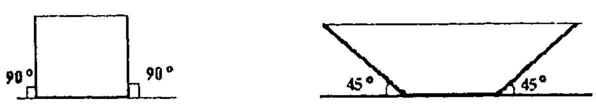


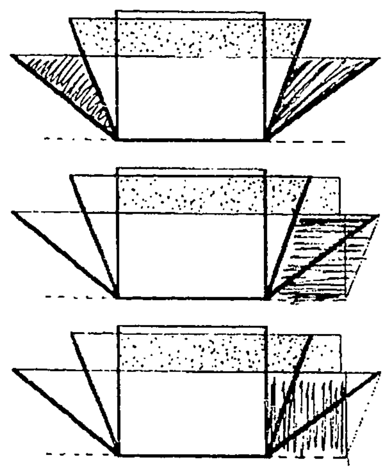
Figure 4

Stage 3

Most students were struggling with the problem of calculating simple areas in their drawing The figure of the trapezoidal transversal section, which is essentially a static object, did not embody by itself dynamic qualities needed to model the processing of changing angle α whereas the trapezoidal area was getting distorted and changing as well. Some considerations were made in terms of how much area was lost and gained by angle α varying continuously What the students found disturbing was obvious based on the incompatibility between the actual picture of the trapezoid drawn by them and the mental picture of the actual continuous variation process involved in the problem.

Stage 4

There was the certainty that once x was fixed, while α varies there was something lost and something gained marked The problem now lies in understanding this correspondence that refers us to the funtional relation between area and angle The following image (fig 5)



embodies geometrical aspects in a very insightful way that can easily be incorporated in the final understanding of the actual problem It is important to underline that this simple and standard problem can be seeing from a totally different perspective One may expect that the "standard" approach will be to **define** the volume of the container as

$V(x,\alpha) = [(a - 2x) (x \operatorname{sen}\alpha) + (x^2 \cos\alpha) (\operatorname{sen}\alpha)] /$ and to calculate $\partial v / \partial x$, $\partial v / \partial \alpha$ to obtain $\frac{\partial v}{\partial x} = \operatorname{sen}\alpha\,[a - 4x + 2x\cos\alpha]$ and $\frac{\partial v}{\partial x} = x[(a-2x)\cos\alpha + x\cos^2\alpha - x\operatorname{sen}^2\alpha]$ and to solve the equation system

$$\operatorname{sen}\alpha[a - 4x + 2x\cos\alpha] = 0$$
$$x[(a - 2x)\cos\alpha + x\cos^2 x] = 0$$

Once the cases $\alpha = 0$ and $x = 0$ were eliminated and with the use of the identity $\operatorname{sen}^2\alpha + \cos^2\alpha = 1$ the final result is $x = a/3$ and $\alpha = 60°$

Some remarks

Through this simple problem setting, one realizes that any interpretation of the problem involves highlighting Furthermore, it makes no sense to speak of the single or the correct interpretation There are a variety of interpretations, and it is even possible to disciminate distinctive interpretations We are dealing with the phenomenon of understanding As Gadamer points out "Understanding must be conceived as a part of the process of the coming into being of meaning, in which the significance of all statements - those of art and those of everything else that has been transmitted - is formed and complete " p 146

The recognition that prejudices enable us to understand, and that understanding is constitutive of what we are in the process of becoming, has some very strong consequences in mathematical education A basic recurrent prejudice is the use of some mathematical algorithms presented in the classroom without really understanding what is acctually being modeled or interpreted

Gadamer does want to make the all important distinction between blind prejudices and justified prejudices productive of knowledge or what may be called enabling prejudices But this does not diminish the significance of the thesis that both types of prejudice are constitutive of what we are But how do we make this crucial distinction? How are we to discriminate which of our prejudices are blind and which are enabling? One answer is clearly ruled out, we cannot do this by an act of pure self-reflection For Gadamer, it is in and through the encounter with problem situations and more general what is handed down to us through tradition that we discover which of our prejudices are blind and which are enabling It is only through the dialogical encounter with what is at once alien to us, makes a claim upon us, and has an affinity with what we are that can open us to risking and testing prejudices This does not mean that we can ever finally complete such a project, that we can ever achieve complete self-transparency, that we can attain perfect or absolute knowledge It make no sense to think that a human being can ever be devoid of prejudices To risk and test our prejudices is a constant task, not a final achievement

Using as a starting point the optimization problem presented above, formulations of the hermeneutical circle of understanding is object oriented, in the sense that it directs us to practices or forms of life that we are seeking to understand It directs us to the sensitive dialectical play between part and whole in the circle of understanding Many standard characterizations of the hermeneutical circle focus exclusively on the relation of part to whole in the phenomena which we seek to understand No essential reference is made to the interpreter, to the individual who is engaged in the process of understanding optimization and questioning, except insofar as he or she must have the insight, imagination, openness, and patience to acquire this art -an art achieved through practice There is no set method for acquiring or pursuing this art, in the sense of explicit rules that are to be followed Yet, we might say that rules here function as heuristic guides that gain their concrete meaning by appealing to exemplars of such hermeneutical interpretation But a full statement and defense of the hermeneutical circle requires us to ask the Kantian question, How is such understanding and interpretation possible?

There is a positive role that fore-having, fore-sight, fore-conception, and prejudgement play in all advanced mathematical understanding The reference to the ontological character of the circle indicates something basic about our very being-in-the world -that we are essentially beings constituted by and engaged in interpretative understanding

The reference to the things themselves is not to be misunderstood as suggesting that these things exist and we must purify ourselves of all forestructures and prejudgments in order to grasp or know them objectively. On the contrary, the meaning of the thingsthemselves can only be grasped through the circle of understanding, a circle that presupposes the forestructures that enable us to understand The most important consequence of the scheme of hermeneutical circle clarifies the relation between the interpreter (student) and what he or she seeks to understand (optimization problem)

We must learn the art of being responsive to situations that we are trying to understand. We must participate or share in them, listen to them, open ourselves to what they are saying and to the claims to truth that they make upon us And we can accomplish this only because of the forestructures and prejudgments that are constitutive of our being

It is true, of course, that understanding requires effort and care, imagination and perceptiveness, but this is directed to the opening of ourselves to what we seek to understand And such receptiveness is possible only by virtue of those justified prejudices that open us to experience

References

Dewey, John 1978 *How We Think* Ed Jo Ann Boydston The Middle Works, Vol 6, *1910-1911* Carbondale and Edwardsville Southern Illinois University Press

Foucault, Michael 1987. *La Arqueologia del Saber* Translated by Aurelio Garzon del Camino Mexico Siglo Veintiuno Editores

Gadamer, Hans 1975 *Truth and Method* Translated and edited by Garret Barden and John Cumming New York Seabury Press

Kuhn, Thomas S 1971 *La Estructura de las Revoluciones Cientificas* Translated by Agustin Contin México Fondo de Cultura Económica

Piaget, Jean and Garcia, Rolando 1987 *Psicogénesis e Historia de la Ciencia* Translated by Rolando Garcia. México. Siglo Veintiuno Editores

Presmeg, Norma C 1992 "Prototypes, Metaphors, Metonymies and Imaginative Rationality in High School Mathematics " *Educational Studies in Mathematics* **23** (6) 595-610

TEACHING PROBLEM SOLVING: A TEACHER'S PERSPECTIVE

Olive Chapman
The University of Calgary

This paper reports on a case study that investigated an experienced elementary teacher's perspective of teaching problem solving in mathematics. Data was collected through interviews, discussions and classroom observations. Naturalistic inquiry processes were used in the analysis. Findings indicate that the teacher uses a 3-stage process, i.e. preparation, collaboration and presentation, to organize her teaching. Embodied in this process is a non-traditional view of the teaching and meaning of problem solving that reflects the teacher's belief of the nature of problems and problem solving and her personal experience as a problem solver.

This paper focuses on an experienced elementary teacher's perspective of the teaching of problem solving in mathematics and the meaning framing that perspective.

Research on problem solving has traditionally focused on learning in the context of what students do or should do in solving problems, but has given no substantial consideration of the teacher. Even when problem solving instruction was being investigated, the focus has been on the students and the "experimental treatment", but not the teacher. Kilpatrick (1985), in reviewing two and a half decades of problem solving research, found that the teacher was always ignored in such studies. He contended that the lack of information on how teachers approach problem solving could be one of the reasons there has not been much success in improving the teaching of problem solving. Other prominent mathematics educators (Grouws, 1985; Lester, 1985; Silver, 1985; Thompson, 1985) have expressed concern about the systematic control of the teacher and teacher-related variables, either consciously for statistical reasons or unconsciously because of the researchers bias towards empirical research perspectives, in research on problem solving instruction. To change this pattern in problem solving research, a shift to more naturalistic research processes has been promoted (Silver, 1985; Eisenhart, 1988). Although this shift may be happening with respect to mathematics teaching in general (Hoyles, 1992, reviews studies on mathematics teachers), there is still a major gap in published information on how teachers conceptualize problem solving or how they attempt to teach it.

While studies on mathematics teachers thinking are only now beginning to receive attention, there exists a growing body of literature on teacher thinking and teacher personal knowledge in the context of teaching in general (for eg. Clandinin, 1986; Elbaz, 1991; Leindhart, 1987; Russel & Munby, 1992; Solas, 1992; Thompson, 1985). This literature indicates that knowing how a teacher makes sense of his/her teaching is critical in understanding teaching and how to create meaningful and effective teacher development programs. Given recent emphasis on problem solving in mathematics education (eg. NCTM, 1989), such understandings are definitely needed to enhance or transcend traditional boundaries in the teaching of problem solving.

This paper deals with one aspect of a larger study[1] in progress that focuses on the teaching of word problems, in general, and non-routine problems, in particular, from the perspective of a beginning teacher and two experienced teachers. The case of one experienced teacher will be presented here. Because of constraint on space, the case will be discussed only in the context of what happens from an instructional perspective when students are assigned a problem.

THE PARTICIPANT

The participant, who will be referred to as Lillian, has been an elementary teacher for 24 years, 3 of which she was also an elementary mathematics consultant for her school board. In the last five years, as a classroom teacher, she has actively participated in mathematics projects conducted by her school board, piloted mathematics text book materials for publishers and made presentations (not on problem solving) at mathematics conferences. During the last 10 years, she has attended problem solving workshops locally and at National Council of Teachers of Mathematics conferences and read some materials on the teaching of problem solving. She is considered to be a very good teacher by peers and students. For the duration of the study, she taught grade 6. At the beginning of the study, she had been using non-routine problems to teach problem solving for about five years. Prior to that, she associated problem solving with routine word problems, not because she believed that, but she felt that that was what was required from a mathematics perspective.

After learning about Polya's model to problem solving in the early 1980's, Lillian started applying it in her teaching, but in a rote manner, drilling students on the various stages. By the late 1980's, she became convinced that this approach was meaningless, not only for her personally, but more for the students. The turning point occurred when, one day, she questioned a group of her grade 6 students on what they were doing as the worked on a problem. They replied, "Carrying out the plan." She asked what was the plan. They replied that they didn't have one. Similar behaviour was reflected by other groups of her students. Since then, Lillian has been trying to figure out and create, based on her personal experience and practical knowledge, what works for her instead of slavishly following what someone else has prescribed. The findings of the study reflects her current perspective (which she believes she always held conceptually) of teaching problem solving.

---

[1] This research project is being funded by the "University Research Grants Committee", The University of Calgary.

RESEARCH PROCESS

Data for the study was collected in the following ways: classroom observations of Lillian's instructional strategies; discussion with Lillian following each observation; audio-taping of 7 problem solving classes; flexible interviews with Lillian to capture her past experiences with problem solving, her belief about problem solving, her understanding of problem solving and her thinking of how she taught problem solving; Lillian's ongoing feedback on the transcripts and the researcher's interpretation of her teaching. All interviews and discussions were tape recorded and transcribed. Copies of teaching documents (eg. text material, written plans, handouts) were also collected.

The data was interpreted by drawing from naturalistic inquiry processes (Lester, 1985), in general, and narrative inquiry processes (Connelly & Clandinin, 1990; Polkinghorne, 1988), in particular. The data was thoroughly examined to determine the nature of and the factors that guide the participant's instructional decisions and behaviours. Meanings and interpretations of prominent behaviours were determined through collaboration between participant and researcher.

LILLIAN'S TEACHING OF PROBLEM SOLVING

In discussing the findings of the study, the focus will be on situations when the students are assigned non-routine problems. The most consistent pattern in Lillian's teaching follows three stages. These I have labelled: the preparation stage, the collaboration stage and the presentation stage. Embodied in these stages are Lillian's perspective of problems, problem solving and herself as a problem solver. The most prominent and consistent features of each stage will be presented here with the apparent meaning that frames them.

STAGE 1: PREPARATION

The preparation stage is where the students try to understand the problem and make it their own, but do not consider or attempt a solution. This stage unfolds in three sub-stages as follows:

(1) Teacher-led discussion

After students read the problem, Lillian acts as chair of a whole class discussion on any initial concerns students may have about the problem. This generally involves dealing with surface features of the problem, for example, meaning of words, ambiguities in the language.

(2) Student Reflection

Following the teacher-led discussion, the students get into their groups (size of 2, 3 or 4) and further reflect on the situation presented in the problem. Lillian describes this as:

> giving them time to rummage around [with the problem] in their heads and in their groups. ... to reflect on what they are thinking, what is this problem, what they don't understand about it, what they understand about it.

(3) Teacher-led sharing

Lillian acts as chair of a whole class sharing during which the groups share their thinking about the problem. Her goal here is not to get everyone to arrive at the same understanding of the problem, but to have the students demonstrate that the problem makes sense to them and they are ready to proceed with a solution. The sharing also allows the groups to get feedback from the class on any misunderstanding reflected in their thinking and to inform the teacher of their starting point.

The preparation stage seems to be intended to accomplish more than simply understanding the problem, when viewed from Lillian's perspective of problems and problem solving. Lillian believes that,

> a problem has to have meaning for the participant, if it doesn't, then there is no problem. ... If you do not take ownership, then you do not care, then it is not a problem.

The preparation stage allows her students to establish their meaning of the problem, to make the problem their own and to take some ownership for what they do with it. The "free play" aspect of the preparation stage allows students to "play around" with the context of the problem to gain a personal understanding and appreciation of the problem. Lillian, through her behaviour, constantly conveys to her students that when she gives them a problem, it is also their problem and not just hers. Thus she allows them to proceed to a solution only after they have indicated what they had interpreted as **their** problem to solve.

STAGE 2: COLLABORATION

In the collaboration stage, students collaborate in their groups to solve the problem. The goal, Lillian points out, is for them

> to bring to the solution whatever ability, skills, ideas, thoughts they have to work toward a solution ... to share and dump ideas towards solutions, ...to take that information into their thinking to help them get to a solution, ... to come to an agreement about how they understand the solution.

This open ended approach is discussed with the students in the first few weeks of the term when problem solving strategies are considered in the context of the students' personal experiences, (eg what they bring to the problem situation) and collaborative strategies considered in the context of cooperative learning. Lillian explains,

> At the beginning of the year I have to teach what is involved in the process and then they just have to take it and do something with it and work through it.

There are brief reviews of those strategies throughout the year.

As the students work in their groups, Lillian circulates among them without becoming attached to their processes or controlling the direction of these processes. She describes her role:

> I move about the classroom to overhear what the interaction is within the group and to move into it to assist, to referee, to suggest, to encourage. ... I am not the focus of the activity or the deciding factor as to what happens.

She is able to accomplish not being the focus or deciding factor of the students' processes because she generally selects and assigns problems she has not actually solved. She explains:

> I am not holding secretly the solution and waiting for the students to get that solution. ... Often times I do not have a clue what the problem is and how it operates.

This apparently unique situation Lillian creates for herself significantly shapes the social context of her instructional approach. The teacher-student interaction that results is not one of bargaining or negotiating, overtly or covertly, to arrive at the teacher's solution, since Lillian does not claim ownership of or have students work towards a specific solution. Instead, it is one of open-dialogue. Lillian describes this process as follows:

> In the dialogue, what I'm doing is I'm hearing what they (the students) are saying, and I'm putting that into my context and then I'm taking that (the way she understands it) and giving it to them, so then they can hear it and put it in their context. So this is a continual back and forth exchange. ... Everybody (teacher and students) is free to join the dialogue, so what you need to do is put into the pot (pool of ideas) what you think and take out what you can use. ... The dialogue from my end is to get what the understanding of the student is so I can get a better sense of how to work with them and what to do. I want them to give me what they have and sometimes I can't give them what I have, because I do not have any. So what I might do is only pick up something from a group and then depart. ... The dialogue from their (the students) end is to understand what it is I'm trying to give them, how they can use it and what they do with it.

There are 2 levels of questioning Lillian initiates in this dialogue. One is to get an understanding of how the students are thinking. The other is to get the students to "throw ideas into the pot", to reflect on their processes if stuck or off track, or to stimulate their thinking. Regarding the latter case, she explains,

> I'll just ask them questions. What I want the questions to do is to start them questioning what they're doing. For example, "Are you sure that's what you want to do?" I don't want to give them specific direction but I want them to re-check, I want them to re-think so that in part it could be their own re-discovery rather than mine.

In general, Lillian's strategy is to make the students live through the consequences of their actions and their thinking to learn from them.

The dialogue approach is what Lillian personally seeks out when she solves problems, generally, both in and out of the classroom. In fact, this is why she stopped solving problems before using them in class. She found it was more meaningful and easier for her to solve the problems if she used her students as her collaborators instead of solving them by herself. So as she listens to her students' ideas as students collaborate in their groups, she draws from them to establish or clarify her understanding and solution of the problem. By the end of the collaboration stage, she may or may not have her own solution and her solution may or may not be the same as any of the groups. The dialogue, then, facilitates her own problem solving needs. But, more important, through it, she is creating for her students a context she would want as a problem solver because she understands the value of it from her personal experience.

STAGE 3: PRESENTATION

The presentation stage is when students get to present and defend their solution and reflect on what they had learned. It is a coming together of the students and teacher to share knowledge as the groups "bring what they see as theirs, both in the problem and in the solution" (Lillian) to present.

I view this stage more like a conference with Lillian as the convenor and the students as the presenters and audience. Lillian summons the students to this conference to present their problem, their approach and their findings. She chairs the sessions by acknowledging speakers (presenters and audience), but also poses questions, summarize discussions and sometimes becomes a presenter herself. At the end of the conference the students should leave with some understanding of, for example, what they brought to the situation, what they learned from others, what needs future consideration, what was resolved.

During the conference, both presenters and audience are always actively involved. Lillian describes the goal of the audience:

> The audience follows along, so they can see the direction they (the presenter) came from, how they got the solution and whether the solution in fact fits.

However, the audience's "following alone" is not passive. Students ask questions to help them make sense of the solution being presented, make comments on what they agree or disagree with regarding the solution and make suggestions to the presenters. Lillian becomes a presenter only if she thinks her solution would add another perspective for the students. For her, closure to the process does not have to be a final answer. Instead, as she explains,

> It is a going away with the understanding of the process and what each of those students brings into his or her own repertoire and how it connects to what he or she had before.

Thus, to end the conference, Lillian asks students to respond to:

> what did you do in your group that made it work or did not work? What did you learn here? What became meaningful? What helped you to understand and solve the problem?

The presentation stage legitimizes the students' processes and Lillian's instructional approach --- particularly giving the students ownership of the problem. In general, this stage contributes to the experiential conception of problem and problem solving reflected in Lillian's teaching.

CONCEPTUALIZING PROBLEMS AND PROBLEM SOLVING IN LILLIAN'S TEACHING

In the context of Lillian's teaching, problems and problem solving take on an experiential meaning. Conceptually, Lillian makes no distinction between problems and problem solving. The problem solving is the problem. The problem is in the solving and not the written text. One does not have a problem until one starts to experience and deal with a barrier in a situation one is curious about or has an interest in. Thus problem solving is the experience of trying to satisfy one's curiosity or to fulfil one's interest in situations when one encounters barriers. In Lillian's teaching, this experience becomes more significant than the final answer to a problem. Thus solutions are viewed as the experiencing of "the process of how to work through the problem" (Lillian). It is not the final answer that will move the students to new levels in their problem solving, but the nature of the experience leading to the final answer.

Lillian's teaching also reflects a different kind of problem teacher and students must face compared to a more traditional classroom. Because students have to identify "their" problem, take control and responsibility for it and deal with it, the "mathematics problem" Lillian assigns becomes the nucleus of a larger problem the students perceive. This larger problem for the student includes, how to work collaboratively, how to dialogue with the teacher, how to present and defend their solutions, how to understand other perspective of the solution. Similarly, the teacher perceives a larger problem that includes, as Lillian notes,

> To know how to dialogue with them (the students), to know how to intervene and contribute without interfering with their process but add enough to help them make sense so that they can go ahead with this process. Also being able to recognize when something is working, not to interfere to get that off track.

Finally, in Lillian's teaching, problem solving is reflected not only as a cognitive endeavour, but also a social endeavour. The latter allows the same problem to be different in different groups of students depending on how it "bounces off" the students in a particular group and the nature of the group dynamics.

CONCLUSION

Lillian's case may not be representative of the way many teachers teach problem solving, but it provides us with a perspective that is worth exploring in its own right. Investigations of this perspective could also be undertaken to examine its actual effect on performance and the type of learning it produces. The case also provides a situation teachers can resonate in to reflect on and understand their own teaching. But, of more importance, it draws attention to the role of teacher's belief and personal experiences in shaping problem solving instruction, a situation that deserves much more attention for us to understand teacher thinking and to provide teachers with the kind of support that will foster meaningful and effective problem solving teaching styles.

REFERENCES

Clandinin, D.J. 1986. Classroom Practices: Teachers Images in Action. London: The Falmer Press.

Connelly, M. & Clandinin, J. 1990. Stories of Experience and Narrative Inquiry. Educational Researcher. 19, 4.

Eisenhart. M.A. 1988. The Ethnographic Research Tradition and Mathematics Education Research. Journal of Research in Mathematics Education, 19 2.

Elbaz, F. 1991. Research on Teacher's Knowledge: The Evolution of a Discourse. Journal of Curriculum Studies, 23.

Grouws, D.A. 1985. The Teacher and Classroom Instruction: Neglected Themes in Problem Solving Research, in E.A.Silver (Ed.) Teaching and Learning Mathematical Problem Solving: Multiple Research Perspective, London: Lawrence Erlbaum Associates, Publishers.

Hoyles, C. 1992. Mathematics Teaching and Mathematics Teachers: A Meta-Case Study. For the Learning of Mathematics. 12, 3.

Kilpatrick, J. 1985. A Retrospective Account of the Past Twenty-five Years of Research on Teaching Mathematical Problem Solving, in E.A.Silver (Ed.) Teaching and Learning Mathematical Problem Solving: Multiple Research Perspective, London: Lawrence Erlbaum Associates, Publishers.

Lester, F.K. 1985. Methodological Considerations in Research on Mathematical Problem-solving Instruction, In E.A.Silver (Ed.) Teaching and Learning Mathematical Problem Solving: Multiple Research Perspective, London: Lawrence Erlbaum Associates, Publishers.

Leinhardt, G. et al. 1987. Introduction and Integration of Classroom Routines by Expert Teachers. Curriculum Inquiry, 17 ,2.

National Council of Teachers of Mathematics. 1989. Curriculum and Evaluation Standards for School Mathematics. Reston, Va.

Polkinghorne,D.E. 1988.Narrative Knowing and the Human Sciences. Albany, N.Y.: State University of New York Press.

Russell, T. & Munby, H. 1992. Teachers and Teaching: From Classroom to Reflection, London: The Falmer Press

Silver, E.A. 1985. Research on Teaching Mathematical Problem Solving: Some Under-represented Themes and Needed Directions, in E.A.Silver (Ed.) Teaching and Learning Mathematical Problem Solving: Multiple Research Perspective, London: Lawrence Erlbaum Associates, Publishers.

Solas, J. 1992. Investigating Teacher and Student thinking, Review of Educational Research,62,2

Thompson, A.G. 1985. Teacher's Conception of Mathematics and the Teaching of Problem Solving, in E A.Silver (Ed.) Teaching and Learning Mathematical Problem Solving: Multiple Re earch Perspective, London: Lawrence Erlbaum Associates, Publishers.

SKETCHING GRAPHS OF AN INDEPENDENT AND A DEPENDENT QUANTITY: DIFFICULTIES IN LEARNING TO MAKE STYLIZED, CONVENTIONAL "PICTURES"

Daniel Chazan
Michigan State University

Sandra Callis Bethell
Holt High School

This paper presents a series of issues which lead students to sketch relationships between quantities non-conventionally. Some issues seem to be associated with the quantities being graphed and others lead to graphs which aren't useful "pictures" of the relationship. An examination of these issues suggest a reinterpretation of the Iconic Translation phenomenon (Monk, 1992)

Like many, we are impressed by the potential of technology to make visual representations of mathematics widely available. At the same time, we are aware of the student difficulties with graphs described in the mathematics education literature (For surveys, see Eisenberg, 1991; Kerslake, 1981; Romberg, Fennema, & Carpenter, 1993.). Rather than approach student difficulties as misconceptions to be uprooted, we have approached them as ideas which can change in the normal course of learning and instruction, and as indicators of conventions to which our training blinds us.

We have been teaching a low-tracked, high-school level Algebra One course and begin this course with the sketching of graphs of dependence relationships between quantities. As we begin the year, we are concious of our own mathematical enculturation (in the sense of Bishop, 1988). The graphs that students produce often seem mathematically incorrect to us, or, at least, non-conventional. These graphs challenge us, as teachers, to articulate, and discuss with our students, how graphs work in mathematics and the reasons behind conventional approaches to creating "pictures" of the relationships between quantities in situations.

In this paper, we present a selection of students' graphs and a structure for understanding how their graphs contravene accepted mathematical practice. We organize our presentation of students' graphs in two categories, by the type of teaching intervention their graphs suggest to us. When students' graphs raise issues about their understandings of "quantity," as teachers, we can help students clarify their options and choose what they want to represent. By way of contrast, when students have questions about how to proceed to create a graph of the relationship that they are representing, we, as teachers enculturated in mathematics, find ourselves arguing for standard ways of creating graphs by pointing out the benefits of the resulting displays--the ways in which they create "good pictures."

COURSE CONTEXT

In the United States, Algebra is taught as a separate mathematical topic, usually in two courses labeled Algebra One and Algebra Two. As teachers of low-track Algebra One students, we are interested in understanding Algebra in a way that would allow us to create a course which could be useful and challenging for a wide range of students--students who are headed to the world of work after high school and students who are headed to college and calculus. We have been exploring an approach based in the mathematics of quantity.

In the beginning of the course, we look for quantities in the world around us. We talk about quantities which can be measured and counted, as opposed to quantities which are computed. After developing what we mean by quantities, we begin to look at relationships between quantities. In the early part of the course, we learn to represent these relationships in verbal descriptions, graphs, sketches of graphs, tables of values, and algebraic expressions, each of which has its own conventions.

In introducing graphing, we walk a tight balance. We emphasize that, even though graphs aren't pictures, this visual representation was created because it has helped people "picture" relationships that frequently can't be seen in other ways. After some introductory activities, we ask our students to write stories for a partner and then to swap stories and make graphs to illustrate the other person's story.

WHAT ARE THE QUANTITIES WHOSE RELATIONSHIP YOU WANT TO "PICTURE?"

In any given situation, there are a plethora of quantities whose relationships might be graphed. Yet, in order to choose to make an interesting "functional" graph on the Cartesian plane, it is useful to choose two quantities which change, where one can be said to depend on the other.

<u>What are quantities?</u>

There are subtle complexities in choosing quantities whose relationship students want to graph. Below are descriptions of a situation and a student's list of the quantities present in the description.

> The QE2 was headed from Martha's Vineyard to NY on a fine night with clear visibility and half moon. Captin Woodall, with more than 40 years experience, was in comand, but a local pilot was controling the navigation. Shortly after the QE2 left Martha's Vineyard, Woodall said, he felt a very heavy vibration. He first asumed that the ship hit something on the water or there was a mechanical failure. After crew members ruled out those possibilities, the only one left was that the ship touched bottom. Charts showed ocean depth of 39 ft and 2 ft due to the tide. The ships hull goes 32 feet below the water line. After the grounding, the 1815 passangers were evacuated from the ship.
>
> Passengers were put on trains to complete their journey to NY. In his first public statement, Captain Woodall indicated that on that day he didn't drink any alchohol. (Berkshire Eagle on 8/14/92)

> I how many crew members there are. how many pass-angers. how Big the hall is
> how deep was the water 34ft.
> when you hit the object.
> how many years did the Captain have time ship left and grounded.

Figure 1--The QE2 story and a student's list of quantities

In the QE2 example, "when you hit the object" and "time ship left and grounded" are each listed by the student as different quantities. These are not seen as points along a single timeline. The student also views the number of years of experience of the captain as a quantity, even though it doesn't vary during the described situation.

This example points out that there are different meanings for the word quantity. The first two definitions of quantity in the American Heritage Dictionary indicate important differences in usage:

> 1. a. A number or amount of anything, either specified or indefinite. b. A sufficient or considerable amount or number. c. The exact amount of anything. 2. The measureable, countable, or comparable property or aspect of a thing.

As enculturated mathematicians, when we look to graph a relationship between quantities, we think about quantity with the second definition, certainly not with definitions 1a or 1c. We do not consider every number as a different quantity. Yet, students may bring this other view of quanitity to their mathematics class.

<u>What is the quantity for this unit?</u>

In addition to students' understandings of "quantity," units add another set of complications. Below are two graphs which differ in the labeling of units and quantities on the axes. One student provides both quantities and units, while the other puts a quantity on one axis and a unit on the other.

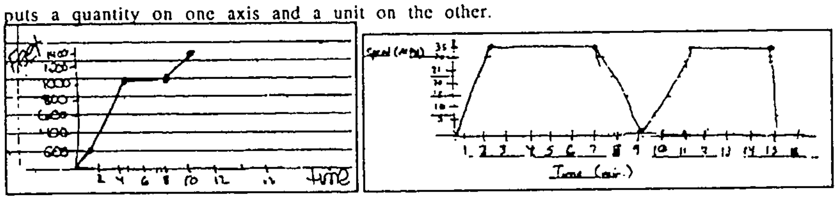


Figure 2--Two graphs and their units.

The first graph comes from a story which mentions the unit, feet, but not the quantity which it measures. Students can then find it difficult to name the quantity separate from its unit.

Using the unit, instead of the quantity, to label a graph can lead to complications. In the following example, the axes are not labeled, but even an indication of units would not help.

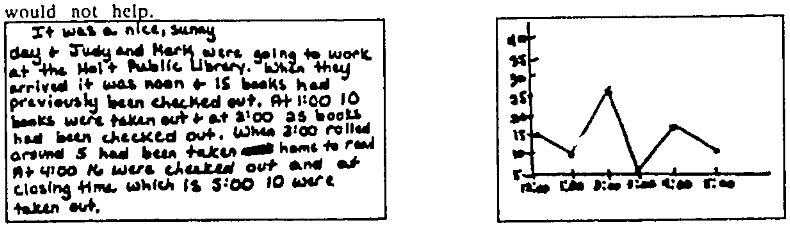


Figure 3--A library story and its graph

The class discussed whether this graph and story indicated how the total number of books checked out of the library varies, as opposed to how the number of books checked out varies, over intervals of time. In the discussion, students talked about whether the decreasing portions of the graph represent books being returned or a decrease in the rate at which books were being checked out.

This example points out that the same unit can be used to indicate different quantities. In particular, the unit for the dependent variable does not indicate whether the value of the dependent variable should be accumulated or not. If the variable is accumulated you have a different quantity than if it is not accumulated.

Where are the quantities?

In a further complication, important quantities may not be mentioned at all (neither directly or by their unit) in a description of a situation. The following two examples were recorded by a student from a software program (Dugdale & Kibbey, 1986) because she did not understand. In each case, the independent quantity is not mentioned in the description of the situation. Nonetheless, as enculturated mathematicians, with the travel description for example, we would not find it strange to be asked to sketch a graph of the gasoline used as a function of the distance traveled, even though the distance traveled isn't mentioned in the description.

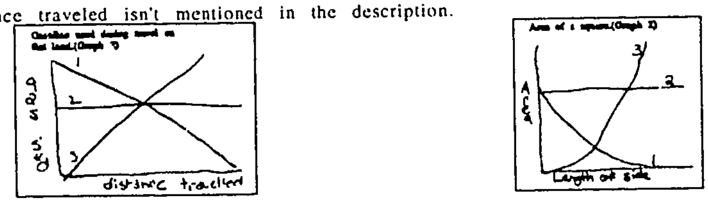


Figure 4--Graphs a student didn't understand

MAKING A GOOD PICTURE

When we introduce the sketching of graphs to our students, we encourage them to think of their graphs as made up of intervals between known points. Modeled on the approach taken in *The Grapher's Sketchbook* (Yerushalmy, Shternberg, & CET, 1992), we present a set of icons and suggest that they patch these icons together to create a graph which is a "good picture" of the relationship between quantities that they are depicting.

Figure 5--A set of icons.

A good picture captures information about the rate of change of a quantity as well as about its value. This emphasis on the graph as a picture of a kind, one which conveys information about the value of the dependent quantity and how that changes, allows students, and us, their teachers, to argue for and against particular ways of making a graph. In particular, thinking about the types of pictures produced by mathematical approaches to graphing suggests reasons for having an equally spaced metric on the axes, reasons for accumulating the independent variable in a situation, and for choosing one quantity to be an independent variable and another to be dependent.

Metrics for the axes?

When beginning to sketch graphs, some students take numerical information out of a description of a situation and do not create a metric along each axis. Instead, they put each value at an equal distance from the next one.

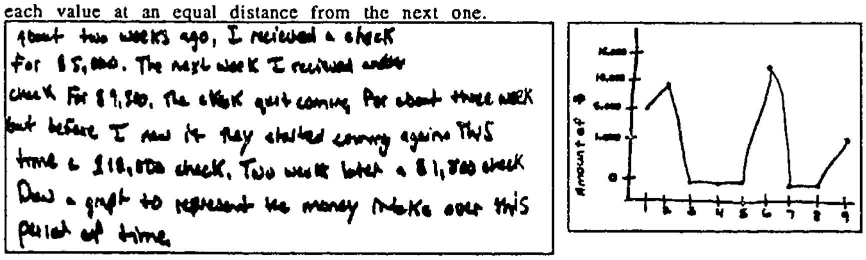


Figure 6--Mysterious checks and their graph

With the notion of creating a "good picture," we can argue for the use of a standard, constant difference metric along the axes, because such a picture allows a better sense of the change in value of a quantity or of its ra·e of change. In this case, one can argue that the provided graph is misleading, because it suggests that the difference between 15,000 dollars and 5,000 is not nearly as big as the difference between 0 dollars and 5000 dollars, though, a counterargument could be made based on a psychological metric.

Continuous lines or points?

The natural numbers are used in many situations in an ordinal sense. When asked to write stories and make graphs of the variation in the relationships between quantities, this raises the question of whether points on the coordinate plane should or shouldn't be connected. In the example given below, the independent variable names the order of the meets. Arrows are drawn to capture the variation, but the points aren't connected, because there were no intermediate values between the different meets. In other examples of this kind, students connected the points.

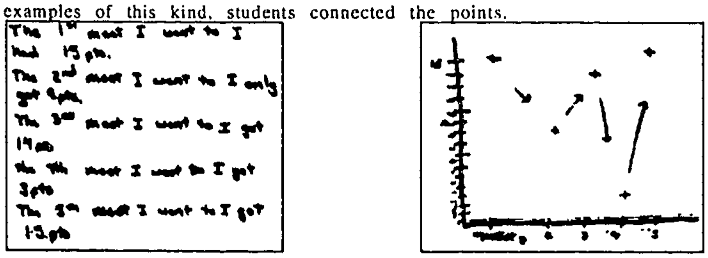


Figure 7--Graphs of points for each meet.

However, in some cases when the natural numbers are used as ordinals, they name an interval rather than a specific moment in time. One student connects the points and the other does not.

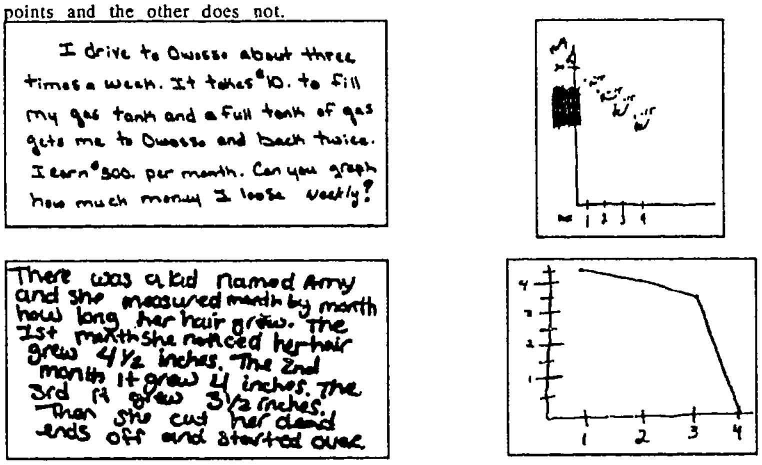


Figure 8--Spending on gas and hair growth

In the gasoline example, one might use a continuous graph with points of discontinuity, though such a graph requires a conjecture about the times of the fillups. Though it looks similar, the hair growth example is complicated. If the dependent quantity is amount of growth per week, then a even a continuous graph with points of discontinuity suggests that it is possible to assign more than one value to this quantity in a week.

As these examples illustrate, the use of lines or points is a complex issue for students, but we must also admit that it is a difficult one even for members of the mathematical community. In theory, the use of lines or points signals the types of values which the variables can take on. A continuous line is used when both variables can take on intermediate, real number values between any two values. In contrast, when the variables represent categorical information, or more generally are conceived of as not being able to take on intermediate real number values, points are used. However, these conventions are not universally observed. In most cases, with variables over the rational numbers--a dense set, a continuous line is used and irrational values are overlooked. Also, if the scale is sufficiently large, a continuous line is sometimes used with variables over the natural numbers.

<u>Accumulation of the independent variable</u>

As noted in the discussion of metrics, when students are just beginning to learn about graphs, there is a tendency to take numbers directly out of a situation without relating them to one another Students might include numbers with the same unit in a table, even though these numbers represent different quantities. For example, students created four different tables for the fanciful bungy-jumping story they created collaboratively.

> Chris bungee jumped from 200 ft. He jumped and at 100 ft. above ground the cord snapped (after 4 seconds). He fell 15 more feet in 2 seconds and then grabbed a bird. The bird carried him 100 ft. up in 7 seconds.

Figure 9-- A Bungie Jumping story

Notice that the 200 feet and 100 feet represent altitudes above ground while the 15 ft and second 100 feet represent changes in altitude. All of the times, except the implicit starting time, represent intervals.

A		B		C		D	
time	distance	time	distance	time	distance	time	distance
0	200	0	200	0	200	0	200
4	100	4	100	4	100	4	100
2	15	6	15	2	85	6	85
7	100	13	100	7	185	13	185

Figure 10--Students' Bungy Jumping Tables

In tables C and D, the "distance" (distance above ground) is accumulated; in tables B and D, the time intervals are accumulated. As enculturated members of the mathematics community, table D looks appropriate and table B might be fixed to record changes in distance above ground over intervals of time. However, tables A and C are both problematic. In seeking to explain why we felt these tables were problematic, we asked students to graph their tables and see which gave them the "best picture" of the relationship between the quantities in the story.

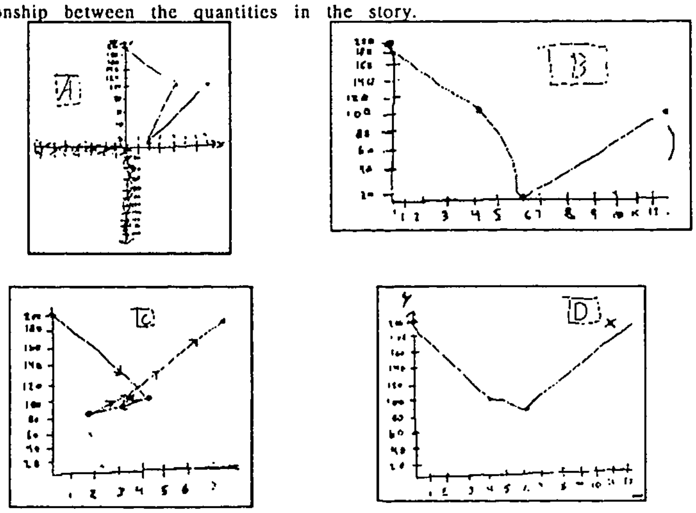


Figure 11--Students Bungy Jumping Graphs

Tables A and C then stood out as creating graphs that double back on themselves and, unlike text in Romance languages, can not be read continuously from left to right. In fact, in part C, the student whose work is above added arrows to indicate how the graph should be read. This suggested that graphs B and D were "better pictures" of the relationships described in the situation.

Choosing which quantity is the independent one

Similarly, an appeal to the desire to read from left to right can suggest which quantity is the dependent quantity and which is the independent one. Students were given a table with age and weight measurements for a growing child. Many wanted the age to be a function of the weight, rather than the other way around.

As soon as we started to think about circumstances in which the weight of the person began to decrease, the desire to have a graph that doesn't double back suggested that weight should be depicted as a function of age.

CONCLUSION

In closing, we would like to offer a reinterpretion of the phenomenon of *Iconic Translation* (Monk, 1992) which has been identified in the literature. When we first began to teach about sketching graphs, we wanted our students to understand that a graph was not "merely" a picture of a situation; we did wanted our students to overcome this "misconception." Yet, the more we thought about the task, we realized that we did indeed want our students to think of graphs as pictures, but as pictures which must be drawn in certain conventional ways.

This perspective on graphs suggests that the problem with *Iconic Translation* is not to be found in its pictorial literalness, but elsewhere. We suggest that when students create such graphs, they make correct graphs for a different relationship betwe quantities than the ones they are being asked about. For example, in a speed/tin graph, students are instead drawing a picture of the relationship between distance above some reference height and time. In the example in figure 4, the student prefers a picture of the constant rate of consumption of fuel over time to one of the amount of fuel used as a function of distance traveled.

Our purpose in offering this reinterpretation is that it suggests alternative pedagogical moves. Rather than suggest to students that graphs are not pictures, this reinterpretation suggests that graphs are pictures, but that we must understand what it is that they are indeed pictures of.

As one of our students said when helping a student who was new to the class learn about sketching graphs from the program *Interpreting Graphs*: (Dugdale & Kibbey, 1986)

New: "Oh Yeah, I don't look at those." (*points at the quantities written on the axes*)

Experienced: "You got to remember those (*points to the quantities*). You got to remember, that's the most important thing on the graph."

References

Bell, A. & Janvier, C. (1981) Interpreting Graphs. For the Learning of Mathematics 2(1) 34-42.

Bishop, A. (1988) Mathematical Enculturation: A cultural perspective on mathematics education. Dordrecht: Kluwer.

Dugdale, S. & Kibbey, D. (1986) Interpreting Graphs (computer software). Pleasantville, NY: Sunburst.

Eisenberg, T. (1991) Functions and associated learning difficulties. In D. Tall (Ed.) Advanced Mathematical Thinking (pp. 140-151). Dordrecht: Kluwer.

Kerslake, D. (1981) Graphs. In K. M. Hart (Ed.) Children's understanding of mathematics: 11-16. (pp. 120-136). London: John Murray.

Monk, S., (1992) Students understanding of a function given by a physical model. In E. Dubinsky & G. Harel, The concept of function: aspects of epistemology and pedagogy. pp. 175-195 Washington, DC. MAA.

Romberg, T., Fennema, E., & Carpenter, T., (1993) Integrating Research on the Graphical Representation of Functions. Hilldale, NJ: Lawerence Erlbaum.

Yerushalmy, M, Shternberg, B, and Center for Educational Technology: (1992). The Grapher's Sketchbook (computer software). Scotts Valley, CA: Wings for Learning.

AN INQUIRY INTO THE CONCEPT IMAGES OF THE CONTINUUM.TRYING A RESEARCH TOOL

Carles Romero i Chesa & Carmen Azcárate Giménez
Department of Mathematics and Sciences Education
Universitat Autònoma de Barcelona

Abstract: In order to survey how students conceive the idea of continuum as they begin their training on Calculus, we have established a questionnaire and an analytical approach to responses, fully rooted in networks. The theoretical framework is based on the Dreyfus and Vinner idea of "concept image", where we attune the expression "perception of a property". We have tested the questionnaire on 74 students in the range of 16-17 years old, acquiring valuable informa tion in return about their perception on non-discrete order ing. On the other hand, we have established a draft ordered system for models of geometric continuum, verifying the fact that, for non-integers, their written expression comes out to be the most representative property of the various sets of numbers. At the end, we put forth some open questions, hinting the need for a deeper re search on the notion of belonging ($\in$), from a cognitive point of view.

I.- Introduction.

The mathematical modelling of nature, as it was undertaken in the Renaissance, has been, from the very beginning, closely linked to the invention and development of *Infinitesimal Calculus*. Being this still true nowadays, to attain an efficient teaching of Calculus is, consequently, one of the most relevant issues we have to deal with today. But all the basic concepts of this specialty are, in two ways, built on top of the idea of *continuum*, both numeric and geometric: i) the first way deals with the *logical and formal* construction of *continuum*, soundly establish since Weierstrass, ii) the second refers to the *cognitive* arena, related to the apprehension and construction mechanisms of concepts and methods on the student side. This second aspect is the one we are interested in now.

The research we are presenting is part of a wider project which aims to at a complete knowledge of how students develop their concepts of *continuum* while they follow the basic training on the foundations of *Calculus*. Nevertheless, in this presentation, we are only concerned about two goals:

1.- Building a research means: that is, a questionnaire and it related annexes, to be submitted to a testing process.

2.- Establishing a first approach to the conceptualisation the surveyed individuals make of the *continuum*, in order to design further scrutinies.

II.- The Continuum as a Congnoscitive Problem.

Mathematicians enjoy today of a satisfactory *formal* characterisation of the *continuum*. As far as *numeric continuum* is concerned, the task was accomplished by Weierstrass, Cantor, Méray and Dedekind, while Hilbert [Hilbert, 1930], among others, did the work on the *geometric continuum* area, as he enounced explicit continuity axioms for the straight line. Furthermore, a *logical* equivalence between both views was established, in the same way Descartes and Fermat had proposed three centuries earlier. But, each *individual's knowledge* of the continuum is not likely to be built on his logical and formal characterisation, through the way people learn, for instance, the Group Theory. Let's see why: a group is *defined* as a system which conforms some given axioms, while as we build a formal structure for real numbers, since the students already know more or less vaguely about them, we only succeed to show the existence of a completely ranked body, but not the fact that real numbers make up one [Steiner, 1984].

If, for any reason, it is advisable to avoid giving the students an axiomatic or constructive introduction to the straight line, it is common use to bring up a *soi-disant* "intuitive view" of its characteristics. Dedekind himself ex presses it in a paradigm form [Dedekind, 1901, pg. 1] which supports this technique "if one does not wish to lose much time". But then we are implicitly admitting two hypothesis: i) that the *geometric continuum* is a universal intuitive concept which is, additionally, easily grasped by individuals; ii) that the numbers-geometry linkage is also *intuitive*, that is, immediate and universally apprehensible without the need for any conceptual elements which belong to the *discursive mode of thought*. Contrarily, the experiences we are about to present will show that, at least from a didactic point of view, there are clear indications that those two hypothesis are false.

Consequently, the first approach to the problem would include the study the individual's *intuition about the continuum*. But, for the targets established in our research, the *philosophical* concept of "intuition", whatever version may it be, comes out to be obviously insufficient since, among other reasons, it does not contain any reference to the linkage between the "intuitive object" and the "intellect". On the side, some recent epistemologic works insist on the fact that the *continuum*, far from being a simple intuitive determination, is the result of a deeply ontologic cogitation and of some mathematic conceptualisation (see Caveing, 1982).

Keeping this in mind, in order to *confine* our research, we have borrowed from Vinner, 1983, Vinner & Dreyfus, 1989, Dreyfus, 1990, Tall, 1991a and Vinner, 1991, the notion of "*concept image*". On doing so, we have followed Azcárate, 1990, and the main reason for this option is based on the fact that the *concept image* definition traces a clear shape of the concept and, furthermore, it fits perfectly well inside the constructivist perspective of cognitive science. On adopting the hypothesis that, beyond the concept's definition, the individuals deal with the "properties" in connection to the concept's *concept image* itself, rather than because of their logical implications, we are convinced to be coherently in line with the notion of *concept image*. Starting from different concept images, the same verbal enuncia tion of a property may hint different individuals to differ ent actions, frequently without any mutual cohesion. We shall use the expression "*perception of a property*" to il lustrate this interaction between *enunciation* of a property and the *concept image* of the individual who uses it.

The concept of *geometric continuum* has been clearly estab ished by Eudox in his *Elements, V, def. 5*, but it is only a geometric concept within the Euclidean logic and deductive paradigm. Diversely, the concept of *numeric continuum* comes as a result of a long lasting evolution, of which the first signs can be retraced from Babylonian mathematics, vigorously penetrating the West by means of the widespread *calculus* over decimal fractions, since Simon Stevin's La Disme, of 1855; we are in front of an *algorithmic* mode of reasoning which eludes the logic foundation. In everyday "practical" use, real numbers make the mathematical entity created in order to fix the more or less vague and *para-mathematical* (as Chevallard, 1991, states it), notion of *magnitude*. As far as the paradigm magnitude is concerned, that is *length*, we require that the *scaled straight line* contain a sufficient amount of points as for the previously defined unit to be able to measure any other length established by *any other means*. The problem arouses if this "other means" is *numeric*, that is, the result of *calculus algorithms*: we then face the so called *"failure to link geometry with numbers"* [Cornu, 1991, pgs. 159-160], and it is in this context, before the *cognitive* issue, where our research breeds its own rationale; we are persuaded that at non-university level, the teaching of mathematics emphasises the *algorithmic way of thinking*, probably under the pressure of its own dynamics on the acquisition and development of concepts (see Hiebert & Wearne, 1986, and Sfard, 1991).

III.- The Questionnaire.

The present project is designed as a research on *concept images* and the *perception of properties*, and consequently of qualitative data Therefore, both the *tool* and the *data analysis approach* have been designed to be able to deal with this kind of data.

In order to prepare the questionnaire, we have reviewed the experience described in Robinet, 1986, and, on the other hand, we have previously tested the outcome of the questions which made up the questionnaire. To do so, we have interviewed people of different ages for whom Mathematics is an essential proffessional item. As a result of the testing and from the theoretical concerns already mentioned, we have embraced the criteria the our research has to be focused on the *mental models* and on the *perception of the properties* of continous magnitudes, avoiding more academic categories of number sets. Our questionnaire includes the following questions:

1.- In order to dig into the *perceptions* of the properties and characteristics of each sort of numbers, we suggest a taskwork which put into play those K and based on proposing the individuals to look for and to establish different *ranking criteria* for numbers. Therefore, we ask each interviewed individual to classify the following numbers:

$$9. \quad 217 \quad 735 \quad -3 \quad \frac{3}{4} \quad \frac{2}{7} \quad 0.75 \quad 6\,325 \quad 3.1416$$
$$5.3 \cdot 10^{1} \quad 0.1919\ldots \quad 2.9999\ldots \quad \frac{5}{4} \quad -7\,75$$
$$-4.3333\ldots \quad \sqrt{2} \quad 1.101001000100\ldots \quad 1.23456\ldots \quad \pi \quad \sqrt{-4}$$

s/he will find randomly distributed in a chart. We hint to avoid scholastic criteria and to clearly highlight the actual criteria used. In order to further validate the responses, we require two more sample numbers from *outside* the given list, for each ranking criteria. Let us point out that the given set of numbers includes "hidden couples" (like: 3/4 & 0.75, or 3.1416 & π, etc.), and that $\sqrt{-4}$ is not encompassed by the "magnitudes" scope.

2.- The second question asks for a *description* of a continous object - a straight line. The question is presented following the hypothesis that *concept images* for geometric objects are basically constructed from *visualisation* experiences. The question is stated as follows:

"Imagine you have a very powerful microscope. Even more, imagine this microscope can magnify as much as you want and that you focus on a straight line. Give a description of what you see and explain what happens as you increase the microscope's magnifying. Can you describe what you see when the microscope reaches infinite magnification?"

This question is, in fact, a reshaped version of Robinet's, 1986 fourth of his questionnaire, suitably modified to allow seising the dynamic aspect of the fictitious experience we propose and also to dig into the notion the individuals have of *infinity*.

3.- Through our third question we put forward the task of *handling* a *bound* and *complete* object. We propose to consider a piece of rope, one yard long (traditional representation of a segment), and to use a "perfetc" cutting tool to submit the rope to the following recurrent process:

"Cut the rope with the "perfect" scissors in two pieces, identical or not. Discard one of the two pieces and throw it to the waste bin. Cut the kept piece with the "perfect" scissors in two new pieces, identical or not Discard one of the two pieces and throw it to the waste bin. Cut the kept piece..."

We then ask the individual to explain whether he believes the process can go on for ever and, depending on his option, to describe the object he had in his hands *before* the last possible cut or else, *after* performing the unlimited cuts. What we aim here is to enquire into the *model* used for *the straight line*, keeping in mind three hypotetical ones: i) the atomistic straight line, ii) the "non-discrete" straight line, and iii) the continous straight line.

IV.- Analysis of the Data.

In order to compile the responses we have fully used the "*network analysis*" technique [Bliss et al., 1979]. Each response in the *network* is considered as an option acquiring its significance from being a particular configuration of the structured netting of independent options. Each possible *option* configuration is a *paradigm* (if we use Bliss's own term), and each one of the *significances* matches one of the possible paradigms. Therefore, tha analysis of the *concept images* is carried by dealing them a set of *options for representation models*, within various semantic environments allowing various option possibilities. The legitimacy of the approach lies on the hypothesis that, within each environment, the *rellevant* significance of the available options does not depend on the relationship between the representation model and the represented object but on the contrast against other different options within the same environment. From the teacher point of view, the *relevant significance* of a given mental representation for a concept is the action, the response an individual develops when trying to solve a problematic situation related to the concept. But, what the inquirer tries to analyse is the rationale which brings the responder to this and not to any other, among a range of available actions, corresponding to an option, and not to any other, within the set of possible representations.

V.- About the Collected Data. Results.

The analysis of the responses to the first question in the questionnaire shows with an overwhelming evidence that for the surveyed individuals, the most significant characteristic of non-integers is their written expression: the number of times each non-integer in the list is classified following each criteria, yields significant maximums to the their *written expression*, against other different criteria. Consequently, numbers with different *written expressions* come to be numbers of different species: 9 and 735 were linked together 116 times, while 9 and 5.3 x 104 were linked only 49 times, and 9 and 2.9999... just 4 times. But, if those numbers were connected to a specific use (i.e. 3.1416 and π) they would be considered as "equal".

The analysis of the second question grants a first ranking for the *concept images* of the straight line. In fact, apart from a 6.75% of the individuals whom we will characterise as *"naive realistic"*, since they did not manage to "play the fictitious game" they were faced to, the vast majority, even thouh they showed various levels of tension, depicted the interaction between their *perceptions* of the abstract properties of the straight line as a "pure mental construction" and the *images* of the straight line as a "physical" object. The straight line is viewed as a sort of tape or as a set of points, frequently conceived as discs or as little spheres. We have engaged a study about the *stability* of the images during the three periods of the fictitious experience (looking at the straight line, increasing the microscope magnifying and reaching the infinite magnifying): if from the very start, a 47.3% view the straight line as a whole and a 29.7% already see *elements* in it, as they arrive to the infinite magnifying, the corresponding share waves to 20.3% and 37.8%. By contrast against the "*naive realism*", we have here the "*naive abstraction*". On the other hand, the "fine structure" views of the straight line allow us to rank the individuals among "*continuists*" and "*atomists*". We must point out that the non-discrete order structure does not appear in any of the *continuists'* descriptions, while, the *atomists*, are equally divided in two groups, those who do not seem to be imbeding the non-discrete order structure into their *concept image*, and those who struggle to offer an image coherent with this property. Finally, the responses give us some hints to dissect the kind of *infinity* each individual accepts. The following tree represents a summary of what we have so far stated:

Concept Images of a Straight Line
- *Abstraction*
  - Naive Realism.
  - Naive Abstraction
- *Structure*
  - Continuists
  - Atomists
    - Discrete order not included
    - Discrete order allowed
- *Infinity*
  - Potentialists
  - Actualists

As for the third question, a 21.6% of the individuals state that the subdivision process is *necessarily finite*, while a 58.1% explicitly consider it to be *necessarily unlimited*, although this does not show any significant correlation with the *semantic scope* of the response which comes out not to be a discrimination trait for the responses. Furthermore: *we have not found any significant correlation* with the *models of the straight line* obtained from the analysis of the second question, as displayed in the previous tree, either. Contrarily, we have found a 95% significant correlation between the *response option* and the *need* sensed by the individuals *to offer some explanations* about what causes the outcome they obtain after the divison process of the rope: those who state that the process will be unlimited do not give any explanation to it, while those who believe the process is always finite sense this need to explain it. It is hard to justify this fact but we risk a hypothesis which would need to be tested by different means and which denies "*Dedekind hypothesis*": generally speaking, the various *concept images* of the continuum are no more than a very loose aggregate of mental images and properties acquired by many different ways, among which the *intimidation* is included (see Lang, 1985).

VI.- About the Questionnaire. Results.

The first question has yielded a quite rich and complex collection of data: the inspection of the ranking criteria used to group the numbers render significant information about the relevance imputed to the properties and characteristics of the different sets of numbers. Consequently, it has come out to be a useful tool towards the target we were aiming to. If we expect the data from the second question to be signif icant it is necessary that the individuals accept (or manage) "to play the abstraction game" in the *semantic universe* we are putting forward: but, even with this constraint, the "game" has worked out for the great majority of the cases (83.8%) and we can take the question a perfectly valid. For the *atomist* individuals, the question grants a sound scrutiny into the *perception* of the non-discrete order, while the *continuists* can coherently respond bypassing it. Consequently, to further investigate this point requires to submit some other kind of activity which would allow, on the one hand, to study the relationship the individuals establish between the straight line and the *numeric continuum*, and, on the other hand, to survey the *perceptions* of the properties of the numeric continuum in itself as a set with a non-discrete order structure.

Most likely due to the level of mathematical knowledge of the surveyed individuals, the third questios has not alloed us to scrutinise the *perceptions* of the completness property and, consequently, it should be validated by means of a sample of higher level individuals. Contrarily, we positively believe it yields significant results about the *perceptions* of the properties of the *infinity*; that is: the description of the remaining piece of rope never corresponds to a *finite* objecte. It is always "very small" or "infinitesimal" be cause as *infinity* actuates on a finite object it anihilates it. Therefore, tha value of this question derives from the fact it allows to analyse the *articulating level* of the "*mental pictures*" and the "*properties characterising the concept*" in the individuals' *concept image.*

VII.- Conclusions.

We are persuaded that the analysis we have presented provides sound grounds to assert that, from a didactic point of view, the "Dedekind hypothesis" about the *immediate intuitiveness* of the continuum have no founding base; in fact: as proven by the research which led to the definition of the notion of "concept image" (see *cit. bibliography*), "intuition", taken here as an *inductive* process over previous individuals' experiences, is no more than resorting to the *concept image* itself, which at the same time is generated in part through experiences, as well. The global analysis has proven for the individuals in our sample that the *concept image* of the continuum is a loose aggregate of images and enunciations of properties which has caused an erratic behaviour when facing the questions we had submited to them. Also, we have not discerned any kind of *non-trivial* link between *geometry* and *numbers*. This circumstance is obviously connected to the already mentioned factor that the most relevant characteristic of numbers has come to be, as a matter of fact, their written expression.

VIII.- Some Open Questions.

While developing the present project, we have put forth the following issues:

i) We have found out that the surveyed *concept images*, mental images, "parts of the concept images of the continuum", have properties logically incoherent with "the properties which characterise the concept". Putting aside the fact that we have take a sample of *non-expert* individuals, we ask ourselves if the *concept images* of "expert" individuals about a given mathematic concept can get to be a set of *mental representations* free from any kind of logical *incoherence*. If the answer is no, then we have to raise the issue of locating the difference between an "expert" individual and a non-expert individual, difference which can no be exclusively nor mainly based on the coherence of their own *concept images*.

ii) While surveying the perceptions of topological properties, we have been pushed to consider the perceptions of the *way to be* of points and numbers inside the sets, the straight line or the real body to which they belong. The "$\in$" relation, *logically* primitive, is a target for perceptions and we ask ourselves whether there are diverse perceptions of the "$x \in A$" relation, depending on the perception of the structure of set A. Could the origin of the *obstacle* which impedes the articulation of the property of non-discrete order with the mental representations of continous objects be the generalisation of a certain *mode of perception*, suitable for a set, but not for the set in which it is immersed (the case of $Z \hookrightarrow Q$ immersion)?

References

1. [Azcárate, 1990] Azcárate , C. *La velocidad: introducción al concepto de derivada*. In *Doctoral Thesis*. 374 p.p. Bellaterra, 1990.

2. [Bliss et al., 1979] Bliss, J. & Ogborn, J. in colaboration with Grize, F. L'Anàlisi de Dades Qualitatives. In *European Journal of Science Education. Vol. 1*. no.4. Pag. 427-440. 1979.

3. [Caveing, 1982] Caveing, M. Algunas observaciones sobre el trato que recibe el contínuo en los Elementos de Euclides y en la Física de Aristoteles. In *Pensar la Matematica*. Guénard, F.& Lelièvre, G. (Ed.) "Cuadernos ínfimos", no.114. *Pag*. 17-44. Tusquets, Editores, Barcelona, 1984.

4. [Chevallard, 1991] Chevallard, Y. *La Transposition Didactique du savoir savant au savoir enseigné*, suivie *de* Un *exemple de Transposition Didactique*. 240 p.p. Editions La Pensée Sauvage, Grenoble, 1991

5. [Cornu, 1991] Cornu, B. Limits. In *Advanced Mathematical Thinking*. Tall, D (Ed.) "Mathematics Education Library", no.II. Pag. 153-166. Kluwer Academic Publishers, Dordrecht, 1991.

6. [Dedekind, 1901] Dedekind, R. Continuity and Irrational Numbers. In *Essays on the Theory of Numbers*. Pag. 1-27. Dover Publications, Inc., New York, 1963.

7. [Dreyfus, 1990] Dreyfus, T. & col.. Advanced Mathematical Thinking. In *Mathematics and Cognition*. Nesher, P. & Kilpatrick, J. (Ed.) Pag. 113-134. Cambridge University Press, Cambridge, 1990.

8. [Hiebert & Wearne, 1986] Hiebert, J. & Wearne, D.. Procedures over Concepts: the Acquisition of Decimal Number Knowledge. In *Conceptual and Procedural Knowledge: The Case of Mathematics*. Hiebert, J (Ed.) p.p. 199-223. Lawrence Erlbaum Associates, Hillsdale NJ, 1986.

9. [Hilbert, 1930] Hilbert, D. *Les fondements de la Géométrie* Rossier, P.(Ed.) 311 p.p. Dunod, Paris, 1971.

10. [Lang, 1985] Lang, S. *Math! Encounters with High School Students*. 138 p.p. Springer-Verlag, New York, 1985.

11. [Robinet, 1986] Robinet, J. Les réels: quels modèles en ont les élèves?. In *Educational Studies in Mathematics*. Vol. 17. Pag. 359-386. D. Reidel Publishing Company, 1986.

12. [Sfard, 1991] Sfard, A. On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Dides of the Same Coin. In *Educational Studies in Mathematics*. Vol. 22. Pag. 1-36. Kluwer Academic Publishers, Dordrecht, 1991.

13. [Steiner, 1984] Steiner, R. Teaching About the Real Numbers. In *American Mathematical Monthly* Vol. 91, no.3. Pag. 202-203. The Mathematical Association of America, 1984.

14. [Tall, 1991] Tall, D. The Psichology of Advanced Mathematical Thinking. In *Advanced Mathematical Thinking*. Tall, D. (Ed.) "Mathematics Education Library", no.II. Pag. 3-21. Kluwer Academic Publishers, Dordrecht, 1991.

15. [Vinner, 1983] Vinner, S. Concept Definition, Concept Image and the Notion of Function. In *International Journal for Mathematical Education in Science and Technology. Vol. 14*. no.3. Pag. 293-305. 1983

16. [Vinner, 1991] Vinner, S. The Role of Definitions in the Teaching and Learning of Mathematics. In *Advanced Mathematical Thinking* Tall, D. (Ed.) "Mathematics Education Library",, no.II. Pag. 65-81. Kluwer Academic Publishers, Dordrecht, 1991.

17. [Vinner & Dreyfus, 1989] Vinner, S. & Dreyfus, T. Images and Definitions for the Concept of Function. In *Journal For Research in Mathematics Education*. Vol. 20. no.4. Pag. 356-366. 1989.

THE METAPHORICAL MODELLING OF "COMING TO KNOW"

David J. Clarke
Mathematics Teaching and Learning Centre
Australian Catholic University (Victoria)
Oakleigh, Victoria 3166, Australia

ABSTRACT

Our theorizing in educational contexts, and in particular in relation to the process of learning, labours under the burden of an inheritance of metaphors which may be both inappropriate and inhibiting with respect to our continued theorizing about cognition. In particular, the metaphors of: reflection and negotiation should be subjected to immediate scrutiny. These metaphors have been employed as implicit models of aspects of the process of "coming to know" in socially-located situations such as classrooms. The unguarded use of each of these metaphors has enabled theorists to gloss over the need to specify any mechanism whereby an interaction in the social domain is translated into a corresponding perturbation in the cognitive domain. It is the purpose of this paper to suggest how these and other metaphors might be incorporated with legitimacy into a coherent model of the process of coming to know.

INTRODUCTION

The challenge in seeking to understand the process of "coming to know" is the challenge of modelling the social-personal interface. The approach adopted in here is that of a critical examination of the metaphors which are found within existing theories of learning. Historically, the use of particular metaphors has committed the mathematics education community to particular positions with regard to knowledge and the process of coming to know, and our freedom to consider alternatives in our modelling of the social-personal interface has been unnecessarily constrained by the limitations of this burden of inherited methaphors.

In clarification: the use of the term "social-personal" is not intended to exclude the "cultural" but, rather, conceives of culture as a key element of Social Context. Social Context is seen as a dynamic combination of elements, such as Cultural, Societal, Institutional (Bishop, 1985), and Situational (Clarke & Helme, 1993), each of which is always present in any social interaction. We now find a universal recognition of the significance of social context in research on classroom learning: whether the classroom is situated in Australia, Britain, Germany, New Zealand, or the USA.

REFLECTION OR REFRACTION

The first point of discussion is the personal construction of sense. Two processes should be distinguished, and we must question the validity with which the metaphor of reflection can be applied to either. It is proposed that these two processes are:

- the process of sense-making; and,
- the process of constructing meaning.

This distinction is not an arbitrary one, and in many ways mirrors the Piagetian distinction between assimilation and accommodation. The process of sense-making I associate with "construal", and I take this to refer to the sense which an individual is able to associate with experiential input such as "authority", "triangle", or "turn on the light". The term "construal" is used in preference to the term "comprehension", which unlike "construal" does not, from this writer's perspective, suggest a sufficiently active role for the individual's cognitive framework.

In relation to the somewhat indiscriminant use of the metaphor of reflection, I suggest that the first of the above two processes is better called *refraction*: the reconstruction of an object's image in a form which reveals certain aspects of its structure in a representation characteristic of both the refracting medium and the original object. To elaborate the metaphor beyond all reasonable bounds: we refract our sensory impressions of experience through the prism of our personal theories and conceptual frameworks, and the resulting spectra, that is, the reconstructed cognitive representations of our experiences, possess pattern and structure which are a consequence of both the originating experiences and those personal theories by which we make sense of the experience.

An individual's actions together with subjectively-significant features of the social situation in which those actions are undertaken must be cognitively represented by the individual, that is, they must be subject to a process of construal or refraction, before they can be considered accessible for the purposes of cognitive reflection. The resultant cognitive representations, that is, the products of the *refractive* process, then provide the objects for the *reflective* process.

Yackel and Cobb (1993) provide a useful endorsement of this distinction in their discussion of the acceptability of student explanations and justifications within a particular classroom microculture.

> It was not sufficient for a student to merely describe personally-real mathematical actions. Crucially, to be acceptable, other students had to be able to interpret the explanation in terms of actions on mathematical objects that were experientially real to them.
>
> (Yackel & Cobb, 1993, p. 2)

Voigt (1993) has argued that the symbolic interactionist approach to social interaction is useful when studying children's classroom learning because it emphasizes the individual's sense making processes as well as the social processes. This is precisely the issue which we must now address. Equipped with one model of the individual's sense making processes, we can now examine that aspect of the process of "coming to know" which most explicitly relates social and cognitive phenomena, that is, the "Negotiation of Meaning".

THE NEGOTIATION OF MEANING

Negotiation as a form of social behaviour is a process whose goal-state is the achievement of consensus. The process is mediated by two overt response-types: agreement or "elaborating statement". Simply, if I am trying to establish a state of shared meaning, that is, consensus, with you, I can respond to an utterance of yours with one of two response types: a nod of the head, or "But . . ." followed by a statement of my meaning in a form which distinguishes it from the meaning I have construed from your statement.

The process of negotiation of meaning can be modelled in the following fashion: In order to sustain this process, I need to *construe* your statement. That is, I need to create an internal representation of your statement in a form which employs elements already present in my previously existing cognitive framework. I can then assess the degree of isomorphism of this cognitive representation with any other cognitive representation of the same subject matter which I might construct from pre-existing cognitive schema, or which I already possess and can call to mind. If the two representations are of sufficient similarity for me to adjudge that any discrepancy is insignificant with respect to the realization of my immediate social purpose (which will be a consequence of our social interaction and situation), then I will nod my head or display some equivalent social signal. If, however, the discrepancy between my construal of your meaning and my "pre-existing" representation is too great for the realization of my social purpose, then I will probably identify the most significant area of discrepancy and say, "But ...", followed by an *articulation* (typically a verbal representation) of my meaning. You do likewise and the negotiation continues until one of us signifies agreement, that is, the achievement of mutual consensus as to the form of the shared meaning and the closure of the negotiation. The process of construal is crucial to this exchange and, as suggested in the preceding section, one model of this process is that of *refraction*.

In institutionalized learning situations, the process of "coming to know" functions as the process of "coming to agree". Consensus is the goal state of this process of "coming to agree". Von Glasersfeld's use of the term "consensual domain" (Von Glasersfeld, 1991) usefully identifies the standards of tolerance within which collegial agreement might be meaningfully conceived, and within which a community of discourse can be said to exist. The process of "coming to agree" is enacted within both the social and cognitive domains, but there is no reason to suppose that the mechanisms operating in the two domains are at all similar; only that their outcomes are mutually sustaining and mutually constraining.

To reiterate: The individual's existing cognitive framework is doubly significant. First, in the process of refraction, which has the virtue of suggesting the construction of an image which is simultaneously a consequence of characteristics of the incoming signal (your statement, for instance) and the refracting medium (that is, my pre-existing cognitive framework). Second, in the acknowledgement that negotiation involves the representation of my pre-existing knowledge in a form that can be compared by me with my construal of your statement.

In Voigt (1993), and in other discussions of negotiated meanings, "meaning" is apparently used in the sense that the meaning of a word, for instance, is the conceptual construct which the word is taken to denote. The centrality of this "negotiation of meaning" in the constructivist view of classroom learning is generally acknowledged (Voigt, 1993; Yackel & Cobb, 1993). The detail of this negotiation process is seldom discussed. To assert that the negotiation takes place between learners behaving as self-organizing systems in interaction with themselves and others (Steffe, 1993, p. 2 and pp. 16 - 17) is to associate the negotiation simultaneously with the possible interpersonal classroom interactions and with some internal dialogue within the learner. The simultaneous use of the same metaphorical process to represent both a negotiative process located in social interaction and a related process occurring within the conceptual framework of the individual constitutes a serious difficulty in the use of negotiation as a metaphor for the process of coming to know. Since negotiation as a process is social in origin, use of the metaphor retains some legitimacy when applied as a description of phenomena in the social domain. It remains to be demonstrated that the "internal" process is more appropriately modelled by the metaphor of *negotiation* rather than, for instance, the metaphor of *arbitration*, where some executive metacognitive capability functions in an evaluative capacity in judgement over competing cognitive representations.

The use of the same metaphor to represent both the social interactive process and the associated internal accommodative process has enabled constructivist theorists to gloss over the nature of the interface between the social and the personal domains. Importantly, it is a characteristic of the process of negotiation in social contexts that negotiation occurs between different individuals offering competing representations of some social situation. As a consequence, the use of negotiation as a metaphor to describe the cognitive process whereby the individual constructs knowledge from the interaction between the cognitive representations of sensory data and pre-existing knowledge embedded in the individual's conceptual framework, necessarily requires that the individual speak with "multiple voices" within their internal cognitive domain. By contrast, use of the metaphor of arbitration requires only that the individual arbitrate through some metacognitive capability (*reflection*) between competing cognitive representations. In this second case, there is no explicit requirement that the individual operate cognitively with more than a single "voice". I am not proposing this second view as a definitive one, but rather offering it as a tenable alternative, which must be explicitly rejected if we are to continue to use negotiation as a metaphor for some cognitive process. The relevance of negotiation as an appropriate term for the social interaction whereby meanings are exchanged is not challenged. It is from social contexts that the construct "negotiation" derives its meaning, and in which it finds its most appropriate application.

In formulating a model of the process of "coming to know", we have identified "coming to know" with "coming to agree" in school-like situations. We have distinguished refraction from reflection, and negotiation from arbitration, invoked articulation, and located all of these metaphors within the emergent model. Any attempt to model learning must deal with "intersubjectivity". It is intersubjectivity that has been invoked most recently by those arguing for the irreconciliability of the Radical Constructivist and Vygotskian perspectives (Lerman, 1993). In the model being elaborated here, intersubjectivity is represented as those "taken-as-shared" meanings which facilitate and arise from the process of "coming to agree". As such, intersubjectivity constitutes a form of agreement. This representation of intersubjectivity is not incompatible with existing representations (see, for example, Leiter, 1980, p. 162).

Central to this paper is the proposal that identifying intersubjectivity with agreement locates social interaction within the process of learning, while the necessity to locate cognitive representations within the negotiative process identifies the role of the individual's cognitive framework as the basis for the construction of these cognitive representations, either for the purposes of construal (input) or articulation (output).

So, how are we to relate "construal" and "coming to agree" to the practices of the classroom and the actions of the social group in which the process of "coming to know" is being enacted? For this we invoke the constructs "Corporate Meaning" and "Consensus". Without proposing, in any form (including metaphorically), the existence of a corporate mind, I would like to suggest that the object of classroom social negotiation is the construction of "corporate meaning". This corporate meaning is the evolving consequence of social consensus, where consensus is the goal state of the process of "coming to agree". The corporate meaning at any given point in an extended series of social interactions is the combination of those statements or discourse elements which are common to the personal meanings of all individuals participant in the social situation. That is, the corporate meaning at any moment is the combination of those elements with which all participants agree: the combination of mathematical and social meanings which are "taken-as-shared" within the classroom community. This agreement may be tacit rather than explicitly stated, and it may be that the corporate meaning is never articulated throughout the particular period of social interaction. One aspect of the individual's construction of meaning is the personal construal of the corporate meaning, and individuals contribute to the construction of the corporate meaning through the communicative (eg verbal) enactment of their personally constructed meanings. Inclusion of this dialectical relationship in our model of the process of coming to know significantly elaborates the emergent character of the social-personal interface.

Classrooms, from this perspective, are the sites for the communicative enactment of individuals' personal meanings (especially the teacher's personal meanings). A particular individual, participant in the social situation, must construe the enactment of another's personal meaning. This construal involves the representation of the other's meaning in a form comprehensible within the conceptual framework of the recipient. The invocation of "corporate meaning" within this process is a recognition that there are meanings which reside in the social conglomerate, rather than in the statements of any single individual, and these meanings also require construal. Corporate meaning is the matter for which the consensual domain denotes the region, its boundaries and its tolerances. The form taken by the consensual domain and the obligations imposed by subscription to the corporate meaning constrain the individual's articulation; that is, their ability to participate in the social interaction of negotiation, and to contribute to evolving corporate meaning. The outcome of this process at the level of the individual may be the construction of knowledge (the outcome of the process of coming to know) that is, accommodation, or it may not.

Agreement construed as consensus can act within classrooms, and other social situations, to cue closure of social interactive episodes. Specifically, if time constraints do not intervene, closure in classroom discussion commonly occurs when the corporate meaning is adjudged by the teacher to have acquired a sufficient level of complexity to approximate the teacher's model of appropriate knowledge of that topic for that class.

The centrality of language in the negotiation of meaning in classrooms is nowhere more evident than in mathematics classrooms, where the possession of a mathematical vocabulary privileges certain members of the community (most notably the teacher) and disenfranchises others. Language is culturally-specific and constrains both construal and articulation, as well as conception (that is, the construction of new (novel) cognitive representations) while facilitating all three processes. Mathematical activity is engaged in by individuals as members of intersecting communities of discourse and practice. It is membership of a community of practice that defines the context and the purpose which an individual associates with a task. It is as a member of a community of discourse that an individual construes the meaning of a task and frames an appropriate response.

The recognition that we enact the constructive process of coming to know with building blocks that are culturally-constrained and that our resultant constructions are, to some extent, pre-determined, is one of the consequences of the social-cultural modelling stimulated by the rediscovery of Vygotsky's work and the attention now being paid to the role of language.

SUMMARY

This paper has examined the process of "coming to know" from the explicit perspective of the metaphors by which the process has been modelled and those by which it might be modelled more effectively. In the course of this discussion, distinctions were drawn between *refraction* as a model of the process of construal, and *reflection* as a process of evaluative comparison of competing cognitive representations. Communication requires the *articulation* of individual cognitive representations. *Negotiation* is relocated within the social situations from which the metaphor derives its meaning, and *arbitration* is suggested as a possible metaphor for the process whereby an individual discriminates between competing cognitive representations. I have argued for a recognition of the metaphorical nature of the constructs employed in current theories of learning, and for an explicit reconceptualization of the character of each, and of its domain of valid application. In combination, these re-conceived metaphors must form the basis of our attempts to model the social-cultural interface and thereby come to a coherent theory of the process of "coming to know".

ACKNOWLEDGEMENTS

The contents of this paper benefitted from discussion with many people at the following institutions: The University of Hawaii; the University of Georgia - Athens; Australian Catholic University - Oakleigh; University of California - Berkeley; South Bank University - London; the Freudenthal Institute - University of Utrecht; and the Institute for the Didactics of Mathematics - University of Bielefeld.

REFERENCES

Bishop, A. (1985). Social context and mathematics education. Presentation to the Faculty of Education, Monash University, Clayton, Victoria, Australia.

Clarke, D.J. & Helme, S. (1993). Context as construct. Paper presented at the 16th Annual Conference of the Mathematics Education Group of Australasia, Brisbane, Queensland, July 9 - 13, 1993.

Glasersfeld, E.v. (1991). *Radical Constructivism in Mathematics Education.* Mathematics Education Library, 7, Bishop, A.J. Dordrecht/NL: Kluwer Academic Publishers. Cited in Bauersfeld, 1992.

Leiter, K. (1980). *A primer on ethnomethodology.* New York: Oxford University Press. Cited in Voigt, 1993.

Lerman, S. (1993). The 'problem' of intersubjectivity in mathematics learning: Extension or rejection of the Constructivist paradigm. Unpublished manuscript, South Bank University, London, UK.

Steffe, L.P. (1993). Interaction and Children's Mathematics. Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, Georgia, April, 1993.

Voigt, J. (1993). Negotiation of mathematical meaning and learning mathematics. Manuscript of a paper to be published in *Educational Studies in Mathematics.*

Yackel, E. & Cobb, P. (1993). Sociomathematical Norms, Argumentation and Autonomy in Mathematics. Paper presented at the 1993 Annual Meeting of the American Educational Research Association, Atlanta, GA.

A SUMMARY OF FOUR CASE STUDIES OF MATHEMATICAL LEARNING AND SMALL-GROUP INTERACTION

Paul Cobb
Vanderbilt University

The longitudinal analysis summarized in this paper focuses on four pairs of second-grade students' small-group activity over a ten-week period. Theoretically, the issue that motivated the case studies was that of clarifying the relationship between students' situated conceptual capabilities and their small-group interactions. The view that emerged in the course of the analysis was that of a reflexive relationship between individual students' mathematical activity and the small-group relationships they established. Pragmatically, the case studies were conducted to relate the various types of interactions in which the children engaged to the occurrence of learning opportunities. The findings are summarized by discussing two aspects of students' social relationships that appear to be crucial for productive small-group activity. The paper then concludes with a consideration of the instructional implications of the analysis.

Purpose

The longitudinal analysis reported in this paper focuses on four pairs of second-grade students' small-group activity over a ten-week period. The *theoretical* reason for analyzing the four pairs of children's small-group activity stems from perceived limitations of neo-Piagetian and Vygotskian perspectives. From the neo-Piagetian perspective, social interaction is considered to stimulate individual cognitive development, but is not viewed as integral to either this constructive process or to its products, increasingly sophisticated mathematical conceptions. From the Vygotskian perspective, conceptual development occurs as the child internalizes interpersonal, social processes. Here, the social dimension of consciousness is primary, and the individual dimension is secondary (Vygotsky, 1979). Against the background of these competing perspectives, I sought to develop an approach that acknowledges the importance of both cognitive and social processes without subordinating one to the other.

The *pragmatic* reason for conducting the analysis was to investigate how and to what exent small-group collaborative activity facilitates children's mathematical learning. The findings of several previous investigations indicate that small-group interactions can give rise to learning opportunities that do not typically arise in traditional classroom interactions (e. g., Good, Mulryan, & McCaslin, 1992; Shimizu, 1993; Yackel, Cobb, & Wood, 1991). The case studies extend this line of inquiry by relating the occurrence of learning opportunities to the different types of interaction in which the children engaged. As a consequence, they indicate the extent to which the various types of interaction were productive for mathematical learning.

Data

The data were collected in the course of a year-long teaching experiment in which instruction was generally compatible with current American reform recommendations. The general instructional strategy used was that of small-group problem solving followed by a teacher-orchestrated whole-class discussion of children's interpretations and solutions. All mathematics lessons were video-recorded throughout the entire school year. One camera was used to record the whole-class discussions, and two were used to record the children's small-group work. In the latter case, four pairs of children turned to face the cameras, and it was possible to record approximately half of each group's mathematical activity on a daily basis.

The data analyzed for the case studies consisted of the video-recordings of the small-group portions of the 27 lessons that involved arithmetical activities conducted between January 23 and March 13 of the school year. Additional data consisted of video-recorded individual interviews conducted with all eight children just prior to the first of these lessons. The decision to limit the analysis to activities involving arithmetic was made so that established psychological models of children's place-value conceptions and their non-standard computational algorithms could be used to account for their mathematical development (Cobb & Wheatley, 1988; Steffe, Cobb, & von Glasersfeld, 1988). The data set was restricted to the second half of the school year because the children had not previously been expected to collaborate to learn in school. Classroom observations and prior analyses indicate that, by January, most of the children could resolve small-group social conflicts and thus develop interpersonal relationships that satisfied the teacher's expectations without assistance (Cobb, Yackel, & Wood, 1989).

Methodology

The methodology used to analyzed the data followed the constant comparison method of Glaser and Strauss (1967) and involved coordinating cognitive accounts of the individual children's mathematical activity with microsociological accounts of the small-group participation structures. A detailed description of the methodology and a discussion of the viability of the case studies can be found in Cobb (in press).

For current puposes, it suffices to note that the task of identifying stable structures that might characterize the children's social relationships initially proved difficult because their interactions frequently varied markedly from one session to the next. However, an exploratory analysis of one pair of children's activity across five consecutive sessions indicated that the expectations the children had for each other's activity and the obligations implicit in their own activity appeared to be consistent across situations provided the relative sophistication of their individual mathematical interpretations was taken into account. In other words, although there were dramatic differences in the nature of the children's interactions both within and across sessions, their expectations and obligations appeared to be stable once their construals of tasks and of each other's mathematical actions were considered. Further, these inferred obligations and expectations suggested a way to characterize the social relationships they established for doing mathematics in an empirically-grounded way. The viability of this approach was confirmed when conducting the four case studies in that it was possible to identify cognitively-situated expectations and obligations for each of the four pairs of children that held across the ten week period covered by the video-recordings.

For each of the four case studies, the analysis was conducted in the following three phases:

1. The interview and classroom video-recordings were analyzed in chronological order and, within each small-group session, children's attempts to solve individual tasks were treated as distinct but related episodes. The analysis of each of these episodes involved inferring:

a. The children's expectations for and obligations to the small group partner.

b. The mathematical meaning the children gave to their own activity, the partner's activity, and the task at hand.

c. The learning opportunities that arose for each child.

d. The conceptual reorganizations that each child made (i.e., their mathematical learning).

2. The inferences and conjectures made while conducting the above episode-by-episode analysis themselves became "data" and were (meta-)analyzed to develop interrelated chronologies of :

   a. The children's obligations and expectations (i.e., the social relationships).

   b. Each individual child's mathematical activity and learning.

3. In the final phase of the analysis, these chronologies were further synthesized to create outlines for the written case studies.

Theoretical Constructs

Due to space limitations, I will outline only the sociological constructs used to account for the pairs of children's collective activity. A discussion of the cognitive constructs used to account for individual children's socially-situated activity can be found in Cobb (in press).

As has been noted, each group's social relationship was characterized in terms of regularities in children's obligations and expectations. Additional constructs that emerged in the course of the analysis deal with the types of interactions in which the children engaged, and the relations of power and authority established in the groups.

Types of Interaction

Differences in the types of interactions in which the children engaged influenced the learning opportunities that arose for them. This was the case both when a pair was still in the process of solving a task and when one or both children had arrived at a solution. In the latter case, either one child attempted to explain his or her thinking to the other, or the children attempted to resolve conflicts between their interpretations, solutions, and answers. The relevant distinction in these situations where one or both children have already developed a solution is that between interactions that involve *univocal explanation*, and those that involve *multivocal explanation*. In the first of these types of interactions, one child judges that the partner either does not understand or has made a mistake, and the partner accepts this judgment. The interaction proceeds smoothly as the first child explains his or her solution and the partner attempts to make sense of the explanation. The term "univocal" is used to emphasize that the perspective of one child dominates. It should be noted that even in these interactions, explaining is a joint activity in that the partner has to play his or her part by accepting the first child's judgment and attempting to understand the explanation.

Interactions involving multivocal explanation often occurred when a conflict had become apparent and both children insisted that their own reasoning is valid. One child might again assume that the partner has made a mistake. However, in contrast to univocal interactions, the partner challenges this assumption by explicitly questioning the explanation. In general, multivocal interactions are constituted when both children attempt to advance their perspectives by explicating their own thinking and challenging that of the partner.

With regard to the interactions that occurred while both children were still in the process of developing solutions, the relevant distinction is that between *direct* and *indirect collaboration*. The children engaged in direct collaboration when they explicitly coordinated their attempts to solve a task. For example, to solve a multiplication task corresponding to 6 X 6 = _ one child might count modules of six while the partner puts up fingers to record how many modules have been counted. In general, interactions involving direct collaboration only occurred when the children made taken-as-shared interpretations of a task and of each other's mathematical activity. Interactions of this type can be contrasted with those that involve indirect collaboration. Here, one or both children think aloud while apparently solving the task independently. Although neither child is obliged to listen to the other, the way in which they frequently capitalize on the other's comments indicates that they are monitoring what the partner is saying and doing to some extent.

Mathematical Authority and Social Authority

The discussion of interactions involving univocal explanation illustrate an instance in which one child was the established *mathematical authority* of the group. There, one child judged that the partner either did not understand or had made a mistake, and the partner accepted this judgment without question. It is important to stress that the notion of mathematical authority refers to the relationship the children have interactively constituted rather than to a single child's beliefs about his or her own role. For example, a child might believe that he or she is the mathematically more advanced and routinely attempt to help the other understand. However, regardless of what the child believes, he or she is not the mathematical authority of the group unless the partner accepts the child's judgment. Thus, in this account, both children participate in the interactive constitution of one child as the mathematical authority. In those instances in which such a relationship was observed, there was a clear power imbalance between the children in that one child was obliged to adapt to the other's mathematical activity in order to be effective in the group.

A second type of power imbalance can arise when one child regulates the way in which the children interact as they do and talk about mathematics. Power, as the term is used here, refers to whose interpretation of a situation wins out and becomes taken-as-shared. Thus, for example, it might be observed that two children engage in multivocal interactions only when a discussion of conflicting solutions fits with one child's personal agenda. The manner in which the child controls the emergence of interactions of this type indicate that he or she is the *social authority* of the group.

Findings

Learning and Interaction

As was stated at the outset, a primary reason for conducting the case studies was to develop a theoretical approach that acknowledges the importance of both the cognitive and social aspects of children's small-group activity without subordinating one to the other. The view that emerged in the course of the analysis was that of a reflexive relationship between individual children's mathematical activity and the social relationships they established. On the one hand, the children's situated cognitive capabilities, as inferred from both the interviews and their activity in the classroom, appeared to constrain the possible forms that their small-group relationships could take. Although it was possible to imagine that a particular pair of children might have established a variety of alternative relationships in other circumstances, a wide range of possibilities did not seem plausible. On the other hand, the relationships that the pairs actually established constrained the types of learning opportunities that arose and thus profoundly influenced the children's construction of increasingly sophisticated mathematical ways of knowing. These developing mathematical capabilities in turn constrained the ways in which their small group relationships could evolve, and so forth.

Learning Opportunities

The case studies indicate that the social relationships the four groups established were generally consistent with the teacher's expectations and yet differed markedly from each other. The findings concerning the relationship between the occurrence of learning opportunities and the four identified types of interactions are summarized below. The names of the children in the four groups are: Holly and Michael, Ryan and Katie, Andrea and Andy, and Jack and Jamie.

Univocal explanation. Michael and Holly's case study indicated that they routinely engaged in interactions that involved univocal explanation both when their solutions were in conflict and when Holly judged that Michael did not understand. In these interactions, Holly, as the mathematical authority, was obliged to explain her solutions. Michael, for his part, was obliged to try to understand her mathematical activity. Even though Holly attempted to fulfill her obligation of helping Michael understand, there was no indication that these exchanges were productive for either child. This conclusion indicates that the mere act of explaining does not necessarily give rise to learning opportunities. In this regard, Webb (1989) contends that giving help facilitates a student's learning only if the helper clarifies and organizes his or her thinking while explaining a solution in new and different ways. Further, Pirie (personal communication) reports that interactions involving univocal explanations can be productive when one student is an active, participatory listener. However, Michael rarely asked clarifying questions or indicated which aspects of Holly's solutions he did not understand. Consequently, Holly merely described how she had solved a task when she gave an explanation. Although it was not possible to unequivocally explain why Michael did not play a more active role, there was some indication that the differences in his and Holly's conceptual possibilities may have been such that it was difficult for them to establish a viable basis for mathematical communication. In any event, it seems premature to conclude from Michael and Holly's case study that small group relationships in which one child is the established mathematical authority must inevitably be unproductive.

Multivocal explanation. The case study analyses indicate that Ryan and Katy, and Andrea and Andy both engaged in multivocal explanations with some regularity. In Ryan and Katy's case, exchanges characterized by argument and counter argument appeared to give rise to learning opportunities for both children. In addition, they sometimes developed novel joint solutions. Further, they both seemed to believe that they had solved a task for themselves when they participated in these exchanges. This was not the case when Ryan responded to a sequence of questions that Katy posed in an attempt to lead him through her solutions. The comparison of Katy and Ryan's and Michael and Holly's case studies indicates the need to take account of the social situations that children establish when assessing the role that particular activities such as explaining can play in their mathematical development.

In contrast to Ryan and Katy, interactions involving multivocal explanation did not appear to be particularly productive for either Andrea or Andy. They had difficulty in establishing a taken-as-shared basis for communication and, as a consequence, frequently talked past each other. The difficulties they experienced appeared to reflect differences in both their beliefs about mathematical activ' y in school, and their interpretations of particular tasks. For example, there was every indication that an explanation for Andrea could involve stating the steps of a procedural instruction whereas, for Andy, it had to carry numerical significance. As a consequence, there were occasions when Andrea felt that she had adequately explained a solution and Andy persisted in asking her to explain what she was doing. Further, in accounting for the persistence of Andrea's instrumental beliefs, it became apparent that her interpretations of tasks were sometimes at odds with those of Andy, the teacher, and the other member of the class. As a consequence, exchanges that might have given rise to learning opportunities for both children instead often resulted in feelings of frustration.

Direct collaboration. Interactions that involved direct collaboration did not appear to be particularly productive for any of the children. Consider, for example, a collaborative solution in which one child reads the problem statement, the other counts to solve it on the hundreds board, and the first child records the answer. This example is paradigmatic in that the children typically engaged in direct collaboration when a task was routine for both of them and the intended solution was taken-as-shared. As a consequence of this compatibility in their interpretations, the possibility that learning opportunities might arise was relatively remote.

Indirect collaboration. Jack and Jamie were the only group who engaged in indirect collaboration with any regularity. In these situations, they thought aloud while apparently solving tasks independently However, the way in which they frequently capitalized on each other's comments indicates that they were in fact monitoring each other's activity to some extent. Learning opportunities arose when what one child said and did happened to be significant for the other at that particular moment within the context of his ongoing activity. The frequent occurrence of such seemingly fortuitous opportunities of course requires a reasonably well established basis for communication. In its absence, Jack and Jamie would merely have engaged in independent activity.

Summary. A comparison of the learning opportunities that arose for the four pairs of children indicates that interactions involving both indirect collaboration and multivocal explanation can be productive. The former type of interaction occurs when both children are still attempting to arrive at a solution, and the latter type occurs when they attempt to resolve conflicting interpretations, solutions, and answers. In both cases, the establishment of a taken-as-shared basis for communication seems essential if learning opportunities are to occur. Significantly, both types of interaction involve a reciprocity in the children's roles, with neither being the mathematical authority. In contrast, learning opportunities were relatively infrequent for the one pair of children who routinely engaged in univocal explanation. As a caveat, it should be noted that situations of this type might be productive if one child is an active participatory listener.

Implications

The two aspects of children's social relationships that seem critical for their mathematical learning in a classroom where an inquiry mathematics microculture has been established are:

1) The development of a taken-as-shared basis for mathematical communication.
2) The routine engagement in interactions in which neither child is the mathematical authority, namely those involving indirect collaboration and multivocal explanation.

In this regard, Steffe and Wiegel (1992) argue that cooperation among children

> should involve give-and-take and a genuine exchange of ideas, information, and viewpoints. To engage in such cooperation, a child must be able to assimilate the activity of the other, and in an attempt to understand what the other is doing, modify the assimilated activity as a result of interactive communication until there is an agreement about what the activity means (p. 459)

In other words, the children should have developed a taken-as-shared basis for communication. However, this is not

> enough to ensure genuine cooperation. Cooperative learning should not be reduced to one child attempting to understand the activity of the other. Rather, each child also has the obligation to become involved in his or her productive activity that might yield a result, and then be willing to correlate the result with the result of the other. Cooperation is possible only if the involved children are striving to reach a common goal. (pp. 459-460)

Thus, neither child should be the mathematical authority in the group. The only caveat to add to this argument is that it might be premature to dismiss as unproductive interactions involving univocal explanation in which one child is an active participatory listener.

At first glance, the two criteria listed above seem to indicate that the differences in the children's conceptual possibilities should be relatively small. Such a conclusion, of course, leads to difficulties in that homogeneous grouping clashes with a variety of other agendas that many teach[illegible] rightly consider important including those that pertain to issues of equity and diversity. Further, it can be noted that the strongest predictor of children's eventual reading ability is the reading group to which they are assigned to in first grade. At the very least, it should be clear that general rules or prescriptions for organizing small group activity cannot be derived from the case studies. Lest this outcome seems unsatisfactory, it should be noted that general principles of conduct are almost invariably too extreme and must be balanced by conflicting principles (Billig, 1987). This is clearly the case in a complex activity such as teaching which is fraught with dilemmas, tensions, and uncertainties (Clark, 1988). Within the context of such activity, the proposed criteria suggest ways in which teachers might look at and interpret small group interactions when making professional judgments. In this regard, it appears that neither harmonious, on-task activity nor the mere occurrence of explanations are good indicators of interactions that are productive for mathematical learning. Instead, the criteria indicate that teachers should monitor the extent to which children engage in genuine argumentation when they solve tasks and discuss their solutions. Further, teachers should intervene as necessary to guide the development of small group norms that make genuine argumentation possible.

References

Billig, M. (1987). Arguing and thinking: A rhetorical approach to social psychology. Cambridge: Cambridge University Press.

Clark, C. M. (1988). Asking the right questions about teacher preparation: Contributions of research on teacher thinking. Educational Researcher, 17(2), 5-12.

Cobb, P. (in press). Mathematical learning and small group interaction: Four case studies. In P. Cobb & H. Bauersfeld (Eds.), Emergence of mathematical meaning: Interaction in classroom cultures. Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Wheatley, G. (1988). Children's initial understandings of ten. Focus on Learning Problems in Mathematics, 10(3), 1-28

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D.B. McCleod & V.M. Adams (Eds.), Affect and mathematical problem solving: A new perspective (pp. 117-148). New York: Springer-Verlag.

Glaser, B. G., & Strauss, A. L. (1967). The discovery of grounded theory: Strategies for qualitative research. New York: Aldine.

Good, T. L., Mulryan, C., & McCaslin, M. (1992). Grouping for instruction in mathematics: A call for programmatic research on small-group process. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 165–196). New York: Macmillan.

Shimizu, Y. (1993). The development of collaborative dialogue in paired mathematical investigation. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F-L. Lin (Eds.), Proceedings of the Seventeenth Conference of the International Group for the Psychology of Mathematics Education (pp. 73-80). University of Tsukuba, Japan: Program Committee of 17th PME Conference.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). Construction of arithmetical meanings and strategies. New York: Springer-Verlag.

Steffe, L. P., & Wiegel, H. G (1992). On reforming practice in mathematics education. Educational Studies in Mathematics, 23, 445-466.

Vygotsky, L. S. (1979). Consciousness as a problem in the psychology of behavior Soviet Psychology, 17(4), 3-35.

Webb, N. (1989). Peer interaction and learning in small groups. International Journal of Educational Research, 13, 21-39.

Yackel, E., Cobb, P., & Wood, T. (1991). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education, 22, 390-408.

MODELLING UN-FORESEEN EVENTS IN THE CLASSROOM SITUATION

Claude Comiti et Denise Grenier

DidaTech , L.S.D.2., CNRS and Université Joseph Fourier, Grenoble

et Institut Universitaire de Formation des Maîtres de Grenoble

Abstract

We present here some results of a research aimed at obtaining descriptions and models of how the ordinary didactic situations work in the classroom. Our study especially points out certain events which could a priori appear as casual events in a class situation but reveal necessary for the process of teaching and learning. After having described our theoretical framework and the questions our research is concerned with and then presented and justified our methodology we detail the analysis of a particular type of event which we analyse as a "milieu breakdown". For this, we take an example related to the introduction of the notion of square root with 14 to 15 year olds students.

Nous présentons ici quelques-uns des résultats d'une recherche ayant pour objectif général de décrire et de modéliser le fonctionnement de situations didactiques "ordinaires". Notre travail met en particulier en évidence comment certains évènements, qui pouvent apparaître comme contingents dans le déroulement de la classe, se révèlent , à l'analyse, nécessaires pour l'apprentissage et l'enseignement. Après avoir décrit notre cadre théorique, les questions de notre recherche et présenté et justifié notre méthodologie, nous détaillerons l'analyse d'un type d'évènement particulier, que nous avons appelé "panne du milieu", en prenant un exemple concernant l'introduction de la racine carrée en classe de Troisième (élèves de 14 à 15 ans).

Introduction

Our study places us alongside researchers who, far from thinking the theoretical/practical link in the unilateral sense of applying theory to practice, use theory in order to describe and interpret practices (COMITI, 1987; GRENIER, 1993; TOCHON, 1992). From this point of view, the classroom is not simply a site of application but also becomes a site of development for the researcher. But what forms and directions might this development take ?

In this presentation, we will show that certain precautions of methodology should be taken when analysing classroom practice scientifically ; we will present the theoretical devices developed to conduct this analysis and the initial results obtained.

The research problem

The research presented here is the second part of several years'work, the first part of which (1989-1991) was concentrated on mathematic's teachers' beliefs on their discipline, their job, their students and the way in which the latter learn (BONNEVILLE et al, 1991). These second part focuses on the study of "practices" in the classroom situation. An isolated classroom study carried out within this framework was briefly presented at PME17 (COMITI, 1993).

In their teaching plan, teachers define their forthcoming practice taking the representations they have of their discipline into account, the mathematics to be taught, the phenomena of the transfer of knowledge and their students' means of learning. In practice, gaps appear between this plan and its realisation in the classroom situation, gaps provoked by trigger events of different types, from those far removed from the learning situation to those closely linked to the contents at stake.

In this context, our research proposes to develop theoretical and methodological devices in order to create models in terms of didactic phenomena :

- un-foreseen events in the classroom situation identified as important for learning,
- the factors at the origin of these events,
- the process underlying decisions made by the mathematics teacher in the classroom situation.

Theoretical framework

This work overlaps two fields of research :

- the study of the way the teacher functions, whether it be what he/she "thinks" and his/her representations (CLARK-PETERSON,1987 ; ERNEST,1989 ; ROBERT,1989 ; THOMPSON, 1992) or his/her practices and decision-making (ARSAC et al, 1992 ; CLARK-YINGER, 1987 ; ROMBERG, 1988 ; SCHOENFELD, 1988 ; TOCHON, 1989) ;
- the study of the "milieu" defined as the whole set of the objects (concepts, rules...) which have achieved stability for the time being for the student (BROUSSEAU, 1989 ; CHEVALLARD, 1992 ; MARGOLINAS, 1993).

The research questions

We can resume them as follows :

- the existence of regular or variable elements concerning the events identified in the practices analysed ;
- the identification of different types of factors at the origin of these events ;
- the links between the means of integrating these events into the teacher's original plan and his/her beliefs about the mathematical notion at work, the learning situation, the knowledge available and the way the students function.

Methodology

We are confronted with the problem of how to observe ordinary classes, a problem which differs from that of observing situations set up for research purposes. How should the problem of classroom observation be raised in terms compatible with scientific practice ? It is a problem which has concerned us for a long time (COMITI, 1988). Let us briefly resume our point of view on the subject.

The class being observed is a "system". Observing this system involves taking information concerning its "state". The taking of information from the classroom taken as a system depends on what we mean by observation and also, obviously, on the theory available for identifying the "relevant" information.

The data collected by observing a system are not neutral, they are in fact construed. If we do not construe them ourselves, we only collect what the institution itself presents to us. But there is no reason why this institutional data be relevant to our research problem: that is why one cannot reduce observation "of" the classroom to observation "in" the classroom.

In order to overcome these problems as best as possible, we set up an experimental protocol (COMITI, GRENIER, 1993) described below.

It involves on the one hand the contract agreed with the teachers concerned and on the other hand the setting up of a means of gathering data which allows the taking of "mixed" information.

The contract with the teachers :

1 - they are willing to participate in the research, which means that they agree with the question, they participate in the setting up of the protocol defined from the transcription of recordings and notes of the observers and contribute towards the analysis of the data gathered on their lessons.

2 - the choice of the mathematical content on which their practices are observed is made in common but devising and managing the teaching sessions is their responsibility entirely.

The collecting of data. The data collected is of several types :

1 - the scenario of the sequence of the different learning stages, written before the beginning of the lesson by each of the teachers concerned ;

2 - interviews with each teacher :

* an interview prior to the sequence, about the scenario of a planned lesson, on the notion to be taught, the activities planned, the teaching methods envisaged, the objectives,

* an interview after the teaching sequence aimed at collecting what the teacher has to say on what happened during the lessons as well as his/her analysis of any differences with what had been planned ;

3 - the recording of discussions at the very end of the sequence ;

4 - observations and sound recording of sessions in the classroom.

The study

The study presented here is on the sequence of about ten sessions of 55 minutes in three classes of 14 to 15 year olds from different schools. The sequence is on the teaching of the square root.

Some preliminary findings

We will leave aside, in what follows, the events which are not relevant to the teaching/learning project, events which could occur in any teaching, not because they are without interest (they would be to the psychologist of sociologist), but because they are not relevant to our didactic research question.

Our study reveals different types of events, unplanned by the teacher but which take place in the classroom situation which we will identify as didactic phenomena and that we will classify as follows : the factors which trigger them, the role they play in the students' learning and whether what happens in the classroom is casual or necessary.

Our study also enables us to analyse the different possible means for the teacher to control this or that type of event.

In what follows we have chosen to analyse a type of didactic phenomenon which seems essential as it corresponds to a type of event in our typology that we very often observe in classrooms : the appearance of student production that seems to have no connection with the question asked.

The "milieu breakdown" event

Our hypothesis is that the student placed in a problem-situation reacts by activating old knowledge which will determine the objects (the "milieu") which will enable him/her to "read" the situation and the personal devices he/she can activate on these objects in order to understand the questions asked and to answer them. The teacher builds his/her teaching around the hypotheses that the students function with the milieu appropriate to the situation he/she proposes.

Numerous un-foreseen events might be explained by the gap between the "milieu" supposed by the teacher and that in which some students actually function. Here are two examples.

ex.1 : a milieu of signs and writing rules versus a milieu of positive or negative numbers equipped with multiplication

The aim of the problem situation is to be sure that the students know how to find the square of positive or negative integers or of decimals, rationals and to institutionalize the properties of squares, in order to introduce afterwards the square root of a number (condition of existence, definition and properties).

The following questions are asked :

- *"Are there numbers whose square is -1?"*
- *"Can two different numbers have the same square?"*

The M milieu on which students are supposed to be able to act is formed of sets of numbers **N**, **Z**, **D**, **Q**, equipped with multiplication.

The devices presumed to be available are :

- the definition of the square of a number,
- the properties of multiplication (rule of signs) in Z

These devices functioning in the M milieu should allow the students to produce couples (a,b) where b is positive and equal to a^2.

In fact, the M' milieu on which several students will work is defined by the set N and signs such as : ( ) ; - ; / ; . , ... ; $.^2$.

The devices which function naturally in this "milieu" are the usual rules of writing these signs, (knowledge of position) of the type a ; -a ; a/b ; a,b ; a^2 ; $-a^2$; $(-a)^2$; the productions are thus writings obtained by a combination of numbers and signs available.

Therefore, to the question *"Are there numbers whose square is -1?"* Michaël replies : "*yes, because* $-(1)^2 = -1$. " This answer leads to a discussion which confirms that other students function like Michaël : we see the sign minus (-) arise as a subtraction, or the absolute value as writing having the same effect on (-1) as the square, the confusion between the square of (-1) and the opposed to 1^2.

The teacher can only come back to his/her initial objective with the complicity of "good" students, those who work with the M milieu.

This does not solve the problem as, on several occasions, we will see arguments arise from M' and even, at the end of the session, whilst the teacher is writing on the board the property to learn : "two opposite numbers have the same square, for example, the square of -5 and 5 is 25", several students protest. For us the square of (-5) is 25 "because $(-5)^2 = 25$" , but the square of -5 (without ( )) is -25 "because $-5^2 = -25$".

ex. 2 : a "drawing/measurement milieu" versus a "construction/geometric object milieu"

The problem raised is the geometric construction of $\sqrt{8}$ (as in the length of a segment). There are two objectives :

- to show that this number exists as one can construct a segment of its size, even though one cannot give its value in the sets of numbers known to the students **N**, $\mathbf{D}^+$, $\mathbf{Q}^+$;
- to show that one cannot write its exact value without introducing the symbol $\sqrt{}$.

The following exercise is proposed :

"Construct a square with sides measuring 2cm. Let this square be ABCD.
What is its area ?
From this square construct a square whose area is double that of ABCD.
Let this new square be VOLE".

The M milieu on which the students are supposed to be able to act is the following :

- the square geometric object, its graphic representation whatever its position on the sheet of paper, its sides, its diagonals, its area...,
- the instruments of construction and measurement
- the sets of numbers $\mathbf{N}$, $\mathbf{D}^+$, $\mathbf{Q}^+$, equipped with multiplication

The devices supposedly available are essentially the construction of a square the side of which is a given integer and Pythagoras' theorem.

The expected productions are the tracing of the diagonal of the ABCD square, the square of the size of this diagonal recognised as double the area of the square, and the construction of the VOLE square with this diagonal as its side.

In fact the M' milieu on which most students work consisted of drawings of squares with sides parallel to the side of the sheet of paper, positive numbers as measurements of size in centimetres and millimetres, the graduate ruler and the calculator.

The tools which function naturally in this milieu are measuring with the help of a ruler, the relationship between side and area of a square.

The productions of the students are thus :

- the calculation of the area of the ABCD square ($4cm^2$) and the double of this area ($8cm^2$),
- the calculation, with the $\sqrt{}$ key on the calculator, of the measurement of the side of the VOLE square of an area of $8cm^2$.
- the tracing of a square with sides of 2.82cm (rounded from the value given by the calculator).

As in example 1, the teacher will rely on the "good" student, who, although he gives a wrong answer, is recognised by the teacher as functioning with the M milieu.

To sum up, the event that we call "milieu breakdown" generally emerges, in the classroom situation with, from the teacher's point of view, an unexpected and inexplicable production from one or more students.

In the first example, the production of $-(1)^2 = -1$ is, for the teacher, not relevant because, as far as he or she is concerned there is no connection for him/her with the problem being treated.

In the second case, the production by students of 2.82, a number from which they construct a square, is unacceptable for the teacher who, on the contrary, wants to justify the existence of $\sqrt{8}$ based on a geometric construction.

These productions show the objects on which the students work effectively. The analysis shows how these possibly chance interventions create a necessary event in order for the students to accomplish the learning aimed at.

To manage this event, the teacher has no other possibility than relying on the "good" student, the one who functions with the milieu anticipated for the situation.

Discussion

The first analyses carried out show how connecting the "a posteriori analysis" of didactic situations with the study of the teaching project of the teacher

- enables one to step back from the situations observed,
- is a relevant and efficient way of interpreting events which perturb the teacher's project in the classroom situation.

This connection appears to be a relevant model setting device as it allows one not only to interpret but also to predict events likely to arise in a didactic situation.

Our analysis and in particular the study of the event developed above shows that the work which really takes place in the classroom (sensitive objects) is often concentrated on other knowledge than that which the teacher wishes to convey. It demonstrates that the objects which, according to the teacher, are supposed to be known by the students and thus considered previously grasped, will in fact be issues to be learnt by the latter.

This phenomenon is well known but our work provides a sharper analysis by showing how the disturbance created by a students' intervention will show objects on which the latter is actually working, creating an indispensable event for the teacher to carry out his/her project (a casual answer which proves necessary for the process of teaching/learning). If this event does not occur, there will be a loss of meaning of what is happening of the students' side.

Other analyses, not developed here, show how, in managing the classroom, the teacher seizes certain errors (and not others which may to an outside observer seem just as important) because they reveal lack of knowledge which must be compensated for in order to complete the learning programme.

We are now in a position to make hypotheses on the existence of consistencies which are beyond the teacher concerned and beyond even the notion being taught.

References

ARSAC G., BALACHEFF N, MANTE M., Teacher's role and reproducibility of didactical situation, *Educational Studies in Mathematics*, vol 23 pp.5-29

BONNEVILLE J.F., COMITI C., GRENIER D., LAPIERRE G., 1991, Représentations d'enseignants de mathématiques, *in Cahier du Séminaire de didactique des mathématiques*, LSD2, Université Grenoble I

BROUSSEAU G., 1989, Le contrat didactique : le milieu, *Recherches en Didactiques des Mathématiques*, Vol 9.3, pp. 309-336

CHEVALLARD Y., 1992, Fundamental concepts in didactics : perspective provided by an anthropological approach, in *Researche in " Didactique" of mathematics*, Ed. R.Douady and A.Mercier, La Pensée Sauvage, Grenoble,

CLARK C.M. and PETERSON P.L.,1986, Teachers'thought processes, in M.C.WITTROCK 5ed), *Handbook of research on teaching*, 3°édition, NewYork, Macmillan, pp.255-296

CLARK, C.M. and YINGER, 1987, Teacher planing in CALDERHEAD J.(Ed) *Exploring teacher thinking* London, Cassell Press, pp.84-103

COMITI C., 1987, Histoire d'une collaboration chercheur-enseignant ou comment développer les liens entre recherche en didactique et pratiques enseignantes ? Actes du*Colloque Franco-Allemand de didactique des mathématiques et de l'informatique*, La Pensée Sauvage (Ed), Grenoble, pp.317-328

COMITI C., 1988, How and Why shall we observe students in the classroom ? *6e Congrès International sur l'Enseignement des Mathématiques* (ICME VI), Budapest, in Actes du Groupe Preservice Teacher Education, Mathematics Departement, Illinois University (Ed.).

COMITI C. and al.,1993, Focusing on specific factors occuring in classroom situation that lead the teacher to change his practice and make him modify his original plans, oral communication in *Proceedings of the 17th International Conference of PME*, Tsukuba.

COMITI C., GRENIER D., 1993, L'observation, outil de recherche dans un travail de modélisation de l'enseignant acteur du système didactique, in Actes de la *VII° Ecole d'été de Didactique des Mathématiques*, NOIRFALISE (Ed.), Institut de Recherche sur l'Enseignement des Mathématiques de l'Université de Clermont-Ferrand

ERNEST P., 1989, The Knowledge, Beliefs and Attitudes to the Mathematics Teacher : a model, *Journal of Education for Teaching*, Vol 15, n°1, pp.13-33

GRENIER D., 1993, Question of methodology for a study of the teacher in the didactic system, *The Journal of Mathematical Behavior*, R.B.Davis (Ed.)

MARGOLINAS C., STEINBRING J., à paraître, Double analyse d'un épisode : cercle épistémologique et structuration du milieu, *Actes du colloque "20 ans de didactique des mathématiques en France*, Ed. La Pensée Sauvage, Grenoble

ROBERT A. et ROBINET J., 1989, Représentations des enseignants de mathématiques sur les mathématiques et leur enseignement, *Cahier de DIDIREM n°1*, Ed. IREM Paris7

ROMBERG T., 1988, Can Teachers be Professionals ?, in Grouws D.A., Cooney T., Jones D; (eds) *Perspectives on research on effective mathematics teaching*, NCTM, Lawrence Erlbaum, pp.224-244

SCHOENFELD A.H., 1988, When good teaching leads to bad results : the disaster of well taught mathematics courses, *Educational psychologist* Vol 23-2

THOMPSON A.G., 1992, Teachers' Believes and Conceptions : A Synthesis of the Research, in Grouws D.A. (Ed) *Handbook of Research on Mathematics Teaching and Learning*, New York, Macmillan, pp.207-213

TOCHON F., 1989, A quoi pensent les enseignants quand ils planifient leurs cours?, *Revue Française de Pédagogie*, n°86, pp. 23-34

TOCHON, F., 1992, A quoi pensent les chercheurs quand ils pensent aux enseignants Les cadres conceptuels de la recherche sur la connaissance pratique des enseignants, *Revue Française de Pédagogie*, n°99, pp. 89-113

Six Approaches to Transformation of Functions Using Multi-Representational Software

Jere Confrey
Cornell University

This paper reports on six approaches to teaching transformations of functions. The approaches are embedded in a framework for functions involving multi-representational software, contextual problems and vertical and horizontal translations, stretches and reflections. The six approaches include: substitution to a template, function building, a symmetric approach, visual matching, horseshoe display and scaling.

This paper reports on the results of five years of teaching transformations in a variety of experimental settings: 1) a university precalculus course; 2) summer workshops for teachers; 3) high school precalculus courses (grade 11); 4) teaching interviews. The teaching of transformations was done within a framework for teaching functions which relied on the use of contextual problems[1], prototypic families of functions in multiple representations, and transformations. The class of transformations that we used were limited to vertical and horizontal translations, dilations (we called these stretches) and reflections in a Cartesian coordinate system. For a more complete discussion of the framework, see Confrey and Smith (1991).

Transformations are an important part of the framework for they provide the tools to move among functions of the same family In doing so, they are useful in modeling activities. Typically, students in our courses learn to examine a contextual problem and identify the characteristic family. Transformations provide the means for tailoring the function to the particular data Having a unified set of approaches to transformations across families of functions also serves to provide the "glue" to unite the functional families.

Function Probe

In all of these settings, the approach to functions involved the use of multiple representations: graphs, tables, algebra and calculator keystroke notation. Students used a software called *Function Probe* (Confrey, 1992) which allows the use of the four representations, and was designed to highlight meanings of transformations in different settings. For instance, in the graph window, students can stretch, translate and/or reflect a graph through mouse actions. They can also sample points and send the points to the table. A history of the student's actions is also available for examination. In the table window, a variety of tools are available for the students' use. They can fill a column arithmetically or geometrically through a simple command; they can link columns without an explicit algebraic dependency; and they can sort values, insert means, and display differences and accumulations. The calculator window keeps a record of their keystroke procedures and allows them to build buttons to generalize their procedures

[1] The problems in the course are described in Learning about Functions through Problem Solving (Confrey, 1991)

We have claimed that algebraic notation has repeatedly been used in modern times to dominate other forms of representation, which has led to a loss in breadth of thinking. Historically, this was not the case, in that curve-drawing devices and data tables provided equal contributions to symbolic forms in the development of functions. Hence, we have argued for an "epistemology of multiple representations" in which contrast between representations is recognized as significant as convergence in establishing meaning (Confrey, 1991). By identifying multiple approaches to transformations, we can encourage students to find multiple ways to make sense of their results and to develop their sense of flexibility and elegance. Multiple approaches support the diversity in students' preferences and provide alternative approaches to use when faced with cognitive obstacles. Multiple approaches have also proven valuable in highlighting the differences among the behavior of different families of functions and among different contextual problems. As a result of supporting the use of multiple representations, our students have given us many opportunities to rethink our understanding of transformations in light of their approaches. In this paper, I seek to share six of the most prevalent approaches.

Algebraically, the class of transformations we have confined ourselves to can be described one way as $y = A\,f(Bx + C) + D$ varying A, B, C, and D, or as a set of linear transformations on x and on y. Graphically, one could describe this as a set of vertical and horizontal translations (VT, HT), vertical and horizontal stretches (VS, HS), and reflections[2] (R) Tabular descriptions of transformations include the use of adding /subtracting a constant to all entries in a column, multiplying/dividing the column entries by a constant, or multiplying by -1. Coordinating the actions in the different representations is a major goal of our instructional program. It requires the coordination of visual transformations, numerical operations and algebraic symbol manipulation.

All of the students we worked with had previously been introduced to algebraic manipulation through introductory course work which focused on solving and simplifying equations, factoring, working with trigonometric identities, and plotting graphs from equations. In this paper, I will present the six approaches. A longer paper including examples of student data will be available at the conference, and for further reading, project papers on the topics are available (Borba and Confrey, 1992; Smith and Confrey, 1992; Borba, 1993, Borba, 1994).

Approach 1: Substitution to a Template. In this approach to functions, one uses the algebraic form as a template and learns to identify the actions associated with each parameter. Thus, in the form, $y = A\,f(Bx + C) + D$, the student learns that:

A is the magnitude of the vertical stretch factor

1/B is the magnitude of the horizontal stretch factor

-C/B is the magnitude of the horizontal translation

[2]Although reflections can be thought of as stretches by -1 we chose to treat these generally as a separate class. However, students often curtailed this into a single action using a negative value for stretch

D is the magnitude of the vertical translation.

Although this approach is arguably the most concise approach, students often experience significant difficulties with it. They have little difficulty with the parameters A and D which act visually in the same way as the parameters appear in the equation (Artigue and Dagher, 1993). However, the fact that the coefficient 2 in the equation $y = \sin 2x$ acts as a horizontal shrink confuses them for its apparent inconsistency with its effect on $y = 2 \sin x$ as vertical stretch. Similarly, students have some difficulty understanding why $y = \sin x + 2$ moves in the positive direction vertically while $y = \sin(x + 2)$ moves in the negative direction horizontally. Finally, they are confused by the impact of the stretch factor on the translation factor, such as in $y = \sin(2x + 3)$, where the translation is 3/2 to the left, not the 3 units they expected. Thus, both of the horizontal actions (HS and HT) confuse them using this approach.

One alternative approach within the substitution perspective is to alter the template to $y = A' f((x - C')/B') + D'$. In this case, the parameters become simply A', B', C', and D', where

A' is the magnitude of the vertical stretch factor;

B' is the magnitude of the horizontal stretch factor;

C' is the magnitude of the horizontal translation;

D' is the magnitude of the vertical translation.

However, when this approach is used, the meanings of the parameters are simplified because the complexity is shifted to the equation. In this case, students experience difficulties remembering where the sets of parentheses lie, and when to add or subtract and when to multiply or divide.

In general, this strategy has only been repeatedly successful with stronger students. For these students, the variations in symbolic form can be coordinated with their understanding of the transformations; however, for less prepared students the use of the template often results in errors they cannot easily correct.

Approach 2: Function Building. With to this approach, all functions begin from the identity function $y = x$. Functions are then built through a series of actions. First, a vertical stretch is applied to create $y = mx$. (note: one could either anticipate a final form by using A, B, C, and D consistently to produce the form $y = A f(Bx+C) + D$ or use notation that describes the series of moves in a consistent way with one's actions. I have chosen the second approach.) The first action could equally have been a horizontally shrink by 1/m, but most students prefer to rely on vertical actions, as this approach allows them to reduce all methods to vertical actions only. Secondly, a vertical translation is applied to produce $y = mx + b$. Using this approach, students are able to begin the problem using the more familiar family of linear functions.

At this point, the function is applied to the line. This can be described as a composite action, that of $f(g(x))$, where $g(x)$ is linear and $f(x)$ is the function one seeks to create. Applying a function to a line is a relatively unfamiliar idea, and it is more easily imagined for some functions than others. When "function building", we apply the method at first to using the absolute value function, where the idea of "absolute valuing" a linear equation such as 2x+ 1 is discussed. The major point of discussion concerns where to "bend" the line, to create the "significant points" (usually the two intercepts). Most students explain this in terms of absolute value not producing any negative outputs; hence, all y's must be greater than zero, and the bend is at the x axis.

Function building can also be applied to the greatest integer function with relative ease. Using this approach to construct quadratic functions has also been relatively successful, and can be used to emphasize how the rate of change of the quadratic differs from that of absolute value.

An advantage of this approach is that it anticipates the introduction of composing functions and gives the student the sense of the function as an action. Function building highlights the order of operations. Rather than experiencing a function such as y= 3 abs (2x - 1) + 5 as a whole, students learn to see it as a series of operational actions. Not only does such an approach work well in the graph window, but also in the table window, where the movement from column to column is the result of a binary operation (each column is changed from the previous set of x or y values by a single operation of +,-, x or / except when the prototypic function is applied). Applying this approach to trigonometric or exponential functions is considerably more difficult for students, though some do find a consistent way to make the interpretation across functional families.

The pedagogical implications of function building are compelling. It provides a solid conceptual base for the template approach, but depends on a careful choice of problems and functional families. Prematurely giving students problems such as f(x) = abs (2(x-3)) can challenge the approach, requiring students to algebraically transform the actions inside the parentheses to 2x - 6 or to apply horizontal rather than vertical arguments to 2(x-3) Polynomial functions such as f(x) = x^2 + 2x + 1 also present students with difficulties, but can provide strong motivation to learn to complete the square.

Approach 3: Symmetric Actions. A major challenge in the substitution to a template approach results from the lack of symmetry between the actions on x and those on f(x). That is, A and D do not act in the same way as B and C. Students can learn why this is so as they examine the location of the parameters in relation to the x and y variables, but f(x) notation in fact obscures this fact by eliminating the y from view. An alternative approach to this is to allow the actions to be taken directly on x and y, and to shift from functional form to relational form. Thus, an equation that would be written as y= 3 sin (2x - 1) + 4 in conventional functional form would appear as (y-4)/3= sin (2x - 1) or perhaps y/3 - 4/3 = sin (2x - 1) or even (y-4)/3 = sin (2 (x - 1/2)). In this final form, which we can write as A'(y-D')= f(B'(x-C') where 1/A' is the vertical stretch, D' is the vertical translation, 1/B' is the horizontal stretch and C' is the horizontal translation

The symmetric approach typically takes some getting used to, but its elegance lies in the parallel structure between the vertical and horizontal transformations In this representational form, the actions on x can be tied to the parameters B and C in the same way that actions on y can be tied to A and D. The unfamiliarity of such an approach makes it appear awkward, however, its fit with the table representation is notable. In the table, there is no clear rationale for numerically transforming only the x variable until there is a functional action. The symmetric approach also challenges the over reliance on f(x) notation, where the equation is solved for f(x). Such over reliance on this form makes students have difficulty recognizing relations such as $x^2 + y = 1$ as functions. Historically, when curves were generated by curve-making devices, and algebra was emerging as a means to describe those curves, input and output views of function did not exist. In fact, using the tools of similarity and proportion and the Pythagorean theorem, descriptions for curves were generated that would be very awkward to describe in f(x) form (See Smith, Dennis, and Confrey, 1992 for a discussion of Descartes' approaches to functions).

Approach 4: Visual curve matching. In this approach, students learn to approach functions visually, such as when one displays a data set and wants to fit a curve to it. Approaching functions visually may be an unfamiliar task, yet is very compelling and satisfying for many students. And in a scientific world in which super computing relies more and more on graphical display, the skills needed for visual curve matching are arguably important for students' mathematical development.

Visual curve matching is less procedural than the other approaches. It depends heavily on the graphs involved and it highlights the use of the stretch icon. For instance, when asked to match two absolute value graphs, most students use the translations to match the vertex. They then endeavor to stretch the functions to match the slopes. In Function Probe, the line of invariance for a VS or HS is movable and called "the anchor line"[3]. The students learn to move this anchor line to the vertex and to stretch away from the line.

In general, this approach accomplishes some important goals in student learning. We find that students typically understand translations intuitively; however, understanding stretching is more complicated. As they learn to use stretching as a tool, they learn what it means to hold a line of the plane invariant. And they puzzle on what the operational meaning of stretching is. By including a Function Probe resource for sampling, we extend this approach by examining the numeric implications of such actions. For an extensive discussion of this approach, see Borba (1993).

Approach five: Horseshoe Display. Extending the visual curve matching approach to allow one to generate the algebraic equation for the transformed curve posed a new challenge to both the students and us and led to a new approach. In an example in class, teachers were trying to find equations to describe different configurations of parking garage rates. Consider the following case, when the costs to park are:

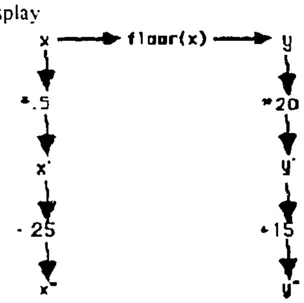
less than .25 hours: 15¢
.25 hr or more but less than .75 hr: 35¢
.75 hr or more but less than 1.25 hr: 55¢

[3]Some software restricts the anchor line to the axes for doing so makes the algebraic notation easier. We see this as an example of privileging one representation over another. Furthermore, many contextual problems are more directly solved with a movable anchor line.

A data table is constructed and values are sent to the graph. The teachers were seeking to fit the function y= floor(x) (the greatest integer function in Function Probe) onto the displayed data values. They quickly recognized that the vertical transformations were a VS of .20 and a VT of .15. Thus, they knew their equation would be some modification of y= .2 floor (x) + .15. They compressed the function horizontally by .5 to get the steps to be a half-hour long. This gave them the equation y= .2 floor (x/.5) + .15. Then they translated horizontally by -.25. The equation was displayed as y= .2 floor ((x+.25)/.5) + .15. See Smith and Confrey (1992) for a more complete description of this episode.

They were totally perplexed by the result. They had carried out the transformations on the graph window in the order of a HS (by .5) followed by a HT (by -.25) but the algebraic display seemed to record it as a translation followed by a stretch. And, it was clear that the two results were not algebraically equivalent. The visual approach had successfully led them to a matched graph, but the algebraic display did not seem to fit the order of the graphical actions. This led to a genuine conflict for all of us in terms of our understanding of transformation. To explore the problem, we worked further with the visual approach. We sampled points after each graph action and sent these to the table. Each of the changes was recorded and described in words. Finally, we sought to explain the algebraic changes in light of the series of table transformations.

Resolving the apparent contradiction led to a new horseshoe shaped display, a fifth approach to functions. This approach drew on two sources: 1) ways in which transformations are described in topology; and 2) ways in which a student in a teaching experiment chose to describe changes in tabular values (Borba, 1993). A student looking at the changes in each variable of the table wanted to describe these as $y_{new} = f(y_{old})$ while $x_{new} = f(x_{old})$. Borba described these as covariational equations consistent with the covariational approach to functions developed by our research group (Confrey and Smith, in press). Drawing on these two sources, I developed the following horseshoe display



Horseshoe Display

The question becomes how to relate x'' to y''. If this is to be written in traditional f(x) = y form, then one must begin at the bottom left of the display with x'' and work one's way around to y''. Thus one takes x and adds .25, divides by .5, floors the result, and then multiplies by .20 and adds .15. This gives the desired equation.

This approach to transformation possesses numerous advantages. In it, one can "see" the visual approaches sequence of actions. The vertical ones proceed from y to y' to y'' and the horizontal ones proceed from x to x' to x''. However, one can also "see" the resulting equation in terms of movement around the horseshoe so as to relate x'' to y''.

A horseshoe display also supports a number of the other approaches. The display itself shows the symmetry of the symmetric approach. It supports function building. And, it allows one to move smoothly between the graph actions and the algebraic representations where the movement in the display varies with the representation.

Also appealing in the horseshoe display is the way in which it builds upon the matrix display for proportional reasoning in the work of Vergnaud (1988). This connection is more than analógical, for if one considers the identity function to be the simplest display in this form (y = x), then direct variation is arguably the second simplest (y = mx). In our future work introducing functions through design projects, we plan to investigate how to build functions from proportional reasoning.

However, as with any approach, transformational matrices possesses certain disadvantages. They lack the compactness of "substitution into a template"; they require more space to display the notation; and they are more complex, in order to be able to accommodate both the directionality of the operations and the multiplicity of desired forms. But horseshoe displays are very elegant representations, for they make interesting use of space (a characteristic of algebraic notation often not given enough attention): the movement in space down a column describes changes in one variable, while the connection across the equation highlights the functional family.

Approach Six: Scaling the Axes. Not all treatments of transformations depend on carrying out a series of graph actions. Some students prefer to leave the curve in place while transforming the axes (scaling). Although we have not used such an approach extensively in our project, we have witnessed episodes where students are inclined toward this method, and have considered design initiatives that would make such an approach a more viable option.

However, we offer a simple twist to the scaling approach. In all treatment of scaling as a form of transformation that we have seen, scaling is used to apply transformations on both x and y axes. In contrast, we have begun to consider the use of scaling on either one or both actions, and as a possible alternative to transformations of the curve. For instance, we ask students to describe a ride on a Ferris wheel by giving the height off the ground as a function of time. If the wheel is 5 meters in radius and its base is 1 foot off the ground, the students quickly find the equation y= 5 sin(x) + 1. What is more difficult for them is to figure out how to adjust for the time parameters. We have witnessed students who see themselves faced with an independent puzzle when trying to account for the fact that the Ferris wheel completes one revolution every 24 seconds. How should they switch from inputs of x in radians to inputs in seconds? These students express confusion in viewing this as a transformation of the sin function, for they must really switch from radians to seconds. For these students, it would be useful to be able to rescale the x axis as a scaling action, while still carrying out the vertical transformations as graph actions.

This approach adds another piece to the transformational puzzle for it defines the shape of a graph as a curve-scale interaction. It suggests that functions be considered as the juxtaposition of a curve on a scale, and introduces a much deeper implication for how scaling is used in investigating functions. In future design work, we plan to consider how to implement scale changes as mappings from a one-dimensional line to another, while keeping the graph intact, and then to consider how to display these results symbolically while also maintaining the option for applying transformations to the curve.

Conclusions. In this paper, I have outlined six approaches to functions within a multi-representational and dynamic environment using contextual problems. These six approaches come from extensive experimentation with students. Moreover, it is through the careful examination of student method on transformations that we have gained significant new mathematical insights. Furthermore, I have suggested that the different approaches should not be reduced to a single approach for each offers its own advantages and disadvantages. And, it may the case, that a particular approach is best for a particular functional family, context, or student preference.

Bibliography

Arugue, M. (1993) and A Dagher The use of computers in learning to correlate algebraic and graphic representations of functions *Proceedings of the Seventeenth International Conference of the Psychology of Mathematics Education* University of Tsukuba, Japan Vol 2 p.1-9.

Borba, Marcelo (1993). Students' understanding of transformations of functions using multi-representational software. Unpublished doctoral dissertation, Cornell University, Ithaca, NY.

Borba, Marcelo, and Confrey, Jere (1992) Transformations of functions using multi-representational software: Visualization and discrete points. *A paper presented at the Sixteenth Annual Meeting of Psychology of Mathematics Education-NA.* Durham, NH, August 6-11, 1992.

Confrey, Jere (1991). Using computers to promote students' inventions on the function concept. In Shirley Malcom, Linda Roberts and Karen Sheingold (Eds.), *This Year in School Science 1991* Washington, DC: American Association for the Advancement of Science

Confrey, Jere (1992). Function Probe© [Computer Program] (v.2.3.5). Santa Barbara, CA: Intellimation Library for the Macintosh.

Confrey, Jere and Smith, Erick (1988). Student centered design for educational software. *Proceedings of the First Annual Conference on Technology in Collegiate Mathematics.* Reading, MA: Addison-Wesley. Columbus, OH, October 27-29, 1988.

Confrey, Jere and Smith, Erick (1991). A framework for functions: Prototypes, multiple representations, and transformations. In Robert Underhill and Catherine Brown (Eds.), *Proceedings of the Thirteenth Annual Meeting of Psychology of Mathematics Education-NA* Blacksburg, VA, October 16-19, 1991. (pp. 57-63).

Confrey J and E. Smith (in press) Splitting, Covariation and their Role in the development of exponential functions Journal of Research in Mathematics Education, NCTM, Reston, Virginia

Dreyfus, Tommy and Eisenberg, Theodore (1987) On the deep structure of functions. In J. Bergeron, N. Herscovics and C. Kieran (Eds.), *Proceedings of the Eleventh International Conference on the Psychology of Mathematics Education.* Montreal, July, 1987 (pp 190-196).

Smith, Erick and J.Confrey (1992) Using a dynamic software tool to teach transformation of functions *Proceedings of the Fifth Annual International Conference on Technology in Collegiate Mathematics* Rosemont, Il Nov. 12-15, Reading, MA: Addison-Wesley Publishing Com Inc.

Smith, E., D.Dennis and J Confrey.(1992) Rethinking functions: Cartesian constructions A paper presented at the *Second International Conference on the History and Philosophy of Science in Science Teaching* Kingston, Ontario, May11-15

Vergnaud, G (1988) Multiplicative Structures Hiebert and Behr, ed Number Concepts and Operations in the Middle Grades Lawrence Erlbaum Associates and National Council of Teachers of Mathematics, Reston, VA

CONCEPTUALIZING TEACHER EDUCATION AS FIELD OF INQUIRY: THEORETICAL AND PRACTICAL IMPLICATIONS*

Thomas J. Cooney
Barry E. Shealy
University of Georgia, USA

Abstract

In this examination of teacher education as a research and development activity, we consider the historical precepts of teacher education as they relate to reform and present a means by which the professional growth of teachers can be conceptualized. Our work is rooted in the notion of authority as we consider the evolution of a teacher's notion of authority from that of being external to that of being internal as an indicator of growth. If we believe reform is related to teachers becoming adaptive and reflective agents in the classroom, then we must consider ways of promoting and monitoring this development in our teacher education programs. We include implications of our theoretical orientation for the case studies we have conducted with secondary preservice teachers and for the practical dimension of designing activities that promote reflection.

Historically, teacher education has been treated as a practical field in which the primary efforts of the teacher educator was to create activities that would promote a certain type of teaching. This condition is still dominant today but because of research we have come to realize that the enterprise of teacher education is far more complex than was at first imagined. In fact, there is very little research on teacher education per se prior to the mid 1980s. Cooney's (1980) review of research on teaching and teacher education lays out a perspective for research on teacher education but is rooted in the body of literature on research on teaching. But research on teaching prior to the mid 1980s was limited primarily to the perspective of the realist in which the goal of empirical work was to establish significant generalizations that could guide the education of teachers. We could argue that mathematics educators were quick to realize the fallacy of developing a general empirical base for teacher education without taking into consideration the content being taught.

Nevertheless, mathematics educators can hardly take the position that we were treating teacher education as a field of inquiry. Indeed, in the 1960s and 1970s the modus operandi of most teacher education programs was to train the teacher to be a reasonably competent mathematician and introduce a little pedagogy on the side. We assumed that giving teachers the mathematics we thought they needed and few pedagogical techniques would suffice for effective teaching.

Part of the difficulty lay with our conception of what constituted science. Kuhn (1970) was instrumental in warning us about the fallacies of thinking of science as devoid of human intention albeit the warning was largely unheeded by educational researchers for

* This paper was developed as part of the NSF-funded teacher preparation project *Integrating Mathematics Pedagogy and Content in Pre-service Teacher Education*, TPE-9050016, Thomas J. Cooney, Director.

at least a decade. In their review of research on teacher education, Brown, Cooney, and Jones (1990) contrasted research from an analytic perspective in which the researcher is detached from the subject and the conception of mathematics is that of received knowledge with that of the humanistic perspective in which the researcher is fundamentally connected to the informant and the view of mathematics is as a subject constructed by humans. The humanistic approach, consistent with the constructivist epistemology, places an emphasis on how teachers, or whomever, come to make meaning about the events in their lives.

Such a shift in orientation has led us to appreciate what it is that teachers believe and how they come to construct those beliefs. Presently, we see much in the literature about the importance of teachers' beliefs (see, e.g., Thompson, 1992). What we need to realize is that this importance has to do with the substance of the beliefs (i.e., what is believed) and the nature of the belief (i.e., how they are held and organized).

The Emerging Foci of Teachers' Beliefs

The studies by Thompson (1984) and (Cooney, 1985) marked the trail of coming to understand how teachers construct their conceptions of mathematics and the teaching of mathematics. It is now commonplace to encounter arguments that what a teacher believes affects how and what mathematics gets taught. What is less common is the identification of lenses through which we can conceptualize the dynamic process of constructing beliefs. What theoretical constructs are there to support a conceptualization of the process by which a teacher can become a reflective practitioner (Schön, 1983)?

We need to recognize that the very notion of being reflective and its corollary of being adaptive is based on the ability of a person to see themselves operating in a particular context, that is, the ability to "step outside of themselves" in order to reorient themselves. This ability stems from Bauersfeld's (1988) notion of fundamental relativism which requires that an individual be able to "see" events from a multitude of perspectives. (While Bauersfeld was speaking about mathematics educators more generally, his notion applies equally well to teachers.) But to develop such a perspective one must sense that they are the experts from which perspectives develop. To say it another way, if a teacher's orientation is entirely external, then the cues for what is considered important necessarily comes from that external, authoritarian agent.

There are two works that address the issue of one's relationship to authority: Perry's Forms of Intellectual and Ethical Development in the College Years (1970) and Belenky, Clinchy, Goldberger, and Tarule's Women's Ways of Knowing (1986). While these two works differ (e.g., Perry developed his scheme by working only with men, Belenky et al, only with women), they both describe the extent to which people rely on external authorities for a validation of what they believe. Perry talks about stages such as dualistic (right versus wrong), multiplistic (there are multiple ways of thinking about situations), relativistic (these different ways are not uniformly equal in their value), and commitment (a person commits themselves to one of the positions). Belenky, et al., use the metaphor of voice to describe ways that women orient themselves toward authority. For example, the position of "silence" represents a women who is totally submissive to another being-- perhaps a spouse, for example. In contrast a women who is at the position of "constructed knowledge" is capable of integrating her own voice with that of others, dealing with each in a fundamental way. The preservice teachers whom we have studied fall between these two extremes.

We could say much about these positions. What is important to this discussion is that in each of these schemes there is a certain "transition" point at which a person moves from viewing authority as an external entity to that of being internal. Perry (1970) puts it the following way in describing the fifth of his nine positions.

> Our scheme must now account for a drastic revolution. Up to this point the students have been able to assimilate the new, in one way or another, to the fundamental dualistic with which they began. (p. 109)

Perry goes on to describe the characteristics of this shift as one moves from thinking of authority as an external agent to that of being an internal agent.

There is an analogous position in Belenky's scheme, viz., Subjective Knowledge: The Inner Voice. This position describes a revolutionary step in that truth eventuates from a personal perspective. It is here that a woman begins crossing the bridge from a submissive orientation to one in which her voice is a significant determiner of what she believes. Once a woman begins to accept herself as a legitimate authority, she begins to lay the groundwork for the acceptance of contextuality and relativism which fosters a sense of self and control-- expunging the notion that authority (a professor, a textbook, guidelines from the school district, etc.) is omnipotent.

Why is considering such an orientation important? First, there is considerable evidence that many teachers teach mathematics from a cut and dried, dualistic perspective. (See Brown, Cooney, and Jones (1990) and Thompson (1992).) The orientation suggested here can provide a way of conceptualizing how such teachers view mathematics and their roles as teachers of mathematics. Further, recall that the current reforms in mathematics education call for a reflective and adaptive teacher. But the ability to be reflective and adaptive <u>requires</u> that an individual has the capacity to see the world as contextual, that is, as a world <u>in which one tries to understand how others (e.g., students) come to know and believe as they do</u>. Such a constructivist orientation can not be achieved when the world is seen in absolute terms.

Green (1971) and Rokeach (1960), in their discussions of belief systems both point to the fact that the extent to which beliefs are isolated and the individual fails to see the world as a connected place is the extent to which the person relies on an external authority for verification of truth. Green differentiates between beliefs that are evidentially held from those that are nonevidentially held. The former is based on rationality; the latter is based on the acceptance of what an authority dictates. Rokeach's notion of dogmatism is similarly focused on one's relationship to authority. The dogmatic teacher teaches mathematics from a cut and dried perspective thus indoctrinating kids with certain beliefs about mathematics--which may account for the fact that many students in the United States think of mathematics as a subject best learned through memorization. Belenky, et al, point out that the woman who is a "received knower" cannot tolerate ambiguity. For example, such a person believes that for every poem there is only one correct interpretation; their interpretation is literal. While Belenky, et al, are addressing issues related to poetry, it is not a long reach to think of the analogous problem of trying to help students appreciate and use mathematical processes such as problem solving, communicating, reasoning, and making connections--processes that are fundamental to the NCTM Standards (1989).

The point of this brief analysis is not that women think one way and men think another. Rather it has to do with providing an orientation toward thinking of teachers as developing beings. We consider the implications of these theoretical orientations for our research program in the next section.

Implications for Research: The cases of Greg, Henry, and Nancy

We have used various schemes that focus on authority, e.g., Perry (1970), Belenky, et al. (1986), Green (1971), and Rokeach (1960) as ways of conceptualizing teacher development as we see the issue of authority as central to what one considers mathematics to be and how one conceives of their role as a teacher of mathematics. Our research is longitudinal in nature as we study preservice teachers through the use of surveys, interviews, and classroom observations. We have worked with the teachers over the past year and have generated a number of case studies. Consider the cases of Greg, Henry and Nancy who we began studying as they entered the formal part of their mathematics education program at the beginning of their senior year at the University of Georgia. In their first mathematics education course Greg and Henry were particularly negative toward the use of technology as a means to teach mathematics--a focal point of the first course. Nancy was quietly accepting. After four months of participating in the teacher education program, Greg become excited about technology and open-ended activities; Henry became confused and despondent and wondered if he should be a teacher. In student teaching, Greg incorporated open-ended activities, particularly those involving technology, into his teaching while realizing for the first time how difficult a job it was to be a teacher. Henry, however, confirmed his beliefs during student teaching that teaching is the efficient telling of a story. He was once again excited about being a teacher and thinks the university mathematics education people have been out of the classroom too long.

Perry's (1971) scheme for describing a person's orientation to authority provides one means of conceptualizing Greg and Henry's development. Both Greg and Henry professed beliefs that reflected their successes as students. The Perry's stage of dualism fit both Greg and Henry. As Greg confronted new ideas, however, no matter how vehemently he objected, he considered these ideas and incorporated them into his belief system through accommodation. Henry, however, rejected and refused to consider alternative views. Greg was more open to new ideas, recognized views other than his own, and evaluated these views, thus reflecting Perry's relativistic stages. Henry remained steadfastly dualistic. Another way to describe the difference in Henry and Greg's belief systems is Kelly's (1955) idea of permeability. Greg's constructs were more open to accommodation in the face of new experiences; that is, they were permeable. Henry, on the other hand, continually ignored the potential of new experiences that did not fit his existing constructs; his constructs were impermeable.

It is our intent not only to characterize their beliefs but also understand more deeply why they are permeable or impermeable. Green's (1971) idea of evidentiality provides additional insight. Henry justifies his beliefs in a circular way--what was good enough for him as a student was good enough for his students. Greg's beliefs about teaching mathematics arise from core or more central beliefs (Green, 1971; Rokeach, 1960) about being prepared for life. His beliefs about the practice of teaching are based in his beliefs about the aims of education. Thus, changing beliefs about practice for Greg involves connecting the new practice to these more central beliefs or challenging the old practice in terms of these beliefs. Greg's beliefs are evidentially-held while Henry's beliefs are non-evidentially-held and thus "cannot be modified by introducing new evidence or reasons" (Green, 1971, p. 48).

Nancy was a very conscientious student, committed to becoming a teacher. She eagerly participated in every activity and admonished her classmates to "think like a teacher" while participating in the activities. During the interviews, Nancy always referred to others for her views--her instructors, a past teacher, her mother. Nancy wanted to know the right way to teach and was looking for that way from an external authority. We might say that Nancy was, like Henry, in Perry's stage of dualism or Belenky's, et al., position of received knowledge. On the surface Nancy and Henry were alike in that they have an externally oriented view of authority. But a deeper analysis reveals that they are fundamentally different. Nancy did not ignore or reject new ideas, but seemed to assimilate ideas without accommodation. The perspective of Belenky et al. (1986) provides what seems to be a better way of thinking about Beth's beliefs, however. As a "received knower," Nancy looked to others as the source of truth. She found strength in having shared beliefs with her friends and when disagreement occurred, she deferred to the instructors or to the majority opinion of her friends. Henry was confident in his self as aligned with authority. Nancy lacked that confidence. For example, Nancy believed it was very important for students to "use their imagination" and "think up their own ways" in mathematics, but she adds, "I'm not sure I have an imagination, I mean, a real strong imagination that will think up new ways to answer problems."

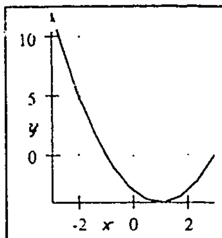
The theoretical frameworks described above are very important for understanding the development of teachers' beliefs and the process of teacher education. The theoretical orientations provide us with dimensions and characteristics to look for and hypotheses for where to look. We can consider how they respond to new ideas and situations (permeability), their basis for belief (evidentiality), connections among their beliefs (quasi-logical system), clustering and isolation of beliefs, and where they go for truth and how they see and respond to authority (Perry and Belenky). This understanding helps elaborate and inform the teacher education process.

Implications for Practice: The Case of the Function Unit

As we develop and use sophisticated ways of understanding the professional development of teachers, it is natural to translate this understanding into improving the practice of teacher education. Consider such a student as Henry who opposed the ideas presented in his teacher education program and left feeling teacher education was a waste of his time. Then, consider Nancy who absorbed all we had to give her but then struggled miserably in her first year of teaching not wanting to let her parents, instructors, peers, and administrators down and wanting her students to like and affirm her. How does a way of understanding their beliefs and development help us provide experiences that will benefit and prepare them for becoming reflective teachers? We have tried to address this question over the past four years by designing and piloting teacher education materials based on research on teachers' beliefs.

If a teacher's beliefs and the nature of those beliefs so affect the teacher's subsequent development, it is important that the teacher educator and the teacher him or herself elaborate those beliefs. If some teachers' beliefs are not open to the challenge of rational argument, those teachers needs active involvement in contexts that will lead them to reconsider their own beliefs. If some teachers lack confidence in themselves as authority, experience with self-defined and open-ended problem-solving situations may help.

As one example of a possible approach we describe the materials developed in our project. The materials are focused on the subject of mathematical function. To help the teachers become more reflective, we begin by having them read and discuss a classroom vignette. Then we have the teacher develop an informal, intuitive understanding by considering dependent-independent variable and correlational relationships in the media and everyday language, e.g., a doctor discusses the effect of a person's cholesterol level on their risk of heart disease. As the teachers think about characteristics of functional relationships, they participate in a classification activity based on Kelly's (1955) repertoire grid technique (see following Figure). (The activity asks the students to classify the six function representations into two piles of three functions each and to describe the basis for their classification.)

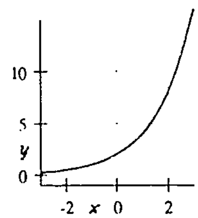


Fred is considering which size pizza is the better buy. He wonders what happens to the area of the circular pizza as the diameter of the pizza changes

The 1990 census shows that Central City has a population of 40,000 people. Recent studies indicate that over the past 20 years the population has steadily increased at an annual rate of 2%. Social scientists predict Central City will experience this same growth rate over the next 20 years. Ms. Callahan has been asked to predict Central City's population for each of the next 15 years

x	-2	-1	1	2	4
y	1	2	-2	-1	-1/4

x	-2	-1	0	1	2
y	1	0	1	4	9



The teachers build on this intuitive understanding of function characteristics through modeling and data collection activities. We give the students an opportunity to challenge preconceived notions about functions by investigating common school mathematics topics (e.g., solving equations and inequalities and such geometric relationships as area and similarity) from a functional approach. The teachers reflect on these experiences through journal writing, and small group and class discussion. Student-conducted interviews is an important activity in our project as we try to help eachers better understand another persons' beliefs as well as reflect on their own understandings.

The open-ended and intuitive activities were particularly important for Greg. As he began to see these activities as important contexts for developing reasoning skills he also saw them as important to his emphasis on preparing people for life. Nancy developed a view of the importance of "imagination" and having students develop "their own ways" of solving problems, possibly indicating the beginnings of a shift from Belenky's stage of received knowledge to subjective knowledge. After participating in the activities, Henry became very uncomfortable with his beliefs, deciding that maybe he had no understanding of how to become a teacher.

Conclusion

It is clear that the activities we developed were not uniformly effective. They enabled Greg to reconceptualize his beliefs about mathematics and the teaching of mathematics. For Henry, they caused considerable consternation as he was never really willing to internalize the activities. Nancy demonstrated growth, albeit hampered by her willingness to acquiesce to authority--a condition that impeded her development in her first year of teaching. What we considered important is that we have an orientation that allows us to conceptualize teacher development and provides us with a basis to reconsider and reshape future teacher education activities. It is that orientation that elevates the art of conducting teacher education to that of a disciplined field of inquiry.

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D. A. Grouws & T. J. Cooney (Eds.), Perspectives on Research in Effective Mathematics Teaching (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics, Lawrence Erlbaum.

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). Women's ways of knowing: The development of self, voice, and mind. New York: Basic Books.

Brown S., Cooney, T., and Jones, D. (1990). Mathematics teacher education. In W. R. Houston (Ed.), Handbook of research on teacher education (pp. 639-656). New York: Macmillan.

Cooney, T. J. (1980). Research on teaching and teacher education. In R. Shumway (Ed.), Research in mathematics education (pp. 433-474). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. (1985). A beginning teacher's view of problem solving. Journal for Research in Mathematics Education, 16, 324-336.

Green, T. F. (1971). The activities of teaching. New York: McGraw-Hill.

Kelly, G. A. (1955). The psychology of personal constructs (Vol. 1). New York: Norton.

Kuhn, T. S. (1970). The structure of scientific revolutions (2nd ed.). Chicago: University of Chicago Press.

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: The Council.

Perry, W. G. (1970). Forms of intellectual and ethical development in the college years. New York: Holt, Rinehart, & Winston.

Rokeach, M. (1960). The open and closed mind. New York: Basic Books.

Schön, D. A. (1983). The reflective practitioner: How professionals think in action. New York: Basic Books.

Thompson, A. (1984). The relation of teachers' conceptions of mathematics and mathematics teaching to instructional practice. Educational Studies in Mathematics, 15, 105-127.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 127-146). New York: Macmillan.

STUDENTS' REPORTS OF THEIR LEARNING ABOUT FUNCTIONS

Crawford, K., Gordon, S., Nicholas, J. and Prosser, M.
University of Sydney La Trobe University

Abstract

The concept of a function is fundamental to mathematics. This paper reports on a study that explores the qualitatively different ways that over five hundred first year mathematics students at the University of Sydney conceptualised functions. An exposition of the phenomenographic approach taken and the methods used to categories the students' responses are reported. The range and distribution of their expressions raises questions about the nature of students' mathematical experiences.

Introduction

This paper reports on part of a larger interdisciplinary project which aims to investigate students' conceptions of mathematics and their approaches to learning it on entering university. The study also explores the relationships between these prior conceptions and approaches and students' experiences of mathematics at university.

In stage one of the project it was found that students entering university held a range of conceptions of mathematics and approaches to learning it (Crawford, Gordon, Nicholas & Prosser, 1993). The students' conceptions of mathematics ranged from viewing it as a fragmented collection of formulae, rules and procedures, to viewing mathematics as a cohesive theory which provides insights into the physical world. The students' approaches to learning mathematics ranged from rote memorisation with the intention of reproducing knowledge to seeking to understand the ideas of abstract mathematics and the situations in which the theory could apply. An analysis of these responses revealed that the vast majority of students viewed mathematics as a necessary set of rules and procedures to be learned by rote. The survey results also indicated a relationship between students' conceptions of mathematics and their approaches to studying mathematics at university level. There was no evidence of gender differences in either conception of mathematics or approach to learning.

The second stage of the project, in addition to clarifying information about students' conceptions of mathematics and their approaches to learning it on entering university, also collected data about their mathematical experiences at university. In particular this paper repc ts on the data analysis and investigations of that part of the project concerned with students' understandings of a fundamental mathematical concept, namely that of a function.

Theoretical background

Experiences with mathematics at school are widely believed to prepare students for later learning at university level. Research, at school and university levels, indicates that learners' previous experiences influence the quality of their approaches to learning, attitudes to and outcomes in learning mathematics (Clarke, 1985; Crawford 1983, 1986, 1990; Resnick, 1987; Ramsden, 1991). There are also indications that students' conceptions of mathematics and their orientations to study affect the quality of cognitive activity and of learning outcomes (Cobb, Yackel & Wood, 1992; Crawford, 1986, 1990; Solomon, 1989). It would be expected that students would enter university with a range of conceptions of mathematics and general predispositions to mathematics learning derived from their school activities.

The interest in the student's awareness, that is the focus of this paper, stems from an acknowledgment that learning is a human activity that occurs in a cultural context. Thinking does not occur in isolation. Phenomenographic research, pioneered by the Gothenburg School (Marton, 1988), describes the qualitatively different ways subjects relate to phenomena. This approach views phenomena systemically and avoids the boundaries in traditional psychology between, thinking, feeling and acting - between person and context. Like Vygotsky (1978), but unlike the constructivist position, there is no assumption of a cartesian duality between self and context; between thinking and acting. That is, socio-cultural and cognitive processes are interdependent and interactive (Vygotsky, 1962).

This relational view of learning has formed the basis of much of the research into student learning in higher education (Gordon, in press; Marton, Hounsell, & Entwistle, 1984; Prosser & Millar, 1989; Trigwell & Prosser, 1991a, 1991b). In summary, that research indicates that learners' experiences should be considered as involving their orientations in context, the ways they relate to the learning environment, their goals or intentions, as well as their learning strategies.

The phenomenographic research approach developed by Marton (Marton and Saljo, 1984; Marton, 1988) has been used in this study to analyse students' conceptions of the notion of a function in their responses to assessment tasks at university level.

Research Method and Results

Over five hundred of the students enrolled in the mainstream first year mathematics course were surveyed. Students from this course are mainly enrolled in degrees in the Faculties of Arts, Engineering and Science, and are majoring in mathematics or may be studying mathematics as a core part of their degree. This course constitutes one quarter of their first year program.

These students were given the following question as part of a "take home" assignment in the second week of the academic year:

Each of the following sets of pairs (x, y) may or may not represent a function from $\{1, 2, 3\}$ to $\{a, b, c, d\}$. Here x comes from the set $\{1, 2, 3\}$ and y comes from the set $\{a, b, c, d\}$.

(i) $\{(1, d), (2, b), (3, d)\}$ (ii) $\{(1, c), (2, a), (3, b)\}$ (iii) $\{(1, a), (3, c)\}$

(iv) $\{(1, a), (1, c), (3, d)\}$ (v) $\{(2, b), (3, c), (1, d)\}$.

(a) Identify the sets that represent functions and state which of these are one-to-one.

(b) Explain clearly why each of the five sets does or does not represent a function.

(c) Explain clearly why each of the five sets does or does not represent a one-to-one function.

The concept of 'function' was selected as this concept is familiar to students from their school studies and is a core topic of the first year university mathematics course. A similar question was asked in the examination at the end of the first semester. This was in an effort to find out if any changes in thinking had occurred after some experience of mathematical activity at university level.

In accord with the phenomenographic approach, particular attention was paid to the ways in which the phenomenon under investigation was categorised and interpreted. A particular concern for qualitative researchers is to develop methods for reaching agreement on shared meanings while recognising that all interpretations are subjective. For this reason considerable time and effort was expended in achieving the major result; the set of categories that describe the data.

An initial meeting was held to discuss preliminary ideas about how this analysis should proceed. The analysis was developed during a number of further meetings. The analysis was done as follows:

A sample of twenty papers was taken away by each of the five researchers, three of whom are mathematicians, and two educators. Student responses were investigated independently by each of the researchers in order to identify a provisional set of categories of description.

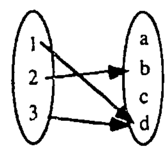
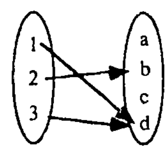
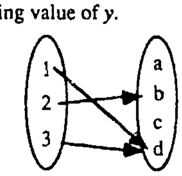
At each meeting the emerging categories were discussed and clarified and 'inter subjective' agreement among the researchers reached. The researchers then independently recategorised the sample data according to the agreed draft categories to test that the meanings of the categories were, in fact, shared. In this way the categories were refined in a cyclical process.

One researcher then categorised all assignment and examination papers. Any paper that was found to be difficult to categorise was discussed by the group and difficulties resolved.

The analysis of the students' written responses to the assessment tasks reported above resulted in the development of a set of categories of description of students' understandings of the concept of a function. Brief descriptions of the categories and illustrations of typical student responses are given in Table 1. It should be emphasised that the categorisation was not affected by whether the student's answer was correct. This focus is fundamentally different from the usual judgements made about a student's mathematical work.

Table 1

Categories of Description of Student Responses to Assessment Tasks about Functions

Category	Illustrative example of student response
A. Choosing and reproducing a representation of a function only	b(i)  Is a function because all numbers in set A have arrows
B. Choosing and reproducing a representation of a function but also showing an awareness of the concept	c(i)  Each element of set x is a unique element of set B, $\therefore$ is a function. ie., $f{:}x \rightarrow y$, where x is a unique element $f(x)$ of y.
C. Expresses the idea of the concept of a function and relates the representation to the concept	c(i) Let $x \in$ A and $y \in$ B. Therefore set (i) does not represent a one-to-one function since when $x = 1$, $y = d$ and when $x = 3$ y is also equal to d. Therefore each value of x doesn't have a different corresponding value of y. 
D. Expresses the idea of the general concept of a function in abstract mathematical terms	c) For f to be a one-to-one function, it first must be a function. So the only sets to consider are those that represent functions: (i), (ii), and (v). (i) This function is not 1:1 as $f(1) = f(3) = d$. (ii) and (v) Both are 1:1 functions as they satisfy $f(1) \neq f(2) \neq f(3)$. In other words, $\forall\, x, y \in$ A, $x \neq y \Leftrightarrow f(x) \neq f(y)$.

In categories A and B, the focus of the student's response was on a representation of the function, for example, an arrow diagram. In categories C and D, the focus was on the general concept of the function itself with possibly the use of a representation for clarification. Hence, the categories can be regarded as inclusive in terms of a growing clarity of the students' expression of the concept and a consequential distancing of the representation.

A quantitative analysis is currently in progress, which will specify the distribution of responses in the categories, both for students at the beginning of their mathematics course and after one semester of university mathematics. However, an initial quantitative analysis indicates that the vast majority of students' responses were in categories A and B.

Discussion

This paper has focussed on both the process of developing categories for the students' responses and on the categories themselves.

The phenomenographic method has provided insights into students' conceptions of a function. The method used for developing categories of description from the data, rather than imposing categories based on theories of learning, for example a SOLO taxonomy, has improved the ability of the researchers to explore students' awarenesses. The use of students' explanations of a mathematical concept, although limited to their awarenesses and abilities to express these in written form during the assessment task, has provided a rich source of information beyond second order interpretation of student behaviour. In addition, the method used to achieve agreement about the categories of description of student explanations provided the interdisciplinary researchers with opportunities for deep reflection and enhanced understanding about student learning or mathematical ideas.

The idea of a function in mathematics is fundamental to the study of mathematics at all levels at university. Therefore, a deep conceptual understanding of a function is essential to the further development of mathematical thinking. The above results indicate that students exhibit a wide range of awarenesses of this concept. At one end of the spectrum, some students' notions of a function are expressed only in terms of a single aspect of it, namely a representation. At the other end of the spectrum, some students are able to abstract the essence of a function and are able to apply this generality to a specific context by using appropriate representations. Further, the use of the language of mathematics by these students is an indication that their concept development is well underway. This range of student awarenesses is not likely to be apparent if the assessment tasks given to students only require the reproduction of mathematical techniques.

Furthermore, the results also indicate that the majority of the university students surveyed focus on the representation of a function rather than the essential meaning. This raises concerns that students' experiences of mathematics predispose them to a fragmented operational approach to mathematical activities. That is, students focus on the immediate mathematical task rather than trying to make sense of the mathematical idea underlying the task. Moreover, a concentration only on mathematical techniques in assessment tasks may inhibit students from developing a meaningful understanding of mathematical ideas. The challenge for university educators is to find ways of providing experiences that encourage students to seek a cohesive and meaningful interpretation of mathematical ideas.

Acknowledgments

This research project was financially supported by a University of Sydney Research Grant, and a grant from the Faculty of Science, University of Sydney.

The authors thank the staff of the School of Mathematics, University of Sydney, for their cooperation and interest, and Mr Duncan Turpie for his valuable research assistance.

References

Clarke, D.: 1985, 'The impact of secondary schooling and secondary mathematics on student mathematical behaviour', *Educational Studies in Mathematics*, 16, 231–51.

Cobb, P., Yackel, E. and Wood, T.: 1992, 'The constructivist alternative to the representational view of mind in mathematics education', *Journal for Research in Mathematics Education*, 23, 2–33.

Crawford, K.: 1983, 'Past experiences and mathematical problems solving', *Proceedings of the Seventh Annual Conference of the International Group for the Psychology of Mathematics Education*, Israel.

Crawford, K.: 1986, 'Cognitive and social factors in problem solving behaviour', *Proceedings of the Tenth Annual Conference of the International Group for the Psychology in Mathematics Education*, London.

Crawford, K.: 1990, 'School definitions of work: their impact on the use of technology in mathematics education', *Proceedings of the Conference on the Political Dimensions of Mathematics Education*, London Institute of Education.

Crawford, K., Gordon, S., Nicholas, J. and Prosser, M.: 1993, 'Learning mathematics at the university level: Initial conceptions of mathematics', *Proceedings of the 16th Annual Conference of the Mathematics Education Research Group of Australasia*, Brisbane, 209-14.

Gordon, S.: in press, 'Mature students learning statistics: the activity theory perspective', *Journal of the Mathematics Education Research Group of Australasia*.

Marton, F.: 1988, 'Describing and improving learning', in R. Schmeck, (ed.), *Learning Strategies and Learning Styles*, Plenum Press, New York.

Marton, F., Hounsell, D. and Entwistle, N.: 1984, *The Experience of Learning*, Scottish Academic Press, Edinburgh.

Marton, F. & Saljo, R: 1984, 'Approaches to learning', in F. Marton, D. Hounsell and N. Entwistle, (eds.), *The Experience of Learning*, Scottish Academic Press, Edinburgh.

Prosser, M. and Millar, R.: 1989, 'The how and what of learning physics', *The European Journal of Psychology of Education*, 4, 513–28.

Ramsden, P.: 1991, *Learning to Teach in Higher Education*, Routledge, London.

Resnick, L.: 1987, 'Learning in and out of school', *Educational Researcher*, 16 (9), 13–20.

Solomon, Y.: 1989, *The Practice of Mathematics*, Routledge, London.

Trigwell, K. and Prosser, M.: 1991a, 'Relating approaches to study and the quality of learning outcomes at the course level', *British Journal of Educational Psychology*, 61, 265–75.

Trigwell, K. and Prosser, M.: 1991b, 'Relating learning approaches, perceptions of context and learning outcomes', *Higher Education (Special Edition on Student Learning)*, 22, 251–66.

Vygotsky, L.S.: 1962, *Thought and Language*, MIT press, Cambridge, Mass.

Vygotsky, L.S.: 1978, *Mind in Society*, M.Cole et al, (eds.), Harvard University Press, Cambridge, Mass.

Algebra, Symbols, and Translation of Meaning

Lillie Crowley
Engineering Division
Lexington Community College
Kentucky, USA

Michael Thomas
Department of Education
University of Auckland
New Zealand

David Tall
Mathematics Education Research Centre
University of Warwick
United Kingdom

In translating problems from a verbal description to algebraic notation, we consider what children will do with statements that do not have the syntactic constructions that may provoke the reversal errors of the "students–professors problem." MacGregor & Stacey (1993) show that children do not follow a simple word-matching order and postulate the existence of mental models that can be accessed in any order. We hypothesise that the children's responses are a natural consequence of their previous development and show that, as the verbal problems become more complex, children are more likely to revert to a process-orientation with the arithmetic operation on the left, rather than an assignment order with the variable on the left. We place this in a theory of cognitive compression as the children grow in mathematical sophistication.

Introduction

The "students and professors" problem of Clement, Lochhead and Monk (1981) has unleashed a profusion of articles formulating and testing theories about conceptualisation of symbols in algebra (see Laborde 1990 for a summary). Given the problem of translating "There are six times as many students as professors" into symbols using S for the number of students and P for the number of professors, 37% of college students sampled were in error and two thirds of these wrote $6S = P$, rather than $S = 6P$. This *reversal error*, has been identified by some as occurring because the letters are thought to stand for the objects (students, professors) rather than the *number* of objects and Kaput (1987) suggested that this was strongly influenced by the underlying language *syntax* where the "$6S$" suggested an adjective-noun structure meaning "six students". This leads to a *word order matching error* (or *syntactic translation*) which essentially follows the word order of the original problem (e.g. Schoenfeld, 1985; Mestre, 1988). Herscovics (1989) contrasted syntactic translation with *semantic translation*, which attempts to interpret the underlying meaning.

MacGregor & Stacey (1993) asked students to translate more simple statements into algebra and reported that there were responses, including errors which could not be adequately described by syntactic translation. They concluded that the students must have "cognitive models" including some easily described in natural language (such as "I have $6 more than you") but less amenable to translation into mathematical code:

> The variety and form of students' responses leads us to infer some properties of their cognitive models and to postulate that information from these cognitive models can be retrieved in any order. Such a retrieval process would explain the apparently random choice of responses that match or do not match the word order. (MacGregor & Stacey, 1993, p. 228.)

The evidence that we have partly supports this view. However, we do not consider that *all* children have cognitive models which can be retrieved in any order. Instead we suggest that the models developed by children are a result of their previous experience which requires several reconstructions to allow them to develop models with a flexible use of order of access. This relates to their learning experiences in mathematics, their use of symbols, and the natural process of compression of knowledge with growing maturity.

A theory of learning in arithmetic and algebra

We begin with the simple observation that the human brain has a huge long-term memory, but a limited short-term working store. It can therefore store vast quantities of information, but can only manipulate a small number of items consciously at any time. To cope with this limitation in mathematics, two strategies are adopted, one to routinise procedures so that they can be performed using little conscious memory, and a more powerful one using symbols to label complex information allowing a small number of symbols to be the . cus of attention at any one time. These symbols are used by experts in a particularly flexible way in arithmetic and algebra by labelling a *process* with a *symbol* which then is also used for the *concept* produced by that process.

Perceiving a symbol as either process or concept gives great power to the individual, for the *process* enables him or her to *do* mathematics, but the *concept* allows him or her to *think* about it and manipulate it mentally. A symbol standing for both process and the output of that process is called a *procept*, with the additional factor that two different symbols which represent the same object can be regarded as the same procept (Gray & Tall, 1994). Mental manipulation of procepts gives the thinker great power. The flexibility in thought grows over time as the individual compresses the process to allow it to be thought of as an object. The proceptual child grows to regard different symbols for the same thing as being essentially the same object, whilst the procedural child persists longer in regarding them as different processes (Gray 1993).

Early arithmetic involves the *process of counting* which becomes compressed into the *concept of number* with *number symbols* fulfilling the pivotal role. The expression "5+4" can mean different things to different children at different stages, including:

(a) count-all (count 5 objects, then 4, then count them all),

(b) count-on (count on four starting after 5 ("six, seven, eight, *nine*"),

(c) known fact (the answer is 9),

(d) derived fact (e g. "I know 5 and 5 is 10, so 5 and 4 is one less").

A child may have one or more of these interpretations at a given time. Those who only count-all may know a few facts, but are unlikely to derive facts (Gray & Tall, 1994). Some children remain procedural and prefer the security of counting-on to solve numerical problems. More successful children leave count-all behind and compress knowledge with great flexibility and fluency using a combination of (b), (c) and (d).

Depending on the development of the child in this sequence of learning, the following equations may have very different meanings:

(i) $5+4=\square$, (ii) $5+\square=9$, (iii) $\square+4=9$, (iv) $\square=5+4$.

The first of these can be answered by any of the four named processes of addition, the second is difficult with count-all, but straightforward with the other techniques (for instance, count-on from 5, and count how many are required to reach 9.) The third is difficult with count-on because the child does not know where to start the count (Foster in prep.). The fourth may make little sense to any child who reads an equation as a left-to-right process to given an answer (Kieran, 1981). For such children there is a preference to express the equation 9=5+4 as an addition process 5+4=9 (5 and 4 *makes* 9). In practice, the difference between (ii) and (iii) is ephemeral, and children soon see it to be equivalent to a subtraction, but (iv) continues to surface strongly later in algebra.

Many children have difficulty with a symbol such as "$x+3$" which they will not accept as an answer because they expect a number (Kuchemann, 1981). From our viewpoint, such children see the symbol $x+3$ as a process and not a mental object – a process they cannot carry out because they do not know what x is. To be able to cope with such a symbol requires not only that it be given a meaning, but that the meaning should cope with it both as a process (of evaluation when x is known) and also as an object which can be manipulated as it stands. It requires the flexibility which views the symbol as a procept.

Such a view is not available to a child who regards number operations only as counting procedures. Regrettably, in traditional teaching, faced with children in difficulty and lacking in comprehension, the way out is often "fruit salad" algebra. The symbol "3a+4b" is explained to stand for "three apples and four bananas". Some children who play along with this delusion are able to sort out "3a+4b+2a" as "three apples and four bananas and two apples", which is "five apples and four bananas", or "5a+4b". So they appear to be able to simplify expressions. But they now have an image of a letter as representing an object, set up ready to fall foul of the student-professor problem. Other children might simply conclude that "3a+4b+2a" is "nine apples and bananas" and – since they have no mathematical symbol for "and" – they may write the letters one after another, *conjoining* them as "9 a b". (see, for example, Booth, 1984).

Algebra teaching further exacerbates the differences between proceptual thinkers with their flexible use of symbolism and procedural thinkers who try to give it a temporal, process meaning. The equation symbolism is reversed to express y as a function of x in the form $y=x+4$. This *assignment order* causes no problems to the proceptual thinker but violates the meaning of the "equals" sign for the procedural child who may prefer the *process-oriented order* $x+4=y$. Aesthetic preferences are introduced – for instance, it is "usual" to write $y=3x+4$ rather than $y=4+3x$, although it is preferable to write $y=4-3x$ rather than $y=-3x+4$, to avoid starting an expression with a minus sign. Flexible thinkers are more likely to take such things in their stride, coping with both the flexibility of meaning of the symbols and arbitrary matters of taste.

We hypothesise that proceptual thinkers will have cognitive models that show flexibility in meaningful re-ordering of algebraic symbols, coloured by personal preferences based on aesthetics and personal experiences. But procedural thinkers are more likely to prefer a *process-oriented order* which we hypothesise will become more dominant as the complexity of problems increases.

Empirical Data

The first problem of MacGregor and Stacey (1993) is:

> z is equal to the sum of 3 and y. Write this information in mathematical symbols.

The responses from 255 Year 9 students (aged 13 to 14) of mixed ability in Australia and their interpretation of the responses as (possibly) syntactic or not are as follows.

Response (possibly syntactic)	Frequency
(a) $z = 3 + y$	70
(c) $z = y + 3$	0
(e) $z = 3y$	37
(g) [unclassified]	27

Response (not syntactic)	Frequency
(b) $3 + y = z$	66
(d) $y + 3 = z$	9
(f) $3y = z$	36
(h) [no attempt]	10

Table 1

Given that the problem is phrased in the same word order as a possible algebraic solution, our viewpoint would interpret that response (a) is a natural direct translation but (b) and (d) correspond either to flexible proceptual equivalents or to process orientation. Given the total lack of responses in (c), which represents the aesthetic order of linear equations, we infer that (b) and (d) are more likely process orientation rather than random proceptual flexibility. We would conclude that items (e) and (f) are more likely to involve primitive conjoining rather than mistaken multiplication. The number of responses in the last four (43%) show a horrendously large number of children who seem be seriously handicapped in learning algebra.

We designed an investigation in which three addition and three multiplication questions increased in difficulty, with the first two in each case having syntax coinciding with algebraic assignment order and the third was more complex so that the syntax did not easily suggest a specific order. The problems were given to two groups whom we hoped would be less likely to produce conjoining and non-responses. One was a group of 75 Year 9 students (age 13 to 14) in a highly selected school in Britain representing roughly the top 35% of the total population. The other was a group of 128 second year university students training to be teachers of children aged 4 to 12, representing the top 20% of the population overall. (These teachers are not mathematics specialists and the vast majority will not have studied mathematics for over three years.) The responses of three questions similar to those of McGregor and Stacey are given in Table 2, the codes s, **a**, **p**, **r**, ✔, ✘, standing for syntactic, assignment-oriented, process-oriented, reversal, correct and incorrect, respectively.

Item	Response		School	University
(1) y is equal to x plus four.	$y=x+4$ [$y=(x+4)$]	as✓	65	107
	$y=4+x$	a✓	0	0
	$x+4=y$	p✓	9	15
	$4+x=y$	p✓	0	2
	$y=x+4$ or $x+4=y$	a✓	1	3
	other	✗	0	0
	no response	✗	0	1
(2) w is equal to the sum of 3 and n	$w=3+n$ [$w=(3+n)$]	as✓	52	91
	$w=n+3$	a✓	4	2
	$3+n=w$ [$(3+n)=w$]	p✓	10	30
	$n+3=w$	p✓	2	2
	$w=3+n$ or $3+n=w$	a✓	0	1
	other [e.g. $3n=w$, $w=3$]	✗	7	2
	no response	✗	0	0
(3) A school has v girls and t boys. There are ten more girls than boys. Write an equation relating v and t.	$v=t+10$ [$v=10+t$, $t=10-v$]	a✓	41	33
	$t+10=v$ [$10+t=v$, $v-10=t$]	p✓	24	44
	$v+10=t$ [$10+t=v$]	pr✗	7	24
	$t=v+10$ [$v=t-10$]	ar✗	1	4
	other [e.g. $10t=v$, $t=10v$]	✗	2	20
	no response	✗	0	3

Table 2 Translations of verbal addition problems into algebra

Note that, in the first two questions the vast majority follow the given word order $y=x+4$ and $w=3+n$. *None* reverse the right hand side of (1) as $y=4+x$, but a small number reverse the word order in (2) to give $w=n+3$ (the aesthetic order). There are also *no reversals* of the roles of the variables (the syntax does not encourage reversal). The majority of the remainder in both questions keep the word order ($x+4$ or $3+n$) of the sum, but follow the procedural order. None of the students fail to answer question (1) correctly (it uses the familiar x and y in the correct order) but a small minority produce incorrect solutions to question (2) including conjoining. Notice too that the number reverting to the procedural solution increases as the problem becomes less familiar. The verbal problem (3) is more complex. There is no clear equation syntax and the solver must relate several different pieces of information in the mind at once – the number of girls (v), the number of boys (t) and deduce that the second sentence means t is $10+v$. This will place stress on short-term working memory and there are more errors.

Tables 3 and 4 show the diminution in algebraic assignment and the increase in process-orientation as the complexity increases. Using a χ^2 test with continuity correction, the change in total process-oriented from (1) to (2) is statistically significant ($p<0.05$) amongst the university students and from (2) to (3) is highly significant ($p<0.01$) in both groups. When the first errors occur in (3), the process reversals exceed the assignment reversals. The difference is significant ($p<0.05$) calculated as a subset of the total school population, highly significant ($p<0.01$) of the corresponding university population. In addition, the proportion of those using the algebraic assignment making an error is smaller than the proportion of those using the process orientation. This difference is highly significant in the school students and significant in the university students.

Assignment oriented	School (N=75)			University (N=128)		
	All (as+a+ar)	as✓+a✓	ar✗	All (as+a+ar)	as✓+a✓	ar✗
(1)	65 (87%)	65	0	107 (84%)	65	0
(2)	56 (75%)	56	0	94 (73%)	94	0
(3)	42 (56%)	41	1	37 (29%)	33	4

Table 3 . assignment oriented notation and reversals in (1). (2). (3)

Process oriented	School (N=75)			University (N=128)		
	All (p+pr)	p✓	pr✗	All (p+pr)	p✓	pr✗
(1)	9 (12%)	9	0	17 (13%)	17	0
(2)	12 (16%)	12	0	32 (25%)	32	0
(3)	31 (41%)	24	7	68 (53%)	44	24

Table 4 .process oriented notation and reversals in (1). (2). (3)

Table 5 has three multiplicative questions increasing in order of difficulty. The simple multiplication problem (4) induces more errors than the addition problems (1) and (2) the majority are pr ess-oriented in the form $5m=n$ which multiplies the first two items in the sentence (m and 5). but turns them into the accepted ordering $5m$. Question (5). though complex uses pop groups familiar to students and has the word order in the same order as a possible syntactic solution. Question (6) does not have a simple syntactic translation into algebra and causes even more errors.

Item	Response			School	University
(4) m is 5 times n.	$m=5n$	$[m=n\text{x}5]$	as✓	49	89
	$5n=m$	$[n\text{x}3=m]$	p✓	19	26
	$5m=n$	$[m\text{x}3=n]$	pr✗	5	13
	$n=5m$		ar✗	1	0
	other		✗	1	0
	no response		✗	0	0
(5) A record by Take That is h minutes long. A record by Kriss Kross is g minutes long. The Take That record is three times as long as the Kriss Kross. Write an equation relating h and g.	$h=3g$	$[h=3\text{x}g, h=g\text{x}3]$	as✓	33	45
	$3g=h$	$[3\text{x}g=h]$	p✓	31	50
	$3h=g$	$[3\text{x}h=g, h\text{x}3=g]$	pr✗	9	25
	$g=3h$	$[g=h\text{x}3]$	ar✗	1	5
	other		✗	1	3
	no response		✗	0	0
(6) A Band makes four times as many singles as albums. It makes q albums and z singles Write an equation relating q and z.	$z=4q$	$[z=q\text{x}4, z=q4]$	a✓	20	13
	$4q=z$	$[4\text{x}q=z, q\text{x}4=z]$	p✓	23	59
	$4z=q$	$[4\text{x}z=q, z\text{x}4=q]$	pr✗	20	38
	$q=4z$	$[q=4\text{x}z]$	ar✗	12	13
	other		✗	0	2
	no response		✗	0	3

Table 5 Translations of verbal multiplication problems into algebra

Tables 5 and 6 show the decrease in algebraic assignment and the increase in process-orientation as the problems become more complex. Once again, as difficulty increases, the numbers in all of these categories increase. The change in total process-oriented from (3) to (4) is highly significant ($p<0.01$) in both groups and from (5) to (6) is highly

Assignment oriented	School (N=75)			University (N=128)		
	All (as+a+ar)	as✓+a✓	ar✗	All (as+a+ar)	as✓+a✓	ar✗
(4)	50 (7%)	49	1	89 (70%)	89	0
(5)	34 (45%)	33	1	50 (39%)	45	5
(6)	25 (33%)	20	5	26 (20%)	13	13

Table 5 Assignment oriented notation and reversals in (1). (2). (3)

Process oriented	School (N=75)			University (N=128)		
	All (p+pr)	p✓	pr✗	All (p+pr)	p✓	pr✗
(4)	24 (32%)	19	5	39 (30%)	26	13
(5)	40 (53%)	31	9	75 (59%)	50	25
(6)	43 (57%)	23	20	97 (76%)	59	38

Table 6 . Process oriented notation and reversals in (1). (2). (3)

significant amongst the university students. The changes in the process-reversal numbers is significant from (4) to (5) in the university students and from (5) to (6) in both. The number of assignment reversals is always smaller than the number of process reversals, at a level which is significant in school in (4) and highly significant in all other cases. As proportions of those using the assignment order or process order, the errors on the latter are highly significant in (4), significant in school in (5), highly significant in university in (5), but not significant in either in (6). The last statistic simply states that, although the figures *look* different, this difference could have occurred by chance in more than one trial in twenty.

Discussion

As problem statements move from being syntactically equivalent to the algebraic assignment formulation to more complex statements, the student responses increase in their use of process-oriented statements with the operation on the left and the answer on the right. In cases where the syntax becomes too complicated to support a straight translation and the words are used in a way which does not encourage the use of letters as objects, errors which reverse the roles of letters still occur. Such errors may occur because of the cognitive complexity rather than a specific syntactic misconception.

A more detailed case analysis will be required to distinguish between those who think in a flexible proceptual manner and those who are more procedural. Flexible thinkers may respond in either the assignment form or a process-oriented equivalent, whilst the procedural child is likely to learn the algebraic assignment mode as a given procedure However, the regression to process-oriented representation with its higher level of failure is consistent with the fact that the method corresponding to earlier experiences in arithmetic is evoked in times of stress and proves to be more prone to failure. We see children responding to the translation process not only in purely syntactic word order, but also by attempting to make sense of the data using arithmetic and algebraic constructs related to their stage of development in algebraic sophistication.

References

Booth, L. R. (1984). *Algebra: Children's Strategies and Errors*. Windsor, Berks: NFER-Nelson.

Clement, J., Lochhead, J. & Monk, G.S. (1981). Translation Difficulties in Learning Mathematics, *American Mathematics Monthly*, 4, 286-290.

Dubinsky E. (1993). Computers in teaching and learning discrete mathematics and abstract algebra. *Advanced Technologies in the Teaching of Mathematics and Science* (ed. D. L. Ferguson). Springer-Verlag, 525–563.

Foster, R. (in preparation): *It's that goat page!*, available from Math. Ed. Res. Centre, Warwick.

Gray, E. M. & Tall, D. O. (1993). Success and Failure in Mathematics: The Flexible Meaning of Symbols as Process and Concept, *Mathematics Teaching*, **142**, 6–10.

Gray, E.M. & Tall D.O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *The Journal for Research in Mathematics Education*, 26 (2), 115–141.

Gray, E. M. (1993). Count-on: The parting of the ways for simple arithmetic, *Proceedings of the Seventeenth International Conference of PME*, Tsukuba, Japan, 2, 204–211.

Herscovics, N. (1989). Cognitive Obstacles Encountered in the Learning of Algebra. In S. Wagner & C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*, NCTM, Reston, Virginia, 60-86.

Kaput, J. J. (1987). Towards a Theory of Symbol Use in Mathematics. In C. Janvier (Ed.) *Problems of Representation in the Teaching and Learning of Mathematics*, Hillsdale NJ: Erlbaum, 159–195.

Kieran, C. (1981). Concepts Associated with the Equality Symbol, *Educational Studies in Mathematics*, 12, 317-326.

Kuchemann D. E. (1981). Algebra. In K. M. Hart (Ed), *Children's Understanding of Mathematics : 11-16*, London: John Murray, 102–119.

Macgregor, M. & Stacey, K. (1993). Cognitive Models Underlying students' Formulation of simple Linear Equations, *Journal for Research in Mathematics Education*, 24 (3), 217-232.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*, New York: Academic Press.

Laborde, C. (1990). Language and Mathematics, in P.Nesher & J. Kilpatrick (Eds), *Mathematics and Cognition*, CUP, Cambridge, 53–69.

Mestre, J.P. (1988). The Role of Language Comprehension in Mathematics and Problem Solving. In R. Cocking & J. Mestre (Eds.), *Linguistic and Cultural Influences on Learning Mathematics*, Erlbaum, Hillsdale, NJ, 201–220.

CHILDREN'S USE OF ALTERNATIVE STRUCTURES[1]

Alice Alston, Robert B. Davis, Carolyn A. Maher, and Amy Martino
Rutgers University
New Brunswick, NJ, USA

Students simultaneously engage in building up mental representations for pieces of mathematics, extending these representations, selecting among them, mapping problem data into a representation, and making mappings between one representation and another. Data from task-based interviews with fourth and sixth grade children demonstrate the complexity of this process, especially in the coexistence of alternative representations that are simultaneously being constructed and may be preferentially selected for use in any specific task situation. (The data collected includes videotapes, student written work, and observer notes.)

It is widely recognized that students build up a sequence of different ways of thinking about mathematical situations. Fractions, for example, are typically thought of as operators for an extended period of time, often years, before they come to be thought of as numbers -- that is, a fourth or fifth grader, asked to explain what "fractions" are, will typically answer in terms of "one half of" something, or "one fourth of" something. Careful study of videotapes of fourth, fifth, and sixth graders working on mathematical tasks show two modifications to this traditional picture: first, there are many very small developments that represent a kind of "fine structure", a set of smaller advances sandwiched in between the more familiar stages; second, there is not a one-directional development, but on the contrary many different kinds of representations are being developed (and used) at the same time. Although the work reported here deals with fractions and combinatorics, we do not doubt that similar developments could be found with younger students, if simpler mathematical content were used.

"Counting" vs. "quantity". Our tapes show instances of fourth graders working with Cuisenaire rods, who will say that a white rod (1 cm. long) is "one-fifth" if the yellow rod (5 cm. long) is called "one", and two white rods placed end to end are "two fifths", but they refuse to call a red rod (two cm. long) "two fifths", on the grounds that "there aren't two, there's only one [thing]." We identify this as a counting stage, which quickly (often by the next lesson) gives way to a quantity stage, where the child recognizes that a train of two white rods has the same length as a train consisting of a single red rod; the child changes from the criterion of counting separate pieces, to the more abstract notion of focusing on the quantity of over-all length. While we have not gone back to study a similar phenomenon with younger children, there must surely be a stage where a young child will accept seven pennies as "seven cents", based on actual counting, but will not accept a nickel and 2 pennies as "seven cents", because there aren't seven observable separate entities. This is not an isolated case. Careful analysis of videotaped work by children continues to show us small "mini-stages" that are sandwiched in between any stages we had previously identified. These "mini-stages" go by quickly, often by the next lesson, and sometimes within a minute or two.

But the language of "stages" may be misleading, since it suggests a one-directional monotonic development. In fact, many different ways of thinking about a situation are being developed more or less simultaneously, although the development of some will proceed more quickly than the development of others. In any specific task, there is consequently a need for the student to select which representation he or she will use.

This simultaneity, and the need for choice, is illustrated in the work of M., a sixth-grade boy, when (during a videotaped task-based interview) he was asked to explain what a "fraction" is, and to mark some fractions on a number line.

M.: Fractions -- fractions are almost like dividing -- and if you have -- like one-fourth -- one fourth -- and the whole total is about 100 -- so you say like one-fourth of 100 ... one-fourth of 100 is 25.

M. did all of this easily and confidently, and (as we shall see) his confidence was not misplaced. He had a good command of many alternative ways of thinking about fractions. In this part of the interview, however, we would classify this as <u>thinking about fractions as operators</u>, because he explained "one-fourth" in terms of "one-fourth <u>of</u> 100".

Asked if he could use any of the available manipulatable materials to say more about fractions, M. put down a yellow rod, and used white rods to mark off fifths [see Figure 1].

Figure 1. M.'s drawing to show fifths, including a red rod called "two fifths", and 1 2/5.

M. did not hesitate to label the red rod as "two-fifths", despite the fact that there was only a single piece of wood; he was clearly thinking in terms of the quantity <u>length</u>, and not counting actual pieces of wood. He also had no difficulty in deciding to name the black rod "one and two-fifths". All of this seems to show the beginnings of another way of thinking about fractions, as number-names for accumulations of the quantity length -- a step toward being able to think about fractions as names for points on a number line, without regard for any explicit inclusion of the operator meaning. (Of course, as we shall see presently, the operator meaning is always present implicitly, as it is even for sophisticated adults.)

Asked to mark some points on a number line, M. drew the picture shown in Figure 2, explaining that the line went on forever, in the positive direction to the right, and in the negative direction to the left. It was on his own initiative that M. chose to mark negative numbers as well as positive ones.

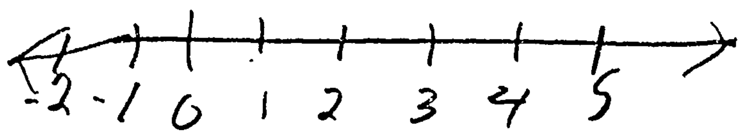


Figure 2. M.'s first drawing of a number line. The inclusion of negatives came from his own initiative.

We interpret a student's ability to mark points on the number line with apparent ease as evidence that the student is, to some extent at least, able to think of numbers rather than merely operators. But what will happen next in this interview, when M. is asked to mark some fractions on his number line?

Interviewer: Can you show me one-half on this number line [referring to Figure 2]?
At first M. declined to mark any fractions on his original number line, saying quietly:
Since it [the line] goes on [as he had previously pointed out], you can't find one-half.

Here we get two powerful clues that M. is thinking about fractions differently from integers; first, he does not say "You can't mark one-half." Instead, he says "...you can't find one-half", language that you might use in "finding one-half of something". But even more important is his refusal to mark 1/2. In many tapes we see children think about "1/2" not as a point that can be marked on the line in the same way that integers can be, but as a fractional part of the entire line segment itself. One might suspect that M. is following this same line of thought -- witness his objection that you can't find half of an infinite line. But even clearer clues are soon to follow.

Instead of marking fractions on his original number line, M. started a new number line, on which he first marked a clear "beginning" - so that, in effect, he was drawing a ray - and on this ray he marked a point labeled "1". He then proceeded to use these two points (the unlabeled "beginning" and the point labeled "1") much as an experienced adult would - that is to say, he made use of the operator meaning to locate 1/5, 2/5, etc., as fractional parts of the unit segment defined by the "beginning" and the point labeled "1". The interviewer asked about what M. was calling "the beginning [of the ray]", and M. said it should be labeled "0", and marked it accordingly. (See Figure 3.)

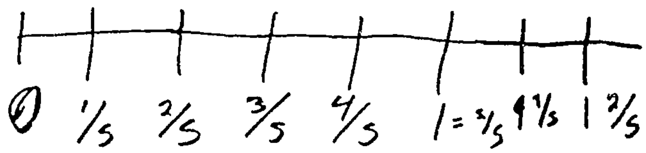


Figure 3. In order to mark fractions, M. drew this new number line. Note the absence of arrows. M. refers to the left-hand end of this ray as the "beginning", and later labeled it "0".

Is M. treating fractions any differently from integers? Thus far in the interview the evidence might still be seen as inconclusive. The interviewer, however, was persistent:

> Interviewer: That means you can't put fractions on that one [referring back to the original number line, shown in Figure 2]? Why not? Or can you?

M. now draws a third number line, and labels it in a quite remarkable way -- see Figure 4.

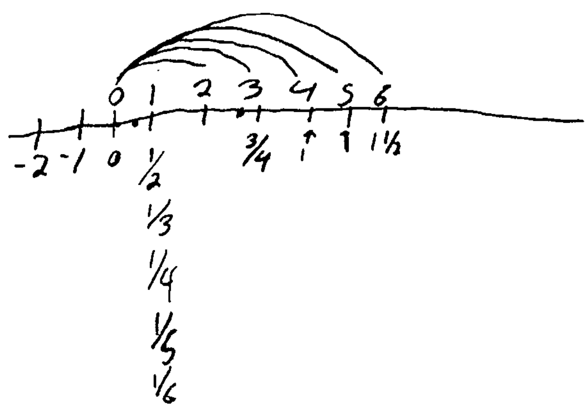


Figure 4. In this single diagram, M. treats integers in one way, and fractions entirely differently. He does not relate one system to the other, nor (apparently) see any need to do so.

M. explained this picture by saying: "This [indicating the point labeled zero] represents zero. ... Between **this** [the point "0"] and **this** [the point 2], **this** [the point labeled "1"] would be one-half. " [M. draws an arc linking "0" and "2".]

But now M. goes on to extend this idea: "Between **zero** and **three**, **this** [the point labeled "1"] would be one-third. And between **zero** and **four**, it [that same point, labeled "1"] would be one-fourth." [M. draws arcs corresponding to each pair of points that he names.] "And 5 -- one-fifth. And 6 [drawing an arc from "0" to "6"], one sixth. And seven -- ... you keep on adding one ..."

So we see that, within a single picture, M. treats integers differently from the way he treats fractions. The integers are true numbers, in the sense that he is comfortable using them as labels for points on the line; fractions, by contrast, are still operators, and he locates them in reference to an assumed segment, not as marks on an infinite line.

The tape shows M. at first refusing to mark fractions at all on his original line. When pushed by the interviewer, M. pauses to consider, then sees a new possibility: he can choose "0" as the left-hand end of a segment, and choose a right-hand end as any positive integer. With this segment as his unit, he can use the operator meaning of a fraction in order to mark fractions on the number line. Of course, this gives him, on one and the same line, a multiplicity of different labeling systems. The interview shows M.'s great resourcefulness in creating a whole infinity of different answers to the request to mark fractions on the line, but it also shows him not yet ready to consider the relations between these different systems. (The interviewer pursued this, which we report in a longer version of this paper, to appear elsewhere.)

Children using -- simultaneously! -- many different representations is not uncommon. Indeed, as one analyzes many instances, it seems clear that this multiplicity has to be the rule, not the exception. The reader may ask themselves how they would locate, say, 2 1/3 on a number line - surely by making use of the operator meaning, applied to the interval (2,3). Learning mathematics involves building up many different representational capabilities, and using many of them more-or-less simultaneously, even while this act, itself, needs to lead to extensions and modifications of the various representations.

Our taped interviews show even more elaborate instances of children using different representations, and making mappings from one to another. Brandon, a 4th grader, had completed the task of finding all possible towers, 4 cubes tall, if cubes were available in 2 different colors. Five months later he was asked to find how many different pizzas could be made, if 4 different toppings were available, and one could also order a "plain" pizza with no topping at all. To solve the pizza problem, Brandon made a table, listing the toppings - "P" for peppers, "M" for mushrooms, "S" for sausage, and "Pepperoni" for pepperoni - as column headings. He then represented each possible pizza as a row in his table, using "0" to indicate the absence of that topping, or "1" to indicate its presence (Figure 5).

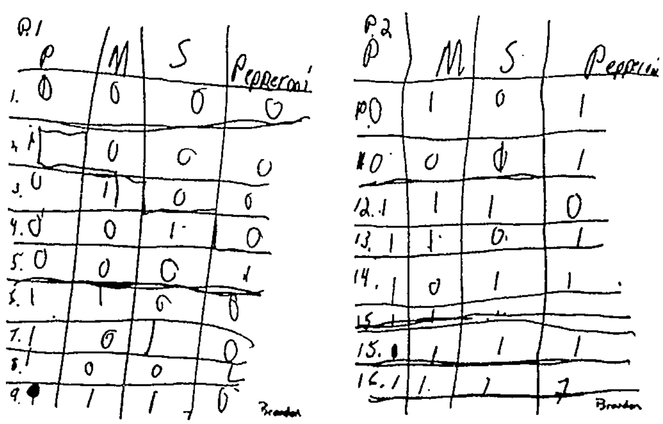


Figure 5. Brandon's chart, showing every possible pizza. Each pizza appears as a row; columns show toppings. In any row, a zero means that that topping is not present, a "1" means that it is.

The interviewer asked Brandon if this reminded him of anything; Brandon responded that it seemed similar to the tower problem [that he had worked on months earlier]. Brandon was not, at first, able to say how the two problems were similar, because he had, indeed, solved them by quite different methods (see Maher et al., 1993). The interviewer was persistent, and Brandon reworked the tower puzzle, using red and yellow cubes. After some intermediate steps, Brandon saw that he could establish a correspondence, with red cubes in towers corresponding to "0's" in the table, and with yellow cubes corresponding to "1's". As Brandon said: "I finally found out what the red would be ... the red's the 'zero guy'." Brandon had established a correct isomorphism between the set of possible pizzas, represented as rows in his chart, and the set of possible towers that were 4 cubes tall - a feat that was possible because of the "0, 1" notation that Brandon had invented to keep track of all possible pizzas. In dealing with two different combinatorics tasks, Brandon had invented two different representational structures, and was then able to show that they were, in fact, equivalent. As with M.'s attempts to mark fractions on a number line, Brandon's correspondence was not achieved without effort; he went through many different selections of possibly-suitable representations and correspondences between representations -- for example, at one point seeming to invoke the notion of zero as "nothing", and arguing that, in the case of the towers, (0, 0, 0, 0) would correspond to no tower whatsoever, before changing the way he mapped the reality of towers into an abstract scheme, and concluding that "the red's the 'zero guy'." Consequently, (0, 0, 0, 0) would mean a tower that contained no yellow cubes -- i.e., a tower that consisted only of red cubes.

Brandon's ingenuity may be unusual, but the general shape of his work is not. Students typically construct or retrieve many alternative representations, and make many trial mappings between reality and representations, or between different representations, as they work to develop better ways of thinking about mathematical situations. Careful readers may have noted many more schemas than we have space to identify -- for example, the "mid-point schema", relating "one-half" to a variety of different segments. Not only "large" schemas are involved, but also a great many "smaller" ones. (The general theory of selecting among alternative mental representations is presented in detail in Davis, 1984. For relevant work on fractions [or rational numbers], see Freudenthal, 1983; Lesh & Landau, 1983; and Streefland, 1991. For the study of the minute details of a student's thinking, see Schoenfeld et al., 1993.)

Note: 1. This work was supported in part by grant MDR 9053597 from the National Science Foundation. The opinions expressed are not necessarily those of the sponsoring agency, and no endorsement should be inferred. The authors wish to acknowledge, with deep appreciation, the contribution of Anita Adamopoulus, M.'s teacher, who conducted the interview with M.

References

Alston, A., Davis, R. B., Maher, C. A., & Martino, A. (to appear). Children's construction of mathematical structures.

Davis, R. B. (1984). Learning Mathematics: The Cognitive Science Approach to Mathematics Education. London: Routledge.

Freudenthal, H. (1983). Didactical Phenomenology of Mathematical Structures. Dordrecht, Holland: D. Reidel.

Lesh, R., & Landau, M., Eds. (1983). Acquisition of Mathematics Concepts and Processes. New York: Academic Press.

Maher C. A., Martino, A., & Alston, A. (1993). Children's construction of mathematical ideas. In: B. Atweh, C. Kases, M. Carss, & G. Booker, Eds., Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australia. (Brisbane, Australia: MERGA) pp. 7-39.

Schoenfeld, A., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's understanding of a complex subject matter domain. In: R. Glaser, Ed., Advances in Instructional Psychology, (Vol. 4). Hillsdale, NJ: Erlbaum.

Streefland, L. (1991). Fractions in Realistic Mathematics Education. Dordrecht, Holland: Kluwer Academic Publishers.

USING STUDENT-GENERATED WORD PROBLEMS TO FURTHER UNRAVEL THE DIFFICULTY OF MULTIPLICATIVE STRUCTURES

Erik De Corte & Lieven Verschaffel
Center for Instructional Psychology and Technology (CIP&T)
University of Leuven, Belgium

Abstract
There is substantial empirical evidence showing the impact of number type on the solution of problems with a multiplicative structure. The theory of the intuitive models of arithmetic operations is today the most plausible theoretical account for these findings. In an attempt to contribute to a better understanding of the number type effects, a study was carried out in which student-generated word problems were used as data to test systematically a series of predictions derived from the theory of the intuitive models.

Introduction

Research has shown that the type of numbers in multiplicative word problems is an important determinant of their difficulty level. More specifically, students of different ages have difficulties with decimal multipliers and divisors, especially when the decimal is smaller than 1, and with divisors that are larger than the dividend (see Greer, 1992). For example, De Corte, Verschaffel, and Van Coillie (1988) have shown that with respect to multiplication problems 12-year-old students are better at choosing the correct operation from a series of six alternatives when the multiplier is an integer than when it is a decimal larger than 1; but problems with a decimal multiplier smaller than 1 are still more difficult. The most frequently occurring error is dividing instead of multiplying. Similar number type effects have been reported for division problems. In a study with large groups of 10-, 12-, and 14- year-olds, Fischbein, Deri, Nello, and Marino (1985) found that only a small percentage of pupils succeeded in selecting the correct operation for problems with a decimal divisor smaller than 1; the most common error consisted of multiplying both numbers. Problems with a divisor larger than the dividend caused also many errors; most of the mistakes originated from dividing the larger number by the smaller one.

The most plausible theoretical account today for these findings is provided by the concept of intervening primitive intuitive models of arithmetic operations introduced by Fischbein et al. (1985). Research relating to this concept is mostly based on multiple-choice tests in which the subjects were asked to choose the correct operation. This work provides only partial support for the theory of Fischbein (Greer, 1992). This article reports a study in which an alternative technique was used, namely asking students to generate multiplicative verbal problems as a window to their difficulties with this kind of tasks, and, consequently, as a contribution to the further validation of this theory.

The theory of primitive intuitive models of operations

According to Fischbein et al. (1985) pupils acquire a primitive intuitive model for every arithmetic operation. The choice of an operation to solve a word problem is mediated by these intuitive models, especially by their implied constraints. When the numbers in a word problem are incongruent with one or more of the constraints of the underlying model, the probability that a wrong arithmetic operation is chosen increases.

The primitive model associated with multiplication is repeated addition: a number of sets of the same size are put together. Under the repeated addition interpretation, one number (i.e. the number of equivalent sets) is taken as the multiplier, the other (i.e. the size of each set) as the multiplicand. Two numerical constraints derive from this model: 1. the multiplier must be an integer; 2. multiplication always produces a result that is bigger than the multiplicand.

In a similar way Fischbein et al. hypothesize the existence of two primitive models for division: the partitive and the quotitive models. In the partitive model a given quantity is divided into a specified number of equal subsets, and one has to determine the size of each subset. This model implies three constraints: 1. the dividend must be larger than the divisor; 2. the divisor must be an integer; 3. the quotient must be smaller than the dividend. In the quotitive model one has to find how many times a certain quantity is contained in a larger quantity. This model involves only one numerical constraint: the dividend must be larger than the divisor.

This theory of the primitive intuitive models accounts for a number of the empirical findings reported in the literature. For instance, the robust multiplier effect mentioned above involves that problems with a decimal multiplier larger than 1 are more difficult than those with an integer as multiplier, and that problems with a decimal multiplier smaller than 1 are even harder. This is in line with the theory: in the first case one numerical constraint of the repeated addition model is violated, namely that the multiplier must be an integer; in the second case the additional constraint - the outcome must be larger than the multiplicand - is also violated.

But, some other observations with respect to multiplication problems are less consistent with the theory. For example, the multiplier effect is much weaker when students are asked to solve problems as compared to the choice of operation task (De Corte et al., 1988). And, with respect to division the explanatory power of the constraints of the partitive and quotitive models has so far remained rather limited (Greer, 1992). For instance, remarkably good results have been reported on partitive problems with a decimal divisor larger than 1 when the dividend is also a decimal larger than 1 (e.g., 11.44 : 4.51) (Bell, Greer, Grimison, & Mangan, 1989), and problems with an integer dividend smaller than the divisor (e.g., 5 : 15) have been found to be more difficult than those with a decimal dividend smaller than a whole divisor (e.g., 4.5 :

6) (Fischbein et al., 1985). The theory of the intuitive models offers no sound explanation for both results.

Referring to the latter finding as well as to their own work, Harel, Behr, Post, and Lesh (in press) have put forward the idea that the "dividend larger than divisor" (D > d) constraint is not as robust as the other two constraints of the partitive model. Fischbein et al. found that in the case mentioned above of a decimal dividend smaller than a whole divisor, very few students reversed the numbers in solving such problems, as they frequently did when both were integers; an explanation for this observation is that they avoided the decimal divisor resulting from such a reversal. In their own study Harel et al. observed that the majority of the pre-service and in-service teachers who failed on a problem that had to be solved by the operation 11 : 2.53 chose the inverse operation 2.53 : 11. This supports the differential robustness idea: while the correct operation violates the constraint that the divisor must be an integer, the applied inverse expression breaks the larger-dividend constraint.

In an attempt to further unravel the number type effects on the solution processes of multiplicative problems, we carried out a study in which we used student-generated verbal problems to test systematically a series of hypotheses and predictions derived from the theory of intuitive models.

Method

Subjects were 107 12-year-olds, 107 15-year-olds, and 99 elementary student teachers. They were given a collective test consisting of 12 multiplicative number sentences, 6 multiplications and 6 divisions (see Table 1). Two versions were developed differing with respect to the sequence of the number sentences, and, for the multiplications, with regard to the order of the numbers. The students were asked to write for each sentence a story problem that could be solved with the given operation; however, they were warned that "pseudo word problems" like "Mother had to divide 0.6 by 0.8. Can you help her?", were not allowed. To restrict the possible influence of computational skills, the outcome of each number sentence was mentioned in brackets.

Table 1. Multiplicative number sentences used in the present study

Multiplication		Division	
a) 9 x 3	= 27	g) 24 : 3	= 8
b) 6 x 2.8	= 16.8	h) 6.3 : 9	= 0.7
c) 8 x 0.9	= 7.2	i) 5 : 25	= 0.2
d) 7.4 x 3.8	= 28.12	j) 6 : 4.8	= 1.25
e) 5.3 x 0.6	= 3.18	k) 4 : 0.8	= 5
f) 0.7 x 0.2	= 0.14	l) 0.6 : 0.8	= 0.75

On the basis of a pilot study a schema in the format of a flow chart consisting of a series of seven questions, was designed to separate appropriate story problems from statements that cannot be accepted as appropriate problems, and to classify the latter into categories. Out of the 3756 collected statements, 2089 were considered as appropriate; only 61 inappropriate statements did not fit in one of the designed categories. An interrater reliability of 95% obtained on a random sample of 118 problem statements, showed the utility of the schema.

Hypotheses, predictions, and results

Because of space restrictions, the findings of the study can only be reported partially in this article. First, the results concerning the influence of the type of number sentence on performance are reviewed. Next, the congruence of the generated problems with the primitive models is discussed.

Influence of the type of number sentence on performance

The general hypothesis was that the nature of the multiplicative number sentence would strongly influence students' performance on the problem-generating task. More specifically, we expected an increase in the level of difficulty of the task depending on the amount of constraints of the underlying intuitive model that needed to be violated. From this hypothesis a series of predictions were derived. To test these predictions chi-square values were calculated for the subgroups of subjects separately as well as for the total group; to assess significance alpha was set at the usual level of 0.5.

Multiplication

1. A significantly higher number of appropriate problems was predicted for the sentences a, b, and c (see Table 1) which are congruent with the repeated addition model, than for d, e, and f. This prediction was confirmed in all groups.

2. The presence of an integer in b and c allowed to state a problem that conforms to the repeated addition model. Therefore, no difference was expected between those sentences and a. The results support this prediction in the groups of 15-year-olds and student teachers. Apparently, the presence of a decimal in b and c raised problems for the 12-year-olds.

3. In line with the intuitive model and with the empirically robust multiplier effect, it was also expected that the multiplications d and e would yield more appropriate story problems than f. Indeed, whereas stating a good problem for d and e requires the violation of the first constraint only (the multiplier must be an integer), with respect to f the second constraint has to be violated also (multiplication makes always bigger). In none of the groups a significant difference between e and f was observed. A significant better performance was found for d than for f in the 12- and 15-year-olds only.

4. The presence of a decimal larger than 1 in sentence e makes it possible to formulate a word problem that violates only the first constraint of the intuitive model (multiplier must be an integer). Therefore, no difference between d and e was anticipated. In contrast to this prediction e yielded significantly less good problems in all groups. Apparently, the presence of a decimal smaller than 1 in e complicated the task seriously.

Division

1. Sentence g is the only one that satisfies the constraints of the quotitive and especially the partitive division models. Therefore, a significantly higher number of appropriate problems was predicted for g than for all other sentences. This prediction was confirmed in all groups, however, with two exceptions. Indeed, in the 15-year-olds and in the student teachers groups no difference occurred between the congruent division g and sentence h which contains a decimal dividend larger than 1 and a whole divisor larger than this dividend. These observations are in line with the idea of Harel et al. (in press) that the D > d constraint of the partitive model is less robust than the other two.

2. On the basis of this differential robustness hypothesis it was anticipated that h and i would both yield significantly more good problems than j, k, and l. While h and i violate the D > d constraint, j, k, and l break one or two of the other intuitive rules. The data provide support

for this prediction in all cases but one; there was no difference between i and k in the group of student teachers.

3. Again in line with the idea that the D>d constraint is less robust than the other intuitive rules, a significantly higher number of appropriate problems was expected for h than for i. The basis of this prediction is that the presence of a decimal dividend in h will reduce the number of reversals of dividend and divisor as compared to i. The data confirm also this prediction, and, thus, provide additional support for the differential robustness hypothesis.

4. Finally, it was anticipated that l which is most incongruent with the constraints of the intuitive model, would yield significantly more inappropriate statements than j and k. The results are in line with this prediction too.

Congruence of generated problems with the intuitive models

We hypothesized that students would try to generate as much as possible word problems that are congruent with the intuitive models. This hypothesis was tested in two ways.

1. It was predicted that, when confronted with a multiplication sentence involving an integer and a decimal number (the sentences b and c in Table 1), students would strongly avoid to use the decimal number as multiplier of their word problem. The data strongly support this prediction. For both number sentences there was an overwhelming and, thus, statistically significant preference for the integer as multiplier. For instance, for sentence b (6 x 2.8) in 253 out of the 260 appropriate problems the integer was used as the multiplier. The results for sentence c were analogous.

2. A second expectation was that a qualitative analysis of the inappropriate statements would reveal a substantial amount of
unallowed modifications of the given numbers and calculations to conform the numerical data to the constraints of the intuitive models. The following findings confirm this expectation.

To avoid violation of the constraint that the multiplier or divisor must be an integer, students rather often changed a decimal into an integer. This error type was very typical for sentence f (0.7 x 0.2): 10% of the inappropriate statements belong to this category (e.g., "One candy weighs 0.2 gr. What is the weight of 7 candies?").

Another incorrect modification consists of reversing the role of the given dividend and divisor. Of all inappropriate statements for sentences j (6 : 4.8) and e (4 : 0.8), 22% and 10% respectively were of this type (e.g., "Four grandchildren are visiting grandma. They all like milk. But grandma has only 0.8 litre of milk. How much does each child get?"). In both cases such a reversal avoids violation of the constraint that the divisor must be an integer; and, for sentence e it allows to bypass the intuitive rule that the quotient should be smaller than the dividend. This finding also supports the differential robustness hypothesis of Harel et al. (in press): apparently, students prefer to state a problem with a divisor larger than the dividend rather than one with a decimal divisor.

Reversing dividend and divisor was also observed in 10% of the inappropriate statements for sentence i (5 : 25) (e.g., "Five friends have together 25 sandwiches. How many does each of them have?"). This reversal avoids violation of the D>d constraint. Interestingly, not even one such a reversal was found for sentence h (6.3 : 9) which would result in having a decimal divisor. This finding involves additional evidence for the differential robustness hypothesis concerning the constraints of the intuitive division model.

Finally, another representative error consisted of using decimals to refer to objects that can only be denoted accurately by integers. By doing so, students used decimal numbers in a way that is incongruent with the intuitive models. For instance, this error type accounted for 25% of the inappropriate statements for sentence d (e.g., "A chicken lays 3.8 eggs a week. How many eggs do 7.4 chickens lay?"), and for 28% of the incorrect problems for sentence j (e.g., "4.8 people measure together 6 m. How much for each of them?").

Conclusions

Several findings of this study support the basic hypothesis of the theory of Fischbein et al. (1985), that the selection of an operation for solving multiplicative problems is mediated by primitive intuitive models of the arithmetic operations. In the three age groups students generated significantly more appropriate story problems for number sentences that are congruent with the intuitive models than for the incongruent ones. The impact of the models was also manifested in students' tendency to generate problems that conform to their constraints.

But, the study also shows that the theory of the intuitive models cannot appropriately account for all the available empirical data. Indeed, several findings with respect to multiplication were not in line with the predictions derived from the theory. And, for division, the present results largely support the idea put forward by Harel et al. (in press) that the D>d constraint is less robust than the other intuitive rules of the partitive model. These findings even raise the question whether this constraint is an essential feature of the primitive division model (see also Greer, 1987).

In summary, while the results of this study are fairly consistent with the the theory of Fischbein et al. (1985), they also suggest that in order to fully account for the available empirical data, it is necessary to aim at the design of a more comprehensive theory which takes into account that solving problems with a multiplicative structure is influenced by a large variety of factors interacting in multiple and complex ways. In view of the development of such a theory, an attempt towards combining the different approaches in research on multiplicative structures described by Nesher (1992), could possibly open new and productive research perspectives.

References

Bell, A., Greer, B., Grimison, L., & Mangan, C. (1989). Children's performance on multiplicative word problems: Elements of descriptive theory. Journal for Research in Mathematics Education, 20, 434-449.

De Corte, E., Verschaffel, L., & Van Coillie, V. (1988). Influence of number size, problem structure, and response mode on children's solutions of multiplication problems. Journal of Mathematical Behavior, 7, 197-216.

Fischbein, E., Deri, M., Nello, M.S., & Marino, M.S. (1985). The role of implicit models in solving verbal problems in multiplication and division. Journal for Research in Mathematics Education, 16, 3-17.

Greer, B. (1987). Understanding of arithmetical operations as models of situations. In J.A. Sloboda, & D. Rogers (Eds.), Cognitive processes in mathematics (pp. 60-80). Oxford: Clarendon.

Greer, B. (1992). Multiplication and division as models of situations. In D.A. Grouws (Ed.), Handbook or research on mathematics teaching and learning (pp. 276-295). New York: Macmillan.

Nesher, P. (1992). Solving multiplication word problems. In G. Leinhardt, R. Putnam, & R.A. Hattrup (Eds.), Analysis of arithmetic for mathematics teaching (pp. 189-219). Hillsdale, NJ: Lawrence Erlbaum Associates.

A MODELING APPROACH TO UNDERSTANDING THE TRIGONOMETRY OF FORCES: A CLASSROOM STUDY

Helen M. Doerr and Jere Confrey
Cornell University

We present an approach to mathematical modeling that integrates physical experimentation, computer simulation, and multi-representational analytic tools. Using this approach, a curriculum unit was designed to develop the relationship of a force exerted at an angle to its vertical and horizontal components. In this paper, we report on the models built by five groups of students in an integrated mathematics and science classroom. Through their models, the students bridge the trigonometric relationships, the Pythagorean Theorem and the notion of force components. The results of this classroom study suggest that this modeling approach is both productive and empowering for students and leads to the formation of robust and flexible concepts.

Introduction

Over the past decade, computer modeling for solving problems in science and engineering has become a well-established practice in the academic research community and for industrial research and development. Somewhat more recently, computer modeling has become of interest to mathematics and science educators. The availability of greater computer power at lower costs and a growing interest in constructivist models of teaching and learning have given impetus to the study of computer modeling. Typical use of computer modeling in the secondary curriculum involves the manipulation of a previously built model (an expert's model) within some set of parameters. What is less well understood is the potential effectiveness of engaging students in the actual process of building their own models.

Modeling, simulation and discrete mathematics have all been identified by the National Council of Teachers of Mathematics (1989) and by the Mathematical Sciences Education Board (1989,1990) and other professional mathematics education organizations as important areas for secondary school study. As noted by the MSEB, the interplay between mathematics and computer technology opens up new areas for investigation and study: "Mathematics provides abstract models for natural phenomena as well as algorithms for implementing these models in computer languages. Applications, computers and mathematics form a tightly coupled system producing results never before possible and ideas never before imagined" (1989, p.36). To reach this potential, we need to gain a better understanding of the processes involved in the use of modeling and how these might be most effectively used to improve students' problem-solving skills. This study explores students' understanding of the concept of the relationships among the components of a force vector, using trigonometric functions.

Theoretical Framework

We begin with a consideration of the conceptual framework for the modeling process and its tools. We can differentiate between exploratory models and expressive models (Bliss & Ogborn, 1989). Exploratory models are those models that are constructed by experts to represent knowledge in some content domain. Learners typically explore consequences of their actions within the boundaries of these content domain models. Typical activities would include varying input parameters, observing changes in output, explaining the consequences of certain actions and conditions. The microworlds developed by Papert (1980) and others are good examples of this type of modeling activity. Exploratory models provide a way of asking "can you understand an expert's way of thinking about a problem?" Sabelli (1993) and others (Forrester, 1991; Mandinach, Thorpe, & Lahart, 1988) argue that exploratory models provide powerful entry points for novices to enter into expert understandings. By providing students with realistic models of complex phenomena, they can explore the knowledge domains of experts through the variation of parameters and testing of hypotheses.

Expressive models, on the other hand, provide learners with the opportunity to express their own concepts and to learn through the process of representing their concepts, defining relationships among components, and exploring the consequences of their conceptions. Tools such as spreadsheets, Function Probe, MatLab, STELLA, and Interactive Physics are examples of the kind of software that can be used in expressing and developing student-conceived models of physical phenomena. Expressive tools provide a way of asking "can you understand your own way of thinking about a problem?" Clement (1993) argues that powerful models can be built by students by beginning with what he calls "anchoring concepts." These anchors are correct concepts about which many students have strong intuitions. For example, most students will agree that two hands pressed together exert equal and opposite forces against each other and likewise when both hands pull on a spring. But most students will incorrectly assert that a book resting on a table exerts a downward force on the table, but that the table does not exert a corresponding upward force on the book. Clement goes on to argue that computer modeling can provide a bridge between these anchor concepts and the concepts of Newtonian physics. Others (Feurzig, 1988; Roberts & Barclay, 1988; Roth, 1992; Tinker, 1993; Webb & Hassell, 1988) similarly argue that models are potentially powerful tools for students to act on and reflect on contextual problems, thus building their own understanding and their own skills in problem-solving and analysis. This study is grounded in this second view, which suggests a focus on the tools for modeling and on how students use the tools to construct their own understandings of contextual problems in the specific content area of vectors and forces.

The modeling process is distinguished in three ways from the typical problem-solving activity in the mathematics classroom today. First, the building of computer models by students forces them to make explicit their own understandings and to explore the consequences of those understandings. Second, the kinds of mathematical representations that are possible with computer-based tools extend far beyond algebraic equations to include tables, graphs, calculator algorithms, flow and control diagrams and animations. Third, the iterative nature of the modeling process provides the opportunity for approximate solutions which are refined through analysis and evaluation by the problem solver. This iterative process includes both reflection on the part of the individual ("what do I think") and the communication of that understanding among peers ("what do others think"). This process of iteration is not necessarily linear. But rather, as Bell (1993) argues, students as modelers need to spend time in each of several activities or nodes: confronting and defining the problem, deciding on a model and operating, evaluating and interpreting, and gathering data and information. Students develop their understanding as they move back and forth between these nodes.

Description of Study

In this study, we examine a modeling process which integrates three component activities: the action of building a model from physical phenomena, the use of simulation and multiple representations, and the analysis, refinement and validation of potential solutions. In particular, we explore how and to what extent these components of the modeling process can lead to the improvement of student skills in solving mathematical problems in the content area of forces and vectors. This paper will address the first sub-unit, the resolution of a force vector into its horizontal and vertical components, posed as part of the larger research project on an integrated modeling approach for building student understanding of the concept of force and enhancing problem-solving skills.

Curricular Unit

The instructional approach to the unit is based on the notion of providing an essential, driving question to motivate the inquiry and guide the students. The essential question for this unit is: "How will an object behave if it is traveling down a frictionless inclined plane? How can you predict the behavior of such a object for any randomly chosen angle of incline?" This question in turn generates four sub-units, each with its own essential question: the resolution of a vector into its horizontal and vertical components; the effect of multiple vectors (e.g. forces) acting on an object; the relationship between force, mass and acceleration, and the role of friction as it affects the motion of an object. These four units are then integrated in a final problem which is the analysis of the behavior of an object moving down an inclined plane. The essential question for the first unit is: "When a force is exerted at an angle, how much of it is horizontal and how much of it is vertical? How are these forces related? What relationship can be used to make predictions and analyze components?"

The overall curricular unit is designed around three activities: (1) the gathering of data from a physical experiment, (2) the development and exploration of a computer simulation, and (3) the mathematical (algebraic, graphical, and tabular) analysis of the data. The second and third components of this unit are supported through Interactive Physics and Function Probe, respectively. Critical to the overall philosophy of the researchers and the high school teachers, this unit is designed to include extensive student discussion and reflection, collaborative work, small and large group tasks, and individual assignments. This approach to the design of the unit builds on earlier work of the research group (Noble, Flerlage, & Confrey, 1993).

Setting

The setting for this study was the Alternative Community School, part of the public school system in Ithaca, New York. The school is a member of the Coalition of Essential Schools and has a well-established relationship with the mathematics education research group at Cornell University. The administration and teachers are actively engaged in curricular and instructional change in mathematics and science. The classrooms within the school are an open, flexible environment where small group work is common and the expression of student ideas is encouraged and nurtured. This study took place in an integrated algebra, trigonometry, and physics class with 17 students in grades 9 through 12, who had elected to take the course. Seven computers were available for student use. The class had used both Function Probe and Interactive Physics in an earlier unit, so that the effect of learning to use the software was minimized in this portion of the study. The class met for a double period of 1.5 hours for four days per week. The complete unit lasted approximately ten weeks, including time for final assessment of both small groups and individuals by the classroom teachers.

The class was "team taught" by two experienced mathematics and physics teachers, who are experienced and familiar with computer technology and the particular software. This was the third year that they taught this course. One of the most important aspects of the classroom was the role which the teachers took as guides and facilitators for student inquiry. Students were consistently encouraged to explore their own ideas and to make sense of physical phenomena in a context of interactions with their small group, the entire class and their teachers. There were no textbooks used by the students nor did the teachers give lectures.

The class was divided into five small groups of 3-4 students. The small groups provided a setting within which to analyze and observe how the students go about interpreting the posed essential questions, generating and negotiating their conjectures, devising their strategies for analyzing the data, confirming the sense of one or more conjectures, and using the tools and their empirical data.

Data Sources and Analysis

Each class session of this unit was video-taped, and during small group work, one selected group was video-taped throughout. Written work and computer work done by the focus group was made available to the researchers. In addition, copies of the computer models generated by the students were collected for analysis. Extensive field notes were taken by the researcher during the class sessions. The video-tapes of class sessions were reviewed and selected portions were transcribed for more detailed analysis.

Description of Experimental Setups

The unit began with a simple physical experiment: an object of a given mass was hung from a pulley suspended on a rope that was fixed at one end and attached to a spring scale at the other. The students were asked to check the relationship between the two angles made by the rope with the horizontal and between the pull forces on each side of the pulley. They were asked to look for a relationship between the force in the rope and the weight of the mass and the pulley. The students investigated these questions in small groups of 3 to 4 students. After some initial explorations, the class then turned to an alternative setup for measuring a force exerted at an angle. This time three ropes pulled at a single point, one rope at an angle, one vertical and one horizontal. After discussing this setup and how it compared to the first experiment, two sets of measurements were taken.

Results

The students began by analyzing the pulley setup. They convinced themselves that the angles formed by the rope and the horizontal are in fact always equal, that the length of the string doesn't matter, that the forces are the same on both sides of the pulley, and that the force changes depending on the angle. The students soon realized that as the angle with the horizontal decreased, the force in the rope increased dramatically. Several groups of students moved to graph their data and they began to hypothesize that the relationship between the weight and the force in the string could be tangential, parabolic, or exponential. The second experimental setup was demonstrated to the class and the students began to offer some conjectures about the relationships among the forces: Mark suggested that the sum of the two component forces might a constant. Art hypothesized that subtracting the vertical force from the angular force should give the horizontal force. The students broke into small groups to discuss the data and relationships of the forces involved, but came to no conclusions.

Through a whole class discussion, led by the teachers, the students realized that they did not have convincing evidence about the horizontal forces that are present when the mass on the pulley is being held in place. The class then addressed the question of how to proceed to gather sufficient evidence. Kris argued that the key piece of evidence that they need is evidence for the existence and magnitude of the horizontal force. Karen questioned how they would know that their answer was right. Kris asserted that they can verify a claim about relationships by finding the same answers in different ways.

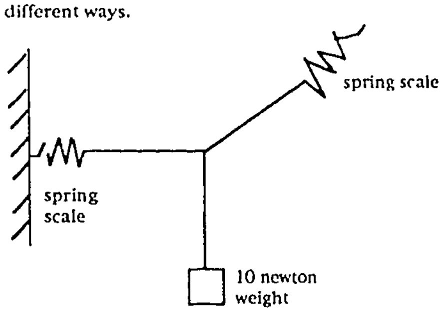


Figure 1

The students suggested three possible ways to proceed: (1) play with the numbers that they already had on their calculators; (2) recreate the experiment, taking new data, measuring the horizontal force; and (3) use Interactive Physics to set up an object and put a force on it. Cal suggested an experimental setup (Figure 1) that would allow them to measure the horizontal force. Mary claimed that this setup would convince them about the horizontal force because the vertical and horizontal forces need to balance the force at an angle. The class broke into small groups each of which chose an approach. Surprisingly, all the groups elected to use the simulation environment. This environment provided the students with the ability to assemble mass objects, ropes and forces, along with a set of meters to measure time, position, tension and the magnitude of forces. The students showed considerable diversity of approaches within the simulation environment. However, one common element to all the simulations was that the students elected to set gravity to zero, thereby having to create explicit forces to represent the weight of objects. Two groups set up models (Figure 2) by placing a vertical, horizontal and an angular force on a mass object.

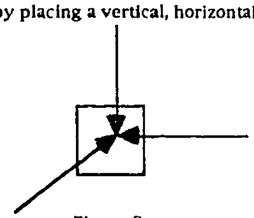


Figure 2

Both groups verified that the vertical and horizontal components were equal in magnitude when the angle for the third force was at 45 degrees. For a given horizontal and vertical force, they were able to find the magnitude of the third force by resetting its value until the object remained motionless. The first group looked at one other setting for the angular force at 60 degrees, but did not proceed beyond that to any type of generalization. The second group, on the other hand, proceeded to take several other readings as data and then turned to their calculators to find that the vertical force equaled the diagonal force times the sin of the angle and that the horizontal force equaled the vertical force divided by the tangent of the angle: $Fv=F\theta * \sin\theta$ and $Fh=Fv/\tan\theta$.

The third group built a simulation similar to what was proposed by Cal in the class discussion. Their model had a mass held by two ropes, one horizontal and one at an angle that could be varied. They then hung a second mass from the first and applied a downward force of 10 newtons to that mass. They measured the tension in each of the ropes and verified that the Pythagorean relationship held for those three values. From this, the group concluded that the forces must be in a right triangle relationship and therefore they can use trig relationships to find unknown forces. They went on to vary the diagonal force and verify that the trig relationships hold. Thus, their simulation model, along with trig relationships for right triangles, exactly answered the problematic posed in the original pulley system: that the diagonal pull on the object must have a vertical component that is equal to half the weight of the object. Now that they have verified that the trig ratios hold for this situation, they can calculate the magnitude of the horizontal force which they couldn't directly measure. This group had no generalized algebraic relationships for the force components, but was quite satisfied with their ability to calculate any specific set of forces using right triangle trigonometry.

The fourth group had a considerably more complex model (Figure 3) than any of the others and their model most nearly fit the original pulley setup.

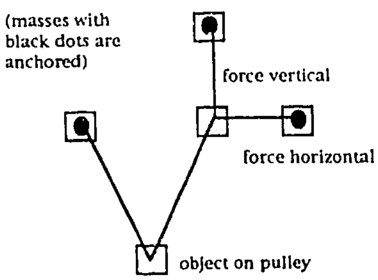


Figure 3

They were able to move their object up and down, thus varying the angle of the force (or tension) in the rope. The tension in the vertical and horizontal ropes at the upper right had to equal the vertical and horizontal tension in the rope that holds the object on the pulley. The fact that the tension in the vertical rope was always 5 was consistent with the weight of the object. They (like the other groups) verified that at an angle of 45 degrees the vertical and horizontal forces were equal. They then proceeded to set the object at various positions along the y-axis and created a table recording the position of the object, the horizontal tension and the diagonal force. They brought this data into Function Probe and correctly computed the angle of the force, given the object's position, but did not formalize either a graphic relationship or equations for this data. It was not clear if this was due to the lack of time or their need for more specific guidance.

The fifth group, with some guidance from the teacher, constructed yet another model. This model (Figure 4) differed significantly from all the others in that the diagonal force (not the vertical) is kept constant while varying the angle. Like the other groups, they verified that the vertical and horizontal forces were equal for an angle of 45. They kept their force at a constant 10 newtons, and varied the angle from 0 to 90 degrees, recording the vertic·, tension for each setting. They immediately took this information into the table window of Function Probe and created a graph. In the graph window, they were able to establish the relationship that the vertical force equaled 10 times the sin of the angle. From this, they became convinced of the following general relationships: $\sin\theta = Fv/F\theta$, $\cos\theta = Fh/F\theta$, and $\tan\theta = Fv/Fh$.

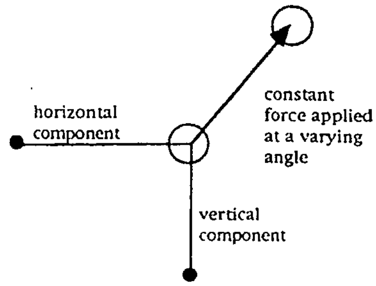


Figure 4

Discussion

The open-ended nature of the essential question provided the students with opportunities to direct their own inquiry into the nature of the relationship between and among the components of a force exerted at an angle. Beginning with an initial physical setup, the students generated hypotheses and possible procedures for gathering evidence to verify their hypotheses. It should be noted that in using Interactive Physics, none of the students simply applied a force to an object at an angle and used the built-in displays to measure the vertical and horizontal components in an information gathering approach. But rather each group devised their own way of building an analog to a physical setup and then directly measured the component forces. Further, each group carefully verified that their model gave results that they were convinced were true in the physical world: that at a 45 degree angle, the vertical and horizontal components must be equal. Finally, several of the groups ran into the limits of the simulation environment when trying to hold the mass object at an angle of zero degrees. Most students were convinced of the impossibility of holding an object up solely by applying horizontal forces.

The students were able to move the data from their simulation experiments into the analytic environment of Function Probe. There, the flexibility of tables and graphs enabled one group to define both equations and related graphs that described the relationships between the force exerted at an angle and its components. The use of a physical experiment, computer simulation, and analytic tools enabled the students to convince themselves through multiple and varied approaches of the validity of the relationship among the components of a force exerted at an angle. We believe that this study provides evidence that an integrated approach to modeling is both productive and empowering for students and leads to the formation of robust and flexible concepts.

References

Bell, M. (1993). *Modeling and the uses of mathematics in the K-6 curriculum.* Paper presented at the K-12 Modeling and Simulation Conference in Boston, MA.

Bliss, J. & Ogborn, J. (1989). Tools for exploratory learning. *Journal of Computer Assisted Learning, 5*(1), 37-50.

Confrey, J. (1991). *Function Probe.* [computer program]. Ithaca, NY: Cornell University.

Feurzig, W. (1988). Apprentice tools: students as practitioners. In R. S. Nickerson & P. P. Zodhiates (Eds.), *Technology in education: Looking toward 2020.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Forrester, J. (1991). Systems dynamics—adding structure and relevance to pre-college education. In K. R. Manning (Ed.), *Shaping the Future.* Boston, MA: MIT Press.

Interactive Physics II. (1989). [computer program]. San Francisco, CA: Knowledge Revolution.

Mandinach, E., Thorpe, M., & Lahart, C. (1988). *The Impact of the Systems Thinking Approach on Teaching and Learning Activities.* Princeton, NJ: Educational Testing Services.

Mathematical Sciences Education Board, National Research Council. (1989). *Everybody Counts: A Report to the Nation on the Future of Mathematics Education.* Washington, DC : National Academy Press.

Mathematical Sciences Education Board, National Research Council. (1990). *Reshaping School Mathematics A Philosophy and Framework for Curriculum.* Washington, DC: National Academy Press.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: The Council.

Noble, T., Flerlage, D., & Confrey J. (1993). *Experiment, simulation and analysis: teaching and learning in an integrated physics and mathematics class.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA.

Papert, S. (1980). *Mindstorms.* New York, NY: Basic Books, Inc.

Roberts, N. & Barclay, T. (1988). Teaching model building to high schools students: theory and reality. *Journal of Computers in Mathematics and Science Teaching, Fall,* 13-24.

Roth, W-M. (1992). Bridging the gap between school and real life: toward an integration of science, mathematics and technology in the context of authentic practice. *School Science and Mathematics, 92*(6), 307-317.

Sabelli, N. (1993). *Understanding science: metaphors for exploring a non-linear space.* Unpublished paper, Applications of Advanced Technologies, National Science Foundation, Washington, DC.

STELLA II An Introduction to Systems Thinking. (1992). Hanover, NH: High Performance Systems, Inc.

Tinker, R. (1993). *Modelling and theory building: technology in support of student theorizing.* Unpublished paper, The Technology Education Research Center, Inc., Cambridge, MA.

Webb, M. & Hassell, D. (1988). Opportunities for computer based modelling and simulation in secondary education. *Computers in Education, proceedings of the First European Conference on Computers in Education, International Federation for Information Processing, July,* 271-277.

PROSPECTIVE TEACHERS: SIGNIFICANT EVENTS IN THEIR MATHEMATICAL LIVES

Brian Doig

The Australian Council for Educational Research

The research described here is part of a larger investigation of prospective primary teachers attitudes and beliefs about mathematics, its teaching and learning. Whilst the overall study investigates attitude via a questionnaire this paper relates to a single question. This question was free response, and simply asked respondents to relate specific events in their mathematical lives that they felt had contributed to their attitude to mathematics. Responses were categorised using McLeod's affective domain categories and the beginnings of a continuum describing these teacher trainees' attitude-causes established. While the methodology is relatively crude, the results underscore the primal rôle played by teachers in the formation of student attitudes to mathematics.

Introduction

The evidence of research into the effects of learner beliefs in science shows that they are indeed critical to the outcomes of instruction and have been well documented (Adams, Doig and Rosier, 1991). Borasi's description of the effects of dysfunctional beliefs in mathematics lends weight to any argument for research into the affective domain.

Recently research into the affective domain with respect to mathematics has gained more prominence with McLeod's 1992 review and Leder's keynote address to the 1993 PME meeting in Japan. While some PME papers have described the construction of instruments for measuring affective variables (Relich and Way, 1993) few appear to have heeded Schoenfeld's call for research into *how* affect is constructed (Schoenfeld, 1992).

Notable exceptions to this can be found in recent PME papers. For example Imai (1993) found that the a good personal relationship between teachers and junior high school students did influence positively the students' attitude towards mathematics. Other causal links were found to exist between the teacher's methodology and student enjoyment of mathematics. In a study exploring the link between curriculum and student attitudes Ponte, Matos, Guimarães, Leal and Canavarro (1992) found that there is some relationship between the attitudes of teachers and that of their students.

In his review McLeod (1992) categorizes affective research into the following categories: beliefs about mathematics; beliefs about self; beliefs about mathematics teaching and beliefs about the social context. With these as starting points it seemed possible to investigate attitude causality and perhaps answer Schoenfeld's *how*?

Research questions

The investigation reported here forms a small part of a larger study. While the larger study sought to create a validated, reliable attitude to mathematics inventory with a user-friendly interpretative and reporting scheme, the focus selected for this report is the final free-response question that asks for specific events that may have helped to form the respondent's attitude to mathematics. Thus the main question investigated was whether it is possible to 'map' those events that help to form attitude to mathematics. A subsidiary question is whether such a 'map' can orient educators in any fruitful directions.

Subjects

The subjects were from a Victoria (Australia) university that trains elementary school teachers. The entire entry cohort for two sequential years were surveyed. The cohorts were predominantly female, and most had just completed their secondary schooling before entering the training course. Tables 1 and 2 give details of the students surveyed.

Table 1: Details of students with respect to cohort and sex.

Cohort 1		Cohort 2	
Males	Females	Males	Females
17	53	5	46
Total = 70		Total = 51	
Total subjects = 121			

Table 2 Details of students with respect to age.

Age group	Number
Under 20	87
20 to 30	15
Over 30	19

Data collection

The two groups of primary teacher-trainees were administered the Boyd-Doig Mathematics Survey instrument (Boyd and Doig, 1990). The focus of this report concerns the final question of the instrument, which simply asks: *Can you describe some experience(s) that could explain why you feel the way you do about maths?*

The respondents were provided with nearly an entire page in which to write their response. It is these responses that form the data for this report.

The completed surveys were sorted and some thirtyfour scripts were discarded on the basis that they either contained no response to the question, or the response did not address the question. All remaining scripts were analysed as set out below.

Analysis

As outlined earlier, the methodology used was that of categorising written responses to a leading question. The procedure used was that used by Adams and Doig (Adams et al, 1992) in their study of science beliefs. All responses to the question were read to give an overall 'feel' for the range of responses. Theoretically each response is unique, but in practice responses tend to 'group' themselves in a qualitative sense. From the basic foci provided by McLeod's analysis and a reading of student responses, all responses were placed into one or more of the defined categories. Where necessary, more categories were defined until all responses were accommodated. In all 127 separate statements were categorised in this manner. The final set of nine categories used to analyse the responses and typical responses were as follows:

Mathematics

Its content
Sample statement: *'Three years of logarithms'*
Its complexity
Sample statement: *'I find problem-solving quite difficult and this turned me off'*

Self

Achievement in mathematics
Sample statement: *'I have positive feelings about maths ... I tended to do well'*
Confidence in mathematics
Sample statement: *'I always believed that there had to be some magic formula that I was too dumb to know'*

Mathematics teaching

Methodology
Sample statement: *'He was prepared to sit down and explain things and slow down ... not rush through because they wanted so much learnt before the exam'*
Teachers' attitude to the subject
Sample statement: *'Teachers who used maths as a punishment especially times tables'*

Social contexts

Usefulness of mathematics

Sample statement: *'If you were good at other subjects ... then it didn't matter if you weren't too crash hot at maths'*

Peer attitudes to mathematics

Sample statement: *'My dad says it's important'*

Teachers' personal attributes.

Sample statement: *'Having a primary teacher who understood the difficulties her class were having ... she met the needs of the children'*

Analysis

Table 3 contains the number of responses in each of the four major categories. In each case responses were divided into those which gave a response which fitted only a single category (labelled below as 'single responses') of which there were 91, and responses which contained multiply categorisable statements (labelled below as 'all responses'). In all there were 127 responses.

Table 3: Number of responses in each of four major categories (percentages rounded to one place)

	Single responses (91)		All responses (127)	
	Number	Percentage	Number	Percentage
Mathematics	15	16.5	24	18.9
Self	26	28 6	26	20 5
Mathematics teaching	33	36.3	41	32.3
Social	17	18.7	36	28.3

The first stage of the analysis was to look at responses in the four major categories only, and to ignore whether the associated attitude was positive or negative. Those respondents who gave a single category response claim that the most common cause of their attitude (whether positive or negative) was to do with teaching methods (36.3%), with self (28.6%) a clear second cause. Social reasons (18.7%) and mathematics (16.5%) form the lesser groups of reasons. When responses containing more than one reason are included, teaching methods remain the most common cause claimed (32.3%), but social reasons exchange with self as the second most claimed cause (28.3%). In both instances (single and multiple causes) over one third of responses point to teaching methodologies as critical in the formation of attitudes towards mathematics. While self and social causes occupy a second ranking of causes individually, together they represent about one half of all reasons given for forming attitudes.

While these figures are interesting, they do not give a detailed picture of the causes given by these responden.s. In order to gain a clearer picture it is necessary to look further. In this instance all 127 responses were examined and Table 4 details the responses for all nine categories.

Table 4: Responses in all nine categories (percentages rounded to one place)

	Number of responses	Percentage of responses
Maths content	14	11.0
Maths complexity	10	7.9
Self - achievement	12	9.4
Self - confidence	14	11.0
Teaching methodology	22	17.3
Teaching - teacher attitude	19	14.9
Social - usefulness	10	7.9
Social - peers	3	2.4
Social - teacher personal qualities	23	18.1

It is clear from Table 4 that teaching methodology is believed to play a major role in attitude formation. Nearly as crucial are the teacher's personal qualities, which may well be linked in some way with teaching methodology. In fact 5 responses from this group combined teacher qualities with teaching methodology in the same response (which represents 21.7% of the responses on teacher qualities). Interestingly, peers appear to have little influence bèing mentioned in only 2.4% of all responses, and represent only 8.3% of all social causes given.

In order to determine whether these believed causes lead to positive or negative attitudes each category was divided on this basis. Tables 5 and 6 give the break-down of percentage responses for this analysis, ordered by frequency.

Table 5: Positive attitude responses ordered by frequency of occurrence (percentages rounded to one place)

	Positive attitude (%)
Maths content	3 1
Maths complexity	3 1
Social - peers	3 1
Self - confidence	9 4
Social - usefulness	9.4
Teaching - teacher attitude	12 5
Teaching methodology	15 6
Self - achievement	18.7
Social - teacher personal qualities	25.0

Table 6: Negative attitude responses ordered by frequency of occurrence (percentages rounded to one place)

	Negative attitude (%)
Self - achievement	2.1
Social - peers	6.3
Teaching methodology	7.4
Maths complexity	9.5
Self - confidence	11.6
Maths content	13.7
Teaching - teacher attitude	15.8
Social - teacher personal qualities	15.8
Social - usefulness	17.9

Which causes are the most frequently cited as attitude forming, positive or negative, can be readily seen in the above tables. With regard to causes of positive attitude the most frequently claimed is teachers' personal qualities (25% of responses). Typical of the responses that fall into this category are:

'I had teachers ...[whose]... enthusiasm shined through and rubbed off on me.'

'...having helpful teachers always there to guide me ...'

Causes of negative attitude are not as clearly separated as those for positive attitude. While usefulness is the most cited cause (17.9% of responses), teacher qualities (15.8%), teacher attitude (15.8%), mathematics content (13.7%) and self-confidence (11.6%) would all seem to be well supported causes. Typical responses in the usefulness category are:

'...when we asked her how could we use these methods [to] help us in life she couldn't answer. ... it seems to me that I learnt a lot ... that I will never use again.'

'I got very frustrated when we learned meaningless formulas [sic] all the time.'

As a final analytical aid, two attitude-cause response continua were constructed to give a visual perspective of causes which were believed by the respondents to have had effect on them. Figures 1 and 2 show each category of response on a continuum representing the percentage of responses in that category for which the associated attitude is either positive (Figure 1) or negative (Figure 2).

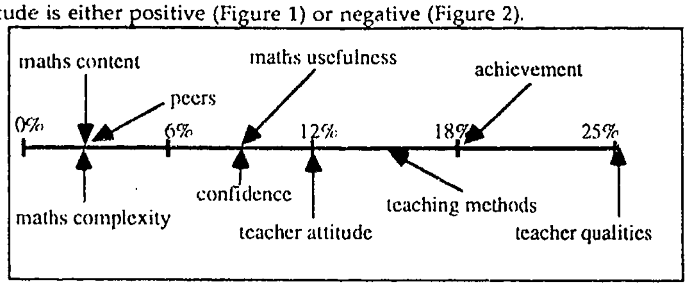


Figure 1: Positive attitude-cause continuum (% of all positive responses)

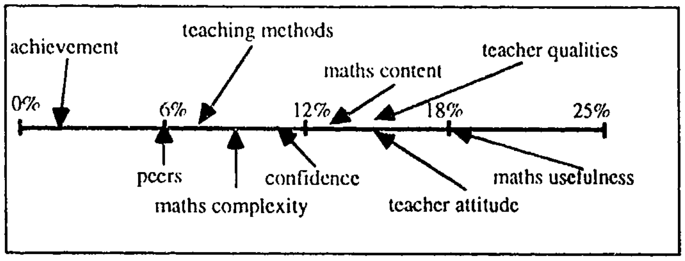


Figure 2: Negative attitude-cause continuum (% of all negative responses)

Discussion

It is clear that analysing people's replies to open questions about what they believe influenced their attitude to mathematics is an informative exercise. Using categorisations, such as those suggested by McLeod's review with suitable additions, appears to satisfactorily deal with all responses and forms a useful basis for analysis. To what extent one can generalise from this analysis is dependent on the sampling and questions asked, rather than the analytic procedure. However bearing in mind the limitations of sampling, the following points can be made. First, the question of mapping causes of attitude seems to be answered in the affirmative. It appears possible to establish a map of causation in the sense that those influences perceived as important by respondents can be placed relative to one another in such a way as to make clear priority foci for further investigation or classroom action. Whether all causes have been canvassed is open to question; larger and more intensive studies need to be conducted. Follow-up interviews to assist in validating written responses would seem a possible next step. Secondly, in regard to priorities for classroom action, this study makes quite clear that any attempt to treat all students as (attitudinally) similar is doomed to failure. Despite the similarity of causes given by both positive and negative attitude respondents (namely teacher qualities and teacher attitude) there exists at least one clear distinction between the two groups. Those respondents with a positive attitude are supported by their success (achievements) while those who hold negative attitudes focus not on achievement (or lack of it) but on the perceived uselessness of mathematics. A further difference in the responses concerns teachers themselves. Many of the negative attitude respondents cited teachers who used mathematics as a punishment, who taught *'rote'*, and had few positive personal relations with their students. By comparison, those with a positive attitude cited warm, friendly teachers who taught using concrete approaches and *'understood the difficulties we were having'*. Such cited differences indicate possible actions for the improvement of student attitudes in the future.

References

Adams, R., Doig, B. and Rosier, M. (1991). *Science learning in Victorian schools 1990.* Hawthorn: The Australian Council for Educational Research.

Borasi, R. (1990). The invisible hand operating in mathematics instruction: students' conceptions and expectations. In Cooney, T. and Hirsch, C. (eds) (1990). *Teaching and learning mathematics in the 1990s.* Reston, Virginia: National Council of Teachers of Mathematics. p 174-182.

Imai, T. (1993). Causal relationships between mathematics achievement, attitudes towards mathematics, and perceptions of mathematics teacher on secondary school students. In Hirabayashi, I., Nohda, N., Shigematsu, K. and Lin, F-L. (eds) (1993). *Proceedings of the seventeenth PME conference, volume ii*. Tsukuba: IGPME. p 244-251.

Ponte, J., Matos, J., Guimarães, H., Leal, L. and Canavarro, A. (1992). Students' views and attitudes towards mathematics teaching ar earning: a case study of a curriculum experience. In Geeslin, W. and Graham, K. (eds) (1992). *Proc ngs of the sixteenth PME conference, volume ii.* Durham: IGPME. p 218-225.

Leder, G. (1993). Reconciling affective and cognitive aspects of mathematics learning: reality or a pious hope? In Hirabayashi, I., Nohda, N., Shigematsu, K. and Lin, F-L. (eds) (1993). *Proceedings of the seventeenth PME conference, volume ii*. Tsukuba: IGPME. p 46-65.

McLeod, D. B. (1992). Research on affect in mathematics education: a reconceptualization. In Grouws, D. A. (ed) (1992). *Handbook of research on mathematics teaching and learning.* New York: Macmillan Publishing. p 334-370.

Relich, J. and Way, J. (1993). Attitudes to teaching mathematics: the development of an attitudes questionnaire. In Hirabayashi, I., Nohda, N., Shigematsu, K. and Lin, F-L. (eds) (1993). *Proceedings of the seventeenth PME conference, volume i*. Tsukuba: IGPME. p 276-283.

Schoenfeld, A. H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In Grouws, D. A. (ed) (1992). *Handbook of research on mathematics teaching and learning.* New York: Macmillan Publishing. p 575-596.

ENGINEERING CURRICULUM TASKS ON THE BASIS OF THEORETICAL AND EMPIRICAL FINDINGS

Abraham Arcavi, Nurit Hadas and Tommy Dreyfus*
The Weizmann Institute of Science, Israel

In this paper we describe and illustrate the process of engineering a curriculum, for a specific population, on the basis of students' cognitive characteristics and learning styles.

Introduction

The purpose of this paper is to illustrate the process of engineering a curriculum, taking into account theoretical views and empirical findings related to students' learning and understanding. Broadly speaking, our theoretical inspiration is constructivist. How do we envision constructivist design of curriculum tasks? We attempt to design tasks and problems which can potentially engage students' sense making, which build on what students know and which respect students' points of view and transitional conceptions (Moschkovich, 1992). We view learning as a long term, non-linear process, which is highly dependent on the context, and during which students should have meaningful experiences seeking to engage both the mathematical and common sense tools they have available. Taking into account that view of learning, we design the curriculum as an environment with learning opportunities which promote the creation of connections between pieces of knowledge. These learning opportunities are afforded by tasks and problems, engineered on the basis of research on learning and extensive experience in classroom implementation within a long term curriculum development project (Dreyfus, Hershkowitz and Bruckheimer, 1987). Detailed observations of students and classrooms are the main source for the redesign of the tasks according to the kinds of learning behaviors they promote.

The target population

We define our target population in two different ways: firstly, according to external criteria (established by the school system), and secondly, according to learning characteristics.

Our students are defined by the system as "low ability" 10th, 11th and 12th graders. Under the Israeli system students are streamed at the beginning of 10th grade into four main tracks: three of them leading to a national matriculation examination (at different levels), and the fourth leading to a non-matriculation certificate. Students streamed into the fourth option in grade 10 are deprived at a

* Faculty member at the Center for Technological Education, Holon, Israel.

young age (about 15) of a fair chance to achieve matriculation which is a necessary condition for acceptance to any institution of higher education. The mathematics these students learn in grades 10, 11 and 12 is minimal, mostly based on drill and practice around algebraic manipulation, and is not highly regarded either by the teachers or by the students themselves. Moreover, it is rare (and very difficult) for a student streamed into the non-matriculation track to be able to return to the matriculation track: the gap is too wide in terms of content and depth.

The student population streamed into the non-matriculation track is not homogeneous in terms of their capabilities. Nevertheless, we can point to some characteristics our students share (some of which are shared to a lesser extent by the whole population). Even at the risk of oversimplifying a complex picture, we briefly state these characteristics:

- *Short lived memory.* Students tend to forget very quickly both, procedural techniques and conceptual links.
- *Sensitivity and lack of sensitivity to contextual changes.* In some tasks, students tend to over-generalize ideas or procedures without examining whether the generalization can be done. On other occasions, precisely the opposite happens: as soon as a contextual feature is different, they are unable to recognize that their knowledge can transfer to the new situation.
- *Low frustration threshold.* Students usually do not persist in the solution of problems, they tend to give up as soon as they encounter minimal difficulties. Often this abandonment is accompanied by anger and frustration.
- *Length and scope of engagement with problems.* Students have difficulties to deal with problems in which the solution process involves several stages. For example, they tend to get lost when they need to devise a global solution plan, follow its subgoals in detail, and periodically return to the global plan in order to pursue the next subgoal. They fail to distinguish the nature of global aspects from that of technical details. This and the next point are linked to the students' short lived memory and low frustration threshold mentioned above.
- *Length and scope of engagement in discussions.* Students are capable to work on tasks adapted to their previous knowledge and capabilities for quite a long time, but have a very short attention span for classroom discussions which can easily disintegrate beyond an initial period of about ten minutes.
- *Lack of confidence.* Students are extremely unsure of what they do, know and understand. A "poker face" teacher reaction towards students' suggestions may induce instantaneous derailment from a promising track.
- *Stereotypical views of mathematics and mathematical activity.* Students usually view mathematical activity as a technical task, and initially they tend to distrust and dislike problems in which the answer is an "idea" rather than a "number". They are not used to rely on common sense, observation, and visual skills;

however in the long run they enjoy being relieved from the burden of techniques they have difficulty to master, and the possibility to make sense of what they do.

- *Fragmented knowledge*. Students' focus and activities are very "local" in nature, and they have difficulty to connect different pieces of knowledge.
- *Difficulties with communication*. Students have great difficulties to express themselves, either orally or in writing.

The '2U' Curriculum Project[1]

The goal of our curriculum project is to offer non-matriculation students the opportunity to learn the same meaningful mathematics their friends learn at the lower matriculation level (3 units), but with an appropriate approach and by building on what these students are strong at. The approach is based on teaching the same topics but stressing their intuitive, qualitative, graphical, common sense aspects while downplaying the technical and formal manipulations which are difficult to the students and easily cause them to detach the activities from the underlying meanings. The general objective is an "open door" policy to enable them to return to a matriculation track with a reasonable effort if they decide so.

The principles upon which we base our approach feed on the characteristics of our target population as described above. We depart from situations in which common sense is engaged, basing learning experiences on qualitative, visual and graphical reasoning, and integrating multiple representations as different ways to envision the same ideas. An attempt is made to link students' thinking to their acting (on sketches and physical devices such as graphs on transparencies which can be easily moved around), in order to help them make sense out of what they do. The use of any thinking tool they have available is stimulated, and they are explicitly encouraged to build connections. Estimated results, visual and numerical approximations, are legitimized and qualitative and informal reasoning is preferred over technical and symbolical treatments. The problems are built in such a way that students are encouraged to go back and forth between the "big picture" and the "small details" (see Tall and Thomas, 1991, for a discussion of flexibility between global and local processing); the solution to some problems is no more than the design of a course of action or a plan. We try to keep our predisposition towards being surprised regarding what these students can do and thus rarely dismiss a problem a priori because it seems to us they will fail; instead, problems are presented in a manner in which they might be able to deal with them. The purpose is to convey the message that they can do mathematics and that we have high expectations from them.

---

[1] The name '2U' is a shorthand for 'two units' which is the English translation of its Hebrew name. The number of units is the identification of the level of mathematics students take in the last years of their high school education: 3, 4 and 5 units are matriculation tracks.

Engineering the curriculum - an example

The process of engineering the curriculum as we envision it, feeds on three main components: our beliefs about what mathematics is and what mathematical activity should be, our theoretical biases about learning and teaching issues, and our knowledge of the population's characteristics. It is on that basis that we delineate the scope of the mathematics our students should be exposed to and the internal consistency and flow of each learning module (or as we like to ask: 'what is the story line of the module?'). These inspire the actual design of the specific problem situations.

The design process itself consists of two major phases: the *in-vitro* creation of a problem (on the basis of the components listed above) followed by *in-vivo* (and *in-situ*) classroom trials. These two phases are undertaken by us as developers and teachers for every single module we develop. Since we can only guess what students' performance in a particular problem situation will be, each module is pilot tested in at least two classes, and the feedback is analyzed and used in the redesign of the module. The second phase thus involves modifications and adjustments which are then re-tried in yet another classroom. It is only after this that we embark in large scale implementation. In this section we illustrate the process of engineering a specific curriculum activity.

As mentioned above the first step is to "inspect" the mathematics. In our example, we started by considering a central theme in algebra, analysis and analytic geometry: the use of at least two main representations, the symbolic and the graphical one (including basic geometrical facts) and the "Cartesian connection" between them (Schoenfeld et al., 1993). In plain words, the Cartesian connection sounds deceptively simple: "A point whose coordinates are (x_0,y_0) is on the graph of the function $y=f(x)$ if and only if the pair (x_0,y_0) satisfies the equation $y_0=f(x_0)$." This statement may seem trivial to experts in the domain, but it is the source of a host of learning difficulties for students.

The following is a rather traditional task which requires some understanding of the Cartesian connection: "Determine an equation of the line that is parallel to $y=2x-5$ and that goes through the point $(1,4)$." We would like students to be able to solve this problem without major difficulties after studying algebra or analytic geometry. Moschkovich et al. (1993) analyze extensively the kinds of difficulties this task involves as an example of how, beyond the Cartesian connection, the perspective under which one envisions the mathematical entities one deals with, either as objects or as processes, influences the solution process. The authors report that the success rate with tasks of this nature, as reflected in US national surveys, was extremely low.

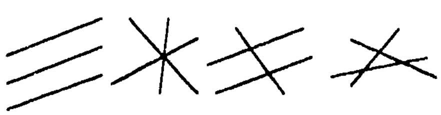
The problem as is has an analytic-symbolic departure point and thus could be very difficult to our students. At this point, our view of learning and our design principles come into play. If we want our students to achieve some understanding of the translation between representations in general, and the Cartesian connection in particular, we must build on their previous knowledge (including, of course, the learning trajectory of previous sections in the module) and their particular ways of knowing. Briefly stated, when we consider our students to be ready to meaningfully face such tasks, they know:

- the graphical meaning of the slope of a linear function - rise over run - by counting on the Cartesian grid;
- how to physically "put" a line (drawn on a transparency and which they carry with them to almost all the problems) on a Cartesian grid in such a way that its equation becomes, say, y=3x+4;
- to identify the b in y=ax+b as the translation (upwards or downwards) of a whole line of the form y=ax; namely, on the basis of a table of values of, say, y=2x and y=2x+3, they can "physically" slide the line representing y=2x three units upwards, and identify "three" as the y-coordinate of the point which was originally at (0,0);
- to draw any line through a single given point, and to read its equation from the graph (by looking for the slope - rise over run - and the y-intercept).

The following is the "in-vitro" version of the problem situation which emerged as a consequence of all the above considerations: First, present students with the two possible mutual positions of two lines in a plane, namely, parallel or intersecting. Then request students to produce two pairs of equations, one for each situation. Finally, and most importantly, request them to draw all the different possible situations for three different lines in the plane, and write down corresponding triples of equations for each situation.

When we went to the classroom, students had no problem with two lines: since they knew from their manipulative experiences that parallel lines have the same slope, they had no problem to write the equations for two lines which have the same "**m**" but a different "**b**". Similarly, since they knew that the **b** represents the y-intercept, they had no problem to produce two equations with the same **b** and different **m**.

For the three lines situation, students collectively found that the four possible situations are:



Finding triples of equations for the first three situations is in some sense an extension of the previous activity with two lines, and thus there were no major problems. However, the fourth case (three intersection points) was not as easy.

Many students gave three equations with three different **m**'s and three different **b**'s and thought they had solved the problem. The teacher then asked whether they are sure that with different parameters they would always get that graphical situation, and they seemed to be sure. The teacher, who perceived that this was one of the situations in which she could make a lot out of those "10 minutes at most" available for collective discussions, requested to write the equation of a line through (2,3) - a point not on the y-axis - and they were unable to do so right away. What they were able to do is to "put" a line through that point, read its slope and y-intercept from the graph, and then produce the equation. This way of producing an equation is some kind of "surrogate" Cartesian connection which we believe is a step towards building deeper connections. (In a similar way students can approach the initial problem above of producing an equation of a line parallel to a given line through a given point.) By repeating this procedure three times through the same point, they were led to obtain the second situation, namely three lines through one point, and realized that different **m**'s and different **b**'s may lead to the second as well as the fourth situation.

The classroom trial made it clear that students could use what they knew ("putting" a line through a point) to revise the inaccurate hypothesis they suggested. Moreover, in that class the teacher took advantage of this in order to deal with the Cartesian connection. What she did was to request once more the equation of a line through a given point, and then she asked to check whether the coordinates of the point satisfy the equation, in order for the students to get a feeling what that means. Then she requested to produce the equation of yet another line through the same point, but this time without drawing it. Since she was aware of the contextual dependence of the knowledge, she felt the need for asking at this point, whether the two equations have a common solution. The question was not at all trivial for the students (because it involved a switch of representation) and was used as introduction to the meaning of the Cartesian connection in the context of two intersecting lines.

The design idea, which began *in-vitro* was followed by the *in-vivo* trial. During the trial the teacher felt the need to redesign the task and to adapt it to the students' learning experience she was facing. This task, redesigned and adapted in the way described, made its way into the textbook for large scale implementation.

We do not wish to give the impression that we believe the translation between representations and the Cartesian connection will be easily understood by just working on a few tasks of this nature. What we do claim is that students who share our students' characteristics and background can be exposed to problems in which they encounter aspects of the connections which they are able to handle and partially understand. We found that the way to produce tasks of that nature is to undergo the engineering process we described. The process has the following characteristics:

- One of our students' difficulties, as mentioned earlier, is to focus on global issues. However, our design is not aimed at building general conclusions from "atomic" logically organized small tasks, because such an approach does not ensure that students will take the leap to a generalization and/or to knowledge transfer. Instead tasks are designed to expose students to a global graphical situation (e.g., the four possibilities for three lines to lie on a plane) from which they are able to engage in finding the "atoms" (e.g., equations, intersection points etc.) for themselves, which in this way are more meaningful and contextualized.
- Instead of using the symbolic perspective as a point of departure, problems can be presented qualitatively, graphically and geometrically. The graphs and the geometry can become the springboard towards different levels of analytic activity, some of it symbolic, according to the students' "breaking points".
- The engineering is not meant to produce tasks to help students to "achieve" results. The end process should be an activity which engages students and supports them in building the connections which are critical to understand the issues.
- Once the solution is reached, the problems should always include some question asking about aspects of the task from a slightly different perspective. Even questions which may seem trivial to the teacher, may not be easy for the students (see the example above, in which the teacher requested to find the intersection point of two lines through a given point).

Conclusions

The design process exemplified above is used throughout the development of the entire three year curriculum. The engineering of tasks for the curriculum is characterized by four main aspects, three of which have been discussed in this paper:

- The constructivist-based principles on which all phases of the design process are consciously built, including engaging students' intuition and common sense, making them go back and forth between the "big picture" and the "small details", linking their thinking to their acting in order to connect their actions to the underlying meanings; generally, developing flexible thinking and explicitly encouraging them to build connections.
- The direct influence of classroom trials on the design within a circular process which includes the in-vitro design phase and at least two sequential classroom-based revisions.
- The essential manner, in which the design takes into account the characteristics of the target students including the context dependence and the fragmentation of

their knowledge, their limited scope of engagement and their low frustration threshold.

The fourth main aspect could not be discussed in this paper. It is the spiral recurrence of basic themes such as the Cartesian connection throughout the curriculum. Different curricular topics are not only approached on the basis of the same principles and engineered by the same methods, but they are carried by the same fundamental ideas; for example, the Cartesian connection in the situation of intersecting graphs appears in a similar manner as it was presented above for linear functions, also in the modules on quadratic functions, on trigonometric functions, on analysis and on linear programming and repeatedly in the module on analytic geometry. In this manner, we hope that the students are given the opportunity to make these fundamental ideas their own in a meaningful and lasting manner.

References

Dreyfus, T., Hershkowitz, R. and Bruckheimer, M. 1987. "Processes in the transition from syllabus to curriculum". *Zentralblatt für Didaktik der Mathematik* 19(1), 19-25.

Moschkovich, J. N. 1992. *Making Sense of Linear Equations and Graphs: An Analysis of Students' Conceptions and Language Use*. Doctoral Dissertation, Graduate Division, University of California at Berkeley.

Moschkovich, J.N., Schoenfeld A. H. and Arcavi A. 1993. "Aspects of Understanding: On Multiple Perspectives of Linear Relations, and Connections Among Them". In T.A. Romberg, E. Fennema and T.P. Carpenter (Eds.), *Integrating Research on the Graphical Representation of Function*. Erlbaum, NJ., pp. 69-100.

Schoenfeld, A. H., Smith, J.P. and Arcavi, A. 1993. "Learning - The Microgenetic Analysis of One Student's Evolving Understanding of a Complex Subject Matter Domain". In R. Glaser (ed.) *Advances in Instructional Psychology*, Vol 4. Erlbaum, NJ. 1993, pp. 55-175.

Tall, D. and Thomas, M. 1991. "Encouraging versatile thinking in algebra using the computer". *Educational Studies in Mathematics* 22(2), 125-147.

TOWARDS A THEORY OF LEARNING

Janet Duffin and Adrian Simpson

University of Hull, U.K.

We summarise some of the ideas behind a personal theory of learning we have built up over the past three or four years. These came about in response to our individual needs to account for the learning, and non-learning, observed in our students, as well as that experienced directly ourselves. The development of the theory began when we tried to make sense of just two or three incidents in our shared experience but has grown to become the central core of our research and the framework on which we hang our understanding of much of the teaching and learning we do. During its growth we have constantly reflected on the implications of the theory for our own classroom practice in particular, and for educational research in general. The development of the theory, as a learning experience of each of us, is a prime example of the learning that the theory itself accounts for.

Our reasons for wanting to present this paper are twofold. We want to open our theory up to outside scrutiny to help us to develop it further. In addition we hope that the ideas of others will enable us to move towards its application to learning and teaching at a variety of levels.

As is often the case in building a theory, much of the essence of the development will no longer show through in any bald statement of the tenets which now form its basis, rather in the way that a mathematics textbook cannot show what experiences the writer went through in writing it and, for the student, cannot show the mathematical thought that comes between the lines of the text.

The avenues we explored and got stuck in, the key breakthroughs (or cognitive shifts, Mason and Davis, 1988), the emotional commitments and the failures are hidden from view in the definitions of the fundamental concepts of the theory. Though we will not retrace all the intermediate steps of the development, we will give the genesis of the concepts and the purpose we had in exploring them and we will discuss some of the major breakthroughs we have made as we have developed the theory.

We started working together on this theory as a result of having shared our reactions to a piece of work done by an eight year old girl (Duffin and Simpson, 1990). We felt that this girl's work was creative and this led us to seek a definition of creativity which could include her work. From this experience we began to see the value of sharing our reflections on our own experiences of learning. This in turn led us to feel that there were a number of aspects of these experiences which our ways of thinking about education at that time did not account for to our own satisfaction.

Conscious that we were drawing on basic ideas of constructivism in the way that we had approached the girl's work and our shared experience of it, we began to think about the kinds of experience that learners go through in order to build their own knowledge. In doing so, we were taking as axiomatic the notion that all learners are trying to make sense of their world of experience.

Initially we came to a first perception of the concepts within our theory from our attempts to analyse critical classroom incidents we had observed or been otherwise involved with (Duffin and Simpson, 1993a). These first examples came from our own recent experiences with, respectively, primary children and their teachers, and mathematics undergraduates. We came to believe that, disparate as these incidents appeared to be on the surface, they had common features which we wanted to explore.

Having formulated our first categorisation of a learner's experience from examining these incidents, we then looked for other instances from our experience which exhibited similar characteristics. It is from this initial exploration and analysis that our theory has been developed and expanded.

The theory

We postulated that all learners, as they interact with their world, build up internal structures to account for what they experience. We came to see three basic types of experience that a learner meets, which we termed **natural**, **conflicting** and **alien**. Which category a particular experience falls into depends on the extent to which the experience fits, or fails to fit, the individual's internal mental structures at that time.

So a natural experience is one that fits a learner's current mental structures, it is not surprising, and not unexpected, and is perceived as obvious and uncomplicated.

A conflicting experience, however, is one that the learner sees is inconsistent with their mental structures. It jars with their expectations and highlights possible contradictions or limitations in their current ways of thinking. It may bring two previously disparate mental structures into play in one situation: one may contradict the experience or may not cope with an experience it had been expected to be able to deal with.

An alien experience is one that the learner finds has no connection with any of their internal mental structures. It is seen as meaningless because of its lack of either fit or perceived inconsistency with other ways of thinking. No mental structure can cope with it, nor is there a mental structure that was expected to do so.

Distinction between conflicting and alien

The distinction between conflicting and alien was one of the main breakthroughs in the development of the theory. We had previously restricted ourselves to just the notions of natural (fitting) and conflicting (non-fitting and therefore a situation in need of resolution) when we recalled instances of situations in which we observed what we felt were obvious contradictions in something a learner had done. In some of these situations, however, there was no evidence of an internal conflict: of the learner having to rethink their current mental structures to cope with this contradiction.

The incident which first drew our attention to this was that of a six year old boy (Duffin and Simpson, 1993a) who had written down a subtraction sum:

```
526
-249
---
323
```

and had written beside the answer, "But the real answer is 277"

Although initially we accepted this statement, later we felt that our theory, with only natural and conflicting defined, did not account for it. He appeared to be oblivious to the contradiction an observer could see in the two different answers he had produced. His comment suggested that he had some reason for thinking that the original answer he had produced was not correct but he did not feel the need to investigate it.

From this, and other similar incidents we were then able to identify as being in the same category, we realised that it was important to make a distinction between a contradiction that an observer may perceive and the learner's perception of the experience. Thus each of our definitions concerns the learner's perception in terms of the fit with their mental structures.

The consequent distinction between the two concepts serves to highlight the experiences as belonging to the learners themselves and, for us, reinforced the constructivist basis of our work. It also drew our first attention to what was later to become an important element in what we were doing, namely the importance of the learner's own perspective in a piece of learning, something which became manifest in our work through our study of our own learning as well as that of others we came into contact with.

Response to experience

We also began to realise that the internal mental structures that determined whether particular experiences were natural, conflicting or alien were themselves formed, and subsequently shaped, by the results of responding to previous natural, conflicting and alien experiences. While the forms of these responses were implicit in our earlier work, we have since come to separate them out in order to highlight the complementary, but highly complex, two-way interaction between the learner's internal structures and their world of experiences (Duffin and Simpson, 1993b).

We saw the learner altering their internal mental structures by strengthening, merging, limiting or destroying already formed ways of thinking, or forming new ones.

A response to a natural experience is to add that experience to the mental structure it fits with. Thus the mental structure grows and becomes stronger as a result of this extra experience, or class of experiences, it can now cope with.

There might be any of three types of response to a conflict which is perceived by the learner. It might

- effect the merger of the experience with old inconsistent mental structures in such a way that a new mental structure, with some aspect of the old ones, is formed which can account for both old and new experiences in a more consistent way. Thus the mental structures are changed to make the experience fit, i.e. the experience becomes natural,

- alter the mental structure so as to limit it, changing the structure so that, while not accounting for the experience as such, recognises that there is a class of experiences that the structure cannot cope with. In this way the conflict is resolved by changing the structures so that the experience no longer jars with any internal mental structure and hence becomes alien.
- destroy a way of thinking. The structure may be destroyed if the conflict highlights that all (or a fundamental part of) the class of experiences, that a way of thinking was expected to cope with, cannot be accounted for. Again, this resolves the conflict by ensuring that there is no structure with which the experience can jar and that experience (together with some previously natural ones) become alien.

In reflecting on the incidents we had observed and on aspects of our own learning, we also found that alien experiences can cause a variety of responses. Having no connection with the learner's current mental structures, the experience may essentially go unnoticed and so be ignored. Alternatively, a learner may notice that there is an experience which they have no way of coping with at that time and, having no way of accounting for the incident, they may actively avoid it by resorting to a familiar procedure or situation in which they expect more comfortable and easily handled experiences may arise.

Instead it may be that, having noticed the experience, the learner stores it away - not as an addition to an already formed structure as in the case of a natural experience, but as an isolated structure on its own. The learner constructs the seed of a new mental structure, with no substantial connections to any others currently held. Such a new structure, while weak, can subsequently be strengthened, merged, limited or destroyed by further responses to new natural, conflicting and alien responses.

With the identification of the distinction between conflicting and alien experiences, and our later development of the responses to them, we were thus able to account more fully for the behaviour of the six year old boy described above. In getting an answer which was not in accord with his knowledge of what it should be, he was responding by ignoring what to the observer appears to be a contradiction but which, for him, is alien.

Implications of the theory

Reflecting on the separation of the two interacting parts of the theory came to be another key breakthrough in its development. We began to see more clearly the nature of the interaction between the individual and their experiences - on the one hand, the internal mental structures *determine* the way in which the learner responds to the experiences (or the type of experience the learner perceives them to be). On the other hand, the way in which the learner responds *determines* the way in which the internal mental structures grow and are modified.

This form of reflection also led us to another important realisation. We had come to build the theory, initially, in order to account for incidents which we had observed in classrooms and lecture rooms. We had become aware quite early that we were also bringing in incidents from observing both trainee teachers and teachers on in-service courses and, more significantly, we found that we were using incidents in which we had been the learners, responding to natural, conflicting and alien experiences ourselves. So we had an indistinct feeling that there were analogies between the ways in which pupils, teachers and ourselves all learn.

Our reflection on the theory and on how it had developed, however, gave us a significant insight into this. As it stands, the theory does not mention *who* the learner is, what particular situations the learner's experiences arise from or what the learner is learning. In the basic theory, all learners are the same - individuals who respond to natural, conflicting and alien experiences (though, of course, the experiences and the internal mental structures through which they are interpreted, and which are modified as a result, can differ significantly from individual to individual).

So we saw that the fundamental processes that a pupil goes through in developing their mathematics are the same as those which a teacher goes through in developing their teaching. These, in turn are the same processes that education researchers go through in developing their theories of education and are also precisely the processes we have gone through (and are still going through) in developing this theory.

It has been suggested (Mason, 1987) that mathematics researchers are of two kinds: "We are all trying to model or describe the inner world of experience. Some of us proceed by contemplating and studying other people, or by studying ourselves as if from outside; others proceed by contemplating and studying ourselves from inside." As our work developed we realised more and more that we were using our own learning, from the inside, to further our thinking, so that introspection was becoming a research tool for us as we developed our theory. The breakthrough to seeing that, within our theory, all learners are the same gave us a powerful way of justifying this which the Mason reference reinforced.

If, at the level of responding to experiences, everyone is the same, it means that examining any group of learners can provide insights into learning in general which can then be applied to others. So in examining our own learning, we can try to develop general insights into the learning of others.

We also realised that, in examining the learning of others, we are merely interpreting the natural, conflicting and alien experiences provided by our observations through our internal mental structures, and building up an image of the way in which we think those others are learning. Constructivist theories suggest that we have no direct access to the learning of others. Our own learning, however, is open to as direct a view as possible and, albeit with the benefit of hindsight, we can carefully examine the ways in which our own mental structures are formed.

The theory's future

When we began our work on this theory, we were looking for a way of accounting for just three incidents from primary, secondary and university mathematics education. While the number of such incidents on which the theory has been based has grown considerably and the number we have used the theory to explain is tremendous and growing all the time, it is valuable to take this opportunity to examine where we think the theory is going and ask ourselves the question "Why are we still working on it?"

The theory has come to be a very powerful one for us, colouring much of our thinking about our own teaching and learning. While it is still very personal to us, a number of others have commented on the strong resonances it has for them. So the theory is influencing our own practice, has long acted as a useful language for our own discussions and we are beginning to see it as a way of communicating some of our ideas to others. Our goal remains that of using and building the theory in order to give ourselves a framework on which to place our thinking about teaching and learning, but in looking to the future we are beginning to see just how much has yet to be done.

It is inevitable that, in writing about where we think we are going, we will find much uncertainty and many conflicts to be resolved. Indeed, we find that, while we agree on much of the substance of what needs to be done in the medium term and strongly agree on our goals, we have not yet resolved a conflict between us about which direction to go first. In any case, as we have found in the past, the avenues of our interests change and (as we found with our discovery of the notion of alien experiences) seemingly minor conflicts can be resolved into major new parts of the theory which strongly influence the directions we are then able to take.

In the long term, however, we are hoping to develop new layers of the theory up to the point where we can begin to account for as much of what we see happening in our classrooms and lecture rooms as possible. We do this with a view to seeking out the implications of such a developed theory for our own teaching, as well as to encourage others to do the same for their own teaching.

Much needs to be done before that. In the short term, we realise that the detail of the theory will become unwieldy when we come to the stage of trying to examine the long term effects of our teaching within a classroom of 30 pupils, or a lecture theatre of 130. If we only have the language of individual response to individually determined natural, conflicting and alien experiences, trying to account for the simplest situation will become highly complex and perhaps also tedious, Undoubtedly, within our theory, the situation where many learners are involved *is* complex, but we hope that some of this complexity can be absorbed by using broader terms which, themselves, can be based on the fundamentals of the theory.

For us, there are many aspects of education that need to be well-defined in the theory before we can move on. If we use words like 'knowledge', 'understanding' and 'communication', we need to be clear what we mean by them and, in our case, we need to find our own definitions of these words in terms of our own theory: in terms of the learner's internal mental structures and the natural, conflicting and alien experiences that influence them.

We have already begun a tentative exploration of the notion of understanding within our theory. It seems to us that understanding may be defined as the awareness of connections between internal mental structures. This appears to begin to fit some of the criteria that other authors confer on understanding such as being able 'to make use of in various ways', 'understanding as a continuum' etc. (See Byers, 1980, for a more comprehensive account of some criteria of understanding). It also appears to echo aspects of what has been said by Skemp (1976).

Such definitions, however, need to be explored more fully: we need to ask such questions as "Does our definition fulfil the criteria that others expect of understanding?", "What are the implications (for teaching and learning) of this definition of understanding?" etc.

We also need to look at the large-scale particular mental structures. So far, the theory deals with mental structures in general terms and, quite deliberately, does not refer to the specific content of what is being learned and what particular experiences the structures relate to.

Some particular structures, however, need to be examined. For example, we suggest that the vast majority of people build up a structure that deals with interacting with other people - a 'social structure' which determines what kind of experiences other individuals provide for them and which contains their notions of what kind of experiences they provide for other individuals. This particular structure, we feel, may contain ways of dealing with the problem that others have raised concerning general constructivist theories - the problem of intersubjectivity and an apparent shared meaning between individuals.

There are other such structures that need to be explored - that which people build to deal with their preferred mode of learning (do they like to learn in alien ways, building many separate mental structures and, perhaps, connecting them through conflicts later? Do they prefer to learn in natural ways, trying to make sense of each new experience as it occurs, looking always for natural experiences or conflicts that they can resolve before they move on?). What about the structure that deals with an individual's 'world view' and what they believe knowledge is (views highlighted by Perry, 1970)?

These and other large-scale particular structures need to be explored. We need to see how they are built and what effect they have on an individual's learning and understanding.

A further level is to explore 'topic structures'. We need to look at particular detailed structures and how they are built up and what effect they have. For example, what kind of structures are built up that deal with arithmetic?

There is no doubt that the scope of our theory has grown and our aims for it are ambitious. We feel that we currently have a theory that can account for individual learning incidents and it is already beginning to influence our everyday teaching in a variety of ways. Our medium term aim is to face up to some of the complexities, like intersubjectivity, inherent in a theory that starts at as basic a level as the individual's experience and, in the long term, to build a theory which can be the mainstay of our own daily practice.

References.

Byers, V. (1980), "What Does it Mean to Understand Mathematics?", *International Journal of Mathematical Education in Science and Technology*, 11(1), pp1-10.

Duffin, J. and Simpson, A. (1991), "Interacting Reflections on a Young Pupil's Work", *For the Learning of Mathematics*, 11(3), pp10-15.

Duffin, J. and Simpson, A. (1993a), "Natural, Conflicting and Alien", *Journal of Mathematical Behavior*, 12(4), pp313-328.

Duffin, J. and Simpson, A. (1993b), "Natural, Conflicting and Alien: Concepts for Analysing the Learning Process", *Proceedings of BCME-2*, Leeds.

Mason, J. (1987), "Representing Representing: Notes following the Conference" in Janvier, C. (ed.) *Problems of Representation in the Teaching and Learning of Mathematics*, Lawrence Erlbaum: Hillsdale.

Mason, J. and Davis, P.J. (1988), "Cognitive and Metacognitive Shifts", *Proceedings of PME XII*, Hungary.

Perry, W. (1970), *Forms of Intellectual and Ethical Development in the College Years: A Scheme*, Holt, Rinehart and Winston: New York.

Skemp, R. (1976), "Relational Understanding and Instrumental Understanding", *Mathematics Teaching*, 77, pp20-26.

MAKING SENSE OF A MATHEMATICAL MICROWORLD: A PILOT STUDY FROM A LOGO PROJECT IN COSTA RICA[1]

Laurie D. Edwards

University of California at Santa Cruz

A pilot study of sixth grade students engaged in mathematical explorations was carried out in the context of a national Logo project in Costa Rica. Thirty-four students, approximately 12 years old, worked in pairs investigating a mathematical microworld written in Logo. The investigation had three phases: (1) Open exploration of the microworld, during which the students recorded their observations and formulated hypotheses about how the program worked; (2) Group discussion and sharing of hypotheses; and (3) Additional guided discovery and problem-solving. The students were successful in discovering certain functions of the microworld during the first phase, but their hypotheses were improved after discussion with their peers, guided by the instructor. The students were successful in applying their knowledge of the computer microworld in the problem-solving tasks during the third phase.

Introduction

The purpose of this report is to describe work with Costa Rican students in the area of computer-based mathematical explorations, setting this work in the context of recent research on social and cognitive factors involved in the construction of mathematical knowledge, and on the role that new interactive technologies can play in the learning process. The report will describe the results of a pilot study in which a simple Logo microworld was introduced to upper-elementary students, who worked together in pairs to make sense of the environment, discussing and writing hypotheses and completing guided discovery worksheets. The students had never carried out mathematical explorations within the kind of collaborative social setting used in the pilot study; however, they made good progress overall in being able to <u>use</u> the microworld to create or match patterns, and in beginning to <u>discriminate</u> its distinctive features, both of which have been described as important aspects in building a mathematical understanding of a microworld (Hoyles & Noss, 1987).

Related work and theoretical framework

The research reported here was carried out in the context of related work in two important areas: the use of computer-based learning environments in mathematics instruction, and the role of social interaction in the construction of mathematical knowledge. A wide range of computer environments labeled "microworlds" have been created, and a body of research has accumulated on their use in the learning of mathematics and science (*e.g.*, diSessa, 1990; Edwards, 1992a; Hoyles &

[1] The work described in this report was supported by the Fundación Omar Dengo, San José, Costa Rica. Information about the Computers in Elementary Education Project can be obtained by writing to the FOD, Apartado 1032-2050, San José, Costa Rica. I would like to thank my collaborators, Anny Gonzalez and Efraim Lopez for their help and support.

Noss, 1987, 1992; Thompson, 1987) Pea defines a microworld as "a structured environment that allows the learner to explore and manipulate a rule-governed universe, subject to specific assumptions and constraints" (Pea, 1987, p. 137). Students learn from microworlds by interacting with the mathematical or scientific concepts embodied in the computer programs, and by "debugging" their understanding of the rules which govern how objects behave in the environment (Edwards, 1992b). However, the learning which takes place depends not only on the design of the technological artifact but also on the social context, on the way in which the computer environment is introduced and on the activities structured around it.

The social nature of knowledge construction has been addressed both within the mathematics education community and more broadly (*cf.*, Resnick, Levine & Teasley, 1991). Cobb, Wood and Yaekel (1992), for example, have investigated in great detail how young learners co-construct understandings of mathematical ideas and operations while interacting in pairs and small groups. These children are taught in classroom settings where they are encouraged to work together to solve mathematical problems, and then to verbalize their solutions to the whole group. In such settings, the children engage in processes which include:

> ...resolving obstacles or contradictions that arise when they use their current concepts and procedures, accounting for a surprising outcome..., verbalizing their mathematical thinking, explaining or justifying a solution, resolving conflicting points of view, and constructing a consensual domain in which to talk about mathematics with others (Cobb, Wood & Yaekel, 1992, p. 158)

One goal of the pilot study reported here was to create a social setting for upper elementary Costa Rican students in which they had the opportunity to engage in the learning processes described above, but one in which the mathematical problem solving would take place while exploring computer microworlds.

The context of the project

The Mathematical Explorations in Logo study was carried out as one part of the Omar Dengo Foundation's Computers in Elementary Education Project. This project, ongoing since 1988, has placed computer laboratories (a network of 20 IBM PS/2 computers plus printer) into more than 160 Costa Rican schools, primarily in rural and marginal urban settings. The study reported here thus took place within the context of an established and successful Logo education project. The students and teachers were comfortable with computers, and all students, from kindergarten through sixth grade, were accustomed to spending at least 80 minutes a week on Logo activities. However, although the students had an adequate level of programming knowledge for creating their Logo projects, there was no explicit attempt to relate mathematics to programming. Furthermore, as in many places, Costa Rican students were often taught mathematics in a somewhat rote and decontextualized fashion. Collaboration and discussion did not typically occur in their regular mathematics classes. Thus, a new goal was set, that of extending the use of Logo into the teaching and learning of mathematics through the creation of Logo-based microworlds, and of utilizing these microworlds in a social setting which encouraged collaboration, discussion and verbalization of mathematical discoveries.

The CUERDAS microworld

The microworld, called "CUERDAS" ("Strings"[2]) is illustrated in Figure 1. (This microworld is an adaptation of a Logo procedure which has been used within the Logo community for some time). The procedure CUERDAS takes two inputs, and creates either polygons or "star" figures by connecting equidistant points on the circumference of a circle. The first input gives the number of evenly-spaced points around the perimeter of the circle, and the second input is used to "count off" the distance between successively-connected points. The figure created depends on a fairly simple mathematical pattern, specifically, whether the ratio of the first to the second input results in a whole number or not. As seen in Figure 1, whole number ratios create simple polygons; ratios involving fractions result in stars.

These and other mathematical regularities or "rules" are not immediately obvious, yet learners can discover them through experimentation with the microworld. The first task set for the students in the study was to find out the purpose of each input in the CUERDAS procedure. Further guided explorations focused on which inputs create polygons and which create stars, and on the fact that pairs of inputs having a common ratio result the same figure (for example, both CUERDAS 6 2 and CUERDAS 3 1 make an equilateral triangle).

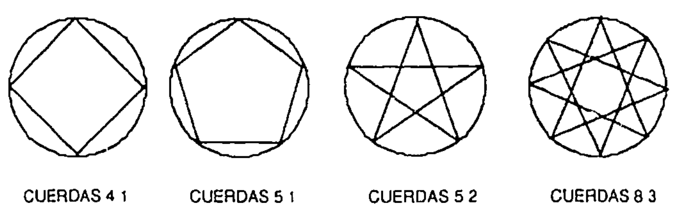


FIGURE 1: The microworld CUERDAS with four sets of inputs

[2] The microworld was called "CUERDAS" or "Strings" because a concrete version of this activity involves creating "string pictures" by physically connecting pins or nails on the circumference of a hoop with lengths of string.

Methodology

The study took place in an urban school in a poor neighborhood in San José, the capital city of Costa Rica. A class of 34 sixth-grade students (approximate age, 12 years) participated in the session, which was set up to pilot-test the software. Sixteen boys and 18 girls took part in the pilot study during an 80 minute afternoon session in the school's computer lab. The students' regular computer teacher was present, as well as the investigator and a specialist in computer education from the Omar Dengo Foundation, who served as the instructor during the pilot session.

Data were collected through observations of the students as they worked in pairs and participated in group discussion, and by collecting the written worksheets they completed during the session. There were two kinds of worksheets: a "data sheet" and a follow-up set of questions and problems to solve using the microworld. For the students, the "data sheets" were used as a way to gather information so that they could write a hypothesis about how the microworld functioned. For the investigator, the data sheets and the problem sets served as written records of the students' work during the session, and were used in conjunction with written notes to analyze the students' learning as they interacted with the microworld and with each other.

Procedure:

The session began with all students gathered in a circle, where the instructor spent about 10 minutes introducing the activity. She stated that the children would be working that day as scientists or detectives, gathering data in order to come up with a hypothesis for how the computer program worked. Data sheets were distributed to pairs of students; each data sheet had two columns, one headed "What I Tried" and the other "What I Found Out." The students were instructed to record each "experiment" as they entered various inputs into the computer, and to sketch or describe the result. At the bottom of the sheet the students were asked to write out their hypothesis by answering the question: "How does the program work? My hypothesis:__________." The purpose of the data sheet was both to help students remember and record their explorations in a systematic way, and also to allow them to reflect on the results of their experiments with the microworld.

The students then moved to the computers in self-selected pairs, and set to work, one pair per computer, with the computers arranged side-by-side around the perimeter of the room. The students were able to talk freely with their partners and neighboring pairs of students. They spent approximately 30 minutes exploring the microworld and filling in their data sheets. Near the end of this time period, they were asked to spend five more minutes working and then to write down their final hypotheses. They were then gathered into a small group, and asked to describe or read out their hypotheses. A discussion followed, during which the students were encouraged to question each other and to clarify their hypotheses to the group (the instructor did not evaluate the hypotheses;

instead, the process of deciding on the best hypothesis was carried out by the group). Finally, the students were given a second worksheet, which asked specific questions about the function of each input to the procedure (offering, in essence, an opportunity to correct or refine their hypotheses) and presented a set of problems and challenges to solve using the program (i.e., further applications and explorations of the microworld). The students spent an additional 20 minutes filling out this worksheet, working until the end of the 80 minute session.

Results

The students quickly understood the goal of the activity and worked actively during the allotted time with their partners, experimenting with the microworld, recording their results, discussing possible hypotheses about how the program worked, and calling over the instructors or classmates to show them particularly interesting figures on the screen. During the last five minutes of the initial period of exploration, the instructor reminded the students that they needed to come up with a written hypothesis. This prompt resulted in a final burst of activity and conversation among the students. After writing their hypotheses, the students were gathered into a circle again and asked to read their hypotheses, one at a time. The group then discussed various hypotheses, some of which they decided were incorrect or only partially-correct. The correct hypothesis was demonstrated by one pair of students who had discovered it during their initial exploration of the microworld. The students then returned in their pairs to the computers, and most of them confirmed the correct hypothesis by entering additional examples, and then went on to complete the problems and explorations on the second worksheet.

An analysis of the written worksheets revealed that almost all of the students (14 out of 17 pairs, or 82%) were able to correctly describe the function of the first input to the procedure in their initial written hypothesis. However, only one pair of students (6%) was able to correctly determine the precise function of the second input within the time allowed for this task. The other pairs gave a variety of hypotheses. In general, these hypotheses stated that the second input had something to do with the shape of the figure inside the circle, but were not specific about the precise mathematical relationship. For example, two hypotheses taken from the written worksheets were: "El segundo # sirve para hacer la figura que ésta inscrita" ("The second number serves to make the inscribed figure"); and "El segundo es el numero de figura" ("The second is the number of the figure"). As can be seen, the students' hypotheses were vague and difficult to interpret. For the most part, the students were not able to discover the specific function of the second input, at least not within the amount of time they were allowed for the initial exploration of the environment.

As noted above, after the students shared their hypotheses, the pair of students who had discovered the correct functioning of both inputs demonstrated their solution on the computer. The other students at this point watched and carefully questioned the successful pair. After this demonstration and discussion, the majority of the students (13 out of 17 pairs, or 76%) were able to accurately describe in writing the function of the second input. For example, one pair wrote, in answer to the question "Qué significa la segunda entrada?" (What does the second input mean?), "El espacio entre vertice y vertice para ser la figura dentro la circunferéncia" ("The space between vertex and vertex to make the figure inside the circumference").

Most of the children successfully completed the remainder of the second worksheet, which consisted of challenges to create specific figures using the microworld, or to discover general "rules" such as how to create stars rather than polygons. The students worked very productively during this third phase. By the end of the session, the children were able to use the CUERDAS procedure with confidence both to create shapes specified on the worksheet and to make new designs of their own. This facility in using the microworld to solve problems is evidence of an understanding of the program which apparently surpassed what the students could put into words. Although not all of the students could initially state in precise language how the microworld functioned, all could correctly select inputs in order to create specific figures, that is, they could successfully use the microworld in solving problems and creating interesting patterns of their own.

Discussion

According to one model of mathematical understanding (Hoyles & Noss, 1987), a learning sequence which reflects increasing mathematical power includes the following "stages:" Using, Discriminating, Generalizing and Synthesizing. "Using" indicates that the learner knows what to do with a mathematical operation or procedure, while a child who is at the "discriminating" stage can describe features of the procedure which make it distinct, and knows how it works. At the "generalizing" stage, a learner can test for mathematical regularities, and when "synthesizing" can make connections with other mathematical topics. The children working with the CUERDAS microworld were able both to use the procedure and to discriminate at certain aspects of its features, writing hypotheses which reflected their understanding of how the procedure worked. They were able to reach this level of understanding by experimenting with the microworld and talking about the results with their partners, by sharing their hypotheses with the whole group, and then returning to the computer for further exploration and problem-solving.

It is important to note that these children had never been asked to carry out an activity like the one described above, either in their Logo class or as part of their regular mathematics instruction. The initial description of the purpose of the project, the introduction of the idea of being a detective or scientist, was very important in helping these students to understand the idea of a mathematical exploration. Yet, aside from helping the children to start the program and answering a few questions, the teacher and investigator did not intervene during the session. The students consulted among themselves, argued, debated, questioned, looked at other screens and formulated their written answers together. This kind of activity, in which students engage in authentic debate and conversation, contrasts sharply with the traditional "recitation script" in which the teacher provides information to the whole class and then tests the students' recall through individual question and answer (Tharp and Gallimore, 1988). In the work described here, the students focused on the computer as a "conversation piece", one which embodied a mathematical puzzle that they worked together to solve. The teacher's role became one of coaching and guiding the students to make their discoveries explicit and to express them in words.

This pilot study, although limited in scope, contributes to the growing body of research showing that mathematically-rich computational environments can provide a basis for learning which goes beyond memorization and rote procedural knowledge. The students in this study acted, during the pilot session, as members of a community of fellow explorers, sharing a common goal of making sense of a new mathematical phenomenon, embodied in a computer microworld. In future work with these students and others, it is hoped that this pedagogical context will become a familiar and accepted way to support deep and significant learning through the exploration of computer environments.

References

Cobb, P., Wood, T. & Yaekel, E. (1992). A constructivist approach to second grade mathematics. In E. Von Glasersfeld (ed.). Radical constructivism in mathematics education. (pp. 157-176). Dordrecht, Holland: Kluwer.

diSessa, A. (1990). Social niches for future software. In A. diSessa, M. Gardner, J. Greeno, A. Schoenfeld & E. Stage (Eds.), Toward a scientific practice of science education (pp. 301-322). Hillsdale, NJ: Lawrence Erlbaum.

Edwards, L. (1992a). A Logo microworld for transformation geometry. In C. Hoyles & R. Noss (Eds.), Learning Logo and mathematics (pp. 127-155). Cambridge, MA: MIT Press.

Edwards, L. (1992b). A comparison of children's learning in two interactive computer environments. Journal of Mathematical Behavior, 11 (1). 73-82.

Fonseca, C. (1991). Computadoras en la escuela pública costarricense: La puesta en marcha de una decisión. San José, Costa Rica: Fundación Omar Dengo.

Hoyles C. & Noss R. (1987). Children working in a structured Logo environment: from doing to understanding. Recherches en Didactique des Mathématiques, 2 (12), 131-174.

Hoyles, C. and Noss, R. (1992). A pedagogy for mathematical microworlds. Educational Studies in Mathematics (23), 1, 31-57.

Mendez B., Z. and Johanson, R. (April, 1990). Evaluation of a national program of Logo instruction: Progress report from Costa Rica. Paper presented at the Annual Meeting of the American Educational Research Association, Boston, MA.

Pea. R. (1987). Integrating human and computer intelligence. In R. Pea & K. Sheingold (Eds.), Mirrors of minds: Patterns of Experience in Educational Computing (pp. 128-146). Norwood, NJ: Ablex.

Resnick, L., Levine, J. and Teasley, S. (Eds.) (1991). Perspectives on socially shared cognition. Washington, DC: American Psychological Association.

Tharp, R. & Gallimore, R. (1988). Rousing minds to life: Teaching, learning, and schooling in social context. Cambridge: Cambridge University Press.

Thompson, P. (1987). Mathematical microworlds and intelligent computer-assisted instruction. In G. Kearsley (Ed.), Artificial intelligence and instruction: Applications and methods (pp. 83-109). Reading, MA: Addison Wesley.

WHAT IS SOCIAL CONSTRUCTIVISM IN THE PSYCHOLOGY OF MATHEMATICS EDUCATION?

Paul Ernest

University of Exeter, United Kingdom

Social constructivism is appearing in the work of some researchers in the psychology of mathematics education as an alternative perspective to radical constructivism However there are widespread disparities in what is meant by social constructivism To overcome this ambiguity, the roots of social constructivism in sociology, symbolic interactionism, philosophy and social psychology are traced, and two types of social constructivism are distinguished One is based on a radical constructivist (Piagetian) theory of mind, and either bolts on aspects of the social, or adds it or as an alternative complementary perspective The other type is based on Vygotskian theory of mind, and is more thoroughly social

INTRODUCTION: THE PROBLEM

The central problem for the psychology of mathematics education is to provide a theory of learning mathematics that facilitates interventions in the processes of its teaching and learning Thus, for example, Piaget's Stage Theory inspired a substantial body of research on hierarchical theories of conceptual development in the learning of mathematics in the 1970s and 1980s (e g Hart, 1981) Piaget's constructivism also led to the currently fashionable radical constructivist theory of learning mathematics, which accounts for the individual idiosyncratic construction of meaning, and thus for systematic errors, misconceptions, and alternative conceptions in the learning of mathematics It does this in terms of individual cognitive schemas, which it describes as growing and developing to give viable theories of experience by means of Piaget's twin processes of equilibration, assimilation and accommodation

Although a number of different forms of constructivism exist, the radical version most strongly prioritises the individual aspects of learning It thus regards other aspects, such as the social, to be merely a part of, or reducible to, the individual A number of authors have criticised this approach for its neglect of the social (Ernest 1991b, 1993d, Goldin 1991, Lerman, 1992, 1994) Thus in claiming to solve one of the problems of the psychology of mathematics education, radical constructivism has raised another how to account for the social aspects of learning mathematics? This is not a trivial problem, because the social domain includes linguistic factors, cultural factors, interpersonal interactions such as peer interaction, and teaching and the role of the teacher Thus another of the fundamental problems faced by the psychology of mathematics education is how to reconcile the private mathematical knowledge, skills, learning, and conceptual development of the individual with the social nature of school mathematics and its context, influences and teaching? In other words how to reconcile the private and the public, the individual and the collective or social, the psychological and the sociological aspects of the learning (and teaching) of mathematics?

One approach to this problem (and there are of course others not discussed here) is to propose a social constructivist theory of learning mathematics On the face of it, this is a theory which acknowledges that both social processes and individual sense making have central and essential parts to play in the learning of mathematics Consequently, social constructivism is gaining in popularity

However a problem that needs to be addressed is that of specifying more precisely the nature of this perspective. A number of authors attribute different characteristics to what they term social constructivism Others are developing theoretical perspectives under other names which might usefully be characterised as social constructivist Thus there is a lack of consensus about what is meant by the term, and what are its underpinning theoretical bases and assumptions The aim of this paper is to begin to clear up this confusion by clarifying the origins and nature of social constructivism, and indicating some of the major differences underlying the use of the name

BACKGROUND TRADITIONS

Although there are few explicit references to social construction in the work of symbolic interactionists and ethnomethodologists such as Mead, Blumer, Wright Mills, Goffman and Garfinkel, their work is centrally concerned with the social construction of persons and interpersonal relationships They emphasise conversation and the types of interpersonal negotiation that underpin everyday roles and functionings, such as those of the teacher in the classroom. Indeed, Mead (1934) even offers a conversation based social theory of mind Following on from this tradition, a milestone was reached when Berger and Luckmann (1966) published their seminal sociological text 'The social construction of reality'. Drawing on the work of Schutz, Mead, Goffman and others, this elaborated the theory that our knowledge and perceptions of reality are socially constructed, and that we are socialised in our upbringing to share aspects of that received view They describe the socialisation of an individual as "an ongoing dialectical process composed of the three moments of externalization, objectivation and internalization [and] the beginning point of this process is internalization." (Berger and Luckmann 1966 149)

From the late 1960s or early 1970s, social constructivism became a term applied to the work of sociologists of science and sociologists of knowledge including Barnes, Bloor, Knorr-Cetina, Latour, Restivo, and others This tradition drew upon the work of Durkheim, Mannheim, and others, and its primary object is to account for the social construction of scientific knowledge, including mathematics (Restivo 1988) Recently, there has been work in this tradition (e g. by Restivo and Collins) in developing a social theory of mind (drawing on the work of Mead and Vygotsky).

Not long after the development of these sociological traditions, in the 1970s social constructionism became a recognised movement in social psychology through the work of Coulter, Gergen, Harré, Secord, Shotter, and others These authors have been concerned with a broad range of social psychological issues such as the social construction of the self, personal identity, emotions, gender, and so on (Gergen, 1985) A shared starting point, elaborated by different researchers in different ways, is that of Vygotskian theory Consequently, one of the special features of social constructionism in social psychology is the explicit central use of the metaphor of conversation for mind, as well as for interpersonal interaction

Within psychology there are other inter-related traditions which build on the work of Vygotsky, and which propose more less well developed social theories of mind These include both Soviet Activity Theorists (Vygotsky, Luria, Leont'ev, Gal'perin, Davydov), what might be termed 'dialogists', including Volosinov, Bakhtin, Lotman, Wertsch, and sociocultural theorists such as Lave, Wenger, Rogoff, Cole and Saxe

The term 'social constructivism' was not applied in philosophy, to the best of my knowledge, until the late 1980s, when the growing interdisciplinarity of sociological and social psychological studies, and their terminology, spilled over into philosophy. However, a social constructivist tradition in philosophy can be identified, with its basis in the late work of Wittgenstein, although some scholars, such as Shotter, trace it back to Vico There are strands in various branches of philosophy which might be termed social constructivist This includes the tradition of ordinary language and speech act philosophy, following on from Wittgenstein and Ryle, including the work of Austin, Geach, Grice, Searle and others In the philosophy of science, a mainstream social constructivist strand includes the work of Hanson, Kuhn, Feyerabend, Hesse and others In continental European philosophy there is a older tradition including Enriques, Bachelard, Canguilhem, Foucault which has explored the formative relations between knowledge, especially scientific knowledge, and social structure. In social epistemology there is the work of Toulmin, Fuller and others In the philosophy of mathematics there is a tradition including Wittgenstein, Lakatos, Bloor, Davis, Hersh, and Kitcher Ernest (1991a, In press) surveys this tradition, and represents one of the few specifically philosophical approaches to mathematics to adopt the title of social constructivism

In the early 1970s the social construction (of knowledge) of reality thesis became widespread in educational work based on sociological perspectives, such as Esland, Young, Bernstein, and others. By the 1980s theories of learning based on Vygotsky were also sometimes termed social constructivist, and although we might now wish to draw distinctions between their positions, researchers such as Andrew Pollard (1987) identified Bruner, Vygotsky, Edwards and Mercer, and Walkerdine as contributing to a social constructivist view of the child and learning

To the best of my knowledge, the term 'social constructivism' appeared in mathematics education from two sources The first is the social constructivist sociology of mathematics of Restivo, which he explicitly related to mathematics education in Restivo (1988) The second is the social constructivist theory of learning mathematics of Weinberg and Gavelek (1987) The latter is based on the theories of both Wittgenstein and Vygotsky, but also mentions the work of Saxe, Bauersfeld and Bishop as important contributions to the area, even though they might not have explicitly drawn called themselves social constructivist Unfortunately Weinberg and Gavelek never developed their ideas in print (to my knowledge) Bishop (1985) made a more powerful impact with his paper on the social construction of meaning in mathematics education, but he did not develop an explicit theory of learning mathematics, and focused more on its social and cultural contexts. Social constructivism became a more widely recognised position following Ernest (1990, 1991a, 1991b), but a number of authors have used and continue to use the word in different ways, such as Bauersfeld (1992) and Bartolini-Bussi (1991) There are also a number of contributions to mathematics education which might be termed social constructivist, in one sense or other, even though they do not use this title (e g the socioconstructivism discussed below)

In summary, social constructivism originated in sociology and philosophy, with inputs also from symbolic interactionism and Soviet psychology, and subsequently it influenced modern developments in social psychology and educational studies, before filtering through to mathematics education. Because of the diverse routes of entry, and doubtless because of the varying paradigms and perspectives into which it was assimilated in mathematics education, social constructivism is used to refer to widely divergent positions What they share is the notion that the social domain impacts on the developing individual in some crucially formative way, and that the individual constructs (or appropriates) her meanings in response to her experiences in social contexts. This description is vague enough to accommodate a range of positions from a slightly socialised version of radical constructivism, through sociocultural and sociological perspectives, all the way to fully-fledged post-structuralist views of the subject and of learning.

The *problematique* of social constructivism for mathematics education may be characterised as twofold It comprises, first, an attempt to answer the question how to account for the nature of mathematical knowledge as socially constructed? Second, how to give a social constructivist account of the individual's learning and construction of mathematics? Answers to these questions need to accommodate both the personal reconstruction of knowledge, and personal contributions to 'objective' (i e socially accepted) mathematical knowledge. An important issue implicated in the second question is that of the centrality of language to knowing and thought

Elsewhere I have focused on the first more overtly epistemological question, concerning mathematical knowledge (Ernest 1991a, 1993b, In-press). However, from the perspective of the psychology of mathematics education, the second question is the all-important one It is also the source of a major controversy in the mathematics education community. In simplified terms, the key distinction among social constructivist theories of learning mathematics is that between individualistic or cognitively based theories (e g Piagetian or radical constructivist theories) and socially based theories (e g Vygotskian theories of learning mathematics).

Although this is a significant distinction, an important feature shared by radical constructivism and the varieties of social constructivism discussed here is a commitment to a fallibilist view of knowledge in general, and mathematical knowledge in particular. (This will not be discussed further here, but see e g Ernest 1991a, In press.)

SOCIAL CONSTRUCTIVISM WITH A PIAGETIAN THEORY OF MIND

A number of authors have attempted to develop a form of social constructivism based on what might be termed a Piagetian or neo-Piagetian constructivist theory of mind. Two main strategies have been adopted First, to start from a radical constructivist position and add on social aspects of classroom interaction That is, to prioritise the individual aspects of knowledge construction, but to acknowledge the important if secondary place of social interaction This is apparently the strategy of Yackel, Cobb and Wood (In press), who claim to be radical constructivist, but also lay a special emphasis on the social negotiation of classroom norms. Indeed these researchers adopted the term socioconstructivist for their position, but have since reverted to the term 'constructivist'. However it is possible that these researchers should be interpreted as having adopted a complementarist position (the second strategy, discussed below), for certainly in some publications (e g. Cobb, 1989) they explicitly write of the adoption of multiple theoretical perspectives and of their complementarity Overall, a number of developments in radical constructivism would seem to fall under this category, in all but name (e.g. Richards 1991)

The second strategy is to adopt two complementary and interacting but disparate theoretical frameworks One framework is intra-individual and concerns the individual construction of meanings and knowledge, following the radical constructivist model The other is inter-personal, and concerns social interaction and negotiation between persons It can also extend far enough to account for cultural items, such as mathematical knowledge. A number of researchers have adopted this complementarist version of social constructivism, including Driver (In press), who accommodatc, both personal and interpersonal construction of knowledge in science education. Likewise Murray (1992) and her colleagues argue that mathematical knowledge is both an individual and a social construction. Bauersfeld (1992: 467) explicitly espouses a social constructivist position based on "radical constructivist principles... and an integrated and compatible elaboration of the role of the social dimension in individual processes of construction as well as the processes of social interaction in the classroom". Most recently Bauersfeld (1994. 467) describes his social constructivist perspective as interactionist, sitting between individualist perspectives, such as cognitive psychology and collectivist perspectives, such as Activity Theory. Thus he explicitly relates it to the symbolic interactionist position mentioned above, but he retains a cognitive (radical constructivist) theory of mind complementing his interactionist theory of interpersonal relations

In Ernest (1991a) I proposed a version of social constructivism, which although intended as a philosophy of mathematics, also included a detailed account of subjective knowledge construction. This combined a radical constructivist view of the construction of individual knowledge (with an added special emphasis on the acquisition and use of language) with Conventionalism; a fallibilist social theory of mathematics originating with Wittgenstein, Lakatos, Bloor and others.

> The two key features of the account are as follows. First of all, there is the active construction of knowledge, typically concepts and hypotheses, on the basis of experiences and previous knowledge. These provide a basis for understanding and serve the purpose of guiding future actions. Secondly, there is the essential role played by experience and interaction with the physical and social worlds, in both the physical action and speech modes. This experience constitutes the intended use of the knowledge, but it provides the conflicts between intended and perceived outcomes which lead to the restructuring of knowledge, to improve its fit with experience. The shaping effect of experience, to use Quine's metaphor, must not be underestimated. For this is where the full impact of human culture occurs, and where the rules and conventions of language use are constructed by individuals, with the extensive functional outcomes manifested around us in human society.
>
> Ernest (1991a: 72)

> However this conjunction [of social and radical constructivist theories] raises the question as to their mutual consistency. In answer it can be said that they treat different domains, and both involve social negotiation at their boundaries (as Figure 4.1 illustrates). Thus inconsistency seems unlikely, for it could only come about from their straying over the interface of social interaction, into each other's domains. ...there are unifying concepts (or metaphors) which unite the private and social realms, namely construction and negotiation.
>
> Ernest (1991a: 86-87)

In commenting on work that combines a (radical) constructivist perspective with an analysis of classroom interaction and the wider social context, Bartolini-Bussi (1991: 3) remarks that "Coordination between different theoretical frameworks might be considered as a form of *complementarity* as described in Steiner's proposal for TME the principle of complementarity requires simultaneous use of descriptive models that are theoretically incompatible " However, Lerman (1994) argues that there is an inconsistency between the subsumed social theories of knowledge and interaction, and radical constructivism, in this (or any) complementarist version of social constructivism

I, too, now think that there are severe difficulties associated with the form of social constructivism which builds on radical constructivism. There are first of all many of the problems associated with the assumption of an isolated cognizing subject (Ernest, 1991b). Radical constructivism can be described as being based on the metaphor of an evolving and adapting, but isolated organism – a cognitive alien in hostile environment. Its world-model is that of the cognising subject's private domain of experience (Ernest 1993c, 1993d). Any form of social constructivism that retains radical constructivism at its core retains these metaphors, at least in some part. Given the separation of the social and individual domain that a complementarist approach assumes, there are also the linked problems of language, semiotic mediation, and the relationship between private and public knowledge. If these are ontologically disparate realms, how can transfer from one to the other take place?

Lerman (1992) made an interesting attempt to rescue (as he saw it) radical constructivism by replacing its Piagetian theory of mind and conceptual development by a Vygotskian theory of mind and language, in what might be seen as a form of social constructivism. However, in taking leave of radical constructivism Lerman (1994) has recently extended his critique, and now argues that any form of social constructivism which retains a radical constructivism account of individual learning of mathematics inevitably fails to account adequately for language and the social dimension. Bartolini-Bussi (1994), however, remains committed to a complementarist approach, and although espousing a Vygotskian position, argues for the value of the co-existence of a Piagetian form of social constructivism, and the necessity for multiple perspectives.

SOCIAL CONSTRUCTIVISM WITH A VYGOTSKIAN THEORY OF MIND

In a survey of social constructivist research in the psychology of mathematics education Bartolini-Bussi (1991) distinguishes complementarist work combining constructivist with social perspectives from what she terms social constructionist work based on a fully integrated social perspective. Some of her attributions of individual projects to these approaches might be questioned. For example, I would locate the diagnostic teaching approach of Alan Bell in a cognitively-based post-Piagetian framework, not one of social constructivism. Nevertheless, the distinction made is important. It supports the definition of a second group of social constructivist perspectives based on a Vygotskian or social theory of mind, as opposed to the constructivist and complementarist approaches described in the previous section.

Weinberg and Gavelek's (1987) proposal falls within this category, since it is a social constructivist theory of learning mathematics explicitly based on Vygotsky's theory of mind. A more fully developed form of social constructivism based on Vygotsky and Activity Theory is that of Bartolini-Bussi (1991, 1994), who emphasises mind, interaction, conversation, activity and social context as forming an interrelated whole, and indicates a broad range of classroom and research implications and applications.

In Ernest (1993a, 1993c, 1993d, 1994, In-press) I have been developing a form of social constructivism differing from my earlier version (Ernest 1990, 1991a) because it similarly draws on Vygotskian roots instead of Piagetian constructivism in accounting for the learning of mathematics. This approach views individual subjects and the realm of the social as indissolubly interconnected, with human subjects formed through their interactions with each other (as well as by their individual processes) in social contexts. These contexts are shared *forms-of-life* and located in them, shared *language-games* (Wittgenstein). This version of social constructivism has no underlying metaphor for the wholly isolated individual mind, drawing instead upon the metaphor of *conversation*, comprising persons in meaningful linguistic and extra-linguistic interaction. (This metaphor for mind is widespread among 'dialogists' e g Bakhtin, Wertsch and social constructionists e.g. Harré, Gergen, Shotter)

Mind is viewed as social and conversational because of the following assumptions. First of all, individual thinking of any complexity originates with and is formed by internalised conversation, second, all subsequent individual thinking is structured and natured by this origin, and third, some mental functioning is collective (e.g group problem solving). Adopting a Vygotskian perspective means that language and semiotic mediation are accommodated Through play the basic semiotic fraction of signifier/signified begins to become a powerful factor in the social (and hence personal) construction of meaning (Vygotsky, 1978)

Conversation also offers a powerful way of accounting for both mind and mathematics Harré (1979) has elaborated a cyclic Vygotskian theory of the development of mind, personal identity, language acquisition, and the creation and testing of public knowledge, all in one cyclic pattern of appropriation, transformation, publication, conventionalisation This provides descriptions of both the development of personal knowledge of mathematics in the context of mathematics education (paralleling Berger and Luckmann's socialisation cycle), and describes the formative relation between personal and 'objective' mathematical knowledge in the context of academic research mathematics (Ernest 1991a, 1993b, 1994, In press) Such a theory has the potential to overcome the problems of complementarity discussed above

CONCLUSION

It is important to distinguish Vygotskian from radical constructivist varieties of social constructivism, for progress to be made in theoretical aspects of the psychology of mathematics education However, the adoption of a Vygotskian version is not a panacea Undoubtedly Piagetian and post-Piagetian work on the cognitive aspects of the psychology of mathematics education is at a far more advanced stage and with a more complete theorisation, research methodology and set of practical applications. Nevertheless, Vygotskian versions of social constructivism suggest the importance of a number of fruitful avenues of research, including the following.

- the acquisition of transformation skills in working with semiotic representations in school mathematics;
- the learning of the accepted rhetorical forms of school mathematical language, both spoken and written,
- the crucial role of the teacher in correcting learner knowledge productions and warranting learner knowledge; and
- the import of the overall social context of the mathematics classroom as a complex, organised form of life including (a) persons, relationships and roles, (b) material resources, (c) the discourse of school mathematics, both content and modes of communication (Ernest 1993a, In press).

REFERENCES

Bauersfeld, H. (1992) Classroom Cultures from a Social Constructivist's Perspective, *Educational Studies in Mathematics*, 23, 467-481.

Bauersfeld, H. (1994) Theoretical Perspectives on Interaction in the Mathematics Classroom, in Biehler, R. Scholz, R. W., Straesser, R. and Winkelmann. Eds *The Didactics of Mathematics as a Scientific Discipline*, Dordrecht: Kluwer, 133-146.

Bartolini-Bussi, M. G. (1991) Social interaction and Mathematical Knowledge, in *Proceedings of PME-15* Assisi, Italy, 1, 1-16.

Bartolini-Bussi, M. G (1994) Theoretical and Empirical Approaches to Classroom Interaction, in Biehler, R. Scholz, R. W., Straesser, R. and Winkelmann. Eds *The Didactics of Mathematics as a Scientific Discipline*, Dordrecht: Kluwer, 121-132.

Berger, P. and Luckmann, T. (1966) *The Social Construction of Reality*, London: Penguin Books.

Bishop, A. J. (1985) The Social Construction of Meaning: A Significant Development for Mathematics Education? *For the Learning of Mathematics*, 5, 1, 24-28.

Cobb, P. (1989) Experiential, Cognitive, and Anthropological Perspectives in Mathematics Education, *For the Learning of Mathematics*, 9, 2, 32-42.

Driver (In press) Constructivist Approaches to Science Teaching, in L. P. Steffe, Ed., *Alternative Epistemologies in Education*, London: Erlbaum.

Ernest, P. (1990) Social Constructivism as a Philosophy of Mathematics Radical Constructivism Rehabilitated?, Poster paper, *PME 14 Conference*, Mexico, July 1990.

Ernest, P (1991a) *The Philosophy of Mathematics Education*, London: Falmer.

Ernest, P. (1991b) Constructivism, The Psychology of Learning, and the Nature of Mathematics· Some Critical Issues, in *Proceedings of PME-15* (Italy), 2, 25-32. (Reprinted in *Science and Education*, 2 (2), 1993, 87-93.)

Ernest, P. (1993a) Mathematical Activity and Rhetoric: Towards a Social Constructivist Account, *Proceedings of PME 17 Conference*, Tsukuba, Japan, 2, 238-245

Ernest, P. (1993b) Dialectics in Mathematics, Paper presented at 19*th International Conference on the History of Science*, Zaragoza, Spain, August 1993

Ernest, P. (1993c) Metaphors for Mind and World, *Chreods* 6, 3-10.

Ernest, P. (1993d) Constructivism and the Problem of the Social, in C. Julie, D. Angelis, Z. Davis, Eds, *Political Dimensions of Mathematics Education: Curriculum Reconstruction for Society in Transition*, Johannesburg, South Africa. Maskew, Miller and Longman, 1993, 121-130.

Ernest, P. (1994) Conversation as a Metaphor for Mathematics and Learning, *Proceedings of BSRLM Annual Conference*, Manchester, November, 1993

Ernest, P. (In press) *Social Constructivism as a Philosophy of Mathematics*, Albany, NY: SUNY Press

Gergen, K. (1985) The social constructionist movement in modern psychology, *American Psychologist*, 40, 266-275.

Goldin, G. A. (1991) Epistemology, Constructivism and Discovery Learning Mathematics, in Davis, R. B., Maher, C. A. and Noddings, N. Eds (1990) *Constructivist Views on the Teaching and Learning of Mathematics*, Reston, Virginia: National Council of Teachers of Mathematics.

Harré, R. (1979) *Personal Being*, Oxford: Blackwell.

Hart, K. Ed. (1981) *Children's Understanding of Mathematics: 11-16*, London: John Murray.

Lerman, S. (1992) The Function of Language in Radical Constructivism: A Vygotskian Perspective, *Proceedings of PME-16 Conference*, Durham, New Hampshire, USA, 2: 40-47.

Lerman, S. (1994) Can we talk about constructivism? *Proceedings of BSRLM Conference*, Manchester, UK, November, 1993.

Mead, G. H. (1934) *Mind, Self and Society*, Chicago: University of Chicago Press.

Murray, H. (1992) Learning Mathematics Through Social Interaction, paper presented to Working Group 4, *ICME 7 Conference*, Quebec, August 1992.

Pollard, A. Ed. (1987) *Children and their Primary Schools*, Lewes, East Sussex· Falmer Press.

Restivo, S. (1988) The Social Construction of Mathematics, *ZDM*, 20, 1, 15-19.

Richards, J. (1991), Mathematical Discussions, in Glasersfeld, E. von, Ed. (1991) *Radical Constructivism in Mathematics Education*, Dordrecht: Kluwer, 13-51.

Vygotsky, L. (1978) *Mind in Society*, Cambridge, MA.: Harvard University Press.

Weinberg, D. and Gavelek, J (1987) A Social Constructivist Theory of Instruction and the Development of Mathematical Cognition, *Proceedings of PME 11*, Montreal,. 3, 346-352.

Yackel, E., Cobb, P. and Wood, T. (In press) Instructional Development and Assessment From a Socioconstructivist Perspective, in G. Leder, Ed., *Curriculum + Planning = Learning*.

JUDGMENTS OF ASSOCIATION IN CONTINGENCY TABLES:

AN EMPIRICAL STUDY OF STUDENTS' STRATEGIES AND PRECONCEPTIONS

A. Estepa, *University of Jaén (Spain).*

D. R. Green, *Loughborough University of Technology (U. K.).*

C. Batanero and J. D. Godino, *University of Granada (Spain).*

SUMMARY

In this paper an experimental study of students' strategies in solving a judgment of association in contingency tables is presented. The classification of these strategies from a mathematical point of view allows us to determine concepts and theorems in action and to identify students' conceptions concerning statistical association in contingency tables. Finally, correspondence analysis is used to show the effect of task variables of the items on students' strategies.

PSYCHOLOGICAL RESEARCH ON CONTINGENCY TABLES

A contingency table or cross-tabulation is used to present, in a summarized way, the frequencies in a population or sample, classified by two statistical variables. In its simplest form, when the variables only involve two different categories, it takes the format presented in Table 1:

Table 1: Typical format for the 2x2 contingency table

	A	NOT A	Total
B	a	b	a+b
NOT B	c	d	c+d

The study of reasoning about statistical association began with Inhelder and Piaget (1955), who considered that the understanding of the idea of association has as prerequisites the concepts of proportionality, probability and the combinatorial capacity. For this reason, they only studied reasoning about association with pupils in their formal operation stage (IIIa) and (IIIb), proposing to the subjects the problem of the association between eye colour and hair colour.

They found that, at stage (IIIa), some adolescents only analyze the relationship between the favorable positive cases to the association (cell a in Table 1) and the total number of data. In other cases adolescents only compare the cells two by two. Once they admit that the cases in cell d (absence-absence) are also related to the existence of association, they do not understand that the cells a and d have the same meaning concerning the association, comparing a with b or c with d instead. This fact is explained because, although (IIIa) subjects are able to compute single probabilities, understanding association needs the consideration of the quantity (a+b) as favourable to the association and of (c+d) as opposed to it, and also it is necessary to consider the relation :

$$R = \frac{(a+d) - (b+c)}{a+b+c+d}$$

between the difference of cases confirming the association (a+d) and the other cases (b+c) and all the possibilities. This is only produced at 15 years age (stage IIIb), according to Ihhelder and Piaget.

After Inhelder and Piaget, many psychologists have studied the judgment of association in 2x2 contingency tables in adults, using various kinds of tasks and, as a consequence, it has been noted that subjects have a poor capacity in establishing a correct judgment about association. For example, Smedslund (1963) and Shaklee and Mins (1982) noted that many adult students base their judgment only using cell a or comparing a with b.

The difficulty of this type of task is shown by the fact that, as Jenkins and Ward (1965) pointed out, even the strategy of comparing the diagonals in the table, considered as correct by Piaget and Inhelder, is only valid in the case of tables having equal marginal frequencies for the independent variable. For the general case, Jenkins and Ward have proposed as the correct strategy the comparison of the difference between the two conditional probabilities, $P(A|B)$ and $P(\bar{A}|B)$:

$$\delta = a/(a+c) - b/(b+d)$$

In addition to the difficulty of this topic, Chapman and Chapman (1967) showed that there are common expectations and beliefs about the relationship between the variables that cause the impression of empirical contingencies. This phenomenon has been described as "illusory correlation" (Tversky and Kahneman, 1974), because people maintain their beliefs in spite of evidence of the independence of variables. As Nisbett and Ross (1980) and Scholz (1991) have described, posterior studies have shown that for the same association problem structure, strategies not only vary inter-personally, but even intra-personally. People shift strategies depending on task characteristics.

EXPERIMENTAL STUDY

Aims of the study

Psychological research provides us with valuable information concerning students' performance and strategies in judging association in 2x2 tables. Nevertheless, as Vergnaud (1987) pointed out "*competency is always related to* *conceptions, however weak these conceptions may be, or even wrong"* (pg. 33) and, from an educational point of view, the identification of students' preconceptions (Artigue, 1990; Confrey, 1990) is needed in order to plan an adequate instruction.

The aim of this research is to analyze students' strategies in 2x2, 2x3 and 3x3 contingency tables from a mathematical point of view, in order to identify students' preconceptions concerning statistical association in contingency tables. As an initial step we also have identified concepts and theorems in action as described by Vergnaud (1982) who consider *"the essential purpose for a cognitive analysis of tasks and behaviors is to identify such theorems in action"* (pg. 35).

Sample

The sample consisted of 213 students in the last year of secondary school (18 year old students). It is in this level where the topic association is introduced in the Spanish syllabus. The questionnaire was given to the students before the instruction was started. About half of the students (113) were males and half (110) females. This study has a quasi- experimental character, because of the non-random character of the samples of students and problems.

Questionnaire

The whole questionnaire included 5 items concerning contingency tables. A pilot study of the questionnaire was performed with a separate sample of 53 students, which served to check the reliability and the coding system of the students' answers and to improve the final version. The items were similar to item 1 presented in Fig. 1. The following task variables were considered:

ITEM 1: In a medical center 250 people have been observed in order to determine whetner the habit of smoking has some relationship with a bronchial disease. The following results have been obtained

	Bronchial disease	No Bronchial disease	Total
Smoke	90	60	150
Doesn't smoke	60	40	100
Total	150	100	150

Using the information contained in this table, would you think that for this sample of people bronchial disease depends on taking the drug? Explain your answer.

Figure 1

VI. Type of table: Because most research which has been done in psychology has used 2x2 tables, we have included 3 items of this kind. We have also included a 2x3 table and a 3x3 table, in order to assess the similarities in difficulty and in student strategies between 2x2 tables and other types of tables.

V2: Sign of the association: Direct association, inverse association and independence were used in 2x2 tables In the 2x3 table the association was direct. The sign of association was not applicable to the 3x3 table which refers to nominal variables.

V3: Relationship between context and prior belief: the association suggested by the context of the problem and the empirical association presented in the table may coincide (theory agreeing with data) or not coincide (theory contradicting data). The specific values assigned in each one of these variables in the different items are shown in Table 2

Table 2: Values of task variables in the different items

	Item 1	Item 2	Item 3	Item 4	Item 5
V1	2x2	2x2	2x2	2x3	3x3
V2	Independent	Inverse	Direct	Direct	Independent
V3	Theory contradicted	Unfamiliar context	Unfamiliar context	Theory contradicted	Unfamiliar context

DISCUSSION

Once the data were collected, the arguments expressed by the students were categorized. Two dependent variables have been considered in each one of the items: the type of association perceived by the student (direct association, inverse association or no association) and the procedure employed by them to solve the proposed task.

Table 3: Frequency and percent of type of association perceived by the students

Item	Independence	Direct	Inverse	No answer
1	* 84 (39.4)	118 (55 4)	1 (0.5)	10 (4.7)
2	65 (30.5)	25 (11.7)	*108 (50 7)	15 (7.0)
3	9 (4 2)	*194 (91.1)	3 (1 4)	7 (3.3)
4	19 (8 9)	*186 (87.3)		8 (3.8)

(*): Correct answer

In Table 3 we present the frequencies and percentages of the type of association perceived by the students in items 1 to 4 In item 5 in which the sign of association was not applicable, because the two variables were nominal and had three different categories, 129 students (60 6% of the total number of students) gave the correct answer (independence) , 23 students (10.8%)

considered the existence of association and 61 students (28.6%) provided no answer. The difficulty was very low in items 3 and 4, which correspond to direct association and to problems in which the students' expectations about the type of association coincides with the contingency presented in the data (item 4) or to problems in which the student has no previous theory, but the strength of the association is high (item 3). Item 5 had a moderate difficulty, although it is a 3x3 table and corresponds to independence; Nevertheless a high number of students has provided no answer. Comparing with item 1 which also refers to independence, the difference in difficulty is notable; in item 1 most students have considered the association as direct, because this is the expected type of association suggested by the context of the problem (smoke and cough). Finally, we refer to item 2 which has a moderate difficulty. Nevertheless, a significant number of students have considered as a case of independence the fact that the presence of A implies no B, that is the inverse association.

The correct judgment of association by a student is not sufficient to evaluate a student's primary conceptions concerning statistical association, due to the fact that many authors have noticed that it is possible to get a correct judgment with an incorrect procedure. From the point of view of statistical education both correct judgment and correct strategy are needed for an adequate understanding of this concept.

In order to evaluate this second aspect, the procedures employed by the students were classified according the implicit mathematical concepts and theorems in action (Vergnaud, 1982), in case of correct or partially correct procedures and according to the types of errors in incorrect procedures. Taking this analysis as a base for classification, we have identified the following students' strategies:

<u>Correct strategies</u>

S1: Comparison of all the conditional relative frequency distributions: These students compare the relative frequency $h(A_i|B_j)$ of every value A_i for the different values of B_j. As a consequence, they are implicitly using the idea that the dependence of a variable A on another variable B implies the variation of the relative conditional frequencies $h(A_i|B_j)$ when B_j varies. This involves comparison of columns. The case of comparing rows is also included here.

S2: Comparison of one conditional relative frequency $h(A_i|B_j)$ i fixed, in each possible value B_j with the marginal frequency h_i. They implicitly use the property of invariance of the distribution of A when conditioned with a value B_j (rows and columns could be interchanged).

S3: Comparing the frequencies of cases in favour and against A (or comparing the ratio of these frequencies) in each value of B. In this procedure the correspondence between the probability of an event and the odds ratio, which is valid when the table only has two rows or two columns, is implicit. It is this idea which serves as a base for the definition of the odds ratio as a measure of association.

Partially correct strategies

S4: This is similar to strategy S1, but the students do not use relative frequencies explicitly. Instead, they perform qualitative or additive comparisons, and for this reason they do not quantify properly the difference of probability of the values of one of the variables as a function of the other one.

S5: This is similar to S2, but the students only use one of the frequencies in each conditional distribution. This strategy is correct when the variable has only two different values, but not in the general case.

S6: Comparison of the sum of the frequencies in the diagonal. That is, comparing a+d with c+b in the 2x2 table. This is a comparison of the cases confirming and disconfirming the association. This strategy is valid for a table with equal marginal frequencies, but not for the general case.

Incorrect strategies:

S7: Using only one cell, often the cell whose frequency is maximal.

S8: Using only one conditional distribution: These students do not realize that they have a problem in which a comparison of two probabilities is needed. They base the judgment only in the ratio or on the difference of frequencies of the different values of A for a fixed value of B.

S9: Comparing absolute frequencies with the total number of cases. The students use one or several relative frequencies of different pairs of values (A_i, B_j). They try to link the frequencies in different cells to the total number of cases. They do not realize the necessity to compare the frequencies of a value of A in the different values of B. The strategies S7, S8, S9 corresponds to students who present a local conception of association, because in judging the association they do not take into account the complete distribution but only the most salient information.

S10: Using marginal frequencies. Some students consider that is not possible to solve the problem, because of the difference in marginal frequencies in the different values of A or B.

S11: Other procedures. In this category we have included students who express their previous theories about the nature of the association between the variables, the students that give incomplete arguments, and other procedures, such as trying to solve a system of equations.

Finally, we have grouped the students who did not provide a judgment for the association and cases where the arguments were not sufficiently clear to be interpreted In general this group of students is not very large, except in item 5 (3x3 table) in which a great number of non-answers was obtained. In Table 4 the frequencies and percentages of strategies used by the students in the different items are presented

Table 4: Frequency and percent of procedures by implicit mathematical concept

Procedure	Item 1	Item 2	Item 3	Item 4	Item 5
1:All h_{ij}	8(3.8)	16(7.5)	2(0.9)	20(9.4)	18(8.4)
2:h_{ij} with $h_{i.}$	33(15.5)	27(12.6)	35(16.4)	21 (9.9)	
3:Odds comparison	13(6.1)	39(18 3)	34(16.0)	62(29.1)	5(2.3)
4:two, three f_{ij}	3(1 4)	19(8 9)	21(8 9)	15(7.0)	76(35.6)
5:f_{ij} fixed i with $h_{.j}$	2(0.9)	19(8.9)	17(8,0)		
6:Sum of diagonals	5(2.3)	11(5.2)	18(8.4)	7(3.3)	4(1.9)
7:One cell	29(13.6)	26(12.2)	22(10.3)	6(2 8)	
8:f_{ij} or h_{ij} fixed i	56(26.3)	28(13 2)	30(14 1)	47(22.1)	24(11.3)
9:f_{ij} with n	1(0 5)	6(2.8)	7(3 3)	2(0.9)	
10:$f_{i.}$ or $f_{.j}$	8(3.8)		1(0 5)	6(2.8)	
11. Other	13(6 1)	2(0.9)	3(1.4)	7(3.3)	4(1.9)
No or confused answer	42(19.7)	20(9 8)	23(10.8)	20(9 4)	82 (38.5)
Total	213(100)	213(100)	213(100)	213(100)	213(100)

Correspondence analysis:

A correspondence analysis (Greenacre, 1984) of this last table has been performed, in order to synthesize the possible relationships between strategies and items Task variables of the items have been used as supplementary variables, in order to improve the interpretation. The analysis have shown a quasi-unidimensional structure the first eigenvalue explains 60.7 percent of the total inertia of table 4 and the second eigenvalue only 21.9 percent; these two factors together explain 82.6 percent of the total inertia

In the first axis the items concerning 2x3 and 3x3 tables were opposed to the items concerning 2x2 tables; in fact the following ordering of the items was shown in the axis 3x3, 2x3; 2x2(independence), 2x2(inverse association) and 2x2 (direct association). The strategies (S1) and (S4), which consist of using all the conditional relative distributions or all the conditional absolute frequencies distributions, are opposed to the rest of the strategies. So, we could interpret this factor, which explains most of the variability in the strategies, as the opposition in taking into account all the different conditional distributions or using only part of the information contained in the table Although these last types of strategies may be used for solving some of

569

the 2x2 contingency table problems, they seem inadequate for obtaining the solution to more complex situations. Concerning the task variables, this factor also opposes the independence to the existence of association, in particular to the inverse association. So, in independence problems a more elaborate strategy seems to be needed in order to provide a correct judgment.

The second factor opposes the incorrect strategies and the partially correct strategies. Concerning the task variables, the most notable influence is that of having theories against the data (in association with incorrect strategies) as compared with theories in favour or no theory (in association with partially correct strategies). As a consequence, we deduce that some of the incorrect strategies have been used as a means to justify the previous theories about association when there is not agreement between those theories and the contingency given in the data.

Acknowledgement: *This report forms part of the Project PS91-0114.1, granted by Direccion General de Investigacion Cientifica y Tecnica (M.E.C.). Madrid.*

REFERENCES

Artigue, M. (1990). Epistemologie et didactique. *Recherches en Didactique des Mathématiques*, 10(2)-3, 241-286.

Chapman, L. J. and Chapman, J. P. (1967). Illusory correlation as an obstacle to the use of valid psychodiagnostic signs. *Journal of abnormal psychology*, 74, 271-280.

Confrey, J. (1991). A review of the research in students conceptions in mathematics, science and programming. *Review of Research in Education*, 16, 3-35.

Greenacre, M J. (1984). *Theory and applications of correspondence analysis*. London: Academic Press.

Inhelder, B. and Piaget, J. (1955). *De la logique de l'enfant a la logique de l'adolescent*. Paris P.U.F.

Jenkins, H. M. and Ward, W. C. (1965) Judgement of contingency between responses and outcomes. *Psychological monographs*, 79.

Kahneman, D.; Slovic, P.; Tversky, A. (Eds.) (1985).*Judgment under uncertainty: Heuristics and biases*.(New York: Cambridge University Press).

Nisbett. R. and Ross, L. (1980). *Human inference: strategies and shortcomings of social judgment*. New Jersey: Prentice Hall

Scholz, R. W. (1991). Psychological research in probabilistic understanding In R. Kapadia and M. Borovnick (Eds.). *Chance encounters: probability in education*, 213-249. Dordrecht: Kluwer.

Shaklee, H. y Mins, M. (1982). Sources of error in judging event covariations: effects of memory demands. *Journal of experimental psychology: Learning, memory and cognition*, 8(3), 208-224.

Smedslund, J, 1963, Le concept of correlation in adults. *Scandinavian journal of psychology*, 4, 165-174.

Tversky, A. and Kahneman, D. (1974) Judgment under uncertainty: heuristics and biases *Science*, 185, 1124-1131 In D. Kahmenan, P. Lovic and A. Tversky (Eds.). *Judgment under uncertainty: heuristics and biases*. New York: Cambridge University Press

Vergnaud, G, 1982, Cognitive and developmental psychology and research in mathematics education some theoretical and methodological issues. *For the learning of Mathematics*, 3(2), 31-41

QUANTITATIVE AND QUALITATIVE RESEARCH METHODOLOGIES: RIVALRY OR COOPERATION?

Jeff Evans
School of Mathematics and Statistics
Middlesex University, Enfield, UK

The terms quantitative and qualitative, often used to distinguish research methodologies, are ambiguous. Here I discuss several possible meanings. Drawing on a study of cognition and affect in the context of numerate activity among adult students, I show the strengths of each of three strategies - quantitative, qualitative cross-sectional, and qualitative case study - for critically considering claims about gender differences in mathematical performance.

Introduction

Perhaps the most obvious way to distinguish types of research is by *data type*: quantitative research produces numerical data, analysed statistically, and qualitative produces textual data, analysed non-statistically. (Indeed this is one basis for classifying PME research reports.) But almost immediately we find dilemmas: for example, how should we classify data based on classification, analysed by counts and cross-tabulations (e.g. Table 2 below)?

What about *method type*? Thus experiments, tests and surveys can be seen as "structured" (by the researcher), and many types of interview (e.g. life history), case studies and document use are seen as semi-structured or "unstructured". But this breaks down too: case studies may use test and questionnaire results, and experimental de-briefing may often be semi-structured.

Can we distinguish research by its *aims*, or its *epistemology*? Cook and Reichardt (1979) claim that quantitative research aims at the *testing* of theory, whereas qualitative emphasises *discovery*. Many would agree, arguing the need for research to be based on subjects' perspectives, but the researcher's "avoidance of preconceptions" may lead to a minimal acknowledgement of the role of theory.

The most promising basis, in my view, is the distinction between *types of explanation*: research using *causal* or *deductive-nomological* explanation an be distinguished from that using *purposes* or *reasons*. Put roughly, the former tends towards determinism - based on forces (e.g. socialization), instincts, characteristics and/or attitudes - and aims for generality, whereas the latter emphasises subjects' freedom and often celebrates particularity.

Illustrative material comes from a study of thinking and affect in the context of "practical" activity among adults (Evans, 1993; Evans, 1991; Evans and Tsatsaroni, 1993). The subjects were students on a social science degree at a London Polytechnic in the mid-1980s. Most were studying a compulsory maths (pre-calculus) / statistics course, and most were familiar with a range of numerate practices outside of school. Here I focus on methods and findings related to gender differences in mathematical performance.

The Quantitative Research: Theory and Methods

At the start of the course, subjects (n > 900) were asked to complete the questionnaire at one of their lectures. It included questions on gender, age, qualifications in school maths, and (in later versions) social class measured by occupation; performance; and maths anxiety. Here performance was considered to be explained (causally) by the above characteristics, by qualifications in maths, and by measures of anxiety or affect.

The maths anxiety items were selected from the Mathematics Anxiety Rating Scale (MARS) analysed by Rounds and Hendel (1980) into Maths Course / Test Anxiety (MTA) and Numerical Anxiety (NA). MTA and NA were considered to relate to school and everyday contexts respectively, parallel to the distinction between school maths and practical maths; the latter included questions used in the survey of 3000 adults done for the Cockcroft Report (ACACE, 1982).Thus, in the quantitative phase of this resesarch, *context* was specified by the *wording* of performance or anxiety questions.

Results from the Quantitative Research

We can scrutinise the sometimes mythic claims about gender differences in maths performance (Walkerdine *et al.*, 1989), using the quantitative data. First, the men's and women's average score on School Maths performance (10 items) were compared, yielding an uncontrolled difference of 3/4 of a question (statistically significant; p<.001) in favour of the male students; see Table 1.

In order to control for qualification in maths (i.e. maths course-taking), age (since more of the women were mature students), maths test anxiety and confidence (self-rating) in maths, a multiple regression model was constructed. Now the difference for younger students (aged 18-20), about 1/6 of a question, was no longer statistically significant, and that for mature students (21+), just over half a question, was borderline; see Table 1.

Table 1. Gender Differences in School Maths Performance: Uncontrolled Group Means and Estimates Controlled for Qualification in Maths, Age, etc.

	Number of questions correct		
	Men's Average	Women's Average	Difference
Uncontrolled	8.78	8.07	0.71 (Std. Error 0.11)
Controlled			
Young (18-20)			0.17 (S.E. 0.17)
Mature (21+)			0.57 (S.E. 0.24)

Thus the gender differences in performance, which are fairly substantial, become much less so after controlling not only for maths qualification, but also for age, affective variables, and relevant interactions, and the quantitative analysis allows us to do this. (Quantitative research with similar aims is presented in Chipman, Brush and Wilson (1985)).

Qualitative Research: Theory and Methods

We can think of *discourses* as sets of ideas, goals, values and techniques, "competing ways of giving meaning to the world and of organising social institutions and processes." (Weedon, 1987, p.35). Since all our activities are based in discourses, we sometimes speak of *discursive practices*.

Basing our analysis on discourses allows us to develop a much fuller idea of the context of mathematical thinking or affect, to include: the crucial role of *language;* the *goals* of the activity, (Newman Griffin and Cole, 1989); the *relations of power* exercised (e.g. Weedon, 1987) and the *material and institutional resources* (e.g. training, equipment, professional associations) made available for the activity.

In my approach, the context is understood as *positioning in practice(s).* Each practice produces positions which subjects take up; in some contexts / practices, the availability of a particular position may depend on the subject's gender, age, etc. (cf. Hollway, 1989). For example, in the activity of feeding children, the child is positioned as demanding ("more"); the parent is positioned as having to regulate the child's consumption; the form of this regulation is likely to depend on the parent's income and social class. For a discussion of how positioning in such practices might provide the context for children's thinking in one topic of primary school mathematics, see Walkerdine and Girls and Mathematics Unit (1989, pp.52-53).

In my project, I wanted to avoid the tendencies towards excessive determinism that I found in the idea of subjects' *being positioned by* practices. Thus I argue that, in a given setting, subjects in general are *positioned by* the practices which are at play in the setting, but that a particular subject will *call up* a specific practice (or mix of practices) which may differ from those called up by other subjects, and which will provide the context for that subject's thinking and affect in that setting (Evans, 1993).

Semi-structured interviews were conducted in my office with a randomly designed sample (n=25) of students at the end of their first year. The interviews had "life history" and "problem solving" phases; in the latter, subjects were presented with a number of "practical" problems - e.g. deciding how much (if at all) they would tip after a restaurant meal (see below), deciding which bottle of tomato sauce they would buy, etc. (cf. Sewell, 1981). But this interview differed in its use of *contexting questions*: when the student was first shown the "props" for the problem - e.g. a facsimile of a restaurant menu in Qu.4 - before being asked anything "mathematical" - s/he was asked: "Does this remind you of anything you currently do?" And after discussing the question: "Does this remind you of any earlier experiences?" Subjects' responses to these questions were to help me judge the context of their thinking about the problem, etc.

My analysis of positioning in the interview setting was that subjects would tend to be *positioned* in two main practices:
academic maths (AM), with positions teacher / student; and
research interviewing (RI), with positions researcher / respondent.
To the extent that RI, rather than AM, was called up by the subject, s/he would be able to *call up* ways of thinking, emotions, etc. from further non-academic practices, with numerate aspects I called "practical maths" (PM). For Qu.4, I expected these to be practices of "eating out", with several configurations of related positions possible: host / guest; friends each paying their own share; or perhaps customer / wait(e)r(ess). Each position in practice will support different ways of thinking and feeling, including different kinds of numerate or mathematical thinking.

In order to judge which practice the subject called up, I drew on various indicators (Walkerdine, 1988, Ch.3; Evans, 1993):
(i) the "script": e.g. how the interview, problems, etc. were introduced - as "research", "views", "numbers", rather than "test", "maths", etc.;
(ii) unscripted aspects of the researcher's performance;
(iii) the subject's talk, especially the responses to the contexting questions; and
(iv) reflexive accounts: e.g. whether I had been in the position of maths teacher to each student, before the interview.

Two "qualitative" approaches were used. First, a *cross-subject* approach, based on that of Miles and Huberman (1984), aimed to consider, in a comparative way, the results from all of the interviews. Second, each interview was considered as a *case study* of a particular subject's thinking and affect.

Results from the Qualitative Cross-sectional Research

Let us consider the results for all interviewees for problem 4, which presented a restaurant menu. For this analysis, I aimed to record the practice that *predominated* in each subject's positioning. Indicators for a predominant academic maths (AM) positioning were considered to be:

- the use of written calculations; or
- the giving of an answer which involved a fraction of 1p.

Indicators for a predominant positioning in "practical maths" (PM) were: - the use of mental calculations; or
- the formulating of an answer in practical, i.e. money, units.

The results from the cross-subject analysis of the relationship between gender, predominant positioning and performance judged as correct for problem 4 (a 10% tip when eating out) are in Table 2.

Table 2. Gender, Positioning and Performance: Cross-tabulation of number of questions judged correct by gender and predominant positioning for problem 4

Predominant Positioning	Number (and Percentage) Correct		
	Women	Men	Total
Practical Maths (eating out)	6 / 7 (84%)	10/11 (91%)	16/18 (89%)
Academic Maths	3 / 5 (60%)	-	3 / 5 (60%)
Total	9 /12 (75%)	10/11 (91%)	19/23 (83%)

We must first stress the small numbers involved - 23 students in all. (Two students did not reach Qu.4, because other life history material intervened.) Also, the level of performance, among both women and men, was rather high. And the gender differences were very small (especially remembering the small numbers). However, there are two findings, which - if they can be replicated - would be interesting. First, a slightly lower level of performance among those judged to have a predominant positioning in academic maths than among those positioned in practical maths / eating out (60% - 3 out of 5 - compared with 89%). Second, almost half of the women - but no men - called up academic maths. Taken together, these very tentative findings pose the question as to whether some of the gender differences in performance could be explained by differences in *positioning*. And whether this might be true in other settings, e.g. in large scale testing?

Results from the Qualitative Case Study Research

One case study will illustrate the potential of this analysis. "Jean" was aged 18 at entry, with CSEs (a "less academic" exam than O-level, taken at 16+) in both Mathematics and Arithmetic. I judged her to be working class, on the basis of a noticeable regional accent, and an often-voiced concern about never having enough money (see below).

A major issue in considering her performance is: Why does she "always get %'s the wrong way round"? In one of her attempts at Qu.2 [10% of 6.65], she tries {10/6.65 x 100}, realises that is incorrect, and then answers "0.65 ... just a guess". For Qu. 4, when asked about a "10% tip on £3.75", she first tries {10/3.75 x 100}, as for Qu.2, and then she tries {3.75/10 x 100}. But she doesn't really know: "it goes something like that".

How can we explain these errors? We might say she has a "conceptual problem". Or we might refer to affect: she constantly expresses (or exhibits) anxiety - about percentages; and also about the interview ("I sound horrible on tape.") But she expresses even more anxiety about money; e.g. "I've never, ever got enough money" (for tips); about the level of tipping required in the U.S. to which she is about to depart, and about being able to afford 15% tips there. She is also anxious about being able to afford the trip at all (see below). When shopping, "I do always follow the prices,... for fear of being ripped off". What appeared at first to be "maths anxiety" seems now to relate to her constant worry about money and financial constraint.

Thus we can read her errors as based in a complex of factors:
- a conceptual problem about percentages; along with
- beliefs about herself as a solver of maths problems;
- maths anxiety, especially about percentages;
- some anxiety about the interview; and
- anxiety about the relevant practice, viz. tipping in restaurants, and/or tipping in the USA.

The latter anxiety also relates to an apparently chronic anxiety about money. This may relate to a positioning in social class terms, as a member of a family with money problems, possibly in poverty. (Cf. Ellen, a middle class woman of about the same age, whose anxiety about calculating a tip can be read as relating to anxiety about the *relationship* which has provided the site for her affectively charged experiences of "eating out" (Evans, 1991).) Thus we might analyse Jean's interview in a second straightforward way, by focussing on a determining factor, namely social class.

Can we produce different readings? There is something else which might attract our attention as meaningful. Listen to her talk: "it goes something like that"; "I sound horrible on tape". Then concerning her CSE Grade 3 in Maths: "I wasn't very good at all". She also expresses much anxiety about percentages; e.g. "I always get the formula wrong..."; "I always mix it up". We can hear these as "confusion", "anxiety", but all we have are indices of something other than the talk. We could quickly disregard them, as "self-defeating self-talk" (Tobias, 1978).

Or we might attend to these signifiers marking her talk. This would lead us to e.g. the discourse of norms (right / wrong), of aesthetics (good / bad), of science (truth / falsity), all "mixed up" in her talk. It is the following up of this thread of signifiers that blocks our clear criteria of true / false that normally directs judgments on the use of mathematics. This would take us into the discourse of Kantian divisions. But her talk has confused our "clear" conceptual categories.

So if we continue listening to her narrative about her trip to the USA, and her anxiety about it: "I haven't had a holiday for three years, so this will get us to America: it's the only way I can ever make it..." We could be led from a language of conceptual divisions to a psychoanalytic language of desire. This cannot be discussed in detail here, but see e.g. Evans and Tsatsaroni (1993); Walkerdine (1988).

Conclusions

In this research, both quantitative and (two types of) qualitative methodology have been used. Rather than polarising the discussion by asking which method is "best", we can note the relative strengths of each, and attempt to combine the different approaches in a way that is effective for the problem at hand.

The quantitative approach is useful when we wish to make comparisons across subjects, or groups of subjects, and we aim for some degree of generality: we have seen the importance, and the power, of the controls used above. For example, this approach is useful for studying gender differences in participation or performance, and outcomes of policy interest more generally.

The qualitative case study approach is useful when we wish to explore the richness, coherence (i.e. not being separated into variables) and process of development of a limited number of cases. Thus we can display episodes of problem solving, in order to understand the process - for research purposes or to improve teaching and learning. The sort of semi-structured life-history / problem solving interview with contexting questions used here allows the tracing of multiple signification (as with "expense"), and the making of judgments about the positioning of subjects (Evans and Tsatsaroni, 1993). In order to produce different readings, it is advantageous that (some of) the research method(s) should produce "semi-structured" talk which can be reread in the manner illustrated above.

The qualitative cross-subject approach provides an intermediate approach, for cases where it may be challenging to produce comparability across subjects (as in judgments about "predominant positioning"), but where some generality in findings is sought.

References

ACACE (1982). *Adults' mathematical ability and performance.* Leicester: Advisory Council for Adult and Continuing Education.

Chipman S., Brush L. and Wilson D. (Eds.) (1985), *Women and mathematics: Balancing the equation.* Hillsdale, NJ: LEA.

Cook T. and Reichardt C. (Eds.) (1979), Qualitative and Quantitative Methods in Evaluation Research. Newbury Park: Sage.

Evans J.(1991), Cognition, affect, context in numerical activity among adults. In F. Furinghetti (Ed.), *PME XV: Proceedings of the fifteenth PME conference,* (vol. II, pp.33-39), Assisi, Italy.

Evans J. (1993), *Adults and Numeracy.* PhD thesis, University of London Library.

Evans J. and Tsatsaroni A. (1993), Linking the cognitive and the affective: A comparison of models for research. In I. Hirabayashi et al. (Eds.), *PME XVII: Proceedings of the seventeenth PME conference,* (vol. III, pp.210-217), Tsukuba, Japan.

Evans J. and Tsatsaroni A. (1994), Language and subjectivity in the mathematics classroom. In S. Lerman (Ed.), *The culture of the*

mathematics classroom. Dordrecht, NL: Kluwer.
Hollway W. (1989), *Subjectivity and method in psychology: Gender, meaning and science*. London: Sage.
Miles M. and Huberman M.(1984), *Qualitative Data Analysis*. London: Sage.
Newman D., Griffin P. and Cole M. (1989). *The construction zone: Working for cognitive change in school*. Cambridge: CUP.
Rounds J. and Hendel D. (1980). Measurement and dimensionality of mathematics anxiety. *Journal of Counselling Psychology, 27*, 138-149.
Sewell B. (1981). *Use of mathematics by adults in everyday life*. Leicester: Advisory Council for Adult and Continuing Education.
Tobias S. (1978), *Overcoming math anxiety*. Boston: Houghton Mifflin.
Walkerdine V. (1988), *The mastery of reason*. London: RKP.
Walkerdine V. and Girls & Mathematics Unit(1989), *Counting girls out*. London: Virago.
Weedon C. (1987). *Feminist practice and poststructuralist theory*. Oxford: Basil Blackwell.

TWO YOUNG TEACHERS' CONCEPTIONS AND PRACTICES ABOUT PROBLEM SOLVING

Domingos Fernandes
University of Aveiro and Instituto de Inovação Educacional, Portugal

Isabel Vale
Escola Superior de Educação de Viana do Castelo, Portugal

Abstract

For two consecutive academic years the conceptions and practices of two young teachers were investigated through a qualitative case study methodological approach. Study findings show that, as students in a school of higher education, both participants revealed very similar conceptions towards mathematics and problem solving. However, as beginning teachers, their practices differed quite substancially. In one case, mathematical problem solving was integrated in the curriculum and there was a consistency between the participant's discourse and her practice. In the other case, contradictions emerged between what the participant claimed to be his ideas and intentions and what actually happened in his classrooms. Several reflections are made concerning the complex relationships that the participants seemed to establish with their teacher education program, with the course syllabus and with the schools' organization.

Introduction

It is generally accepted that problem solving is one of the most important goals of mathematics teaching because it contributes to develop students' thinking capacities and competencies (e.g., Charles, 1982; Fernandes, 1992; Lester, 1980a, 1980b, 1983, 1985). However, there are several complex and interdependent factors (e.g., cognitive, metacognitive, cultural, social, emotional) that affect the development of those capacities and competencies (Lester & Charles, 1992).

Teachers' conceptions and attitudes towards mathematics and mathematical problem solving seem to play a determinant role in the actions they undertake in the classrooms during the teaching/learning process. Therefore, research efforts have been directed to understand the nature of teachers' conceptions about mathematics and about problem solving and its teaching and to analyse the extent to which teachers' practices are influenced by those conceptions and by other related factors (Cooney, 1985; Thompson, 1989; Clark & Peterson, 1986; Grouws, Good & Dougherty, 1991; Romberg & Carpenter, 1986; Vale, 1993).

For many years, as mathematics educators, we have been realising that in contrast with the optimism that our senior students usually reveal concerning the teaching of mathematics and, particularly, the teaching of problem solving, there is a somewhat discouraging situation in the current state of Portuguese elementary schools (Grades 1-6). Our experience tells us that mathematical problem solving is one of the most neglected subjects in these schools. Usually, students are not asked or encouraged to solve problems. Therefore, we were interested in finding out what happens to those optimistic and idealistic visions of our students about mathematics and mathematical problem solving when they arrive in public schools. That is why we felt that we needed to thoroughly understand how teachers' conceptions evolve and how they are related with teacher's practices and with their teacher education program at a school of higher education.

This research was guided by the following questions: (1) What conceptions about mathematical problem solving are revealed by the participants while they are senior students in a school of higher education? (2) What conceptions and practices about problem solving and its teaching are revealed by the participants while they are in their first year as mathematics teachers? (3) What relationships could be established between these teachers' conceptions and other factors such as their preservice education?

Research Framework and Methodology

Goetz & LeCompte (1984), Merriam (1988) and Yin (1989, 1993) assert that interpretative research is a process which allows one to analyse and to interpret meaning as constructed by the participants instead of a process which aims at describing an objective reality. The holistic case study is a methodology which helps one to intensively describe the particularities of social and educational phenomena and to interpret them as well. Several researchers who work in the problem-solving area and/or with students' and teachers' conceptions have used this kind of methodological approach (e.g., Cooney, 1985; Matos, 1991; Thompson, 1982). As stated by Yin (1989) case study methodology favours the analysis and interpretation of phenomena in real-life contexts; it allows one to interpret meanings as developed by the participants within the case.

Taking into account the nature and the overall purpose of this research we decided to conduct two qualitative case studies which could provide the grounds for an understanding of two young teachers' conceptions and practices and their relationships with other factors.

Participants

Two participants were involved in this study: Rui and Maria. They were both seen as good informants because they were open enough to allow a collection of a rich set of data. Besides, they were seen as being able to reflect upon their own practice and actions and seemed quite involved and committed to their personal and professional development. Also they were purposefully selected for this study because they were known among their colleagues as enthusiastic about becoming mathematics teachers and as mature, responsible students. According to their former teachers, Rui was considered an "average" student in mathematics and Maria a "good" student.

Data Collection and Data Analysis

In the process of data collection several data sources were used: (a) Interviews; (b) Observations; (c) Field Notes; and (d) Artefacts. As stated by several authors (Erickson, 1986; Johnson, 1980, Yin, 1989), the idea was to diversify the utilization of instruments and sources of information so that we could come up with a grounded and consistent set of descriptions, interpretations and conclusions about the phenomena under study.

For almost two years each participant was interviewed in 11 different occasions and contexts and systematically observed both as a senior student and as a teacher. The interviews were of informal or semi-structured nature and were supported by a questionnaire (to help in the collection of biographical data), scripts, and a set of problem-solving related tasks. The tasks were adapted from the ones developed by Thompson (1982) and Kloosterman & Stage (1989). All interviews were transcribed from audiotape recordings. Instruments developed by Putt (1978) and by Charles, Lester & O'Daffer (1987) were adapted to conduct observations of problem-solving related activities in the classrooms.

Data collection was programmed to take place during an extended period of time to augment the possibility for us to observe chains of similar situations which might occur in different occasions throughout the two phases of the study.

Data analysis was performed inductively; that is, repeated analysis of data collected through the interview transcripts, field notes, observations and artefacts enabled us to come up with categories which were used to organize them. (Goetz & LeCompte,1984; Merriam, 1988; Thompson, 1982; Yin, 1989). Also, as suggested by Erickson (1986), different data sources were used to check the validity of assertions generated by the researchers.

Phases of the Study

This study was developed in two phases. The first ocurred in the 1990-1991 academic year while the participants were senior students in a school of higher education. All the observations were performed in the context of a methods course during which students were exposed to a problem-solving module. The participants were interviewed in six different occasions.

In the second phase, which ocurred in the 1991-1992 academic year, both participants were teaching fifth grade mathematics for the first time in two different "preparatory" schools (Grades 5-6). Five interviews and twelve class-observations were conducted during that phase.

Main Findings and Discussion

Rui's Conceptions as a Senior Student

Rui's views and perspectives about problem solving and its teaching seemed very similar to the ideas presented and discussed in the methods course that he was attending. He saw problem solving as an important means to develop reasoning and higher order thinking and, accordingly, asserted that problems must challenge and motivate students for learning. Besides, Rui stated that it was desirable and possible to provide students with a set of "things" that they can learn such as problem-solving strategies and types of problems.

Teaching mathematics, teaching reasoning and teaching problem solving were identified as the same thing. Rui stated that problem solving can be naturally and easily integrated in the development of the curriculum in mathematics classrooms. Furthermore, asserted that mathematics content could be taught through a problem-solving approach. However, when asked to explain how to put these ideas into practice Rui referred some constraints, like time and the contents listed in the syllabus that needed to be taught, to justify the integration of problem solving in extra-curricular activitities rather than in the daily classroom teaching.

There is a contradiction in Rui's views. On one hand, he identifies mathematics with problem solving but, on the other hand, he sees problem solving as something that can be handled as an extra-curricular activity, out of the "normal" mathematics classroom. For him, problem solving seems to have a lower status than the mathematics content listed in the syllabus.

Rui's Conceptions and Practices as a Beginning Teacher

In his very first year as a mathematics teacher Rui was assigned to teach in a school located in a poor and rural in-land city attended by many "at risk" students whose parents participated very little in school-related activities. In general, teachers in the school were quite young, not certified, and living in other cities. There were four mathematics teachers and two of them did not hold a certification. The group did not meet frequently and, according to an annual plan presented by the teacher responsible for the group, the major concern had to do with teaching all the contents listed in the syllabus. Rui's fifth grade class had several students who tended to misbehave during class time.

Regardless this detrimental situation, Rui felt and adjusted well and soon became one of the most active persons responsible for the computers' room where he initiated students to the LOGO language and prepared materials that he needed for his classes. However, Rui did not include problem solving in his teaching plans because, in his opinion:

> "problems must be connected to the contents you are teaching and it is not easy to find them in these conditions. Having students solving problems which are not content-related is a waste of time and students might become unmotivated."

Rui didn't put into practice the pedagogic principles, ideas, and materials discussed in the methods course he attended one year ago, with which he claimed to agree. He basically followed the textbook and the course syllabus and seemed to make his decisions about what and how to teach based upon these documents. This is quite different from Rui's "talk" which, for example, referred to the active involvement of his students in their own learning process or to small groups of students engaged in problem-solving experiences.

Several factors may explain this situation. One may have to do with the low expectations that teachers, Rui included, in this school held about their students. Another may be related to the fact that Rui could feel insecure if he used an approach which calls for more students' participation and interaction. Still another reason can be linked to the functioning of the mathematics group. In fact, teachers in the group did not discuss teaching issues, did not reflect upon their own work and did not plan together.

Professional isolation, lack of stimuli, absence of a strong school's pedagogic culture, absence of mechanisms to integrate beginning teachers into the school and the community, and low participation of parents in school life, may be factors which did not help Rui to put into practice the views and perspectives about teaching problem solving revealed in his discourse.

Maria's Conceptions as a Senior Student

Maria saw problem solving as playing a major role in the development of students' critical thinking which enables them to analyse, to select, and to interpret information. At the same time she claimed that it was important and possible to teach problem solving based upon a model which helped students to organise and systematise their thinking. Nevertheless, like Rui, she also considered that it was difficult to teach mathematics through a problem-solving approach. Maria seemed to hold a contradictory view of problem solving: on one hand she seems to see it as essential for the development of mathematical ideas but, on the other, she seems to identify it as a mathematical activity which has nothing to do with the integral development of the curriculum.

Maria's Conceptions and Practices as a Beginning Teacher

Maria was assigned to teach Grade 5 mathematics in a "preparatory" school located in a poor zone of a litoral urban area. Most of the students in the school came from poor economic and social backgrounds and parents' participation in school-related activities was seen as very low. The teachers were all certified and most of them lived in town. Maria mentioned that the school, as a whole, was active and dynamic because it was involved in several projects (journal, stamps club, computers' cclub, and the like) and hosted students from the local school of higher education for their student-teaching activities. However, the math group was not used to meet on a regular basis for planning, reflecting, or discussing teaching issues. Maria seemed to adjust to this situation and she decided to put the essential of her opinions about problem solving and about new mathematics education trends into practice. Although, like Rui, she felt that the contents listed in the syllabus must be given priority, she managed to teach problem solving to her students. She actually did it frequently and in a diversified way: during classes about topics under study, as motivation and introduction to new topics listed in the syllabus, as homework and still as extra-curricular activities.

Gradually, Maria became more confident in her own work because she started to realise that she was integrating problem solving into the curriculum and that students were highly motivated. She learned that problem solving was far from preventing students from learning mathematics; in fact, in her clas, all students but one were successful.

Some Reflections

After the first year of this investigation we could say that the performances of the participants as teachers would be quite similar. In fact, they were educated in the same institution, received the same preparation on problem solving and had the same opportunity to get acquainted with teaching methods more consistent with new trends on mathematics education (e.g., using calculators, using manipulative materials, solving different kinds of problems). On the other hand, their conceptions seemed also similar: both revealed a dual view of mathematics (computation and reasoning), both saw problem solving as fundamental for the development of students' thinking capacities, and both claimed a willingness to teach problem solving in their future classrooms.

The second year of research demonstrated that the participants performed in very different ways. In Rui's case a contradiction emerged between what he claimed as his ideas and intentions and what happened in his classes. In Maria's case it was possible to perceive a consistency or coherence between her claimed ideas and intentions and her practices as a teacher. One can say that in this case it was possible to verify that, as time passed by, the relationship between Maria's conceptions and practices became stronger and stronger.

Many factors, external and internal to the participants, could explain the differences in their teaching styles. The following reflections are an attempt to understand the complex relationships that the participants seemed to establish with some of those factors.

The Participants and Their Teacher Education Program

The preparation in mathematical problem solving was determinant in providing both participants with a vocabulary, a "language", and a pedagogy of problem solving. They both seemed to acknowledge, because they both experienced it, that solving problems contributes to develop one's mathematical and critical thinking. However, there were different levels of "assimilation" of that preparation. In Rui's case there was a reliable replication of the "discourse" or the views expressed during his methods classes which, however, was not put into practice. In Maria's case there was a strong relationship between the views discussed in her methods classes and her practices.

The Participants and the Course Syllabus

Undoubtedly, for both participants the official syllabus is a framework that seems to strongly interfere with their planning and teaching. Both participants tended to overemphasize the contents listed that are supposed to be taught and, at the same time, seemed to "ignore" that problem solving is the proposed central aspect of the mathematics curriculum. Problem solving is seen as "something that we can do if we have the time". It was interesting to notice that both participants "read" the syllabus ignoring that problem solving was there.

Regardless those similar views the participants "handled the syllabus" in very different ways. Rui followed the lists of contents very closely. His "problem-solving classes" consisted in practicing techniques and the use of concepts through exercises listed in the textbook. Mary was also committed to the contents but she found and selected a variety of problems which allowed her to meet the syllabus requirements. She managed to integrate problem solving into the curriculum.

The Participants and the School Organization

The schools where this research took place did not possess a pedagogic culture which facilitated a dynamic and innovative development of the curriculum. There was no systematic group work among teachers, no reflection about teaching practices, and no motivation to do different things. In sum, there was no an appropriate atmosphere to support and help young beginning teachers. However, once again, the participants reacted in quite different ways. Maria invested most of her time, knowledge, and capacities in preparing and teaching her classes. She created a relaxed atmosphere in her classroom and her students became enthusiastic about problem solving. She even involved parents in problem-solving activities! Rui channelled his willingness to do something different to the computer lab where, in a quite informal environment, he could organize teaching and learning activities that he did not do in his classroom. He invested little in his classes and in his teaching.

A Retrospective View of the Study

This study helped us to provide some explanations to some of our initial questions. Today we are more confident that students need to be given opportunities to solve a wide range of problems, to make use of problem-solving strategies, to think about the solutions they find, and to monitor and control their knowledge and solution processes more effectively. Today we are still more convinced that this can only happen if teachers hold a set of visions and perspectives about mathematics, problem solving and their teaching which is consistent with the development of students' critical thinking, with a teaching style that facilitates the comunication of mathematical ideas, with different teaching dynamics in the classrooms and with a curriculum approach not dominated by lists of contents to be delivered.

But we also could identify contradictions, dilemmas, and questions that are still far from being answered. What is the actual impact of the preservice education of teachers on the development of their conceptions and practices? What teacher education experiences could be more favourable of practices that are more consistent with the "discourse" that is generated during the preservice years? What factors may help young teachers to develop a wider vision of the curriculum? What are the factors that, ultimately, have more influence on the teaching decisions made by young teachers? Their conceptions? Their knowledge of mathematics, education and pedagogy? The context in which they operate? Their own experiences as students?

We would like to have learned more to provide answers for those and other questions that this study brought up. We think that one of the ways to gain a better understanding of all these complex issues is through the development and implementation of research projects jointly participated by researchers and teachers and positively valued by both communities.

References

Charles, R. (1982). A instructional system for mathematical problem solving. *Problem Solving in the Mathematics Classroom.* Alberta: MCATA.

Charles, R., Lester, F. & O'Daffer (1987). *How to evaluate progress in problem solving.* Reston, VA: NCTM.

Clark, C. & Peterson, P. (1986). Teacher's thouhgt processes. In M. C. Wittrock (Ed), *Handbook of research on teaching*. New York: Macmillan.

Cooney, T. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16*(5), 324-336.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching*. New York: Macmillan.

Fernandes, D. (1992). Resolução de problemas: Investigação, ensino, avaliação e formação de professores. In M. Brown, D. Fernandes, J. F. Matos & J. P. Ponte (Eds.), *Educação Matemática: Temas de investigação*. Lisboa: IIE.

Grouws, D., Good, T. & Dougherty, B. (1991). Teacher conceptions about problem solving and problem-solving instruction. *Proceedings 14th Mathematics Education Conference*, México.

Goetz, J. & LeCompte, M. (1984). *Ethnography and qualitative design in educational research.*. Orlando: Academic Press.

Johnson, S. (1980). The research process. In NCTM (Ed.), *Research in mathematics education.*. Reston: NCTM.

Kloostermann, P. & Stage, F. (1989). *Measuring beliefs about mathematical problem solving.* Trabalho apresentado no annual meeting of the American Educational Research Association. San Francisco.

Lester, F. (1980a). Research on mathematical problem solving. In R. J. Shumway (Ed), *Research in mathematics education*. Reston: NCTM.

Lester, F. (1980b). Problem solving: Is it a problem?. In M. Lindquist (Ed.), *Research in mathematics education.*. Reston: NCTM:

Lester, F. (1983). Trends and issues in mathematical problem solving research. In R. Lesh & M. Landau (Eds.). *Acquisition of mathematics concepts and processes.* New York: Academic Press.

Lester, F. (1985). Methodological considerations in research on mathematical problem-solving instruction. In E. A. Silver (Ed), *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale: Lawrence Erlbaum Associates.

Lester, F. & Charles, R. (1992). A framework for research on problem-solving instruction. In J. P. Ponte, J. F. Matos, J. M. Matos & D. Fernandes (Eds.), *Mathematical Problem Solving and New Information Technologies: Research in contexts of practice.* Berlim: Spinger-Verlag.

Matos, J F. (1991). *Logo na educação matemática: Um estudo sobre as concepções e atitudes dos alunos.* (Tese de doutoramento). Lisboa: Faculdade de Ciências da Universidade de Lisboa.

Merriam, B. (1988). *Case study research in education: A qualitative approach.* San Francisco: Jossey-Bass Publishers.

Putt, I. (1978). *An exploratory investigation of two methods of instrution in mathematical problem solving at the fifth grade level* (Tese de doutoramento). Indiana: Indiana University.

Romberg, T. & Carpenter, T. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In M.C. Wittroch (Ed.), *Handbook of research on teaching.* New York: Macmillan.

Thompson, A. (1982). *Teachers' conceptions of mathematics and mathematics teaching: Three case studies* (Tese de doutoramento). Athens: University of Georgia.

Thompson, A. (1989). Learning to teach mathematical problem solving: Changes in teachers' conceptions and beliefs. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving.* Reston: NCTM & LEA.

Vale, I. (1993). *Concepções e práticas de jovens professores perante a resolução de problemas de matemática: Um estudo longitudinal de dois casos.* Tese de Mestrado não publicada. Departamento de Educação da Faculdade de Ciências da Universidade de Lisboa.

Yin, R. (1989). *Case study research: Design and methods.* London: Sage.

Yin, R. (1993). *Applications of case study research.* London: Sage.

HEMISPHERICITY AND THE LEARNING OF ARITHMETIC BY PRESCHOOLERS: POSPECTS AND PROBLEMS

URI FIDELMAN

DEPARTMENT OF EDUCATION IN TECHNOLOGY AND SCIENCE

TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL

ABSTRACT

The distinction between ordinal and cardinal numbers was explained to kindergarten children in the experiment of Fidelman (1992a). It was found that young children can distinguish between ordinal and cardinal numbers, when properly presented. It was also found that the learning of ordinal and cardinal numbers are related to the left- and right-hemispheres, respectively. Problems concerning the application of this finding to education, which were not discussed in Fidelman (1992a), are presented, and solutions are suggested.

1. INTRODUCTION

According to Levy-Agresti & Sperry (1984) there are two data processing mechanisms in the brain. One mechanism processes one datum after another analytically, and is partially lateralized to the left hemisphere. The other mechanism synthesizes a new whole out of several data, and is partially lateralized to the right hemisphere. Ordinal numbers are properties of individual elements perceived one after another temporally, while cardinal numbers are properties of whole sets synthesized from individual elements. Therefore we may expect a relation between ordinal- and cardinal-numbers and the left- and right-hemisphere, respectively. Such relation was found in Fidelman (1984, 1990) in experiments with adult students.

An experiment performed with preschooler during the last 3 months in kindergarten before entering school is described in Fidelman (1992a,b). The experiment was financed by Israel's Ministry of Education, which is not responsible for the results. The children of two kindergartens learned both ordinal and cardinal numbers in an inverse order. The children were tested on their knowledge of ordinal- and cardinal-numbers concepts before and after the teaching. They were tested during the experiment three times in tests for the efficiencies of their left- and right-hemispheric mechanisms. The test for the left hemisphere in the counting of signs presented very fast one after another temporally. The test for the right hemisphere is subitizing, i.e., the enumeration of several signs presented simultaneously for 180 milliseconds. The relation of these tests to the hemispheres is shown in Fidelman (1990). At the beginning of the experiment the children were tested on counting, and it was found that all the children knew the five principles of how to count of Gelman & Gallistel (1978). The children were tested also three times in tests determining their Piaget stage regarding the principle of conservation of number. It was found that part of the children were at the lower Piaget stage, and other children were at the higher Piaget stage. Piaget stage of some children changed during the experiment.

The difference between the standardized scores of the children on their initial knowledge (before the teaching) about cardinal and ordinal numbers was correlated with the difference between the standardized scores on the right- and left-hemispheric scores. This correlation was significant at $p<0.05$ in a 1-tailed test. That is, it is possible to predict the success of each child relatively to himself (or herself) in the two arithmetical approaches.

Two or three days after the lesson on cardinal or ordinal numbers the children were tested about this lesson. Few days after the test the children received the second lesson, and two or three days later they were tested about this lesson. The correlations between the scores on cardinal and ordinal numbers and the hemispheric tests depend on the order of teaching. We expect that the scores on ordinal numbers correlate positively with the left-hemispheric score. This occured only when the lesson (and the test) on ordinal numbers was given after the lesson on cardinal numbers. When the lesson on ordinal numbers was given first, the correlations between the scores on ordinal numbers and the left-hemispheric scores were negative. This phenomenon was significant statistically. A similar phenomenon, though non-significant statistically, was found regarding the correlations between the scores on cardinal numbers and the scores on the right-hemispheric tests.

The explanation of this phenomenon is that there is a permanent creation of new synapses. Only those synapses which are applied become permanent, see Greenough (1985). Learning is related to the creation of longer-term memory, which is related to permanent changes in synapses. These permanent changes are the result of the application of the synapsees. The lesson and test on ordinal numbers included questions entirely new to the children, like: What is the ordinal number of the second after the third? Therefore this lesson created entirely new synapses, which at the first days after the learling were not yet efficient. These new synapses were created in the brains of the children having efficient left hemisphere, who understood the ordinal concepts. However, the techniques of enumerating ordinal and cardinal numbers are similar, i.e., counting one after another. The difference is only lingual and conceptual. Therefore the children having a right-hemispheric brain may solve the problems related to ordinal numbers applying the old permanent and efficient synapses related to cardinal numbers. This explains the negative correlations between the scores on the left-hemispheric tests and on ordinal numbers when they are taugh first.

The lesson on cardinal numbers included an explanation of the difference between ordinal and cardinal numbers. Therefore when ordinal numbers were taught after cardinal numbers, the synapsesd related to the learning of ordinal numbers were applied more before the test on ordinal numbers. This, and the additional elapse of time, caused a larger efficiency of the synapses related to ordinal numbers in the brains of children who understood the lesson on ordinal numbers. A left-hemispheric technique is more suitable than a right-hemispheric technique to solve problems about ordinal numbers. Thus when cardinal numbera were taught first, positive correlations were obtained between the left-hemispheric scores and the scores on ordinal numbers.

The ordinal-numbers concepts were entirely new to the children, while they were more acquainted with the cardinal number concepts. This may explain why the similar phenomenon regarding the learning of cardinal numbers was smaller, and not significant statistically.

2. POSSIBLE APPLICATIONS

The findings may be applied to classify children according to their relative hemispheric efficiencies in order to teach each child arithmetic in an approach more suiting the child s brain. However, it may be argued that the correlations found in this experiment are due to the fact that the hemispheric tests apply the perception of ordinal and cardinal numbers, and therefore these tests may indicate reliability and not validity.

There are two arguments contradicting this argument. The first argument is that all the children who participated in this experiment knew all the five principles of how to count of Gelman & Gallistel (1978). Therefore the hemispheric tests tested perception rather than cognition. The second argument is that the negative correlations described in the previous section contradict the hypothesis that the correlations are related to reliability. These negative correlations are explained by the hypothesis that the enumeration tests are related to the hemispheric mechanisms.

3. PROBLEMS CONCERNING APPLICATIONS

There are two problems which disturb the application of these findings to education. The first is that the correlations presented in Fidelman (1992a) are not large enough, and they vary too much. This phenomenon may be explained by the observation that only one lesson on each ordinal- and cardinal-numbers was given. The changing of the negative correlations into positive and sometimes significant correlations may indicate that if the number of lessons would be increased, the correlations (and the explained variance) may be increased. However, partial reason for the relatively small correlations may be related to the relatively small reliabilities of the hemispheric tests, inferred from the correlations presented in Table 1, which may indicate that these tests are not accurate enough.

The hemispheric scores were computed by two modes. The first is by true/false (T/F), i.e., a correct answer received the score 1 and a wrong answer received the score 0. The second mode is by the percentage of the mistake (%), the score was computed by the formula: $\max\{ 0 \ ; \ 1- |n-m|/n \}$
where n is the correct answer, and m is the answer of the subject.

Table 1: Pearson's Correlation Coefficients between Scores on Identical Hemispheric tests on the Three Times (n=39)

Times	Right-Hemispheric Tests				Left Hemispheric Tests			
	Simultaneous Counting of Dots		Simultaneous Counting of Forms		Ordinal Counting of Dots in Space		Ordinal Counting of Dots at A Point	
	T/F	%	T/F	%	T/F	%	T/F	%
1st & 2nd	.427	.574	.415	.302	.702	.681	.338	.673
1st & 3rd	.309	.464	.306	.061	.657	.577	.245	.347
2nd & 3rd	.686	.829	.447	.233	.670	.741	.293	.291

5. SUGGESTED SOLUTIONS TO THE PROBLEMS

The small reliabilities may be related to the fast development of the brain of children, which may change the efficiencies of their hemispheric mechanisms. We can see in Table 1 that the temporally most remote tests, the first and the third, provide the smallest correlations between identical tests. The change of Piaget stage is related to the development of the brain. Piaget stage of part of the children changed during the experiment (Fidelman 1992b). This change was found to be related to the scores on the hemispheric tests. The test of simultaneous counting of forms was found to be the most sensitive to the change of Piaget stage. We can see in Table 1 that correlation between the scores on this test in the first and third times in the % mode is the only correlation which is near zero. This may indicate that the change of Piaget stage is a possible reason for the relatively small reliabilities.

There is a possibility that the hemispheric scores after the change of Piaget stage can be predicted by the hemispheric scores before the change. Additional research is required in order to test this hypothesis. If this hypothesis is correct, it may contribute to the solution of the reliability problem. Otherwise, the children should be classified twice, before and after the change of Piaget stage.

Klein & Armitage (1979) found 1 1/2 hour oscillations in the efficiency of the hemispheric mechanisms. This cycle may influence the hemispheric scores and reduce the reliabilities of hemispheric tests. A research should be performed in order to find whether such a cycle exists also in children. If a cycle exists, we should find its lenght and whether this lengh depends on Piaget stage.

In the experiments of Fidelman (1984, 1990) the hemispheric battery included 3 cycles, the duration of each 1 2 hour. In each cycle both hemispheres were tested. Thus a cycle of 1 1 2 hour in the activity of the hemisphres does not bias the hemispheric scores. Suppose that the lengh of the cycle in children is 1 1/2 hour. It is impossible to test preschoolers during 1 1 2 hour. However, we can test a child during 15 minutes, which was the duration of the testing in Fidelman (1992), let the child rest 30 minutes, and then test the child again during 15 minutes. The mean of these two hemispheric tests will provide us with the exact mean of the hemispheric test during the entire cycle. The second test will always be during the antisimmetric position to that of the first.

REFERENCES

Fidelman U. (1984). The hemispheres of the brain and the learning of Peano s and Frege's arithmetics. Focus on Learning Problems in Mathematics, Vol. 6, No. 4, 77-83.

Fidelman U. (1990). Cerebral hemispheres, sex, handedness and the learning of arithmetic: a theory and initial experiments. Cybernetica, 33, 293-327.

Fidelman U. (1992a). The hemispheres and arithmetic learning by children. In: Proceedings of the 13th International Congress on Cybernetics, 339-343. Namur, Belgium: Association Internationale De Cybernetique.

Fidelman U. (1992b). The role of hormones in cognitive and motor Behavior: Piaget stages and the efficiency of the corpus callosum in children. In: Proceedings of the 13th International Congress on Cybernetics, 335-338. Namur, Belgium: Association Internationale De Cybernetique.

Gelman R. & Gallistel, R. (1978). The Child's Conception of Number. Cambridge, Mass.: Harvad University Press.

Greenough W.T. (1985). The possible role of experience-dependent synaptogenesis, or synapses on demand, in the memory process. In: Memory Systems of the Brain, N.M. Weinberger, J.L. McGaugh, & G. Lynch (Eds.). New York: Guilford Press, pp. 77-103.

Klein R. & Armitage R. (1979). Rhythm in human performance: 1 1/2 hour oscillation in cognitive style. Science, 204, 1326-1328.

Levy-Agresti J. & Sperry R.W. (1966). Differential perceptual capacities in major and minor hemispheres. Procedings of the National Academy of Science, 61, p. 1151.

FIRST YEAR MATHEMATICS STUDENTS' NOTIONS OF THE ROLE OF INFORMAL PROOF AND EXAMPLES

Keir Finlow-Bates
South Bank University, London.

This paper presents the result of a series of video-taped interviews held to investigate first year mathematics students' notions of informal proof. In the context of their usual classroom experience of working in small discussion groups, students were asked to rank four solutions to a familiar problem on closure of a set under addition. The solution consisting of an informal proof followed by some examples was ranked higher than any of the other solutions by all the interviewees. Although failing to refer to the informal proof as a "proof", or comment on its role in convincing the reader of the truth of the initial conjecture, students consistently used relevant words such as 'explanation' and 'clarification' to describe some of its functions.

Introduction

Although the criteria used by first year mathematics undergraduates in evaluating proof mirror those of the practising mathematician as described by Hanna (1989), in a previous study (Finlow-Bates et al., 1993) it was found that students have difficulty applying these criteria in distinguishing empirical arguments, fallacious mathematical arguments, and proof. Schoenfeld (1988) argues that the reason for this inability to distinguish between proof and non-proof may lie with the way proof is taught, and the aims of the student within the learning environment.

In the right environment students can construct a reasoned justification for a conjecture, or even an informal proof (Hanna, 1983), but may not have the knowledge to assess, or the correct vocabulary to describe what they have produced. This paper aims to investigate the value attributed by first year mathematics undergraduates to proof and examples, with particular reference to the roles they play in explaining a mathematical statement to students, and convincing them of its truth.

In this paper I present an analysis of interviews conducted to investigate the views held by first year undergraduate students at South Bank University on the roles of informal proof and examples in the structure of solutions to mathematical problems. This analysis is part of an ongoing project, under the direction of Dr. Stephen Lerman, designed to develop and evaluate materials for teaching the notions of proof to sixth form and first year undergraduate students .

Methodology

This study was of first year mathematics students on the Mathematical Contexts and Strategies Unit, a one semester course intended to "develop confident, flexible and self-aware approaches to mathematical thinking and problem solving through the study of fundamental mathematical topics" (South Bank University, 1993). During classes students work in small groups using a variety of worksheets. A strong emphasis is placed on the importance of working in groups rather than as individuals. The course is assessed by a course work folder containing work produced in class and at home.

During their very first class the students worked on problems concerning the closure of a variety of sets under the operations of addition and multiplication. The procedure adopted by the groups was to try a few examples to obtain an initial conjecture as to the closure of the set under the given operation. The students were then encouraged to confirm or reject this initial conjecture through the use of mathematical generalisation and justification. Proof was not formally introduced.

After the start of the course, the six participants were selected on a voluntary basis, and were interviewed separately. The interviews were designed using an adapted form of the discourse-based interview introduced by Odell & Goswami (1982) and used in an academic setting by Herrington (1985). The key features adopted from the discourse-based interview were: the interviews were based around a selection of work familiar to the interviewees, the interview was conducted in a context familiar to the interviewee, and the interviewee was assured that he/she was considered the expert. By this last feature it was hoped that the responses would be personal, rather than what the interviewee might have thought the interviewer expected.

After piloting the initial interview design on one student it became evident that the problem to be considered was too complicated. Although the few results that were obtained from this initial interview corresponded with the results of the final series of interviews they are not considered in this paper, as the initial interview was conducted as a 'test-run'.

The final interview design was as follows: the interviewee was given a sheet with the following question printed on it:

> *Consider the following set:*
> *M = {x such that x is an integer greater than 4}.*
> *Is the set closed under addition?*

The interviewee was also presented with four solution sheets, A, B, C, and D. Solution A consisted of an informal proof similar to ones produced by some of the students in class (see fig. 1), followed by some examples. Solution B consisted of the same examples followed by the informal proof. Solution C consisted only of an expanded version of the informal proof. Solution D consisted of examples only (see fig. 2).

Let y and z be in M. Let a and b be positive whole numbers such that $y = a + 4$ and $z = b + 4$.

Then $y + z = (a + 4) + (b + 4)$

$= a + b + 4 + 4$

$= a + b + 8$ which is greater than 8 and therefore greater than 4.

fig 1 - The Informal Proof included in Solutions A and B

<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
Proof **Examples**	**Examples** **Proof**	**Proof**	**Examples**

fig 2 - The Structure of the Four Sample Solutions

The interviewee was told to imagine that he/she was working in a group in class on the problem, and that the four solution sheets were answers produced by different members of the group. The interviewee had the task of ranking the solutions from best to worst, to select the best solution to be included in the course work folder, explaining his/her reasons for the order selected. Then the interviewee was questioned on what he/she thought the purpose of the examples and the informal proof, in the solutions given, was. Finally, a copy of a piece of work produced by the interviewee in class was produced, and its structure was discussed in comparison to the structure of the solution selected as the best at the beginning of the interview.

All the course work produced in class on the topic of closure was collected from the participants and photo-copied. Of the five students involved in the final interview, two (R,T) had only used examples to answer questions in their course work, in a similar style to solution D in the interview. Two (S,U) had produced informal proofs followed by examples, as in solution A. The final student (V) had answered all questions in his course work using only informal proofs, as in solution C. None of the participants had produced solutions similar to solution B.

Results

The participants all presented an initial ordering of the solutions from best to worst, but when asked to justify their ordering during the interview all except subject T re-evaluated their ranking. Tables 1 and 2 show initial and final choices:

Subject	Best	2nd best	3rd best	Worst
R	C	A	B	D
S	A	C	B	D
T	A	B	D	C
U	C	A	B	D
V	C	A	B	D

Table 1 - Initial Ranking of Solutions

Subject	Best	2nd best	3rd best	Worst
R	A	B	C	D
S	A	B	C	D
T	A	B	D	C
U	A	B	C	D
V	A	B	C	D

Table 2 - Final Ranking of Solutions

There are three points worth noting here:

- Solution A, the informal proof followed by the examples, was consistently ranked higher than solution B, the examples followed by the informal proof. Some of the reasons for this are discussed below.
- Solution D, consisting of examples only, was ranked worst by all interviewees except T, whose course work contained no proofs or justifications.
- The response to solution C, the informal proof with no examples, varied the most during the interviews. Initially ranked quite highly,

solution C was later down-graded due to its lack of examples. This variation appears to reflect student confusion about the value of proof.

The interviewees all eventually ranked A as the best solution, followed by B. The following reasons were given for preferring solution A to solution B:

> **R**: Because, they have, I mean, they clarify what they're doing first, and then they give some examples, of ... of the ... the solution, I mean.
> **K**: OK, and you think that's better than giving some examples and then clarifying?
> **R**: Yeah. Well, you need to understand it first, I mean, what they're doing and then the examples.
>
> **T**: So you know what you're on about, what you're on about for the examples. I believe in explaining what you're gonna do. If, when you give examples. I don't think it should be the other way around.
>
> **U**: ... I just like to have a bit of writing beforehand 'cause its explaining to me... If I look at examples before writing I wouldn't really take much notice. But if it was the other way you've read it so you want to see if it's true so you'd read it, then you take more notice of the examples.
>
> **V**: Well, I'd say A is structured better, and you can follow through much more easily what they are doing. B is basically saying the same thing, but it is not as obvious.

The reason give by all the interviewees quoted above for preferring A to B, was because A had the "explanation" before the examples, and this was considered a preferred structure for the solution. None of the interviewees referred to the initial part of A as a proof. Subjects T and U did use the word proof later on during their interview, but applied it to the examples:

> **U**: Yeah, because I like the way it talks about it (points at first section of A) and then *proves* it (points at second part of A), and that one (points at B) is *proving*, then talking about it, where I think it should be the other way.
> **K**: Right.
> **U**: That one is, it's not *proving* it, you know, it's just got some writing (indicating C).
>
> **K**: And the examples, what are they there for?
> **T**: Just to *prove, prove* the statement.
> **K**: What does that mean, "they prove the statement"?
> **T**: They *prove*, that means they make it true.
> **K**: So the ...
> **T**: (interrupting) under all conditions.

I found the use of the word "proof" to describe the examples surprising, and asked subject U to explain what she meant:

> **K**: What do you think the word "proof" means?
> **U**: That's quite hard. Um. (Pause) I found the condition and I thought, the condition isn't just, you can't just put it down, because you're not saying if it is right or wrong. See, my proof is, like, saying if it is going to be right or wrong, that's what I think proof ... when you're proving it, you're saying that the rule is right or wrong.
> **K**: Right, OK. So why do the examples show that the rule is right or wrong?

U: Because you're putting in, like numbers, numbers into this, and you can prove that it's right, 'cause if the answer comes out right, you know, like, if it's in the set.

U is aware that proof is connected with "saying if it is going to be right or wrong", but seems to believe that the examples are better at this than the initial "explanation".

The interviewees had no difficulty labelling the second section of A as "examples". The first section posed more problems:

K: OK. You call the second section "examples". What would you call the first section?
V: (Pause) Erm, I'd call it an explanation of what the set is.

K: So you called these 'examples'. What would you call this section? (indicating second half of B)
R: The comment, you know. This? (indicating second half of B)
K: Yes.
R: What would I call this part?
K: Yes.
R: The summary of ... of ... of ... I don't know. It's just a summary.

K: What would you call this on its own? (covering the first part of B) Would you still call it a summary, or comments?
R: No, um, some notes.

The words "examples", "comments", "summary", and "notes" were all used to describe the informal proof. Although the students named the examples without hesitation, finding a phrase to label the informal proof caused the students, particularly R, some difficulty.

Discussion and Conclusions

Although initially ranking informal proof on its own highly, when asked to justify their choices, students re-evaluated their ordering and selected informal proof followed by examples as the best structure for a solution to the problem. The solution consisting of examples only was considered overall to be the worst solution. Students believed the role of the informal proof to be that of explanation and clarification of the mathematical conjecture, whereas the examples were seen to fulfil the purpose of convincing them of its truth. The students had difficulty in finding suitable words to describe the section of the solution consisting of an informal proof, and the words applied to this section varied depending on its position in the overall solution.

More research needs to be carried out to determine on what basis the students made their decisions. The question remains of how the students' ideas developed, which could be determined by investigation of secondary school classrooms. Another possible influence is the structure of school mathematics textbooks, which typically introduce a topic with a piece of explanatory writing, followed by a set of exercises; this structure has parallels with solution A. The following excerpt from the transcript of sub'ect T's interview supports this:

> T: Just to let you know how (pause), um, just to let you know how, just to let whoever's reading the *book* where, how the answer, how the examples make this possible.

The student has changed the context of the solution here, from the situation of working in a group in class to that of a 'book' aimed at an unspecified reader.

What is clear is that the students interviewed do attribute value to an informal proof as an explanation or clarification of a mathematical problem, especially as an introduction to worked examples, to the extent that this explanation is valued more highly than the examples. And yet, ironically, they seem to be more convinced of the truth of a mathematical statement by the examples than by the informal proof. The problem remains to encourage them to recognise the importance of proof as "an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing" (Hanna, 1989) above empirical evidence in the form of examples. This has important implications for the teaching of reasoning, justification and proof.

References

Finlow-Bates, K., Lerman, S. and Morgan, C. (1993); A Survey of Current Concepts of Proof Held by First Year Mathematics Students, *Proceedings of the Seventeenth Conference of the International Group for the Psychology of Mathematics Education*, Japan, Vol. I, p252-259.

Hanna, G. (1983); Rigorous Proof in Mathematics Education, Toronto: OISE Press.

Hanna, G. (1989); More than Formal Proof, *For the Learning of Mathematics* 9, Montreal: FLM Publishing Association, Feb.

Herrington, A. (1985); Writing in Academic Settings: A Study of the Contexts for Writing in Two College Chemical Engineering Courses, *Research in the Teaching of English*, 19(4), 331-359.

Odell, L. and Goswami, D. (1982); Writing in a non-academic setting, *Research in the Teaching of English*, 16(3), 201-224.

Schoenfeld, A. H. (1988); When Good Teaching Leads to Bad Results: The Disasters of "Well-Taught" Mathematics Courses, *Educational Psychologist* 23(2), 145-166.

South Bank University (1993); *Unit Guide: Mathematical Contexts and Strategies*, School of Computing, Information Systems and Mathematics.

THE IRRATIONAL NUMBERS AND THE CORRESPONDING EPISTEMOLOGICAL OBSTACLES

Efraim Fischbein, Ruth Jehiam and Dorit Cohen
Tel Aviv University
School of Education, Tel Aviv

Abstract

It has been assumed, on historical and psychological grounds, that the concept of irrational numbers faces two major intuitive obstacles: a) the difficulty to accept that two magnitudes (two line segments) may be incommensurable (no common unit may be found); and b) the difficulty to accept that the set of rational numbers, though everywhere dense, does not cover all the points in an interval: one has to consider also the more "rich" infinity of irrational points. In order to assess the presence and the effects of these obstacles, three groups of subjects were investigated: students in grades 9 and 10 and prospective teachers.

The results did not confirm these hypotheses. Many students are ignorant when asked to classify various numbers (rational, irrational, real) but only a small part of them (20 - 30%) are really disturbed by the assumed obstacles. It has been concluded that such erroneous intuitions (a common unit can always be found by indefinitely decreasing it" and "in an interval it is impossible to have two different infinite sets of points [or numbers]") have not a primitive nature. They imply a certain intellectual development.

Little attention is paid to the irrational numbers in school mathematics. The main reason, in our opinion, is that school mathematics is essentially conceived as an ensemble of solving techniques. The idea of mathematics as a coherent, structurally organized body of knowledge, is not systematically conveyed to the student. Certainly, this is didactically a difficult task, but curricula should not avoid it. We would like our students to get the feeling of the grandeur, the beauty of mathematics as a fundamental human achievement, not only its utility for practical matters. Certainly, theorems and proofs are taught - not only solving procedures - but the image of an organized whole, the image of the infinitely ingenious endeavors spent by the human mind in thousands of years, in order to create this dynamic, coherent and harmonious structure is mainly lost in the day to day teaching process.

Not only school students seem to have deficiencies in this matter. The results of a research project recently performed by us (together with Dr. Dina Tirosh) have shown how vague, how incoherent and fragmentary are the students' (prospective teachers) mathematical notions - especially those referring to the system of numbers (unpublished Research Report).

If one intends to convey to the students the feeling of the structurality of mathematics, one has to emphasize, first of all, the coherent picture of the number system with its strict hierarchy. Let us confine our discussion to the system of real numbers. If we pass from the natural numbers to the set of integers and from it to the set of rational numbers, the term "rational number" itself imposes the opposite concept of irrational numbers. How would it be possible to pass from the rational numbers to the set of real numbers without describing the set of irrational numbers?

The irrational numbers are a part of the system and without them the concept of real numbers is incomplete. It suffices to neglect the irrational numbers and the whole system falls apart. This is what happens today.

But the understanding of the irrational numbers raises a severe epistemological problem: The irrational numbers seem to be essentially counter-intuitive. This has been the basic assumption of the present research. The main aim of the investigation has been to prove that assumption.

First, we assumed that the situation of incommensurability which, in fact, generates the class of irrational numbers is, naturally, counter intuitive. Let us consider two segment lines, AB and CD, of different lengths. If one assumes that one may decrease at will indefinitely the unit measuring one segment, one may naturally assume that one may always find a unit which will fit both segments (that is a segment unit which will cover an entire number of times both segments). The idea that two line segments may be, a priori, incommensurable - that is, no common unit is to be found - that idea seemed to us inacceptable intuitively. Consequently, we assumed that, most of the individuals, with no appropriate mathematical training, will claim spontaneously, that two magnitudes - two line segments - are, in principle, always commensurable. Incommensurability will appear to them, intuitively, surprising.

The second hypothesis which inspired the present research, was the following: We assumed that individuals without appropriate training, will not be ready to accept that in an interval, no matter how small, there is an infinity of rational points (numbers) and nevertheless in the same interval there is room for another infinity of a different kind of points (numbers), namely the irrational ones (see Courant & Robbins, 1941/1978, p. 60).

Briefly speaking, we assumed that, because of these two intuitive obstacles, the teaching and learning of irrational numbers constitute difficult tasks.

As a matter of fact, as one will see, the above hypotheses have not been confirmed. Only about a quarter of the subjects questioned, hold the intuitive views -- assumed by us to be quasi general.

The Method

Considering the psychological interest of incommensurability and irrational numbers, a research has been conducted referring to two basic aspects: the formal knowledge and the intuitive understanding. The main hypothesis of the research has been that the notions of incommensurability and irrational numbers are counter-intuitive. It has been assumed that the intuitive feelings of subjects, at various ages, concerning the concepts of measuring and infinity constitute natural epistemological obstacles opposed to the acceptance of incommensurability and the existence of irrational numbers.

It has also been assumed that, as an effect, students of various ages will encounter difficulties in identifying, manipulating, and defining the irrational numbers. It has been assumed that age has little influence on the respective obstacles, while mathematical knowledge and experience may have a positive effect.

The Subjects were pupils and preservice teachers enrolled in schools and colleges in the Tel Aviv area. The following grade levels were considered: Thirty students in grade 9, 32 students in grade 10 and 29 college students.

The Instrument used was a questionnaire which considered:

a) the formal knowledge of the students (definitions, hierarchy of the system of real numbers, the location of various numbers in that hierarchy),

b) the reactions of the subjects to questions exploring the intuitive attitudes towards density and continuity of sets, towards the nature of infinity, and the operation of measuring.

Procedure. The questionnaire was administrated in the usual classroom conditions. The time allowed was about one hour.

Results

The Formal Knowledge

A. The first question presented 15 numbers of various types and the students had to determine their membership in diverse classes of numbers. We will refer only to the most striking results. The symbol (*) indicates correct answers.

With regard to number π, the following results were obtained (see Table 1).

Table 1: The Nature of π (correct and incorrect answers in percentages)

π	Is a Number*	Rational	Irrational*	Real*
Grade 9	43	13	17	3
Grade 10	59	13	19	28
Preservice Teachers	90	7	79	76

It is absolutely surprising that in grades 9 and 10, most of the students are not yet aware of the irrational character of π. Many did not even know that π is a number, let alone that it is a real number.

B. The nature of - \f(22,7). See Table 2

Table 2: Percentages of Correct and Incorrect Answers

- \f(22,7)	Is a Number*	Rational*	Irrational	Real*	Negative*
Grade 9	70	13	20	7	73
Grade 10	91	31	31	28	84
Preservice Teachers	100	90	10	76	100

Again, we see that the terms, rational, irrational, real numbers are unknown to most of the students in grades 9 and 10. Most of the preservice teachers answered correctly, but there were still 24% who were not able to identify - \f(22,7) as a real number!

C. The Nature of Number 0.121221...

Table 3: Percentages of Correct and Incorrect Answers

0.12122...	Is a Number*	Rational	Irrational*	Real*
Grade 9	27	13	57	17
Grade 10	81	9	66	44
Preservice Teachers	97	17	69	79

Many students, at all age levels, do not identify 0.12112... as an irrational number. Even at the college level, more than thirty percent are in that situation. The notion of real numbers is unknown to most of the high school students and to more than 20 percent of the preservice teachers.

The Nature of $3\sqrt{8}$: (See Table 4)

Table 4: Percentages of Correct and Incorrect Answers

$3\sqrt{8}$	Is a Number*	Is a Whole Number	Rational	Irrational*	Real*
Grade 9	32	10	7	17	10
Grade 10	78	-	13	38	38
Preservice Teachers	97	-	10	86	76

The irrational number $3\sqrt{8}$ has been identified as such by a relatively few students in grades 9 and 10. There were still 14% of preservice teachers who did not recognize the irrational character of the number. The identification of $3\sqrt{8}$ as a real number follows about the same pattern (10% in grade 9 and 38% in grade 10). Among the preservice teachers, 24% do not know that $3\sqrt{8}$ is a real number!

The Nature of 0.0555... (See Table 5)

Table 5: Percentages of Correct and Incorrect Answers

0.0555...	Is a Number*	Rational*	Irrational	Real*
Grade 9	83	50	23	13
Grade 10	97	53	25	41
Preservice Teachers	100	97	-	83

About half of the students in grades 9 and 10 recognize 0.0555... as a rational number. Almost all the preservice teachers do so as well. A quarter of the ninth and tenth graders consider 0.0555... as an irrational number. The notion of real number is not applied correctly to the number 0.0555... by most of the 9th and 10th graders.

The Nature of $\sqrt{16}$. See Table 6.

Table 6: Percentages of Correct and Incorrect Answers

$\sqrt{16}$	Is a Number*	Is a Whole Number*	Rational*	Irrational	Real*
Grade 9	60	57	30	-	32
Grade 10	84	56	44	3	59
Preservice Teachers	97	76	69	3	90

Less than half of the 9th and 10th graders consider $\sqrt{16}$ to be a rational number. According to the other students (except one in each grade) $\sqrt{16}$ is also not an irrational number. This shows again, that for most of the high school students (9th and 10th graders) the terms "rational" and "irrational" numbers are totally obscure.

The situation is certainly better in the preservice teachers, but there are still about 30% who do not identify $\sqrt{16}$ as a rational number (and not as an irrational one).

Let us consider a final example: the number 34.2727... (see Table 7). In the former examples, we have seen that, for many students, a number can be neither rational nor irrational. In the above example (34.2727...), we have a case in which a number may be considered both rational and irrational!

Table 7: Percentages of Correct and Incorrect Answers

34.2727...	Is a Number*	Rational*	Irrational	Real*
Grade 9	83	83	70	13
Grade 10	84	13	56	44
Preservice Teachers	97	59	28	83

Firstly, one can see that for most of the 9th and 10th graders and for more of a quarter of the preservice teachers, 34.2727... is an irrational number! Secondly, for most of the 9th graders, 34.2727... is both rational and irrational. Again, we can see that for most of the 9th and 10th graders, the term "real number" has no meaning.

<u>The Infinity of Points in an Interval A B</u>

The subjects were asked: "Given two points A and B on a straight line, how many points corresponding to rational numbers are there in the interval?" A similar question was put in regard to irrational points.

With regard to the rational points the answer "an infinity of points" was given by 54% of the students in grade 9, 50% in grade 10 and 90% college students. Very few indicated various finite numbers, and 30% (grade 9), 34% (grade 10) and 7% (college students) did not answer at all.

With regard to irrational points, the answer "an infinity" was given by 64% of the subjects in grade 9, 69% in grade 10 and 97% college students. Thirty percent in grade 9, 22% in grade 10 and only 1 student among the college students did not answer at all.

Thus, more than half of the students in grades 9 and 10, and almost all the college students affirm that, in an interval AB on a straight line, there are an infinity of rational points <u>and</u> an infinity of irrational points.

<u>The Intuitive Background: Rejection of our Hypotheses</u>

a) With regard to one of our hypotheses (the intuitive difficulty to accept that in the same interval, there are <u>two</u> infinite sets of elements of a different type) we have to emphasize that <u>no subject mentioned explicitly this difficulty</u>. The hypothesis concerning this difficulty has not been confirmed. As we have seen, most of the students, at all age levels, consider that in an interval AB, on a straight line, there are an infinity of rational points <u>and</u> an infinity of irrational points. The intuitive difficulty, assumed by us, did not manifest itself in the subjects' reactions.

b) A second hypothesis was that the concept of incommensurability is counter-intuitive. It is this idea which we would like to discuss in continuation. There were two questions devoted to this problem.

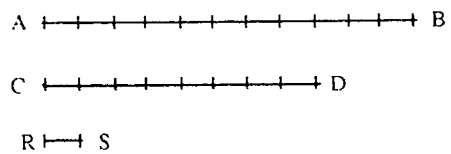
The first one asked whether it is always possible to find, for two line segments, AB and CD, of different lengths, a common unit (that is, a segment which could cover by iteration, exactly the two given segments). The second question asked whether it is possible to find a common unit for the side of a square and its diagonal.

Let us start with the first question. The exact wording of it was the following:

"A line segment is called 'a common unit' of two given line segments, if it covers a whole number of times the two line segments.

607

For instance:



RS is a common unit for the two line segments (since it covers eleven times the segment AB and 8 times the segment CD).
But:
P |_____| Q is not a common unit for both because it is not contained a whole number of times in AB.

Question: Is it always possible to find, for every two line segments, a common unit?"

The data show that 37% in grade 9, 50% in grade 10 and 31% among the preservice teachers, answered affirmatively, which is the wrong answer. The correct answer: "It is sometimes possible" was given by 27% in grade 9; 28% in grade 10 and 38% by the college students. As a matter of fact, only a few among those who chose the correct answer - "Sometimes" - gave an acceptable justification ("only if both segments have the measure expressed by a rational number: (3% - one student - in grade 9; 6% - 2 students - in grade 10; and 17% college students).

Some of those who gave the incorrect answer ("yes") justified it by claiming that one may decrease the unit as much as one wants, until we find a common unit (24% in grade 9; 22% in grade 10; and 28% in college students). This type of answer expresses clearly, the intuition assumed by us. But, as a matter of fact, as we see, only about a quarter of the subjects possess this, mathematically incorrect, intuition.

The answer "No" is ambiguous, but from the respective justifications, one can learn that it expresses, generally, an incorrect understanding. Let us quote some of these justifications: "One cannot always find a common unit as one cannot always find a common divisor for two numbers". The analogy is incorrect. The reasons are fundamentally different; "It is possible that the length of the segments are not whole numbers". Forty-seven percent in grade 9; 31% in grade 10, and 17% among the college students, gave such answers. Finally, many students were simply perplexed by the question and did not answer at all (23% in grade 9; 41% in grade 10; and 28% of college students).

Briefly speaking, only one student in grade 9, only two students in grade 10 and only three college students from about 30 in each group) were able to accept the idea that two segments may be incommensurable! The justification given by these students was that it is possible that when attempting to measure two line segments by the same unit, one may never obtain a rational number for one of them.

One of our hypotheses has been that most of the subjects would not accept the possibility of incommensurability because they would believe, intuitively, that it is always possible to decrease the magnitude of the unit until one would get one which would fit for both segments.

That hypothesis has, also, not been confirmed As we have seen above, only a quarter of the subjects at all three age/grade levels, expressed this claim. The incorrect answers were based on various inadequate or irrelevant explanations.

The intuitive attitude (a common unit can, always, be found by decreasing it indefinitely), seems to be much more elaborate than assumed by us, initially. This is not a primitive, intuitive cognition. As a matter of fact, we assume, it includes the formal, explicit understanding of the concept of unit and its use in the operation of measuring. Because of the lack of a genuine understanding of these concepts, many students encounter difficulties in dealing with the quantitative expression of various types of empirical magnitudes (for instance, in physics, in chemistry). On the other hand, as an effect of this lack of understanding, many students do not see the connection between irrational numbers and incommensurability.

As a result of all these, the wrong intuitive belief that "a common unit can always be found", implies a certain level of intellectual development which, as we have seen, is not to be found in most of the high school students who answered incorrectly.

This explanation leads us to a more general hypothesis: apparently primitive, wrong intuitions, may sometimes be less primitive than one may assume. They may imply a certain preliminary intellectual development.

The same problem related to the incommensurability of two segments has been put in relation to the side and the diagonal of a square: "Is it possible to find a common unit for both the side and the diagonal of a square?". The correct answer ("never") was given by 30% in grade 9, 16% in grade 10 and 49% by the preservice teachers.

The correct answer was justified in an acceptable manner only by 3% (one student) in grades 9 and 10 and by all the preservice teachers who answered correctly (that is, about half of the students). This acceptable justification was: "The length of the side being 1 (a rational number) the length of the diagonal is $\sqrt{2}$, which is an irrational number".

Those who gave the incorrect answers "It is always possible", or "sometimes", justified their solutions in the following manners: "It is possible to find a common measure at will" or "we may decrease the length of the unit as much as we want in order to obtain a common unit". Again, the number of those who offered such justifications was relatively small, contrary to our initial hypothesis: 3% (1 student) in grade 9, 25% in grade 10 and 17% in preservice teachers.

As a matter of fact, most of the students were not able to find any justification for their incorrect answers or gave irrelevant justifications (94% in grade 9, 65% in grade 10, and 28% among the college students).

Briefly speaking, the idea of incommensurability, even when referring to the specific question of the side and the diagonal of a square, remains unclear to most of the 9th and 10th graders and to half of the preservice teachers. But only a few students, contrary to our hypothesis, consider explicitly that a common unit can always be found. This corroborates what has been said above concerning the two line segments of different lengths.

In order to check this finding, the same question (the possibility to find a common unit for two line segments) was asked in a class of 60 students participating in a course of psychology. Half of the students did not answer at all. Among those who answered (29 students) only six - contrary to our expectations - expressed the belief that a common unit can always be found. Seven students answered correctly that "it depends". Two of them explained that it depends whether one of the segments has a rational length. The rest of 5 students gave adequate examples of pairs of segments for which one may find a common unit and pairs of segments for which one cannot find a common unit. The rest of 17 students gave irrelevant answers or did not answer at all. All the students who answered correctly held a BA degree in mathematics or physics.

Concluding Remarks

We have found that most of the students do not possess the assumed wrong intuition "it is always possible to find a common unit". Our explanation is that such an intuition may exist in a mathematically trained mind. It is not a natural belief.

In many cases, the difficulties are of a different origin: the confusion between infinity and irrationality, between negative numbers and the term of irrationality, etc. The Pythagoreans were astonished when discovering that two segments may be incommensurable, because they were mathematicians. (For the historical difficulties in accepting the irrational numbers in mathematics, see: Arcavi, Bruckheimer and Ben-Zwi, 1987.)

A second source of difficulty, not assumed initially by us, is that, generally speaking, the students do not see the connection between irrational numbers and incommensurability. Very few students would justify the possibility of two segments to be incommensurable by referring to the notion of irrational numbers (or vice-versa).

Thirdly, many students know that in an interval, no matter how small, there is an infinity of rational numbers and an infinity of irrational numbers as well. But they are not shocked by the apparent difficulty of these two assertions. How is it possible to have, in the same interval, <u>two</u> different sets of points (the rationals and the irrationals) <u>each</u> of them infinite? This would imply that the infinite set of rational numbers in an interval, do not exhaust that interval. This seems to be intuitively unacceptable. Nevertheless, our subjects were not surprised by that apparent contradiction. It is, again, a mathematically trained mind that feels the difficulty and tries to overcome it.

In short: When initiating the present research, we assumed that the concept of irrational numbers encounters natural intuitive obstacles which would render difficult their understanding and acceptance as it happened in the history of mathematics.

Our findings show a different image. Such obstacles exist. We know it from the history of mathematics and by introspection. <u>But they are not as primitive and natural as we supposed</u>. They imply an intellectual development, they appear in a mathematically trained mind. These are not secondary intuitions because they are not acquired directly, intentionally, through extensive training. These intuitions appear as a natural by-product of a more general development of intelligence.

Should we be satisfied with the fact that most of the students, contrary to our initial hypotheses, are not disturbed, intuitively, by the ideas of incommensurability and by the distinction between density and continuity in an interval? Our answer is negative. We assume that the teacher should find the means to awaken the student's feeling of difficulty, and thus create the cognitive roots for accepting the solutions, which are necessarily formal.

References

Arcavi, A., Bruckheimer, M., Ben-Zwi, R. (1987). History of mathematics for teachers. *For the Learning of Mathematics*, 7 (2), 18-23.

Courant, R., & Robbins, H. (1941/1978). *What is mathematics?* Oxford: Oxford University Press.

Counting on Success in Simple Addition Tasks

Robin Foster

Mathematics Education Research Centre
University of Warwick
Coventry CV4 7AL
UK

The way in which children develop procedures for simple addition takes up a vast time in schooling. How the conceptual ideas develop from initial procedures is often related both to arithmetical development and algebra. This study investigates the methods used by a sample of children in the age group five to eight years. Written tasks were used to identify groups of children for interview. It was found that children demonstrated a wide range of knowledge to assist in the solution of apparently simple arithmetic tasks.

Introduction

The learning of arithmetic and algebra is known to involve compression of knowledge from the early procedures such as of count-all and count-on through the use of known facts and derived facts and on to the generalisation of arithmetic to algebra. An interesting and crucial stage occurs when children meet what some regard as the first step to algebra using addition sums with missing items, including:

1) Missing *total*, $a+b = \square$,
2) Missing *addend*, $a+\square = c$,
3) Missing *augend*, $\square + b = c$.

Individuals competent at arithmetic see all three as being the same thing. But the child who has yet to compress knowledge in this sophisticated way is likely to see these as very different tasks. Counting is at the centre of many of the early solution methods adopted (Gelman and Gallistel, 1982; Ginsburg, 1982), but the solution methods will change as the child grows in sophistication. Fuson (1992) summarises the research on the cognitive growth of whole number concepts. Children first learn to add two numbers together by the procedure of ***count-all*** (counting one set, then the other, then counting the combination of the two). They then compress this procedure so that they only ***count-on*** the second number, starting from first. If the second number is larger than the first some children introduce a further refinement. They change the order of the numbers and ***count on from larger*** to get a more economical way of performing the addition task. This is related to a more flexible ***count on from either*** procedure where the children 'turn around' the numbers whether or not one is larger than the other. Baroody and Ginsburg (1986) term this *protocommutativity*. Beyond these procedures others will use ***known facts*** to obtain a result. The known facts may be acquired by familiarity brought about by frequent use, by extensive practice, or by exploiting relational knowledge about number. A further modification occurs when known facts are used to obtain ***derived facts***.

These various stages of compression, from count-all to known and derived facts are likely to lead to different levels of success in coping with problems with missing numbers. For instance, although any method may successfully lead to a solution of $a+b=\square$, the problem $a+\square=c$ becomes difficult for the child who can only count-all (how can the second set be counted when it is not known?), whilst the child who can perform count-on can solve it by counting (starting at the first number, how many do I count on to get the total?) The third problem $\square+b=c$ is difficult by this method, because the child does not know where to start the counting process. Therefore, a link may be hypothesised between the stage of compression achieved by the child and the level of success in solving these problems as in figure 1.

	$a+b=\square$	$a+\square=c$	$\square+b=c$
count all	✓	✗	✗
count on	✓	✓	✗
count on from either	✓	✓	✓
known/derived fact	✓	✓	✓

Figure 1 : Hypothesised success related to arithmetical sophistication

The difficulties encountered by children with this adaptation are reported by several researchers. Kamii (1985) concludes that missing addend problems are too difficult to give to first-graders. This is consistent with the observation in Figure 1 that children at this stage may cope only with count all. Level One of the English National Curriculum (DES, 1991) can be interpreted as requiring children to be familiar with count all for numbers up to ten.

Hughes (1986) working with very young children notes that the symbolism required in school arithmetic is problematic, and children have problems with the presentation rather than the basic ideas.

Gray and Tall (1991, 1994) see the flexibility of symbolism at the heart of mathematical compression from process to concept. They introduce the term *procept* to help formulate the manner in which a process is compressed to be conceived as a concept. They saw count-all as a combination of three processes, but count-on conceives the first number as a concept and the second number as a counting-on process, whilst a known fact would be a combination of concepts. However, in their analysis, interviews were required to distinguish between the various levels of compression. Figure 1 suggests a way in which a measure of compression may be found which distinguishes between count-all, count-on and count-on-from-either (and higher levels of compression) using only a written test.

The study described in this article analyses the written responses given to the three types of missing number problems by children aged 5 to 8, and reports the relationship with the nature of thinking processes observed in interviews with selected children.

Method

Data for this study were gathered in three schools. Two schools were in the English Midlands and the third was in the North West of England. One hundred and forty seven children in Years One (age five to six years), Two (age six to seven years) and Three (age seven to eight years) were given fifteen question to answer. Their responses to different presentations of the tasks were analysed and investigated. To investigate the qualitatively different performances of children of differing levels of attainment, the children were divided into three equal groups on their written performance, the top third (Higher Attainers), the middle third (Middle Attainers) and the rest (Lower Attainers).

In addition twenty four children were observed in video taped interviews. The subjects for interview were selected by their responses to the written tasks. Two main groups were selected: a *more successful group* characterised by getting all of the tasks correct (who came from the Higher Attainers); and a *less successful group* (coming mainly from upper portion of the Lower Attainers), characterised by those who got all of the missing total tasks correct and a substantial number of the others incorrect.

The Written Task

The fifteen questions were administered to the children in normal classroom time. Class teachers were requested to ask the children to perform the tasks with no assistance, but with reference to counting aids if necessary.

Type of presentation	Items	Year One n=47	Year Two n=54	Year 3 n=47
(1) Missing Total	2+3=□ 5+5=□ 3+5=□ 7+6=□ 4+8=□	82%	89%	97%
(2) Missing Addend	5+□=10 2+□=5 7+□=13 3+□=8 4+□=12	37%	70%	90%
(3) Missing Augend	□+3=5 □+5=8 □+5=10 □+6=13 □+8=12	37%	65%	83%

Figure 2 Percentage of correct responses to items

The first impression from the results is the high degree of accuracy of each year on the missing total items increasing over the years from 82% to 98%. These contrast markedly with performances on the other two tasks, especially in the first year.

However, the table fails to distinguish the performances of the higher attainers from the lower attainers who, according to our theory would be expected to perform very differently on the three types of question. To see this we took the children and entered their results on a spreadsheet, ordering them according to their totals (highest first) in each year and where two totals were equal, the ordering was performed using random numbers. Each year list was then divided into three (almost) equal parts in order to give three groups: Higher Attainers, Middle Attainers and Lower Attainers. (Years 1 and 3 had 15 Higher, 16 Middle, 16 Lower; Year 2 had 18 in each group.) The results of the three groups in each year are given in figure 3.

Year 1	(1)	(2)	(3)
Higher	99%	*81%*	*83%*
Middle	86%	29%	29%
Lower	59%	5%	1%

Year 2	(1)	(2)	(3)
Higher	100%	96%	96%
Middle	93%	*76%*	*80%*
Lower	74%	38%	20%

Year 3	(1)	(2)	(3)
Higher	100%	100%	100%
Middle	100%	98%	98%
Lower	93%	73%	53%

Figure 3 Percentage of correct responses in each category by year at different levels of attainment

According to the theory, one would expect the totals to be greater in category (1) (missing total) than category (2) (missing addend), which in turn will exceed category (3) (missing augend). In fact there are small reversals in performance between categories (2) and (3) in Year 1 Higher and Year 2 Middle (shown in *italics*). In interview such children used a fair proportion of counting strategies and demonstrated willingness to interchange addend and augend (protocommutativity), hence little difference would be expected. In addition it transpires that *if* count-on was to be used, the numbers to be counted are slightly larger in category (3) (e. g. $4+\square=12$ which requires a count of eight) than category (2) (where $\square+8=12$, turned around to $8+\square=12$ requires only a count of 4).

Comparing category (1) and (2) in order of size of missing number gives:

(1) missing addend: $2+\boxed{3}=5$ $3+\boxed{5}=8$ $5+\boxed{5}=10$ $7+\boxed{6}=13$ $4+\boxed{8}=12$

(2) Missing augend: $\boxed{2}+3=5$ $\boxed{3}+5=8$ $\boxed{4}+8=12$ $\boxed{5}+5=10$ $\boxed{7}+6=13$

showing smaller numbers in each case. Hence *if* the children can handle the change in order, and proceed to count-on, counting is slightly more prone to error in (3) than (2). In retrospect, therefore, the perturbations in the percentages due to executive errors may reverse categories (2) and (3) when they are close.

The Higher Attainers in Year 1 show a small difference between category (1) and categories (2) and (3). In the interviews the selected children in this group proved to use a variety of strategies, involving counting, known facts and derived facts, showing that they were flexible and *proceptual* in the sense of Tall & Gray (1994). Year 2 Higher Attainers were almost completely successful, and Year 3 totally successful, responding with more known facts..

The Middle Attainers in Year 1 show a marked difference between category (1) and categories (2) and (3). The difference is far less in Year 2 and insignificant in Year (3). As categories (2) and (3) differ little from Year 1 onwards, this suggests that protocommutativity is available to Middle Attainers from an early stage but that counting persists to a greater extent, for a longer period than the Higher Attainers.

Lower Attainers show a different pattern from Middle and Higher Attainers. In Year 1 they have almost no success with categories (2) and (3), although they average approximately 3 questions correct out of the five in category (1) (straight addition). According to the theory, this is consistent with a wide use of count-all by Lower

Attainers in year 1. Although the percentage success increases in every category in each year, in years 2 and 3 there remains a clear difference between all three categories. Given that the actual counting involved is *easier* in category (3) than category (2), this suggests that the reversal of order causes some difficulty. As we shall see in the interviews, the reasons behind this are quite subtle. For instance, Rebecca (Year 3) was able to 'turn around' the numbers in some situations but did not do so in the missing augend examples.

The Interview tasks

The twenty four children were interviewed in pairs. The problems of obtaining information by requesting children to 'talk-aloud' have been addressed by Schoenfeld (1985). He indicated that in one to one interviews, the results could be seen as unreliable. For this study the pairing of children was introduced so as avoid such problems and to allow children the opportunity to discuss with each other. Any prompting from the interviewer would be minimised as they spoke to each other. At the same time these exchanges could be subsequently analysed to discover aspects of the procedures they had adopted to solve the problems. Twelve questions were prepared to give to the children. They were written on individual cards so that they could be copied by the children on to paper. In most cases all questions were given to the children, but in some instances especially where it was obvious to me as the interviewer that they would present difficulties alternative question were asked. Also in two interviews, where the children were clearly succeeding additional more difficult question were asked. This did not affect the data obtained, in these cases it confirmed what was being discovered. The questions on the cards are as follows:

$3+4=\square$	$3+\square=7$	$\square+4=7$
$4+2=\square$	$4+\square=6$	$\square+2=6$
$5+6=\square$	$5+\square=11$	$\square+6=11$
$6+8=\square$	$6+\square=14$	$\square+8=14$

Ten of the children were from Year One (five were in the more successful group – those who got all 15 of the written questions correct and five were in the less successful group- those who got the first five correct and appeared to have problems with the last ten questions). Eight of the children were from Year Two (four in each category) and six were from Year Three (four assigned to the more successful group and two to the less successful group).

Results of the interviews

The overwhelming difference between the groups is the way in which the more successful group were less dependent on counting as a basis for their solution methods. Counting procedures were not totally absent from the children in the more successful group. Several children who were successful in all the tasks used counting procedures first before applying other methods. The younger children in the successful group were more likely to use counting methods to solve the missing total questions. Quite young children (5 years) were actually using known facts (e.g. 'five and five is ten and one more is eleven' for $5+6=\square$). The older children in the more successful group were using known facts with ease and did not seem to be using counting at all. The speed at which they responded seemed to suggest that no counting was involved, although when questioned two of them, Christopher B (7 years) and Sarah (8 years) said they were counting and thinking in their head to obtain the results for all presentations of the tasks. On deeper questioning they did appear to be using known facts and derived facts to obtain solutions. Two others in this category – James D (7 years) and Sally (7 years) were in no doubt that they were 'using their heads' to obtain an answer. In fact they appeared to see the tasks as being most trivial, James requested some harder ones and answered very quickly to $\square+8=100$, saying '92, because 100 subtract 8 is 92.'

Even quite young children in the more successful group were willing to work out missing augend problems by turning them around. Christopher A (5 years) used a count-all method with his fingers for the missing total questions, but had difficulty with $5+6=\square$, until Leon (5 years), his fellow interviewee said the answer quickly. An interesting exchange took place.

Christopher	How did you know that?
Leon	Five and five makes ten, so five and six is eleven.
Christopher	I can't use my fingers for that.

Later the role reversed when Christopher used a his fingers to represent the total in the missing addend and missing augend examples and subtracting. Christopher was quite happy to 'turn the numbers around' when necessary.

None of the younger children in the more successful group could really be described as fully conceptual as they all at some point resorted to procedure, even though they seemed to be giving general rules in specific cases. The children in the less successful group of the sample used solution methods dominated by counting. Four of the children (James C (5 years) Year 1; Claire (5 years) Year One; Jacquie (7 years) Year 2 and James A (6 years) Year Two) were good at answering the missing totals but found the others difficult and were less successful at them.

Some simply added the numbers in both the missing addend and missing augend examples. This behaviour is regularly noted by teachers and researchers (see for example Steffe, Thompson & Richards, 1982). The children add the numbers according to Kamii (1985) because 'addition is natural for them and that cannot make the hierarchical relationship necessary to *read into the equation*.'

The missing augend questions seemed to take longer for them to complete and were often answered incorrectly as were the missing addend questions. One child Rebecca (aged 7) in Year Three used counting procedures. She found the missing addend uniquely difficult. She used count-on for $3+4=\square$ and $4+2=\square$. For $5+6=\square$ and $6+8=\square$ she 'turned them around.' For the missing addend types she said she, 'tried to remember how many I counted between the first number and the answer.' She later said that she could not do the last ones because she did not know where to start her counting.

James D, one of the oldest children in the less successful group was able to talk in clear terms about how he was working. In the interview he demonstrated a relational understanding which was similar to many younger children in the more successful group, but his inaccuracies seemed a result of unsureness on his part.

Discussion

The study has indicated for the given sample there is an increase in skill as the children get older. This may be self evident, but within this there is the finding that for all three presentations of the task there is an improvement in success which is greater for the two presentation of the tasks in the form of missing addend and missing augend.

There appears to be no difference in success between the missing addend and missing augend presentation. This is surprising and runs counter to the prediction in figure 1. It would appear if asked the questions in this form, children seem quite naturally turn them around. This might suggest that the procedural methods for missing augend are the same as for missing addend. The interviews confirm this, but they also indicate that these question took longer to perform. Once children were competent at all of the available procedures they select one which best fits what they require. The turning around of a statement does not imply a full understanding of commutativity, rather it seems to confirm the observations of *protocommutativity* made by Baroody and Ginsburg (1986).

The behaviour demonstrated by Rebecca confirms that the procedure of count-on does not by itself allow for solutions of missing augends even though she was able to turn around the order to solve missing total problems. The compression of thought represented by an ability to solve all of the types of presentation described in this study is complex. When actually solving a particular example children resort to the whole battery of skills and knowledge at their disposal. The left to right reading of a number statement is quickly over-ridden by the convenience of counting on from larger or counting on from either for the majority of children. As procedures these are probably more important than count-on itself. This study demonstrates that the required versatility is present in large measure in the successful group, and is not entirely absent from the less successful group.

In terms of the performance on the written test alone, the top third of each year perform increasingly well starting from a high standard in Year One which already shows protocommutativity and a wide range of strategies, to total success in year three. The middle third are far less successful in missing addend and missing augend tasks in Year One, but by Year Three they perform almost as well as the top third in these paraticular tasks. Although the bottom third of the class show increasingly better performances year on year, there continues to be a marked difference between their average performance on all three tasks, consistent with the "proceptual divide of Tall & Gray (1994) which shows a qualitative difference in performance between flexible use of number concepts on the one hand and counting procedures on the other.

References

Baroody, A. & Ginsburg, H., (1986) *The Relationship between Initial Meaningful and Mechanical Knowledge of Arithmetic*. In Hiebert, J., (Ed.), *Conceptual and Procedural Knowledge: The case of Mathematics* (pp. 75–112), Hillsdale, NJ: Lawrence Erlbaum Associates.

DES (1991) *Mathematics in the National Curriculum*, London: HMSO.

Foster, R. (in preparation) *It's that goat page!* Pre-print available from Mathematics Education Research Centre, University of Warwick.

Fuson, K. (1992) *Research on Whole Number Addition and Subtraction*. In Grouws, D. A. (Ed.) , *Handbook of Research on Mathematics Teaching and Learning*, (pp. 243–275). New York: Macmillan.

Gelman, R. & Gallistel, C. R. (1986). *The Child's Understanding of Number*. Harvard University Press.

Ginsburg, H. (1982) *Children's Arithmetic*. New York: D Van Nostrand..

Gray, E. M. & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *The Journal for Research in Mathematics Education*, 26 (2), 115–141.

Gray, E. M. & Tall, D. O. (1991). *Success and Failure in Mathematics: Procept and Procedure: A Primary Perspective*. Mathematics Education Research Centre, University of Warwick.

Hughes, M. (1986). *Children and Number*. Oxford: Blackwell.

Kamii, C. K. with DeClark, G. (1985). *Young Children Reinvent Arithmetic*. New York: Teachers College Press.

Schoenfeld, A. H. (1985). Making Sense of 'Out Loud' Problem Solving Protocols, *The Journal of Mathematical Behavior*, **4**, 171-191.

Steffe, L. P., Thompson, P. & Richards, J., (1982). *Children's Counting in Arithmetical Problem Solving*. In Carpenter, T. P., Moser, J. M. & Romberg, T., (Editors), *Addition and Subtraction: A Cognitive Perspective*. Hillsdale, NJ: Lawrence Erlbaum Associates.

PARAMETERS, UNKNOWNS AND VARIABLES: A LITTLE DIFFERENCE?

Fulvia Furinghetti* and Domingo Paola**

*Dipartimento di Matematica - Università di Genova, **Liceo Scientifico 'G. Bruno' - Albenga (Italy)

ABSTRACT. *In this paper we report on a research concerning algebra learning in secondary school; the focus is on parameters and their relation to unknowns and variables. In developing our study we at first analysed the notion (in its manipulative and conceptual aspects) using a methodology we had already tested in other studies on algebra learning which consists in singling out what lies behind to a given notion and n constructing a tree of notions related to the initial one. We then prepared a questionnaire to establish how students perceive the differences between parameters, unknowns and variables and deal with algebraic situations where these notions intervene. The questionnaire was handed out to 199 students aged 16-17 of 3 schools. The results offer us useful insights for the analysis of fundamental aspects of algebraic thinking.*

INTRODUCTION

One of the recommendations to teachers by Augustus de Morgan in his treatise *On study and difficulties of mathematics* published in 1831 by the *Society for the diffusion of useful knowledge* is that of avoiding 'purely metaphysical considerations' in teaching algebra. This idea of filtering the knowledge to be transmitted to students is still present: in (Hanna, 1989) the author observes that in school practice it may be convenient that some notions be left in shade.

Among the notions mentioned by Hanna we find that of *parameter*, an elusive concept that carries with it the difficulties of literal symbols and the ambiguities of its analogy-difference with the concepts of variable and unknown. As it happens with many algebraic notions the difficulties encountered by students are of a dual nature: syntactic-manipulative and semantic-conceptual.

In school practice we have often noticed a prevalence of interest by teachers on the former aspect (synthatic-manipulative); consider, for example the problems concerning the domain of a function in which students have to use theorems, to deal with algebraic expressions, to discuss certain operations and find solutions. These problems test logical failures and wrong strategies in problem solving, but, in general, do not contribute significantly to clarify the nature of parameters and their role in problems.

Also in educational research parameters are rather neglected: they are usually mentioned in algebraic articles as a side feature in the discussion about variable. As far as we know in the journals *For the learning of mathematics* and *Educational studies in mathematics* the only article containing in the title the word parameter is (Šedivy', 1976). The focus of this article is mainly on the difficulties linked with manipulation and on the didactic remedies to be introduced in teaching; this is an important feature considered also in (Wenger, 1988).

Since we have often observed that students, even if able to manage a satisfying 'syntactic manipulation' of formulas containing parameters, do not grasp the 'underlying semantics', in our research report we shall investigate this last point, paying attention in particular to the links with the concepts of variable and unknown. The importance of these links is stressed by the history of mathematics: the use of letters allowing a generalization of problems is considered a fundamental step in the passage from arithmetic to algebra, but letters representing quantities known or unknown were already used by Euclid; what determined the onward leap was Viète's idea of a convention to distinguish in a problem the numbers given (the parameters, for which he uses consonants) from those to be found (the unknowns, for which he uses vocals).

In line with (Harper, 1987) it is reasonable to think that this fruitful convention has taken such a long time to emerge because of epistemological obstacles. The problem we face in the present research is to establish what happens to students, whether they have completely overcome the obstacles and what degree of mastery they have in dealing with algebraic situations in which parameters, unknowns and variables intervene. This research on parameters is also an occasion to obtain significant insights on different aspects of algebraic thinking that we shall allude to in the following.

METHODOLOGY

The study was carried out through a questionnaire distributed to 199 students aged 16 to 17 from three schools (in a big town, in a little town, in a big village respectively); the number of boys and girls was nearly the same. The teachers involved were 8. All the students studied algebra (including functions) and basic elements of analytical geometry. The mathematics programs in the schools considered are exacting and mathematics is one of the most important subjects taught.

In the questionnaire there are 11 questions with one or more correct answers, with the possibility of adding a comment. Students are asked to grade their answers on a 3-step scale according to the degree of confidence they have in their answer (1 = low, 2 = medium, 3 = high). The last question is an open question concerning students' views on the differences among variables, unknowns, parameters. The time allowed was 50 minutes. Students were not informed of the objective of the questionnaire, in particular of the fact that is was focused on parameters.

In preparing the questions we considered the difficulties hidden behind parameters by following our methodology applied for the first time in the research reported in (Chiarugi, Fracassina & Furinghetti, 1990) which consists in investigating what lies 'behind the notion' under consideration. This methodology enlarges the domain of the study to other mathematical notions linked with the initial one and originates a tree of notions and related difficulties. In this way we can get to what we can term 'atoms of notions' and 'atoms of difficulties', that is to say the primitive components of the students' knowledge of a certain notion. At the end of this process it is further possible to single out the pre-existing concept images present in students' mind when a certain notion is introduced for the first time, as already discussed in (Furinghetti & Paola, 1988; Furinghetti, 1993). We point out that for us the term 'notion' includes concepts, techniques and applications.

The significant notions we singled out behind the notion of parameter (apart from those of variable and unknown) were: context, generalization, infinity, symbol, stereotypes, language, logic (in particular use of quantifiers). The result of this preparatory work is the questionnaire which tests the students' understanding on:

• concept of parameter as a variable independent from the other variables of the problem, namely difference in the role of parameters and unknown, of parameters and variables (questions 1, 6, 8, 9)
• influence of the context in interpreting this role (questions 2, 3, 4, 10) and, in particular, the parameter as a variable which determines the positions of geometrical entities (questions 2, 4)
• use and interpretation of mathematical notations and language, with particular reference to the language of logic (questions 5, 7)
• autonomous use of basic mathematical ideas such as generalization, infinity, quantifiers in the explanations of the nature of parameters, variables, unknowns (question 11).

THE FINDINGS OF THE QUESTIONNAIRE

The results of the questionnaire with the added information of the degree of students' confidence gives interesting insights into students' behavior in different directions since the students' performance in algebra involves many factors. Here we shall confine ourselves to the features linked to parameters and report only in a few cases data on students' confidence (indicated by A_1, B_1 and so on). Our considerations rest on all these data and the comments written by students that orient and reinforce the feeling arising from the data. The questions of the questionnaire are reported integrally and the correct answers are underlined. The sums of the percentages may amount to less or more than 100 since there are multiple choices or no answers at all.

ROLE OF PARAMETERS, UNKNOWNS AND VARIABLES

•••1. Given the equation (x is the unknown)

$$(1 - k)x^2 + 2kx + 3 = 0,$$

is the proposition «the equation has no solution if $k = 1$» true or false?

[A] 1 2 3	the proposition is true since..	8% (A_3 56%)
<u>[B] 1 2 3</u>	the proposition is false since..	<u>92%</u> (B_3 84%)

6. Given the exercise: «Find x solving the inequality $kx>0$», is it correct the solution «if $x>0$ the inequality is solved by $k>0$, if $x = 0$ solutions do not exist; if $x<0$ the solutions are the negative values of k »?

[A] 1 2 3	yes	64% (A_3 55%)
<u>[B] 1 2 3</u>	no, since..	<u>32%</u> (B_3 76%)

8. In the expression $y = f(x) = 2kx + 3x^2$

[A] 1 2 3	x is function of y	20%
[B] 1 2 3	x and y are each function of the other	9%
[C] 1 2 3	k and y are function of x	5%
<u>[D] 1 2 3</u>	only y is function of x	<u>61%</u>
<u>[E] 1 2 3</u>	the variation of k does not depend on the variation of x	<u>19%</u>
[F] 1 2 3	....	4%

•••9. In a problem of the type «Given the equation (*) (x is the unknown)

$$(*) \quad 3ax - 1 = 5x + 3(a + 1)$$

find a such that 2 is solution of the equation (*)»

[A] 1 2 3	a and x are unknowns of the problem	5%
[B] 1 2 3	a is parameter of the equation (*) and x is unknown of the problem	54%
[C] 1 2 3	a is another unknown of the equation (*)	2%
<u>[D] 1 2 3</u>	a is unknown of the problem and x is unknown of the equation (*)	<u>36%</u>
[E] 1 2 3	....	3%

The preceding questions concern situations in which parameters appear together with to unknowns or variables. In question 1 the strategy of solution is prevalently based on manipulating symbols, while in the other questions it rests on the interpretation of symbols. The equation in question 1 is similar to that solved by Sylvestre-François Lacroix (1765-1843), making the 'famous' mistake of remarking that for $k = 1$ the equation has no roots. In our questionnaire students are induced by the form of the statement (see: «if $k = 1$») to first substitute the given value 1 to k in the equation and then solve the equation in which the role of unknown is well defined. This is what they did in the great majority (92%) and with a high degree of confidence (B_3 84%). The source of confidence lies in the opportunity to develop a computational process which has a visible and concrete end in the solution of the equation; all the solvers reach the solution even if not required. We point out that the wrong solvers too show a good confidence in their answer (A_3 56%). This feature of the question (i. e. *the presence of an terminating process*), provides a kind of 'reference point' for students and we conjecture that its absence is the cause of some of the failures we find in algebraic students' performances. In other questions, indeed, where this reference point is missed, i. e. there is not explicitly a terminating process to develop, the results are worse; in question 6, for example, students showed difficulties (wrong answers are 64% and many of the correct answers have an unsatisfying justification). In this question the two objects k and x are in a symmetrical situation: this make it difficult to render explicit the difference (which we could ascribe to a 'metalinguistic' level) between them. If we formulate the answer according to the following schema in which the presence and the role of quantifiers are made explicit

$$\forall\, k \in \mathbb{R}^+ \; \exists\, x \in \mathbb{R}^+ \text{ such that } kx > 0$$

we understand the students' difficulties over parameter: to introduce parameters in equations or inequalities means to pass to more complex formulae in which a universal quantifier followed by an existential quantifier appears. Moreover we observe that the concept of infinity is hidden behind quantifiers.

In addition to this error we find a secondary error which is rather frequent in school practice concerning the discussion of the value $k = 0$): it may be ascribed to students' attitude of regarding number 0 with suspicion (0 is a useless number since $a + 0 = a$, 0 is a dangerous number since it is not possible to divide by 0). Looking back once again at the history of mathematics we see that 0 was accepted with difficulty by the ancient and later authors of manuals were aware of the difficulties hidden by 0; as a significant example we quote an excellent book (Agnesi, 1748) written with declared educational purposes in which the cases of parameters equal zero seem to be treated with a certain caution.

We considered it particularly significant to point out the high rate of confidence students show in answering these questions (both in the case of correct answers B of question 1 and the incorrect A of question 6) which we ascribe to the computational side of the questions proposed. The data we have on the other questions where focus shifts to more critical aspects show a greater level of doubt.

Questions 8 and 9 show the importance of the notations and symbols in mathematics; only 38% fail question 8 while 64% do so in the case of question 9. Their schema is quite the same: in a given expression the role of parameters, unknowns and variables has to be specified. But in question 8 the notation $y = f(x)$ is 'self-explaining' (k does not appear as argument of f and the letter f suggests the dependence) while in question 9 the interpretation of symbols suffers the ambiguity induced by the convention that usually (since Viète) the first letters (here a) indicate parameters and the last ones (here x) unknowns. Moreover, while in question 8 the context is well defined, in question 9 there is a superposition of contexts and the role changes according to the context: the parameter a in the equation (*) becomes the unknown in the problem, the unknown x becomes a given number i. e. a constant. In this case the gap between the syntactic level (the manipulation of the algebraic formula, such as in solving an equation) in which students showed little doubt (see question 1) and the semantic level of the other questions in which students are uncertain, is evident.

The scant interest of students in option [E] of question 8 stresses a problem of parameters that will emerge in the tentative definitions given by students in question 11: the dependence or independence of parameters, unknowns and variables.

USE AND INTERPRETATION OF MATHEMATICAL NOTATIONS AND LANGUAGE

5. Ariel has to indicate to Kaliban that real values of k exist for which the equation (x is the unknown) (*) has not real solution

$$(^*)\ x^2 + 2kx + 1 = 0.$$

Among the following expressions which seems to you the most suitable to indicate the values of k for which the equation has not real solutions?

[A] 1 2 3	$-1<k<1$	22%
[B] 1 2 3	k exists between -1 and 1 such that x is not a real number	15%
[C] 1 2 3	for k between -1 and 1 the equation has not real solution	35%
[D] 1 2 3	for k belonging to the interval (-1, 1) x real does not exist such that $x^2 + 2kx + 1 = 0$	37%

Briefly justify your answer.................................... ....

[E] 1 2 3	....	5%

7. In your opinion in the literal expression $x^2 + y^2 + c^2 + 2xy + 2yc + 2xc$

x is:

[A] 1 2 3	variable	27%
[B] 1 2 3	constant	9%
[C] 1 2 3	parameter	7%
[D] 1 2 3	unknown	75%
[E] 1 2 3	....	1%

y is:

[A] 1 2 3	variable	30%
[B] 1 2 3	constant	9%
[C] 1 2 3	parameter	10%
[D] 1 2 3	unknown	67%
[E] 1 2 3	....	1%

c is:

[A] 1 2 3	variable	30%
[B] 1 2 3	constant	18%
[C] 1 2 3	parameter	46%
[D] 1 2 3	unknown	19%
[E] 1 2 3	....	3%

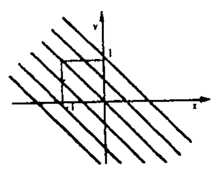
If we observe the results of question 5 we observe an increasing students' preference towards the a form of the statement in which as much information as possible is made explicit, where in particular quantifiers appear in option [D]. We also note student's 'reassuring' habit of repeating in the answer the same words of the question. In our paper on the understanding of mathematical statements (Furinghetti & Paola, 1991) we found the same attitude of telling not *the truth*, but *all the truth*, as already stressed by Freudenthal. The explanations given by students for justifying their choice reveal that they are not really aware of the importance of language in mathematics and, even more, they are unable to interpret what is asked in the given statement: many of them check the correctness of the interval (-1, 1) through computations instead of justifying their preference. The irresistible impulse to calculate that we pointed out in (Furinghetti & Paola, 1991) prevails on other critical reflections.

Concerning question 7 the data show the strong presence of stereotypes in notations: it seems that students loose completely their control of the semantic situation and put themselves at the mercy of the syntactic aspect. We observe that x and y are classified by almost the same percentage of students as unknown, for c there a balanced share of attributions with some preference for the role of parameter. We can say that these data confirm our initial observation about the two levels (the manipulative and the conceptual) one; interpreting x and y as unknowns means to stress the algorithmic-solving aspect rather than the formal aspect evoked by variable. As to this last term there is a quite constant percentage (27%, 30%, 30%) of students ascribing to the three letters this meaning, maybe because variable is perceived as a multipurpose object. In this question very few students observe that the role to be given to the letters should not be stated *a priori*.

We observe that some students before answering perform computations to reduce the given expression to $(x + y + c)^2$: again the irresistible impulse to calculate emerges. But the stereotype on notations is so strong that it prevails on the good [illegible] given by the representation $(x + y + c)^2$ which has the advantage of stressing further the symmetry of the situation.

INFLUENCE OF THE CONTEXT IN DEALING WITH PARAMETERS

2. Which equation is associated with the following right lines?

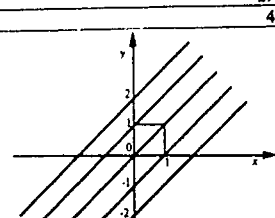


[A] 1 2 3	$y = kx$	17%
[B] 1 2 3	$x + y = k$	55% (B1 10%; B2 16%; B3 75%)
[C] 1 2 3	$y = x + k$	12%
[D] 1 2 3	....	12%

3. In a Cartesian plane the equation $a(x - 1) + b(y - 1) = 0$ represents the pencil of right lines with center (1, 1) (set of right lines passing through (1, 1)). Which sets a, b, x, y belong to?

[A] 1 2 3	a and b belong to $\mathbb{N}$, x and y belong to $\mathbb{R}$	14%
[B] 1 2 3	to $\mathbb{R}$	53% (B1 37%; B2 42%; B3 21%)
[C] 1 2 3	a and b are constants, x and y belong to $\mathbb{R}$	29%
[D] 1 2 3	....	4%

4. The following five right lines are represented by equations of the type $y = x + k$. Which sets x, y and k belong to?



[A] 1 2 3	x and y belong $\mathbb{R}$, k belongs to $\mathbb{N}$	4%
[B] 1 2 3	x and y belong to $\mathbb{R}$, k belongs to the set of reals between -2 and 2	36%
[C] 1 2 3	to $\Re$	16%
[D] 1 2 3	x and y belong to $\mathbb{R}$, k assumes values in the set {-2, -1, 0, 1, 2}	52% (D1 15%; D2 41%; D3 45%)
[E] 1 2 3	....	3%

•••10. I heard about parameters

[A]	in algebra	68%
[B]	in geometry	20%
[C]	in informatics	15%
[D]	....	1%

The first impression arising from the data is that if the parameter is 'contextualized', i. e. has a specific operative meaning, it is better managed by students. We have chosen analytical geometry as the context in which to operate, but the teachers who have participated to the experiment think that also physics is a good framework to give meaning to parameters.

The best results are achieved in question 2 where students have to understand the meaning of the parameter as an element that determines the position of geometrical entities. Since the wrong answers [C] and [D] are due to errors relating to the slope of the right lines we may conclude that about 79% grasp this meaning. The high degree of confidence in comparison to that of question 3 is remarkable. In question 3 the mistakes do not depend on misunderstanding on geometrical features, but on the meaning of parameters. In particular, the results of [C] and the comments in [D] show that students come across difficulties in the duality between parameters that may vary in $\mathbb{R}$, as variables do, but are fixed in a given problem. The problem of the domain of parameters emerges in question 4: students who avoid considering a discrete set as domain of parameters are 63%. One cause of the numerous choices for option [B] is also due to the ambiguity students experience in interpreting the graphical representation we gave (they think about infinite right lines).

Question 10 shows that students recognize the word parameter as a specific term peculiar to algebra; this does not occur in respect of geometry, where it is much used in practice, but introduced implicitly. Almost all the students who have studied informatics (which is introduced only in classes that experiment new mathematical programs) relate the term parameter to this domain, owing to the fact that in this domain the term parameter is also introduced explicitly.

CONCLUSIONS

The answers to the last question

•••11. Do you think that there are significant differences between parameters, unknowns and variables? If «yes» try to explain the differences, if «no» state why. You can use examples to illustrate your point of view.

offer material for the summing up of the situation about parameters, since all the trends emerged in the preceding questions are catalysed in the students' tentative explanations of differences.

The answers are only 73%, plus 2% of answers of the type «I do not know», while almost all students have answered the other questions. We can understand this behavior since students are not used to write about mathematics nor to explain their mathematical opinions; among the answers, indeed, some are simply «Yes». The first feeling emerging is that the sentence by William Shakespeare «What's in a name? That which we call a rose by any other name would smell as sweet» (*Romeo and Juliet*, act II, scene II) does not always apply to algebra, as students perceive different names as labels for different 'objects'. Only 2% of students states that there are no differences among parameters, unknowns and variables, 69% notices a difference. The following naïve answer «Of course there is a difference otherwise they would have the same name» suggests that this perception rests more on the authority of owners of knowledge (teachers, books, ...) than on a real understanding of the differences.

We observe that students accept hints at explaining by examples; their favourite context for examples are equations and geometry (chiefly right lines); analysis (function) is not much used to explain the nature of variables, only one student uses (and poorly) informatics. Of course equations are used to explain unknowns: this occurs rather effectively in equations with only one unknown x, but if the equation contains also y explanations become muddled. There is a trend to *associate the concepts of unknown and variable* (even if without clear justifications). The first is acceptably described by many students as «a value to be found to solve an equation», its main difference with variable lies in the fact that the unknown is *single* number (indicated by a letter) while variable is rather a *set* of numbers. This set of number may be infinite, while in the case of parameters it is often pointed out that there is an *interval of variation*, (this interval being a recollection of the exercises to find the condition for the existence of the solution of an equation) or that parameter is a *fixed number* or a *fixed point*.

Not more than 20 students are able to form sentences really explaining anything and those who do so also omit the basic features we pointed out, such as generalization or the hidden presence of quantifiers, not even in the 'approximate' form used for some other concepts. Only one student uses quantifiers (in a naïve way). The basic idea that context has relevance in answering this question is neglected, only one student points out that «the differences between parameters, unknown and variables have to be stated by the problem». Variable, for its less operative character, maintains the elusive meaning pointed out in the paper (Schoenfeld & Arcavi, 1988), while the word unknown is explained best and students also show to have an idea of parameters. In both cases they are helped by the operative side of the concepts that offers a concrete context to construct meaning.

REFERENCES

Agnesi, M. G.: 1748, *Instituzioni analitiche*, Milano.

Chiarugi, I., Fracassina, G. & Furinghetti, F.: 1990, 'Learning difficulties behind the notion of absolute value', in Booker, G., Cobb, P. & De Mendicuti, T. N. (editors) *Proceedings of the PME XIV*, v.III, 231-238.

Furinghetti, F.: 1993, 'Images of mathematics outside the community of mathematicians: evidence and explanations', *For the learning of mathematics*, v.XIII, n.2, 33-38.

Furinghetti, F. & Paola, D.: 1988, 'Wrong beliefs and misunderstandings about basic concepts of calculus (age 16-19)', in Goupille, C., Thérien, L. & others (editors) *Proceedings of the 39th CIEAEM's meeting*, 173-177.

Furinghetti, F. & Paola, D.: 1991 'On some obstacles in understanding mathematical texts', in Furinghetti, F. (editor) *Proceedings of the PME XV*, v.II, 56-63.

Hanna, G.: 1989, 'More than formal proof', *For the learning of mathematics*, v.IX, n.1, 20-23.

Harper, E.: 1987, 'Gost of Diophantus', *Educational studies in mathematics*, v.18, 75-90.

Schoenfeld, A. & Arcavi, A.: 1988, 'On the meaning of variable', *Mathematics teacher*, v.81, 420-427.

Sedivy', J.: 1976, 'A note on the role of parameters in mathematics teaching', *Educational studies in mathematics*, v.7, 121-126.

Wenger, R. H.: 1988, 'Cognitive science and algebra' in De Lange, J. & Doorman, M. (editors) *Seniors secondary mathematics education*, ICME 6, Budapest, A4, OW&OC, Utrecht, 69-103.

NEGATIVE NUMBERS IN ALGEBRA. THE USE OF A TEACHING MODEL

AURORA GALLARDO

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL IPN

MEXICO.

Abstract

The extension of the numerical domain from natural to whole numbers, which occurs as the secondary school student undergoes the process of acquiring algebraic language, constitutes an element which is essential for achieving algebraic competence in the solution of problems and equations. This article reports the results of a clinical study where a concrete model of teaching was used. The main objective was that of exploring the operative difficulties encountered by secondary school students in the domain of whole numbers. The results reveal the existence of cognitive tendencies which exhibit the various levels of acceptance of negative numbers by the subjects. The research is preceded by a historical-epistemological study of negative numbers in the solution of algebraic equations on which the present empirical analysis is based.

Resumen

La extensión del dominio numérico de los naturales a los enteros, durante el proceso de adquisición del lenguaje algebraico por el estudiante de secundaria, constituye un elemento esencial para lograr la competencia algebraica en la resolución de problemas y ecuaciones. Este artículo reporta los resultados de un estudio clínico donde se utilizó un modelo concreto de enseñanza, cuya finalidad principal era indagar las dificultades operativas en el dominio de los enteros presentadas por estudiantes de secundaria. Los resultados obtenidos revelaron la existencia de tendencias cognitivas que a su vez, exhibieron los diversos niveles de aceptación del número negativo por los sujetos. Esta investigación está precedida por un estudio histórico epistemológico de los números negativos en la resolución de ecuaciones algebraicas que fundamenta el análisis empírico.

Introduction

As the individual acquires algebraic language, the extension of the numerical domain from natural to whole numbers becomes a crucial element for achieving algebraic competence in the solution of problems and equations. Studies have been carried out in the field of the teaching of mathematics, such as those of Freudenthal (1973), Glaeser (1981), Bell (1982), Janvier (1985), Fischbein (1987), Dreyfus/Thompson (1988), Resnik (1989), Vergnaud (1989), Gallardo/Rojano (1990, 1993), which support the thesis to be demonstrated: the **acceptance of negative numbers** by secondary school students goes through various stages of conceptualization, these are: **subtraction**, where the notion of number is subordinated to the magnitude; **signed number**, when a plus or minus sign is associated with the number; **relative number (or directed number)**, where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domain; **isolated number** where there are two levels, that of the result of an operation or as the solution to a problem or equation. Finally, the formal mathematical concept of **negative number** is reached, where the same status as that of positive number is acquired.

Questions such as the following arise from the above: in equations and problems, which is the numerical domain the secondary school student confers on the constitutive parts of the equation during the process of solution? Which numerical domain is accepted for the solution? What is the relationship between the numerical domain assigned to an equation and the type of language associated with the equation? Which methods or strategies obstruct or facilitate evolution towards the notion of number?

These questions point to the need for research such as that described here[1] whose central concern is the study of the interrelationships between the components: processes of acquisition and use of algebraic language; methods for solving word problems and linear equations; status of the negative number in word problems and linear equations. The general methodology of the project contemplates the interaction of these three components on two levels of analysis, the historical-epistemological level (evolution of meaning) and the didactic level (teaching-learning- cognition). With regard to the first of these, we found that the terms of subtraction, the laws of signs as well as certain elements necessary for operativity with negative numbers appeared in remote historical times in the context of the solution of algebraic equations. The opposing concepts of gain and loss, property and debt, future and past, sale and purchase, are adequate interpretations for positive and negative. A crucial step towards the **acceptance of the negative number** was to admit negative solutions to equations. The main difficulty facing medieval mathematicians in the solution of concrete problems was precisely the interpretation of negative solutions. It is the exploration of these facts which has allowed us to locate the historical component of our work in the medieval period in Europe, between the XII and XV centuries. The basic content of school algebra at a secondary school level today corresponds to this historical period. One of the most important conclusions of this study at a historical level is that the acceptance of the first negative solutions presupposed advanced syncopated language, as well as complete operativity and evolved levels of interpretation of negative numbers. Conclusions such as these provide a basis for the formulation of hypotheses at an ontological level, concerning the conditions in which is it feasible to pass from primitive stages of conceptualization to stages of consolidation and formalization of the notion of negative number (see Gallardo, Rojano, Carrion, 1993). This historical analysis provided guidelines for a later study

1 Research project in process. Departamento de Matemática Educativa, CINVESTAV, Mexico.

with subjects and allowed some of the conditions that propitiate the acceptance by secondary school students of a negative solution to word problems to be established. At an ontological level, we can thus say that ad hoc methods of solution have to be used, as well as the construction of sources of meaning appropriate to the context of the problem and a fluid operativity, in order that the student accept the negative solution. In the case of some problems, the use of algebraic language becomes indispensable for the possible arrival at such a solution (see Gallardo and Rojano, 1993).

The facts mentioned above justify the methodology used in the research.

The Teaching Model

This article describes and reports the results of the use of a concrete teaching model whose main purpose is to explore the operative difficulties that secondary school students have in the domain of whole numbers. The analysis is carried out in the context of the project described in the introduction. The specific methodology of this piece of research consisted of asking 25 second-year students from a secondary school in Mexico City to answer a questionnaire; individual clinical interviews were video-recorded and analyzed. The blocks of items in the interview dealt with the following topics:

1) Operativity in the domain of whole numbers. Use of teaching models.
2) Processes of acquisition and use of languages. Resolution of linear equations.
3) Resolution of word problems.

In this article we report the results of the clinical interview regarding the first topic. The student population selected was familiar with the model of the number line and the syntactic rules of operativity of whole numbers. These students showed conflict in the cases $a - (-b)$ and $-a - (-b)$ with a and b as natural numbers. It was then decided to use a model which would give concrete existence to negative numbers. The historical analysis carried out in the first stage of the research suggested the use of the so-called "Chinese model". In effect, Chinese mathematics operated with negative numbers in the solution of problems. The eighth chapter of the "Mathematical Treatise of Nine Chapters", 250 a.n.e., contains the oldest general method for solving linear equations. Tabulating the coefficients and independent terms in a rectangular arrangement, the calculation board, they could obtain the value of the unknowns of the system. The application of this method to a multitude of problems led them to consider negative numbers. As a consequence, these numbers emerged from a language of calculation free from the meanings associated with them in the context of specific word problems. They used sticks colored red to designate positive numbers and black-colored sticks for negative ones. The operativity employed in the teaching model corresponds to that of the Chinese mathematicians, that is, positive numbers are opposite to negative ones. The central concept arises, the sum of opposites is zero, which gives foundation to all the operations carried out within the model. The Chinese model is based on: 1) the counting of positive numbers is extended to negative numbers; 2) in the process of subtraction there are cases in which an alternative representation of the minuend is required in order to carry out the operation of taking away. Then, the adequate addition of zeros is employed, according to each case. The students were presented with a diagrammatical version of the model. In the world of paper and pencils, positive numbers were white balls and negative numbers, black balls. Zero was represented by the simultaneity of a white and a black ball. Operativity was carried out in the additive domain. For example, the addition 3 + (-2) is presented; the numbers are described in the model ○○○ ●●. They join together, provoking the formation of zeros ○○○●●. The result is a white ball which represents the number 1. Analogously, for the subtraction 2 - 3. The numbers are described in the model. Three cannot be subtracted from 2. A zero is added to the number 2. The representation of 2 + 0: ○○○● is obtained. Then the subtraction can be carried out, crossing out. ⊠⊠⊠● A black ball is obtained, the representation of -1.

During the period of instruction in the model, the majority of the students inferred the rules of the signs in the domain of multiplication. However, the model becomes complicated for the case (-a)(-b) and is a long way from being paradigmatic. It should be said that the main interest in modeling in this analysis is not its usefulness in teaching[2] but as a resource which exhibits the peculiarities of the phenomenon and shows the different cognitive tendencies of the subjects when faced with new concepts and mathematical operations (see Filloy, 1991). These tendencies, observed when the Chinese model is taught in a clinical interview situation, also exhibit the level of acceptance of negative numbers by the student. We will now describe some of the issues identified by our analysis and present some examples taken from the interviews with the students[3].

---

2 Since the 1970's, references to this model in its different forms have been made in the literature on the subject; its advantages and disadvantages in the teaching of whole numbers has been analyzed. See, for example, Janvier (1985), Rowland (1982), Kohn (1978), Bartolin (1976).

3 The concept of cognitive tendency is taken from the work of Eugenio Filloy, "Cognitive Tendencies and Abstraction Processes in Algebra Learning", PME XV, 1991.

Abbreviation of texts. Representation of the number with a minimum of zeros or else a tendency to abbreviate the actions of taking away. Some students reduce the size of the white balls and show the black balls as dots.

Endowment of intermediate meanings. Adding as subtracting, lack of differentiation between the operations of adding and taking away. At the moment of carrying out the actions, the minus sign predominates and they interpret a + (-b) as a - (+b) or else as a - (-b); a, b natural[4]. **Subtracting as adding,** the representation of the number gets in the way of subtraction. For example, 1 - 2 is written o ooand it is stated that the result is 3. 2 - (-3), it is represented as o • o • • and zeros are formed. Instead of executing the action of taking away, the addition of opposites is effected. Identification of the rule of signs at the level of results, when a - (-b), -a - (-b) appears, the rule of signs is not brought into play, it is just stated that the results are equal. **Confusion of the domains of addition and multiplication,** simultaneous use of the binary operation and the rule of signs. It is stated that 3 - (-5) = 2. This result is obtained by operating 3 - 5 = 2 and then using (-)(-) = +.

Operations the student was previously familiar with are not brought into play,

- a + (-b) as -a - (+b)	a - (b) as a - (+b)
a + (-b) as a - (-b)	a - (-b) as a - b
	-a - (+b) as -a - (-b)

Centering of readings, preference for some of the following readings:

Horizontal reading o o • 1 + 0
Vertical reading o o • + 2, - 1. No identified with + 1.
Diagonal reading o o o • + 3, -1. Not identified with 2.
Positional reading o o o • + 10.

Generalizations and erroneous processes. The use of too many signs.
"This sign [-a - (b)] is to subtract; the other [-a - (-b)] is not necessary".

Rule of the minus sign.
"When the minus signs are separated by a number, [-a-b], the result is negative"

Rule of brackets
When there is a bracket [a - (b)] between two minuses it's multiplication"

---

[4] In the interview schedule, specific numbers always appear.

Multiple rule of signs

"Minus [-a - b] with minus [-a - b] is plus".

"Minus [-a - (-b)] with minus [-a - (-b)] is plus, and plus with minus [-a - (-b)] is minus". All signs are used and the numbers added like natural numbers.

Syntax-semantics interaction. The existence of a negative converts the result into a negative. In the model -2-(-3) = 1. At the syntactic level -2-(-3) = -5. This result is more common and the loss of the minus sign is not seen as possible.

Production of personal codes. Simultaneous use of two codes, that of the model and that of arithmetic language: -2 + 1 is represented as ● ● + ○ = 1.

Alteration of the codes in the model. 4 - 2 is described as ○ ○ / ● ● instead of crossing out ○ ○ / ⊠ ⊠. Invention of new codes. The act of taking away represented simultaneously by crossing out and filling in: 2 - 3 is expressed as ⊠ ⊠ / ■ ■

Conclusions

The following can be concluded from the analysis of the cognitive tendencies exhibited by the students while they were taught the Chinese model: the competent users of the model showed,

- a clear idea that the sum of opposites is zero.
- good use of the dual nature of zero, as a null element and formed by opposing elements.
- non-use of the equivalent representation of the number (3 = 4 - 1 = - 2 = ...) but the appropriate addition of zero (3 = 3 + 0 = 3 + 0 + 0 = ...).
- permanence in the additive domain and non-application of the multiplication rule of signs.
- clear differentiation of the actions of adding and subtracting.
- distinction between the languages used (language of the model, arithmetic language).
- recognition of negative numbers (no separation of the sign from the number).

It is important to point out that even when he has a concrete model, the student tends to decode expressions of the type a - (-b), a + (-b), a - (+b), focussing only on the syntactic form and ignoring the meanings associated with the symbols in the model. There is a strong tendency on the part of the subjects to consider negatives as "more powerful than positives" and they do not accept positive results when they operate with negatives. This teaching model is immersed in the realm of arithmetic. It is hoped by the end of the project to demonstrate that negative numbers "do not manage to be numbers in the full sense of the term". Their genesis is found in the realm of algebra where they acquire meaning as solutions to problems and equations. In effect, the consolidation of algebraic language is determined in a fundamental way by evolution towards more advanced levels of conceptualization of negative number (as generalized number, unknown, variable).

Final Discussion

In the historical-epistemological and didactic areas of the research project presented in the introduction, it can be seen that acceptance of negative numbers passes through various levels of conceptualization before becoming the formal mathematical notion of whole number. In this article, the use of a concrete model of teaching revealed the existence of cognitive tendencies that exhibited these levels of acceptance of negatives by the students. In other topics dealt with in the clinical interview the same levels appeared. Thus, regarding topic 2, solution of linear equations, inhibitory mechanisms come into play when the subject perceives a **"possible negative solution"**. On the other hand, methods of solution of equations that the subject was familiar with before for positive solutions do not come into play: method of inversion of operations, transposition of terms, approximation. In some cases, the equation to be solved is re-formulated. For example, $x + 1568 = 392$ becomes $x \times 1568 = 392$ which, by carrying out the erroneous division 1568 392, leads to the solution $x = 4$ whole positive "more natural for the student" than the correct negative solution. With respect to topic 3, regarding problems with a negative solution, when the subject cannot find a positive solution, he resorts to his own methods and language, assuming the different levels of acceptance of the negative (see XV PME-NA, 1993). The above shows the need to consider the mutual interrelationships between the processes of acquisition and use of algebraic language, the methods of solving word problems and linear equations and, the status of the negative number in teaching situations. As we have said before, the study of these interrelations constitutes the central objective of the project as a whole.

References

- Bartolin, P. (1976) "Addition and substraction of directed numbers". Mathematics Teaching.
- Bell, A (1982). "Looking at children at directed numbers". Mathematics Teaching No. 100.

- Carraher, N. (1990). "Negative numbers without the minus sign". Proceedings of Psychology of Mathematics Education XIV. Oaxtepec, Morelos, México.
- Filloy, E. (1991). "Cognitive tendencies and abstraction processes in algebra learning". Proceedings of the XV Psychology of Mathematics Education.
- Fischbein, E. (1987). "Intuition in Science and Mathematics. An Educational Approach". Reidel, Holland.
- Freudenthal, H. (1973). "Mathematics as an educational task", Reidel, Dordrecht.
- Gallardo, A. (1987). "Habilidades prealgebraicas en los niños de menor rendimiento escolar". México: Centro de Investigación y de Estudios Avanzados del IPN, Sección de Matemática Educativa, (tesis de maestría).
- Gallardo, A. y Rojano, T. (1990). "Avoidance and acknowledgement of negative numbers in the context of linear equations". Proceedings of the International Group for the Psychology of Mathematics Education. XIV PME-Conference with the XX North American Chapter (PME-NA), México, Vol. II, pp. 43-49.
- Gallardo, A. y Rojano, T. (1993). "Negative solutions in the context of algebraic word problems". Proceedings of the XV International Group for the Psychology of Mathematics Education and North American Chapter, PME-NA.
- Gallardo, A.; Rojano, T. y Carrión, V. (1993). "Los números negativos en el contexto de resolución de ecuaciones". Sent to Recherches en Didáctique des Mathématiques for publication.
- Glaeser, G. (1981). "Epistemologie des nombres relatifs. Recherches en Didáctique des Mathématiques. Vol. 2, Nc. 3, pp. 303-346
- Janvier, C. (1985). "Comparison of models aimed at teaching signed integers". Proceedings of the IX Psychology of Mathematics Education.
- Kohn, J. (1978). A physical model for operations with integers. Mathematics Teacher.
- Peled, I.; Mukhopadhyay, D. y Resnick, L. (1989). Formal and informal sources of mental models for negatives numbers". Proceedings of the XII Psychology of Mathematics Education.
- Rowland. T. (1982). "Teaching directed numbers. An experiment". Mathemtics in School.
- Thompson, P. & Dreyfus, T. (1988). "Integers as transformations". Journal for Research in Mathematics Education. Vol. 19, No. 2, pp. 115-133.
- Vergnaud, G. (1989). "L'Obstacle des nombres négatifs et l'introduction à l'algèbre. Construction des savoirs. Colloque International Osbtacle Epistémologique et Conflict Socio-Cognitif, CIRADE.

Acknowledgements

I wish to thank Hermanos Revueltas School in Mexico City for providing the setting for our research study. I also wish to acknowledge the support of the Department of Mathemática Educativa of CINVESTAV (México).

MATHEMATICAL MODELLING OF THE ELONGATION OF A SPRING: GIVEN A DOUBLE LENGTH SPRING .....

Rossella Garuti — Institute for Applied Mathematics, CNR, Genoa

Paolo Boero — Department of Mathematics, Genoa University

This report analyzes how, in facing a specific problem of mathematical modelling, the students of two classes of grade VIII have used resources such as their conception of the phenomenon, the mathematical tools available, their previous modelling experience as well as some general "principles". This report also compares two different ways of managing the operations of verificatica of the modelling hypotheses produced.

1. Introduction

A number of studies conducted in the last fifteen years (see Blum & Niss,1991) together with our previous studies (Boero & al.,1993b), have shown that the problem of mathematical modelling of a natural phenomenon requires that rather complex operations be performed by a student (hypothesis formulation and verification, considering, on the one hand, the constraints and relation inherent in the phenomenon, and, on the other hand, their own experience and the mathematical tools available). In the modelling and the model verification process a student should therefore use various resources: firstly, he must possess qualitative knowledge of the phenc nenon (possibly including identification of relationships among any relevant variables, awareness of the cause and effect relationships governing the phenomenon, etc.). If this knowledge is lacking it is very difficult for a student to formulate a reasonable hypothesis and to motivate it consistently with the phenomenon, especially if the modelling problem in question appears as an actual "open problem". Generally, however, this knowledge of the phenomenon is not by itself sufficient to come to the formulation of a mathematical model hypothesis. Good mastery of the mathematical tools is also required in order to enable the translation of the "qualitative idea" on the phenomenon into a coherent hypothesis of quantitative "law"; if this is lacking, the modelling hypothesis formulation process may be "stuck" on formal analogies with other situations only superficially similar or not translate into a mathematical model coherent with the phenomenon. Moreover, previous experience of mathematical modelling may suggest useful ways of thinking and intellectual attitudes in the formulation and subsequently in the verification of the modelling hypothesis. The modelling experience of phenomena which may show some analogy with the proposed situation may also be useful for the student to detect possible common aspects and to guide him in the choice of the mathematical tools to be used.

Finally, the modelling process may involve principles of various kinds. By "principle" we mean any organizing criterion of the formulation operations of an hypotesis, existing before the analysis of the specific problem examined, which is applied to the phenomenon based on some of its peculiar features, recognized as relevant and sufficient to establish the relation among the variables, coming from the organizing criterion itself. Instances of "principles" which are often applied (suitably or unsuitably) in the modelling process are the principle of temporal continuity (enabling the extrapolation in the future of the behaviours and trends observed in the past), the identity principle (phenomena depending in the same way on the same set of variables, have the same development),etc.

As regards the verification process, the two presumptive modes, the empirical verification (mainly based on ad hoc experiment) and the argumentative verification (mainly based on a line of reasoning) appear more or less accessible according to the phenomenon and the preparation of the students to perform and evaluate the two types of verification.
Generally, it is possible to observe that the empirical verification (on the educational ground) has the advantage of establishing a privileged as well as direct relationship with the experimental data, and may also produce, in some cases, a clear answer on the validity of the formulated hypotheses. It is however possible to observe how, in many cases (both in the classroom and in the work of scientists) the empirical verification entails the analysis of data not easy to interpret, or impaired by big experimental mistakes, or too partial, etc. It should also be noticed how, from the point of view of the motivation to investigate a phenomenon thoroughly, the empirical verification process may block (especially if its result is univocal) any further process of interpretation, since it satisfies the proposers of a given hypothesis with a valid result, and frustrates the proposers of a given hypothesis disproved by facts, even though a part of the reasoning proposed in support of the different hypotheses (including those disproved), may be valid for a deeper interpretation of the phenomenon.
On the other hand, the argumentative verification process is generally more oriented towards the interpretation of the phenomenon, but it may, in some cases, not lead to a sufficiently sure and univocal conclusion, and therefore the empirical verification may still be necessary. The argumentative verification process is generally not easy to manage in the classroom if the aim is to get most of the students involved, and to obtain a sufficiently deep and convincing interpretation of the phenomenon.
Still on the verification, previous studies (see Boero & al.,1993a) suggest that the involvement in the classroom in the verification process through the discussion directed by the teacher may enable the class to produce an interpretation of the phenomenon which integrates and positively surpasses the contributions of the individual students, and may help them to interiorize intellectual values and attitudes practised in the classroom under the direction of the teacher.

This report aims at verifying the suitability of the previous analysis to provide guidelines for both planning a teaching experiment and investigating the behaviours of the students from grade VIII facing a mathematical modelling problem, also by comparing two different ways of managing the process of verification of the hypotheses produced.

2. Problem choice, educational context

The modelling process proposed concerns the mathematical model of the elongation of a spring to which variable weights are applied, knowing that the spring is initially twice as long (and, subsequently, triple ...) as a spring with the same physical features (material, section, diameter of the coils, etc.) the elongation law of which is known (and expressed under the form $y=Kx+L$, where x is the weight, K is the coefficient of elongation, L is the unstretched length of the spring). A student should therefore detect the proportionality between K and the unstretched length of the spring. Indeed, a preliminary experiment performed three years earlier (see Garuti, 1992) had shown that the "double length problem"enables the formulation of different hypotheses,so as to allow a productive discussion as well as motivate the verification process; it offers the students various possibilities of approach in the formulation of hypotheses; it also allows a smooth experimental verification as well as an argumentative verification to be performed in the classroom (cf. § 1).

The teaching experiment was performed in two classes of grade VIII (of 15 and 21 students respectively) under the guide of the same teacher who had taught mathematics and science in grade VI and grade VII. The preparation of the students was deemed adequate to face it as regards the following points:

- modelling experience: the work on approaching the mathematical model of direct proportionality was particularly thorough and extensive in grade VI and grade VII (see Garuti & Boero, 1992 on the first portion of this work). Almost all the students had mastered the mathematical model of direct proportionality to solve various proportionality problems in different domains. As to the linear model, modelling activities on springs had already been performed in grade VIII, considering springs with different features, but an equal unstretched length, coming to the equation $y=Kx+L$;
- necessary mathematical prerequisites: formulae of the type $y=ax+b$ and relating graphics in the Cartesian plane, with an awareness of the geometrical meaning of a and b, and of their physical meaning for a given spring; mathematical relation of direct proportionality with different representations (formula $y=kx$; graphic in the Cartesian plane; geometric representation through the theorem of Thales); and
- the work method, particularly as regards the habit of individually formulating motivated hypotheses, and subsequently classifying and discussing them in the classroom under the theacher's guide.

In this "ideal" situation the students could be considered free to use all the resources previously described and both verification modes could be considered feasible.

As far as the <u>monitoring of the students</u> is concerned, a habit of individually expressing their reasoning in writing has produced extensive documentation material. This has been integrated with the recordings of the most important dicussions and with the teacher's notes on the class work.

3. The teaching experiment

(i) The following problem was posed in both classrooms (to be solved individually):

"Let us imagine we have two springs made of the same material, but one double the length of the other (15 cm and 30 cm). If we attach the same number of nuts to both springs, what will the elongation of the two springs be? Detect the possible hypotheses and choose the one you deem more convincing".

Subsequently, both classes were asked to classify and compare in a discussion, the hypotheses they had produced.

From now on, the course of the experiment will take different directions in the two classrooms.

The exploratory study made three years before on the same problem had supplied us with useful indications about experimental verification: performed immediately after the formulation of the

hypotheses, it proved useful only in the rough determination of which hypothesis was correct, but caused the teacher trouble in directing the transition to the interpretation of the phenomenon. This is why we chose to deepen the discussion of the hypotheses in the classroom as much as possible by achieving an argumentative verification before passing on to the experimental stage.
This was not possible in the first class (of 15 students) since the discussion had led to a deadlock: the hypotheses formulated in the beginning had been repeated with no further investigation. Besides, the brighter students were among the supporters of hypothesis A thus preventing a real confrontation. In this case, only the experimental verification appeared to be suitable to solve the deadlock produced.

ii) In the other class (of 21 students) the discussion had engendered a lively confrontation between the supporters of the different hypotheses as well as an effort of interpretation of the nature of the phenomenon; but no substantial progress had been realized. In order to prevent a standstill situation in the discussion, an individual work stage was then accomplished with the following request: *"Try and produce a drawing relating to the problem posed so as to support your hypothesis with various elements"*. This request produced a new stage in the discussion during which useful elements for the argumentative verification were brought up. Only after this discussion was the experiment performed.

Subsequently, in both classes, each student was requested:

iii) to *"give a physical explanation of the result of the experiment"* ; and finally:

(iv) to *"foresee the characteristics of the elongationof a spring made of the same material as the previous ones, but measuring 45 cm in length, and motivate the answer"*. With this task a student may confine himself to solving a specific problem (even at a qualitative level only) without involving the model; it is thus possible to ascertain how many students prefer thinking in terms of "model".

4. Analysis of the students' behaviours (see Annexe, TABLE I and II)

(i) The students consider all three possible modelling hypotheses:
- hypothesis A: both springs have the same elongation;
- hypothesis B: the 30 cm spring stretches more than the 15 cm one;
- hypothesis C: the 30 cm spring stretches less than the 15 cm one.

The students tend to choose the first two hypotheses: particularly 10 students from the first class chose hypothesis A and 5 chose hypothesis B, while in the second class 12 students chose hypothesis A and 9 chose hypothesis B. An analysis of the students' papers supplies interesting elements of investigation as regards the resources used to make this choice. In this case the behaviours are substantially similar in the two classes and will therefore be analysed together.
Hypothesis A (same elongation) is chosen with the following motivation: « *since the two springs are made of the same material and their coils have the same diameter, also their elongation will be the same* ».[18] Those students proceeding in this way resort to the previous modelling experience, where the elongation of the spring depended (given the same initial length) on the material of the spring,

and thus select some meaningful variables. Since in the new problem these variables are the same in both springs, it will be evinced that the extension is also the same. These students therefore use an identity principle to relate the previous experience to the forecast hypothesis to be formulated.
Hypothesis B (longer elongation) is mainly chosen based on a physical qualitative conception of our phenomenon: the student tries to imagine what will happen, by keeping close to the physical aspects of the phenomenon: « *The 30 cm spring will stretch more than the 15 cm one, because it is longer and there is therefore more material to be stretched* » [2] These students apparently "see" the springs as though these were formed by many coils, each one of them stretching; they show a potential "local" vision of the phenomenon.
These two ways of approaching the formulation of hypotheses look different and a conflicting dialectic between them is produced in the students who take both into account, as is clearly shown in this case: « *I think that hypothesis B is correct even though theoretically hypothesis A should be the correct one simply because the two springs are of the same material. It is however impossible for the smaller spring to stretch as much as the 30 cm one!* » [23].

In the discussion and confrontation stage between the two hypotheses, further elements emerge: particularly some supporters of hypothesis B resort to experience made outside the school «*when someone uses a catapult with a very short elastic band, shooting is harder work than if the elastic band were longer. In this case the catapult is more supple.*» [6]. This argument is obviously convincing only for students with a direct experience in this field. As a matter of fact, only two students ([12] and [14]) pass to the hypothesis group B, after analysing it.
In this first stage of the discussion we can clearly notice how the positions of the two groups remain very far away from one another: the argument based on principle is very convincing and clear, while the physical argument is often confused.

(ii) In the second class, the students' level was more or less the same in both groups supporting the two hypotheses, and efforts were made to expand on their arguments to convince the others. The request to produce drawings to illustrate their hypothesis allowed them to deepen their motivations and to reflect on the others', thus paving the way for a substantial confrontation between the two groups.
New elements emerge among the arguments supporting hypothesis B:
- a number of students refer to the activities performed in the previous years by making the proportionality relationship between the coefficient of elongationand the initial length explicit: «*In my opinion they stretch proportionally, that is K is in function of the initial length* » [21];
- another student produces an image of springs "in series": « *It is my opinion that in order to prove that the 30 cm spring stretches more, it is possible to divide it in half to obtain two 15 cm springs, that is of the same size as the other spring. We know for certain that the 15 cm spring stretches as much as 1.4 cm for each nut , therefore even the springs cut in halves should stretch up to 1.4 cm. By adding the two increases of length, I obtain 2.8, that is the increase of length of the 30 cm spring* » [32] This appears to be real proof, and much more convincing than the initial physical interpretations.
However, the most interesting aspects are to be found in the groups supporting hypothesis A.

The drawings produced show that very often those who proceed based on the "identity principle" have a static idea of the elongation phenomenon of the two springs, and almost cannot imagine that the spring deforms: a student [16] draws the two springs (one double the other), "attaches" the nuts to them and represents the stretched length by adding the same number of coils (Fig. 1); another speaks of « *coils which must not stretch* » (!?) [27]. In other cases, in the graphic representation elements of dynamism are shown through a picture of the initial and the final aspects of the phenomenon . In these cases we have a conflict between the identity principle on which the hypothesis is based and its iconographic representation. A student, for instance, does not deform his springs, but in order to represent the final aspect of the longer spring, he draws twice as many coils and so comments: « *I have made a mistake because I have increased the length of the spring twice, thus agreeing with the others*» [18]. Another student draws the two springs proportionally deformed according to the weight applied to them; his representation is positively in contradiction with the principle: «*at this moment I could not tell which is correct because in my drawing one extension is twice as long as the other. This might be understood by argument, but I think I would need a different mode to prove it* ».[30](Fig.2)

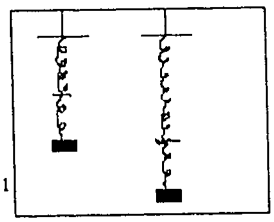


Fig. 1

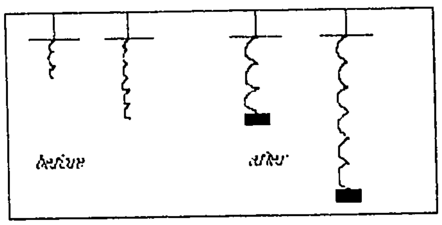


Fig 2

(iii) In the two classes, we can observe a different behaviour as regards the physical interpretation of the phenomenon: in the first class we only find a "local" interpretation which emerged in the first part of the discussion; in the other class the interpretation of the phenomenon is richer since there we also have an interpretation of the "springs in series" kind, which emerged in the second discussion and appears to be rather convincing as regards the students reasoning according to the principle.

(iv) In the two classes, behaviours also differ as far as the transition to the mathematical modelling of the phenomenon is concerned: in the first class, only 5 students (out of 14) foresee, for the 45 cm spring, a coefficient K three times higher than that of the 15 cm spring; the other students confine themselves to foreseeing a longer elongation with no references to the proportionality relationship.
In the other class, 14 students (out of 21) foresee, for the 45 cm spring, a coefficient K three times higher than that of the 15 cm spring, and especially their reflections on the model are more exhaustive. Seven students make explicit the general proportionality relationship: « ... *if the second spring is twice as long as the first, K will double too. If it is three times as long K will treble and so on.* [...] *K varies in relation to the length of the spring.*» [18].Some of them express a more complex and appropriate regularity than the initial one: « *I have noticed that with 20 nuts the two springs stretch as much as their initial length. Generalizing, I can say there is a constant relationship between the initial length of the spring and its increase of length (with the same nuts). If the spring is 7 cm long, we will have a*

7 cm elongation, and if it is 45 cm long we will have a 45 cm elongation. In this case it is quite easy since the relationship is 1:1, but this method is always valid » [30].

5. Discussion

The teaching experiment confirms the variety of the resources students may refer to (in various ways) in producing the modelling hypotheses; it also confirms the productivity of the argumentative verification which, in a case like the one examined, may lead a student to a direct interpretation of the phenomenon. We have also seen how some argumentative verifications of the "springs in series" kind have adopted the role of real mental experiments as well as of "proof" (see Koyré , 1966 about the mental experiments described by Galileo in his works).

The teaching experiment emphasizes the role of drawings as particularly useful tools in hypothesis discussion and in outdoing inadequate hypotheses. Drawings, as global representations of the physical phenomenon, play a role of their own in bringing out the contradiction between "identity principle" and phenomenon. Considering TABLE II, we can see that 5 drawings (out of a total of 8) have been produced by students who had chosen hypothesis A, following the "identity principle". The drawings produced enable some of these students to make their point of view explicit and render it more accessible to other people's critical judgement, and others to develop a conflict with the physical intuition, and others still to modify the hypothesis they initially formulated. We see also that 5 out of 8 students who had produced a drawing, succeed in making explicit the general proportionality relationship between K and the unstretched length of the spring (7 students were successful in it).

However, a problem is still unsolved: how to overcome a deadlock in the transition to the argumentative verification such as the one occurred in the first class. On the one hand, it seems necessary to thoroughly investigate the emotional dynamics within the discussion (role of the bright students), on the other hand, the mediation work performed by the teacher. What would have happened in the first class, if instead of passing immediately on to the "concrete" experiment, the teacher had let the students into the mental experiment of the "springs in series" performed in the other class, and asked them to take a position on this?

References

Blum, W. and Niss, M.: 1991, 'Applied Mathematical Problem Solving, Modelling, Applications, and Links to Other Subjects', *Educational Studies in Mathematics* 22, 37-68

Boero P.: 1989, 'Mathematical literacy for all: ..' , *Proceedings of PME XIII*, Paris

Boero, P., Ferrari, P.L., Ferrero, E. and Shapiro, L.: 1993a, 'Some Aspects Regarding the Social Construction of Hypotheses .....', *Journal of Mathematical Behaviour* (to appear)

Boero, P., Chiappini, G.P. and Garuti, Rossella: 1993b, *'Towards proportional modelling: an exploratory study about context sensitivity'*, (preliminary report), DI.MA., Univ. Genoa.

Garuti, R. and Boero,P.: 1992, 'A sequence of proportionality problems: an exploratory study', *Proceedings of P.M.E.-XVI* , Durhan

Garuti, R.: 1992, 'Funzioni come trasformazioni associate a formule, grafici e modelli di fenomeni', *L'insegnamento della matematica e delle scienze integrate*

Koiré, A. : 1966, *Etudes galiléennes* , Hermann, Paris

Markovits Z., Eylon B.S. and Bruckheimer M.: 1986, 'Functions Today and Yesterday', *For the Learning of Mathematics*

Meira, L.: 1991, 'On the Transparency of Material Displays: the Case of Learning Linear Functions', *Proceedings AERA Symposium*

Vinner S.& Dreyfus T.: 1989, 'Images and Definitions for the Concept of Function', *Journal for Research in Mathematics Education*

641

Annexe: these tables summarize the students' behaviours in the two classrooms:

TABLE I

Cl.1	HYPOT.	MOTIVATIONS	PHYS. INT.	MATH. REL.
1	A		local	
2	B	phys. conception	local	proportional
3	A		local	
4	A	principle	local	
5	B			proportional
6	B	phys.conception	local	
7	B		local	proportional
8	A	principle		proportional
9	A		local	
10	B			
11	A		*	*
12	A(--->B)	principle(->ph.c.)		
13	A	principle	local	proportional
14	A(--->B)	principle(->ph.c.)	local	
15	A	principle	local	

* =absent

TABLE II

Cl.2	HYPOT.	MOTIVATIONS	REPR.	PHYS. INT.	MATH.REL.	GENER.
16	A	principle	drawing		proportional	#
17	A	principle	drawing	series	proportional	#
18	A	principle	drawing	series	proportional	#
19	A	principle	drawing		proportional	
20	B	phys. conception				
21	B	phys. conception	drawing		proportional	#
22	A		drawing	local	proportional	
23	B	phys. conception		local	proportional	
24	A	principle		local		
25	B				proportional	#
26	B					
27	A			local		
28	A	principle		local	proportional	
29	A	principle			proportional	
30	A	principle	drawing	series	proportional	#
31	B	phis. conception		local	proportional	
32	B	phis. conception		series	proportional	
33	A	principle		series		
34	B	phis. conception		series		
35	B	phis. conception	drawing		proportional	#
36	A	principle				

Conceptions About Mathematics Teaching of Preservice Elementary and High-School Teachers

Linda Gattuso, Université du Québec à Montréal
Nicole Mailloux, Université du Québec à Hull

The goal of this research was to find out the declared conceptions of two groups of preservice teachers, one in Elementary school teaching and one in High-School teaching and compare the outcomes. The results showed that the preservice Elementary school teachers are mainly interested in motivating their pupils while preservice High-School teachers give more importance to the learning of the pupils.

Statement of the problem

Recent reviews of educational research (Kagan, 1992; Reynolds, 1992) come to the conclusion that the student teacher's beliefs and conceptions about mathematics, their learning and their teaching are extremely stable and inflexible. The different studies show that "despite course work and field experiences, the candidate' beliefs about teaching and themselves as teachers remained unchanged" (Kagan, 1992: 140) and are transferred in their teaching practice. Therefore, the novice teachers are quickly disillusioned and without sufficient procedural knowledge, they rapidly reproduce a rigid teaching that leaves little place to the student and isn't favorable to the development of conceptual knowledge.

As teacher trainers, it is essential for us to question the impact of the training on the preservice teachers' conceptions and eventually on their instructional practice so we can eventually consider it in their training.

On the one hand, in the mathematics teaching field, it is acknowledge that the teacher's conceptions of mathematics and mathematics teaching play a significant role in shaping their instructional practice (Gonzalez Thompson, 1982, 1984; Gattuso, 1992). On the other hand, the "pedagogical content knowledge"(Shulman, 1987), (i.e., questions related to the teaching of the discipline: different teaching strategies, learning difficulties connected to different concepts, links between concepts, various interventions, means of diagnosis and evaluation, etc.) as an important influence on teaching practice (Grossman, Wilson & Shulman, 1989; Dörfler, McLone, 1986; Ernest, 1989). From there on, to explore a possible link between "pedagogical content knowledge", as communicated in the preservice training courses, and the conceptions about the teaching of mathematics, we decided to compare the conceptions about teaching of preservice teachers having two different training. This exploratory study is the first step of a longer study that wants to investigate the preservice teachers' conceptions before, during, and after their three years of training, looking parallely into their teaching practice during the years of trainings and onto their first teaching experience.

The principal aims of this preliminary study are:

—to find out the declared conceptions of the preservice teachers

— to compare the conceptions of two groups relatively to their training

At the same time, a secondary aim is

—to experiment with a tool allowing the investigation of the conceptions of preservice teachers.

The Study

We wanted to see if students having intensive instruction in "pedagogical content knowledge" have different conceptions from those of students having a more generalist training. To do so, we looked at the declared conceptions about the teaching of mathematics of a group of preservice Elementary school teachers (EST) and of a group of preservice High-School teachers (HST). Traditionally, to enter a preservice Elementary school teaching program, there is no prerequisite mathematics courses. Most students entering the program followed only the basic High-School courses in mathematics and no mathematics course during the two years of junior college. In University*, their training is mostly pedagogical and general, although they follow two or three courses in the teaching of mathematics. Students entering the high school teacher training program come from a scientific path, so they have successfully completed at least three mathematics courses in junior college. In University, their curriculum contains mathematics courses and teaching of mathematics courses, their teaching training is mostly focused on mathematics, their learning and their teaching.

The two groups selected for this preliminary study were one of each kind: one group of Elementary preservice teachers (EST) and one group of secondary preservice teachers (HST). Both groups were in their second year of training and were experimenting their first teaching practice in school. The 51 students of the first group (EST) had followed two courses in the teaching of mathematics amongst their

* In Quebec, teacher training is supplied by Universities

pedagogical studies. To be allowed to go for their first teaching experiment in school, the 38 students of the second group (HST) had to complete, in the previous semesters, at least three courses in mathematics teaching, the others being mathematics, computer science and few education courses. Although we did not consider the sex variable in our study, let us mention that there were only five male students amongst the Elementary preservice teachers and 17, in the High-School preservice teacher students.

One of the first problem encountered was to find a way to investigate the declared conceptions. We left out interviews because we wanted a tool that could be easily used in a larger survey. We also wanted a tool with a more open configuration than (that of) a questionnaire. For these reasons, we have chosen an inventory of conceptions about the teaching of mathematics that was elaborated in a previous study (Gattuso, 1992).

The 50 conceptions proposed in the inventory were formulated as sentences starting with: "A teacher of mathematics should... ". Many of the conceptions were inspired by items presented in two questionnaires of previous studies (Ernest, unpublished; Thompson, 1982). They were meant to reflect different philosophies about the teaching of mathematics (Ernest, 1991). This tool prepared for experienced teachers was tested for the first time with preservice students.

The students were asked to choose 5 to 10 conceptions in the inventory that contained 50 statements. We wanted to restrict the choices so as to force the subject to choose conceptions that were the most important to him/her. The compilation of the results are presented in the next section.

Results

The data of each group were compiled separately. We will first present the more important results for each group and we will compare them afterwards. Discussion follows.

In table I, we can see the conceptions that cover 52% of all the choices of the Elementary student-teachers group (51). Table II shows the conceptions that represent 53% of the choices of the High-School student-teachers group (38). In table III, we can find the combination of the previous results for both groups.

TABLE I: CONCEPTIONS MAINLY CHOSEN BY EST

Statements describing conceptions	Nb/51	%
Present mathematics from real situations	30	58.8
Place the pupils in situations where they can explore and discover	24	47.1
Encourage pupils to find more than one way to solve a problem	23	45.1
Present mathematics in interesting ways so the pupils will be motivated	22	43.1
Place the pupils in problem solving situations	21	41.2
Encourage the development of logical reasoning	19	37.3
Encourage the development of creativity	17	33.3
Integrate mathematics to other disciplines	15	29.4
Plan his teaching well	12	23.5

TABLE II: CONCEPTIONS MAINLY CHOSEN BY HST

Statements describing conceptions	Nb/38	%
Question the pupils to check their understanding	21	55.3
Present mathematics in interesting ways so the pupils will be motivated	20	52.6
Plan his teaching well	17	44.7
Encourage the development of logical reasoning	15	39.5
Diagnose the misunderstanding of the pupils	15	39.5
Place the pupils in situations where they can explore and discover	14	36.8
Encourage the pupils to find a meaning behind the procedure	14	36.8
Present mathematics from real situations	12	31.6

A priori, we did not expect such a concentration of the choices. In the first group (EST) 52% of the choices involve only 9 conceptions. If we examine the statements, we can see that for the Elementary school teacher-students, it is important

to present mathematics - from real situations
- through exploring and discovering situations
- in interesting and motivating ways ;

to encourage - research of diverse solutions

- problem solving
- logical reasoning
- creativity

to plan their teaching
and to integrate mathematics to other disciplines.

We can see that the focus is mostly put on motivating strategies of teaching which could be a result of their training. In fact in many courses (micro-teaching and teaching of mathematics) importance is given to developing strategies that motivate pupils. Moreover the EST were in there first teaching practice and they saw the importance given to problem-solving and the use of real situations.

For the Secondary School teacher-students, it is important
to question the pupils in order to verify their understanding,
to diagnose the lack of understanding,
to plan teaching,
to encourage the development of logical reasoning
to present mathematics - in interesting and motivating ways
- through exploring and discovering situations
- from real situations,

to encourage pupils to find a meaning behind the procedure.

The results of a chi-square test comparing the choice of both groups (Table III) revealed a significant difference ($\alpha = 1.0 \times 10^{-4}$) in the distribution of the choices. The High-School student-teachers (HST) are mainly interested in how the pupils learn.. They question, diagnose and emphasize the meanings. In fact, this is the characteristic that distinguishes the most the HST from the EST. The statements concerning the questioning, the diagnosis and the importance of meaning are chosen twice as often by the High-School student-teachers. For the Elementary student-teachers (EST), we can see that the students believe in an open form of teaching, they express their willingness to insure development of creativity and divergent thinking almost twice as often as the HST. On the other hand they are preoccupied with certain Quebec Elementary school practices such as the integration of mathematics to other subjects and problem-solving.

We can also see that there is some overlapping of the choices of both groups. One statement: "Encourage the development of logical reasoning" is chosen by almost the same proportion of students of EST and HST groups (37,3% for the first group and 39,5% for the second group). For these students, logical reasoning seams to be associated with mathematics. The use of everyday situations and exploring situations is selected considerably by both groups but more by the EST group. On the other hand, planning the lessons and giving interesting presentations are preferred by the HST group although these statements have been picked out by a large percentage of each group. Although it is not possible at this point to link every difference to the students training, we can see that each group has some specific characteristic. The HST conceptions seem oriented towards the results of their teaching and the EST more towards the creation of motivating situations. Each statement mentioned here separately implies at least 25% of the total choices. Other statements reveal interesting information. For example, around 20% of the choices of each group concerned the statement "maintain discipline in the classroom".

TABLE III: RESULTS OF THE TWO GROUPS

Statements describing conceptions	Nb EST	% EST	Nb HST	% HST
Diagnose the misunderstanding of the pupils	9	17.6	15	39.5
Encourage pupils to find more than one way to solve a problem	23	45.1	10	26 3
Encourage the development of creativity	17	33.3	7	18.4
Encourage the development of logical reasoning	19	37.3	15	39.5
Encourage the pupils to find a meaning behind the procedure	9	17.6	14	36.8
Integrate mathematics to other disciplines	15	29.4	3	7.9
Plan his teaching well	12	23.5	17	44.7
Present mathematics from real situations	30	58.8	12	31.6
Present mathematics in interesting ways so the pupils will be motivated	22	43.1	20	52.6
Place the pupils in problem solving situations	21	41.2	8	21.1
Place the pupils in situations where they can explore and discover	24	47.1	14	36.8
Question the pupils to check their understanding	8	15.7	21	55.3

The following four statements were completely rejected by both groups:

TABLE IV: REJECTED CONCEPTIONS

Statements describing conceptions	Nb
Be sure that each pupil rapidly knows if his answer is right	0
Have the pupils memorize the laws	0
Insist on the fact that there is only one way of solving a problem	0
Strictly follow a textbook.	0

We also found that the four statements completely rejected by both groups gave an interesting information. They all concern a very restrictive way of seeing mathematics, a way Ernest (1991) would call "authoritarian": rules, one answer, one text book and only the right answer. However, these conceptions are still widely part of today's reality in mathematics teaching in our Elementary and Secondary Schools and it will be interesting to see if the novice teachers will still reject these conceptions when they are confronted with real school situations.

Discussion

One of the aims of the study was to find out the declared conceptions of the preservice teachers. Not wanting to impose any conceptions, we had chosen a long list that included opposing views of teaching mathematics. It was surprising to see that the choices of each group concentrated on a relatively small sample of these statements. However, they are not exactly the same and the ones chosen by both groups are almost never chosen in the same proportion. Differences between the two groups cannot at this point be tied with any certainty to the difference in their training but some characteristics are more specific to a teaching level than to the other.

The tool chosen is easily implemented and can lead to an interesting analysis but a deeper examination of the results in process (by hierarchical cluster analysis) may lead to some improvement. This analysis will enable us to discover the preservice teachers' conceptions but overall it is essential to see if these conceptions are transfered in teaching practice. In the future, a longitudinal study will examine the conceptions and the teaching practice of the preservice teacher trough their three years of training

REFERENCES

DÖRFLER, W., MCLONE, R.R. (1986). "Mathematics as a school subject." In B. Christiansen, A.G. Howson, M. Otte (Eds.), Perspectives on Mathematics Education. Dordrecht: D. Reidel Publishing company. 49-97.

ERNEST, P. (1989). "The impact of beliefs on the teaching of mathematics". In Ernest, P. (Ed.). (1989). *Mathematics Teaching The State of the Art.* London: Falmer. 249-254.

ERNEST, P. (1991). *The Philosophy of Mathematics Education.* London: Falmer Press.

GATTUSO, L. (1992) *Les conceptions personnelles au sujet de l'enseignement des mathématiques et leur reflet dans la pratique, un essai d'autoanalyse.* Doctoral dissertation. Université de Montréal. Montréal.

GROSSMAN, P. L., WILSON, S. M., & SHULMAN, L. S. (1989) "Teachers of substance: Subject matter knowledge for teaching". In M. C. Reynolds (ed.), *Knowledge base for the beginning teacher.* New York: Pergamon Press. 23-36.

KAGAN, D. (1992). "Professional growth among preservice and beginning teachers." *Review of Educational Research* . (62) 2: 129-169.

LEINHARDT, G. (1990). "Capturing craft knowledge in teaching" *Educational researcher* 19: 18-25.

REYNOLDS, A. (1992). "What is competent beginning teaching? A review of the literature." *Review of Educational Research* . (62) 2: 129-169.

SHULMAN, LEE S. (1987). "Knowledge and Teaching: Foundation of the New Reform." *Harvard Educational Review* 57, 1-22

THOMPSON, A. (1982). *Teachers' conceptions of mathematics and mathematics teaching: three case studies.* Ph.D. Dissertation Athens, Georgia: UMI 8228729 02950.

THOMPSON, A. (1984). "The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice". *Educational Studies in Mathematics, 15,* 105-127.

650

SE

Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

PME XVIII

29 July - 3 August 1994

Vol. III

University of Lisbon
Lisbon • Portugal

BEST COPY AVAILABLE

Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

PME XVIII

29 July — 3 August 1994

Volume III

University of Lisbon
Lisbon — Portugal

Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

Volume III

Published by the Program Committee of the 18th PME Conference, Lisbon, Portugal

Editors: João Pedro da Ponte and João Filipe Matos

Cover: Eduardo Veloso

Depósito legal nº 78568 94

ISBN 972-8161-02-6

Printed by GRAFIS, Coop. de Artes Gráficas, CRL

Cover: The background is a reproduction of the *Cantino* chart. This Portuguese map was smuggled from Portugal in 1502, by the spy of the Duke of Ferrara, Alberto Cantino. It is the first world map of the modern times, and it takes into account the findings of the voyage of Vasco da Gama to India.
Cover overleaf: The astrolabe and the quadrant, the most used nautical instruments of the beginning of astronomical navigation, in the 15th century, are represented by two drawings included in the world map of Diogo Pereira, 1529.

Departamento de Educação
Faculdade de Ciências da Universidade de Lisboa
Edifício C1 — Campo Grande
1700 Lisbon — Portugal

Contents of Volume III

CHILDREN'S REPRESENTATION OF THE COUNTING SEQUENCE 1-100: STUDY AND THEORETICAL INTERPRETATION

Noel Thomas, Charles Sturt University, Bathurst, Australia,
Joanne Mulligan, Macquarie University, Sydney, Australia, and
Gerald A. Goldin, Rutgers University, New Brunswick, NJ, USA

In a cross-sectional exploratory study of 166 children in Grades K-6 and an additional 79 high ability 3rd-6th graders, we seek to infer children's internal imagistic representations from their explanations and drawings of the numbers 1-100. Our observations are interpreted with respect to developing theoretical models for mathematical learning and problem solving based on characterisitics of representations.

Introduction and Theoretical Bases

In this paper we report part of a broad study designed to explore the relationship between children's counting, grouping and place value knowledge, and their understanding of the structural development of the number system. Our aim in the overall study is to describe in the greatest possible detail children's internal representational capabilities evidenced in solving a wide range of tasks related to counting and numeration. In this report, we consider one aspect of children's developing conceptual knowledge: their spontaneous representations of the numbers from 1-100.

A body of related research (e.g. Bednarz and Janvier, 1988; Denvir and Brown, 1986 a,b; Fuson, 1990; Hiebert and Wearne, 1992; Kamii, 1989; Rubin and Russell, 1992; Steffe, Cobb and Richards, 1988; Steffe, 1991; Wright, 1991) focuses on children's counting strategies and conceptual development of numeration, and identifies counting, grouping, estimating and notating skills as essential elements in developing numerical structure. Rubin and Russell consider additive and multiplicative structure, the generation and analysis of mathematical patterns and mathematical definitions. Hiebert and Wearne describe children's understanding of numeration as "building connections between key ideas of place value, such as quantifying sets of objects by grouping by ten, treating the groups as units... and using the structure of the written notation to capture the information about grouping" (p.99). Other researchers (e.g. Davis, Maher and Noddings, 1990; Goldin and Herscovics, 1991; Goldin, 1993; Thomas, 1992) consider children's physical, pictorial, or notational representations of number in analyzing the development of their conceptual understanding. In a pilot study of 40 children in Grades K-4, Thomas reported the wide variety of their "mental pictures" of the number sequence 1-100.

The analysis described here is based on a model for children's problem-solving competency structures proposed by Goldin (1987a,b; 1988; 1992). We draw also on studies of the role of imagery in representation and in the construction of relational understanding in mathematics (Brown and Presmeg, 1993; Brown and Wheatley, 1989; Mason, 1992; Presmeg, 1986). Goldin's model distinguishes cognitive representational systems internal to problem solvers (a theoretical construct to describe the child's inner cognitive processing) from (external) task variables and task structures (cf.

Goldin and McClintock, 1984). We consider three of the five types of internal representational systems discussed by Goldin: (a) verbal/syntactic systems (using mathematical vocabulary, developing precision of language, self-reflective descriptions); (b) imagistic systems (non-verbal, non-notational representations, e.g. visual or kinaesthetic); and (c) formal notational systems (using notation, relating notation to conceptual understanding, creating new notations). These systems develop over time through three stages of construction: (i) inventive/semiotic, in which characters in a new system are first given meaning in relation to previously-constructed representations; (ii) structural development, where the new system is "driven" in its development by a previously existing system; and (iii) autonomous, where the new system of representation can function independently of its precursor.

In the structural development of the number system, the system of representing units (1's) must serve to drive the representation of assemblages partitioned into groupings of ten. The "ten", while still remaining ten ones, becomes an iterable "unit of ten". Similarly a system of "hundreds" is later constructed on the system on tens, and so forth (recursively). Children's conceptual structures for number words are now "multiunit conceptual structures in which the meanings or referents of the number words are collections of entities ....or a collection of collections of objects" (Fuson, 1990, p.273). This process is not just a verbal or notational one; the role of imagery in it is essential.

Presmeg (1986) identifies five types of visual imagery used by students: (i) concrete, pictorial imagery (pictures in the mind); (ii) pattern imagery depicting pure relationships; (iii) memory images of formulae; (iv) kinaesthetic imagery involving muscular activity, and (v) dynamic (moving) imagery. Mason (1992) distinguishes between images that are eidetic (fully formed from something presented), and those that are constructed (i.e. built up from other images). He suggests that for students to access images they must actively process them, "looking through" rather than "looking at" the "mental screen", regardless of the mode of external representation.

With these theoretical bases, we next describe the study and some of its outcomes.

Method

Two samples of children were selected for comparative purposes, and administered task-based problem-solving interviews. A cross-sectional sample consisted of 166 children in Grades K-6, randomly chosen from 8 State schools in the Western Region of New South Wales (NSW). This sample represented a wide range of mathematical abilities. A high ability sample consisted of 79 children from Grades 3-6, assessed by teachers for participation in a Program for Gifted and Talented students from 75 country and city schools in NSW. The children in the cross-sectional sample were interviewed individually over two sessions to ascertain their understanding of numeration, using 25 tasks. The children in the high ability sample were interviewed individually once on selected numeraton and visualization tasks. The numeration tasks for both samples were categorised into five groups: counting; grouping/partitioning; place value; structure of numeration; and visualization. In one of the visualization tasks children were asked to close their eyes and to imagine the numbers from one to one hundred. Then they were asked to draw the pictures that they saw in their minds. They were also asked explain the image and their drawing. The visualization task was asked first, prior to other numeration tasks, so that responses could not be influenced by representations used by the researcher in other tasks.

In this way 245 interview transcripts, together with the external pictorial and notational representations produced by the children, were obtained and analyzed. The external representations were considered with respect to three dimensions, illustrated in greater detail below: (a) the type of representation (pictorial, ikonic and notational), from which we sought to infer characteristics of each child's internal imagistic representation; (b) the level of structural development of the number system evidenced in the representation; and (c) evidence of the static or dynamic nature of the image.

Discussion and Analysis of Children's External Representations

Here we discuss features of selected examples of the children's representations, making reference to the theoretical perspectives described above. Evidence of internal imagistic repesentations, structural development of the number system, and dynamic imagery were found across both samples. Examples discussed here are drawn from the cross-sectional sample, except for two examples of dynamic images from the high ability group (see below).

(i) Types of external representations: Pictorial recordings were defined as pictures drawn, or oral descriptions of objects given by a child, e.g. a drawing of a truck, a dinosaur labelled with the numeral 100, a description (with some drawings) of one hundred people each labelled with the numerals 1 through 100, and a description of one hundred objects lying on the floor. Ikonic recordings were defined to include drawings of tally marks, squares, circles or dots that represented the counting sequence. Notational recordings were distinguished by the predominant use of numerals drawn in various formations such as a number line, array, 100 cm ruler or vertical column. From these characteristics of externally produced representations, we infer the child's construction of internal configurations in meaningful semantic relationships.

Figures 1-4 show the drawings of Anthony (Grade1), Andrew (Grade 1), Candice (Grade 3), and Timothy (Grade 4) respectively.

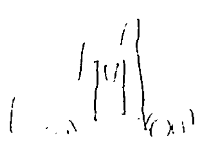


Figure 1 Anthony (Grade 1)

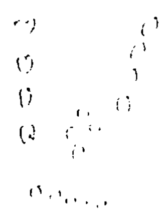


Figure 2 Andrew (Grade 1)

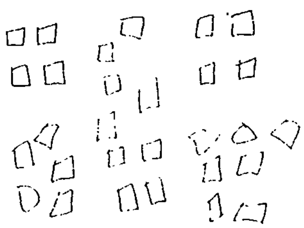


Figure 3 Candice (Grade 3)

Figure 4 Timothy (Grade 4)

The truck drawn by Anthony (Figure 1) reflected the association of an image of his dad's truck with the number 100. Anthony's reason for the image was verbalised as "cause my Dad's truck does a hundred". This suggests we can infer an inventive semiotic internal representation relating the truck-image to speed. This is highly idiosyncratic, but quite meaningful. Andrew saw a picture of 100 shells (Figure 2) and explained as he drew, "some were in rows, some were in diagonals and some across like that". We inferred an ikonic internal representation of number with some evidence of structural development of the number system. Analysis of Andrew's protocol showed some partial development of grouping, with capability of dealing with three or ten as a unit, but without the recursive capability of keeping track of how many units. Candice also drew an ikonic representation using squares, with some evidence of developing structure. Her internal representation was evoked by her prior experience of using square counters, and she attempted to draw these in groups representing the numbers four, five or six. Candice explained that her drawing was "square counters to count with" and further evidence showed that she was unable to treat numbers as iterable units. Candice used a highly imagistic representation to explain how she saw numbers only "as squares" rather than notational symbols. Timothy's representation (Figure 4) similarly displayed little structure but in this case was notational, focusing on just the one numeral 9. This example alone gave insufficient evidence to interpret Timothy's level of structural development. However, in other portions of this protocol Timothy revealed his counting capabilities explaining that "his numbers stopped at 9 and this was the biggest number". We inferred from this idiosyncratic example a relatively undeveloped understanding of numeration.

(ii) The level of structural development of the number system was inferred from structural elements (i.e. grouping, regrouping, partitioning and patterning) found in the recordings of the numbers 1 to 100. Evidence of emerging structural development of number was found across a range of recordings. There were a number of cases where children showed no evidence of structure and these were typified by drawings showing a single object, a random pattern of dots or a single numeral. Emerging structure was typified by numerals organised in a counting sequence, recorded continuously in a horizontal, vertical, curved or spiral formation. Children showing evidence of a more developed multiplicative system recorded a multiple counting sequence, and marks or pictures in a partial or complete ten-by-ten array structure.

Figures 5-8 show the drawings of Warren (Grade 2), Joshua (Grade 2), Cassie (Grade 4) and Kimberley (Grade 2) respectively.

Figure 5 Warren (Grade 2)

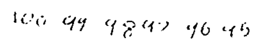


Figure 6 Joshua (Grade 2)

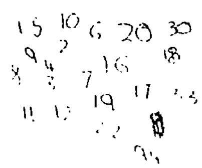


Figure 7 Cassie (Grade 4)

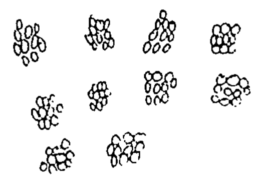


Figure 8 Kimberley (Grade 2)

Warren, and Joshua (Figures 5, 6) produced horizontal, linear structured representations of numbers. Warren's picture of a line of marks was ikonic, whereas Joshua used conventional notation writing the counting sequence of numerals counting-back from 100. Warren's ikonic representation was related to his concentration on counting "by ones" with the marks representing his internal process of counting on by ones. Further evidence from Warren's protocol revealed his ability to count in threes, but his mental image of this remained ikonic rather than seeing numerals. In contrast, Joshua was able to elaborate on his drawing by counting forwards and backwards, grouping in tens and using multiple counting efficiently. We inferred a high level of structural development from these examples. Cassie (Figure7) wrote the numerals in counting sequence in a spiral configuration initially and then became random in sequence and spatial setting. From this we inferred an internal representation with a non-conventional structure of the number sequence. The structure in Kimberley's recording (Figure 8) was more explicit as she saw numbers in groups of ten, but could not identify the general structure explaining her drawing as "just circles".

Figures 9 and 10 show drawings made by Mellissa and Robert, both from Grade 2.

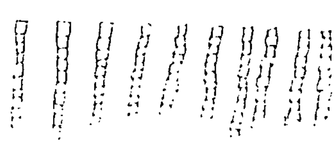


Figure 9 Mellissa (Grade 2)

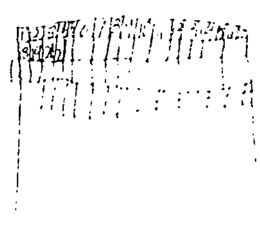


Figure 10 Robert (Grade 2)

Mellissa (Figure 9) drew ten ten-rods to produce another ikonic representation of grouping in tens. This gives evidence of a highly structural imagistic internal representation for the developing numeration system. Robert (Figure 10) drew a square and subdivided rows of separate squares, each square not being aligned to adjacent squares, and then recorded numerals for the numbers in squares, 1 to 17 being in the first row. This partial array displays an emerging notational structure, but Robert showed further evidence of difficulty with using ten as an iterable unit, saying "you just put the numbers in the boxes as far as you can go... and you count in ones".

(iii) The static or dynamic nature of the image was defined according to whether the recordings and the children's explanations of their representations described fixed or moving (or changing) entities. In the cross-sectional sample 3% of the children displayed dynamic images of the number sequence and in the sample of high achieving children 10% had dynamic images. Examples of dynamic images included numerals flashing one at a time, groups of numerals moving around, and numerals rolling down. Figures 11 and 12 show drawings produces by Jane (Grade 1) from the cross-sectional sample, and David (Grade 4) from the high ability sample.

Figure 11 Jane (Grade 1)

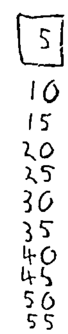


Figure 12 David (Grade 4)

Nik (Grade 4) from the high ability sample explained that "big thick numbers were flashing and flying past, each taking 2 seconds" and indicated that all the numbers would have come if he had closed his eyes for long enough. Jane (Figure 11) recorded the number sequence using conventional notation but explained that she saw the numbers moving in a spiral formation "going on forever". Some evidence of a developing system of grouping by tens was revealed in her segments of number strings in tens (e.g. 71-80, 81-90). David's picture (Figure 12) showed numerals flashing one at a time, multiple counting in fives, up to 100. This dynamic notational model gave evidence of an emerging structure for the system of numeration. The formal notations of Jane, Nik and David were organised imagistically in a non-conventional manner. Analysis of their protocols showed that these images were highly creative and unrelated to these children's conventional experiences in the classroom.

Conclusions and Limitations

We would ultimately like to be able to describe in detail children's internal representations of the numeration system, and how these representations develop. From the external representations produced by the children, we have attempted to infer aspects of their internal representation, and from this in turn to infer some description of the structural development of the system that has taken place. We found in general a wider diversity of representations of the counting sequence 1-100 than might have been expected. This diversity occurred at each grade level, and across both samples. We found evidence that the children's internal representations of numbers are highly imagistic, and that their imagistic configurations embody structural development of the number system to widely varying extents, and often in unconventional ways. We saw instances in which the formal notational symbols are organized imagistically, as in Jane's spiral and David's flashing numerals.

It should be noted that we have described only the children's spontaneous responses to the visualization task as presented to them. Other responses might have occurred if the children had been prompted--for example, to write numbers in rows, to imagine moving numbers, or to group the numbers in tens. Other representations may have been available to the children, with just one of several possible internal image configurations having been selected for recording or elaboration. Thus we are probably gaining but a partial description of each child's internal representational capabilities.

Many questions are raised but unanswered. Why do some children develop powerful systems of numeration, while some develop misconceptions? Why do some have the capability of spontaneously visualizing the counting sequence in a dynamic way? Can static external representations represent ("carry the meaning") of dynamic internal representations? Further research is needed to shed light on how children construct their personal numeration systems, and how they structure them over time. We hypothesize that the further-developed is the structure of a child's internal representational system for the counting numbers (e.g., kinaesthetic, auditory, or visual/spatial representation of the counting sequence that embodies grouping-by-tens), the more coherent and well-organized will be the child's externally-produced representations, and the wider will be its range of numerical understandings.

References

Bednarz, N. and Janvier, B. (1988). A constructivist approach to numeration in primary school: results of a three year intervention with the same group of children. *Educational Studies in Mathematics*, 19, 299-331.

Brown and Presmeg, N. (1993). Types of imagery used by elementary and secondary school students in mathematical reasoning. In Hirabayashi, N. Nohda, K. Shigematsu and F.-L. Lin (Eds.), *Proceedings of the Seventeenth International Conference of PME*, Tsukuba, Japan, Vol. II, 137-144.

Brown and Wheatley, G. (1989). Relationship between spatial ability and mathematical knowledge. In C.A.Maher, G. A. Goldin , and R.B. Davis, *Proceedings of the Eleventh Annual Meeting of PME*: NA, New Brunswick, NJ, Rutgers Uni., Centre for Mathematics, Science and Computer Educ., 143-148.

Davis, R., Maher, C. and Noddings, N. (1990). *Constructivist views on the teaching and learning of mathematics*. Monograph Number 4, *Journal for Research in Mathematics* Education. Reston, VA: National Council of Teachers of Mathematics.

Denvir, B. and Brown, M. (1986a). Understanding of number concepts in low attaining 7-9 year olds: Part I. Development of descriptive framework and diagnostic intrument. *Educational Studies in Mathematics*, 17, 15-36.
Denvir, B. and Brown, M. (1986b). Understanding of number concepts in low attaining 7-9 year olds: Part II. The teaching studies. *Educational Studies in Mathematics*, 17, 143-164.
Fuson, K. (1990). Issues in place-value and multidigit addition and subtraction learning and teaching. *Journal for Research in Mathematics Education*, 21, 4, 273-280.
Goldin, G. A. (1987a). Levels of language in mathematical problem solving. and (1987b). Cognitive representational systems for mathematical problem solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics.* Hillsdale: NJ: Erlbaum59-65 and 125-145.
Goldin, G. A. (1988). The Development of a model for competence in mathematical problem solving. In A. Borbas, (Ed.), *Proceedings of the Twelth International Conference of PME,* Veszprem, Hungry: OOK Printing House, Vol. 2, 358-368.
Goldin, G.A. (1992). On developing a unified model for the psychology of mathematical learning and problem solving. In W.Geeslin and K.Graham (Eds.) *Proceedings of the Sixteenth PME Conference,* Durham, New Hampshire, USA, Vol. 3, Durham, NH: Univ. of New Hampshire Dept. of Mathematics, 235-261.
Goldin, G. A. (1993). Observing mathematical problem solving: perspectives on structured, task-based interviews. In Bill Atweh, C. Kanes, M. Carss and G. Booker (Eds.), *Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia*, Brisbane: Queensland University of Technology, 303-309.
Goldin, G.A. and Herscovics, N. (1991). The conceptual-representational analysis of children's early arithmetic, In R.G.Unde:hill (Ed., *Proceedings of the Thirteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Blacksburg, Virginia, October, 1991, 118-125.
Goldin, G.A. and McClintock, C. E.(Eds., 1984). *Task Variables in Mathematical Problem Solving.* Philadelphia: Franklin Institute Press (now Lawrence Erlbaum Associates, Hillsdale, NJ). ERIC Clearinghouse for Science, Mathematics and Environmental Education, Columbus, Ohio (1979).
Hiebert, J. and Wearne, D. (1992). Links between teaching and learning place value with understading in first grade. *Journal for Research in Mathematics Education*, 23 (2), 98-122.
Kamii, C. K. (1989). *Young Children Continue to Reinvent Arithmetic.* New York: Teachers College Press.
Mason, J. (1992). Doing and construing mathematics in screen-space. In B.Southwell, B.Perry, and K.Owens (Eds.), *Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia* ,Nepean, Australia: University of Western Sydney,1-17.
Presmeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics,* 6(3), 42-46.
Rubin, A. and Russell, S. (1992). *Children's developing concepts of landmarks in the number system.* In W.Geeslin and K.Graham (Eds.) *Proceedings of the Sixteenth PME Conference,* Durham, NH: Univ. of New Hampshire Dept. of Mathematics, Vol. III, 136.
Steffe, L., Cobb, P and Richards, J. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.
Steffe, L. (1991). *Epistemological foundations of mathematical experience.* New York: Springer-Verlag.
Thomas, N. (1992). An analysis of children's understanding of numeration. In In B.Southwell, B.Perry, and K.Owens (Eds.), *Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia* ,Nepean, Australia: University of Western Sydney, 521-540.
Wright, R. J. (1991). The role of counting in children's numerical development. *The Australian Journal of Early Childhood,* 16(2), 43-48.

WORKING TOGETHER FOR CHANGE

Colleen Goldstein, Phillip Mnisi and Pam Rodwell
Maths Centre for Primary Teachers, Johannesburg, South Africa

South Africa in its search for transformation to a more just society, stands at a critical crossroad changes can be controlled by the existing power structures (albeit with a new face) or they can be directed by the needs of those whose lives will be affected This paper outlines a mathematics research programme, which seeks to develop a participatory model for curriculum and professional development that is feasible and sustainable within prevailing South African conditions The project supports junior primary teachers in moving from current ubiquitous authoritarian and rote teaching methods to methods which enable teachers and pupils to construct their own knowledge Action research has guided all levels of the programme, ranging from macro concerns regarding the direction of development of the project as a whole, to micro concerns such as the nature of classroom interventions Video excerpts will be shown

INTRODUCTION

Current attitudes to curriculum policy in South Africa are dominated by rationalist linear approaches (Samuel & Naidoo, 1992) where policy emanates from elite experts (government agencies or universities) They research, evaluate and develop curricula, and expect unquestioned implementation We believe that such approaches are not appropriate or effective, particularly within the current South African situation Our concerns centre around three broad areas

1. Socio-Political Concerns: Need for Redress

Top-down structures function to create and maintain hegemony for the dominant group (Fasheh, 1990) This has been particularly evident in South Africa where apartheid education was explicitly engineered to create minority group control and provide inferior education for the majority in order to sustain their position of social, political and economic subjugation Different sections of the South African community have been exposed to vastly different educational experiences, particularly in terms of access to resources

> *The debacle of Bantu Education has left a legacy of overcrowded under-equipped classes, teachers whose education, training and confidence has been undermined, and children who have been denied the opportunity to develop their full potential These conditions combine with rigid, authoritarian teaching methods to produce a double depression of achievement levels in DET schools.*
>
> *Goldstein, James & Rodwell, 1991*

One response has been to regard apartheid curricula as foreign constructs and an integral part of the structure of oppression This has resulted in rejection and alienation Many teachers and parents in disadvantaged groups see the solution in terms of redress They enshrine the authority of the dominant culture and attempt to appropriate it: they believe that given the same facilities as white schools, all will be well. This leads to a call for good transmission teaching in well resourced classrooms, meeting the demands of existing authoritarian, top-down structures and syllabi A symptom of this belief is that, since the removal of restrictions on racial enrolments in schools, township children are flocking to white schools

However, there is much evidence in Africa that merely increasing educational budgets does not, in itself, improve the education of previously deprived groups (Putsoa, Unpublished Address, S A Association for Research in Maths and Science Ed , 1993)

2. Educational Concerns

Even within the most advantaged sections of the community, there is a great deal of dissatisfaction with the limited educational outcomes of present "expert designed" curricula and teaching practices, particularly in science and mathematics Within marginalised communities failure in these subjects is endemic and they have been "identified as clear markers of the elitist society and the province of the culturally and academically advantaged" (Goldstein Mnisi & Tshongwe 1993).

A major concern has been that top-down curricula centre around the interests of universities and to some extent, industry They do not address the circumstances and needs of learners The exact direction which future practices should take is under debate, but various methodologies based on constructivist thought are gaining increasing support because they hold promise for developing flexible, reflective learners better able to meet the challenges of a changing society

Since constructivist theory holds that individuals must construct their own knowledge, it supports practices which locate responsibility for a great deal of what and how children learn firmly within the classroom itself. This is not to suggest that outside researchers should be excluded but rather that researchers and teachers should become interdependent and link their expertise. Elmore and Sykes (pg 202, 1992) point out

> *....policy is reasonably effective in determining the broad content of the school curriculum But as the aims of policy become more ambitious - to introduce new <u>conceptions</u> of content, to jointly influence curriculum and instruction, to change what teachers and students know, believe, and choose to work on together - then the limits of policy emerge*

3. Teacher Development Concerns

We argue that bureaucratic curricula which dictate to teachers disempower them since they generate a "culture of non-participation" (Samuel & Naidoo, 1992) in curriculum development. This in turn strongly mitigates against the implementation of change. Once again this is particularly evident in South Africa Our discussions with local teachers have highlighted pervasive beliefs which serve as strong barriers to implementation of curriculum change These include beliefs that within the constraints of their authoritarian work structure their under-resourced classrooms, and their poor educational background they are ill-equipped to effect change

Our classroom observations of local teachers indicate that these attitudinal barriers lead them to filter the mandated curriculum through the sieve of their established practices and attitudes unless they understand and support proposed changes they, consciously or unconsciously, remould them to fit established practices For example, teachers expressed great enthusiasm when first introduced to mathematical games and equipment. However, we observed that during their initial classroom implementation, they rigidly structured children's interaction with the equipment, so that it had merely become a medium for new rote practices

MOVING TOWARDS CHANGE

1. Government

In 1978, of the 14 398 lower primary teachers working for the DET (Department of Education and Training, responsible for so-called "black" education under the Apartheid Government) only 207 had at least a matric plus a teaching qualification. As the world's attention focused negatively on such statistics, a major government response was to pressure teachers to upgrade their paper qualifications.

> *The feverish academic paper chase which resulted may have more negative than positive outcomes. Courses remove teachers from their classrooms and deprive them of thinking and preparation time. Teachers popularly choose 'soft option' subjects which bear no relation to classroom practice.*
>
> *Goldstein & Rodwell, 1993*

Gordon, Goldstein & Rodwell (1989) point out that this strategy raises many queries concerning the relationship between qualifications and competence, particularly for experienced teachers who are now considered to be "unqualified" in terms of promotion and wage increases. Hartshorne (1992) sums up present government endeavours on in-service training as follows:

> *From a more fundamental viewpoint of the improvement of classroom practice . it is doubtful whether the centralised, top-down, efficiency approach has contributed positively to the regeneration of either the school or the teacher.*

2. Other Agencies

The violent explosions of disaffection since 1976 have brought an increasing number of players into the field. A groundswell of non-government education organisations (NGEOs) has sprung up The challenge for these NGEOs has been to counter the negative effects of the paper chase. They have sought ways to effect and sustain meaningful change in classroom practice. in the face of limited resources as well as "their lack of authority to institutionalise their work in recognised educational frameworks" (Hofmeyr. 1991)

A further limitation has been that many agencies have come to schools with their own agenda and usually with pre-packaged solutions. Teachers have been overwhelmed, confused and frustrated by this myriad of conflicting interventions which, therefore, do not have grass roots support. Meso (unpublished action research into effective school entry points for NGEOs. 1993) describes how teachers' perceptions and belief systems impact negatively on their identification with programmes She reports that most teachers participate under false pretences, and for wrong reasons. they take part on instruction from above, to gain access to the materials on offer, or out of courtesy. They identify the projects with the existing power structures and perceive them as businesses which come into schools either to make money or to conduct research for their own purposes.

Particularly relevant to the South African situation is Gordon's (1993) examination and critique of the aims and practices of selected (international) mathematics programmes working within the constructivist paradigm. She states:

> ..practices underwritten by certain of the programmes support narrowly defined educational agendas. In the main political aims remain hidden as programmes are silent on the success of their strategies to reduce class, race and gender inequities in access and achievement. It therefore appears that few constructivist programmes have the potential to radically transform classroom practice, at least in the domain of redressing current ineauities regarding race and gender.

An alternative view to finding solutions calls for a total restructuring of society to create an environment which will enable people to develop structures to meet their needs. In a country whose economy has been ravaged by greed and self interest, the implications of such restructuring are overwhelming. "A future democratic South Africa will have fewer resources than necessary to advance all the development needs of the country." (Naidoo & Golombik, 1993). The driving question, therefore, becomes how best to utilise our available resources

So, in order to find a way forward for mathematics education in South Africa, we have to work on four fronts firstly, find a new and enabling way of teaching maths; secondly, make it work within the realities of the prevailing South African context; thirdly, help teachers to make the cognitive and attitudinal shifts necessary to embrace new practices, thereby assuming ownership; and fourthly, redress inequalities.

In the rest of this paper we will present, through a case study, attempts of the Maths Centre for Primary Teachers (MCPT) to develop an alternative in-service model, taking the above considerations into account, and linking practitioner and researcher in curriculum development. We argue that for change in classroom practice to be meaningful and sustained, programmes should be built up in an integrative and participatory manner in order to locate ownership within the community of educators. Together with teachers, we are attempting to develop a "home-grown" set of teaching practices, based on constructivist thought, which will give broad access within current conditions.

THE CASE HISTORY

The MCPT seeks to help junior primary teachers from under-resourced communities improve their own maths competence and teaching practice. During the eight years of its existence the project has made three major changes in its intervention model based on ongoing action research which is an integral aspect of the programme. The progression from one model to the next has, therefore, not been arbitrary but has been informed by observation, reflection and critical analysis on the part of both the MCPT staff and participating teachers "Each new conceptualisation has added another dimension to the approach" (Volmink. 1993)

Our initial computer-based tutorial model, intended to improve teachers' mathematics, was soon abandoned when it became apparent that it could not deal with their conceptual difficulties, with respect to both mathematics and teaching methods On the contrary, the drill-and-practice programmes fed into old didactic approaches

We changed to using modelling strategies which mirrored targeted classroom changes During workshops, in small collaborative groups, teachers were encouraged to grapple with mathematical problems and investigations, and devise and practise methods of dealing with them in the classroom MCPT staff modelled facilitation strategies which helped teachers reconstruct their ideas about mathematics and how to teach it

In terms of personal development this was effective Teachers were excited about their growing mathematical empowerment They became progressively more able to extract and reflect on teaching implications such as the need to replace "telling" practices with "questioning" practices; to set up appropriate problem solving opportunities, use concrete materials value individual problem solving strategies; harness the power of collaborative work and critical exchange of ideas; change classroom organisation; and deal constructively with pupils' misconceptions

However, we observed that these selfsame teachers regressed to established patterns at the chalk front. They could not visualise how to bridge the gap between their workshop insights and their classroom practices, they lacked the confidence to challenge existing structures; they feared that the new methods were overly time consuming and would prevent them from completing the overloaded syllabus. It became apparent that more direct intervention was necessary to carry the changes through to the children. We would have to take the approach into the classrooms to prove to ourselves and the teachers that it could be effective in spite of prevailing conditions.

> *We needed to work alongside the teachers in their classroom, face the problems together and find local solutions. We used the investigative approach itself to surmount difficulties. In this way, we, with the teachers, arrived at solutions and ways of working investigatively*
>
> *Goldstein, James & Rodwell, 1991*

From our reflections on our classroom experiences the MCPT is evolving a model which involves participants simultaneously in personal development and in the growth of a different culture and practice of teaching and learning. Presently the model is conceptualised in four broad stages (which overlap considerably in practice).

MODEL FOR SCHOOL-BASED APPROACH

Phase 1: Information Phase (usually about 2-4 weeks)

Aim:

1. To sensitise teachers to the need for change in learning and teaching styles

2. To expose teachers to investigative methods as possible alternatives to current methods
3. To initiate the establishment of intra-school structures to sustain new practices once the project has withdrawn

Methods used:

1. Modelling, through demonstration lessons to groups of teachers: incorporates alternative methods such as investigations "questioning techniques", group work; uses simple, available equipment; supports individual problem solving strategies
2. Initiating reflective practices, through introducing teachers to the concept of the action cycle, collaborative reflection on class experiences, facilitating journalling.
3. Initiating collaboration between teachers, through sharing of experiences and expertise during discussion and video workshops

Phase 2: Classroom Practice Phase (about 3 months)

Aim: To provide teachers with specific, intensive, practical experiences of investigative work in their own classes in order to

1. Facilitate the creation of the necessary classroom ethos to enable children to construct knowledge
2. Facilitate teachers in evaluating how their own class explores and develops a single mathematics concept/topic
3. Enable teachers to take ownership of the changes

Methods used:

This phase grows out of phase 1 and the methods are similar except that the initiative is progressively shifted to the teacher who ultimately directs classroom activities. Collaborative work among teachers is strongly supported.

Phase 3: Gradual Withdrawal Stage (varies)

Aim To hand over the project to the school, which has access to the MCPT as required.

Methods used:

1. Gradual, negotiated withdrawal: joint decision between teacher and MCPT about nature and extent of future support.
2. Strengthening of intra-school structures by continuing workshops but handing over organisational responsibilities to the teachers.

Phase 4: Building a Local Educative Community

Aim.

1. To draw together local schools in an educative community which will independently sustain and spread change in its area.
2. To offer the MCPT's expertise to the community as teachers feel the need to develop their own maths and teaching skills further

Methods used:

1. Complete withdrawal from classrooms unless invited to deal with a particular issue.
2. MCPT initiates and supports continued contact between a cluster of local schools.
3. Joint organisation (MCPT and community) of local workshops dealing with mathematical content, or pertinent issues such as evaluation.
4. MCPT available to assist with maths days, maths competitions etc.

This classroom based approach is ongoing and we are finding it to be a powerful way of assisting individual teachers to become reflective practitioners In her journal, a staff member points out that, "Teachers move from a fascination with keeping up to date with the syllabus, to a realisation for the need for more time to be spent on concept development" (Volmink, 1993). Speaking of change in one teacher, she comments, "She moved from being bored and authoritarian to being capable of organising group work and asking open questions". One teacher expressed the realisation that she and her children were as capable as any of their more advantaged peers as follows. "I never knew my children could be so clever. They don't need to go to white schools. We are better." Referring to the progress of a group of first grade children previously identified as non-achievers, another staff member says:

> *Both teacher X and I have been amazed at how much the children have extracted from the workcards, and how much independence they have developed The fact that they are coping with the task and understanding its demands seems to have given them a great deal of confidence*

Phases 1 and 2 serve as a research base for developing principles of effective implementation, but are very cost and labour intensive We realise that if we are to become an agent for substantial change within the country, we must find ways to provide broader access to our work. The community building phase is embryonic, having been conceived towards the end of 1993 and introduced in only two schools. However, it is already showing promise of helping to meet the great challenge of sustaining and spreading our work. One of the teachers in these schools informed his colleagues, "This (experience) has changed my way of teaching all subjects. I will never go back to my old ways. I will try and help others to change."

Our other vehicles for replication have proved to be extremely effective, not only in spreading our work but in giving teachers a vision of how they can become active stakeholders and participants in curriculum tranformation. Teachers' and children's ideas are depicted in a series of text booklets being developed by the MCPT Classroom experiences are also shared through our newsletter which is broadly disseminated. Appropriate, cost effective equipment, devised and trialed in conjunction with teachers, is also available from the MCPT

Probably our most powerful tools are our videos. We have argued that for strategies to be effective they should exemplify the constructivist message, both in the way they are made and the way they are used (Goldstein et al, 1993) Our videos (four are currently available) are, therefore, produced collaboratively by volunteer teachers and small MCPT teams. Together they construct an approach to an aspect of the syllabus. No script is written. Rather, the video sets out to capture the teacher's genuine attempt to implement the approach within the constraints of a large class. Since the videos emanate from familiar and relevant contexts, teachers identify strongly with them and we are gathering considerable ethnographic data which indicate that this identification is being translated into action (see Goldstein et al, 1993)

Furthermore our video workshops are creating space for debate on critical educational issues. Volmink (1993), in his evaluation of the MCPT's work, states, "The MCPT videos are amongst the best documentation on participatory curriculum development available in the country".

CONCLUSION

Our experiences during the past eight years have led us to a participatory model of professional and curriculum development which we believe to be pertinent, feasible and sustainable in present South African conditions We also believe that the model has relevance for other countries, particularly developing countries. The challenge for 1994 is to convince a new government that this, rather than old, familiar and seemingly more simple top-down practices, will form an effective basis for educational upliftment in a democratic society.

REFERENCES

Elmore, R. & Sykes, G. (1992) Curriculum Policy In P.W. Jackson (ed) **Handbook of Research Curriculum,** New York: McMillan, 185-215.

Fasheh, M. (1990) Community Education. **Harvard Educational Review**, 60, 19-36.

Goldstein, C., James, N., & Rodwell, P. (1991) I'm a Culprit Because I'm a Victim. **Proceedings, 14th National Convention on Mathematics and Natural Science Education**, 17-26. Cape Town.

Goldstein, C., Mnisi, P. & Tshongwe, T. (1993) Medium and Message. In C Julie, D. Angelis & Z. Davis (eds) **Political Dimensions of Mathematics Education-2,** 232-339

Goldstein, C. & Rodwell, P (1987) Prevention Through Intervention. **Proceedings, Saaled National Conference,** 24-137 Johannesburg: University of the Witwatersrand.

Goldstein, C. & Rodwell, P. (1993) Empower Mathematics Teachers. **Financial Mail Education Supplement,** 2, 10 Johannesburg: Venture Publishing (Pty) Ltd.

Gordon, A. (1993) Constructivism and the Politics of Pedagogy. In C. Julie, D. Angelis & Z. Davis (eds) **Political Dimensions of Mathematics Education-2,** 175-184

Gordon, A., Goldstein, C. & Rodwell, P. (1989) Technology and Pedagogy in Mathematics **Proceedings Inter. 3rd Conference: Children in Information Age.** Bulgaria, 327-341

Hartshorne, K. (1992) **Crises and Challenge: Black Education 1910-1990.** Cape Town Oxford University Press

Hofmeyr, J. (1991) **Policy Change in South African Education.** (Unpublished Ph.D Thesis) Johannesburg University of the Witwatersrand

Naidoo, K & Galombik, N (1993) Key Issues Facing the Democratic Movement with Regard to Electronic Media in Education Unpublished paper at **Electronic Media Education Workshop** Johannesburg.

Volmink, J (1993) **Maths Centre for Primary Teachers** Unpublished evaluation report Durban

SOCIAL NORM: THE KEY TO EFFECTIVENESS IN COOPERATIVE SMALL GROUPS AND WHOLE CLASS DISCUSSIONS IN MATHEMATICS CLASSROOMS

Zahra Gooya, The University of Shahid Beheshty, Iran
Thomas Schroeder, State University of New York at Buffalo, USA

This study examines the social norm of the class in an introduction to mathematics course taught by the first author. The approach to the instruction was teaching mathematics via problem solving, using non-routine problems. The instruction was based on metacognitive strategies including cooperative small groups, whole-class discussions, and journal writing. The study follows the authors' recommendations for more research on the nature and effects of planned and documented interventions by teachers to engage students in classroom discourse in small groups and whole class discussions. Evidence is presented about efforts that were made to establish a suitable social norm or classroom culture to make the cooperative work in small groups and the whole-class discussion effective, and about effects of those efforts on learning outcomes.

In a previous PME research report (Schroeder, Gooya, & Lin, 1993) we argued that research on processes in cooperative small groups should consider not only "naturally occurring" group process but also the effects of planned interventions designed to make group work effective. In this paper we take up that challenge and examine the social norm of a mathematics class in which teaching mathematics via problem solving and metacognition-based instruction were central (Gooya, 1992). We believe that in order to establish a suitable social norm in their classes teachers must make efforts to understand and appreciate their students and must adopt roles that are conducive to and supportive of the classroom culture they wish to establish.

Theoretical Foundations and Previous Research

There is broad agreement that the most suitable environment—or "classroom culture," as Davis (1989) calls it—for teaching and learning mathematics, is one which is natural, relaxed, and friendly, but also business-like, engaging, and thought provoking. A number of mathematics educators who have done classroom-based research on problem solving (e.g., Lester, Garofalo, & Kroll, 1989a, 1989b; Raymond, Santos, & Masingila, 1991; Schoenfeld, 1985, 1987a, 1987b) have concluded that appropriate settings can help students become more aware of their thinking processes and develop positive beliefs about mathematics. For example, it may be beneficial for students to analyze another student's behavior while solving problems, and to associate themselves with that student's thinking when solving a problem. Schoenfeld (1985; 1987a) has argued for the creation of a "microcosm of mathematical practice" in which students "do" mathematics in a way similar to the way that mathematicians do mathematics. Lester, Garofalo and Kroll (1989a) concluded that students' beliefs about mathematics are mainly shaped by classroom instruction and the classroom environment.

The literature is also overflowing with research findings regarding the role of small groups in students' learning (e.g., Dees, 1985; Good, Grouws, & Mason, 1990; Maher, & Alston, 1985; Noddings, 1985; Schoenfeld, 1989). Although there is a great deal of variation in the ways in which different researchers and teachers have used small groups in their studies and their classrooms, Vygotsky's (1976) ideas have given a strong rationale for the role of small groups in students' development. He argued that working in collaboration with others helps students to reach their "Zone of Proximal Development" (ZPD). Vygotsky (1962) explained that a student might be able to function up to a certain level, but working cooperatively with others who are more capable might help him or her to function at a higher level. The potential ability that a child (or person at any age) has and that could be cultivated by some assistance but not in isolation, is the ZPD. (Also see Schoenfeld, 1985, 1987a.)

These theoretical perspectives and research findings, in contrast with reports such as that of Stacey (1992) who found that students working in groups did not necessarily perform better on mathematical problem solving than students working individually, support our views (Schroeder, Gooya, & Lin, 1993) about the need to create an appropriate social norm in order to increase the chances that two heads will be better than one in a mathematics classroom. Whether students are working in cooperative small groups or engaging in a whole-class discussions we believe that the social norm of the classroom can have a crucial impact on its effectiveness for students, both in regard to what they learn and the beliefs they develop. Our purpose in this paper is to examine the efforts made by the instructor of a mathematics class to develop a suitable social norm in her class and to seek evidence concerning the effects of those efforts on both the social norm of the class and the outcomes for individual students.

Research Setting

Gooya (1992) taught an introduction to mathematics course for undergraduates at the University of British Columbia. One thrust of her teaching, which was based on metacognitive strategies and teaching mathematics via problem solving (Schroeder & Lester, 1989), was the creation of a social norm for the class in which the students would be able to do and to enjoy mathematics. Her purpose was to show the students that as the teacher she was not the authority in the class who was always "right." On the contrary, she confronted them as a moderator, facilitator, monitor, friend, and role model who is also fallible. She showed the students that she too experiences the ecstasy of solving a problem after spending lots of time and effort, as well as the agony of not knowing and being frustrated and confused. She talked to the students about the ways in which mathematicians do mathematics to come up with world shaking ideas. She challenged the myth that "some people have it and some people don't" concerning mathematics. Her goal was to lead students to believe that everyone can do mathematics to some extent and that the best way of doing it is to get involved and take responsibility for one's own learning.

To create such an environment required her as teacher to adopt a role very different from the traditional one, but it was not easy for the students to change their expectations concerning the teacher's role in class. Many of them wanted to be told what to do all the time. In the beginning, a majority of them preferred "spoon feeding" instruction since they were used to being taught that way. However, work in small groups and whole-class discussions helped the students to see the dialectical nature of mathematics; it helped them experience mathematics as a social activity that is learned through social interaction. The students were encouraged to create and to do mathematics by and for themselves without relying on an outside authority, the teacher, to tell them what to do; and to value the process of doing mathematics rather than being concerned only about the finished product, the "correct answer." Gooya worked hard to ensure that the students were actively involved in all the class activities and to convince them that their participation in the class discussions was indeed beneficial to themselves and to other students. As in most typical classes, only a few students were really engaged in the class discussions in the beginning. Sometimes the active involvement of those students and the strength of their reasoning intimidated other students, and a number of students reflected on this issue in their journals. They asked why, since those students were better able to solve the problems and explain them to others, should they "waste" the class time with their "wrong" solutions and "inadequate" explanations. Gooya responded that her aim was to show them that there is not always a single correct way of solving a problem, and besides, everyone would learn a great deal from each others' ideas and even their mistakes.

Classroom Interactions

Instruction was based on the students' interactions with each other and the instructor in both the whole-class setting and in small groups. It took a long time and a lot of effort to get the students to accept that it would be worth their while if they actively took part in the discussions. Some of the students expressed annoyance and frustration with the noise level in the classroom. In their journals, some students asked for less interruption, which meant little or no questioning, and more direct instruction and guidance on the teacher's part. However, Gooya believed if the students began to see benefits, they would like the idea of interacting with each other in the class. Sometimes students chose to work by themselves; in such cases, she asked them repeatedly to work in a group. Eventually, working in small groups and participating in whole-class discussions became the norm, and most students felt comfortable participating in these activities, although a few students resisted. The study showed that bringing about changes in classroom processes requires a tremendous effort on the part of both students and instructor.

To illustrate the social norm of the classroom and to portray the interactive nature of the course and a kind of environment that might be suitable for teaching and learning mathematics, we have chosen one episode based on video-tapes of the class and Gooya's reflective journal. In the following excerpts from the interaction in class (Gooya, 1992, pp. 86-89), the teacher/researcher is identified as "Zahra" and various students are referred to using pseudonyms. On that day, in the fifth week of the course, the class began with a problem designed to introduce the fundamental principle of counting. The problem, "In how many ways can you answer a true/false test that has three questions?" was posed, and the students were asked to spend 10 to 15 minutes discussing it in their groups. After the group discussions, every group presented its solution attempts to the class. The instructor recorded all the ideas suggested while the students were presenting them. Then, the class as a whole discussed each solution. The discussion helped the class to reject the solutions that did not make sense to them and retain those which did.

Sandra: 2^6, because there are two ways of answering [each question].

Zahra: What do you think about it [asking the whole class]?

Nina: It *sounds* mathematical [every one laughs].

Jim: I don't want to create friction here. I only have an argument against it. I think that's the number of possible answers that you can have when you consider [pause] like in each case you are only allowed to have three answers out of the six possibilities. That's [referring to 2^6] suggesting that you are able to have six answers out of six possibilities. But you have to eliminate three because of the fact that you can only have three answers in such an exam. You can't have for example, ... the questions cannot be [both] True and False. They either have to be True or False. Does that make any sense? I don't know if that's clear.

This excerpt shows that the students and instructor had by this time established a social norm for the class in which everyone had a voice and they all respected each other's opinions. Jim's response to Sandra, "I don't want to create friction here. I only have an argument against it," is interesting because it expresses that respect while focusing on the meaning of the situation. After further discussion about why 2^6 could not be the answer to the problem posed, Zahra continued.

Zahra: Now, is it okay to say that this one [2^6] is not correct? [interrupted by Jim]

Jim: Sandra! The way of looking at it would be to say that if you want to look at the number of possibilities, then that would be appropriate ... because then you would have six T, F, T, F, T, F. But when you look at the possible answers, then you have to do it with three.

Zahra: How different are they? You said the number of possibilities and [the] number of answers.

Jim: What? You mean the distinction that I made?

Zahra: Yes.

Jim: Well, the fact is that there are six; there are six possible responses: T, F, T, F, T, F ... [pause] ... but you can have [only] one answer for each one. So that's ... yeah!

Nina: So two possibilities for each question.

Jim: Yeah! two possibilities for each question.

Nina: And three questions, so there are only eight possible ... [interrupted by Jim]

Jim: I'm only trying to make a distinction between possibilities and answers. You can only have three answers, but there are six possibilities.

This interaction shows that Jim could relate to Sandra and hypothesize why she thought of 2^6 for the answer. Although it would have been possible simply to tell her that she was wrong and let that be the end of it, the whole-class discussion gave Sandra a chance to explore her confusion, rather than letting those ideas go underground (Cobb, 1991). Jim's contribution was intended to help Sandra realize where she went wrong, and Nina's returned the focus to the reality of True/False questions and the fact that they can only be answered either "True" or "False" and not both. Later Kent extended the problem to an exam with three choices (rather than only two).

Kent: In other words, [if] you can say True, False, or True/False, you could! Then it logically tells us that there are three possible choices and three questions, then it would be 3^3. Am I right?

Class: Yeah!

Kent: Okay, then it seems we've got some principle here. ... You could have multiple choice questions with five possible answers and three questions, then we have 5 x 5 x 5. Three questions and five possible choices.

Lana: Are you saying that there are three choices?

Kent: No! That would be 3 x 3 x 3. [Kent proceeded to explain a problem with five questions and five answer alternatives very clearly. After a good discussion about this problem he continued.] ... Well, I'm just trying to find the underlying structure of the logic of this and seeing if I plugged in other numbers from different questions and still it could be true.

Zahra: So 2^6 doesn't work for the True or False exam with three questions?

Kent: If it were 6 questions, it would!

Patrick: And with two possible responses for each question.

It was interesting to see how the class developed their understanding of the counting principle. Kent tried to find "some principle here," "the underlying structure of the logic," a general pattern that would apply to various situations involving different numbers.

Later in the discussion Patrick drew attention to a difference between the solutions to these problems (involving the number of possible ways of responding to tests) and another problem involving the number of choices that someone could make using five shirts and three pants. Patrick argued that in the latter problem the number of choices was 5 x 3. Although both problems seemed to involve the same principle, the solutions seemed different (because the one involved only multiplication whereas the others used exponentiation). Many students got involved in the discussion to find reasons for the difference. They knew there were two choices to make for each one of the three questions, therefore the total number of choices was 2 x 2 x 2, and the same argument for the number of choices relating to five shirts and three pants was 5 x 3. However, the presentation of a solution for the first problem (2^3, that was also suggested by the students) confused several students, including Patrick. Patrick expected to have the same form for the problem involving shirts and pants since they both had the same underlying structure. The instructor explained to the students that 2^3 was only a short cut for presenting the solution to the problem, and that otherwise they did exactly the same thing in both problems, that was multiplying the number of choices for each question, or for pants and for shirt, to find the total number of choices to be made: $2^3 = 2 \times 2 \times 2$ ways of answering 3 questions with 2 choices for each, and 5 x 3 ways of combining 5 shirts and 3 pants.

Solving problems like these might not take more than five minutes in a traditional class. However, if the class had not discussed the problem, the teacher/investigator would not have realized the difficulties that students were facing in understanding the problem. Although it might be easier to introduce the concept (in this case the counting principle), solve a few examples showing the class how to apply certain rules and formulas, and then ask students to solve a list of similar problems; in such situations Sandra's answer would simply be "wrong" and the "correct" answer would be put on the board for display, and, of course, students would copy it down in their notebooks.

Discussion

Although the metacognition-based instruction and the class environment provided an opportunity for the students to express their ideas freely and discuss them both in small groups and in class, only a few problems were solved in each class. However, many mathematical concepts were developed through those few problems. In this teaching environment, the teacher had a special role to play. She orchestrated the class discussions and monitored the students' work in small groups. Creating an interactive environment in which students actively take part in their own mathematics learning and preparing appropriate activities requires many hours of hard work and continuous reflection about what is going on in the class. However, Gooya's study shows that there are good reasons to believe that the hard work will pay off enormously, and that it can add much joy to mathematics teaching and learning.

There is much evidence in the study leading us to believe that the students left the class feeling more confidence in their mathematical understanding and believing that everyone was capable of doing mathematics to some extent. Increasing self-confidence and self-reliance were associated with increasing participation in the metacognition-based class activities. The students also learned to be critical regarding mathematical problems, and as Ben wrote at the end of the final exam: "I learned to never accept anything without asking why. ... That's the best thing I've learned from this course. I learned to learn. I got my confidence back, and that's just great!"

The students also learned to assess and monitor their work, and to make appropriate decisions, by working cooperatively and discussing the problems in small groups. They were constantly asked the three questions that Schoenfeld (1985a, 1987) asked his students while they were working in the groups: "What are you doing? Why are you doing it? How does it help?" The teacher's role as an external monitor (cf. Lester et al., 1989) was to coach the students in a way similar to that described by Schoenfeld (1987), helping them to become aware of their own resources, to appreciate them, and to use them proficiently. Small groups also afforded opportunities for the instructor to offer the students a variety of problem solving strategies (heuristics).

The students were asked to share their thoughts about given problems with the whole class after spending between ten and twenty minutes discussing them in their groups. The teacher did not try to lead the students to any particular solution. On the contrary, she constantly encouraged them to come up with as many different ideas about solving the problems as they could. They were expected to participate in the class discussions and be actively involved in the process of solving the problems. The teacher's role at this stage was to coordinate the discussions, to help the students to utilize what they knew, and to help them become more reflective. The students' responsibility was to try to make sense of the problems, to speculate, to make conjectures, and to justify them. In these ways, the class discussions helped them a great deal to become more self-regulated and better decision makers.

However, sometimes nothing seemed to work, and no approach yielded a solution. Since this could lead to the students' frustration and disappointment, in such instances Gooya took control of the class discussions by summing up what went on in those sessions. Her intention was to help the students realize why they had gotten nowhere, what possibly had gone wrong, and what could be done. This tactic helped them to see both the pitfalls and the light, through the awareness and the knowledge that they gained by examining the different approaches to the problem. Then they were in a better position to put everything in perspective and decide what to do to solve the problems. All these events served as a vehicle to promote self-regulation.

Concluding remarks

The teacher's role in creating a suitable social norm is crucial, especially in small-group work. The instructor should ensure that every single student is engaged in mathematical activities in the groups. We believe that working in small groups will be effective if and only if all the students are negotiating the meaning of the material they are working on. The students have to take the responsibility to make reasonable decisions after discussing the problems in their groups. Of course, there is always a tendency in group work for some members to follow the others passively without being involved in the meaning-making process. To avoid such situations, we believe that teachers should reserve the right to intervene and to make sure that everyone knows his or her own responsibility. However, it takes even more effort to establish a social norm for whole-class discussions, since every student needs to feel comfortable getting actively involved in the meaning-making processes, and eventually becoming more self-regulated and a better problem solver. Much more research is needed to document the norms for discourse established in mathematics classrooms, and to investigate the ways in which teachers interacting with students can create a classroom culture--a social norm--that makes cooperative small group work and whole class discussions productive, that facilitates students' understanding of mathematics, and that makes the teaching and learning of mathematics more meaningful, fun, and enjoyable.

References

Cobb, P. (1991). Classrooms as learning environments for teachers and researchers. Seminar presented at the University of British Columbia. Vancouver, BC

Davis, R. B. (1989). The culture of mathematics and the culture of schools. *Journal of Mathematical Behavior, 8*, 143-160.

Dees, R. L. (1985). How does working cooperatively help students in increasing problem-solving ability? In S. Damarin & M. Shelton (Eds.), *Proceedings of the Seventh Annual meeting of North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 325-330). Columbus, OH.

Good, T. L., Grouws, D. A., & Mason, D. A. (1990). Teachers' beliefs about small-group instruction in elementary school mathematics. *Journal for Research in Mathematics Education, 21*, 2-15.

Gooya, Z. (1992). *Influences of metacognition-based teaching and teaching via problem solving on students' beliefs about mathematics and mathematical problem solving.* Unpublished doctoral dissertation, University of British Columbia, Vancouver.

Lester, F. K., Jr., Garofalo, J., & Kroll, D. L. (1989a). *The role of metacognition in mathematical problem solving: A study of two grade seven classes, Final Report: A Project of the Mathematics Education Development Center.* Bloomington, IN: School of Education, Indiana University.

Lester, F. K., Jr., Garofalo, J., & Kroll, D. L. (1989b). Self-confidence, interest, beliefs, and metacognition: Key influences on problem solving behavior. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 75-88). New York: Springer-Verlag.

Noddings, N. (1985). What do individuals gain in small group mathematical problem solving? Paper presented at the annual meeting of American Educational Research Association. Chicago.

Raymond, A; Santos, V; & Massingila, J. (1991). The influence of innovative instructional processes on mathematical belief systems. Paper presented at the annual meeting of American Educational Research Association. Chicago.

Schoenfeld, A H. (1985). *Mathematical problem solving.* New York: Academic Press.

Schoenfeld, A H. (1987a). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-216). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (1987b). Confessions of an accidental theorist. *For the Learning of Mathematics*, 7(1), 30-38.

Schoenfeld, A. (1989b). Ideas in the air: Speculations on small group learning, environment and cultural influences on cognition, and epistemology. *International Jou.. nal of Educational Research, 13*(1), 71-78.

Schroeder, T, Gooya, Z, & Lin, G. (1993). Mathematical problem solving in cooperative small groups: How to ensure that two heads will be better than one? *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics education* (vol. 3, pp. 65-72). Tsukuba, Japan.

Schroeder, T. L. & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton & A. P. Shulte (Eds.), *New directions for elementary school mathematics*, [1989 Yearbook] (pp. 31-42). Reston, VA: National Council of Teachers of Mathematics.

Stacey, K. (1992). Mathematical problem solving in groups: Are two heads better than one? *Journal of Mathematical Behavior, 11*, 261-275.

Vygotsky, L. S. (1962). (E. Hanfmann & G. Vakar, Eds. & Trans.). *Thought and language.* Cambridge, MA: MIT Press and Wiley.

Vygotsky, L. S. (1978). (M. Cole, V. John-Steier, S. Scribner, & E. Souberman, Trans.). *Minds in society.* Cambridge, MA: Harvard University Press.

Spectrums of Performance in Two Digit Addition and Subtraction.

<u>Eddie Gray</u>

Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL
UK

This paper presents some qualitative results obtained from a longitudinal study devised to act as a pilot investigation into the relationship between quality of thinking and achievement. It hypothesises that children who extensively use counting procedures for basic number combinations tenuously use related procedures for horizontal two digit addition and subtraction. It shows how children who display a rich interpretation of basic combinations use their flexibility to achieve success with the horizontal combinations through a large number of numerical rearrangements. We see how the former seem to reproduce more naturally the procedural requirements of a standard paper and pencil algorithms. However, the requirement that the latter follow an explicit set of directions for the formal algorithms may force them to think in a way not in tune with flexibility. Temporarily at least, we may see a reversal in levels of achievement.

Introduction

Two digit and other multidigit numbers use notation in a very powerful way. The two 3s in the symbol 33 are used with entirely different meanings; the first being 3 tens, the second 3 units. It requires the procept of number (Gray & Tall, 1991, 1994) (rather than just the process of counting or the knowledge of the sequence of number words) to be able to view place value as both a grouping procedure in which 33 is the procedure of grouping "3 tens and 3 units", and the result of the procedure: the number 33.

Standard pencil and paper algorithms for two digit addition and subtraction may be presented in a formal way which causes many children to cope by memorising a sequence of procedures; adequate competence may be achieved without appreciating the interrelationship between the numbers involved (see, for example, Resnick, 1983; Johnson, 1989). The children learn a procedure which is more reliant on the interpretation of multidigit numbers as adjacent single digit numbers that are divorced from the sense provided by a rationale giving multiunit meanings for digits in different positions (Fuson, 1992). Concentrating instructional activity on the surface characteristics of the numeration system may well foster a perspective of mathematics as instrumental understanding (Oliver, Murray & Human, 1990).

Gray & Tall (1991, 1994) indicate that instrumental or procedural understanding and understanding emanating from knowledge rich in relationships leads to very different forms of thinking. Procedures form a basic part of mathematical development but their importance lies in the cognitive shift from mathematical processes into manipulable mental objects; for example, the concept of sum encapsulated from the compression of a series of counting procedures. Gray & Tall hypothesise that compression provides the potential for the emergence of styles of thinking triggered by different interpretations of mathematical symbolism. On the one hand we see procedural thinking based on routine manipulation of procedures and, on the other, the flexibility of proceptual thinking: procedural interpretations are tempered by the ability to view symbolism as the "representation of a mental object which may be decomposed, recomposed and manipulated at a higher level" (Gray & Tall, 1994, p 124). Relating children's approaches to simple arithmetic over a period of time to the interpretations they place on two digit addition and subtraction provides some further insight into the qualitative difference between procedural and proceptual thinking.

Background

An analysis of children's responses to the context free numerical components of a series of Standard Assessment Tasks (SAT), (SEAC, 1992), was presented in Gray (1993). The SAT's were designed to measure level of achievement within a range of Mathematics Attainment Target's (D.E.S., 1989) associated with the National Curriculum of England and Wales. The analysis of the results identified children at three levels of achievement:

- Level 1 (L1): children who could add and subtract objects where the numbers involved were no greater than ten.
- Level 2 (L2): children achieved Level 1 and were able to "recall the number combinations to ten without calculation" (SEAC, 1992, pp 36-37)
- Level 3 (L3): the children achieved Levels 1 and 2 and were able to "recall the number combinations to twenty without calculation" (SEAC, 1992, pp 39-40).

The analysis concluded that though the SAT's took no account of how achievement was obtained they achieved their purpose of differentiating between children over three levels. It was concluded that levels of achievement were related to quality of thinking: children who displayed flexible proceptual thinking were more successful than those who relied extensively on counting procedures.

Method

This paper brings the earlier research up to date and reports on children's achievement and quality of achievement on two subsequent occasions:

- All children were again interviewed in March 1993 The format of the interview was similar to the first interview. The children were now in Y3 and their ages ranged from 7 years 6 months to 8 years 6 months (8 +)
- A random sample of children within each achievement level, identified as a result of the March 1993 interviews, were interviewed during November 1993. During a two stage interview previously presented basic number combinations were considered and then , on a separate occasion, two digit addition and subtraction presented horizontally and vertically. The latter were mainly from the 1992 SAT (SEAC 1992). Children were first presented with the combinations as a paper and pencil exercise and then approaches discussed. All results were video recorded.

Standard algorithms for two digit addition and subtraction – the latter focusing on the decomposition method – had formed an integral part of classroom teaching from March to July 1993. In the somewhat conventional school that the children attended, there had been whole class introduction to these through demonstration with Stern apparatus. Children were then given opportunities to practice and consolidate through the use of a standard text. Little emphasis was placed upon horizontal addition and subtraction. During the period September to December the main focus of interest had been multiplication and division.

Results

1. Strategies used for basic number combinations.

A comparison of the overall strategies used by the three groups of children in the first and second series of interviews. i.e. when the children were 7+ and 8+, bears out the trends that are reported in snap shots of unrelated samples of children of the same ages reported in Gray (1991).

Most strikingly only those who achieved L3 show the extensive use of derived facts over both series of interviews (see Figure. 1) and there was no evidence of the use of count-all or take-away by this group.

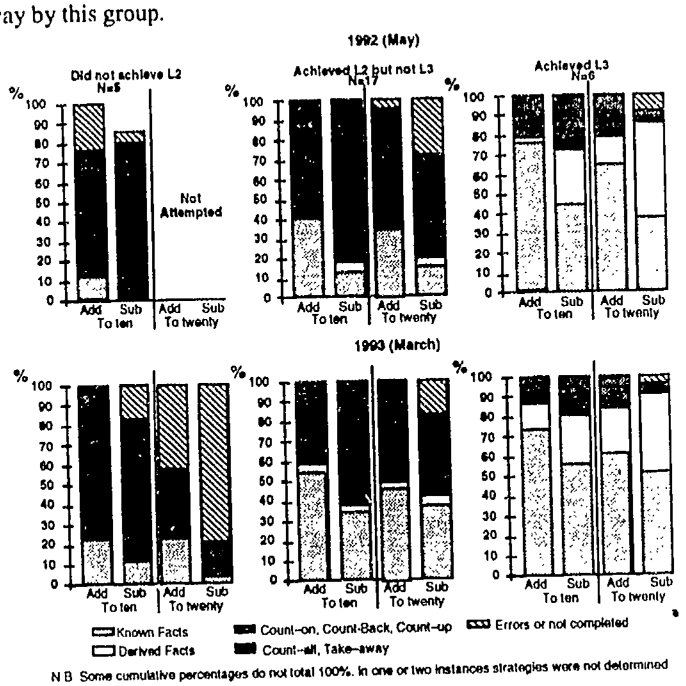


Figure 1 : Strategies used to determine solutions to basic number combinations by groups of children identified by their level of achievement at the ages of 7 +.

We see that counting procedures, frequently very inefficient, totally dominated the strategies used over the two interviews by the children who did not achieve L2. We see too that counting procedures form a very high percentage of the strategies used by child who achieved L2 but not L3. The use of derived facts by children within this group is largely accounted for by 2 children who, as a result of this series of interviews, achieved L3.

None of the group who failed to achieve L2 in 1992 satisfied criteria to achieve it in 1993. At both interviews counting procedures, although frequently very inefficient, dominated their approach. In many instances their procedures were so covert that had they completed these procedures within 5 seconds (the time limit for each combination). During the formal assessment process of April 1992 the class teacher was unable to distinguish between children who used covert counting actions and children who knew or derived solutions.

Overall, children who achieved L2 but not L3 were more successful with the L3 combinations than they had been a year earlier. It is noticeable from Figure 1 that there were no errors for addition and they were almost halved for subtraction. However, though the percentage of combinations known by the group increased, in general there was a remarkable consistency in the solution procedures used by the children during both interviews though this consistency was generally marked by more highly developed procedural competence.

By March 1993, a considerable proportion of this group of 17 children had improved their procedural competence to the point where they were now almost attaining the same level of achievement in the basic number combinations as those who had achieved L3 almost a year earlier. However, apart the two children who were now displaying the same flexibility as these "more able" children, the quality of thinking the children brought to the introduction to two digit addition and subtraction was different. The distinction was manifest through either the extensive use of counting procedures or a the limited use of counting procedures to support an otherwise rich interpretation of basic number combinations.

2. Two Digit Addition and Subtraction

The qualitative analysis which is the focus of this section of the paper stems from random samples drawn from within three groups constructed as a result of level of achievement established in March 1993. Separate samples display the following characteristics:

- Group 1: (n=2 from 5): Did not achieve L2. Achievement characterised by very limited number of facts known and limited procedural competence.
- Group 2: (n=5 from 15): Achieved L2 but not L3. Achievement characterised by extensive use of counting procedures to support of a limited range of known facts.
- Group 3: (n=4 from 8): Achieved L2 and L3 through a spectrum of performance characterised by its flexibility.

The interesting feature of all of the children is that though they were free to use any approach they used mental methods for horizontal combinations and a taught pencil and paper algorithm for vertical combinations.

By the end of 1993 the counting procedures of the children within Group 1, now largely count-on for addition and count-back for subtraction, were sufficiently robust to cope with all of the L 2 combinations. However, although used competently enough to cope with the L3 addition cor' 'nations they remained totally unreliable for the L3 subtraction combinations. Neither child satisfied criteria to achieve Level 2. They attempted to generalise their procedures to deal with the two digit horizontal combinations but they proved to be too difficult to implement and led to an extensive number of errors, signified by miscounting, an inability to keep a check on the amount counted-on or counted-back and a continued need to revert back to the start of the problem. Interviewing ceased during each child's attempts at the second problem in each category. The children had considerable difficulty resolving combinations given in vertical form. Tens and units components were treated as discrete quantities and each combined through counting methods. Though relatively successful when there was no exchange where it was required there was evidence of place value errors in addition and in 'smaller from larger' errors or difficulties with zero emerged with subtraction.

Of the five Group 2 children, only one achieved L3 in November 1993. As with other children who had previously reached this level of achievement this success was characterised by the flexibility associated with proceptual thinking. Consequently, this child will be discussed with Group 3. None of the other children made use of derived facts over the three interviews and their procedural approaches remained very much as previously identified, count-on for addition and some use of take away with either count-back or count-up for subtraction. No one child used both of the latter.

The four children's approaches to the horizontal addition and subtraction combinations showed a remarkable consistency. The addition combinations were usually solved by sequencing in tens and then in ones in the following manner:

24 + 43: <u>40</u>, 50, 60,...64...65, 66, 67

In some instances a less sophisticated approach was used:

39 + 26: <u>30</u>, 40, 50...51, 52, 53, 54, 55, 56, 57, 58, 59...60, 61, 62, 63, 64, 65.

Such methods worked successfully for addition whether or not exchange was involved. Their inverse caused procedural difficulty when subtraction was attempted. 80 – 30 was answered successfully by counting back in tens and 74 – 4 and 84 – 10 by counting back in ones. 29 – 6 and 26 – 19 both caused particular difficulties. All four children attempted the former by decrementing in ones from 29 to 6. All obtained an incorrect solution. One attempted to modify her approach by counting up but "couldn't keep a check on how many were counted". Another, asked to attempt it using a vertical format, dealt with the units relatively quickly but then became confused "because I thought it was times". After some difficulty one child completed 26 – 19 by counting back to 19 and recording the difference. Other solutions were given as follows:

(i) 26 – 19	= 4	(ii) 26 – 19	= 13	(iii) 26 – 19	= 17
Because		*Because*		*Because*	
2 – 1	*= 1*	*2 – 1*	*= 1*	*2 – 1*	*= 1*
9 – 6	*= 3*	*9 – 6*	*= 3*	*9 – 2*	*= 7*
one and three	*= 4*	*Ten and three*	*= 13*	*Ten and seven*	*= 17*

These four children successfully carried out pencil and paper methods for the two digit addition combinations even when exchange was involved. Individual digits were combined through either a known fact or through count-on. A ten, carried as the result of exchange, was always referred to as "one" and tens added as if they were ones. One child gave some evidence of largest from smallest errors with subtraction and all of the children were initially unsuccessful with the vertical subtraction 33 from 80 – each child providing the answer 53.

As the children who make up Group 2 displayed remarkably similar approaches to the horizontal combinations so did those who achieved L3 – the Group 3 children. Their similarity was characterised by proceptual flexibility rather than procedural inflexibility. There was evidence of counting on, for example, "18 count-on 5 is 23", but combinations were more likely to be solve using knowledge of basic number combinations, for example:"18 and 2 is 20 and 3 is 23" and "7 + 2 is 9 so 70 + 20 is 90".

Where required, all children possessed the flexibility to make transformations on both operands in the original numbers, for example:

24 + 43:	20 + 40 = 60;	3 + 4 = 7;	60 + 7 = 67	
39 + 26:	30 + 20 = 50;	9 + 6 = 15;	50 + 15 = 65	
or	30 + 20 = 50;	9 + 6 = 15;	50 + 10 = 60;	60 + 5 = 65

This approach was not observed amongst the Group 2 children.

A major distinction between children within this group and children in the other groups is their success with all of the horizontal subtraction combinations. Once again flexibility was evident. Combinations that did not involve exchange were generally solved through knowledge of basic addition and subtraction facts, tens usually computed first. Counting was evident, for example one child solving 26 – 19 by counting up from 19 to 26 although the same child obtained the solution for 29 – 6 using the following approach:

29 – 6: 6 + 3 = 9 so 9 – 6 = 3 and 20 + 3 = 23

There was some evidence of transformations on both operands, for example:

26 – 19: 27 – 20 = 7

There was a surprise when this group of children attempted the vertical addition combinations. Two of them, one of whom had easily achieved L3 in May 1992 and the other who had achieved L3 in March 1993, gave original solutions to the addition combinations which were in the following form.

(i) $\begin{array}{r} 1\ 9 \\ +7\ 6 \\ \hline 8\ 15 \end{array}$ (ii) $\begin{array}{r} 1\ 9 \\ +7\ 6 \\ \hline 8\ 5 \\ 15 \end{array}$ Represention of the written algorithm was taught as: $\begin{array}{r} 1\ 9 \\ +7\ 6 \\ \hline 9\ 5 \\ 15 \end{array}$

Children had been taught to record units as shown in the above representation. Interestingly, none of the Group 2 children did this, three of the five Group 3 children did; two successfully and one with the modification seen in (i) above. Both children who displayed the above errors very quickly corrected them and none of the children gave incorrect solutions to the subtraction combinations.

Discussion

Gray (1993) suggested that the ways in which young children deal with simple arithmetic illustrates a spectrum of performance that may reflect the limitations of procedural thinking or the flexibility of proceptual thinking. This is not to suggest that the spectrum is linear. It is hypothesised that there may be several dimensions and we may see children moving between the dimensions when dealing with different aspects of the arithmetic. However, it is hypothesised that the more the child moves in one dimension the more that there may be consequences that lead to success or failure.

In the children's approaches to the horizontal combinations we see some of the consequences of the hypothesised bifurcation caused by counting (Gray, 1993). It is almost as if certain aptitudes dominate and place a child on one side of the scale and we can see the effects that this has for some children.

Within the solutions to the two digit combinations we clearly see two kinds of information processing: that based on rearrangements of numbers and that based on the utilisation of explicit instructions.

Within the horizontal two digit combinations we see a form of pattern recognition; the children visualise rearrangements of the individual numbers. Within the processing that is related to these rearrangements we may see those that do not generalise and those easily generalise. The Group 2 children relied extensively on counting procedures to obtain solutions to the basic number combinations and utilise similar procedures to deal with addition and subtraction. The manner in which they do this is at its most sophisticated when they use "*accumulation* or iterative strategies...leading to a gradual approach to the final answer by a series of increasingly better "approximations" "(Oliver et al, 1990, p.301). Interestingly this approach was not in evidence amongst Group 3 children and neither did Group 2 children attempt to use it for subtraction, preferring instead to use relied and trusted methods even when they proved to be too difficult. In contrast, the more successful see a large number of rearrangements that may be applied to the combinations. To solve these the children have not learned any rule-like behaviour but bring together personal knowledge and experience to use a variety of approaches one of which includes a "*replacement* strategy, transformations are made on both operands in the original problem before any attempt at computation" (Oliver et al 1990, p.301).

Within the vertical combinations we see the children attempting to follow explicit directions. We may conjecture that they have been taught to learn algorithms as bits of meaningless information in the hope that it makes things easier for them. In the short term it may but, although the evidence points to the fact that given more practice children may demonstrate improvement , the flexibility may still not be there for longer term success.

An interesting feature, worthy of further investigation is the evidence of a reversal in achievement when children tackle vertical addition and subtraction. It is hypothesised that this reversal stems from the greater difficulty that the more flexible thinkers may have in slipping into a procedure. We see an example of the children meeting with a new scheme of action which has to be adopted to fit the new situation. However, the algorithm does not take account of their flexibility; the "more able" may have some difficulty with a new idea if they are required to follow meaningless procedures – a course of events that the procedural thinkers may take to more naturally.

References

D.E.S. (Department of Education and Science): (1989). *Mathematics in the National Curriculum.* London: H.M.S.O.

Fuson, K. C.: (1992). Research on whole number addition and subtraction. In D. A. Grouws, (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. New Y rk: Macmillan.

Gray, E. M.: (1991). An analysis of diverging approaches to simple arithmetic: preference and its consequences. *Educational Studies in Mathematics*, **22**, 551-574.

Gray, E. M.: (1993). Count-on: The parting of the ways for simple arithmetic. In I. Hirabayashi, N. Hohda, K Shigematsu and Fou-Lai Lin (Eds.), *Proceedings of XVII International Conference for the Psychology of Mathematics Education*. (Vol. I pp. 204-211). Tsukuba, Japan. .

Gray, E.M., & Tall, D.O.: (1991). Duality, ambiguity & flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), *Proceedings of XV International Conference for the Psychology of Mathematics Education*. (Vol. II pp. 72-79). Assisi, Italy.

Gray, E.M., & Tall, D.O.: (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal of Research in Mathematics Education*, **25**, 2, 115–141.

Hiebert, J., & Wearne, D.: (1992). Emerging relationship between teaching and learning arithmetic during the primary grades. In W. Geeslin and K. Graham (Eds.), *Proceedings of XVI International Conference for the Psychology of Mathematics Education*. (Vol. I pp. 273-280). Durham, New Hants. U.S.A.

Johnson, D. (Ed.): (1989). *Children's mathematical frameworks 8–13: A.study of classroom teaching*. Windsor, England: NFER-Neslon.

Oliver, A., Murray, H., & Human P.: (1990). Building on young children's formal arithmetic knowledge. In G. Booker, P. Cobb & T. Mendicutti (Eds.), *Proceedings of XIV International Conference for the Psychology of Mathematics Education*. (Vol. III pp. 297-304). Mexico.

Resnick, L. B.: (198. . A developmental theory of number understanding In H.P. Ginsburg (Ed.), *The development of mathematical thinking*, (pp. 110-152). Orlando, Florida: Academic Press. Inc.

SEAC (Schools Examination and Assessment Council), (1991): *Standard Assessment Task, Key Stage 1, Teacher's Handbook,* London, H. M. S. O.

THE EFFECT OF CALCULATOR USE ON THIRD AND FOURTH GRADERS' COMPUTATION AND CHOICE OF CALCULATING DEVICE

Susie Groves *
Deakin University - Burwood Campus

As part of a study of the long-term effects of calculator use, 110 grade 3 and 4 children from six schools were observed while carrying out a range of computation tasks. Calculators and concrete materials were provided, as well as pencil and paper. Children with long-term experience of calculators performed significantly better overall on the 24 computation items, with an item by item analysis revealing significantly better performance on the five items requiring a knowledge of place value for large numbers, subtraction with a negative answer, division with a remainder, and multiplication and division of money. These children also made more appropriate choices of calculating device and were better able to interpret their answers when using calculators, particularly where decimal answers were involved.

Introduction

Despite research evidence to the contrary (Hembree & Dessart, 1986; 1992; Shuard, 1992) and agreement amongst mathematics educators that calculators should be used at all levels both as a computational tool and as a teaching aid (National Council of Teachers of Mathematics, 1980; 1989; Cockcroft, 1982; Curriculum Development Centre & Australian Association of Mathematics Teachers, 1987; Australian Education Council, 1990), there is widespread fear that calculators will undermine mathematics learning (see, for example, National Research Council, 1989, p. 61; Wheatley & Shumway, 1992, p. 7; and Saxon, 1987 – a publisher's two page advertisement "saying no to calculators" in the February 1987 *Arithmetic Teacher* focus issue on calculators).

The emergence of calculators and computers highlights the lack of congruence between school mathematics and real mathematics – in everyday life mental and calculator computations are the ones typically used, while pencil and paper methods still receive the most emphasis in schools (Willis & Kissane, 1989, p.58).

Data from the USA's Third National Assessment of Educational Progress (National Assessment of Educational Progress, 1983; Reys, 1985) and a more recent Canadian and USA study (cited in Reys, 1992) indicate that fifth to seventh grade children perform poorly on mental computation tasks and make inappropriate choices of calculating device, with the majority preferring pencil and paper or calculators to mental computation, even for straightforward questions such as 1000 x 945 .

* This research was funded by the Australian Research Council, Deakin University and the University of Melbourne. The *Calculators in Primary Mathematics* project team consists of Susie Groves, Jill Cheeseman, Terry Beeby, Graham Ferres (Deakin University); Ron Welsh, Kaye Stacey (Melbourne University); and Paul Carlin (Catholic Education Office).

Recently, powerful attempts have been made to change this situation in Australia. In line with world-wide trends, the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) has an increased emphasis on developing number sense and mental computation, partly in recognition of the role of the calculator. In order to make intelligent use of calculators, children need to make wise choices of calculating devices, have a strong intuitive understanding of number and develop skills such as estimation and approximation (Reys, 1992; Groves, 1992; Jones, 1988).

The *Calculators in Primary Mathematics* project is a long-term investigation into the effects of the introduction of calculators on the learning and teaching of primary mathematics. The project commenced at kindergarten and grade 1 levels in six schools in 1990. Over 60 kindergarten to grade 4 teachers and approximately 1000 children participated in the project during the period 1990 to 1993, with the 1990 kindergarten and grade 1 children participating for the full four years. All children were given their own calculator to use whenever they wished, while teachers were provided with systematic professional support to assist them in using calculators to create a rich mathematical environment for children to explore.

Research has focused on the extent and purpose of calculator use; changes in teachers' expectations of children's mathematical performance and consequent changes in the curriculum; long-term learning outcomes for the children; and changes in teachers' beliefs and teaching practice.

Amongst a number of papers on various aspects of the project, Groves (1993a) reported that children with long-term experience of calculators were better able to tackle "real world" problems, which would normally be beyond their pencil and paper skills, than children without such experience. While the project children did not make more use of calculators, they were more able to attach meaning and interpret their answers.

This paper focuses on the long-term effect of calculator use on children's computation and choice of calculating device.

Method

In 1991 and 1993, as part of the investigation of the long-term learning outcomes for children, interviews were conducted with a stratified random sample of over 10% of all grade 3 and 4 children at the six project schools. Three children were selected at random, subject to achieving gender balance, from each grade 3 and grade 4 project class, while in mixed classes one boy and one girl was selected at each grade level.

At the time of the interviews, the 1993 children had taken part in the project for $3^1/_2$ years – i.e. since entering school for the grade 3 children and since grade 1 for the grade 4 children – while the 1991 children, who formed the control group, had not taken part in the project. For the purposes of this analysis, data from all six grade 3 children at one of the schools was discarded, as grade 3 children at the school had extensive calculator experience in both 1991 and 1993, due to a different pattern of participation in the project. This resulted in sample sizes of 52 in 1991 and 58 in 1993.

The interview was designed to test children's understanding of the number system; their choice of calculating device, for a wide range of computation tasks; and their ability to solve "real world" problems amenable to multiplication and division, with or without calculators (see also Groves, 1993a). Throughout the interview, children were free to use whatever calculating devices they chose. Unifix cubes and multi-base arithmetic (MAB) blocks were provided, as well as pencil and paper and calculators. Many of the questions were expected to be answered mentally.

The 24 computation items referred to throughout this paper are listed in Table 2. As well as their answers, children's choices of calculating device were recorded. For the purpose of this analysis, these have been classified as mental (which may include the use of fingers), calculator, and other (which includes drawing, the use of concrete materials, and the use of standard pencil and paper algorithms). In order to minimise the need to memorise the numbers involved, each of the 24 computation items was presented to the children clearly displayed on a card.

In an attempt to limit the interview to less than 30 minutes, it had initially been the intention to ask each child to nominate how they would attempt each item, but only ask children to actually carry out the computations for the first four items and half of the remaining items, with each child being pre-assigned either the remaining odd or even numbered items. While this had been a successful strategy during fairly extensive trialing of the interview, it soon became apparent during the actual interviews that most children would attempt most items, as they immediately carried out the computation when asked how they would attempt the item. Consequently, in 1993 children were generally encouraged to attempt most items, with 98% of the children attempting each item on average, compared to 93% in 1991.

This presented difficulties in attempting to compare overall performance for the two groups of children. However anecdotal evidence from the interviewers suggests that, in general, it was the more difficult and time-consuming items which were not attempted in 1991. This is supported by an analysis of the results of the interviews. In 1991, for the two items attempted by fewer than 80% of the children, the average percentage of correct answers was 47%, compared to 62% for the five items attempted by 80–89%, and 67% for the 17 items attempted by at least 90% of the children. In 1993, all 24 items were attempted by at least 90% of the children, for an average of 75% correct.

In order to enable a comparison of overall performance to be made, each child's actual score was factored up, if necessary, to give a (rounded) integer score out of 24. While this clearly distorts the data, in view of the discussion above it would seem reasonable to assume that the effect would be to over-estimate the scores of the 1991 children by a greater amount than those of the 1993 children, and hence, in view of the direction of the results (see Table 1), bring the mean scores closer together.

These new scores were used to compare, at each grade level, the overall performance of the 1991 and 1993 children on all 24 items. For the item by item analysis, actual frequencies of correct and incorrect answers were used. Similarly, actual frequencies were used for the item by item analysis of the choice of calculating device. No attempt was made to make a comparison of overall use of mental computation, calculators or other devices, as the desirability of making a particular choice of device depends on the nature of the computation item.

Results

Table 1 compares the scores out of 24 for the grade 3 and grade 4 children in 1991 and 1993. It can be seen from the table that, as would be expected, in each of the years grade 4 children performed better than grade 3 children (significant at the $p \leq 0.01$ level). Furthermore, at each of the grade levels, the 1993 project children performed significantly better than the 1991 control group ($p \leq 0.01$ for grade 3 and $p \leq 0.05$ for grade 4).

Table 1: Comparison of scores for grade 3 and 4 children in 1991 and 1993 on 24 computation items

Year	1991			1993		
	N	Mean	SD	N	Mean	SD
Grade 3	25	14.48 *	3.38	29	16.59 **	4.54
Grade 4	27	16.85 *	3.29	29	19.00 **	2.73

significant difference between 1993 and 1991 at $p \leq 0{\cdot}05$ level (df = 52, t = 1.907)

significant difference between 1993 and 1991 at $p \leq 0{\cdot}01$ level (df = 54, t = 2.668)

* significant difference between grade 3 and grade 4 at $p \leq 0{\cdot}01$ level (df = 50, t = 2.563)

** significant difference between grade 3 and grade 4 at $p \leq 0{\cdot}01$ level (df = 56, t = 2.455)

Table 2 provides an item by item comparison of correct and incorrect answers and choice of calculating device for the combined grade 3 and 4 children in 1991 and 1993.

The four Items 12, 16, 19 and 22 (15 ÷ 4 , 62750 + 50 , \$153 ÷ 4 , 3 – 7) showed a significant difference at the $p \leq 0{\cdot}001$ level using a $\chi 2$ test on the frequencies of correct and incorrect answers, while Item 17 (7 x \$3·53) showed a significant difference at the $p \leq 0{\cdot}05$ level. For all five items the 1993 project children achieved better results than the 1991 control group. There were no items for which the 1993 children performed significantly worse at the $p \leq 0{\cdot}05$ level.

Only two items showed a significantly different pattern of choice of calculating device, Items 19 and 22 (\$153 ÷ 4 , 3 – 7), with $p \leq 0{\cdot}05$ in both cases. Both items were also amongst those for which the 1993 children performed significantly better than the 1991 control group.

For Item 19 (\$153 ÷ 4), there was an increase in use of calculators in 1993 at the expense of both the mental and other categories. This item is one where calculators would appear to be the most reasonable choice of device for many grade 3 and 4 children. It is also an item where the use of a calculator is far from straightforward – in order to obtain a correct answer it is necessary for children to be able to interpret the calculator display correctly. It can be seen from the table that while 27 children attempted this item using a calculator in 1991, only 16 of these obtained a correct answer, compared to 41 correct answers from the 44 children using a calculator in 1993. Closer examination of the data confirms that it was not inability to accurately key in the numbers which caused the errors in 1991, as every incorrect answer using calculators was due to inability to correctly read the display. This result is similar to ones obtained from a comparison of 1991 and 1992 grade 3 children on the "real world" problem solving items from the same interview (Groves, 1993a), where it was shown that division items which resulted in a decimal answer showed a significantly better result for project children, as did other interview items which required children to read and interpret decimals. It should also be noted that for the simpler question Item 12 (15 ÷ 4), which also showed a significant improvement for 1993 project children, there was an increase in the use of mental computation which, although not statistically significant, suggests that the project children make choices based on both the operations and the numbers involved.

Table 2: Frequencies of correct and incorrect answers given and calculating devices used by grade 3 and 4 children in 1991 and 1993 on 24 computation items

Year		1991 (N=52)						1993 (N=58)					
No	Item	n[¢]	√[#]	x[#]	M[§]	C[§]	O[§]	n[¢]	√[#]	x[#]	M[§]	C[§]	O[§]
1.	7 + 5	52	48	4	49	0	3	58	54	4	57	0	1
2.	20 + 30	52	48	4	50	1	1	58	56	2	57	0	1
3.	100 - 10	52	48	4	47	5	0	58	54	4	57	1	0
4.	3 x 5	52	52	0	49	2	1	58	56	2	54	3	1
5.	36 - 13	50	44	6	26	15	9	58	43	15	39	13	6
6.	12 ÷ 3	49	47	2	34	12	3	57	50	7	45	5	7
7.	3 x 99	46	35	11	11	26	9	57	43	14	20	23	14
8.	How many 3's in 30?	50	38	12	38	7	5	56	44	12	44	6	6
9.	1024 - 23	47	34	13	31	13	3	56	48	8	45	10	1
10.	30 ÷ 10	52	48	4	47	2	3	55	52	3	52	2	1
11.	52 - 18	48	25	23	29	12	7	56	37	19	33	15	8
12.	15 ÷ 4	49	8	41	24	15	10	57	31	26	37	14	6
13.	225 + 76	45	31	14	23	9	13	56	37	19	33	12	11
14.	2000 - 1	51	30	21	48	2	1	58	33	25	56	2	0
15.	321 - 164	38	17	21	13	17	8	55	31	24	22	21	12
16.	62750 + 50	46	26	20	32	10	4	55	49	6	44	9	2
17.	7 x $3·53	47	19	28	9	24	14	56	36	20	11	32	13
18.	14 x 20	44	33	11	27	14	3	57	46	11	35	19	3
19.	$153 ÷ 4	46	16	30	5	27	14	54	41	13	1	44	9
20.	34 x 26	38	19	19	7	19	12	56	36	20	9	35	12
21.	20 ÷ 40	51	7	44	30	11	10	57	15	42	41	13	3
22.	3 - 7	52	13	39	40	2	10	58	38	20	46	11	1
23.	$2·50 + $3·50	51	41	10	36	8	7	58	53	5	49	4	5
24.	$12·38 + $2·52	50	28	22	26	13	11	56	33	23	38	11	7

¢ n – number of children who were asked to attempt item
\# √ – number of children giving correct answer; X – number of children giving incorrect answer
§ M – number of children using mental computation; C – number of children using calculators; O – other
significant difference at $p \le 0{\cdot}05$ level using a $\chi 2$ test on frequencies
significant difference at $p \le 0{\cdot}001$ level using a $\chi 2$ test on frequencies

For Item 22 (3 – 7), the only other item showing a significantly different pattern of choice of device, neither of the two children who used a calculator obtained the correct answer in 1991, while all 11 children using a calculator in 1993 were correct. However, it would appear unreasonable to attribute the improved performance for 1993 children on this item to their better use of calculators. Rather, it confirms considerable evidence from other sources that project children, having been exposed to large numbers, negative numbers and decimals at a much earlier age through their use of calculators, have developed more sophisticated conceptual understandings in these areas. It is therefore also not surprising that the other item showing the most significant difference was Item 16 (62750 + 50), where again 1993 children were more inclined to use mental computation, although not to a statistically significant extent.

Space does not permit a full report of the detailed item by item analysis for grade 3 and grade 4. Results were similar to those for the combined grade levels. Again there were no items on which the 1993 children performed significantly worse. For grade 3 children Item 22 (3 – 7) again showed a significant improvement at the $p \leq 0{\cdot}001$ level, while results for Items 16 and 19 (62750 + 50, \$153 ÷ 4) were again significantly better, but at the $p \leq 0{\cdot}05$ level. For grade 4 children, Items 12, 16 and 19 (15 ÷ 4 , 62750 + 50 , \$153 ÷ 4) showed a significant improvement at the $p \leq 0{\cdot}001$ level, while Items 9 and 22 (1024 – 23, 3 – 7) were significantly better at the $p \leq 0{\cdot}05$ level.

For grade 3 children, three items showed a different pattern of choice of calculating device. Items 5, 6 and 12 (36 – 13 , 12 ÷ 3 , 15 ÷ 4) all showed a decrease in calculator use, together with a corresponding increase in mental computation, all significant at least at the $p \leq 0{\cdot}05$ level. This suggests that fears that children who regularly use calculators will come to rely on them even for the simplest calculations are unfounded. For grade 4 children, only two items showed a significant difference at the $p \leq 0{\cdot}05$ level for choice of calculating device. These were Items 21 and 22 (20 ÷ 40, 3 – 7), where more children used both calculators and mental computation instead of other devices.

It should also be noted that in both years only about 2.5% of responses made use of standard written algorithms, with somewhat over a half of these being correct. However, as in other studies (Reys, Reys & Penafiel, 1991; Koyama, 1993), many children were observed mentally applying pencil and paper algorithms, often pointing at imaginary numbers and mumbling statements such as "put down 1 and carry 1". Unfortunately no consistent records were kept of this phenomenon.

Conclusion

Among the original hypotheses of the *Calculators in Primary Mathematics* project was the expectation that children who had been involved in the project would successfully deal with larger numbers and acquire certain concepts (such as place value, negative numbers and understanding of decimal notation) at an earlier age than other children.

The results discussed in this paper show that the children with long-term experience of calculators performed significantly better overall on 24 selected computation items, while the item by item analysis revealed significantly better performance on the five items requiring a knowledge of place value for large numbers, subtraction with a negative answer, division with a remainder, and multiplication and division of money. These children also made more appropriate choices of calculating device and were better able to interpret their answers when using a calculator, particularly where decimal answers were involved.

Sowder (1988, p. 183) characterised number sense as referring to a well organised conceptual network, which enables a person to relate number and operation properties and use flexible, creative ways to solve number problems. She claims that the intuitive nature of number sense makes it difficult to teach directly. Hope and Sherrill (1987) distinguished skilled mental calculators from the unskilled partly by their ability to use specialised numerical information, which seemed to have been memorised through the pursuit of interesting mathematical activity. As well as acting as a computational device, the calculator is a highly versatile teaching aid which has the potential to radically transform mathematics teaching by allowing children to experiment with numbers and construct their own meanings (Corbitt, 1985; Wheatley & Shumway,1992). However, the UK based *Calculator-Aware Number* project (CAN) found that the calculator's full potential could not be realised without a change in teaching style (Duffin, 1989, p.15).

Evidence from the *Calculators in Primary Mathematics* project (Groves, 1993b) suggests that teachers with long-term involvement in the project have adopted a more open-ended teaching style as a result of the increased opportunities for exploration of number presented by the calculator. Teachers speak of their mathematics teaching becoming more like their language teaching and of letting children lead the learning process. The calculator is seen as facilitating more sharing and discussion in the classroom, which in turn allows teachers to become more aware of children's thinking. From a constructivist perspective, an important aspect of learning is communicating and constructing shared understanding (Davis, Maher, & Noddings, 1990, pp. 2-3; Wheatley, 1991; Greeno, 1989, p. 45). The presence of the calculator not only provides children with the opportunity to engage in mathematical investigation, but also enables them to share their discoveries with teachers and other children by providing an object which can become the focus for genuine mathematical discussion.

References

Australian Education Council (1990). *A National Statement on Mathematics for Australian Schools.* Melbourne: Curriculum Corporation (Australia).

Corbitt, M. K. (1985). The impact of computing technology on school mathematics: Report of an NCTM conference. *Arithmetic Teacher, 32* (8), 14-18.

Cockcroft, W.H.(1982). *Mathematics Counts* . London: Her Majesty's Stationery Office.

Curriculum Development Centre & Australian Association of Mathematics Teachers (1987). *A National Statement on the Use of Calculators for Mathematics in Australian Schools.* Canberra: CDC.

Davis, R.B., Maher, C.A. & Noddings, N. (Eds.) (1990). *Constructivist views on the teaching and learning of mathematics. JRME Monograph 4.* Reston, VA: NCTM.

Duffin, J. (1989). *Annual Report of Evaluation of CAN Component of PrIME Project.* National Curriculum Council Primary Initiatives in Mathematics Education Project.

Greeno, J.G. (1989). Some conjectures about number sense. In J.T. Sowder & B.P. Schappelle (Eds.). *Establishing foundations for research on number sense and related topics: Report of a conference* (pp. 43-56). San Diego: San Diego State University Centre for Research in Mathematics and Science Education.

Groves, S. (1992). Processes and strategies of third and fourth graders tackling a real world problem amenable to division. In W. Geeslin & K. Graham (Eds.). *Proceedings of the Sixteenth International Conference for the Psychology of Mathematics*. (Vol. I, pp. 249-256). Durham, NH: University of New Hampshire.

Groves, S. (1993a). The effect of calculator use on third graders' solutions of real world division and multiplication problems. In I. Hirabayashi, N. Nohda, K. Shigematsu & Fou-Lai Lin (Eds.).*Proceedings of the Seventeenth International Conference for the Psychology of Mathematics*. (Vol. II, pp.9-16). Tsukuba, Ibaraki, Japan: University of Tsukuba.

Groves, S. (1993b, November). *Calculators as an agent for change: Teachers' perceptions and practice*. Paper presented at the Annual Conference of the Australian Association for Research in Education , Freemantle, WA.

Hembree, A. & Dessart, D. (1986). Effects of hand-held calculators in pre-college mathematics education: A meta-analysis. *Journal for Research in Mathematics Education, 17*, 83-99.

Hembree, A. & Dessart, D. (1992). Research on calculators in mathematics education. In J.T. Fey (Ed.). *Calculators in mathematics education:1992 yearbook*. (pp. 23–32). Reston, VA: NCTM.Hope, J.A. & Sherrill, J.M. (1987). Characteristics of unskilled and skilled mental calculators. *Journal for Research in Mathematics Education, 18*, 98–111.

Jones, P. (1988, May). Mental mathematics moves ahead. *Mathematics in School*, pp. 42–44.

Koyama, M. (1993). Research into relationship between the computational estimation ability and strategy and the mental computation ability: Analysis of a survey of the fourth, fifth, and sixth graders in Japan. In I. Hirabayashi, N. Nohda, K. Shigematsu & Fou-Lai Lin (Eds.).*Proceedings of the Seventeenth International Conference for the Psychology of Mathematics*. (Vol. II, pp.65-72). Tsukuba, Ibaraki, Japan: University of Tsukuba.

National Assessment of Educational Progress (1983). *The third national mathematics assessment: Results, trends and issues*. Denver, CO: Education Commission of the States.

National Council of Teachers of Mathematics (1980). *An Agenda for Action - Recommendations for School Mathematics of the 1980s*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM.

National Research Council (1989). *Everybody counts:A report to the nation on the future of mathematics education*. Washington, DC: National Academy Press.

Reys, R.E. (1985). Testing mental computation skills. *Arithmetic Teacher, 33* (3), 14–16.

Reys, B. J., Reys, R.E. & Penafiel, A.F. (1991). Estimation performance and strategy use of Mexican 5th and 8th grade student sample. *educational Studies in Mathematics, 22*, 353–375.

Reys, R.E. (1992). Mental computation: Some ideas and directions for teaching. In C.J. Irons (Ed.). *Challenging children to think when they compute: Conference proceedings*. Brisbane: Queensland University of Technology.

Saxon, J. (1987). Why Saxon elementary books say no to calculators. Advertisement in *Arithmetic Teacher, 34* (6), 20–21.

Shuard, H. (1992). CAN: Calculator use in primary grades in England and Wales. In J.T. Fey (Ed.). *Calculators in mathematics education:1992 yearbook*. (pp. 33–45). Reston, VA: NCTM.

Sowder, J.T. (1988). Mental computation and number comparison: Their roles in the development of number sense and computational estimation. In J. Hiebert & M. Behr (Eds.). *Number concepts and operations in the middle grades* (pp. 182–197). Hillsdale, NJ: Lawrence Erblaum.

Wheatley, G.H. (1991). Constructivist perspectives on science and mathematics learning. *Science Education, 75* (1), 9–21.

Wheatley, G.H. & Shumway, R. (1992). The potential for calculators to transform elementary school mathematics. In J.T. Fey (Ed.). *Calculators in mathematics education:1992 yearbook*. (pp. 1–8). Reston, VA: NCTM.

Willis, S. & Kissane, B. (1989). Computing technology in teacher education in mathematics. In Department of Employment, Education and Training *Discipline Review of Teacher Education in Mathematics and Science, Volume 3, Appendixes* (pp. 57–92). Canberra: Australian Government Publishing Service.

A MODEL OF TEST DESIGN TO ASSESS THE VAN HIELE LEVELS

Adela Jaime and Angel Gutiérrez
Universidad de Valencia (Valencia, Spain)

We propose a framework for designing tests to assess the Van Hiele level of reasoning. The framework is based on the consideration of the different key processes involved in each thinking level and the use of open-ended questions. We present a proposal of paper and pencil super-items with a structure approaching that of the clinical interviews, in order to obtain as much information as possible from the students' written answers.

Introduction.

The Van Hiele model describes the evolution of the kind of reasoning of a student in geometry. It establishes a sequence of 5 levels of reasoning, labelled 1 to 5 in this paper. In this paper we do not try to summarize the general characteristics of the Van Hiele levels. Such description can be found, for instance, in Crowley (1987), Hoffer (1983) and Jaime, Gutiérrez (1990).

We can hardly meet any researcher on the Van Hiele model who has not needed to assess the students' Van Hiele level; this implies the use of a test (written or oral). The Usiskin's test (Usiskin, 1982) and the Burger and Shaughnessy's test (Burger, Shaughnessy, 1986) are the most frequently used, but both tests have some objections:

- The Usiskin's test is based on paper and pencil multi-choice items, and there are some doubts about the possibility of measuring reasoning by means of this kind of items (Crowley, 1990 and Wilson, 1990). Nevertheless, this test has as its main advantage that it can be administered to many individuals and it is easy and quick to asses a level of reasoning to the students.

- The Burger and Shaughnessy's test has to be administered by an interview, and it is very time consuming; this makes the test unsuitable for assessing many people. However, the great advantage of this test is that the information obtained from interviews results in a deeper knowledge of the way students reason and, therefore, in a more reliable assessment of the Van Hiele level than that obtained by paper and pencil tests.

Aware of the necessity stated by many researchers of having a Van Hiele test without the inconveniences mentioned above, we have been working for several years in the design of such a test. Some previous results can be found in Shaughnessy et al (1991), where different ways of assessing the Van Hiele levels were analyzed. Now, the objective of our ongoing research is twofold:

1- To offer a procedure enabling the design of reliable and valid tests to measure the Van Hiele levels of reasoning.

2- To implement a pool of items from which one should be able to make several of such tests.

We present here a theoretical model of design of items and tests (first objective) and also some examples of such items (second objective).

A model for the design of Van Hiele tests.

The core of our proposal is the consideration of each Van Hiele level of reasoning as integrated by several *key thinking processes*. Then, to evaluate a student's thinking level means to evaluate the way this student uses each key thinking process characteristic of that level. This interpretation of the assessment of the Van Hiele levels is implemented by means of paper and pencil open-ended items, designed in a way that they provide an amount of information that approaches the obtained by means of clinical interviews.

1. Key processes of the Van Hiele levels.

As a thinking level is integrated by some thinking processes, quite different one from the others, the items in a test should not be intended to assess a whole level, but one or more of the key processes involved in this level. Then an ideal test should contain at least an item able to assess each key process of each Van Hiele level. For instance, as we shall see below, for the assessment of level 3, the items should allow to assess the way students' use the processes of definition, proof, and classification.

The view of considering the kind of reasoning of a Van Hiele level divided into several components is not new. De Villiers (1987) makes this distinction. Also Hoffer (1981) shows a characterization of 5 geometric skills to be considered for the assessment of each Van Hiele level.

In the following paragraphs, we describe the main key processes we have identified for the Van Hiele levels 1 to 4. We does not consider level 5, as our research is directed to primary and secondary school students. It would be possible to make a more detailed list of key process, since some of the processes stated below can be decomposed in sub-processes. For instance, in level 4, the key process of formal proof could be divided into the processes corresponding to the different ways of proving that students should know. However, this would conduct us to a position impossible to be put into practice because of the length of the tests. These key processes characterizing the Van Hiele levels 1 to 4 are:

Identification of the family a geometric object belongs to.

Definition of a concept, understood from two different points of view: To read definitions, that is to use given definitions, and to state definitions, that is to formulate a definition for a class of geometric objects.

Classification of geometrical objects into different families.

Proof of properties or statements, that is ways of convincing someone else of the truth of a statement.

The table below summarizes the key processes characteristics of each Van Hiele level. An X or the name of a process in a cell means that this process is a part of the reasoning of the level, so it has to be assessed in this level. The "---" means that this process is not a part of the reasoning of the level, so it has not to be assessed in this level.

	Identification	Definition	Classification	Proof
Level 1	X	State	X	---
Level 2	X	Read & State	X	X
Level 3	---	Read & State	X	X
Level 4	---	Read & State	---	X

When assessing the Van Hiele level of reasoning of a student, it is important to notice that, in some levels, some of the processes just mentioned do not have their usual mathematical meaning. Then, when analyzing a student's answer, we have to consider the processes from the perspective of the level exhibited by the student. For this purpose, we specify in the following paragraphs the meaning of each process integrating the levels 1 to 4.

Level 1:

Identification: The students recognize figures on the basis of physical global characteristics, like aspect, size of elements, position, etc.

Definition: Students take into consideration only attributes which refer to physical objects in a global way, or non-mathematical properties like "round" for circle, so they are not able to read a mathematical definition. When stating a definition, the students refer to this same kind of attributes. Sometimes the name of the concept is the definition itself; for example, children quite often say that "a square is a square".

Classification: Students use the same kind of properties of the figures as in the previous processes. They do not accept any relationship among two different families nor, many times, among two elements of the same family having quite different physical aspect (for instance, two isosceles triangles having angles of 50°, 50° and 80°, and 82°, 82°, and 16°, respectively).

Level 2:

Identification: Students recognize geometrical figures on the basis of their mathematical properties.

Definition: The students pay attention to mathematical properties but, when reading or stating definitions, they may have problems with some logical particles, such as "at least".

When stating a definition, sometimes the students omit a necessary property, which they are using implicitly. Other times, they provide a list with more properties than needed, even when the dependence among them is easy to realize. For example, some students define a rectangle as "a parallelogram having two pairs of equal sides, being two sides longer than the other two" (they omit the reference to the right angles). Other students define a rectangle as "a parallelogram having two pairs of equal parallel sides, being two sides longer than the other two, four right angles and two equal diagonals (they include an extra property).

Classification: It is exclusive, that is, the students do not relate families based on the attributes provided in the definitions. When they are given a new definition of a concept, different from the one they already knew, the students do not admit the new definition. This happens very often with quadrilaterals, when the students are habituated to use the exclusive definitions and they are given the inclusive definitions.

Proof: A typical proof in this level consists on verifying the truth of the property to be proved in one or a few examples.

Level 3:

Definition: The students are able to interpret and state mathematical definitions, being conscious that a necessary and sufficient set of properties is needed and that adding more properties to the definition does not result in a better one. Therefore, when providing a definition, the students try not to be redundant, although some redundancies may appear when the relationships among the properties do not consists on one-step implications.

Classification: The students may do inclusive classifications, based on the properties stated in the given definitions of the concepts. The students are able to change their mind when a new definitions of a concept is given, even when there is a change from exclusive to inclusive, or vice versa.

Proof: The students may check the property to be proved in some examples, but they look also for some informal explanation based on mathematical properties, or the examples are selected.

Level 4:

Definition: The progress from the level 3 reasoning consists on a better understanding of definitions and the ability to prove the equivalence of different definitions of the same concept.

Proof: Students in this level are able to do standard formal mathematical proofs. Specific figures are used only sometimes to help to choose the adequate properties for the proof, but the students are aware that a figure is only a case and that to prove a statement it is necessary to make a sequence of implications based on already proved properties.

2. Open-ended items for assessing the Van Hiele levels.

Paper and pencil open-ended items, where the students can freely explain the reason for their answer, are more reliable than multi-choice items for assessing the Van Hiele levels of reasoning. On the other side, what defines a student's level of thinking are not the items administered but the student's answer to such items, since most of the questions can be answered according to several levels of thinking. Therefore, we defend the administration of tests based on open-ended items that are not pre-assigned to a specific level, but to a range of the levels in which answers can be given. In this way, an item contributes to the assessment of each level in this range.

A useful characteristic of clinical interviews is the possibility for the interviewer to modify the questions, to give some hint, etc., depending on the previous student's answers and the reflected thinking level. This is what makes interviews so useful for the assessment of the Van Hiele levels and the reason why they provide more information than any paper and pencil test. In order to approach the amount of information obtained by written tests to that obtained by interviews, we have designed super-items divided in several parts. Students are provided with extra information in every new part of the item, in order to help them if they have not been able to answer correctly before (Jaime, 1993).

This technique has proved to be useful, for instance, in items about proof, where it happens quite often that level 4 students cannot answer because they do not find a suitable way to the result. In ordinary items, with only a statement and a question, often these students do not write anything or they erase what they have written, since they believe they are wrong. This behaviour results in the assignation of the student to a Van Hiele level lower than their real one.

The structure of the super-items we have designed is the following:

- The first part of the item just state the problem and the question.
- When this part has been answered, students have to turn the page and they answer the next part of the item, that provides them with some more information and states again the same question. This may happen several times in complex problems.
- When answering these super-items, students are not allowed to go backwards after they have turned a page. That is, when they have answered a part of the item, they are not allowed to answer again a previous part.

In the annex we show, as examples of the notions introduced above, some items taken from the pool of items we have designed for this research. The table below, that refers to those items, summarizes the characteristics of each item, that is what levels and what key processes of each level are evaluated.

V.H. levels / Key proces.	Level 1		Level 2				Level 3				Level 4
Identification	1	2	1	2							
Definition Read				2			2	3			3
State	1	2	1	2	4		2		4		
Classification		2		2			2				
Proof						5		3		5	3

Annex: Examples of super-items.

Item 1

(The students are given several figures). For the following figures, write a T inside of the triangles and a Q inside of the quadrilaterals.

Explain how do you know which shapes are triangles and which are not. (The same question for quadrilaterals).

Write the numbers of the figures which are not triangles and explain, for each of them, why it is not a triangle. (The same question for quadrilaterals).

Item 2

2.1. (The students are given several figures). For each figure, write all the names in the following list that are appropriate for the figure: Square, rectangle, rhombus, parallelogram and rhomboid.

Explain the assignations that you have done for ... (the numbers of several figures are given, among which there should be at least a rectangle or a rhombus, and a square).

2.2. The students are given the inclusive definitions for rhombus and rectangle, and they are requested to use these definitions for answering to the same questions as in 2.1.

Item 3

Here are two definitions of certain polygons:

Definition A: It is a quadrilateral having two pairs of parallel sides.

Definition B: It is a quadrilateral in which the sum of any two consecutive angles is 180°.

Do A and B define the same quadrilaterals? Why, or why not? Give a proof for your answer.

Item 4

(The students are given a list of true properties for rhombi). Write a definition of rhombus down taking properties from the list. Remember that you are asked to write a definition down, so you have not to use more properties than needed.

Is it possible to solve again the same task but using a different set of properties from the list?

Item 5

5.1. Recall that a diagonal of a polygon is a segment joining two non adjacent vertices of the polygon. How many diagonals does an n-sided polygon have? Give a proof for your answer.

5.2. Complete the following statements (you can draw if you want):

In a 5-sided polygon, the number of diagonals which can be drawn from each vertex is . . . . . and the total number of diagonals of the polygon is . . . . .

In a 6-sided polygon, the number of diagonals which can be drawn from each vertex is . . . . . and the total number of diagonals of the polygon is . . . . .

In an n-sided polygon, the number of diagonals which can be drawn from each vertex is . . . . . Justify your answer to the last statement.

Using your answers above, tell how many diagonals an n-sided polygon has. Prove your answer.

References.

Burger, W.F.; Shaughnessy, J.M. (1986): Characterizing the van Hiele levels of development in geometry, *Journal for Research in Mathematics Education* vol. 17, pp. 31-48.

Crowley, M.L. (1987): The Van Hiele Model of the development of geometric thought, in *N.C.T.M. (1987): Learning and teaching geometry, K-12.* (N.C.T.M.: Reston, USA), pp. 1-16.

Crowley, M.L. (1990): Criterion-referenced reliability indices associated with the van Hiele Geometry Test, *Journal for Research in Mathematics Education* vol. 21 n° 3, pp. 238-241.

De Villiers, M.D. (1987): *Research evidence on hierarchical thinking, teaching strategies and the van Hiele theory: Some critical comments.* (Research Unit for Mathematics Education, Univ. of Stellenbosch: Stellenbosch, R. of South Africa).

Hoffer, A. (1981): Geometry is more than proof, *The Mathematics Teacher* vol. 74 n° 1, pp. 11-18.

Hoffer, A. (1983): Van Hiele-based research, in *Lesh, R.; Landau, M. (1983): Acquisition of mathematics concepts and processes.* (Academic Press: N. York, USA), pp. 205-227.

Jaime, A. (1993): *Aportaciones a la interpretación y aplicación del modelo de Van Hiele: La enseñanza de las isometrías del plano. La evaluación del nivel de razonamiento* (Doctoral dissertation). (Univ. de Valencia: Valencia, Spain).

Jaime, A.; Gutiérrez, A. (1990): Una propuesta de fundamentación para la enseñanza de la geometría: El Modelo de Van Hiele, in *Llinares, S.; Sánchez, M.V. (1990): Teoría y práctica en educación matemática.* (Alfar: Sevilla, Spain), pp. 295-384.

Shaughnessy, J.M.; Burger, W.F.; Gutiérrez, A.; Jaime, A.; Fuys, D. (1991): Analyzing and describing students' thinking in geometry: Continuity in the Van Hiele levels, in *Proceedings of the 13th Annual Meeting of the P.M.E.-N.A.* vol. 1, pp. 183-188.

Usiskin, Z. (1982): *Van Hiele levels and achievement in secondary school geometry.* (ERIC: Columbus, USA).

Wilson, M. (1990): Measuring a van Hiele geometry sequence: A reanalysis, *Journal for Research in Mathematics Education* vol. 21 n° 3, pp. 230-237.

A STUDENTS' BELIEF ABOUT THE SOLUTIONS OF THE EQUATION $X=X^{-1}$ IN A GROUP.

Orit Hazzan
Technion -- Israel Institute of Technology

Abstract

This report is a contribution to the ongoing international research on learning Abstract Algebra. This study addresses students knowledge and intuitions in considering the concepts of identity element and inverse element in a group. More specifically, in facing the question: Is it possible to have elements in a group, except the identity element, that are their own inverses? Some students believe that the answer to this question is negative. This phenomenon is demonstrated in a variety of contexts and possible explanations to it are suggested.

1. Introduction

This paper is concerned with students' understanding of group theory concepts: the concepts of the inverse element and the identity element. More specifically, the solutions of the equation $x=x^{-1}$, when x is an element in the group, are discussed. To be precise, I discuss the students' answers to the following questions: How many solutions may this equation have?, What are they?. It turns out that students tend to presume that the above equation has *only* one solution which is the identity element of the group, e. (I shall call this "the belief" or "the phenomenon".)

The above phenomenon appeared during a well spread research which was conducted at the Israel Institute of Technology, dealing with undergraduate students' understanding of concepts in Abstract Algebra, more specifically, group theory. The data was collected using questionnaires and interviews with students participating in the Abstract Algebra course. These two tools were not aimed to check the discussed belief. It just "popped out" in different contexts.

In this paper I would like to present some situations in which this phenomenon emerged. Also, I suggest possible description of students' mental processes which may explain this phenomenon. I also demonstrate how these mental processes might have influenced students' responses in other contexts.

2. Description of the phenomenon

In this section, five situations in which the above phenomenon emerged will be presented.

(A) One task in an interview was to build a "multiplication" table of four elements a, b, c and d so that it will form a table for a group. Most of the students began by choosing a to be the identity element. Next, they filled the first column and the first row. Then, the students determined the *pairs* of inverses. They applied their knowledge that the identity element is its own inverse and they chose another two elements, c and d for example, to be the inverse to each other. At this point, the students had to discuss the question: "What will be b's inverse?"

> **Ethan**: It is clear that $b*b$ should be different than e. (He marked a by e. Recall that e is the identity element.)
>
> **Interviewer**: Why?
>
> **Ethan**: Because otherwise b is its own inverse and this is only permitted for one element.

Later in the interview Ethan made a distinction between the following two situations: $b*b^{-1}=e$ and $b*b=e$.

> **Ethan**: So, $b*b^{-1}=e$ [...] This is by definition.
>
> **Interviewer**: O.K.
>
> **Ethan**: O.K. [...] So, if $b*b=e$, what does it mean logically?
> It implies that b is the identity. [He wrote: $b*b=e$ --> $b=e$]

In the first situation Ethan looked at b^{-1} as a variable - an element of the group that is the inverse of b. In the second situation, b must be its own inverse and "this is *only true* for the identity element".

(B) A problem on one of the questionnaires designed for another purpose, was:

> Let G be a commutative group. Prove that the set $\{a \in G \mid a^2=e\}$ is a subgroup of G.

Some students "saw" in the set only *one* element. One of them explained:

> $a^2 = e$ ==> $a \circ a = e$ ==> $a = a^{-1}$ and this is true for e . Hence every element in the subset will be e. I.e., $\{a \in G \mid a^2 = e\} = \{e\}$ and this is always a subgroup.

(This student chose to use the symbol $\circ$ to signify the group operation. This is equivalent to * used by other students.)

(C) In order to check what exactly leads the students to imply the above, I divided a group of 19 Abstract Algebra students into three sub-groups and each of these sub-groups received a different one-question questionnaire. Each sub-group discussed a different description of an element in a group which satisfies the equation: $x=x^{-1}$.

The three phrases were:

(i) Let G be a group and x be an element in G. Prove or refute:
If x is its own inverse then x is the identity element.

(ii) Let G be a group and x be an element in G. Prove or refute:
If x is of order 2 then x is the identity element.

(iii) Let G be a group, x be an element in G and e be the identity element of G. Prove or refute:
If $x*x=e$ then x is the identity element.

When the questions were so directed all the students except two refuted the phrases by a counter-example. One of these students, in interpreting phrase (iii) revealed his way of thinking as follows:

$$x \circ x = e$$
$$x^{-1} \circ x \circ x = x^{-1} \circ e$$
$$e \circ x = x^{-1} \circ e$$
$$x = x^{-1} \implies x=e$$

This answer and the one illustrated in example (B) are almost the same: The "belief" is that if an element is its own inverse then it is the identity element.

(D) The following comes to demonstrate that even when a student "feels" that an element of a group, other than the identity element, can satisfy the equation $x=x^{-1}$, he prefers not to deal with this situation. Adam begins to fill the 4x4 table by choosing a to be the identity element. Then he continues, explaining:

> **Adam**: O.K. now, we should try. So, let's begin and we will see if something will go wrong. Let's say that $b*b$ equal c. I will not put here a [for $b*b$] because it will definitely cause problems, since a is a special element. So let's put here c. [...]

After several calculations he goes on:

> **Adam**: Let's see if it [$b*b=a$] causes problems. No. It only means that b squared, $b*b$, O.K., b squared equals a. Meanwhile, I don't see that it contradicts anything. So, O.K. [...]. First, I will put here c anyway [for $b*b$] because I prefer not to deal with this problem. I have never seen it, ... I have no... Because I haven't seen it, I have no intuition, I prefer to choose here an element that will not cause problems, like c....

(E) Ron proves that the axiom: "For all a in G there exists an element b in G, so that $ab=ba=e$" holds for Z_n. He feels that he should pay special attention to an element which is its own inverse:

> An inverse exists because for all $a \in Z_n \subset Z$
> <u>there exists</u> $b \in Z_n \subset Z$ so that: $b=n-a$.
> <u>Remark</u>: It is possible that $b=a$.

3. Discussion

Here are some suggested explanations for the discussed phenomenon. For each explanation another situation which demonstrates this way of looking at things, is described.

(A) *Looking at the arbitrary multiplication of the group as multiplication of real numbers.* It is possible that students consider the "multiplication" of group elements as a sort of multiplication of the real numbers. The term "multiplication" in the case of groups is used metaphorically. Lecturers and books use to define a group in the following way:

> "Let G be a nonempty set together with a binary operation
> (usually called multiplication) ..." (Gallian, 1990, p. 34).

Because the group definition is axiomatic, the students need a model (a metaphor) to connect it with. Since they are familiar with the multiplication of real numbers, the real numbers seem to be the most available as a ready-made model. The first real number that students give as a solution to the equation $x=x^{-1}$ is 1, sometimes forgetting -1. They "generalize" the situation to a group: The equation $x=x^{-1}$ has only one solution - e - the identity element of the group.

This is not the only situation where students "borrow properties" from the multiplication of real numbers. For example, some students use the term "to divide" instead of "to multiply by the inverse element"; some students describe the inverse of an element x as "1 over x".

Therefore, it is possible that the limited view of "multiplication" is an outcome of the metaphor used while the subject is taught. Tom and Sarah expressed this idea explicitly.

> **Sarah:** The concept of multiplication is in the air all the time .

Tom admitted that his intuition was determined by familiar examples, which had disturbed him in the abstraction process:

Tom: Suddenly, everything [in Abstract Algebra] looks so strange. I mean, why isn't $a*b$ equal to $b*a$? [...] One tends to cancel [the same expression on both sides of an equation], it is the opposite to the basic rules of the arithmetic.

In my opinion, this is the main explanation to the phenomenon. It deals with the influence of familiar examples and metaphors on the understanding of mathematical concepts. With this point of view, the phenomenon can be view as an example of the nature of metaphors which "deny distinctions between things. Problems often arise from taking structural metaphors too literally. Because unexamined metaphors lead us to assume the identity of unidentical things, conflicts can arise which can be resolved only by understanding the metaphor". (Pimm, 1987, p. 108)

Herstein (1986) refers to this problem and after defining the concept of group, he adds: "The operation * in G is usually called the *product*, but keep in mind that this has nothing to do with product as we know it for the integers, rationals, reals, or complexes [...]. *However, a general group need have no relation whatsoever to a set of numbers*" (p. 47).

The influence of the real numbers multiplication demonstrates how a familiar system influences the understanding of mathematical concepts. Another system which the students freely relate to is that of the natural numbers. They relate to the latter system in a different way. Some students prefer to use a counting argument or theorem which deals with the natural numbers while solving a problem which does not have any connection to the natural numbers. For example: One can not prove that a subset of a group is a subgroup using Lagrange's Theorem (The order of a subgroup divides the order of the group). But, when asked whether Z_3 is a subgroup of Z_6, twenty out of 113 students answered "yes", supported by the following argument:

> "Yes, the statement is true. 3 divides 6 and hence, according to Lagrange's Theorem, Z_3 is a subgroup of Z_6."

(B) *The influence of a language.*

The discussed phenomenon can be a result of the communication between the lecturers and their students. Some of the students give to the term "multiplication" a different meaning than that of the lecturers.

The connection between language and mathematics has already been discussed by many writers (For example: Austin and Howson, 1979; Rin, 1983; Mason and Pimm, 1984; Pimm, 1987; Laborde, 1990). Here is an example from Abstract Algebra.

The formal definition of group homomorphism is:

$f: G \to G'$ is a group homomorphism if for all a,b in G: $f(ab) = f(a)f(b)$.

Writers and lecturers describe it as function "that *preserves the group operation..*" (Gallian, 1990, p. 100).

Sharon was asked if the following statement is true or false: If A and B are isomorphic groups and the operation of A is a function composition, then the operation in B is a function composition.

> **Sharon:** [...] It keeps this property... [pointing to the formula $f(ab) = f(a)f(b)$]. [...] My isomorphism leaves me with the operation that I performed in A, and when I transformed, I transformed the elements with the operation... The isomorphism does not change the operation.
>
> **Interviewer:** Hmmm...
>
> **Sharon:** I mean, if the operation in A is a function composition then my operation in B will be a function composition.

Then, she returned to a previous example in the interview:

> **Sharon:** If I take Z and 3Z, they are isomorphic... The elements are of the same kind but they are not the same elements [...] The elements are of the same type. If I had here remainder classes in Z, then this [3Z] would be remainder classes, too, which I multiply by three.
>
> **Interviewer:** Hmmm...
>
> **Sharon:** [...] They are of the same kind, and the operation is preserved, but they are not the same elements [...]

In Hebrew, the words **preserve** and **keep** sound very similar. Since Sharon did not know what it means to preserve the operation, she adopted the meaning of *keep*, to which she could attach more meaning.

(C) *The influence of an axiom.*

The axiom of the inverses in the definition of a group is: "For every $a \in G$ there exists an element $b \in G$ such that $a*b=b*a=e$. (We write this element b as a^{-1} and call it the inverse of a in G)." (Herstein, 1986, p. 46). Two *different* letters appear in this axiom, a and b. Therefore, some of the students may interpret the inverse of an element as different from itself. The real's influence seems to be stronger, so the students permit e to be its own inverse. The same phenomenon appears in checking the associativity (For all $a,b,c \in G$: $a*(b*c) = (a*b)*c$). Many students check the associativity only for triples with different letters. For example, when Ben checks whether a 3x3 table presented to him is a multiplication table for a group of three elements, he explains:

Ben: I already checked abc . Now, [I have to check] that $a(cb)$ will be equal to $(ac)b$, and $(ca)b=c(ab)$.

In another context, students were asked to find subgroups of Z_6. Sharon decided that the subset $\{0,1\}$ of the elements of Z_6 was a subgroup of Z_6, explaining:

Sharon: [...] Let's assume 0 and 1. It can form a subgroup of Z_6, because every operation that I will do with the two... with the two things here, It [the result] will be included here.

Interviewer: How will you check?

Sharon: If I do 0+1 I' ll get 1 which belongs to the subset.

Sharon checked the closure only for two different elements, maybe because two different letters usually appear in the closure axiom.

(D) *A relation between two objects.*

Another explanation for the students' belief that only the identity element can be its own inverse, is that it is easier to think of two *different* objects which satisfy a relation between them, than of one element which is in a relation with itself. It is easier to conceive that one element is the inverse of another, different element, than that it is the inverse of itself. The tendency to believe that it is more reasonable for two *different* objects to be in a relation between them, appears also in another context.

Students were asked to prove that a function f: A--->A is invertable if and only if it is one-to-one and onto. One student suggested:

f: A--> A (Let's write f: A--> B when A=B, for convenience).

In another example during a class discussion Dan tried to build a subgroup of S_4 of order two. In all his attempts he got $\{e, (1\ 2), (2\ 1)\}$. Here again he could not accept that (1 2) is its own inverse and felt that he should add to the subgroup the permutation (2 1) - a different notation for the same permutation.

(E) *Confusing between a theorem and its converse.*

The students know that if e is the identity element, then $e^2=e$. It may be that they assume that its converse is true: If x is an element which satisfies the equation $x^2=e$ then $x=e$.

This sort of confusion is also a common phenomenon. In the context of Abstract Algebra, it is demonstrated in Leron, Hazzan and Zazkis (in preparation): The students were asked to determine whether two groups, given by their multiplication tables, are isomorphic. Some of them began by calculating the order of each of the elements (a natural number operation), and decided that the two groups were isomorphic when finding that the orders of their elements were the same. It may be the case that some of the students assume that the *false* converse of the *true* statement, "If groups are isomorphic, the orders of their elements are equal respectively", is true.

4. Summary

The relative simplicity of the material which involves in the discussed phenomenon demonstrates very early in the Abstract Algebra course, some of the difficulties that students are facing. There is no need to go as far as advanced algebraic concepts to observe the difficulties. It is a difficult course for the students, and very abstract. One graduate student described the abstract-level of the material by the following remark: "Working with groups is similar in difficulty to learning relativity in physics. None of the normal modes of thought seem to fit."

5. References

Austin, J. L. and Howson, A.G. (1979). Language and Mathematical Education, *Educational Studies in Mathematics* **10**, pp. 161-197.

Gallian, J. S. (1990). *Contemporary Abstract Algebra,* D.C. Heath and Company.

Herstein, I. N (1986). *Abstract Algebra,* Macmillan Publishing Company.

Laborde, C. (1990). Language and Mathematics, *Mathematics and Cognition,* Cambridge University Press, pp. 53-69.

Leron, U., Hazzan, O. and Zazkis, R. (in preparation). Students' Conceptions and Misconceptions of Group Isomorphism.

Mason, J. and Pimm, D. (1984). Generic example: Seeing the general in the particular, *Educational Studies in Mathematics* 15, pp. 277-289.

Pimm, D. (1987). *Speaking Mathematically*, Routledge & Kegan Paul.

Rin, H. (1983). Linguistic Barriers to Students' Understanding of Definitions, *Proceedings of the Seventh International Conference for the Psychology of Mathematics Education,* Weizmann Institute of Science, Rehovot, Israel, pp. 295-300.

RELATIVE AND ABSOLUTE THINKING IN VISUAL ESTIMATION.

Rina Hershkowitz
Weizmann Institute of science
Rehovot, Israel

Zvia Markovits
Oranim school of Education
Tivon, Israel

Abstract: Four strategies were expressed by third grade children in visual estimation situations. After going through visual estimation activities, some changes in children strategies were found. These changes were in agreement with theories of visual information processing. Most of the children had difficulties in judging the quality of the estimate. They showed "additive" behavior without any traces of relative thinking. After a very directed intervention some of them moved towards relative thinking.

Visual estimation processes of discrete quantities combine both: Estimation processes and visual information processing. They contribute not only to develop visual abilities as well as estimation abilities, but they also helps to develop a basis for "feeling" quantities and numbers.

Visual estimation of discrete quantities occurs when one is presented with a large group of objects (e.g. Figure 1) for a short period of time, and is asked to evaluate their number.

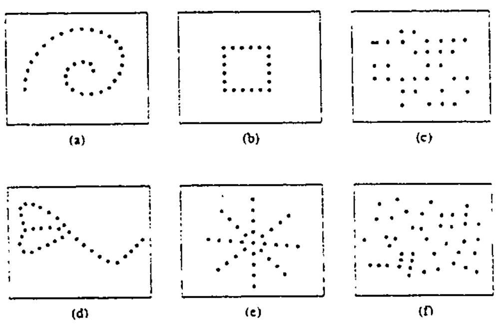


Figure 1

In our study we investigated the visual estimation processes of twelve nine year old children; they were presented with a relatively large number of objects. In such a situation the child is asked to give an **absolute information** about the number of objects shown to him/her.

According to Bryant (1974), human beings are born with the ability of gathering visual information by relative codes only, and not by absolute codes, meaning that children can say that one object is bigger (higher, etc.) than another object, but they cannot say how big it is. With experience, they learn to use external frames of reference and gradually they learn to deal with **absolute information.** In this sense, any previous "chunk of information", that children have, can act as a frame of reference.

One of the units developed as part of the project deals with visual estimation. The unit includes a set of 30 pictures in which the number of objects (mainly dots) varied from 10 to 40 and their arrangement varied as well (Fig. 1). The children were presented with each picture for a short period of time and then asked to write down how many objects they had seen. As immediate feedback, the teacher showed them the exact number of objects, and the children wrote it down next to their answers. After that, similar activities were performed with concrete objects, such as plastic circles, plastic squares, pencils; with pictures in children books and with objects from the classroom and outside the classroom environment.

The Research Questions

The following questions were the basis to our study, but they can serve as a basis for a research which deals with estimation processes of any kind:

a. How close is the estimate to the exact information?

b. What was the strategy or the process that led the subject to her/his estimate?

c. To what extent is the subject aware to the quality of the estimate; or in other words: How does he or she judges if the estimate is good enough, or if one estimate is better than the other?

In each of the above questions the effect of instruction (experience) was investigated.

The research was carried out by interviews, which were done before and after the learning unit and included tasks similar to the tasks of the visual estimation unit. The second part of each interview included tasks developed for the investigation of question c above.

Neither estimation strategies nor the awareness of the quality of the estimate (research questions **b** and **c**) were discussed in the unit.

In the following we will inform briefly on research results concerning questions **a** and **b** (for more detailed information see Markovits and Hershkowitz, 1993) and then focus more on research results for question **c**.

Estimation abilities and estimation processes (research questions a and b)

As for the estimation abilities we found that:

* Most of the answers were underestimates.
* Children were closer to the exact number when the dots were arranged in some geometrical pattern (Fig 1b and 1e).
* Overall, children's estimation improved as a result of the learning experience.
* Children are not used to estimate, they prefer to give exact answers.

The following levels of visual estimation strategies were found:

Counting - children counted as many objects as they could in the short period of time available.

Grouping - children mentally divided the objects into groups, most of the time with equal numbers of objects in each, and then multiplied this number by the number of groups. The size of the group is now used as an external frame of reference.

Comparison - children using this strategy compared the number of objects to something they were familiar with. They used familiar absolute information as an external frame of reference.

Global perception - Children just took a glance and gave their estimate.

In each of the first three strategies, frames of reference were used: In the counting strategy, the "counted quantity" was used as a frame of reference in estimating the remaining dots. In the grouping strategy the size of the group was used as a frame of reference, and in the comparison strategy some external but familiar information was used as a frame of reference.

The most dramatic change in strategy, as an impact of instruction (the unit), was a decrease in the use of the counting strategy , and an increase in the use of the global perception strategy. It seems that by gathering experience - going through the visual estimation unit - children can "educate their eye", meaning that they either become free of using a frame of reference, or a frame of reference is used subconsciously. This is in agreement with Bryant's theory.

It is worthwhile to note that in the explanation of their strategies, some children related to the mental images of the dot pictures, that they had created in their minds, rather than to the real pictures. This observation was found to be independent of the strategy used.

The above findings fit Piaget's theory that "the representation of space is not a perceptual "reading off" the spatial environment, but is the build-up from prior active manipulation of that environment" (Clements and Battista, 1992 p. 422).

Students' awareness of estimates - absolute and relative thinking

In order to investigate students' awareness of the quality of their estimates, a set of tasks was developed. The tasks for the first interview (before the learning experience), which are shown in Table 1, were developed according to the following criteria:

1. Absolute Error Tasks - in which two different estimates of the same quantity were presented. In this situation the errors need to be treated absolutely (tasks 1 and 2 , Table 1). Task 1 was included to make sure that the child understands what we mean by "better answer". Task 2 was added in order to see if the child realizes that an estimate can be an underestimate as well as an overestimate.

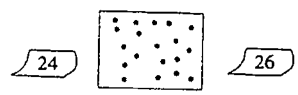
2. Relative Error Tasks - in which the two different estimates relate to different quantities. In this situation the errors need to be treated relatively (tasks 3 and 4). Task 3 is the classic task in which the absolute error is the same (for the two pictures), but the relative error is different. Task 4 is more difficult, since the smaller absolute error turns to be relatively bigger.

3. Visual Conflict Task - task 5 which was given in order to create visual conflict. In tasks 3 and 4 the difference between the number of dots in the two pictures, is not much in evidence - at least not visually. After interviewing four children we noticed that children do not realize that the errors should be treated relatively to the number of dots. We added task 5 in order to create a visual conflict. In this task we presented, the remaining eight children, two pictures in which the visual difference between the number of dots is much more in evidence.

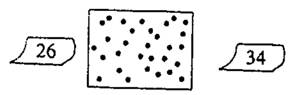
Table 1: *<u>Absolute and relative error tasks in the first interview</u>*

Absolute error tasks

1) *Two children were shown this dot picture for a short period of time and asked how many dots they saw. We know that there are 20 dots in the picture but the children, of course, didn't know it. Noa said that there are 24 dots, Amir said 26 dots. Did one of the children give a better answer than the other or were both answers equally good?*

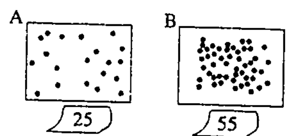


2) *A dot picture was shown to Noa and Amir.This time there were 30 dots. Amir said that there are 34 dots, Noa said 26. Same question as 1.*

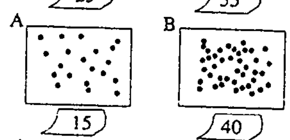


Relative error tasks

3) *Two dot pictures were shown to Noa: 20 dots in picture A and 50 dots in picture B. Noa said that*
there are 25 dots in picture A and 55 in picture B. Was one of Noa's answers better than the other one, or were both answers equally good?

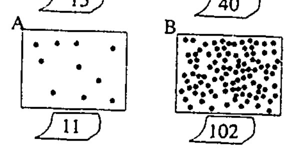


4) *Two dot pictures were shown to Amir: 10 dots in picture A and 30 dots in picture B. Amir said that there are 15 dots in picture A and 40 in B. Same question as in 3.*



Visual Conflict Tasks

5) *Two dot pictures were shown to Noa. 10 dots in picture A and 100 dots in picture B. Noa said that there are 11 dots in picture A and 102 dots in picture B. Same question as in 3.*

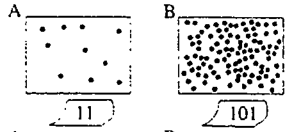


In the second interview (after the learning experience) we repeated tasks 1 - 5 with slight changes in numbers. The intended visual conflict in task 5 is meant to be aggravated in tasks 6 &7 (see Table 2). If the child answered task 5 correctly, the interview was stopped. If not, the child was presented with tasks 6 & 7. If the child did not succeed in either 6 or 7, then the interviewer intervened with task 9. Task 8 served as a check up task, and was presented to children that succeeded in one or both tasks 6 &7 and to children who showed some kind of relative thinking in task 9.

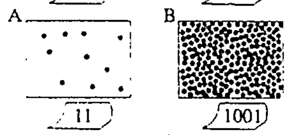
Table 2 . Absolute and relative tasks in the second interview

1 - 5 Similar to those in Table 1.

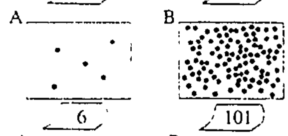
6) Same task as 5 except that Noa said 101 dots for picture A and 11 dots for picture B.



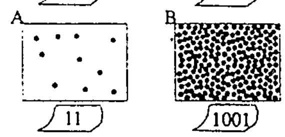
7) Two dot pictures were shown to Amir: 10 dots in picture A and 1000 dots in picture B. Amir said that there are 11 dots in picture A and 1001 dots in picture B. Same question as in 3.



8) Two dot pictures were shown to Noa. 5 dots in picture A and 100 dots in picture B. Noa said that there are 6 dots in picture a and 101 dots in picture B. Same question as in 3.



9) "There was a child in some other class who said that it is true that in both pictures the error was 1, but in picture A there are 10 dots and in picture B 1000 dots. It is much more difficult to say 11 when there are 10, that's why Amir's answer for picture B is better." That's what the child said. What do you think?



In both interviews each task was read to the child while the dot picture or both dot pictures were located in front of him/her. Then the estimates given by Noa or by Amir (see Tables 1 and 2) were located next to the picture while this information was read by the investigator.

From the analysis of the absolute and relative error tasks, it appears that eleven out of the twelve children showed mostly absolute thinking. Their answers were "additive", meaning that they related only to the absolute error, in tasks in which the relative error should have been taken into account. One girl, Dana, showed some traces of relative thinking, although this was not in evidence in all her answers. Thus we will first present the answers given by the other eleven children and then the answers that Dana gave.

<u>*Answers given by the eleven children in the first interview*</u>

Absolute error tasks: The children showed an additive behavior, which turned to be the correct one in these tasks.

In task 1, all children said that Noa's answer was better because *"she is closer"*.

In task 2, nine of the children said that both answers are the same because *"both were 4 dots away"*. One child said that Noa's answer is better, *"because the number is smaller, and the smaller the number, the better the answer"*. Another child said that Amir's answer is better because *"it is in* *the same number as 30"*, which we interpreted that he means that Amir's answer and the exact number, both have the same digit in the tens' place.

Relative error tasks: The children continued to think in terms of absolute differences, although the situation had been changed. Nine of them showed an additive behavior in task 3 and said that both answers are the same. Calculations like *"25-20=5, 55-50=5"*, were done with the conclusion that *"in both it is 5 more"*. In task 4 all eleven children said that Amir's answer in picture A is better *"because the error is only 5, and in picture B the error is 10"*.

Visual conflict task: Since task 5 was added during the interview, only seven out of the eleven children (and Dana) received it. All the seven children said something like: *"Picture A is better because it is closer to the exact number"*. In order to try and put them in a conflict situation, we asked where it is more difficult to get closer to the exact number, in picture A or in picture B. A typical answer was: *"When you have more points"*. But the following sentence was added immediately: *"But it doesn't matter if you have more points or less points, what matters is the difference."*

It seems that the children were not in a conflict at all. They realized that it is much more difficult to get closer to 100 dots, but it had nothing to do with the question they were asked. The mathematical fact, that the difference is smaller, was probably an obstacle for the children. One child seemed at first to change his answer, being influenced by the conflict situation, but then he "solved" the conflict in the following way: *"Maybe picture B is better because it is much more difficult. In principle, picture A is better and picture B is more difficult"*.

Answers given by the eleven children in the second interview

The second interview, which was conducted at the end of the visual estimation unit, did not show evidence of a real change. The answers that the children gave to tasks 1 - 7 (see Table 2) were very similar to those in the first interview. The children continued to be "additive" and looked at the difference between the estimate and the exact number of dots. They did not have any visual conflict in task 5, thus all of them were presented with tasks 6 and 7 in order to sharpen the conflict situation. We hoped that when the children will see two dot pictures, 10 dots in one and 1000 dots in the second, while in both Amir missed by only 1, something will happen. But the answers remained additive as before: *"They are both the same, because in both he missed by 1"*.

So we tried task 9 with all eleven children (see Table 2). For five out of the eleven, this didn't have any impact. One even got angry and answered *"It is the same, I already told you this a thousand times"*. As to the other six, they agreed that "the child from the other class" was right. So all of them received the check up task - task 8. Two of them adopted the following rule: *"whenever it is easier, the answer is better."* As to the remaining four, it seemed that they began to think in relative terms. One of them said: *"The child in the other class is right. It is a disgrace to be wrong in picture A. In picture B, I could do the same mistake"*. Another did not let us finish the story with "the child from the other class" (task 9), and immediately corrected his answer. All four, corrected their answers to task 7, answered correctly 8, and corrected their answers to the previous tasks as well. It seems that although the aggravated visual conflict was not enough to cause any of them to change their answers, four of them were affected by adding the "other child answer" (task 9). Those four children not only changed their answer to task 7, but also answered correctly the check up task (8), and went back to correct all their answers.

Dana Answers

Dana showed different thinking patterns. In task 1, in the first interview, Dana claimed that Noa's and Amir's answers are the same, explaining that *"Noa is more accurate, closer to 20. but Amir goofed only by 2 more, which is not so terrible in situation like this."* This answer, which is not only different from the other children's answers but also unexpected, suggests that Dana means that in estimation one does not have to be exact but has to be close enough. This answer is very interesting if we keep in mind that it was a very tough job for us to convince the children, during the visual estimation unit, that one doesn't have to be exact, and it is all right to give a close estimate.

Dana's answers for the relative tasks (3 and 4 - Table 1) were similar to those of the other. But Dana surprised us in task 5 (Table 1) even in the first interview saying that: *"One of the answers is better. In picture B she is better, even though she missed by 2, and in picture A she only missed by 1. 102 is better, because in picture A it is easier".* It seems that Dana's thinking was influenced by the visual conflict situation and she realized that the absolute error is not suitable to be used in this situation.

We continued to probe Dana and asked what would be her answer if Noa would say 11 in picture A, but 105 dots in picture B. Dana said that still she is better in picture B. We asked what if Noa would say 11 in picture A, but 110 dots in picture B. Dana: *"I am not sure, it is difficult because the numbers are different. One number is around 10 and the other around 100. When Noa said 105, it was still all right, but suddenly when she had 10 more dots it looked different, as if it was raised by 20 dots. That's the way I decided".*

Dana isn't explicitly using the idea of relative difference, but there is no doubt that implicitly this is what she is talking about.. Moreover, she has a "feeling" for the numbers involved. She says that 102 and 105 are "all right" for 100, compared to 11 for 10, but she is not sure that 110 is still "all right". It is not clear whether this has anything to do with her answer to the first task, in which she showed understanding of the idea that in estimation one does not have to be exact, but has to be reasonably close.

In the second interview Dana continued to stick to her idea that the answers in task 1 are the same, because one doesn't have to be exact but has to be close. As others she claimed that Amir's answer is better (task 2) because of the tens digit. This time she used the "relative idea" in task 3, saying that *"55 is better, because it is more difficult when you have more dots".* She came back to absolute thinking in task 4 saying that *"15 is better. A difference of 10 is a much more serious number, also 15 is O.K., because the 1 is as in 10".*

In task 5 Dana said that the estimate for picture B was better because *"it is so much more difficult when you have more dots"*. We continued and presented her with tasks 6 and 7 and in both she claimed that picture B is better for the same reason as in task 5.

It seems that Dana has difficulties when on one hand the difference between the exact number of dots is not very much in evidence, and on the other hand the errors are different and large. She was also distracted by the fact that 10 and 15 have the same digit in the tens' place, while 30 and 40 have different digits. This answer suggests that Dana is not calculating the relative error, she understands the idea intuitively and is successful in those situations, in which calculations are not critical, such as in task 5.

Concluding Remarks

It seems very difficult to cause children to move toward relative thinking, unless something very directed is done. The learning unit was helpful in changing children's strategies, although the strategy was not discussed at all in the unit. But the learning experience did not help them with relative thinking. The visual conflict which we thought will cause the change, did not help either. Only when we "put the cards on the table" using "the child from the other class" some of the children moved toward relative thinking.

We found similar difficulties with the idea of relative thinking, with older children, ages 12 - 15, given the same tasks.

References

Bryant, P., (1974). *Perception and Understanding in Young Children.* Basic Books .

Clements, D. H. and Battista, M. T., (1992). Geometry and spatial reasoning. In Grouws D.A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning*. Macmillan, pp. 420-464.

Hershkowitz, R. and Markovits, Z., (1992). Conquer mathematics concepts by developing visual thinking. *Arithmetic Teacher,* 39 , No. 9, pp. 38 - 41.

Markovits, Z. and Hershkowitz, R., (1993). Visual estimation of discrete quantities. *Zentralblatt fur Didaktik der Mathematik,* pp. 137 - 140.

Razel, M. and Eylon, B., (1990). Development of visual cognition: Transfer effects of the Agam Program . *Journal of Applied Developmental Psychology,* 11, pp. 459 - 485.

ON ONE PERSISTENT MISTAKE IN LINEAR ALGEBRA

Joel Hillel & Anna Sierpinska
Concordia University, Montréal

Abstract: The paper discusses one of the many difficulties that undergraduate students encounter in the linear algebra courses, namely a difficulty related to representing linear operators by matrices in different bases. Students fail to correctly read the values of the vectors of the basis from the matrix representation, very often interpreting the columns of the matrix as, in a way, "the images themselves" rather than as representations of these images in the given basis. We see the source of this persistent difficulty in a more general epistemological obstacle situated on the path from viewing language as part of the world to viewing language as a representation of the world.

Both the teaching and learning of linear algebra at the university level is almost universally regarded as a frustrating experience. Linear algebra is generally the first course that students encounter which is a full-fledged mathematical theory, built systematically from the ground up, with all its fuss about making all assumptions explicit, justifying statements by reference to definitions and already proved facts. Therefore, on one level, students' difficulties with linear algebra stem simply from their inexperience with proofs and proof-based theories. Indeed, Rogalski [1990] asked nearly 380 students in France as to what they found difficult with the subject and dealing with proofs was one of the five main responses.

The other important aspect of being a mathematical theory is its generality. Knowing linear algebra at this level demands that the student starts thinking about the objects and operations of algebra not in terms of relations between particular matrices, vectors and operators but in terms of whole structures of such things: vector spaces over fields, algebras, classes of linear operators, which can be transformed, represented in different ways, considered as isomorphic or not, etc. Referring to Piaget & Garcia's (1989) notion of intra-, inter- and trans-level of knowing something, we see that the level in which students need to operate is the trans-level. Most of our students had already successfully completed a "baby" linear algebra course in their pre-university studies but, as we have shown it elsewhere (Sierpinska, 1992), knowing this elementary course can be done at the inter-level of thinking about mathematical objects such as matrices and linear operators. This is another explanation of the fact that students have such difficulties with the university course and, indeed, in the Rogalski study cited above, statements that the subject is too abstract and that the general notions such as vector spaces, endomorphisms, bases, dimension and kernel are difficult, were also prominent responses.

Students' proof related difficulties, including not understanding the need for proofs nor the various proof techniques, not being able to deal with the often implicit quantifiers, confusing necessary and sufficient conditions, etc. are not generic to linear algebra but surface in most undergraduate courses (see Selden & Selden, [1987]). Among the sources of conceptual difficulties which are specific to linear algebra we would include:

1. The existence of several levels of description: there is a constant shuffling back and forth between the language of the general theory (vector spaces, subspaces, dimension, operators, kernels, etc.), the language of the more specific theory of R^n (n-tuples, matrices, rank, solutions of systems of equations, etc.), and finally the geometric language of R^2 and R^3 orthogonality, etc.). These three languages or levels of description co-exist, are often interchangeable but are certainly not equivalent.

2. Problems of representation: vectors and linear operators have representations which are basis dependent. To add to the potential confusion about representations, two representations of a vector can both be n-tuples with no notational devices relativizing them to a basis. Since this topic will be the main concern of our paper we will elaborate on this point in the next section.

3. Applicability of the general theory: many problems given students in a traditional linear algebra course can often be solved by direct manipulation techniques that do not require the tools of the general theory. This point was already discussed by Dorier [1991].

The problem of representation

In the paper we discuss one specific difficulty; namely the difficulty that students have in understanding the notion of representing a linear operator in a basis, and moving from one such representation to another. This notion becomes an elementary prerequisite for a subsequent topic of the canonical forms of linear operators (e.g. the Jordan and the Rational Canonical Forms). When one comes to study the canonical forms, a view of the language of linear algebra is needed which can be compared to that of a philologist, a theoretician of language: the language of linear algebra must be seen as a network of languages and rules of translation between them. However, the understanding of the operation of representing a concrete linear operator in different bases does not require such a high, "trans-"level of thinking. The inter-level is sufficient. Yet, as it appears, even though students are able to think about objects at the inter-level, some have problems with thinking of language at such a level.

The obstacle that stands between seeing the language at the intra-level and seeing it at the inter-level can be compared to one experienced by a person who has never spoken or even known any other language but his or her mother tongue and believes that the sounds, words and sentences of it naturally depict what is out there in the world: language is part of the world as all other things are. This person suddenly has to learn that there are other languages, as well, that use different sounds, words and sentences to speak about the same things, and that all languages are governed by grammars which can be very different. The difference between these two views of language evokes the one that has been described by Foucault (1973) as separating the *épistémè* of the

Renaissance from the *épistémè* of the Classical Age. Words and things belonged to the same world for the Renaissance man, nature had to be read through its "signs" in the same way as texts had to be interpreted. The meaning of anything could be read from the mark of similitude it bore to some other thing. The seventeenth century breaks with the resemblance as being the main source of meaning, and moves on to order and analysis for which language is an instrument. What gives signs their signifying function is knowledge; they are not anymore there, in the things, waiting to be recognized and uncovered.

The specificity of the problem of understanding the language of linear algebra as a representation rather than as being part of the world lies in the fact that the world of linear algebra is indeed the world of simultaneously used systems of representation. How does one make sure that two externally different representations indeed represent one and the same "thing"? Isn't it then often the case that one of the many possible representations is chosen as being *the thing*, or most *resembling* the thing, and all other representations need to be reported to the chosen one, proved as indeed representing the thing, equivalent, etc.? The choice is a matter of tradition, or of familiarity, first encounter, as in the case of decimal notation for representing numbers, or in the case of representing vectors by strings of numbers, e.g. [0,1,20,0,-5].

As students encounter vectors for the first time mainly in the context of concrete R^n spaces, strings of numbers become the primary representation or even *the thing*, and whatever is called a vector must bear some resemblance to this representation. However, when it comes to representing linear operators in different bases, the identification of vector with a string of numbers becomes very much shaken. One string of numbers represents different vectors depending on the choice of basis. One and the same vector can be represented by different strings of numbers. In someone, for whom vectors are recognized by the strings of numbers which they are supposed to resemble, this ambiguity may cause the whole conception of vector to fall into pieces. Can something that takes on so many forms have any real existence? Strings of numbers, so familiar, so palpable, suddenly feel like "ghosts of departed quantities".

If the old conception is not replaced by a new way of thinking about the language of linear algebra, one that would be closer to viewing it as a representation, students will tend to fall into the traps of "resemblance" and keep making always the same mistakes.

Identification of the problem

Over the past two years at Concordia University where we are teaching, we have been exploring the possibilities of bringing students closer to the "trans-level" of thinking by using various means like, on the one hand, a much closer follow-up of their individual understandings, and, on the other, didactical situations targeted towards overcoming the most persistent of students' difficulties. Some of these situations were based on computer lab activities involving the Computer Algebra System *Maple.*

In this section we shall be somewhat more explicit about the kind of mistakes that students are making relative to the question of matrix representation of linear operators in different bases. We shall provide some evidence that, for us, reveals the existence of a teaching and learning problem. An initial recognition of the problem led to an attempt at remediation (about which only very brief information will be given in this paper); however, students' achievement in this area did not increase significantly in spite of all the attention given to it in teaching.

For us, this means that we have to do here with a conceptual problem that is of a very fundamental type, indeed, with something that we have called an epistemological obstacle. Its overcoming may require a discussion with students on a meta-level of the mathematical problem they are dealing with: an open debate on the nature and status of the language in linear algebra. Passing to meta-level discussions with students in order to deepen their understanding of linear algebra has already been suggested by Dorier (1991).

Our data come from three groups of students: Group I: Fall 1992 Linear Algebra 1 class; Group II: Winter 1993 Linear Algebra 2 class, and Group III: Fall 1993 Linear Algebra II class. Groups I and II were not disjoint; Group III was disjoint with groups I and II.

In the first of the two linear algebra undergraduate courses, students are taught how to represent a linear mapping in a given basis, and the relation between two matrix representations of a linear operator. They learn, in the process, to find the coordinates of a vector in a given basis. However, some of them learn these procedures one way only: they are often unable to "undo" what they have done. For example, having found a matrix representation of a linear operator T in a basis, say, {v,u,w}, they have trouble reading, from this matrix, even the value of T(v). It is tempting, at this point, to use the Piagetian language and say that the activity of representing a linear mapping by a matrix in a basis has not been *internalised as an operation.* However, the persistence, as we shall see later, of mistakes or failure with this kind of problems points to the existence of an obstacle that is of a more conceptual nature and not just related to a difficulty in the operationalisation of a procedure.

One remark is appropriate here: we did not plan a research into this particular problem when we were embarking on our project related to linear algebra in September 1992. Therefore we were not setting tests or questionnaires geared towards collecting data on this subject. The first piece of evidence comes from a regular final examination, given to all sections, in the first course (December 1992). This test contained the following problem (for the sake of brevity we write matrices by rows):

Let L: $\mathbf{R}^2$ ---> $\mathbf{R}^3$ be a linear mapping defined by the matrix A = matrix([[7,-3],[2,1], [8, 0]]) in the bases {(2,1), (5,3)} for $\mathbf{R}^2$ and {(1,0,1), (1,1,0), (1,1,1)} for $\mathbf{R}^3$. (i) Show that L(2,1) = (17,10,15). What is L(5,3)? (ii) Express (1,0) and (0,1) as linear combinations of (2,1) and (5,3). (iii) Find the matrix which represents L with respect to the standard bases of $\mathbf{R}^2$ and $\mathbf{R}^3$.

We only have the data for Group I and not for the other students who wrote the test. Even though the 18 students in Group I generally outperformed the other classes on the final test (having been subject to special attention as part of the experimental teaching already), only 4 had correctly solved the whole problem; 8 students gave correct solution to the first question (this includes the four who got it all correct); 8 students solved only part (ii); 2 students did not answer this question. Thus we have about 44% of students who were able to read information from the matrix representation.

At the beginning of the following term (January 1993), in the second linear algebra course, a class (Group II) of 29 students, including 12 from Group I was given an "Activity" to be done in class, with possibility of consulting with neighbours. It included the following problem, quite similar to part(i) of the previous problem:

Let T: $\mathbf{R}^3$ ---> $\mathbf{R}^3$ be a linear operator represented by the matrix A = matrix([[1,2,3],[3,4,5],[6,7,8]]) relative to the basis {(1,1,1), (2,1,0), (0,5,6)} for $\mathbf{R}^3$. Find T(1,1,1)

There were only 3 correct answers to this problem, all given by students coming from the experimental first course. 17 students left the question unanswered. 3 students multiplied the matrix A by (1,1,1). One student took the first column of A as an answer: "T(1,1,1) = (1,3,6)" and another had the same approach though she hesitated between choosing the first column and the first row and wrote: "T(1,1,1) = (1,3,6) or (1,2,3)". One student found the formula for a coordinate vector in the given basis, obtained, by substitution, (1,0,0) as a coordinate vector for (1,1,1)and claimed that this is T1,1,1). Another embarked on the calculation of the coordinate vector in the given basis but did not finish. One performed th e dot product on vectors (1,1,1) and (1,3,6) obtaining 10 and claiming that this is the answer to the question. A last one obtained a matrix as an answer.

The mistake of taking the first column of the matrix as the image of the first basis vector under the operator seems to be a typical one: many students in Group III made the same kind of error on their class test in a similar problem. It stems from thinking that the columns of the matrix representation of an operator are, so to speak, images of the basis vectors, not *representations* of these images in this basis. The understanding of a vector here is based on its identification with a string of numbers. The problem is compounded by the fact that in the case of the canonical basis, this is a standard and correct procedure.

Attempts at Remediation

In the Fall 1993 term we conducted a second linear algebra course (Group III) accompanied by lab activities with *Maple*. Particular attention in class and through assigned activities was devoted to the question of matrix representations of linear operators. We again report on the results of several tests.

Test 1. This was the first class test, administered in the 8th week of classes, covering the eigenvalue problems, diagonalization and canonical forms. This test contained the following problem related to matrix representation:

Let T: $\mathbf{R}^3$ ---> $\mathbf{R}^3$ be a linear operator represented by the matrix A = matrix([[7,0,1],[0,2,0],[1,3,0]]) relative to the basis {(1,3,0), (0,1,3), (0,0,1)}. Find the matrix of T in the standard basis.

29 students wrote this test. There were 15 correct answers (52%). 10 answers (34%) contained a familiar lapsus: these students would find the coordinate vector formula in the given basis, find the coordinates of each vector e1=(1,0,0), e2=(0,1,0), e3=(0,0,1) in this basis, for example: e1=v1 - 3v2 + 9v3, and then they would go on to calculate the image of e1: T(e1) = T(v1) 3T(v2) + 9T(v3). So far so good, but at this point they would make the mistake of replacing T(v1), T(v2), and T(v3) by the corresponding columns of the matrix A. They thus behave as if the columns were representing T(v1), T(v2), T(v3) in the standard basis, but they probably do not think in terms of representation: rather, as we put it: the columns are *the* images of the basis vectors for them, not the *representations* of these images. The remaining 4 answers (14%) appeared nonsensical.

Test 2. Three weeks after this first test, other tests covering the same material were given for those who wished to improve their marks. During these three weeks, the last 10 minutes of each class were devoted to revision, but problems related explicitly to problems of finding a matrix representation in a different basis were not dealt with. Here is a sample from one such test:

Let T: $\mathbf{R}^3$ ---> $\mathbf{R}^3$ be a linear operator represented by the matrix A = matrix([[1,-1,0],[0,1,-2],[7,0,3]]) in a basis {e1, e2, e3}. Find the matrix of T in the basis {f1,f2,f3}, where f1=e1-e2, f2=e2+e3, f3=e1.

Curiously enough, out of 22 students who wrote this test, 17 answered correctly (77%), 5 got their answers wrong. The correct answers followed the pattern - here we use the kind of language that is often used in class to describe the various substitutions: e's were expressed in terms of f's by simple extraction from the given relations; T of f's were calculated by using linearity of T, representations of T of e's in e's read from the matrix (no mistake this time!), and expressing e's in terms of f's; for example: T(f1) = T(e1) - T(e2)= e1 + 7e3 = f3+7(f3-f1-f2) = -7f1-7f2+8f3. Then [-7,-7,8] was written as the first column of the required matrix.

The reason why the success rate at this problem which gave only a relation between the two bases was so much higher than in the problems of this type which use concrete vectors of the bases, is probably that in solving this one the student is alerted against committing the above mentioned lapsus by the abstract ,"non-string" representation of the vectors of the bases.

<u>The Final Test</u> (2 weeks after the end of classes). The problem, whose solution required representing a linear operator in a different basis, was the following:

Let $T: Q^4 \dashrightarrow Q^4$ be a linear operator given by the formula:
T([x,y,z,t]) = [-2x-y-2z-12t,x+2y+4z+6t,-2z-3t,z+t]. (a) Check that T has its Rational Canonical Form in the basis {[0,1,0,0],[-1,2,0,0],[4,-2,1,0],[-8,4,-2,1]}. (b) etc..

Out of 30 students writing this test, 18 gave correct answers (60%); 7 (23%) still committed the lapsus by calculating the values of the vectors in the given basis and writing these values as such as columns (or rows - in one case) of a matrix without representing them as combinations of the vectors of the given basis. Thus the mistake seems to be quite persistent.

The impact of *Maple* as perceived by students

Students using the *Maple* software were initially in a situation of "forced" collaboration because of the restricted number of computers available but, in time, most of them chose to work collaboratively. As has often been observed in other computer environments, this created a social context for the students to discuss and try to make sense of the confusing notions of linear algebra, particularly for those students who chose to work with *Maple* beyond what was minimally required for the course. But while we have closely observed and taped the lab sessions of six students in Group III, we are not yet in a position to ascertain the impact that *Maple* has had on students' understanding in general and on the problem of representation in particular. However, we can report about the students' own impressions and opinions about the usefulness of *Maple* in the learning of linear algebra.

Out of 26 students who responded to our questionnaire, 14 considered work with *Maple* as a factor enhancing their understanding, in that "*Maple* allowed to check if your understanding was correct" and "it reinforced understanding". Four of these 14 students also mentioned that *Maple* allowed to do more complex examples and do more examples. Sixteen respondents agreed with the statement: "I don't think using any kind of software will make you understand linear algebra by itself; you must understand something before you come to a computer". Eight mentioned that *Maple* was helpful in that it "allowed for quickly looking at different versions of the same problem (experimenting)". Only 13 students agreed with the statement: "I like computers". Three openly agreed with "I hate computers". As many as eight people agreed with the statement "doing those lengthy assignments on *Maple* was confusing".

Final remarks

The question of matrix representation of linear operators and translations from one representation to another must be recognized as difficult for the students. Yet, this concept is the most basic prerequisite to a good understanding of the problems of diagonalization and canonical forms. This understanding must be on the "trans-level": it requires that the student be able not only to find a matrix representation of a given linear operator in a given basis, but also to think about matrix representations of linear operators as objects of inquiry themselves, and about the general conditions under which a linear operator can have such and such particularly desirable representation. However, it seems that while with respect to linear algebra "objects" such as linear operators and matrices, students are at least on the inter-level of thinking about them, for quite a few students, their understanding of the language of linear algebra still lags behind on the intra-level: they, in a way, identify the world of linear algebra with its language and commit the lapsus described above. Viewing language as a representations is still not part of their "nature" even if they can rationally force their minds to work on that level.

References

Dorier, J.-L. (1991): Sur l'enseignement des concepts élémentaires d'algèbre linéaire à l'université. *Recherches en Didactique des Mathématiques 11*, 2.3, 325-364.

Foucault, M. (1973): *The order of things. An archeology of the human sciences.* New York: Vintage Books. (A translation of *Les mots et les choses*).

Piaget, J. & Garcia,R. (1989): *Pschogenesis and the history of science.* New York: Columbia University Press.

Rogalski, M. (1990): Pourquoi un tel échec de l'enseignement de l'algèbre linéaire? *Enseigner autrement les mathématiques en DEUG Première Année.* Commission inter-IREM université. IREM de Lille, 279-291.

Selden, A. & Selden, J. (1987): Errors and misconceptions in college level theorem proving. *Proceedings, Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics,* Cornell Univ. Vol.III, 456-471.

Sierpinska, A. (1992): The diachronic dimension in research on understanding in mathematics - usefulness and limitations of the concept of epistemological obstacle. Talk given at the conference: *Interaction between History of Mathematics and Mathematics learning*, Essen, November 2-5, 1992.

The social construction of classroom knowledge

Barbara Jaworski – University of Oxford, Dept of Educational Studies

This paper uses a case of classroom interaction between a teacher and students in a mathematics lesson to address the social construction of individual and classroom knowledge and issues this raises for the teaching of mathematics.

Background and rationale

This paper is based on a qualitative analysis of classroom transcript from one mathematics lesson. I have extracted part of my analysis of this lesson to emphasise the relationship, as I see it, between teacher and student views of the mathematics involved and the social dimension of the classroom. My observations here were part of a wider study of this teacher in which I judged him to operate from a radical constructivist perspective (Jaworski 1991). From this perspective, teacher and students can be regarded as individual meaning makers, and knowledge constructed as personal to each individual relative to their experience (eg von Glasersfeld 1990). However, a social constructivist view of learning suggests that classroom interaction leads to the development of some level of shared meanings (eg Ernest 1991).

Socio-cultural theorists view classroom learning as a form of enculturation into a community of practice (eg Lave and Wenger, 1991). For the mathematics classroom this suggests a community embracing mathematical processes and topics, ways of working and ways of being in the classroom. Enculturation is then a process where novices are drawn into the ways of the community. However, in parallel with viewing individuals as autonomous meaning-makers, social constructivists see the classroom environment as being constructed by its participants, rather than as perpetuating some pre-existing community of practice.

Here, I shall focus on the sharing of perceptions as individuals interact, and the modification of individual knowledge and understanding as a result. I see classroom interactions predicating a constructed learning environment in which there is a sense of 'common knowledge' through taken-as-shared meanings (along with, for example, Yackel et al, 1990; Voigt, 1992). Communication mainly takes place through language designed to articulate meaning. Whatever sharing takes place, all meanings are individual, but assumptions regarding common meanings are made and tested by participants. The teacher tries to create shared meanings while recognising each student's individuality, and in this issues for teaching arise. In the case described below, Jenny, Lesley and the teacher all have their own understandings and perceptions both of the mathematics being discussed and the learning environment in which the discussion takes place. My focus is on teaching issues arising from a study of the interaction – both in terms of participants' individual knowledge and understanding, and of the development of the learning environment[1].

A mathematics lesson

A class of 29 students, aged 15, were engaged in individual projects based on topics offered by the teacher, Ben, as a basis for GCSEcoursework[2]. Students selected a topic and

---

[1] There are also important research issues to do with the involvement of the researcher and the validity of analyses. These are dealt with in an extended version of this paper obtainable from the author.

[2] General Certificate of Secondary Education. This is a national examination, which all students can take at the age of 16, in a range of curriculum areas. In many areas, coursework is completed and assessed over a two year period prior to the final examination and contributes to the overall assessment.

developed it individually, according to their own questions and interest. I sat at a table with four girls, including Jenny and Lesley, observing their work and recording dialogue on audio tape. Jenny had chosen to work on *picture frames*, and was at the point of developing a general formula for the length of a frame, of unit width, which would surround a given rectangular picture of integer dimensions. She attracted the teacher's attention and he sat down next to her. The other girls were not directly involved in the conversation, although Lesley listened closely and made occasional remarks of some significance to my analysis. I identified 5 phases in the seven-minute, transcribed, interaction from which excerpts are quoted below.

Phase 1: The student's agenda

Jenny began by describing some of her thinking in finding a formula to express the length of frame. Her purpose for calling on the teacher was not clear. She may have been seeking his approval for the formula she had developed. She may have been seeking advice regarding where to go or what to do next. Ben's interjections here were mainly supportive, such as "OK", "Yes", "Yes", with occasional questions such as "What's your formula *for*?"

The teacher seemed to be trying to understand the student's thinking. Ben mainly *listened*, which was common at the beginning of his interactions with students more generally. After reading my transcript, he pointed out that Jenny's contributions were considerably longer than his own. He said, with apparent satisfaction, "She's talking more than I am." He had told me earlier that he wanted to encourage students to talk and to resist his own domination of the conversation and, potentially, the thinking.

Phase 2: Transition from student's agenda to teacher's agenda

The words above, '*to encourage students to talk and to resist his own domination*' imply a detachment on the part of the teacher in which it is possible to observe and control one's interaction while interacting (perhaps 'reflecting *in* action', Schön 1983). This detachment may also be considered a characteristic of Phase 2, beginning at line 18 of the transcript:

Ben ... I don't know where we're going – what happens if I have a 2 by 5?
Jen You add 2 and 2 and 5 and 5. And then I've explained here that the formula..
Ben Hang on, slow down, have you got a bit of paper – let's write down what you've just said – there's too many things going around for me to cope. // [3]
So if I say I've got a 2 by 5 –
Jen Then to try and find a formula for that I've put 2 + 2 + 5 + 5, and then I've as well –
Jen You see what I'm doing there is trying to find a formula for –
Ben How many should it be for 2 by 5?
Jen 2 by 5? That should be 18.
Ben What does that add up to?
Jen That's 4 and that's 10 and add another 4 for the corners – yes.
Ben So if I've got a 3 by 7, what –
Jen 24.
Ben Should be 24? How do you *do* it so *quickly*?
Jen [inaudible] a good brain!

[3] Transcript convention: / pause of approx 2 secs; // 2-5 secs; /// 5-10 secs.

Ben Come on, tell me how you did it quickly, because obviously you've got a method you're not writing down.

Jen Oh, well, no this is what, it's explained here – length of side plus length of side plus length of side plus length of side ...

Ben Yes.

34 Jen Plus 4 for the amount of corners, and the total length.

Ben's interjections here seem more focused than in Phase 1. For example, he drew Jenny's attention to a particular case, the 2 by 5 (line 18); then to '*how many*' (line 23), and then to the operation of adding up lengths of the components of the frame (line 25). Mathematically, he seemed to be exploring whether she could apply her thinking to *any* particular case. The questions seem both evaluative and directive. Here, Jenny would have had to be strongly determined to pursue some other line of thought to resist going with the questions. To this extent the teacher controlled what occurred. This is in contrast to the earlier period, where Jenny led the discussion. It seems from Jenny's replies that she has a rule for the length of a frame related to the sides of a rectangle, and possibly a formula which expresses this.

Of significance here is how the agendas of teacher and student are served in situations of unequal control and the teacher's greater power. The teacher needs to evaluate the student's mathematical progression for various reasons, but in doing so he can influence or subvert her thinking. On the other hand, if her focus is elsewhere and he appears not to address her agenda, how much influence *does* the teacher have?

At two points in this phase he tried overtly to influence her by emphasising learning strategies which he valued and wanted students to value. The first occurred when (at line 20) he said, "Hang on. Slow down!". I thought, as I observed the event, that he genuinely needed to jot down what she was saying in order to understand it, but this, apparently, was not so. It was, as he told me later, a deliberate strategy to emphasise the value of stopping to write things down when they were possibly too complex all to be held mentally. He wanted to exemplify this to Jenny. Similarly (at lines 29 and 31) where he talked of *doing the calculation so quickly,* he said that he wanted to emphasise to her the value of slowing down sometimes to consider what you are actually doing, to express it in words in order to make your methods clearer to yourself. Both of these remarks were about *ways of working* on a mathematical problem. In asking her to explain particular cases of her formula, he placed implicit emphasis on *mathematical processes*, and in refering to a *method* (line 31), he indicated an expectation that she would have some *general rule* in mind.

In this phase, there had been a move from the teacher's efforts to understand Jenny's thinking to his own educational objectives: firstly, through testing out his understanding of, and simulataneously evaluating, her mathematical thinking; and secondly in his use of strategies to develop the mathematical learning environment, and Jenny's understanding of it. Jenny seemed mainly to be involved in her own thinking regarding her formula, and it is hard to know what sense she made of the teacher's remarks.

Phase 3: Differing focuses – the complexity of the learning environment

In what happened next, Ben and Jenny seemed to be negotiate her mathematical formula and her intentions for it. Lesley appeared to interpret the teacher's words as offering advice.

35 Ben Oh, you've now confused me, cause I thought you were trying to shorten *that*!

Jen Nooo!

Ben 'cause I think there's an easier, shorter way.
Jen No, that's me trying to ...
Les That just means there *is* a shorter way, doesn't it.)[4]
40 Ben Justify it. Yes there *is* a shorter way.)
Les There's a shorter way to find it out.
Jen Yes – the four square ...
Ben No.
Jen No.
45 Les [aside] Pythagoras comes into everything.
Ben So you're just trying to explain how you came *to* it.
Jen Yes.
48 Jen Is that all right?

I am struck by the complexity of the interaction here and the seemingly different focuses of its participants. The teacher has seen 'a shorter way'. Jenny has not seen 'a shorter way' and is still focused on her own way. Lesley hears the teacher's reference to a shorter way and interprets this as advice from Ben to Jenny, which Jenny should follow. Lesley's aside about 'Pythagoras' was possibly made to another student as part of a separate conversation. The teacher and Jenny negotiate their positions, and Jenny then seeks the teacher's approval, perhaps sensing his change in focus from what she has done to what he thinks she could have done. How are the teacher's remarks affecting Jenny? Is Jenny starting to doubt her achievements so far? How does Lesley's articulation of her perceptions influence events?

The pace of events in the different phases of the interaction contributes strongly to my analysis. This is where a transcript is unhelpful. As well as being unable to encompass voice tones or emphasis on words, it is very difficult for it to hold a real sense of pausing, of people talking at the same time, or of speed of diction. Brief words and phrases at lines 36 and 37 take up as much time as lines 38 to 47 which go past very quickly. This pacing is crucial to the atmosphere prevailing, and so to the sense which an observer or participant makes of it.

Phase 4: A decision point for the teacher

It seems clear that the teacher had seen other possibilities in Jenny's work, but his first response was to approve what she had done already. Then he seemed unsure of how to proceed. I suggest that he was at a decision point in Line 51.[5]

49 Ben Yes, yes its OK.
50 Jenny Right.
Ben hm, /// I'm just wondering / I don't know, I just don't know whether it's worthwhile, to be honest. If you write it down, you won't lose any marks, but I'm not sure how many I'd have given.
Jen So what would you think would be best? To go into 3D cubes – find out, to put the formula to that – or do you think it would be best to investigate the formula more and try to find a shorter way?
Les I'd try to find a shorter way – a shorter way Jenny.)
Ben That's your choice. /)
Now, er, I'm going to give you a bit of maths you don't know.
55 Jenny Go on then.

---

[4] Transcript convention: brackets like this indicate people speaking at the same time.
[5] See Cooney 1988 and Jaworski 1991(page 290) for a discussion of decision points.

The teacher's *tone* of voice seems important here. At line 51, he started to speak more slowly and hesitantly as if no longer clear himself about how to proceed. He said, later, that he had no recollection at all of what he had been thinking or intending here - which is in complete contrast to the deliberate strategies mentioned above (recall lines 20, 29, 31). I believe he was totally immersed in his own thoughts, to do with the stage that Jenny had reached, to do with what help he might give her, and to do with what marks he would subsequently be able to award for her project. This total immersion excluded the possibility of his being able to account for and influence his own actions. I felt he was thinking aloud, recognizing something of a dilemma, and in consequence his remarks were not totally coherent. He was also unable to recall them later.

Some contextual information is necessary here. This was a lesson in which students were given time to work on GCSE coursework which Ben would eventually mark himself and submit for moderation. The regulations for marking and moderation require that what is presented should be the student's own work. Directly before the lesson, Ben had talked to me about the marking of such work, and of how any 'help' given to the student by the teacher might influence this marking. He had to give less credit for thinking in which he had been substantially involved. This created a dilemma for him. He felt that in talking and working with students, he unavoidably influenced their thinking. However, he also felt that this need not imply that the mathematics they ultimately presented was not their own. This dilemma seemed to be part of the teacher's thinking at line 51.

He later elaborated his own position at that point. In becoming acquainted with Jenny's work he could see a possibility of enhancing the quality of what she might ultimately present if he were to point out to her some algebraic strategies that were at her disposal – thus 'shortening' her expression of generality and making it algebraically more succinct. To do this he had to introduce her to the distributive law which she had not yet encountered. Should he do it?

Jenny's own thoughts during this time are not clear. I tried to ask her about them after the event but was unsuccessful in getting her to reflect on this period. However, Lesley's interjections seem very revealing of her perceptions. She interpreted the teacher's words as giving Jenny instructions which would be of value for Jenny's project (line 39, 41 and 53). She saw the teacher to be saying "You *should* 'shorten' what you have written." None of Ben's tentativity, and personal struggle with his dilemma seemed to be evident to her.

Phase 5: The teacher takes over

In response to Jenny's questions about possible alternatives Ben said, quietly, and almost as an aside "That's your choice." (line 53), then almost immediately, his decision made, he said "I'm going to give you a bit of maths you don't know" (line 54). The next part of the interaction involved Ben explaining to Jenny the distributive law. However, he made a revealing aside to me first (line 56). This was a reference to the dilemma, which indicated his awareness of the issue and his overt decision to involve himself in Jenny's mathematics.

56 Ben [To me] This is what we were talking about isn't it? [laughs]
[To the students] We were talking about this, you know, how much help am I allowed to give people who do coursework. Yes?
[To Jenny] Right, now, there's a thing called the Distributive Law – yes? We've not met the Distributive Law have we?

With the decisively uttered word "Right!" (to Jenny, above), it was as if having decided on a course of action, he was in personal control again and ready to get on with it. He launched into exposition. His tone now was almost avuncular, certainly more assertive than it had been in earlier phases, which might be a feature of teacher exposition. However, even in this expository phase he did not totally dominate the conversation. There were plenty of pauses for the students to think and comment. He dominated the thinking, but Jenny (and Lesley too) responded to what he offered and appeared to make sense of it. (Transcript of this appears in the fuller version of this paper.)

This phase ended (at line 73, below) on a different note. Ben said, "So that might be useful to you.", followed by a rather deprecating laugh. It is as if he was now saying - There you are! Take it or leave it! Perhaps he did not wish Jenny to feel constrained to use what he had offered. Without his contribution she could not have used it, hence his decision to offer, but he seemed to say that there should be no imperative on her part to make use of it. However, did she in practice have this option?

73	Ben	So that might be useful to you. /// [laugh]
	Les	That means you've got to do something with that in your answers [Ben laughs]
75	Jen	But it doesn't have to be that does it? Hm, what about –
	Les	He's just telling you that you see, but / that's the answer.

Lesley again took part in this interaction. During the exposition she had participated in the mathematical argument. However, at lines 74 and 76 her statements are again interpretive of the teacher's intention in his talk with Jenny. She suggests this is advice which Jenny should follow. She may have been playing a provocative role, and perhaps the teacher's laugh indicates that he does not take it seriously.

The learning environment

In speaking of the learning environment in this classroom, it is important to remember that my own knowledge and understanding derives not just from this lesson but from more than six months of studying this teacher and class. During this time I observed Ben deliberately fostering ways of working – for example, students were urged to *listen* when one person was speaking, space was created for contributions from the quieter members of the class, *all* contributions were valued. *Pattern-seeking* and *conjecturing* were part of classroom language, used by the students but deriving originally from the teacher. Cooperative working was openly encouraged. It is possible to view this learning environment's existence as a form of classroom knowledge shared by classroom participants.

This raises questions about the status of this knowledge and its relationship to individual understanding, particularly from a radical constructivist perspective. The ways of working fostered by the teacher encouraged communication through articulation of individual perceptions, including those of the teacher. It seems clear (a) that all understandings of these ways of working were particular to individual participants; (b) that through discussion and negotiation, individuals saw their understandings as similar to those of others in the group. It is on this second point that any notions of common knowledge rest. As an observer, I perceived aspects of the learning environment which seemed to be in common currency. The teacher acted as if there were common understandings whilst himself recognizing the individuality of knowledge. He said on one occasion, "I can't share my mathematical model because that's special to me ... because of my experiences. So, I suppose I'm not a giver of knowledge because I like to let people fit their knowledge into their model, because only then does it make sense to them."

In Phase 2, I talked of the teacher's emphasis on his own particular learning objectives – ways of developing mathematical understanding and use of mathematical processes, for example. At this stage his communication of these objectives was indirect, or by example. In Phase 3, he recognized the value of Jenny presenting her formula in a more concise algebraic form, but was unsure how to communicate this. Here, there was an added complication due to the rules of GCSE coursework, where a consequence of his help might have been to add to the quality of what was presented, but to make it impossible to regard this as Jenny's unaided work. However, his articulation of the possibility of Jenny's *shortening* her expression was seized by Lesley as a cue from the teacher on which Jenny should act. Lesley's reaction to this may have sprung from classroom culture in which the teacher is seen as a repository of knowledge and advice. Jenny herself sought the teacher's advice, although she seemed quite confident in her formula. Should she extend her current work to three dimensions, or would it be better to take up the teacher's suggestion of shortening her formula for two dimensions? The teacher seemed so bound up in his own decision here that he possibly did not consider how *his* choice, to offer the distributive rule, might be of supreme influence in Jenny's choice.

The teacher's language as he decided to offer the distributive rule is interesting, in contrast to his statement quoted above, "I'm not a giver of knowledge". He said, in Phase 4, "I'm going to give you a bit of maths you don't know". Much has been written of utterances of this sort (eg Kilpatrick (1987), Davis & Mason (1989), von Glasersfeld (1990)) We might argue that such words are rarely carefully designed to be consistent with a particular philosophical position, so that, in this case, their utterance does not refute the speaker's position as a radical constructivist. However, classroom discourse is built on such utterances, meanings are construed, and ways of working developed. It is possible, as a result of the *giving* of the distributive rule that the girls saw this rule as some unnegotiable piece of mathematical knowledge, in contrast to the formula which Jenny herself had developed. Was there any possibility of Jenny reciprocally seeing herself as giving the teacher knowledge in offering him her formula? Or did the respective power positions obviate this?

The differential power positions are clearly instrumental in classroom relationships. More than most, I felt, this teacher went out of his way to encourage students to collaborate in classroom mathematics rather than to be mere recipients of it. In the case here, Jenny's responses to the teacher indicated her own participation in the mathematics, and in the discourse, maybe not as an equal of the teacher, but certainly as a valued contributor. At line 52, although seeking his advice, she brings her own ideas into the discourse, and at line 55, her "Go on then" is as if she gives the teacher permission to expound.

We might argue that the unequal power relation is not only inevitable, but also a healthy part of the classroom learning environment. A democratic classroom may be a sought-after ideal, but an anarchic classroom something to avoid. The teacher as experienced practitioner in a leadership role is necessary to classroom culture. It is in the interpretation of this role that issues for teaching arise. If the interpretation is one of student enculturation into a community of practice, teaching decisions may look different from those based on fostering of individual, negotiated (or taken-as-shared) meanings. In the mathematics classroom, classroom culture includes *mathematical* culture. If mathematics is seen as an accumulation of rules like the distributive rule, given by the teacher, the culture is likely to be perceived differently than if students arrive at their own rules through exploration and discussion.

Just to talk of classroom culture is to suggest a belief in forms of knowledge beyond the individual. In order to act in the classroom, and not be consumed in an "infinite regress of constructions upon constructions for each pupil in the class" (Lerman, 1993) the teacher needs to assume some level of intersubjectivity with pupils. However, teaching dilemmas arise from the uncertainties inherent in such assumptions. I have talked elsewhere (eg Jaworski, 1991) about the tensions which arise for a teacher in making decisions about classroom approaches to mathematical concepts. In the case above, what sense is Jenny going to make of the distributive law? How is she going to use it in relation to her project? How might the teacher's offering of it interfere with her own mathematical development and construal? What seems important is that none of this happens in a vacuum. Jenny is not a 'lone organism pitted against nature' (Bruner, 1985), but a member of a discoursive community in which her own meanings develop as a result of interactions and negotiation.

The study of such discourse seems fundamental to gaining insight into relationships between mathematical learning and teaching. However, there are considerable research issues involved, not least power relationships between researcher and researched, and the means which researchers use to elicit knowledge. Where research is seen as an interpretive process, the reflexive nature of such research seems fundamental to its validity. We need to explore development of research methods alongside our reconciliation of philosophical positions in seeking to understand the complexity of the classroom environment.

References

Cooney, T.J. (1988) 'Teachers' decision making', in D. Pimm (Ed.) *Mathematics Teachers and Children.* London: Hodder and Stoughton

Bruner, J.S. (1985) 'Vygotsky: A historical and conceptual perspective', in J.V. Wertsch (Ed.) Culture, Communication and Cognition. Cambridge: Cambridge University Press

Davis, P. J. and Mason, J. H. (1989) 'Notes on a radical constructivist epistemology applied to didactic situations', in *Journal of structured learning.* Vol 10, 157-176

Ernest, P. (1991) *The Philosophy of Mathematics Education.* London. Falmer Press

Jaworski, B. (1991) *Interpretations of a constructivist philosophy in mathematics teaching.* Unpublished PhD Thesis. Milton Keynes: Open University

Kilpatrick, J. (1987) 'What constructivism might be in mathematics education', in *PME XI Proceedings*, Montreal.

Lave, J. and Wenger, E. (1991) *Situated learning: Legitimate peripheral participation.* New York: Cambridge University Press

Lerman, S (1993) *The 'problem' of intersubjectivity in mathematics learning: Extension or rejection of the constructividst paradigm.?* Unpublished paper.

Schön, D. (1983) *The reflective practitioner.* London: Temple Smith

Voigt, J. (1992) 'Negotiation of mathematical meaning in classroom processes: Social interaction and learning mathematics', Paper presented to the *Theory Working Group;* ICME 7, Quebec.

von Glasersfeld, E. (1990) 'An exposition of constructivism: Why some like it radical', in Davis R. et al (Eds) *Constructivist views on the teaching and learning of mathematics.* JRME Monograph no. 4 Reston, Virginia: NCTM

Yackel, E. Cobb, P. Wood, T. Wheatly, G. & Merkel, G. (1990) 'The importance of social interaction in children's construction of mathematical knowledge'. In Cooney, T.J. (Ed.) *Teaching and Learning mathematics in the 1990s.* Reston, Virginia: NCTM

The Roles of Measurement in Proof Problems

- Analysis of Students' Activities in Geometric Computer Environment-

Kyoko Kakihana*, Katsuhiko Shimizu**
Nobuhiko Nohda***, Yoshinori Higasibara***, Kazuhiko Nakayma***
*Tokyo Kasei Gakuin Junior College
**National Institute for Educational Research
***University of Tsukuba

Abstract

The purpose of this research is to investigate activities middle school students who engaged in proof problems in geometric computer environment. For this purpose, two classes are reported in this paper: the class had just started to use the Cabri-Geometry[(1)] and the class had used the software for one month Students' activities were videotaped and their worksheets were collected and some students were interviewed for analysis.

Students in the former class utilized the result of the measurement for proving a statement. On the other hand, the students in the latter class claimed, in interview, that measurement was not sufficient to prove a statement even though they understand its validity from computer measurement. These results showed that the measurement in computer environment played an substantial role for student to understand the validity and that students who were accustomed with geometric software considered the measurement as an demonstration.

I Introduction

Environments where students can investigate properties of figures and theories of geometry by using softwares are emerging very rapidly in the field of mathematics education. Analysis of students' activities and teaching processes are necessary to develop teaching materials, because a tool: geometric software such as *Cabri-Geometry*, could change the nature of learning of geometry.

The constructions in *Cabri-Geometry* is consistent with the construction by ruler, compasses and a protractor. *Cabri-Geometry* also provides the powerful environment that students can move points, figure by holding its geometric properties.

---

(1) Cabri-Geometry was developed by Yves Baulac, Franck Bellemain and Jean M. Laborde at the LSD(IMAG), University Joseph Fourier, Grenoble, 1988. A Japanese version is available for NEC machine.

Moving and measuring figures are effective when students are investigating and discovering properties of figures and relations between angles and sides in figures by themselves (Kakihana,1991).

Using a geometric software helps to "overcome diagram-related learning obstacles that traditionally beset students" (Yerushalmy, M & Chanzan, D, 1993, p53). Yerushalmy(1990) says there are three kinds of obstacles in static diagrams on a paper/blackboard. By using *Gabri-Geometry*, students also overcome these three obstacles as same as by using Supposer(Kakihana,1991).

Is it effective to use a software such as *Gabri-Geometry* when students prove a statement in a proof problem? This question is very important for Japanese mathematics curriculum. In Japanese mathematics curriculum, "syomei (Japanese word corresponding to mathematical proof)" defined as written statements which describes "ronsyo (Japanese word corresponding to demonstration)". "Ronsyo" is defined as an logical sequences from proved statement to a new statement (Hirabayashi,1991). But, from many Japanese surveys, it was very difficult for junior high school students to write "syomei" and to do "ronsyo". A computer geometric software is expected to provide an effective environment to assist students' "syomei" and "ronsyo".

The purpose of this research is to investigate activities of middle school students who engaged in proof problems in geometric computer environment and from these results some advise will be obtained to teach geometrical class by using a software such as *Cabri-geometry.*

II Method

Two classes are reported in this paper: the class had just started to use the software and the class had used the *Cabri-Geometry* for one month. Students' activities were videotaped and their worksheets were collected and some students were interviewed for analysis.

First Case: Class A

Learning stage	Just after learning about operations of *Gabri-Geometry*
Subjects	2nd grade in a junior high school in Ibaraki prefecture (32 students)
Problem	Problem 1 (to apply the midpoint theorem) (Fig.1)
Figure	given
Environment	1 student/1 computer
Activity	50 minutes for writing a worksheet

Second Case: Class B

Learning stage	after using *Gabri-Geometry* in several classes
Subjects	2nd grade in a junior high school in Nagano prefecture(35 students)
Problem	Problem 2 (to apply congruence) (Fig.3)
Figure	given
Environment	2 students/1 computer
Activity	25 minutes for investigating cases of this problem 25 minutes for writing a worksheet

Two problems are taken. Both problems are very popular in the Japanese geometrical course. Problem 1 is given to class A on a worksheet (Fig.1). It is a problem to apply the midpoint theorem. Fig.2 is a typical figure of problem 1

In the following figure points M,N,P,Q are a midpoint of sides AB,AC,DB,DC respectively. What kind of shape is quadrangular ◇MNPQ ?

Name of

Shomei

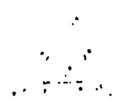


Fig. 1 : Worksheet of problem 1

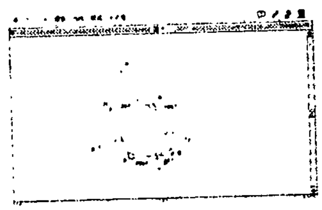


Fig. 2 : One of the figures of problem 1

In the following figures, Both △ABC and △APQ are a equilateral triangles, line segments PQ,QC are equal.

Please write down the reasons why they are equal.

You can start from any figure.

You do not need to write a proof in detail

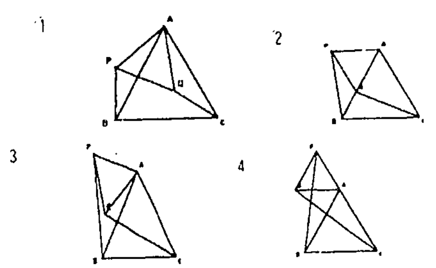


Fig. 3 : Worksheet for problem 2

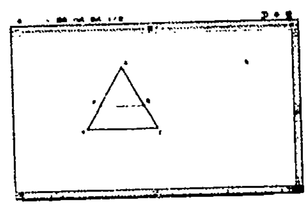


Fig.4 : Given screen for problem 2

which is given in a textbook. Problem 2 is given to class B on a worksheet after students investigate the various cases of figures in a problem depending on the location of a point P(Fig.3). It is a problem to apply the basic properties of congruence. Fig.4 is given on the screen at first. You can move a point P. Usually each cases of problem 2 are treated separately in a textbook.

These problems were chosen by the classroom teacher to fit into usual mathematics lesson. Both problems are to apply the theorem or property to prove a statement in a problem.

III Result

At first all students in both classes measured sides/angles and moved points. They observed the result of measurements to find if there were sides or angles which held the same measurement in movement. In class A students found the properties of quadrangular ◇MNPQ. In Class B students found which triangles are congruent.

Fig. 5 is one of the figures which students measured and moved point D in Problem1 and Fig. 6 is one of the figures which students measured and moved point P in Problem 2.

Students in Class A thought that measurements always showed that the quadrangular ◇MNPQ was a parallelogram even though a point was moved in any ways. They wrote in their worksheet that they thought 'syomei' of the problem was validated and demonstrated by measurements.

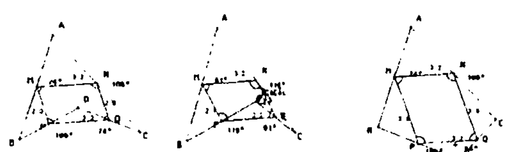


Fig.5 : Measuring and moving in problem 1

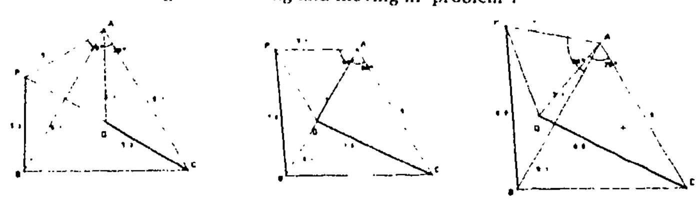


Fig.6 : Measuring and moving in problem 2

Fig. 7 shows the results of analysis of worksheets of Class A. Seventeen students was able to write some sort of "syomei". Sixteen students wrote "syomei" by using the result of measurement (Fig.8) and only one student wrote a mathematical proof (real "syomei") (Fig.9) which is very similar to the proof in a textbook (Fig.10).

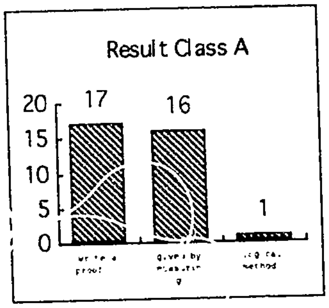


Fig. 7 : Result of worksheets of Class A

MP=NQ=2.0, MN=PQ=3.2
Two sides are equal because of measurements

Fig.8 : Given by measuring

MN=1/2BC, MN//BC (from midpoint theorem)
PQ=1/2BC, PQ//BC (from midpoint theorem)
Then MN=PQ MN//PQ

Fig.9 : Written by logical method

M,N are midpoint of AB,AC
from midpoint theorem
MN=1/2BC, MN//BC
P,Q are midpoint of BD,DC
from midpoint theorem
PQ=1/2BC, PQ//BC
Then MN=PQ MN//PQ
therefore the quadrangular
◇MNPQ is a parallelogram

Fig.10 : proof in a textbook

Fig. 11 shows the results of analysis of worksheets of Class B. Thirty-one students wrote some sort of "syomei". Twenty-six students wrote "syomei" like Fig. 12 for problem 2. In this class, students measured PB and QC. And they knew these were equal. But no student says PB=CQ is true even though the measurements of PB and CQ are always the same. They think the measurements is not sufficient to do "syomei" PQ=CQ. Most of students wrote that triangles were congruence because

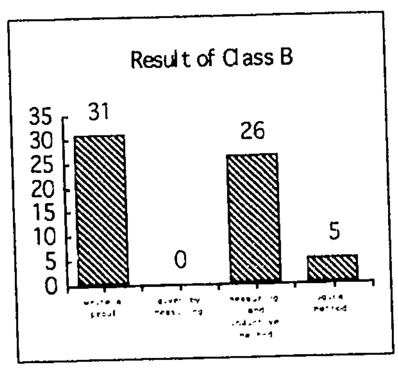


Fig. 11 Results from the worksheets of Class B

AP=AQ, AB=AC, ∠PAB=∠QAC. The reason why they wrote the congruence of triangle was the result of measurement. They especially measured ∠PAB, ∠QAC.

Five students wrote a proof like Fig.13 which is very similar to a proof in a textbook (Fig.14)

AP=AQ
AB=AC
∠PAB=∠QAC
two sides and the angle between them are equal
therefore triangles are congruence
then PB=QC

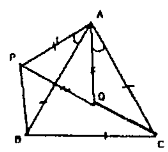


Fig.12 : Measuring and checking the marks

AP=AQ
AB=AC
∠PAB=∠QAC
both angles are 60 - same angle
two sides and the angle between them are equal
therefore triangles are congruence
then PB=QC

Fig.13 : Written by Logical method

In △APB and △AQC
Both △ABC and △APQ are equilateral triangles
so AP=AQ
AB=AC
then ∠PAB=60-∠BAQ
∠QAC=60-∠BAQ
therefore ∠PAB=∠QAC
two sides and the angle between them are equal
therefore △APB ≡ △AQC
then PB=QC

Fig.14 : proof of .1 in a text book

<<Results of Interviews>>

Results of interviews were as following.

Twelve students were interviewed about proof and measurement.

To measure many sides and angles 8
Not to measure so much 4

T: When you measure and move a point what do you know?

Nothing 2
the figures whose shapes are always the same or the side/angle which have the same measurements 5
many cases are showed from the problem 2

T: Do you think you don't need to prove a statement when you see line segments are the same by measurement?

Need proof 11
not need proof 1

T: Why do you need a proof?.

to be sure	6
Teacher says	9
measurement is not sufficient to say a statement is true	4

Two cases of interview are listed as following.

K(girl) low level in geometry. She doesn't like proof problems.
She measures almost all sides and angles.

T: When you measure and move a figure what do you know?
S: The red segments has always the same measurement
T: Do you think you don't need to proof a statement when you see line segments are the same by measurement?
S: I am thinking it is not needed. But,on a worksheet it is said you have to write down a proof . And teacher says so . I wonder.......
T: Are there any merit to use CABRI in a problem?
S: It helps to understand the problem because points are moved and lines are colored..

M(boy) High level in geometry . He likes proof problems.
He wrote a proof for four cases.
He doesn't use measurement so much.

T: When you measure and move a figure what do you know?
S: There are two triangles of the same shape.
T: Why don't you use measurement?
S: Triangles are equilateral triangles. Therefore I don't need to measure. And I don't believe in measurement so much. (On his worksheet Fig. 9 is written. He measured ∠PAB, ∠QAC on his screen)
T: Are there any merit to use CABRI in a problem?
S: It is more understandable than on a paper.

These interviews showes that most students think as following .

(1) "Shomei" is difficult
(2) The results of measurements are not sufficient to do "shomei' because the teacher says.
(3) A computer software helps for them to do "shomei" so much.

IV Conclusion

In this research, Students who used *Cabri-Geometry* several times thought the results of measurement were not sufficient to do "shomei" a statement. But, can I say the 'syomei(proof)' which was written in problem 2 is done by not using measurements? Where is the different from the proof of problem 1 in which the measurements was used directly to say the quadrangular ◇MNPQ is a parallelogram? They considered the measurement as a demonstration.

This research showed the results of measurement were based on understanding of doing "shomei" for students. By using a computer software students are able to measure sides/angles easily and to check the validity of many cases of the problem. It becomes more difficult for school teachers to persuade students that a proof based on measurements is not correct.

Kunimune(1987) reported that 91% students in a second grade and 77% in third grade on junior high school think 'ronsyo' is enough by empirical measurements to be able to explain properties of figures even in the environment without computer.

Hogan(1993) says that there are many mathematicians who think proof has been dead because of computer's developement. Most teachers who attend the seminar titled "Is 'proof' old-fashioned in a high school geometry?" say that most students who belong to a generation of Nintendo/joy stick/MTV think 'proof' is not needed and it is not important for them (Hogan, ibid).

In these days when computers are emerging very rapidly in the field of mathematic education, the teaching of "shomei" in geometric would be expected to change from the teaching of logical proof to the teaching of the understanding of "shomei" based on measurements in geometric computer environment.

Acknowledgments

This research was partly supported by the Science Research Foundation No.05808018 of The Ministry of Education in Japan.

References

Hirabayashi, I (1991) ed : *New Mathematic Guide for Middle School Vol. 3 Geometry*, Kaneko Shobo, 18-19

Hogan, J(1993) : The death of proof, *Scientific American Oct.*, interpreted by Yamazaki Y et, *Nikei science Dec*, 108-119

Kakihana, K (1991) : The effects of Using 'Cabri-Geometry'-The new Learning Environment of Plane Geometry : 'Geo-World', Unpublished Master thesis , University of Tsukuba Japan

Kunimune, S(1987) : The Study on Development of Understanding about "The Significance of Demonstrations in Learning Geometrical Figures" *Jounal of Japan Society of Mathematic Education Vol47*, 3-21

Yerushalmy, M & Chanzan, D(1993) : Overcomming visual obstacle, *THE GEOMETRIC SUPPOSER What Is It a Case of ?*, LEA, 53

Yerushalmy, M & Chanzan, D (1990) : Overcoming visual obstacles with the aid of the Supposer, *Educational studies in Mathematics, 21*, 199-219

ON THE SOCIAL PSYCHOLOGY OF MATHEMATICS INSTRUCTION: CRITICAL FACTORS FOR AN EQUITY AGENDA

Lena Licón Khisty, University of Illinois at Chicago

This paper presents the theoretical argument that there is a need to increase consideration of social psychological factors which affect learning processes, particularly in order to advance reforms which reach all students. Discussion focuses on examples of factors which specifically affect improved instruction for students of Mexican descent. These include the nature and role of 1) preconceived notions about the source of student failure, 2) differential status among students, and 3) cultural inclusion in classrooms. It is suggested that effective attention to these constructs can produce instruction which is culturally and linguistically responsive and which maximize opportunities for academic success for underachieving students in mathematics.

INTRODUCTION

Within the last decade, there has been a radical shift in our definition of the nature of teaching and learning mathematics and in how we expect the respective instructional activities to occur (NCTM, 1989). As a result, questions and concerns regarding the learning of mathematics no longer focus solely on the content. Instead, they now reflect a broader perspective which incorporates questions related to such issues as the setting in which mathematics is learned, the interactive nature of the learning process, and the discourse which transpires in this same process (e.g. Khisty, in press).

The new goals for mathematics also emphasize the need to ensure that improved mathematics instruction "reaches all students" (NCTM, 1989). In the United States, there are particular groups of students who traditionally have been, and continue to be, underachievers in mathematics and underrepresented in professional fields related to the subject. These students are predominantly poor and/or ethnic or language minorities. In light of this, the general questions concerning how to improve instruction in mathematics take on the added dimension of how to accomplish this goal for those groups of students who are educationally at-risk. This paper begins to answer this question by presenting the argument that the nature of the overwhelmingly persistent underachievement among groups of children with unique socioeconomic, ethnic, and/or language characteristics requires equally as much a new perspective of understanding as mathematics itself, and that this new perspective incorporates a broader base of social psychological constructs than previously adopted by mathematics educators. The discussion includes a brief overview of some critical social psychological factors and how they can affect the target students' learning. The discussion will focus on students of Mexican descent who I will refer to as Chicanos and who, like other Latinos, have a strong affiliation with Spanish regardless of their ability with the language. Also, Chicanos present the greatest dilemma to educators because they have the highest dropout rate in the country (e.g., National Council of La Raza, 1988). However, it should be noted that the arguments and concepts put forth herein apply equally well to any minority group.

THEORETICAL FRAMEWORK

Instructional reform in mathematics has adopted ideas that puts learning in a social context, for example, social constructivism (e.g., Steffe & Cobb, 1988), situated cognition (e.g., Rogoff & Lave, 1984), and neo-Vygotskiian theories (e.g., Khisty, McLeod, & Bertilson, 1990). However, as Secada (1993) has pointed out, this still represents too narrow a view in its reliance on psychological constructs and applications; moreover, such a framework does not adequately serve an agenda for equity. Instead, Secada has called for a social psychology of mathematics especially in order to advance the development of this agenda. What constructs would such a perspective entail? How would mathematics educators use these constructs? How would this perspective better inform our understanding of the teaching and learning processes particularly as it concerns Chicanos and other underrepresented groups? The discussion which follows addresses these questions by focusing on three critical factors which need to be considered for more effective instruction with language-minority students: 1) how the problem is defined in the first place; 2) status equalization among students; and 3) appropriate inclusion of culture.

Traditionally in order to understand why some children fail or succeed in mathematics, research has pursued such questions as 1) what is it about mathematics as a subject that makes it easy or difficult, or 2) what is it about students' thinking that produces errors. With this perspective, it has been easy to overlook or take for granted that the classroom is more than a social context where teachers and students interact; it is a *social arena* that mirrors social conditions outside the school, where social-psychological dynamics are continuously at work, and where cultural identities and lived realities are influenced by each other reciprocally.

Consequently, the most fundamental factor that should be addressed as we look at the dynamics of the classroom social arena has to do with how we explain the disproportionate failure of Chicanos in mathematics. The prevailing paradigm for the last thirty years has defined the problem as residing with the student and/or the student's family or culture and has been concerned with the remediation of problems and deficits that these students supposedly bring to school (Moll, & Díaz, 1987). The result has been investigations which concentrate on pinpointing and describing what students do not know, what experiences they presumably do not have, or what language and behavior differences they possess that result in a mismatch with the school. In essence, this model has placed a pathology within the student.

Another result has been that teachers, policy makers, and researchers have accepted the stereotypes and misconceptions that naturally develop from this line of reasoning. There is no shortage of publications that suggests that ethnic and language minorities suffer from particular learning styles, low self-esteem, non-supportive families, and an environment from which they can't escape.

Indeed, socioeconomic and ethnic status matters; but, in what way? Licón (1979) found that preservice teachers (n=230), who had had no real experience in classrooms, were readily willing to make instructional decisions for hypothetical students who only were identified by class and ethnicity. In spite of the fact that there was insufficient information for making a decision, the teachers overwhelmingly determined that poor and Chicano students should be placed in remedial instructional programs. Because of the controlled limited information about the hypothetical students, these novice teachers only could have made this decision if they already had preconceived ideas of the likely abilities of poor and ethnic students. More recently, in a follow-up evaluation of a workshop for teachers and administrators on mathematical instructional innovations for Chicano students, 88% of the attendants rated higher order inquiry teaching as "not being relevant" to students in bilingual school programs since they "hardly know their multiplication tables" (Khisty, 1992). Such thinking regarding the kind of instruction Chicanos should receive is consistent with the assumption that students have some intellectual limitation for one of a number of reasons and that the curriculum can not be too complex. As several researchers have pointed out (e.g., Spring, 1989), the use of lower level curriculum for minority students matches the way teachers define their students, and that this is a systemic problem for it is the rule and not the exception.

What the foregoing suggests is that redefining the nature of the problem and challenging teachers' assumptions about poor, ethnic, and language minority students should go hand-in-hand with changing the nature and delivery of mathematics content. Furthermore, higher expectations have to be more than rhetoric; they have to be specifically implemented in radically different instructional programs and actions that accelerate Chicano students' learning and treats them as gifted (Levin, 1987).

A second social-psychological factor which can affect instruction has to do with the nature of status characteristics among students. Groupwork as an instructional strategy has become an important part of reformed mathematics teaching because of its various intellectual benefits for students. Students who work and talk together gain in conceptual learning and in problem solving abilities among other things. However, as in any other situation when a small group is formed, hierarchies develop where some group members dominate and others are passive (Cohen, 1986). In classrooms, status differential can develop based on a student's perceived expertise in a subject, general academic ability, peer recognition, or simply on membership in one or more of the social categories of gender, class, and ethnicity.

In groupwork, the hierarchies that emerge can reflect these differential statuses. Those students with higher status will tend to dominate and those with lower status may be passive or may not even participate. If these differential statuses go unchecked, then students will gain via groupwork differentially also. As Cohen (1986, p. 31) points out:

> "If you design a good groupwork task, learning emerges from the chance to talk, interact, and contribute to the group discussion. Those who do not participate because they are of low status will learn less than they might have if they had interacted more."

Chicano students may actually be put at a severe disadvantage in groupwork. They already enter the situation with lower status because of language affiliation, lower expectations, and simply minority group membership. The instructional task for a teacher, then, becomes twofold: 1) to design rich and appropriate groupwork problems, and 2) to develop and set norms for equal participation. It is beyond the scope of this discussion to elaborate on the process and activities that can be used to develop groupwork norms. However, it must be noted that setting norms requires specific attention on the part of the teacher so that students understand clearly what is now expected of them. It also requires long-term training since students need to internalize a new way of relating to the teacher and to each other. They need to adopt as second nature that everyone in the group *ought* to have a chance to talk, that everyone's contribution should receive a fair hearing, and that not everyone has the same abilities but all are worthwhile. The same norms, then, are used to evaluate students' work.

There are a number of other things a teacher should do to ensure equal participation by Chicano students. One is to present problems in both Spanish and English so that mathematics is equally accessible to a student even if s/he is limited or non-English proficiency. Also, the use of Spanish establishes that the student's home language and culture have status in the classroom. A second thing is to give written problems and directions. This allows a student to check her/his comprehension of the problem as needed and minimizes having to learn via the weakest skill, listening. A third and last consideration is that groups should be organized by the teacher so that they are heterogenous in terms of both abilities and language. However, there needs to be sensitivity that there should be a bilingual student who can mediate during the discussion.

Unfortunately, most teachers give little attention, or even recognition, to setting norms for groupwork or to status equalization, and teacher training seldom addresses these issues either. Nevertheless, if the sociology of groups and statuses is ignored, then teachers run the risk of reinforcing educational inequalities and even becoming dissatisfied with the instructional strategy as they see students doing other than what was expected or hoped for.

The last factor to be discussed centers around the concept of culture. In the last two decades, there has been a rush by educators to create school environments that are seemingly more open to diversity. Often this takes the form of "celebrating various cultures" by having cultural books, pictures, and special celebrations in the classroom. Diversity or multiculturalism in mathematics frequently takes the form of mathematical artifacts such as the Chinese abacus, the Aztec calendar, or an African tribal game. There even have been attempts to identify a kind of mathematics that is uniquely characteristic of a particular cultural group such as Latinos and Native Americans. While acknowledging the contributions that minorities have made in mathematics, either in the past or present, is very important, most of what is taken as promoting cultural acceptance ends up trivializing those cultures and ignoring those factors which represent and define the most meaningful aspect of students' culture.

In order to begin to effectively make classrooms places where diverse cultures are accepted, it is first necessary to understand what culture is and what it is not. Culture, specifically, may be defined differently by various scholars; nevertheless, we can easily discuss the nature of *surface culture* and *deep culture*. Surface culture refers to superficial indicators or artifacts of culture such as food, dress, music, special customs, or special holidays. Some of these artifacts can have a "folk" quality, for example, as with a sombrero or a Mexican folk dance. Such artifacts may have personal or historical significance for members of the cultural group but may not have much relevance beyond this. Furthermore, because of the superficial nature of this aspect of culture, these artifacts do not exclusively define the cultural group; anyone can make Mexican food, wear Mexican folk dress, or play Mexican music and not be Mexican.

Deep culture, on the other hand, refers to those things which are usually intangible and elusive but that still define a person as being a member of a cultural group. These include values and culturally shared experiences which create a "consciousness of kind". Deep culture can not always be pinpointed because of its psychological, contextual, and multidimensional nature. However, deep culture forms the core of identity, and thus, is what is most meaningful to individual members and what binds the group together. Therefore, deep culture is most powerful when it is manifested through individual expression; however, it loses much of its power when it is put into a generalized abstract characterization of the group. Furthermore, when students' individual expressions of cultural identity is absent within classroom, then there exists a gap between home and school that increases the chance of alienation and lack of engagement. This suggests that meaningful culturally responsive instructional strategies are those that connect the home and school through maximum opportunities for the individual expression of students' cultural identity (e.g., Tharp & Gallimore, 1989). This individual expression can be accomplished via instructional strategies that are interactive and that use student "talk" (in whichever cultural language a student chooses) in meaningful tasks, that facilitate students' "taking over" or appropriating ownership of learning activities, and that use students' own experiences (but not as assumed by the teacher) or questions as the context for mathematical problems.

Another way that Chicano students' cultural identity can be brought into the classroom is by using the students' community, including households, as a "fund" of mathematical knowledge. This, of course, first requires changing the perspective of the nature of working-class and ethnic communities from socially and intellectually limiting to one full of resources. Frequently, this latter perspective is interpreted as using community members and parents as guest speakers in classes to supplement teacher determined learning activities; but, it is much broader than this. Using the community/household as a fund of mathematical knowledge requires that a teacher carefully study these environments to see actually how mathematics is used and how mathematical knowledge is socially constructed within these specific contexts. This activity provides ideas for themes and materials that better place mathematics in a meaningful sociocultural context that more accurately reflects the identity of the student. Since each community and household is intrinsically unique, there is no recipe or prescription for how the community resources can be used. As Moll, Vélez-Ibáñez, and Greenberg (1990, p.2), in discussing how this approach has effectively been used in literacy instruction, point out: "Ultimately, it is the teachers and students themselves that must create the classroom conditions to understand and exploit these resources...." This mode of perceiving and utilizing community resources capitalizes on the student's language and culture, and through this, empowers Chicano students by giving them access to mathematical knowledge since the context of mathematics is not external to them.

IMPLICATIONS AND SIGNIFICANCE

This paper has presented the argument that effective policy and instructional decisions that promote equity in mathematics involve much more than considerations based on subject content or psychological learning factors. In fact, instructional policy making and implementation that do not take into account the social psychology of teaching is highly suspect. Three social-psycho-logical factors were discussed as brief examples of how critical is this perspective for advancing equity.

For too long we have held to a mythology that the learning of mathematics is language, culture, and politically free. One result of this thinking has been that we assume that "good teaching" is simply good teaching; that we can ignore the unique social-psychological and linguistic needs of Chicano and other minority students. However, if we accept this assumption, then we do not alter the conditions that foster failure. Moreover, we run the risk of perpetuating a "blame the victim" model since "good teaching" was used with the target students and they still failed.

The significance of the argument presented is that in order to move forward an agenda for change that benefits all students, we have to begin to ask the following types of questions as we make policy, train teachers, and implement innovations. Do instructional decisions reflect a challenge to the traditional paradigm for why certain groups of students continue to fail, or do they reflect a model that continues to ascribe failure to factors outside the classroom and the instructional act? In the implementation of innovations such as groupwork, are social-psychological dynamics given enough attention and are teachers adequately trained to deal positively with them? Do our mathematical instructional reforms actively and fully capitalize on the working-class and minority student's home, culture, and language, or are they excluded through lack of thought? In essence, the challenge for mathematics education is to acknowledge and carry out a second paradigm shift regarding the influence and nature of social-psychological factors in classrooms that either promote or hinder equity.

WORKS CITED

Cohen, E. G. (1986). *Designing groupwork.* New York: Teachers College Press.

Khisty, L. L. (in press). Making inequality: Issues of language and meanings in mathematics teaching with Hispanic students. In W. Secada, E. Fennema, & L. Byrd (Eds.) *New directions in equity for mathematics education.* New York: Cambridge University Press.

Khisty, L.L. (1992).*Students and teachers acquiring mathematical power.* Unpublished manuscript.

Khisty, L., McLeod, D., & Bertilson, K. (1990). Speaking mathematically in bilingual classrooms: An exploratory study of teacher discourse. In G. Booker, P. Cobb, & T. Mendicutti (Eds.), *Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Education*, (Vol. 3. pp. 105-112). Mexico City: CONACYT

Levin, H.M. (1987).Accelerated schools for disadvantaged students. *Educational Leadership*, 44 (6), 19-21.

Licón, L. (1979). *The effect of social class and ethnic stereotypes on preservice teachers' decision-making*. Unpublished doctoral dissertation, Pullman, WA: Washington State University.

Moll, L. & Diaz, S(1987). Change as the goal of educational research. *Anthropology and Education Quarterly,* 18, 300-311.

Moll, L., Vélez-Ibáñez, C, & Greenberg, J. (1990). *Community knowledge and classroom practice: Combining resources for literacy instruction* (Handbook for Teachers and Planners, Innovative Approaches Research Project). Arlington, VA: Development Associates, Inc.

National Council La Raza. (1988). *Literacy in the Hispanic community*. Washington, D.C.: Policy Analysis Center.

National Council of Teachers of Mathematics. (1989). *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Rogoff, B. & Lave, J (1984). *Everyday cognition: Its development in social context.* Cambridge, MA: Harvard University Press.

Secada, W. (1993). Equity and a social psychology of mathematics education. In J. Becker & B. Pence (Eds), *Proceedings of the 15th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 17-28). SanJosé, CA: The Center for Mathematics and Computer Science Education, San José State University.

Spring, J (1989). *The sorting machine revisited: National education policy since 1945*. New York: Longman.

Steffe, L. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Tharp, R. & Gallimore, R. (1990). *Rousing minds to life: Teaching, learning, and schooling in social context.* New York: Cambridge University Press.

VISUALIZATION AND REASONING ABOUT LINES IN SPACE: SCHOOL AND BEYOND

EVGENY KOPELMAN and SHLOMO VINNER

HEBREW UNIVERSITY of JERUSALEM

Abstract

Several questions from space geometry were suggested to a wide group of subjects: from high school students to teachers and university professors. Their answers were analysed using a cognitive linguistics approach. The analysis revealed a common "earth-biased" nature of spatial concepts and explained the wide-spread failure with certain questions. The study also showed that when the same question was asked using a three dimensional model instead of a picture in most cases the answers were the same. They were determined by the conception of the subject rather than by the amount of visual information.

Introduction

The rising interest in geometry and it's applications at university level is accompanied by attempts to introduce more advanced geometry in school (Malkevitch, 1992). At the same time learning three-dimensional geometry assumes ability to visualize and interpret spatial relations expressed by diagrams. Several authors (Parzysz, 1988; Dreyfus & Hadas, 1991) had pointed on learning difficulties related to that. The initial purpose of our study was to find out how analytical tools learned by Israeli students in advanced high school geometry course are used by them in solving spatial tasks given by means of diagrams.

The test which is the essence of this study is the following:

Two line segments are drawn on two faces of each cube. Do the lines intersect in space ? If not, are the lines parallel in space or not ?

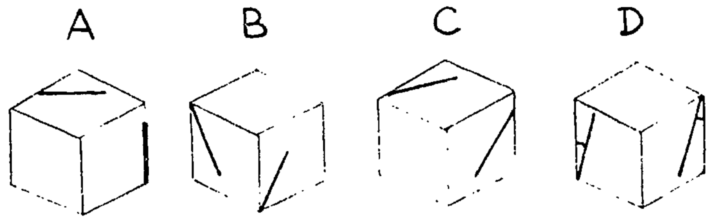


Fig. 1

These questions were first presented to three groups of 11th graders at the end of their one-year "Vectors" course with elements of solid geometry and linear algebra. The students were sampled from two schools in Jerusalem for gifted students. It was quite instructive to observe the students while they were filling the questionnaires. Some were trying to imagine lines in space while others were using fingers, pens and different objects to model the situation. Some students added written explanations to their answers, but no indication of analytical reasoning was found there. The results are given in Table 1 below.

Surprised by the results we decided to examine high school mathematics teachers. The same questionnaire was administered to two groups of high chool mathematics teachers from two different educational systems (Israeli and Jordanian) uring an in-service training. While answering the questions, the teachers, like the students, worked hard trying to imagine or to model lines in space. Their results are also presented in Table 1.

Since these results were not better than the students', we made one step further and presented these questions to a number of academics from two universities - one in the United States and one in Israel. The respondents were either asked to answer the questionnaires or were interviewed. Their results are in Table 1.

Table 1. The number (percent) of correct answers to tasks A - D in different groups of respondents.

	N	Task A	Task B	Task C	Task D
11-th graders	76	63 (77%)	47 (62%)	35 (46%)	41 (54%)
teachers (Is)	29	21 (72%)	13 (45%)	11 (38%)	4 (14%)
teachers (Jr)	25	20 (80%)	15 (60%)	15 (60%)	6 (24%)
chemists	6	6	1	1	0
physicists	8	8	7	4	3
mathematicians	7	7	7	5	4

From observations and interviews we got the impression that there are some common cognitive strategies for solving these spatial problems. The aim of this study is not the valuation or the comparison between the different groups. We are rather interested in the following question:

How do students and mathematically educated people solve these problems?

Observations on mathematicians working on these tasks showed that they solved problem A immediately, usually spent several seconds on task B, spent relatively more time on task C and even more time on D. Being asked to explain their decisions about pictures A and B, usually they answerd like this: "This line belongs to this face and the other line to that face of the cube. So they could intersect only on the edge common to the both faces, but the lines cross that edge in different points, so the lines do not intersect". As to picture C, some respondents claimed that they could not see there a well defined mathematical problem. Trying to solve it "visually", some had failed considering the lines as intersecting but marking their point of intersection far away from the only possible place on the continuation of the edge. Others continued bravely the edge of the cube to show that the lines do not intersect (which is the correct answer).

The problem in picture D was a real test for geometrical knowledge mathematicians. Most of those who succeeded had used the results of classical synthetic geometry. Only in one solution there was something resembling a vector approach: one of the line segments was parallely displaced to the opposite edge of the face and, because they did not occur at the same plane, the fact that the two lines were not parallel became obvious. Leaving mathematicians aside, we could characterize the prevailing approach among the rest of our respondents as intuituve, based on direct perception. Some of them might try to help themselves by physical objects in order to perceive the situation more clearly. Some would even try to justify their conclusions by a certain geometrical argumentation. This was in contrast to the mathematicians in our sample, who tried to avoid visual intuitive reactions.

The following is taken from an interview with a physicist R. specializing in optical features of crystals. After the respondent had carefully analysed all the four pictures and concluded that the lines there did not inersect and were parallel in case D, he was asked to explain his answers.

R.: In case A it is obvious, because the first line is perpendicular to the upper face in the certain point, so the intersection would be possible only if the second line passes through that point. In case B, I see that the line on the right face passes above the second line, so they will not intersect. In case C it seems to me that they will not intersect, because the lines have different slopes. In case D I clearly see that the lines are parallel. (He continued to stare at the picture and suddenly changed his mind.) No, the lines will intersect in cases B and C.

Interviewer: Why?

R.: I make cross-sections of the cube through these line segments (he moved his hand), so the lines will intersect in this cross-section (he marked an intersection point on a picture).

I.: And what about case A?

R.: I think they would intersect... (he looked at the picture) though it is really hard to imagine, how the cross-section goes.

We think that the case of R. examplifies some important features. First, the cited interview demonstrates that solving a spatial problem includes a complex interplay between percepts and concepts which could lead either to right or to wrong conclusions. Second, the solution of R. for the problem A (his first version) strikes with its immediacy and, in fact, laconism when compaired with that of a typical mathematician.

Indeed, the model of the mathematician can be symbolically expressed by

$$L1 \cap L2 = ((F1 \cap F2) \cap L1) \cap L2,$$

where L1, L2 are the relevant lines and F1, F2 - the relevant faces.

The model of a non-mathematician is symbolically expressed by

$$L1 \cap L2 = (F2 \cap L1) \cap L2,$$

where the index 2 corresponds to the top face.

If the last model is effective and used in problem A, then arises a legitimate question: why this model is inactive in other cases? Or in other words:

What goes differently in the solution of problem A?

According to the data gathered in the interviews, many times the fact that the two lines in problem A do not intersect was explained as follows:

(1) because the first line goes up through the face while the second line remains on the (horizontal) face;

(2) the second line is behind the first one

The underlined words express different image schemas (Lakoff, 1987) which provide different models for solution. We may never know, why in different people differnt models are evoked in order to solve a problem. However, we would like to suggest an explanation to the fact that the same person does not use the same model in order to solve a similar problem.

To evoke the above mentioned schemas in the case of picture B we just **turned it clockwise 120°**! When respondents who formerly "saw" in picture B intersecting lines were shown it's "rotated" version they began to see the lines as skew ones (not in the same plane). In several cases the same rotation of picture D had also "cured" the respondent from perceiving the lines as parallel. In these cases the evokation of certain image schemas was clearly related to orientation of the picture. In the absence of these schemas only what might be called "convergent-parallel" scheme remained activated. This led to the wrong model for solution.

Since there are many people who even in the case of picture A conceive the two lines as intersecting, it may be important to examine when people learn to read such pictures correctly. We think that it is a cultural phenomenon and as such might be related to the age of the students and to their educational environment. We were led to this conjecture by testing students of different ages of the same school. The resuls are given below:

Table 2. The number (percent) of correct answers to task A for grades 8-11.

grade 8 (N=21)	grade 9 (N=18)	grade 10 (N=33)	grade 11 (25)
3 (14%)	6 (33%)	28 (85%)	21 (84%)

Do 3-dimensional models help to solve spatial problems?

Another question we addressed in this study was how people solve the four spatial problems shown in Fig.1, provided they have a 3-dimensional model in their disposal. For this purpose a 3 cm cube was made and line segments were drawn on its faces as in pictures B and D of Fig.1. During an interview a person was handed the cube and asked how many pairs of parallel or intersecting lines are among those drawn on the cube. In the case of mathematically educated group this question was asked after the respondent had finished to solve the same spatial problem using a picture. Only two from the 17 respondents in this group managed to realize by means of the model that their previous answers were incorrect. In the case of the school students, the test with the cube was administered a month after they solved the same problems using the pictures. The students who were tested studied in two classes in one of the two schools for gifted students. Only students of the 11-th grade had studied elements of solid geometry. The results of both tests are given in Table 3.

Table 3. The number (percent) of correct answers to tasks B & D for 10th and 11th grades. For each task the numbers (percent) in bold indicate the answers based on the 3-dimentional model, the second numbers (percent) indicate the answers based on the picture.

	N	Task B		Task D	
grade 10	33	**13 (40%)**	8 (24%)	**8 (24%)**	0 (0%)
grade 11	35	**14 (40%)**	23 (66%)	**15 (43%)**	21 (60%)

As stated above, the same cube was used to model both pictures B and D, with three line segments drawn on three different faces. As a result several respondents (including mathematically educated ones) "perceived" additional pair of "intersecting" lines which belonged to parallel faces of the cube. In most cases after the respondent had been given a cube, he or she turned it untill it seemed to them that the line segments fited the vertical plane in front of them After that they came to one of the two folowing conclusions: (1) The lines were parallel. (2) The lines intersected. Only in a few cases respondents continued to turn the cube until they realized that the lines were skew They explained it by noting that (1) the lines go in different directions; (2) the first line passes above (under) the second one. Similarly to the above mentioned situation with pictures, the evokation of the underlined image schemas leading to the right conclusion was a result of of the model orientation. The conclusion that the two lines are skew could come at one glance to anyone had he or she deliberately directed the line segments on the 3-dimensional model along their line of sight.

Conclusion

To understand spatial reasoning of high school students it was found useful to examine another group of people whom we called mathematically educated.

The study of mathematical behaviour of both groups showed that they use similar cognitive strategies for solving spatial geometry tasks.

The analysis of the overall responses revealed an earth-biased nature of spatial concepts which explained the failures in solving certain tasks.

The improvement of the perfomance on spatial tasks with the age supports the view that there is the cultural basis of spatial concepts

The study showes that the substitution of a picture by the 3-dimensional model in spatial tasks is not necessarily helpful to most of the respondents. With some caution it can be claimed that what guides the formation of the mental models (namely visualization) used by the respondents to solve the spatial tasks is not necessary the visual information. It rather the conception which is evoked in the respondents mind as immediate association with a given task.

References

Dreyfus, Tommy and Hadas, Nurit (1991). STEREOMETRIX - a learning tool for spatial geometry. In W.Zimmerman and S.Cunningham (Eds.) Visualization in Teaching and Learning Mathematics (pp.87-94). MAA Notes Series, Vol. 19.

Lakoff, George (1987). Women, Fire and Dangerous Things. The University of Chicago Press, Chicago.

Malkevitch, Joseph (1992). Tomorrow's geometry. In L.A. Steen (Ed.)

Heeding the Call for Change. Suggestions for Curriculum Action (pp.47-82). MAA Notes Series, Vol. 22.

Parzysz, Bernard (1988). "Knowing" vs. "seeing" Problems of the plane representation of space geometry figures. Educational Studies in Mathematics, 19, 79-92.

PFL-MATHEMATICS: A Teacher In-Service Education Course as a Contribution to the Improvement of Professional Practice in Mathematics Instruction

Konrad Krainer
Interuniversitary Institute for Interdisciplinary Research and Continuing Education
University of Klagenfurt, Austria

This article is about the two-year university course "PFL-mathematics" in Austria which aims at special further education of mathematics teachers. Some remarks on the genesis and the philosophy of the PFL programme are made and relating guiding principles are outlined. Using four dimensions of teachers' professional practice, namely action, reflection, autonomy and networking, the article highlights activities of participants and team members within or after a PFL course.

A "mathematical curriculum vitae" as the starting point of an in-service education course

As preparation for the PFL course, a teacher wrote:
"I always wanted to become a teacher - first in a primary school - then my parents made it possible for me to study at the university. The choice of subjects was easy for me: my favourite subjects were mathematics and descriptive geometry. At the university I early realized that purely scientific work in mathematics did not fit my inclinations. What gave me pleasure, was to work as a tutor and practical assistant. After five years of study I looked forward to teaching at school. I had my own conceptions of teaching - derived from my own school days, from the conversations between my parents (both teachers) about their teaching practice, and from my private lessons. Above all I fought against the following image of mathematics: 'Either you have it or you don't! That's fate!' Therefore I tried to stimulate all pupils to cooperate, by proceeding in small steps, carefully preparing the subject matter, pointing to frequent sources of errors, leading support, answering questions etc. ...
[After concretizing some of her beliefs, methods and problems she finally reflected about her re-entry into teaching at school after a four year-leave of absence]
... I realized that the further development of many abilities which I wanted for me and my (own) children, was inhibited by traditional methods of instruction. To broaden and to deepen my experiences I finally decided to participate in this university course. I am looking forward to the first seminar. Kind regards!"
Most of the approximately 30 participants of the in-service education university course *"PFL-mathematics 1988-90"* arrived with a similar *"mathematical curriculum vitae"* at the first one-week seminar. The seminar started in three "regional groups" (parallel working groups, each consisting of about eight participants and two team members) in which all available "personal histories" were discussed. This initial working unit was not only a "get-acquainted session" but also gave a first impression of the broad spectrum of opinons, ideas, beliefs and strengths of the teachers and of our philosophy of using the practical experiences of the participants as a starting point for work within the PFL course and of connecting individual and social learning experiences.

1. Some remarks on the genesis and philosophy of the PFL programme

In the 1970's more and more high school teachers in Austria felt that in addition to their competence in two subjects (e.g. mathematics and geography) further development of their pedagogical and didactic competence would be required for coping successfully with the complexities of teaching and learning. In awareness of this need, an interdisciplinary team of researchers from the University of Klagenfurt began to plan the teacher in-service courses *"Pedagogy and Subject-Specific Methodology for Teachers"* (PFL abbreviation for the German "Pädagogik und Fachdidaktik für LehrerInnen") in the subjects English, German, history and mathematics. After a trial run in 1982 - 84 three PFL courses (for mathematics, English and three years later also for German) were institutionalised as university courses. They are organized by the *Institute for Interdisciplinary Research and Continuing Education (IFF)* of the Austrian Universities of Innsbruck, Klagenfurt and Vienna, in cooperation with some departments of the University of Klagenfurt and regional INSET institutes and are supported by the Ministry of Science and Research (BMWF) and the Ministry for Education and Arts (BMUK).
Some organisational remarks: Each course lasts for *four semesters* and is attended by *about 30 teachers*. Participants are expected to be present at all meetings (e.g. two one-week seminars, two half-week seminars, four one and a half-day regional group meetings) and are required to write case studies on innovations introduced in their schools. After conclusion of the course participants receive a university certificate with a description of their achievements during the course. Most PFL courses are led by a *team of 5 - 6 members* (educationalists, subject-matter specialists, practitioners) who are responsible for the preparation and realization of the course and for follow-up activities (e.g. evaluation), all in all three years of intensive theoretical and practical work. The activities of the team members are seen not only as a contribution to the further education of teachers but also as an experience in interdisciplinary cooperation.

The *guiding principles of the PFL programme* can be described as follows:

- One of the most important principles of PFL is stressing the importance and *interconnectedness of pedagogical and didactic aspects* of teaching and learning. Above all, the complexity of the teacher's task cannot be reduced to content-related considerations. In mathematics instruction, for example, even topics like "proof" involve student motivation, different heuristic strategies, reflections on the nature of proof, discussions on students' understanding of proofs etc.
- In most cases the *starting point* of work within the PFL courses is the *practical experiences of the participants* in order to meet them "where they are", identifying strengths rather than weaknesses.
- *Action research*, understood as *the systematic reflection of practitioners on action* (i.e. their professional activities in order to improve them), is used as a framework to achieve a broader situative understanding and to improve the quality of teaching. Within PFL courses the participants are required to do research work and to write - on the average - two case studies on self selected issues in which they have professional developmental interests.
- Communication among teachers often happens "in passing" and is often felt to be unsatisfactory. An attempt is made to find useful ways towards a *professional exchange of knowledge*, thus promoting the culture of communication on educational

issues. *Communication and cooperation* among teachers is seen as a more and more demanding element of teacher's work. Countless good ideas of teachers exist only in their minds and are therefore not accessible to others. It is an important intention of the programme to make such private ideas public.

- PFL aims at providing opportunities to *connect individual and social learning experiences*. The open atmosphere in PFL courses is used as a basis for initiating and organizing communication and cooperation with colleagues, something which is in many cases difficult to achieve in a teacher's own school. The regional groups are small "professional communities" in which mutual understanding and constructive criticism are conducive processes.
- Promoting the *further development of theory and practice* implies a close cooperation between team members and participants, the former involving themselves in concrete and specific situations of the teachers' practical work, and the latter involving themselves in theoretical and general considerations. For this purpose it is an advantage if people come from different systems, namely "school" and "university", and are able to overcome their restricted perspectives.
- Promoting *active learning processes and reflecting on them* is a basic strategy in PFL courses in a double sense: firstly with regard to an epistemological understanding of learning, which sees the *learner as a producer* (and not consumer) of knowledge and secondly with regard to the conviction that *transfer from the course to the classroom* - where students also should be seen as producers - is more successful if the participants learn such processes through experience. "What can teachers learn from learners?" is a basic question within the courses.
- One aim of the programme is to motivate and qualify the participants *to organize further education for themselves* and for other teachers after conclusion of the course. Thus it is important that the participants be actively involved in planning and realization of the course, and increasingly take charge of their own further education within (and later outside) the course. Therefore the role of the team members within PFL develops - roughly speaking - from providing input and structure (to initiate teachers' activities) to organizing communication among the teachers as experts.

Each PFL course aims at contributing to each of the dimensions sketched above. Its strengths, however, lies in the whole composition of the course, decisively influenced by the people working in it - the team members as well as the teachers. Within this paper it is only possible to sketch some selected activities of teachers within or after a PFL course. This will be done by using *four dimensions of teachers' professional practice*.

2. Promoting action and reflection, autonomy and networking within the PFL course - dimensions of mathematics teachers' professional practice

Altrichter/Krainer (1993, 8-20) describe 4 dimensions of teachers' professional practice:

- Attitude towards and competence in experimental, constructive and goal-directed work (*Action*)
- Attitude towards and competence in reflective, (self-)critical and systematically based work (*Reflection*)

- Attitude towards and competence in autonomous, self-initiative and self-determined work (*Autonomy*)
- Attitude towards and competence in communicative and cooperative work with increasing public relevance (*Networking*)

In the following, a) *a teacher's case study* of her students' errors in algebra will give an impression of the importance of systematic reflection by teachers on their own practice (interconnecting action and reflection), and b) a short view of a *PFL follow-up activity* by a self-organized group of teachers will indicate that autonomy and networking are essential dimensions in teacher in-service education programmes. A more detailed description of activities within PFL-mathematics is given in Fischer et. al. 1985, Krainer/Peschek 1993 or Krainer/Posch 1994.

a) Action and reflection - a teacher's efforts to improve her knowledge about her students' understanding

The first PFL seminar usually contains an introduction to the aims, ideas and methods of *action research* and provides the opportunity for practical experience with some of these methods (e.g. analytic discourse, interview techniques; see e.g. Altrichter/Posch/Somekh 1993). After this seminar each participant selects an issue in which s/he has professional developmental interest and begins to collect data for the first case study. In the regional group meetings (partial) results of these case studies are discussed; some participants present their research at the next seminar. Many participants of PFL begin a second case study which in many cases shows visible progress with regard to e.g. conception, data sampling and interpretation or readability.

In the following we look at a PFL *participant's first case study* (Mayer 1992) on her students' errors in algebra. She aimed at learning more about her students' misconceptions and errors in order to improve her teaching, for example through using new strategies to avoid some errors or finding a constructive way of handling them. A quotation from her case study shows that errors seem to have an essential influence on her actions and reflections concerning mathematics instruction:
"After several years of work as a mathematics teacher one almost gets the feeling that most of the time, instead of explaining, one has to deal with errors."
The teacher studied articles about error analysis in mathematics education and finally began to investigate misconceptions and errors of students from her own classes (15 to 18-year-old girls). She collected different kinds of data including diary notes, interviews with students and *"error-books"*.

Referring to "error books" she writes:
"The students had to keep a so-called 'error-book' which included an identical reproduction of the error, a verbal formulation, if possible a statement about the reason for the error and an exact correction. ... Thus the students had, for the first time, to think about their errors, which for many was a totally new and unusual experience (the verbal formulation was especially problematic) and in this context also numerous systematic errors were discovered ('I have made the same error again')."

One important aim of the teacher's analysis was to work out *"error frames"*. She finally identified about fifty types. Here are two of them (the students' explanations are in parentheses):

- $1/6 > 1/5$ (6 is bigger than 5)
- $x^2 - 2x = x^2 + 2x \implies x^2 = x^2$ (+2x and -2x cancel each other out)

In the concluding chapter of her study she writes:
"Only during such investigations does the teacher realize again, how important it is to scrutinize from time to time one's own teaching and to draw consequences. All too easily one falls back into daily routines."
The teacher sketches a range of general and concrete improvements in her own teaching, for example:
"The students must be given enough time to understand mathematical strategies (the student's learning pace should not be overestimated); as a result of teaching the same subject matter frequently, the teacher tends to increase the working pace or to choose trickier tasks each time."
To quote also one of her more practical consequences:
... *"Side-calculations should not be 'hidden' (removed by an ink-eradicator) to enable the teacher to recognize students' strategies - above all they should not be assessed negatively ..."*.

In her concluding remarks the teacher sums up:
"The appreciation of an achievement from the mathematical point of view - being true or false - and the teacher's interest in investigating the thinking behind the work which leads to a right or wrong result, are important factors for student motivation in also tackling difficult calculations."

This was the teacher's first confrontation with systematic reflection on her own teaching which included writing down her experiences as a crucial element. Of course, many of her results sketched above are well known, investigated and published - but this is not the point! She constructed her knowledge herself; she started at a point which was of real interest to her; she looked for - and partially found - answers with regard to her individual situation; she *produced meaningful "local knowledge"* which cannot be replaced by reading the research results of others.

Systematic reflection by teachers on their own practice - here with regard to student understanding (see Krainer 1993b) - can contribute to an improvement of mathematics instruction in many respects. This leads to the following suppositions which take for granted that improvements take place in various ways and that they are often connected with mutually stimulating processes:

- Interest of teachers in what and how students understand give the latter the feeling that they - with all their questions, answers, interpretations, etc. - are *really taken seriously*. This makes it easier for them to use the given *scope of freedom for experimenting* with ideas and actions, and thus improving the teacher's opportunity for gaining further insights into students' understanding.

- Attempts by teachers to replace assessments like "the student does not understand" with a search for meaningful explanations - even for so-called misconceptions and errors - and appreciation by teachers of even the small achievements of students, can be seen as part of a general attitude *orienting teaching towards strengths* and not towards weaknesses. This makes it easier to regard situations of "not-understanding" as *opportunities to learn - for students and for teachers*. To realize that teachers can learn from learners positively influences the relationship between teachers and students and promotes the further development of strengths.
- Efforts by teachers to analyze and to discuss so-called "errors", help them to get insights which are normally not available to them - strangely enough, precisely because of their expertise. Trying to take the student's point of view is an opportunity for teachers to partly *overcome the constraints on their own way of thinking* and to become aware of "blind spots" caused by their specific involvement in the system of "mathematics instruction". Having the flexibility to change one's point of view promotes a better understanding of the ideas and thought processes of students.
- Efforts by teachers to make the thought processes of students visible, and to reflect on their understanding, increase the abilities of students to reflect on the learning process and *promote self-evaluation and self-organization*. These abilities make it easier to discuss consciously the different meanings of a mathematical concept and differences in student intuitions, and therefore to increase student understanding.
- Better understanding by teachers the ideas and thought processes of students increases the teacher's awareness of new alternatives in teaching and thus also increases their competence and flexibility, which in turn expand *more opportunities for teachers to promote and to assesss students' understanding*.
- Activities of teachers aimed at acquiring knowledge about student understanding and systematic reflection on their own investigation, contribute to improve their scrutinizing of methods of assessing student understanding. This in turn stimulates the *improvement of the ability to evaluate investigation*.
- A teacher's own experiences in investigating student understanding increase their interest in knowing more about the results of other research, for example large-scale projects in mathematics education. Reading research reports in this connection provides teachers with *new ideas for their own small-scale investigations*.
- The efforts of teachers to increase their competence with regard to student understanding promotes self-critical and profound discussion with colleagues and researchers, thus *improving the culture of communication about educational issues*, perhaps motivating other teachers to *undertake self-critical inquiry themselves*.

These issues show that the *interplay between action and reflection* is of great importance. This interplay is an essential characteristic of action research and it is also regarded as an important aspect in the teacher's decision-making processes (Cooney 1988), the student's generalisation processes (Dörfler 1991), or the student's problem solving processes (see e.g. Krainer 1993a).

Teachers' systematic reflections on their own practice can not only improve their own teaching but can also have consequences for the further development of discussion in mathematics education, which is shown by e.g. the (German) book "Man and Mathematics. An Introduction into Didactical Thinking and Acting" of Fischer/Malle (1985): this book is strongly influenced by authors' work with team members and participants within the PFL programme and by other research activities undertaken in cooperation with teachers and students.

b) Autonomy and networking - a group of teachers organizes further education for themselves and others

Fostering *autonomous and self-initiative work by teachers*, and professional communication and cooperation among them, is an important aim of the PFL programme. The participants are invited increasingly to take charge of their own learning process within the course - as individuals and as a group. The statement of a participant after a PFL course - *"I slowly realized that this course cannot be not restricted to 'taking'; but also not to an oral 'giving' of a few personal ideas. The 'giving' should become very concrete, in form of two case studies. This was connected with a very painful process..."* - shows that the transition from viewing learning as consumption of knowledge to viewing learning as production of knowledge is not an easy one. To be able to mediate, and to "live" such an understanding in the classroom, it seems absolutely necessary to experience this transition directly. This holds true for individual learning processes and is even more important for the learning of groups. In PFL this is regarded a crucial point because the participants should be qualified and motivated to act as *"agents of change"* for other teachers.

The following is feedback for the last seminar of the PFL-mathematics course 1988-90 in which groups of participants organized different workshops: *"... formerly I was mostly a consumer in various seminars, not trying to make my own suggestions - concerning organization or content. In this respect I have become more self-confident. Above all I have found it very good that the third seminar has been designed to a large extent by the participants themselves ..."* One of the workshops, *"Elements of an Alternative School"*, gave birth to a self-organized group of teachers. The workshop included a broad spectrum of activities - from general discussion of educational goals to concrete reflections on possible changes in mathematics instruction, and motivated some participants to continue to cooperate. One of them organized an excursion to Tuscany (Italy) where, among other activities, two secondary schools were visited. Eight participants and two team members of the - in the meantime completed - PFL course took part, observed some Italian classroom instruction, held discussions with teachers and principals, and reflected on their observations and impressions. Half a year later, the group met again to discuss "project work" in mathematics. Finally they decided to initiate meetings twice a year and after some time they felt that they - especially the loyal core of seven people - were more than an ordinary group of teachers. In a report about their work in our journal for PFL participants they explained their intentions and the name *"AFL - Aktion Forschender LehrerInnen"* they had given themselves:

"AFL is an abbreviation for 'Researching Teachers' Initiative'. With this name we want to express on the one hand our mutual interest in action research, for which we have been inspired within the PFL course. On the other hand we want to express our interest in actions, in activities in the field of teacher in-service education. Thereby we start from the assumption that teachers have, or can acquire, enough competence to realize teacher in-service education by themselves, that nobody knows as much about school and learning as they themselves, that nobody knows their needs as well as they themselves. And we are not only interested in our own further education but also in putting the idea of

'teachers train teachers' into practice ..."
One related activity was undertaking different interdisciplinary projects as joint research projects of teachers and pupils whereby one member of the group played the role of the project-manager (coming from the "outside"). At present the members of AFL among others are at work on a book(let) with the title "Hands-on Mathematics". In their last meeting in December 1993 they discussed ways of opening the group to other teachers.

Feedback from former PFL participants shows that they have involved themselves in a number of individual and joint activities which contribute to pedagogical and didactic innovation in classrooms and schools, e.g.: *"I have also become braver and more initiative concerning my relations with colleagues. Small contributions at staff meetings, even organizing parts of a staff meeting with colleagues, preparing and carrying out projects together are activities I tackle because I have observed, seen and learned very much ...."* A lot of them act as "agents of change" in their region, are engaged in teacher in-service-courses or in teacher education, and actively participate in conferences in which innovative work by teachers is presented.

References

Altrichter, H./Posch, P./Somekh, B.: *Teachers Investigate their Work. An Introduction to the Methods of Action Research.* Routledge: London 1993. (Revised edition of Altrichter/Posch: Lehrer erforschen ihren Unterricht. Klinkhardt: Bad Heilbrunn 1990)

Altrichter, H./Krainer, K.: *Wandel von Lehrerarbeit und Lehrerfortbildung.* Unpublished paper. IFF/University of Klagenfurt 1993. Appears in Krainer/Posch 1994.

Cooney, T.J.: Teachers' Decision Making. In: Pimm, D. (Ed.): *Mathematics, Teachers and Children.* Open University, Hodder and Stoughton: London 1988.

Dörfler, W.: Forms and Means of Generalization in Mathematics. In: Bishop, A./-Mellin-Olsen, S./Van Dormolen, J.: *Mathematical Knowledge: Its Growth through Teaching.* Kluwer: Dordrecht-Boston-London 1991, 63-85.

Fischer, R. et. al. (Eds.): *Pädagogik und Fachdidaktik für Mathematiklehrer.* Hölder-Pichler-Tempsky: Wien und Teubner: Stuttgart 1985.

Fischer, R./Malle, G.: *Mensch und Mathematik. Eine Einführung in didaktisches Denken und Handeln.* Bibliographisches Institut: Mannheim-Wien-Zürich 1985.

Krainer, K.: Powerful Tasks: A Contribution to a High Level of Acting and Reflecting in Mathematics Instruction. In: *Educational Studies in Mathematics* 24, 1993a, 1-29.

Krainer, K.: Understanding Students' Understanding: On the Importance of Cooperation between Teachers and Researchers. In: Bero, P. (Ed.): *Proceedings of the 3rd Bratislava International Symposium on Mathematical Education.* Comenius University: Bratislava 1993b, 1-22.

Krainer, K./Peschek, W.: *Pedagogy and Subject-Specific Methodology for Mathematics Teachers. Reflections on an In-Service Education University Programme in Austria.* Paper for OECD. IFF/University of Klagenfurt 1993.

Krainer, K./Posch, P. (Eds.): *Lehrerfortbildung zwischen Prozessen und Produkten.* Manuscript of a book on the experiences of the PFL programme. IFF/University of Klagenfurt, to appear 1994.

Mayer, Ch.: Fehler. Unpublished case study of PFL-mathematics 1991-93. IFF/University of Klagenfurt 1992.

Mathematical rationales for students in the mathematics classroom

Koichi KUMAGAI
Joetsu University of Education, Japan

The purpose of this paper is to discuss about mathematical rationales in the mathematical classroom from the view point of social interactionism. Particularly we focus on what mathematical rationales do students construct and how do they use in their mathematics classroom. We observed, recorded and analyzed fifth graders' mathematics classroom. There are thirty-six classes from the beginning of the school year. We find various mathematical rationales in the pattern of interaction. Some of them are accepted in the wider mathematics society and others are unique to this classroom. And students use some knowledge instead of mathematical rationales. These knowledge and rationales help students to participate the pattern of interaction. And the pattern of interaction help students to consider them.

Students talk various ideas, explanations, thinking and so on in the mathematics classroom. Some of them are accepted by a teacher and other students, and others are not. Even if an explanation is accepted in one situation, it is not accepted in other situations. For example, a student often talks "The calculation method is correct because the answer is correct." This explanation is accepted in the situation where a teacher and students infer the validity of method. But when the validity of the methods is discussed, this explanation is not accepted. Students and a teacher share what is accepted or not in interaction. These norms are established through interaction between a teacher and students and a teacher and students use these norms in interaction. As described above, some of these norms have mathematical properties that are accepted in the wider mathematical society (Voigt,1985;1992).

Cobb et al. (1992) and Yackel et al. (1993a) discuss these norms as social norms or sociomathematical norms. Some norms related to the product of mathematics, for example, a different solution, a sophisticated solution, and an efficient solution. Other norms related to process of mathematics, for example explanation and argumentation. We will discuss about norm that relate to process of mathematics and focus on a situation where new problems are formulated.

In this paper, we will discuss about what mathematical rationales do students share and how do they students use them in problem formulating interaction.

Data

We observed and recorded thirty-six mathematics lessons in fifth grade with two video cameras from the beginning of the school year, from 11 in April to 3 in July. And interviews are conducted with the teacher (28 in June) and eight students (in May).

The teacher Mr. Yamagishi has experienced elementary school teacher in 14 years. He is interested in teaching mathematics. There are thirty-six students (15 boys and 21 girls) in a classroom.

We transcribed and analyzed the following data:

-transcription of verbal interaction between the teacher and students
-transcription of blackboards
-transcription of interview with the teacher and students

Problem formulating interaction and pattern

Construction of the pattern of interaction

In some situations, students and the teacher formulated problems that had mathematical appropriateness. In other situations they can not.

At first we will show a typical situation which students and the teacher formulates a problem through interaction. This situation is observed on the 8 of May in 1991.

At the beginning of the lesson, the teacher, Yamagishi posed a problem about multiplication of a decimal fraction, 2.7x1.8= ?, 5.2x3.4 = ?. Students have never experienced the multiplication of a decimal fraction before, but they attempted to solve this problem. Students presented two solutions. The solution and answer by Ya. G. is correct. He did not mention about mathematical rationale for his solution, but explained procedures of calculation, particularly treatments of deciding the place of decimal point. At that moment, Furu.Y. said "My solution is different from others.....", and presented her solution. And the teacher and students interact each other.

[Situation I]

(Ya.G.)		(Furu.Y.)	
	2.7		5.2
	x 1.8		x 3.4
	2 1 6		20 8
	2 7		156
	4.8 6		176.8

P : (Furu.Y; gN) My solution is different from others and wrong. A decimal point, the decimal point, the process of calculation is the same as others. When I decide the place of a decimal point, I did nothing, I did not multiple by 100. I wrote a decimal point simply.

T : Is this a wrong?

P : (Furu.Y;gN) I don't understand well. I wrote the decimal point simply.

T : You wrote the decimal point simply.

N : (Ya.G; gN) We can't decide easily whether your solution is correct or not.

PP(D): (Yama.T.gR) We can not decide easily whether your solution is correct or not.

: (others) That solution is wrong.

T : Now, Furu.Y confused. Is there any comment or so? Yoko.T, come here (6 sec.) please.

P : (Yoko.T; gN) Now, before, every thing is

N : We can't see.

P : (Yoko.T; gN) I'm sorry. At first, we assume that there is a problem of multiplication. The answer is 4.86. When I did this type of calculation before, we could not decide the place of a decimal point simply. But

PP(D) : (Noisy)

P : (Yoko.T; gN) Someone said that, now, a decimal point, we could not write this simply......

A teacher and students formulated a problem about deciding the place of decimal point. When the teacher and students formulate problems, we can find a pattern of interaction.

At first, the teacher poses a problem and expects students to present various solutions that include wrong or naive solutions. Students come up with the teacher's expectation and present various solutions of a problem. For example, the teacher posed problems of multiplication of decimal fraction, and one student Furu. Y. presented her wrong solution. Second, the teacher and students clarify the differences of solutions. In situation I, Furu. Y. clarified that her solution was different from others in deciding the place of a decimal point. Third, the teacher and students formulate a problem depending on the differences clarified before. For example, two students, Ya. G. and Yama. G. mentioned the necessity of discussion whether her solution was correct or not. And the other student, Yoko. T. begun to solve a problem that related to decide the place of a decimal point. Students formulated a problem about the treatment of a decimal fraction.

There is a pattern of interaction, presenting various solutions (P.V.), clarifying differences (C.D.), and posing problems depending on differences (P.P.). And the teacher and students formulate a problem. This pattern has been established through the interaction between the teacher and students from the beginning of school year (Kumagai,1992).

Breakdown of the pattern of interaction

In some situations students and the teacher could not formulate a problem that is mathematically valid. This situation can be observed on the 30 of April.

The teacher and students solved a problem, "32x1.1=." Students approached this problem with various ways. Most of approaches were not generalized easily, for example, 32/10=3.2 32+3.2=35.5. When these approaches have been introduced and shared, one student, Sai. T. posed a problem, "The case of 1.1 or 1.2 is easier than other cases. We must consider other cases, for example, it is the case of 1.3. It is more difficult for us." In this situation the teacher requested students to search difficult cases as problems to discover sophisticated methods.

[Situation II]

T : Sai.T posed a problem. The case of 1.1 is easy. It's the same with 1.2. In the case of 1.1, it is related to 1/10. Therefore it is related to 1/5. But in the case of 1.3 it is difficult to represent with a fraction whose numerator is 1. This case is more difficult than others. Sai.T pointed out this. Today, we are beginning to study. What is a today's theme. 32 multiplied by (12 sec.) various values. 3.2x1.3 We want to calculate such cases. For example the case of 1.3. Sai.T said , this is a difficult case. How about you? Are there any difficult cases? More difficult calculation.

PP(D) : (noisy) (8 sec.)
T : Can you attempt to do every case?
N : Yes.
T : Are there any?
N : (laughing)
T : What do you say?
N : (Toku.N. [gR]) 1.5
T : 1.5?
PP(D) : The case of 1.6 is more difficult.
T : The case of 1.6
N : (Katsu.K. [gR]) 1.7
T : Katsu.K said 1.7.
N : The case of 1.8.
T : Is the case of 1.8?
N : (Ya.G. [gN]) 1.8
T : Are there any cases, more difficult cases?

....

N ; 9.99

T ; 9.99. This case is difficult. Any questions? Now, we try to solve these with our methods. Please try various cases.

On the one hand most students posed values', 1.5, 1.6, 1.8, and they did not pose a mathematically important problem. We can not find the pattern of interaction as described before in this interaction.

On the other hand, one student, Sai. T., posed a mathematically appropriate problem. We infer his thinking process as follows from his explanations. Sai. T. examines the method that is available for the case of 1.1. with various values, 1.2, 1.3, and so on. He calculates each values like this way, 32x1.2=32/5+32=6.4+32=38.4. 32x1.3=32/3+32=?. He can not solve the case of 1.3. It seems that he thinks this method has the idea that numerator is one. He clarifies differences of the result of calculations, 1.1, 1.2, 1.3. The method does not work in some cases. He formulated a problem, "We must consider other cases, for example 1.3. It is more difficult for us." We see the pattern of interaction in his explanations, posing different values, clarifying differences, posing a problem.

In this situation, the teacher inferred his thinking process as described above and expected other each student to follow this process themselves and to pose some values cause problem cases. The teacher did not expect to establish pattern of interaction among them. Most students could not come up with the teacher's expectation. They posed various values, 1.3, 1.5, 1.6, 1.7, 8.9, and so on. It seems that students expect to establish the pattern of interaction explicitly. They are going to clarify differences depending on these various values and to pose problems.

In problem formulating interaction, when we can find a pattern of interaction, students and the teacher formulate a mathematically appropriate problem. But the pattern of interaction is not established explicitly, they can not formulate a problem. The explicit pattern of interaction contributes to formulate mathematically appropriate problem. We will discuss more details of the meaning of contributions.

Mathematical rationales in the pattern of interaction

In general when we formulate problems, we need to consider mathematical rationales: mathematical validity, generality of methods or conclusions, developments of problems or methods, values of problems and so on. In the pattern of interaction, whether do students consider mathematical rationales or not. What mathematical rationales do they consider in problem formulating interaction ?

Students seem to use different mathematical rationales in each situations. The pattern of interaction consists of three parts, posing various solutions, clarifying differences, and posing problems depend on differences. We will discuss mathematical rationales in each part of the pattern.

In the first part, the teacher has accepted all solutions by students and does not pay attention for the sameness and differences from the beginning of school year. But the teacher expects students to present wrong and naive solution. The teacher does not expect to consider mathematical rationales that are accepted to a wider mathematics society, but the teacher's this expectation is a mathematical rationales in this classroom. For example, in situation I, a student posed a wrong solution herself and in situation II, students posed various values.

When we discuss the difference or the sameness of solutions, we need to pay attentions for mathematical rationales. In the second part, the teacher expects students to consider mathematical rationales. The teacher expresses this in the situation observed on the 17 of April. He explained mathematically meaningless differences with student's solution and posed himself mathematically meaningful solution. But it was difficult to understand for students in that situation. Sometime students clarifying every differences that are visible for them. Those visible differences do not always have mathematical rationales. Students seems do not always consider mathematical rationales that are accepted the wider mathematics society.

For example, in situation I, it is possible for students to clarify mathematical differences of solution without consider mathematical rationales that are accepted the wider mathematics society. When students focus on answers and remember the size of the result of multiplication of integer, they find the difference of procedures of deciding the place of decimal fractions. This difference is visible. Students can clarifying differences of solutions without consider mathematical rationales as if they understand them. But students use knowledge about multiplication of integer.

We see more closely this situation, a student, Furu. Y., assumes that her solution is wrong. In other situation we can also see like this assumption by students. When students clarify the differences of solution, they sometimes assume that one solution is correct and others are wrong. This assumption has not been valid mathematically before. Students use such assumption in clarifying differences of solutions.

In some situations students can clarify differences of solutions, as if they consider mathematical rationales. And sometimes they use unique rationales which are not shared in the wider mathematical society.

In general, we need to consider mathematical rationales in posing problem.

Students consider mathematical rationales again. We can find this in situation I. Students considered mathematical rationales and pose a problem. Two students, Ya. G. and Yama. G., insist to consider the reason why the method (b) is incorrect. They mentioned mathematical rationale, that was whether have they discussed and shared the validity of method before or not. They feel that if they use a new method, they need to examine the validity of a method. This mathematical rationale is content-free.

Once mathematically important differences of solution are clarified, students can pose mathematically important problems easily. Because mathematically important differences has mathematical rationale, we can directly see differences as mathematically important problem.

When the pattern of interaction is established, students consider different mathematical rationales in each part of the pattern of interaction. Students use various mathematical rationales and mathematical knowledge to participate problem formulating interaction. Some of rationales are not accepted in the wider society.

Conclusions

For students what is mathematical rationale or validity in the problem formulating interaction? How do they use these to construct problem formulating interaction?

When the pattern of interaction is established explicitly, students consider different mathematical rationales in each part of the pattern.

Students sometimes use mathematical rationale that is not accepted in the wider society, but some of them contribute to maintain the pattern of interaction, for example, posing wrong solution or naive solution in the first part. This rationale is essential to establish pattern of interaction. But this rationale is not accepted in other part of the pattern of interaction.

Students use mathematical knowledge instead of mathematical rationale. In clarifying differences, they use mathematical knowledge that has been shared before. And sometimes they use common sense, when they mention to visible differences. Students use mathematical knowledge that are shared in the pattern of interaction. In posing problem, students use knowledge shared in the interaction of clarifying of differences. This practice of using knowledge depend on the pattern of interaction.

Students use two types of mathematical rationale, one is content-free and the other is content-fixed. Students use easily content-free mathematics rationale.

When students participate in problem formulating interaction, students use knowledge and mathematical rationales that are unique to each part of the pattern. The pattern of interaction helps students to consider mathematical rationales and to formulate mathematical problems.

References

Cobb, P. et al. (1992). Characteristics of classroom mathematics tradition: An interactional analysis. American Educational Research Journal. 29(3). 573-604

Kumagai, K. (1992). The development process of mathematically emphasized interaction in the mathematics classroom. Paper presented a the the ICME, Quebec.

Kumagai, K. (1993). In consistency in levels of interaction -Microscopic analysis of mathematics lesson in Japan-. Proceeding of the seventeenth international conferences PME. III, 218-225

Yackel, E. (1993a). Sociomathemaitlcal Norms, Argumentation and Autonomy in Mathematics. Paper presented at the 1993 Annual Meeting of the American educational Research Association, Atlanta, GA.

Yackel, E. (1993b). Children's talk in mathematics class as a function of context. Proceeding of the seventeenth international conferences PME. II, 199-206

Voigt,J. (1985). Patterns and routines in classroom interaction. Recherches en didactique des mathematique. 6(1). 69-118.

Voigt,J. (1992). Negotiation of mathematical sense making in classroom process -Social infraction and learning mathematics-. Paper presented at ICME,Quebec

COGNITIVE MAP ASSOCIATED TO TWO VARIABLE INTEGRALS

Francisco Cordero Osorio *Arturo Gonzalez Larios*

Departamento de Matemática Educativa, CINVESTAV-IPN.

Universidad de Colima, Mexico

In the frame of compartamentalization, as a component of calculus problematic, in the scholastic context, we studied the knowledge of the double integral We identified two representations, one based on the concept of function and the procedure of approximation and the other based on the concept of quantity and the procedure of variation This led to two situations extention and globalization of the integration concept, elements, which, articulate both the mathematical and cognitive structure of the two variable integral.

INTRODUCTION

We are interested in the problems of teaching calculus, acknowledging that students who efficiently solve problems are those who can easily pass between the different concepts involved in a problem. In a pragmatic sense based upon the conception of the cognitive system, where we intend to understand how the representation of knowledge determines the behavior of the individual exposed to a problem. Pattern recognition and meaning incorporation play no active role (occasionally completely absent) on the students argumentation The central aspects of these research are necessary related in the epistemologic, cognitive and didactic planes. On the other hand, the flexibility of representations is conceived as the type of difficulties the students have at characterizing aspects among the representations.

In this paper we attend to the following

Considerations - in which the representation and study forms are identified.

Discussion frame - in which the role of frames are discussed

Mathematical structure - in which the objects and actions of each representation are linked

Cognitive considerations - in which the type of behavioral response of the students to different problems is observed.

CONSIDERATIONS

a) The way in which we have been identifying the students representations of the two variable integral is through problem solving. We use a case study with videotaped interviews and clinical analysis. b) For the analysis we have established the references of representations favored by the mathematical scholastic language through the review of the textbooks most commonly used by our scholars. Besides, we considered the teachers integral's concept,compiling a catalog of the representations most commonly used to explain the integral concept.

Specifically, we explored in this context, two symbolic expressions:

$$\iint_D f(x,y)dxdy \qquad \iint df(x,y)$$

the representations and meaning associated to each of them, and the type of problems they are applied to

DISCUSSION FRAME

The discussion is based fundamentally on their notion of "cognitive map" which refers to the representational scheme the individuals have of their geographical environment, and which permits them to have an orientation and trace the places they want to go to, distinguishing in their representation some nodal points which they use as reference and through which they can infer spacial relationship of less representative points (De Vega, 1984)

A metaphor of the concept of "cognitive map" with implications on the problem we have outlined is the one of a representational internal format, joined with processes that allow the student to design a strategy to solve the problem situation, depending on the knowledge of his environment, in our case calculus in two variables, or what for Schoenfeld (1985) would be "resources". Therefore the situation is to determine the nodal points and the paths between the concepts that arise in the problems of integration in two variables. Being the nodal points, in this case, the representations of the concept of integral in two variables and the paths being the possible relations between such representations. We only analyzed two of those representations, one R_1, that allows us to view the integral in two variables as a function, the other, R_2 as a quantity, with two parameters, that may vary

$$R_1 \to \iint_D f(x,y)dxdy \qquad R_2 \to \iint df(x,y)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

Two variable function   Quantity that varies

These two representations are present in the scholastic context, the first one, R_1, is commonly found in Calculus books, and often used in the scholastic mathematical discourse The second one, R_2, is often found in explanations of continuous variation phenomena, attended by engineering in a scholastic concept. (Cordero, 1992).

MATHEMATICAL STRUCTURE ELEMENTS.

We have considered three elements of the mathematical structure that distinguish the two representations, R_1 and R_2. Procedures, objects and attributes

PROCEDURES

In the R_1 representation the procedures are to "reach" because from the Riemann integral definition we understand the region and its value in a measurement connotation, then we must only reach that value through a process to the limit, in this sense the understanding of the concept, in the structure, relies on the nature of the two variable function $f(x,y)$.

Continuity, bounded, derivable, discontinuity types, f prototypes, function domain and integration domain

The existency theorems of the integral in terms of the function f have only sense under this perspective. We must also consider the antecedent of the one variable integral where we define an operation to calculate primitives from recognizing the nature of the integrand function, then again the main part relies on the integrand function and not in the primitive of it. The limit is the main procedure in this representation we can note the local features of the function; in this sense the R_1 is located in the argumentational mathematical plane of the **approximation "PA1: Approximation"** (Cordero 1993c).

For the R_2 representation the procedure is to "compare" because we accept that it is a quantity that may vary, then we ask how does it vary in order to recognize its variation (Cordero 1993b). In this case the main part falls in the primitive function and not in the integrand function, and the integration will be a process of joining all the local accumulations of the same nature with the global accumulation. The main part in the process is on one hand **the taking of the differential element**, and on the other the identification of the **invariant.** In this case R_2 is located in the second argumentation plane **PA2** that Cordero calls the **variation** plane

OBJECTS OF THE PROCEDURES.

The objects that characterize each of the representations are, for R_1, component elements and, for R_2, constitutive elements, in the first case we are dealing with well distinguishable elements, in the second case we are dealing with elements that conform features that in turn compose the concept.

OBJECTS

R_1	R_2
↓	↓
NUMBERS	QUANTITIES
↓	↓
DERIVATIVE	DIFFERENTIAL
↓	↓
FLUX	FLUID
↓	↓
SURFACE	VOLUME

In respect to the integral features, both representations conform equivalent mathematical structures, however, being composed and conformed by different objects and procedures (objects and procedures understood as Dubinsky) the actions that the subjects do, derive in the acquisition of different argumentation planes. An element that allows us to identify these differences, in the acquired knowledge, is the type of problems that give sense to each model In particular, the R_2 representation through the accumulation notion allows us to conform a model for the integral as well as a system of significances (Cordero 1993c) The quantities of two parameters related through a functional expression have the particular feature to be able to vary in different forms: partial variation or total variation This difference in the case of one variable allows us to outline two alternatives to join the integration of one and two variables "to extend" or "to include" The first alternative is closer to the first representation, and the process of extension of the integration of one variable to two variables is often found in the actual scholastic media.

COGNITIVE ASPECTS

Cognitive aspects of the students facing the integral concept, that we have observed through the study of cases. The students, whose ages vary from 19 to 20 years, took a normal course of Calculus in various variables in an Industrial Engineering School. They were able to calculate integrals in two variables through the process of iteration and have learned to verbally express meanings associated to the symbol $\iint_D f(x,y)dxdy$, for example, they say that the integral calculates the volume under a surface, however when giving meaning to "*dx*" or "*dy*" or even more to "*dxdy*" they identify them as elements of the surface represented by the two variable function of the integral, that is the taking of the differential element is done on the surface. One of the students appealed to the analogous case of one variable and located the "*dx*" on the curve and the integral would be the sum of the elements that conform the length of the curve, which differs to their first affirmation that the integral is the area under the curve.

A significant aspect of the cognition of the double integral consist on, that the symbolic expression $\iint df(x,y)$, accounts on the variation argumentational plane, that is, the function $f(x,y)$ does not play a main role, even if a surface meaning is assigned to it, there is no recognition, in terms of the function definition, of the number of variables (extension), but the main role consist of discussing the nature of the quantity that varies (globalization), in the taking of the variation, in a specific situation as results in the calculus of an area, length or volume. In this sense the expression $\iint df(x,y)$ represents a structure composed of a set of notions, actions, and factual theorems, all of them encapsulated by the taking of the differential element

FINAL CONSIDERATIONS

The representations treated with the associated argumentation planes determine types of problems, which a priori are not related to each other. A curricular restructure must be guided to identify or construct a type of problems that joins the different representations, that is in order for the student to solve one of these problems he has to appeal to the different representations.

A third plane of argumentation, PA3: tendentious behavior, (Cordero 1993c), in which the central part are the relations between the graphs and the functions, with global and status conceptions and features, may conform a new joining model

From the cognitive point of view, the three argumentation planes and the representations could constitute a disordered conceptual field, understood as the transit between them, with no prefered direction; but from the conceptual development of calculus perspective, these planes can be recognized as stages of the didactical transposition process, where

PA1 conforms the most recent stage, the one with formal validation; in this sense it can be said that the argumentation planes form an ordered conceptual field.

PA1	PA2
R1	R2
$\iint_D f(x,y)dxdy$	$\iint df(x,y)$
Approximation problems	Variation problems

PA3
R3
Tendentious Behaviour problems

REFERENCES.

Cordero, F (1991) "Understanding the Integration Concept by the Teachers of Engineerng Schools", PME-NA, Proceedings of the thirteenth annual meeting, Virginia, USA, pp 91-97

Cordero, F (1993a) "Epistemology and cognition of the integral: elements of reproductiveness in teaching- learning situations", PME-NA, Proceedings of the fifteenth annual meeting, CA, USA, pp. 112-118.

Cordero, F. (1993b) "La integral como modelo de la nocion de acumulación", Lecturas de Cálculo para docentes de Ingenieria, # 4, CINVESTAV-IPN, México

Cordero, F.,(1993c) "Epistemologia, cognición y didáctica del cálculo: una visión de conjunto", Memorias del IV simposio internacional sobre investigación en educación matemática, Universidad de Yucatan, 1993, México.(In press)

Dubinsky, E (1990), "Reflective abstraction in advanced mathematical thinking", ICMI study series, Cambridge.

Shoenfeld, A (1985), "Mathematical Problem Solving", New York Academic Press

De Vega, M (1985), "Introducción a la psicologia cognitiva", Alianza Editorial, México.

Prospective Secondary Mathematics Teachers' Beliefs about "0.999...=1"

Brenda Lee

Wu Feng Institute of Technology and Commerce
Taiwan, Republic of China

Abstract

The purpose of this study is to describe pre-service mathematics teachers' understanding about mathematical subject matter knowledge, pedagogical content knowledge, and beliefs about those knowledge. Eight pre-service mathematics teachers answered questionnaire test item: "Is 0.999... equal to 1 or less than 1? Why?" with an affirmative response "Yes, equal to one"; six of them even provided reasonable proofs or informal explanations. However, through a further probing interview, all of them believed that "0.999...is less than 1." The reasons for this conflict were caused by the epistemological view of the concept of limit: finite view of the reality, intrusion of the potential infinity, and confusion between the process and the product. The concept of limit is central in calculus. Pre-service mathematics teachers' beliefs about mathematics knowledge .night pass to their future students, when they start classroom teaching. This transmission of mathematical concepts is an issue worth further investigation.

Background

Teachers' subject-matter knowledge and its interrelations with pedagogical content knowledge and curriculum knowledge are still very much unknown (Even, 1993; Lee, 1992). The focus of what teachers need to know in order to teach has shifted from quantitatively examining teachers' standardized tests scores to emphasizing knowledge and understanding of facts, concepts, and principles and the ways in which they are organized, as well as knowledge about the disciplines (Ball, 1991; Even, 1993; Lee,1992; Leinhardt & Smith, 1985; Shulman, 1986).

Teachers' subject matter knowledge and pedagogical content knowledge influence not only their teaching but also their students' learning. Although teachers get their professional education from both subject matter and education courses, but we do not know enough about what knowledge they have, what they think about mathematics knowledge , and how they use that knowledge.

Teacher's conceptions of what mathematics is affects way of presenting mathematics subject matter. Teacher's manner of presenting mathematics is an indication of what they believe to be most essential in mathematics (Hersh, 1986). In this study, the focus is not on what is the best knowledge to have, but, on prospective secondary mathematics teachers thought what mathematics is really all about. After all, teaching and learning mathematics is influenced by teachers' conception of mathematics. To understand teaching from teachers' perspective we have to understand the beliefs with which they define their knowledge.

Methodology

Subjects : The subjects in this study were 8 prospective secondary mathematics teachers in the last stage of their professional education. This group was selected because the description of their knowledge would reflect the knowledge they have gained during their professional education, also the knowledge they will bring into their teaching profession.

Procedures : The subjects were volunteered to participate a paper-pencil written questionnaire. The questionnaire was divided into three sections. The first section was to explore the background of the subjects, such as age, sex, GPA, courses taken in mathematics and in education. The second section was a list of problems for testing procedural and computational skills for finding limits of different representations of infinite sequences. The third section was designed to explore prospective secondary mathematics teachers' ability of the conceptual understanding of the limit concepts. After the thorough study of the returned information, the researcher found out that the paper-pencil written questionnaire could not really explicitly express the subjects real thought. Therefore a follow-up interviews were conducted. Unfortunately, there only four subjects were able to participate.

Instrumentations: In this study, we discussed one of the conceptual test item in the questionnaire: "Is 0.999...equal to 1 or less than 1 ? Why ?"

The two interview questions involved the pedagogical knowledge and beliefs about mathematics were discussed here, hoping might shed light on the research of mathematics teaching and learning.

(1) · Do you really think that 0.999...=1 ? Explain why ?

(2) If one of your students said that "my teacher told me that 0.999... = 1, but I don't buy it." How are you going to explain that?

Results

Every one of this group of pre-service teachers answered :" Is 0.999...equal to 1 or less than 1? Why? " with a positive "yes". The affirmative reactions was understandable, because just before the questionnaire was passed out, the professor of the mathematics methods course asked the whole class to read and reacted on the paper written by Lucien (1971). In Lucien' paper, there were six different ways to show that 0.999··· =1. Although everyone of them gave a positive answer "yes", but the reasons "why" were different. Table 1 showed the distribution of the answers of why: " 0.999··· equal to 1." The answers of "why" were categorized into six types. They were as follow: (A) one subject was using the method of infinite series to show 0.999···

=1;(B) three subjects were using the method of converting decimal to fraction; (C) and (D) using the informal approach of explanation; (E) and (F) are two answers with noting much to say.

Table 1. The distribution of " Is 0.999...equal to 1 or less than 1? Why?

A)Equal. (n=1)

$$0.999... = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ... = \sum_{n=1}^{\infty} \frac{9}{10^n}$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{9}{10^k} = 1$$

B) Equal. (n=3)

Proof:

Let $n = 0.999\cdots$

So $10\,n = 9.999\cdots$

So $9n = 10\,n - n = 9$

So $n = \frac{9}{9} = 1$

C) $0.999\cdots = 1$, because if we say that they have a difference n, then as the number of nines goes to infinity n goes to zero. (n=1)

D)Exactly equal because there is no number we could add to $0.999\cdots$ to get 1. (n=1)

E) Equal, by definition. (n=1)

F)Equal. (n=1)

$0.999\cdots = 1$

$1.999\cdots = 2$

$.999\cdots = 1$

$1.0 = 1$

After analyzing the data from the written questionnaire, an interview was decided to be conducted. The researcher was interested in finding out whether this group of pre-service mathematics teachers really agreed with what they read and learned from Lucien's paper. Through the telephone contact, only four subjects were able to participate in the interview survey. Their interview responses were audio-taped and were transcribed. They were the subjects whose written questionnaire were categorized as (B), (C), (D) and (F).

As suspected by the researcher, all the subjects disagreed with their previous written answers. Data from interview question (1) showed the discrepancies between what the pre-service mathematics teachers knowledge and believing about that knowledge. It also indicated pre-service teachers' cognitive views about their academic learned subject matter knowledge. Table 2 demonstrated their responses of the interview question (1): "Do you really think that 0.999...=1 ? Explain why ?"

Table 2 Answers of interview question (1): "Do you really think that 0.999...=1 ? Explain why ?"

(B) pre-service teacher said :

OK, I've seen this proof before h'r class and I have to say that I would still thought that not equals 1. It's infinitely close but not equal to.

(C) pre-service teacher said :

I've seen a lot of examples of proofs or so-called proofs you could use to show these two are actually equal, but I don't know whether I will buy them.

(D) pre-service teacher said :

Just as we talked about like in class, I thought .999... is not equal to 1, because I thought that one was the limit for 0.999..., but there was like entirely a page and half of arguments to show that they are exactly equal, but still you know part of me said "no, they're not."

(F) pre-service teacher said :

You know, I still don't really see how 0.999··· equal to 1, I believe it does and everything, but I just can't, I can't picture it, I guess just because that's what I learned with the definition of a limit.

The researcher was hypothesized that this group of pre-service mathematics teachers is going to transfer what they know to their future students. In order to test this hypothesis, interview question (2): "If one of your studetns said that 'my teacher told me that 0.999... = 1, but I don't buy it' How are you going to explain that?" was intended to find out what this group of pre-service mathematics teachers will do, when provided a teaching situation. Their responses were listed in Table 3. .

Table 3. Answers of interview question (2): "If one of your studetns said that 'my teacher told me that 0.999... = 1 , but I don't buy it' How are you going to explain that?"

(B) pre-service teacher said G
Just to say this works (pause). You don't need this, one thing you can say, I mean, I guess I just have to tell them that.
Algebraically it works out, You can show the kids, I don't know if they're convinced, people in our class still don't convinced.
(C) pre-service teacher said :
Well, if they could see that I like to say that if there is a difference between the two, then that difference, I can show you that the difference between the two is actually smaller, so if I ask, if there is difference give me difference no matter what difference they give me, I can lways make this (0.999...)to be less than this(1), this difference, we can play that for a little while, they would see that, that must be true because they can't tell me what the difference is,(laugh) so I don't know, it's going to be hard
(D) pre-service teacher said :
If it were repeating sequence, you can take that round it off and called that 1, so it seems that what you are doing here rounding it up and called it 1, so actually it is its own number which is just the 9 goes forever and, so it does.
There is always another 9 out there. So as long as you can add those 9's, it's not exactly equal to one, but really it is, but I don't want to except it.
It seems to me that there're always going to be a difference of something, so if took 1 and 1 subtract this (0.999...), I will come out with a ever you know smaller little and it would always keep moving out, but there is always something left over.
(F) pre-service teacher said :
I would say neither do I, but just live with it because the crazy society that we live in that they make us think of things like this.
I guess it would have to be outside the sequence because it's an ∞ , you're dealing with 9 infinitely times, and you can't really have infinity actually in the sequence, it has to be just out of reach, I guess., it's just approaching it.
If this was just a limit of n to ∞, and a_n equal to .999..., you know then there is no variable in there to plug in, so that would equal to .999..., so they are the same.

The findings indicated that if this group of pre-service mathematics teachers were ask to teach, they might transfer their beliefs about mathematic knowledge to their students. As given in Table 3, the data exibited their misconceptions and their teaching pedagogy.

Discussion and Conclusion

Responses from the above three tables showed the discrepancies between this group of pre-service mathematics teachers' subject matter knowledge, pedagogical content knowledge, and their beliefs about these knowledge. Based on researchers' finding (Davis & Vinner, 1986; Lee, 1992, 1993a; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981), pre-service mathematics teachers as well as students had difficulty to accept "0.999...=1", part of the reasons were due to the following epistemological view points of the concept of limit: (1) finite view of the reality, (2) interfering of the potential infinity, and (3) confusing between the process and the product. We will discuss these three view-points.

(1) Finite view of the reality: 0.999... <1, because everything this world we are living in and work with is finite. Therefore, infinite process is impossible to occur in reality. The meanings of limit are twofold: one is daily usage and the other is the mathematical meaning. It seems that ordinary language strongly dominates the thinking of mathematical language. Especially the mathematical limit is considered as a "bound". We could identified these in the following excerpts from the interview transcripts.

(B) pre-service teacher said :

> I always accepted like for all intents and purposes .999... would equal 1 just so that you can use it.

(D) pre-service teacher said :

> (1) you can take that (0.999...) round it off and called that 1
>
> (2) I think, since we used to dealing with finite things, like if you were to give me 99 cents, you know that 99 cents is not equal to one Dallas and so, you know, if you sell something that they said with a lot of 9's people would buy it, if you sell me something for ninety-nine Dallas and 99 cents, it's still not a hundred Dallas, so here're some tie with the finite world I'm used to working with, so whatever this s, this repeating sequence doesn't get passed with the barrier that I have from the practical world.

(F) pre-service teacher said :

> I would say neither do I, but just live with it because the crazy society that we live in that they make us think of things like this.

(2) The intrusion of the potential infinity: 0.999... <1 is because the actual infinity does not exist. Since it is an infinite processing, it has to go on forever and ever. The intrusion of infinity is a severe factor causing learning conflict. In the historical literature review (Lee, 1993b), there are two kinds of infinity, namely, potential infinity and actual infinity. Most people possessed the

potential viewpoint of infinity. That is, potential infinity occurs in a situation in which no matter where an individual is, he/she always can go another step; for example, given any positive integer one can always think of a larger number. In this study, we could identified many of this type of views in the following excerpts.

(B) pre-service teacher said :

(1)It's getting very closer and closer and closer to 1.

(2) It's just you know, a continuing decimal going on to infinity, I just, I really can't picture, picture it well enough.

(3)Obviously, when I told the students that 0.999... equals to 1, they already had misconceptions. Because, you know, 0.999... is just approaching 1.

(D) pre-service teacher said : There is always another 9 out there. So as long as you can add those 9's, it's not exactly equal to one, but really it is, but I don't want to except it.

(F) pre-service teacher said : How do you put on that last 9 to make it all of a sudden urn it into 1?

<u>(3) Confusion between process and product</u>: 0.999··· < 1 is because the confusion of the meaning of this representation. "999..." is not only expressing an infinite process represented by "...", but also indicating a product of that infinite processing represented by the whole thing "0.999...". The three dots indicated an infinite processing, while the whole thing --"0.999..." -- indicated a limiting value which turned out to be 1. Therefore, 1 and "0.999..." are two sides of a coin. They are the same ideas of two different representations (Lee, 1993b). Similarly, the other representation of "0.999..." can be introduced by the limit of an infinite sequence $\{a_n\}$, where $a_n=0.9\cdots9$ (n 9's after the decimal point), i.e. $\lim_{n\to\infty} a_n$. Therefore, $0.999\cdots = \lim_{n\to\infty} a_n = 1$. While viewing from different representations, this group of subjects get different cognitions. As a consequence, they were confused between the process of finding the limit and the actual limit. We could identified these results from the following examples.

(B) pre-service teacher said : It's the value it approaches to but never reaches.

(C) pre-service teacher said : I can show you that the difference between the two is actually smaller.

(D) pre-service teacher said : I thought .999... is not equal to 1, because I thought that one was the limit for .999....

(F) pre-service teacher said: If as just a limit of n to ∞, and a equal to .999..., you know then there is no variable in there to plug in, so that would equal to .999..., so they are the same.

Why the statement of "0.999⋯=1" is so difficulty to accept by this group of pre-service mathematics teachers? From the research literature on the notion of limit (Lee, 1993b), we knew that most of the students had the same difficulty to accept it. Besides the difficulty of the multiple representations of 0.999⋯, there also exists the epistemological conflict implicitly involving the notion of limit. This group of pre-service mathematics teachers were unable to truly understand the mathematical subject matter knowledge. And their disbeliefs about mathematical subject matter knowledge were prevailed . Therefore, we need further study on pre-service mathematics teachers learned academic knowledge with their beliefs about these knowledge.

References

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching*, Vol. 2, 1-48. Greenwich, CT: JAI Press.

Davis, R. B. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5, 281-303.

Even, R. (1993). Subject-matter and Pedagogical content knowledge of function concept. *Journal for Research in Mathematics Education*, Vol. 24, 94-116.

Hersh, R. (1986). Some proposals for revising the philosophy of mathematics. In T. Tymoczko (Ed.), New directions in the philosophy of mathematics. Boston: Birkhauser, 9-22.

Lee, B. S. (1992). An investigation of prospective secondary mathematics teachers' understanding of the mathematical limit concept (Published doctoral dissertation, Michigan State University, 1992). Taiwan: Chiun-Hwa Press.

Lee, B. S. (1993a). Limits of sequences and series: Prospective secondary mathematics teachers' understanding. *Proceedings of the fifteenth annual meeting, North American Chapter of the international group for the psychology of mathematics education*. Vol. 2, 145-151.

Lee, B. S. (1993b). The concept of limit: An historical perspective. *Science Education Monthly*, No.164, 31-37, 46; No. 165, 27-35; No. 166 (in press).

Leinhardt, G. & Smith, D. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology*, 77, 247-271.

Lucien, T. H. (1971). Persuasive arguments: .999... = 1. *Mathematics Teacher*, 749-750.

Shulman, L. (1986). Those who understanding: Knowledge growth in teaching. *Educational Research*, 15 (2), 4-14.

Tall, D. & Schwarzenberger, R. L. E. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, 82, 44-49.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12: 151-169.

PROMOTING ACTIVE CLASSROOM ACTIVITIES THROUGH COOPERATIVE LEARNING OF MATHEMATICS

Roza Leikin and Orit Zaslavsky
Department of Education in Technology and Science
Technion - Israel Institute of Technology, Haifa

This paper discusses the effects of learning mathematics in a particular cooperative small-group setting on students' types of classroom activities. One group of four students from a low level ninth grade class was selected for close observations during sixteen lessons. Eight of these lessons were taught in a whole-class setting and eight were taught in a cooperative small-group setting. Observed classroom activities were first identified and then classified according to the extent of students' active role. Analysis of classroom observations indicated an increase in students' activeness in the lessons taught in a cooperative setting. In particular, growth in students' mathematical communication was exhibited.

Several studies stress the importance of students' active role in the learning process (Brown and Campione, 1986; Fraser et al., 1988). Nevertheless, most mathematics lessons do not sufficiently promote students' activeness (Good, Mulryan, & McCaslin, 1992; Mulryan (1992); Romberg and Carpenter, 1986). In particular, Mulryan (1992) points to the strong effect that whole-class settings have on low achieving students. According to Mulryan (ibid.), such students become progressively more passive classroom participants in whole-class settings as they move through the grades, and by the sixth grade they usually manifest passive classroom behavior.

The effect of cooperative small-group learning on students in general, has shown a positive effect on students' activeness. Students' activeness, in turn, has shown to have an effect on students' achievements, such as in the case of learning to read in the reciprocal-teaching procedure (Brown and Campione, 1986). Thus, finding ways to promote students' activeness in learning mathematics should be regarded as a worthwhile task.

Promoting high status students' activeness in learning mathematics through small-group cooperative settings seems to be more feasible, since sometimes these students tend to dominate and become relatively more active in small-group cooperative settings, than low status students do (Good et al., 1992). Therefore, for the purpose of the study reported in this paper, a low level class was selected in order to explore the effect of a specific method of small-group cooperative learning on the types of classroom activities in which these particular students engage.

As mentioned above, small-group cooperative learning is a strategy that is often recommended for making learners more active (Good et al., 1992). One reason for this is its potential for facilitating students' interactions. According to the constructivist approach, students' interactions are significant activities for effective learning. In particular, task-related verbal interactions are closely related to learning outcomes. As Webb (1991, p. 367) points out: *The content related help that students give each other in small groups might be considered to lie on the continuum according* *to amount of elaboration Detailed explanations would be at the high end of such an elaboration scale, merely stating the answer to a problem or exercise would be at the low end, and providing other kinds of information would fall in between the two extremes.* Thus, in exploring students' classroom activities in the different learning settings implemented in this study, (cooperative sm ll-group vs. conventional whole-class), spe ial attention was given to activities related to students' interactions.

This paper is based on the findings of a larger study (Leikin, 1993; Zaslavsky and Leikin, in preparation) which investigated the effects of a particular cooperative learning method in mathematics on the extent of students' activeness, and on the types of students' activities and interactions. Students' attitudes towards this cooperative learning method in comparison to the customary whole-class setting were also examined. In addition, students' achievements were compared in both learning settings, to make sure that their achievements at least did not drop. In this paper we focus on students' activeness in the cooperative learning setting.

Method

The Cooperative Classroom Setting

In the spirit of the NCTM standards (1989), a method of cooperative learning was developed, based on exchange of knowledge between students in the course of solving mathematical problems. This method is organized through students' work with study cards. Each card consists of worked examples and of new problems of the same type. The learning takes place as follows:

- Most of the time, students study in pairs, within a larger group of four students.
- After completing the work on a card, students change partners within the group.
- Each student is required to explain to his new partner how to solve the worked example in which s/he gained expertise in the previous card, and to listen to the explanations given by this partner on how to deal with a worked example in a new card.
- Each student is required to solve a problem (similar to the last worked example which his partner has explained to him), and is entitled, if needed, to ask his partner (who already tackled the problem earlier) for help in solving it.

Design

The study was carried out in four low level ninth-grade classes, consisting of a total of 98 students. The four classes were divided into two pairs for comparison. In the first pair, students of the first class learned all the material according to the experimental cooperative learning method and students of the second class learned all the material in the conventional way. In the second pair of classes, students experienced learning by both methods, changing from one to another by the end of each learning unit. Altogether, the latter two classes engaged in solving problems during 24 lessons, of which 12 lessons were conducted in the cooperative learning method.

In order to investigate the effect of the experimental learning method on students' classroom activities, one group of four students was chosen for close observation. The group was selected from one of the two classes which encountered both learning methods. One of them was an average achiever while the other three were low achievers. The group of four students were observed during 16 problem solving lessons, 8 in each learning method. During most of the cooperative classroom settings, these four students formed one group.

Research Instruments

Two research instruments were designed for exploring students' activities and interactions.

The first was an observation schedule which served for documenting classroom observations and classifying classroom activities. This schedule consisted of eight classroom activities' categories (see Table 1) which were identified at an earlier phase of the study, based on open, non-structured, observations.

Table 1: Categories of students' classroom activities for observation

Active activities	Passive activities
• Giving an explanation	• Listening to an explanation
• Asking a question or requesting help	• Reading the learning material
• Solving a problem independently	• Other on-task passive activities
• Copying written material into the notebook	• Off-task activities

On the observation schedule the observer was asked to fill every two minutes, for each of the four members of the observed group, the category of the classroom activity in which s/he was engaged. The reliability of this instrument was checked by Pearson's correlation coefficients. For all categories, except for *reading the learning materials* and for *other on-task passive activities*, the reliability was sufficiently high (the values obtained were $.79 < r < .97$, $p < .01$). Thus, the latter two categories were not included separately in the analysis of the results. However, they were taken into account as components of the entire category of *passive activities*, because the reliability of this category as a whole was $r > .79$, $p < .01$.

The second research instrument was a student self-report questionnaire. For each study card, those who learned from it for the first time were supposed to answer the following questions:

1. Did you ask for any kind of help from your partner?

2. Did you receive any help from your partner?

If the answer to the last question is yes, answer the next three questions:

3. Did the help you received include disclosing the correct answer?

4. Did the help you received include detecting a mistake of yours?

5. Did the help you received include verbal explanations?

One questionnaire was given to every group of students at the beginning of each cooperative learning session, and they were all requested to keep track and to report on the same questionnaire the kinds of assistance, if any, received from their partners in the course of solving the problems on the study cards. At the end of the session each group had to turn in its report.

Results

The research findings deal with two main issues: (i) Students' activeness; (ii) Students' interactions. On both issues the results indicate an increase in favor of the small-group cooperative method (i.e., the experimental method).

Students' Activeness

Figures 1 and 2 depict the distribution of types of students' classroom activities for the entire observed group of four students. The data is given (in percentage) by the average number of times which a particular type of activity occurred throughout all observations of the group (eight learning sessions for each method).

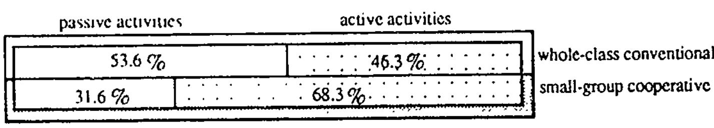


Figure 1: Distribution of active and passive activities

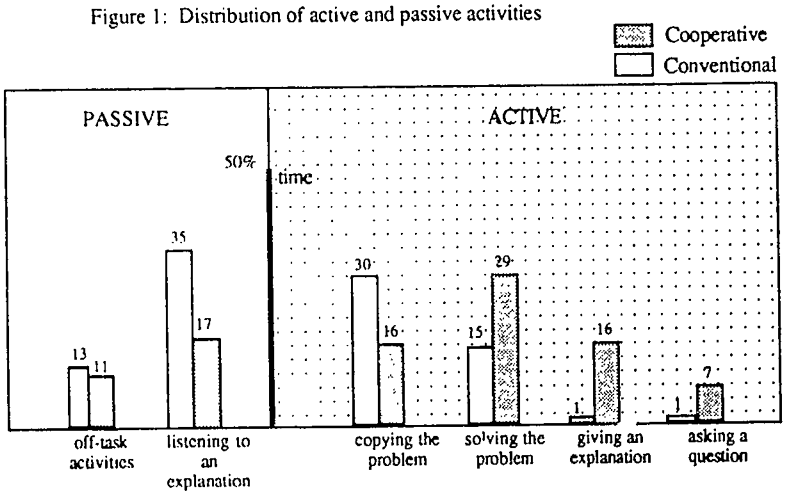


Figure 2: Distribution of different types of students' activities for each learning setting

As presented in Figure 1, the amount of time which students spent engaged in active activities increased by 22% (from 46.3% in whole-class setting to 68.3 in small-group setting). A Wilcoxon test showed a significant level of $p < .068$, which is considered acceptable in such a case of only four subjects. A further look into the specific categories (Figure 2), indicated that on most of them there was an increase of activity level. The only active category in which students decreased their amount of activity related to it was *Copying written material into the notebook*. However, this category seems the least active among the four categories classified as *active*. It should be noted that in all the specific categories, a meaningful difference was found (as done for the two categories: *active* and *passive*), except for the *off-task activities*. On the latter category no difference was noticeable, which could mean that students' slight tendency to off-task activities (around 12% of the time) is not effected by the learning setting.

Students' Interactions

Two main aspects of students' interactions were analyzed. These aspects relate to: (i) Mathematical communication; (ii) Help received.

Mathematical Communication. Webb (1991) discusses task-related students' verbal interactions, and their contribution to learning outcomes. These kinds of interactions fall into two of the categories of activities described in Table 1: *Giving an explanation,* and *Asking a question or requesting help*. We refer to these two categories which are considered very active and desirable as *mathematical communication*. The importance of mathematical communication is also manifested in the NCTM (1989) standards. In Table 2 the mean time (in percentage) which students engaged in *mathematical communication* is presented for the entire group of four, as well as for each member of the group separately.

Table 2: Time spent on mathematical communication

Method / Subjects	small-group cooperative mean time (%) (SD)	whole-class conventional mean time (%) (SD)
Sample group of 4 students	23.5 (6.5)	1.6 (1.9)
Student A	18.7 (11.2)	0
Student B	22.9 (15.1)	3.1 (3.0)
Student C	19.5 (17.1)	0
Student D	32.9 (21.9)	3.5 (4.3)

Table 2 indicates the meaningful difference in time spent on *mathematical communication* in favor of the small-group (experimental) cooperative learning, for the whole group as well as for each of its members. The increase varied from about 19% to 29% of the learning sessions. Note, that two of the four members of the observed group did not exhibit any *mathematical communication* throughout all eight sessions of whole-class learning setting.

Help received. The data related to students' help consisted altogether of 231 students' self-reports to which we refer as 231 cases. Note, that most student reported more than one case. These self-reports were analyzed, and classified first into two types: (i) 166 (72%) cases in which help was received; (ii) 65 (28%) cases in which help was not received. In all the latter 65 cases when help was not received also no help was requested. Furthermore, of the 166 cases in which help was received, in 113 (68%) cases help was also requested.

The first type of cases in which help was received was further analyzed according to the kinds of help received. For each case there could be more than one kind of help received. The categories of help presented in Table 3 correspond to those contained in the self-report questionnaires (see above). Note, that the total number of cases is not the sum of the numbers of occurrences of each kind of category of help.

Table 3: Distribution of the different types of help received by students in the cooperative learning setting

Type of help received	No. of cases in which help was asked for	No. of cases in which help was not asked for	Total
Disclosure of the correct answer	13 (8%)	4 (2%)	17 (10%)
Error detection	32 (19%)	43 (26%)	75 (45%)
Verbal explanations	107 (64%)	8 (5%)	115 (69%)
Other types of help	3 (2%)	8 (5%)	11 (7%)
Total no. of cases	113 (68%)	53 (32%)	166 (100%)

Table 3 points to the high percentage (69%) of cases in which the help received included verbal explanations. Most of the verbal explanations were provided in cases when help was requested (107 cases). These cases constitute 95% of the total amount of cases in which help was requested (113 cases). In addition, it is interesting to note that in 53 (32%) of the 116 cases help was received in spite of the fact that it was not requested. In these cases *error detection* was the dominant kind of help provided (in 43 (81%) of the 53 cases).

Discussion

The findings discussed above support the claim that it is possible to promote low level students' activeness in the mathematics classroom by implementing a small-group cooperative learning setting based on exchange of knowledge between students. From other studies (e.g., Brown & Campione, 1986), it can reasonably be inferred that an increase in students activeness has a positive effect on learning outcomes in general, and students' achievements in particular.

A close examination of the nature of the activities promoted by such cooperative learning method indicated a major increase in students' mathematical communication. This kind of communication is an essential component of the process of making sense in the course of learning mathematics. Thus, it can be assumed that the influence of this change is profoundly related to the quality of learning.

Investigation of the types of help students choose to offer each other in the experimental small-group cooperative learning setting points to *verbal explanations* as a dominating type of help. This type of help is considered by Webb (1991) to be on the high end of a continuum according to the amount of elaboration, and has a critical impact on learning mathematics effectively.

The results of this study provide an answer to the serious question raised by Good et al. (1992, p. 185): *How can the teacher best organize and manage the classroom during cooperative work so that discipline problems do not arise, interactions between students primarily involves tasks, and pupils still have sufficient freedom to contribute to and participate in the group discussion?* The fact, that *off-task activities* did not increase during the cooperative work implemented in this study, shows that discipline problems did not arise and that students' interactions primarily involved tasks. Students freedom to choose when and what sort of help to request or provide, the large amount of help received (in 72% of the self-reported cases), and the increase in *mathematics communication*, support the claim that in the particular experimental cooperative method students, indeed, had the freedom to actively participate in discussions with all members of the group.

References

Brown, A.L., & Campione, J.C. (1986). Psychological theory and the study of learning disabilities. *American Psychologist, 14* (10), 1059-1068.

Fraser, R., Burkhardt, H., Coupland, J., Phillips, R., Pimm, D., & Ridgway, J. (1988). Learning activities and classroom roles with and without computers. *Journal of Mathematical Behavior, 6*,. 305-338.

Good, Th., Mulryan, C., & McCaslin, M. (1992). Grouping for instruction in mathematics: A call for programmatic research on small-group processes. In D.A. Grows (Ed.), *Handbook on research on teaching and learning, a project of the NCTM*, (pp. 165-196), New York, Mc.Millan Publishing Company.

Leikin, R. (1993). Implementation of a cooperative learning method in mathematics, unpublished M.Sc. thesis, Haifa: Technion - Israel Institute of Technology.

Mulryan, C.M. (1992). Student passivity during cooperative small groups on mathematics. *Journal of Educational Research, 85* (5), 261-273.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics,* Reston, V.A.: The Council.

Romberg, T.A., & Carpenter, T.P. (1986). Research on teaching and learning mathematics: Two discipline of scientific inquiry. In M.C. Wittrock (Ed.), *Handbook of Research on Teaching* (pp. 850-873), New York, Mc.Millan Publishing Company.

Webb, N. M. (1991). Task-related verbal interactions and mathematics learning in small groups. *Journal for Research in Mathematics Education, 22,* 390-408.

Zaslavsky, O., & Leikin, R. (in preparation). Students interactions in small-group cooperative learning of mathematics.

Metaphors for Mind and Metaphors for Teaching and Learning Mathematics

Stephen Lerman

Centre for Mathematics Education
South Bank University, London, UK

Abstract *In this paper I address directly the issue of theories in psychology and theories of learning and attempt to offer a meta-discourse in which alternative theories can be compared. I examine theories of the self and of knowledge and the metaphors upon which we draw to conceptualise the process of learning in general and mathematics learning in particular. I argue that radical constructivism and what Gergen terms social constructionism are distinct theories of psychology and outline the theoretical frameworks each offer. The paper draws on simplified caricatures in order to outline such a meta-discourse and to attempt to provoke debate in the PME community. This paper argues for the need to establish a theoretical rationale for links between theories and claimed consequences in classrooms and in interpretations of research in mathematics education.*

As "the study of the mind (induced from behaviour) and how it works" (Oxford Dictionary 1988) psychology, and in particular learning theories, must elaborate notions of the mind upon which they are based and attempt to establish a rationale for the links between that theory and the 'consequent' interpretations of behaviour. In our case this concerns the influence of the teacher, of peers, of texts, of the cultures in and of the classroom and of the semiotics of mathematics. One can slip easily from seductive statements about knowledge and the learner to statements about classroom activities without examining either the theoretical rationale for the links or that alternative theories will enable alternative interpretations. Any classroom transcript or video can be interpreted in different ways depending on which theories are being used although rarely are connections between theories and what constitute exemplars or evidence established in any theoretical way. The mathematics education research community concerned with psychology should, for instance, examine claims that particular classroom structures encourage nch mathematical constructions by children or that learning takes place in or through a zone of proximal development, or that mathematical meanings are contextualised. These studies must be of terms such as 'rich', 'construction', 'learning', 'context' and how they signify in different perspectives, preceding empirical studies.

The mathematics education research community concerned with psychology must also take account of developments in theory in recent decades, in particular how theories constitute discourses of power, or regimes of truth.

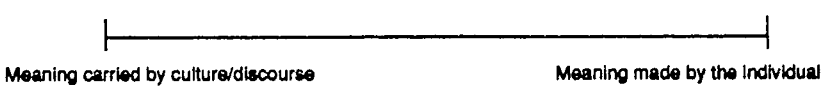
However, comparisons of competing theories are notoriously difficult since proponents position themselves within particular entrenched positions and continue to assert their own. In this paper, which follows a growing criticism of the radical constructivist position in mathematics education (Lerman 1992, 1993, in press, Brown (in press)), I will attempt to offer a framework in which different theories can be elucidated and compared, and metaphors for interpreting classroom behaviours examined. I do not claim more than that I offer a particular interpretation of theories in psychology in an endeavour to engage our community in a meta-discourse. Others are of course possible but if discussion of them is at the level of the meta-discourse then this paper will have achieved its goal.

The framework I will offer is a binary opposition of the possible sources of 'meaning-making' and within that framework I will examine the contrasting metaphors used. 'Meaning' signifies, and what is presented here is an attempt to constitute different **theoretical spaces** through examining that signification, spaces in which other significations may be interpreted. The gloss I give to these spaces, through examining metaphors, is a simplification or caricature along the lines of 'ideal types'. Nevertheless it is intended as a first attempt at creating a practice in which our academic community can engage in comparing theories.

Theories in Psychology

Humans are social animals and nature/nurture discussions are no longer of relevance as even to argue the distinction is to engage in a social practice. The extent to which the individual mind is a product of socialisation is perhaps the fundamental issue for psychology and the extreme positions may be characterised as *the individual as fully social*, and *the individual as fully autonomous.* A focus on 'meaning' may encapsulate this distinction. On the one hand meaning can be taken to be socio-cultural; it is produced in discourse and discourse positions or regulates. Individuals are acculturated into those meanings; the intersubjective becomes the intrasubjective. Individual input into meaning creation is not denied, but manifests in a dialectic of the participants in discourse being changed by, and changing, that discourse. On the other hand meaning is constituted by the individual. Nothing has any meaning until it is given such by the individual although there is an assumption of common rationality. Interactions with others and with reality are not denied, but are seen as outside the subject, with her or his personal construal of those interactions and experiences being the essence of meaning for that individual. In fact to talk of 'outside' makes no sense are there is only the individual's construal. Consequently the following figure may be taken as a first approximation of psychological theories:

Figure 1



In contrasting theories in psychology in terms of the binary opposition of cultural or individual it may appear that I am excluding the possibility of both being part of the theory. On the contrary both positions conceptualise both the 'socio-cultural' and the 'individual', but how they do so is different in each case. I am proposing that taking the source of meaning as characterising the fundamental argument of each may offer a framework through which to compare how 'social' and the 'individual' are elaborated in each. Of course the binary opposition as presented frames the discussion in this paper, as do other elements such as choices of quotes, and the reader's attention is drawn to this.

The left-hand side embeds the individual in social practices. Consciousness is attributed to communication; without social intercourse human consciousness would be animal consciousness. Consequently the individual is acculturated from infancy, if not before, and concepts are absorbed from the specific cultures within which the individual develops. In different contexts, with different sets of social relationships, individuals occupy different *positionings* :

> In general, the subject's positioning will depend on the interplay of a number of factors, including:
> - language and the discursive features of the situation;
> - social differences in, say, class or gender terms; and
> - the subject's "investments of desire" (Hollway, 1989).
> (Evans & Tsatsaroni (in press))

There is a multiplicity of cultures which each individual inhabits and I draw on Evans and Tsatsaroni's notion of 'positioning' rather than 'positioned' as being less determined and also as an entry for the individual. Further, it carries a sense of always incomplete and shifting. Situations, for an individual, offer multiple potential meanings each carrying specific positionings and different ones are called up for different people. Evans uses the term 'particularities' here (Evans 1993). They argue that meanings are displaced along chains of signifiers which might have parts of them in the unconscious. The way that 'socio-cultural' and 'individual' signify is demonstrated in the following:

> Once it is grasped that the unity of the individual is borrowed from the unity of the exterior discourse the true force of the adverb 'culturally', which qualifies any doing, can be appreciated. The mathematics classroom is an exteriority which imprints the identity of the subject just as really as a chicken imprints the object present at its hatching.
> (Waywood 1993).

The term *positioning* is a central metaphor of the mind, emphasising how meaning is carried by social practices through which the individual is regulated.

The right-hand side also acknowledges the centrality of social interactions. However the individual is not positioned by those interactions but *construes* from them. Each individual has a unique private conceptual framework through which all interactions, social and sensori-motor, and graphic representations (von Glasersfeld 1992) are filtered or refracted. It may be the case that an individual is carried along by a particular social practice, but this happens because the individual construes her or his role or understanding in line with that practice. She or he can construe otherwise. Essentially it makes no sense to talk of knowledge existing anywhere but in the individual since every interaction is construed by the individual; there are potentially many worlds and each person constructs a particular world for her or himself. The use of 'individuality' and 'social' are demonstrated in the following:

> . . . we come to see knowledge and competence as products of the individual's conceptual organization of the individual's experience . . . (von Glasersfeld 1983 p. 66)
>
> In this sense it is legitimate to interpret Piaget's work as a social-cultural approach in which he explained the mathematical development of children as self-regulating, autonomous organisms interacting in their environments. He seemed to take the social-

cultural milieu of the children as a given without attempting to alter their most general experiences. (Steffe 1993 p. 3-4)

"Making sense", then can mean to construct ways and means of operating in a medium to neutralise perturbations induced through social interaction. (ibid p. 25)

In this case, *construal* is a central metaphor, emphasising as it does the centrality of the individual as meaning-maker. An issue for radical constructivism, which I take to be at this end of figure 1 is how it appears that a number of people construe the same thing at the same time, as in children creating some mathematics together in the classroom.

Thus each position has a notion of the individual and each has a notion of the social but they are different and distinct. At this juncture one may question the coherence of the social constructivist position in that it ignores the distinctions in understandings of the words social and individual, wanting to have the metaphor of *construal* as the meaning-making process at some times and *positioning* as the meaning-appropriating process at other times. However this is not the place to pursue the issue.

Psychology in Mathematics Education

I have suggested that radical constructivism is situated on the right of figure 1 and I would argue that the cultural psychology of Vygotsky is situated on the left. Another elaboration of the left-hand side would be Gergen's social constructionism. Hence below is an extension of the figure in the context of mathematics education:

Figure 2

social constructionism (Gergen's term) cultural psychology from Vygotsky e.g. Walkerdine	***social constructivism*** *both social and individual psychology* *e.g. Bauersfeld*	**radical constructivism** (von Glasersfeld's term) individual psychology from Piaget e.g. von Glasersfeld

There is an inevitable clumsiness of terms here since all positions appear to want to draw on the metaphor of 'construction' perhaps to avoid both platonist and empiricist views of how knowledge is acquired. These latter are both passive positions in the sense that knowledge is either innate or is transmitted by reality itself through unencumbered observation. Both ends of the diagram above contain notions of the activity of constructing concepts although from different perspectives. Gergen chose the name 'social constructionism' in order to distinguish it from constructivism (Gergen 1985). The expression 'socio-cultural theories' has been used (e.g. Wertsch 1991) and I have used it (Lerman 1994a) to draw together Vygotsky's work with post-structuralism. However, here it is argued that 'socio-cultural' signifies in different ways in both views, and the label social constructionism has been applied to the left side of figure 2. Whilst Vygotsky's work and post-structuralism are by no means synonymous, Vygotsky's choice of 'word', and in some interpretations of the Russian, 'meaning', as the unit of psychology opens the door to other later theories, such as post-structuralism, which examine the ways in which knowledge as power is situated in discursive practices.

I do not view the two positions to be ends of a continuum and consequently the line between the two ends is a broken one. I have set up the two psychological theories as being based in the choice of site of meaning and I have argued that intermediate positions are incoherent. Thus social constructivism is placed, in italics between the two positions but does not have a marker on the line. In the next two sections I will offer metaphors for the teacher and for the learner and examine how those terms carry and constitute the spaces in which the theories operate.

The Teacher

An appropriate choice of metaphor for the role of the teacher in constructivist language is perhaps *facilitator.* This perpetuates the notion that all the teacher can do is to facilitate the learner's constructions. There is no sense in which the teacher conveys, transmits or in any other way actively and directly affects or determines the learner's constructions. At the same time it is argued that the teacher can choose activities, classroom relationships etc. to aid those constructions. Whilst there is clearly a recognition of the teacher's special knowledge of what it is to communicate mathematics, to organise possible mathematical environments and the more general pedagogical knowledge of fostering reflection and abstraction, these skills or understandings do not manifest in terms of power in the classroom, nor do they 'exist' anywhere, as this would imply some pre-existing knowledge which the student would need to acquire. It should also be noted that mathematical communication does not carry any power or regulation of its own. This position on mathematical communication maintains what Piaget says of the semiotic function in general, that it:

> consists in the ability to represent something (a signified something: object, event, conceptual scheme, etc.) by means of a signifier which is differentiated and which serves only a representative purpose. (Piaget, quoted in Gruber and Voneche 1977 p.489)

i.e. it sees the signified as arising extra-discursively (Walkerdine 1988). The metaphor of facilitator carries a function for the teacher, although not a necessary one. There is a sense of the facilitator making something easier, as though the process can take place without her or him. Steffe (1993) talks in terms of 'provocations' by the teacher enducing the children to experience perturbations "which the children must generate", although "there is no necessary correspondence between the two" (p. 28). In what sense, then, can one understand that the teacher's interventions are essential in provoking learning? I would argue that it is in the sense that Piaget talks of interactions, social or otherwise, as producing learning; they have no regulating function, but offer the possibility for the individual to experience perturbations. The sense of social as being distinct from the individual, and the work being done entirely by the individual "because the children must generate the perturbations" contributes to the constitution of the theoretical space at the right side of figure 2.

On the other side of figure 2, an appropriate metaphor may be the teacher as *mediator.* The term acknowledges the position she or he occupies in apprenticing the learner into the particular discourse which constitutes the context; in our case the mathematics classroom. It makes overt the teacher's membership of the community of mathematicians, her or his familiarity with mathematics as a semiotic system, and admits to teaching as a task concerned with an imbalance of power relations by virtue, at least, of that community and that semiotic system. It engages with the notion of the teacher as someone who plays an active part in the students' learning as mediator, as do texts and classroom social relationships. It admits of something pre-existing the learner, or perhaps better expressed as in the environment of the learner, that is the culture(s) in which the learner is placed. It emphasises the necessary function of the teacher and/or other mediators; the mediation is essential for the process to take place. Indeed it makes no sense to speak of autonomous individuals or actions:

> The most central claim I wish to pursue is that human action typically employs "mediational means" such as tools and language, and that these mediational means shape the actions in essential ways . . . the relationship between action and mediational means is so fundamental that it is more appropriate, when referring to the agent involved, to speak of "individual(s)-acting with mediational means" than to speak simply of "individual(s)". (Wertsch 1991 p. 12)

If one thinks of language as such a mediational means the sense of Wertsch's claim becomes clear. In criticism of Vygotsky's argument about tools, Steffe says the following:

> Vygotsky's notion of internalization is an observer's concept in that what the observer regards as external to the child eventually becomes in some way part of the child's knowledge. But Bickhard (in press) has pointed out that there is no explanatory model of the process. (Steffe 1993 p. 30)

Articulated within the theoretical space that sets the individual as meaning-maker, Steffe's claim appears to be right. However in the theoretical space on the left of figure 2 the individual is constituted in the discourse and positioned by it. There is no separation of the individual from social interaction which connection is the process that needs to be explained in the radical constructivist perspective.

Learning

The two metaphors I will choose to exemplify the process of learning within the two spaces are *constructing* and *appropriating*. In the radical constructivist space individuals construct their own knowledge. As quoted above, von Glasersfeld argues for the centrality of the cognizing individual in articulating what is knowledge within the space:

> . . . we come to see knowledge and competence as products of the individual's conceptual organization of the individual's experience . . . (von Glasersfeld 1983 p. 66)

Individuals must do their own constructing. It is an active process which of necessity builds on the particular foundations, schemata, that exist. There are fundamental operations of the mind and, in relation to mathematics, it is these operations from which one conceptualises unit and plurality, the former derived from the conceptual construction of 'objects' and the latter from an awareness of repetition of the recognition of objects. Neither are merely the result of sense impressions but the result of reflective abstraction on those sense impressions. A further notion is needed, that of number, and von Glasersfeld argues (Steffe et al 1983) that it may arise from the activity of counting. Thus mathematics is a matter of internal mental operations and meaning is an association of mental operations with mathematical symbols. This is a private process, which "cannot be witnessed by anyone else"; all one has to go on is the visible results of those mental operations, the writing, speaking or other behaviour of students. Nevertheless, von Glasersfeld argues that the teacher's task is to "stimulate and prod the student's mind to operate mathematically", and that teaching "has to be concerned with understanding" (von Glasersfeld 1992). One does not live in someone else's house, one has to construct one's own.

The notion of *appropriation*, on the other hand, carries a sense of making one's own something which also, and already, belongs to other people. In this way, the metaphor emphasises the function of communication and the cultural interpretation of consciousness. At the same time it indicates the internalisation of, in some senses the positioning by, cultural life and experience. It also carries with it a notion of the need for tools through which appropriation takes place. Vygotsky offers the following:

> . . . examples of psychological tools and their complex systems: language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes; diagrams; maps and mechanical drawings; all kinds of conventional signs" (Vygotsky 1981 p. 137)

Conclusion

Drawing the chosen metaphors together, they may be summarised as in the figure below:

Figure 3

social constructionism
(Gergen's term)
cultural psychology (from Vygotsky)
meaning as *positionings*
teacher as *mediator*
learning as *appropriation*

radical constructivism
(von Glasersfeld's term)
individual psychology (from Piaget)
meaning as *construal*
teacher as *facilitator*
learning as *construction*

I have argued that these two positions can be seen as two distinct theoretical spaces within which notions such as individual, and socio-cultural, teaching and learning are all conceptualised, but differently. In an attempt to create a meta-discourse in which these distinctions and differences can be identified I have drawn on the origins of meaning-making. There are many other aspects that can ~~be examined within the meta-discourse. For example one could examine the historical roots of these~~ positions. Piaget's psychology was developed in a context of a search for individual freedom through rationality and focused on a natural, biological metaphor for development that would argue that the human being can make appropriate autonomous choices in the correct environment, in our case the correct construction of mathematical knowledge. It has been argued that 'natural' rationality has enabled a process of regulation of the child on the basis of what is normal (Walkerdine 1984). Vygotsky's psychology was an attempt to develop a full cultural psychology which would draw on Marx's social theory: "The mode of production of material life conditions the general process of social, political and intellectual life" (Marx (originally 1859) 1970 p. 21). Other aspects to be examined would include the readings of research within the two spaces.

In particular, this paper is an attempt to engage with how learning theories and their metaphors carry and create theoretical spaces within which research activity, interpretation, theorisation and debate are constituted and perpetuated and to provoke discussion amongst our community. Notions such as 'individual' and 'socio-cultural' signify differently in these spaces and to ignore the differences is to encourage incommensurability.

References

Brown T. (in press) "Creating and Knowing Mathematics Through Language and Experience" Educational Studies in Mathematics

Evans J. & Tsatsaroni A. (in press) "Language and "Subjectivity" in the Mathematics Classroom" in S. Lerman (Ed) *The Culture of the Mathematics Classroom* Dordrecht:Kluwer

Evans J. 1993 *Adults and Numeracy* Unpublished Ph.D Thesis, University of London Library.

Gergen K.J. 1985 "The Social Constructionist Movement in Modern Psychology" *American Psychologist* Vol. 40, No. 3, 266-275

Gruber H.E. & Voneche J.J. 1977 *The Essential Piaget* London:Routledge and Kegan Paul

Lerman S. 1992 "The Function Of Language In Radical Constructivism: A Vygotskian Perspective" *Proceedings of Sixteenth Meeting of the International Group for the Psychology of Mathematics Education,* New Hampshire, Vol. 1.

Lerman S. 1993 "The Position of the Individual in Radical Constructivism: In Search of the Subject" in J. Malone & P. Taylor (Eds) *Constructivist Interpretations of Teaching and Learning Mathematics* Curtin University of Technology, Perth.

Lerman S. 1994a "From an Individualistic to a Socio-Cultural View of Learning: The Case of 'Western' Mathematics Education" paper presented at conference "L.S. Vygotsky and School", Eureka Free University, Moscow

Lerman S. 1994b "Playing with the Self: Context and Discourse" *Chreods* No. 7

Lerman S. (in press) "Changing Focus in the Mathematics Classroom" in S. Lerman (Ed) *The Culture of the Mathematics Classroom* Dordrecht:Kluwer

Marx K. 1970 *A Contribution to the Critique of Political Economy* Moscow:Progree

Steffe L. P., von Glasersfeld E., Richards J. & Cobb P. 1983 *Children's Counting Types: Philosophy, Theory and Application* NY:Praeger Scientific

Steffe L. 1993 "Interaction and Children's Mathematics" Paper presented at American Educational Research Association, Atlanta, Georgia

Vygotsky L. S. 1981 "The Instrumental Method in Psychology" in J. Wertsch (Ed) *The Concept of Activity in Soviet Psychology* White Plains NY:Sharpe

von Glasersfeld E.1983 "Learning as a Constructive Activity" *Proceedings of the Fifth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Montreal, Vol. 1, p. 41-69.

von Glasersfeld E.1992 "A Radical Constructivist View of Basic Mathematical Concepts" paper presented at Topic Group 16, ICME7, Quebec

Walkerdine V. 1984 "Developmental psychology and the child-centred pedagogy: The insertion of Piaget into early education" in J. Henriques et al (Eds) *Changing the Subject* London:Methuen

Walkerdine V. 1988 *The Mastery of Reason* London:Routledge

Waywood A. 1993 "Looking for an Appropriate Object for Educational Research: A Reflection on Three Presentations" unpublished paper, Australian Catholic University, Christ Campus, 17 Catlebar Road, Oakleigh VA 3166, Australia

Wittgenstein L. 1958 *Philosophical Investigations* Oxford:Blackwell

Wertsch J.V. 1991 *Voices of the Mind: A Sociocultural Approach to Mediated Action* Cambridge MA:Harvard University Press.

STUDENTS' CONSTRUCTIONS OF GROUP ISOMORPHISM

Uri Leron, Orit Hazzan, Technion - Israel Institute of Technology, Haifa, Israel
Rina Zazkis, Simon Fraser University, Vancouver, BC, Canada

Abstract

This article is concerned with how undergraduate students understand the concept of group isomorphism. Our analysis is based on clinical interviews conducted with students in their first Abstract Algebra course. Students' constructions, difficulties and misconceptions are described and analyzed.

Introduction

This article is part of an ongoing international research on learning Abstract Algebra. It is concerned with how students in their first Abstract Algebra course, construct and discuss the concept of group isomorphism. In a number of recent studies, various topics in the learning of Abstract Algebra were investigated (Dubinsky, Leron, Dautermann & Zazkis, preprint; Hart, in press; Selden & Selden, 1987), but none so far had its main focus on isomorphism .

Informally, two groups are isomorphic if they are "the same except for notation". Thus, if we take any group and rename its elements, we get an isomorphic copy of the same group. The formal definition is quite far removed from this intuitive notion. An isomorphism from a group [G,o] to a group [G',o'] is a one-to-one function f from G onto G', satisfying f(a o b) = f(a) o' f(b) for all a,b in G. Two groups are called isomorphic if there exists an isomorphism from one to the other.

The understanding of isomorphism is built on understanding other concepts in group theory, like the concepts of group, order of element, the group's properties and the relation between properties of a group and properties of an element. Therefore, the way in which students construct the concept of isomorphism can reveal how they perceive additional concepts in Abstract Algebra.

In addition, the concept of isomorphism is combined with other mathematical concepts, like those of function and existential quantifier. These related topics, have been treated in previous studies (Dubinsky, Elterman and Gong, 1989; Dubinsky, 1993; Harel and Dubinsky, 1992; Leinhardt, Zaslavsky and Stein, 1990). Our analysis of students' understanding of the concept of isomorphism may also provide additional insight on these more general topics.

Context - description of the research

The main body of the research consists of in-depth, semi-structured interviews with 5 students, participating in the Abstract Algebra course taught by two of the authors to computer science majors at the Israel Institute of Technology (IIT). The interviews were taken 7 weeks after the final exam, and lasted 60-90 minutes each. All interviews were audiotaped and transcribed. The interview questions were presented to the students verbally, but were based on a pre-prepared questionnaire. The questionnaire, as well as the data from the interviews were translated into English (from the original Hebrew) for this report.

Some research results

In this paper we focus on three observations which deal with students' constructions and discussions of isomorphism. For additional data and a fuller description of the research results consult Leron, Hazzan and Zazkis (preprint). Our first observation concerns students' craving for "canonical" procedures, the other two deal with the concept of existential quantifier and the concept of function-from-G-to-G'. The observations re-occurred in the interviews often enough to convince us that they represented more than just accidental negligence on the part of some students.

Observation a. Students get stuck in constructing an isomorphism between specific groups, when there is more than one way to proceed.

This phenomenon may be explained as a misconception about existential quantifiers: "there exists a function" is taken to mean "there exists a unique function" or, alternatively (and more plausibly), "there is a canonical, algorithmic, way to construct a function".

Attempting to build a correspondence between G and G^o directly, some students are stalled in their efforts because of the variety of possibilities to create such a matching. Having followed the usual sequence of checks, they successfully match the identity of G to the identity of G^o and then turn to calculate the orders of the elements. Upon encountering three elements of order 2 and two elements of order 3 in these groups, they get lost. They can't decide which match to choose and therefore fail to construct any.

The following excerpt from an interview on the question in Figure 1 demonstrates our observation.

> David: [...] It's not going to go like that. Must think of the exact definition.
> Int: What do you mean "it's not going to go like that"?
> David: Start changing now... instead of each letter with circle, to find the corresponding one in G.
> Int: Why? Why shouldn't it go like that?
> David: Because I don't know for example what to put instead of a. Because a instead of a^o is of order 2, but I can put it in three ways. So I don't remember any more if it makes a difference. I think it should make a difference what element of order 2 I choose.

One explanation for this phenomenon can be found in their school experience, where one is always concerned with computational processes (to find a solution, a GCD, etc.) rather than with existence of abstract objects. Proving existence is a more abstract and mature process than performing some computational procedure. Another conjecture of an affective nature may be added: There is much more feeling of security in performing an algorithmic process, where each step is determined by the previous ones, than in trying to construct something from scratch under conditions of vagueness and uncertainty.

Before moving to the next two observations, we attend to some possible mental constructions of the definition of group isomorphism. By its very definition, the isomorphism concept deals with the existence of a certain function on groups. Thus in analyzing students' interviews about isomorphism, we can distinguish levels of understanding of the various concepts involved: group, function and existential quantifier. The following statement, taken from one of our interviews, is a typical example of partial understanding, in which the three conceptions are intertwined: "... these two groups are isomorphic because I can find a one-to-one function from each element in G to each element in G' ".

Theoretically, it is possible to investigate separately the developmental stage of the group, the function and the existential quantifier conceptions. The distinction that turned out to be appropriate for analyzing our data, has been between existential quantifier on the one hand and the compound concept "function from G to G'" on the other.

Following are three group operation tables. Your task is to decide which are isomorphic to each other and which are not.
[Answer: G and G^o are isomorphic to S_3, and G' is isomorphic to Z_6.]

*	a	b	c	d	x	y
a	d	y	x	a	c	b
b	c	d	a	b	y	x
c	b	x	y	c	a	d
d	a	b	c	d	x	y
x	y	c	b	x	d	a
y	x	a	d	y	b	c

G

o	a'	b'	c'	d'	x'	y'
a'	b'	c'	a'	y'	d'	x'
b'	c'	a'	b'	x'	y'	d'
c'	a'	b'	c'	d'	x'	y'
d'	y'	x'	d'	b'	a'	c'
x'	d'	y'	x'	a'	c'	b'
y'	x'	d'	y'	c'	b'	a'

G'

x	a^o	b^o	c^o	d^o	x^o	y^o
a^o	y^o	c^o	b^o	x^o	d^o	a^o
b^o	x^o	y^o	d^o	c^o	a^o	b^o
c^o	d^o	a^o	x^o	b^o	y^o	c^o
d^o	c^o	x^o	a^o	y^o	b^o	d^o
x^o	b^o	d^o	y^o	a^o	c^o	x^o
y^o	a^o	b^o	c^o	d^o	x^o	y^o

G^o

Figure 1

Observation b. We have identified three main types of expressions used by students to refer to the existential quantifier:

(i) "... I can find a function ...",

(ii) "... it is possible to find a function...",

(iii) "... there exists a function...".

These three phrases express different degrees of student's involvement with the mathematics. The first one has a strong component of "me doing something"; it can be characterized as "first person, procedural language". The second expression retains the procedural character, but employs the neutral third person. The last expression employs the fully detached, declarative style of formal mathematics. We conjecture that these three expressions correspond to three levels of development of the existential quantifier conception. Employing the terminology used by researchers who take a developmental perspective on the mental constructions of mathematical concepts, we may conjecture that these three stages correspond to a path from action or process conception of the existential quantifier to its conception as an object. (Douady,1985; Dubinsky, 1991; Sfard, 1987, 1991).

Observation c. This observation parallels the preceding one. Here also we have identified three types of expressions used by students to refer to the compound concept function-from-G-to-G'. The isomorphism function in these expressions maps

(I) "each element of G to each element of G' ",

(II) "the elements of G to the elements of G' ",

(III) "G to G' ".

We note that unlike school mathematics, where functions are often treated without explicit mention of the sets on which they operate, in the context of isomorphism the function cannot be separated from its domain and range: we must talk about isomorphism from one group to another. Thus, the development of the function part of the isomorphism concept must be coordinated with that of the group part, or at least the set of the group.

As in the previous case, using a developmental terminology , these three stages may correspond to the path from action or process conception of function to its conception as a mathematical object.

Observations b and c are usually intertwined in our interviews, and the data for both is presented together. In fact, we show data in which students exhibit various combinations of development for the two concepts involved. Using the numbering of the three expressions in the two observations, we will refer to these combinations as (i-II), (ii-I), etc. All the excerpts shown are taken from students' answers to the question in Figure 2.

> 1. Suppose you are given two groups. How can you tell whether they are isomorphic? How can you convince someone that they are isomorphic? How can you prove it?
> [In case the student doesn't give a full answer including functions, the interviewer proceeds to ask the following question:]
> 2. What is isomorphism?
> [Now question 1 is repeated to see if the answer to 2 has induced any change.]

Figure 2

Ron (ii-III): Two groups are isomorphic if it is possible to find a function which maps from one to the other so that the order of the elements is preserved and the operation is preserved.

Dan (ii-I or ii-III): Two groups are isomorphic if [...] it is possible to find a function between them which connects in a one-to-one way every element to every element, every element in group A to every element in group B.

Saul (ii-I or iii-I): [To show that two groups are isomorphic] in principle it is necessary to find some map between every element here to every element there [...] If such a map exists and the two groups behave similarly then [...]

Dan (i-II) : [Two isomorphic groups are] a pair of groups which I can match a full map, a one-to-one map between their elements [...]

Ben (i-I): I take an element from one group and find its image in... exactly as with function in mathematics.

Conclusion

One interesting feature of research on learning a particular topic is the interplay between the specific and the general. Indeed, the very concept of isomorphism is but a formal expression of many general ideas about similarity and difference, most notably the idea that two things which are different, may be viewed as similar under an appropriate act of abstraction. Here are two more examples:

First, the phenomenon of students getting stuck in constructing a specific isomorphism may be seen as a special case of the general phenomenon of students' craving for canonical procedures and their fear of loose or uncertain procedures, in fact, procedures with any degree of freedom.

Second, difficulties with formulating definitions related to isomorphism may be mere special cases of difficulties with functions and quantifiers; more specifically, difficulties with *quantifying over functions*. This in turn may be related to the even more general developmental issues of the processes and objects involved in constructing the mathematical conceptions involved.

Acknowledgment: The authors would like to express their gratitude to Ed Dubinsky and to Anna Sfard for their helpful comments.

References

Douady, R.: 1985, 'The interplay between different settings: Tool-object dialectic in the extension of mathematical ability - Examples from elementary school teaching, in L. Streefland (ed.), *Proceedings of the Ninth International Conference of PME*, Vol. 2, pp. 33-52. Utrecht, The Netherlands: State University of Utrecht, Sub faculty of mathematics, OW & CO.

Dubinsky, E., F. Elterman, and C. Gong: 1989, 'The student's construction of Quantification', *For the Learning of Mathematics* 8(2), 44-51.

Dubinsky, E.: 1991, 'Reflective abstraction in advanced mathematical thinking', in D. Tall (ed.), *Advanced Mathematical Thinking*, Kluwer Academic Publishers.

Dubinsky, E.: 1993, 'On learning quantification', in M. J. Arora (ed.), *Mathematics Education into the 21st Century*, UNESCO.

Dubinsky, E., U. Leron, J. Dautermann and R. Zazkis: preprint, 'On learning fundamental concepts of group theory'

Harel, G. and E. Dubinsky (eds.): 1992, *The Concept of Function: Aspects of Epistemology and Pedagogy*, MAA Notes.

Hart, E. W.: in press,. 'A conceptual analysis of the proof-writing performance of expert and novice students in elementary group theory', in E. Dubinsky and J. Kaput (eds.), *Research Issues in Math Learning: Preliminary Analyses and Results*, MAA Notes.

Leinhardt, G., O. Zaslavsky and M. K. Stein: 1990, 'Functions, Graphs, and graphing: Tasks, learning and teaching', *Review of Educational Research* 60(1), 1-64.

Selden, A. and J. Selden: 1987, 'Errors and misconceptions in college level theorem proving', in *Proceedings of the Second International Seminar - Misconceptions and Educational Strategies in Science and Mathematics*, Cornell University, NY.

Sfard, A.: 1987, 'Two conceptions of mathematical notions: Operational and structural', in *Proceedings of the Eleventh International Conference of PME*, Montreal, Vol III, pp. 162-169.

Sfard, A.: 1991, 'On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides on the same coin', *Educational Studies in Mathematics* 22, 1-36.

CHARACTERISTICS OF EFFECTIVE MODEL-ELICITING PROBLEMS

Miriam Amit	Eamonn Kelly	Richard Lesh
Israel Ministry of Education	Rutgers University	Educational Testing Service

How can we democratize access to powerful elementary-but-deep math constructs? How can we recognize and reward a broader range of students' mathematical abilities? A key to success is to focus on a small number of elementary-but-powerful constructs (student-constructed conceptual models) and to focus on tasks which require deeper and higher-order understandings of these constructs. Then, using principles which will be described in this report, we create "model eliciting activities" in which students construct one or more of the preceding "big ideas" (or models for making sense of experiences within complex systems). - - - The preceding principles were developed by expert teachers in a series of "three-tiered teaching experiments" (which is a powerful new constructivist research methodology which will be described in the session.

Introduction: The essence of an *age of information* is that many of the most important "things" that influence peoples' daily lives are systems: communication systems, social systems, economic systems, education systems, and other systems which are created by humans based on elementary-but-deep constructs (i.e., models, structural metaphors) which students also must develop. structure the world at the same time they structure the student's interpretations of their experiences. In technology-based societies, these constructs tend to be embedded in powerful conceptual technologies which are used on a daily basis in fields ranging from the sciences to the arts, in professions ranging from agriculture to business and engineering, and in employment positions ranging from entry-level to the highest levels of leadership. Consequently, these tools radically expand: (i) the kinds of knowledge and abilities which are needed for success in a technology-based society, and (ii) the kinds of problem-solving/decision-making situations which are priorities to address in instruction and assessment. For example:

> When business managers use graphing calculators (or spreadsheets) to make predictions about maximizing cost-benefit trends, these tools not only amplify the manager's conceptual and procedural capabilities when dealing with old decision-making issues, they also enable the manager to create completely new types of business systems which did not exist before the tools were available; and, completely new types of problems and issues arise as priorities to address.

In a world where new conceptual and procedural tools are being used for new purposes in new kinds of problem solving situations, past conceptions of mathematical ability are often far too narrow, low-level, and restricted. Therefore, it is misleading to speak of treating students fairly on a given test (or textbook, or teaching program) if the test as a whole reflects obsolete, superficial, and instructionally-unproductive biases about the nature of mathematics, problem solving, teaching, and learning. But, how can a broader range of mathematically capable students be identified and encouraged? Our research suggests that one of the main keys to success is to focus on a small number of "big ideas" which are based on constructs (i.e., student-constructed models, or structural metaphors) which have proven to have the greatest power and usefulness for success in for describing, explaining, predicting, manipulating, and controlling complex systems in "real life" problem solving situations. Then, using principles which will be described in this section, we create "model eliciting activities" in which students construct one or more of the preceding models for making sense of experiences within complex systems).

Based on research with hundreds of students in the preceding kinds of "real life" problem solving situations, the following results have emerged consistently: (1) Even students whose prior experiences in school suggested that they are far "below average" in mathematical ability, their performance on such activities routinely shows that they are able to invent (or significantly extend or refine) mathematical models which provide the foundations for the small number of "big ideas" that lie at the heart of the mathematics courses in which they were enrolled, (2) the mathematical models that they construct (which psychologists might refer to as "cognitive structures) are often far more complex and sophisticated than those that previous teaching and testing experiences suggested they were unable to be taught, and (3) new students are not likely to emerge if modeling is treated as another attempt to teach content-independent Polya-style heuristics, strategies, and processes, (POs: process objectives), or if the applications are used mainly as devices to increase motivation and interest (AOs: affective objectives), or if the applications are used only as contexts for tedious chains of low-level facts and skills (BOs: behavioral objectives).

Theoretical Framework: The theoretical framework for our studies is based on a form of constructivism in which attention is focused on the nature of the constructs that students develop ... at least as much as it is focused on the processes that students use to develop and refine these models. Like other constructivist approaches, however, our theoretical perspectives are strongly influenced by the following shift from mechanistic to organic/systemic assumptions about the nature of mathematics, problem solving, learning, and teaching.

	Mechanistic Perspectives	Organic/Systemic Perspectives
The Nature of Mathematics	The objectives of instruction are stated in he form (*Given ... the student will ...*); and, knowledge is described using a list of mechanistic condition-action rules (definitions, facts, skills) ... some of which may include higher-order metacognitive rules for making decisions about: (i) which lower-level rules should be used in a particular situation, and (ii) how lower-level rules should be stored and retrieved when needed.	Knowledge is likened, not to a machine, but to a living organism. Many of the most important *cognitive objectives* of mathematics instruction are descriptive or explanatory systems (i.e., mathematical models) which are used to generate predictions, constructions, or manipulations in real life problem solving situations ... or whose underlying patterns can be explored for their own sakes.
The Nature of Problem Solving	Problem solving is described as *getting from givens to goals when the path is not obvious.* But, in mathematics classrooms, problem solving is generally restricted to answering questions which are posed by others, within situations that are described by others, to get from givens to goals which are specified by others, using strings of facts and rules which are restricted in ways that are artificial and unrealistic. In this way, students' responses can be evaluated by making simple comparisons to the responses expected by the authority (the teacher). - - - Problems in textbooks and tests tend to emphasize the ability to create meanings to explain symbolic descriptions.	Important aspects of real life problem solving involve developing useful ways to interpret the nature of givens, goals, possible solution paths, and patterns and regularities beneath the surface of things. Solutions typically involve several "modeling cycles" in which descriptions, explanations and predictions are gradually refined and elaborated. Therefore, several levels and types of responses are nearly always acceptable (depending on purposes & circumstances); and, students themselves must be able to judge the relative usefulness of alternative models. - - - Real problems more often emphasize the ability to create symbolic descriptions to explain (manipulate, predict, or control) meaningful situations.

The Nature of Experts	Humans are characterized as information processors; and, outstanding students (teachers, experts) are those who flawlessly remember and execute factual and procedural rules .. and who are clever at assembling these facts and rules in ritualized settings.	Humans are model builders, theory builders, and system builders; and experts are people who have developed powerful models for constructing, manipulating, and making sense of structurally interesting systems. Experts also are people who are proficient at adapting, and extending, or refining their models to fit new situations.
The Nature of Learning	Learning is viewed as a process of gradually adding, deleting, and debugging mechanistic condition-action rules (definitions, facts, or skills). If the precise state of knowledge is known for an expert (E) and for a given novice (N), then the difference between these two states is portrayed as the subtracted difference (E-N).	The models students construct develop along dimensions such as concrete-to-abstract, particular-to-general, undifferentiated-to-refined, intuitive-to-analytic-to-axiomatic, situated-to-decontextualized, and fragmented-to-integrated. So, development often involves conceptual discontinuities and reorganizations ... such as when students go beyond thinking WITH a given model to also think ABOUT it. - - - Experts not only know more than novices, they also know differently.
The Nature of Teaching	Teaching is considered to involve mainly: (i) demonstrating relevant facts, rules, skills, and processes, (ii) monitoring activities in which students repeat and practice the preceding items, and (iii) correcting errors that occur.	The teaching is shifting from authoritarian models based on "transmission of knowledge" to a student-centered practice featuring stimulation of learning. Teachers are catalysts who help students learn to think for themselves. They should not act solely as trainers whose role is to show the "right way" to solve problems.

Methodology: In the past, to investigate the nature of students' developing mathematical knowledge and abilities, we have found it useful to focus on *teaching experiments* which can be characterized as *longitudinal studies of conceptual development within mathematically rich environments* in which it is possible to simultaneously stimulate, facilitate, and investigate the evolution of particular knowledge and abilities. Similarly, to investigate the development of teachers' mathematical-psychological-instructional knowledge, teaching experiments with students often provide excellent contexts for teaching experiments with teachers where, once again, it is possible to create conditions which optimize the chances that development will occur without dictating the direction of development. We refer to such studies as two tiered teaching experiments, and, for both tiers, the goal of the researcher is to create a context in which the participants simultaneously learn and document what they are learning. To accomplish this goal, the environments at both tiers must stimulate the development, testing, and refinement of new constructs.

In the case of the studies that will be reported here, the activities for both students and teachers centered around tasks that it is currently fashionable to refer to as involving *authentic performance*. That is, are authentic in the sense that they are not just surrogates for activities that are meaningful in "real life" situations, they involve actual work samples taken from a representative collection of activities which are important in themselves; and they are performance activities in the sense that they focus complex holistic performances rather than simply piecemeal chains of isolated facts and skills. - - - But, what are the key characteristics of an effective authentic mathematical problem solving activity for a middle school student? And, what is a similarly authentic performance for their teachers?

Answering the first of the preceding questions was in fact the goal of the teacher-tier of the project. That is, participating teachers worked together as co-researchers to formulate a useful answer to the first of the preceding question. The result is the principles that will described in the next section of this paper. Then, these evolving conceptions of effective performance assessment activities were used to help teachers clarify their own conceptions about the nature of: (i) real mathematics, (ii) "real life" situations, (iii) realistic tasks, questions, or decision-making issues, and (iv) realistic tools and resources. Therefore, in the case of the studies that will be reported here, authentic performance activities for teachers involved developing the following resources to be used in their classrooms.

- Examples of high quality performance assessment activities in which the chances are high that students will construct mathematically significant conceptual models, as well as revealing conceptual strengths and weaknesses related to this model.
- Examples of students' alternative ways of thinking about the preceding activities, and examples of various levels of quality for the performances that are required.
- Classroom observation forms which teachers can use to observe (i) roles that various students play during solution attempts, (ii) conceptual strengths and weaknesses that can be taken into account in follow-up instruction, and (iii) strategies and procedures that are useful at various stages during solution attempts
- Quality assessment guidelines and procedures to help teachers recognize the most important criteria for determining the strengths and weaknesses of alternative approaches to the problems.

Results: Results of the preceding activities will provide the data (in the form of videotapes, and written student portfolios) which will be presented at the PME research reporting sessions on which this paper is based. This remainder of this section briefly describes six principles which can be used to assess the quality of model-eliciting activities which are intended to be used for purposes such as "performance assessments" in curriculum reform efforts. These principles were developed by expert teachers during a series of NSF-funded research projects in which teachers, parents, administrators, and community leaders worked together for periods of fifteen weeks to develop: (i) a small library of exemplary "real life" projects for children, and (ii) principles for writing such projects, and for assessing their quality - - - In each case, we began with the assumption that the kind of model-eliciting activities we were seeking to define would have at the following characteristics

- Solutions require approximately 15-60 minutes to construct, and they provide powerful prototypes for dealing with issues that are important to the students or others they would like to impress.
- Issues fit the interests and experiences of targeted students, and they encourage students to engage their personal knowledge, experience, and sense-making abilities.
- Solution procedures encourage students to use realistic tools and resources, including calculators, computers, consultants, colleagues, and "how to" manuals.
- Evaluation procedures recognize more than a single type and level of correct response.
- Overall activities contribute to both learning and assessment ... because student simultaneously learn and document what they are learning

The principles that were identified included the following.

- The Reality Principle: Beyond referring to real objects and events, would this question really occur in a "real life" situation? Will the student know who is asking for the result and why? (If not, how can appropriate decisions be made about the relative importance of speed, accuracy, precision, risks, benefits?) Will students be encouraged to make sense of the situation based on extensions of their own personal knowledge and experiences? Or, as in the case of the example that is given in Appendix C, must they "turn off" their real life knowledge and experience in order to give the response that conforms to the teacher/textbook/test's (often perverse) notion of the (only) "correct" solution process?

- The Model Construction Principle: Does the task create the need for a model to be constructed, or modified, or extended, or refined? Is the goal similar to those in case studies in professional schools ... where the task is not simply to answer some question, but also involves constructing a structurally significant system... or constructing a description or explanation to manipulate, predict, or control a structurally significant system? Is attention focused on underlying patterns and regularities rather than on surface-level characteristics?

- The Simple Prototype Principle: Is the situation as simple as possible, while still creating the need for a significant model? Will the solution provide a useful prototype (or metaphor) for interpreting a variety of other structurally similar situations? Will it be useful to refer back to these metaphors ... as students go on to learn new things (or solve new problems) in school or in their everyday lives?

- The Model-Documentation Principle: Will the response require students to explicitly reveal how they are thinking about the situation (givens, goals, possible solution paths)? If so, then it should be possible to look at the responses that they generate and name what kind of system (mathematical objects, relations, operations, patterns, regularities) they were using to think about the situation?

- The Self-Evaluation Principle: Are the criteria clear for assessing the usefulness of alternative responses? Will students be able to judge for themselves when their responses are good enough (and for what purpose the results are needed, and by whom, and when)? [note: One reason why self-assessment is so important is to give students a sense of power; but, another reason is that, when responses are constructed for model eliciting activities, students usually need to go through a series of modeling cycles in which givens, goals, and possible solution steps tend to be interpreted in a variety of alternative ways So, if students are unable to assess the relative strengths and weaknesses of a given interpretation, then it will be unlikely that they will ever go sense the need to go beyond their first interpretation which usually tends to be rather primitive, barren, distorted compared with later interpretations which would have developed.]

- The Model Generalization Principle: Does the solution involve more than simply generating specific answers to isolated questions. Does the purpose involve constructing a model which can be applied to a broader range of situations?

Throughout the preceding activities, our goals have been: (i) to democratize access to powerful elementary-but-deep "big ideas" in mathematics and science, (ii) to create activities in which students will be able to (simultaneously) learn and document their achievements, and (iii) to recognize and reward a broader range of "real life" mathematical or scientific abilities ... and, in so doing, to recognize and reward a broader range of students. To recognize and reward a broader range of students, it is important to avoid problems where the notion of the (only) "correct" answers require students to "turn off" their real life knowledge and experiences. It is important to encourage them to make sense of the problem situations based on extensions of their real life knowledge and experiences; and, one effective way that we have found to do this is to use familiar "stories" to create the context for each problem.

Topics for Discussion: In addition to giving examples of cases where teachers used the preceding principles to develop, critique, and refine effective model-eliciting activities, this session will examine samples of students work, and explain brief answers to the following four discussion questions (which teachers commonly ask about using model eliciting" activities in instruction and assessment.

- How can I afford to spend whole class periods on single problem solving situations?
- When it took professional mathematicians so many centuries to invent the ideas that are addressed in modern textbooks, how can students be expected to construct these ideas during a small, open-ended, unstructured problem-solving activities?
- Why is it reasonable to expect my students to invent mathematics?
- What role(s) should teachers play when "model eliciting" activities are used in novel ways ... such as when they are used at the beginning of a unit of instruction, when the main purpose is to identify conceptual strengths and weaknesses that exist in a given group of students?

ON ANALYZING PROBLEM-POSING PROCESSES: A STUDY OF PROSPECTIVE ELEMENTARY TEACHERS DIFFERING IN MATHEMATICS KNOWLEDGE

Shukkwan S. Leung

National Chia Yi Teachers' College
Taiwan, R.O.C.

The purpose of this study was to extend the quantitative evaluation of problem-posing products using the Test of Arithmetic Problem Posing (TAPP) by including a qualitative evaluation of problem-posing processes. Process referred to the process of posing a sequence of problems from a given situation described in a story form. Based on each sequence of problems given by an individual the experimenter prepared GPS behavior graphs to represent their solutions. Then the experimenter referred to the graphs and evaluated process according to "order and complexity" and "relatedness". The evaluation method was then used in the comparative study of prospective elementary teacher differing in mathematics knowledge.

Statement of the Problem

Until recently mathematical problem posing was identified as a critical important topic (NCTM, 1989) that was underrepresented in mathematics education research (Kilpatrick, 1987; Silver, 1993). Nevertheless, preliminary studies on various kinds of problem posing were conducted in countries all over the world (see studies cited in Silver, 1993). Specifically, researchers analyzed the end-products of problem posing (Leung, 1993; Mamona, 1993). In this study, the focus was on problem-posing processes. Leung (1993) reported the development of the Test of Arithmetic Problem Posing (TAPP) to quantitatively measure products. This report explained a further development of TAPP to qualitatively analyzed processes of prospective elementary teachers differing in mathematics knowledge.

Theoretical/Conceptual Framework

As suggested in prior classification of research variables in problem solving (Kilpatrick; 1978) process variable was also important in problem posing. Researchers in problem solving already suggested ways to study process. When only written solutions were available researchers have to make inference about process from the finished products. According to TAPP, subjects posed a sequence of problems and so we defined process as the process of posing a sequence of problems.

To study problem-posing process the GPS behavior graphing technique (Newell & Simon, 1972) can be adapted to display solutions of posed problems in fine grained detail. GPS graphs displayed solutions obtained by using means-ends-analysis (MEA), a problem solving strategy which solves problems by a stepwise reduction of difference between the given state and the goal state. MEA is an effective problem-solving strategy for solving arithmetic problems, products of TAPP and so GPS graphs can be used in out study of process.

We also used literature on problem solving to suggest two ways to investigate the process of posing a sequence of problems. The first is to see if the order of the posed problem increases with its complexity.

In a study on asking children to solve a problem again children who switched a strategy when they solved the problem again tended to switch from simpler sophisticated strategies (Silver, Leung & Cai, 1991). The second is to see if subjects "look at a related problem" (Polya, 1954); or, if the posed problems of the same sequence given by the same person are related to one another in problem structure.

If problem-posing process can be evaluated accordingly then we can revisit our data set of posed problems given by prospective elementary teachers. Our prior analyses on products already indicated quantitative differences in the production of products relating to a difference in mathematics knowledge (Leung, 1993). Here, it seemed reasonable to include as well a comparative analyses of process of subjects differing in mathematics knowledge. An investigation of qualitative differences in process, together with previous results on products, will add to our understanding of the role of mathematics knowledge in mathematical problem posing.

Methodology

The Instrument: Test of Arithmetic Problem Posing (TAPP). TAPP consisted of two problem situations: the House problem and, the Pool problem. The problem situations were modified tasks from the instrument on creativity used by Getzels and Jackson (1962). Each problem situation was presented in a paragraph form. Figure 1 shows one item of TAPP. Again, the evaluation of process according to TAPP includes two independent aspects: order and complexity, and. relatedness.

Figure 1

Example Item in Test of Arithmetic Problem Posing

> The Park District installs a swimming pool which holds a total capacity of 20,000 cubic feet. To fill the pool, two inlets with flow rates of 20 and 10 cubic feet per minute respectively are available. A drain will remove water at the rate of 25 cubic feet per minute. A circulating pump is provided which moves the water in the pool through a filtration system at the rate of 5 cubic feet per minute. When the pool is to be cleaned, as it is done once every week, the water is drained and the sides of the pool are scrubbed. The draining and scrubbing together require 15 hours.
>
> TIME: 20 minutes
> INSTRUCTIONS: Consider possible combinations of the pieces of information given and pose mathematical problems involving the operation of the pool. Do not ask questions like "Where is the pool located?" because this is not a mathematical problem.
>
> - Set up as many problems as you can think of.
>
> Think of problems with a variety of difficulty levels. Do not solve them.
>
> - Set up a variety of problems rather than many problems of the same kind.
> - Include also unusual problems that your peers might not be able to create.
> - You can change the given information and/or supply more information.
>
> When you do so, note the changes in the box with the problem to which they apply.
>
> - Write only one problem in each box.
>
> If you think of more problems than the number of boxes provided, write the others on the back of the sheet.

Subject. The subjects were eight (4 top and 4 bottom scores out of 49) prospective elementary school teachers. We chose to study them because they played a vital role in the implementation of problem posing into the classroom as suggested by the *Curriculum and Evaluation Standards* (NCTM, 1989).

Procedure and Task Administration. Two pilot studies were conducted to test the feasibility and appropriateness of the evaluation method. In the actual study, subjects first took PPST, a pre-test on mathematics developed especially for prospective elementary teachers (Educational Testing Service, 1986). The scores represented the subjects' mathematical knowledge. Subjects then took TAPP after the mid-term examination of the course. The test required them to pose a sequence of problems that can be attached to given problem situations in story forms. Each subject did both items of TAPP. In each item there were eight answer spaces provided.

Data Coding and Data Analysis. The experimenter first solved all problems in a sequence and drew the GPS graphs of the solutions. Each GPS graph shows the objects and goals specified by the poser and the operators and solutions supplied by the experimenter. The operators and solution paths indicate a possible solution by the experimenter. For the problem solution in each case, in order to have a single solution and to insure consistency across problems, the experimenter used the minimal path solution given by the experimenter. The experimenter checked the degree of agreement was checked with a second rater.

Since subjects generated the problems in sequence, it is reasonable to assume that they did not always repeat information they intended to be part of a later problem. When considering the objects requiring in solving but not stated in the space that contained the problem the experimenter assumed that they a) follow from the prior posed problems, or, b) come from the paragraph of information given in the task. If there were more than two prior posed problems that contain the objects, then the experimenter used the objects that appeared in the latest posed problems. In the accompanying figure that shows GPS graphs, the given objects given in TAPP are represented by dark circles (●) whereas the supplied objects by subjects and implied objects are represented by hollow circles (○) and dark squares (■) respectively. Implied objects were information not given in TAPP nor provided by subjects but required to solve the problem (e.g. 52 weeks in a year). Also, the subgoals and goals are represented by (SG) and (G) respectively. The operators are clearly marked and the direction of subtraction and division is indicated by an arrow. Figure 2 shows an example of GPS graph.

Figure 2.

Example of a GPS Graph of One Problem Posed by One Subject

"How long does it take to fill the pool if both inlets are functional and if there is no drainage?"

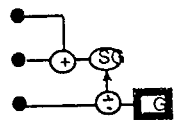
Objects

Flow rate 1 = 20

Flow rate 2 = 10

Capacity of pool = 20,000

No drainage



Goal

Time to fill capacity

By referring to the set of GPS graphs of a sequence of problems the experimenter evaluated the complexity by counting the number of steps one required to solve the problem to see if the complexity grew as the number in which it appeared in the sequence increased. Relatedness was evaluated according to how the subjects used conditions or goals of prior posed problems to pose subsequent problems. For example, if the second problem wrote "How long does it take to fill the pool of the 10 cubic ft/min. drain [inlet is clogged?" then the second problem is related to the first in that the other than clogging one inlet, the rest of the information followed.

Results and Discussions

In this section, there is a only a summary of the comparison of the high group to the low group, each with one example of a subject's process with respect to one problem (we chose the Pool problem because it was given in Figure 1); the full descriptions of each subject's process with respect to each item of TAPP was not given.

Example Evaluation of One High Subject (Subject H). Subject H worked on the Pool problem (refer to Figure 1) and posed 12 problems given in Figure 3. The represented GPS graphs of their solutions can be found in Figure 4.

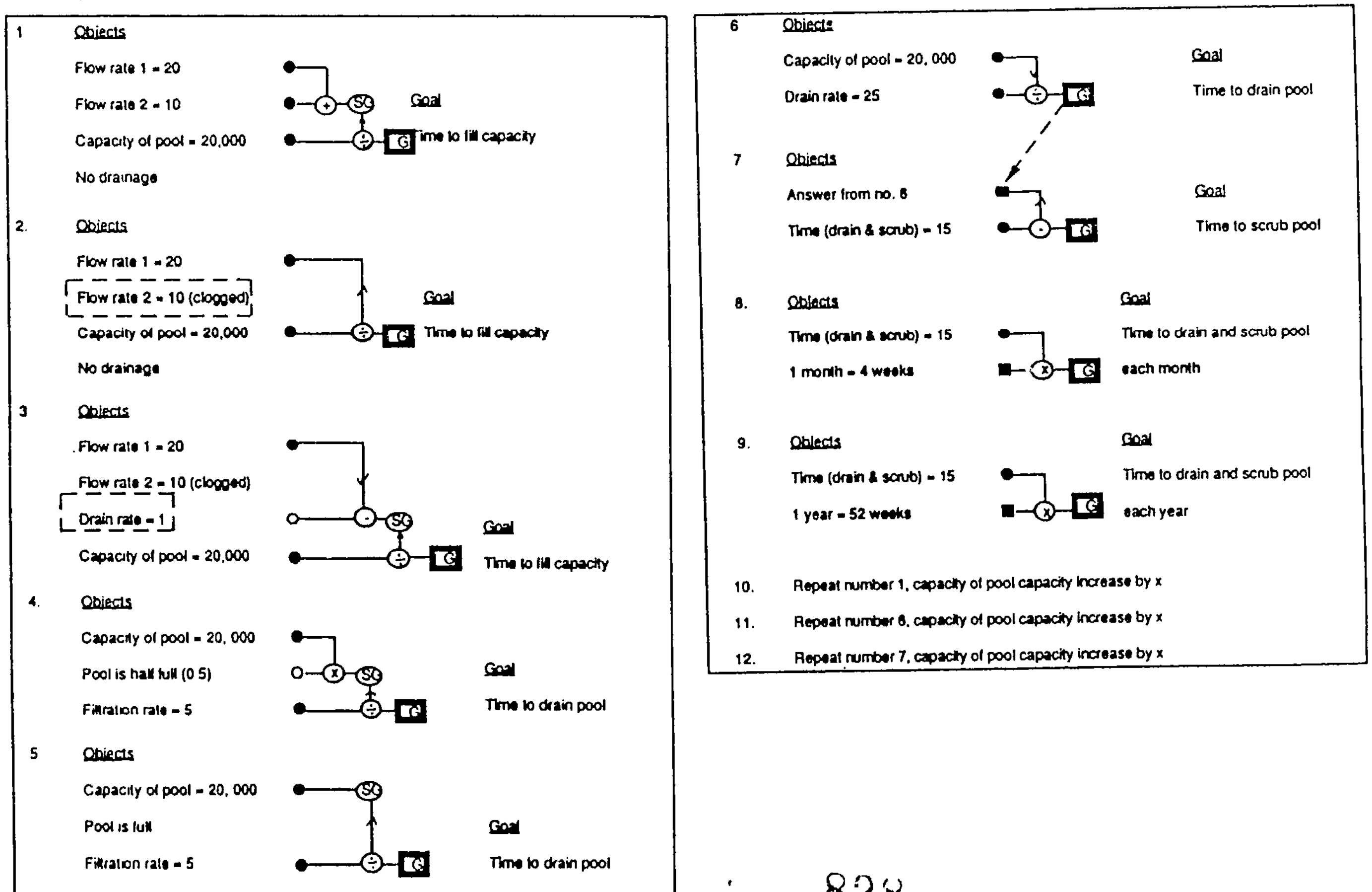
Figure 3

Set of Problems Posed by a Subject H

1 How long does it take to fill the pool if both inlets are functional and if there is no drainage?
2 How long does it take to fill the pool if the 10 cubic ft/min. drain [inlet] is clogged?
3 If the pool loses water through the drain at 1 cubic ft/min., how long will it take to fill the pool.
4 How long does it take to move the H_2O with the filtration system if the pool is half full?
5 If the pool if completely full?
6 How long does it take to drain the pool?
7 How long does it take to scrub the pool?
8 How much time is spent per month draining and scrubbing the pool?
9 How much time is spent per year?
10 If the carrying capacity of the pool was increased by x, how long would it take to fill the pool?
11 to drain the pool?
12 to scrub the pool?

Figure 4

GPS Graph Representation of Subject H



Referring to the above two figures we found that subject H might not posed a more complex problems when the problem number increased. For example, the second problems was less complex than the first problem. However, the sequence of problems were related in problem structure. As shown in the Figure 4 subject H made use of the given objects and systematically posed problems in four groups. The first group asked about the time taken to fill the pool (no. 1, 2, 3) under different conditions of the inlets and drain. The second group referred to the time used to move water out (no. 4, 5, 6). The third was about the cleaning of the pool (no. 7, 8, 9). The last three questions (no. 10, 11, 12) were repetitions of question number 1, 6 and 7 with the given pool capacity increased by x. Though numerical information content was present in the Pool problem, she supplied more to pose two of the problems (no. 3: drain rate=1; no. 4: pool is half full).

Example Evaluation of One Low Subject (Subject L). Subject L did the Pool problem and posed 10 problems in three categories; 7 non-math; 2 zero-step problem and 1 with insufficient information for solving. Four of the non-math problems asked for information about the amount of water being pumped (no. 1); the depth of the pool to be drilled (no. 7); the pressure of the water (no. 8); and, if the earth below and surrounding the pool have, the weight of the pool (no. 9). The other three non-math problems asked opinions on: the favorable size of the enlarged pool (no. 2); the feasibility of moving the pool (no. 6); and the methods to alleviate residents' fears that the pool may be washed away (no. 10). The second category, zero-step problems, referred to the size of the pool, the answer was in the problem. Last of all, the problem with insufficient information required in solving (no. 5) asked for the additional drainage in holding tanks when water in pool was recycled. These ten problems posed by Subject L could not be mathematically solved and represented by GPS graphs.

High and Low Group: A Comparison. First, for both high and low groups there were no pattern of relationship between complexity of posed problems and the order in which they appeared. While the high subjects were more likely to pose multi-step problems the low subjects produced problems that were mostly irrelevant, non-math, and insufficien and so the complexity of each solution structure could not determined. The different findings of this study from that of Silver et al (1990) suggested a difference of requirement in solving and in posing. In posing multiple problems the later posed problems might not be more complex than the earlier posed problems. In solving a problem again after knowing the answer the later solutions were more sophisticated.

Second, there was high relatedness of solutions of posed problems from the high subjects. They manipulated the given objects systematically to pose new problems. There was cross-referencing of information from one product to another. In several cases, the goal of prior posed problem was the given object for subsequent posed problems. Other than just the relatedness of problems within one posing activity, there was also a connection of the first posing activity to the second posing activity. One high subject used the idea of installing insulation to save heat from the House Problem she did during the first posing activity, to pose the problem of installing a new pump to save energy in the Pool problem. In the low mathematics knowledge group, however, the relatedness of solution structure of one problem with each other was not so commonly seen. In fact, most of the problems were not solved and graphed it was impossible to examine the relatedness of solution structure. Of all the posed problems (n=61), only eight could be solved and graphed.

Conclusions and Implications

On Analyzing Problem-Posing Process. The evaluation of problem-posing process was conducted with the definition of problem-posing process being the process of posing a sequence of problems in each problem-posing activity. The evaluation of each process is made by making inferences to the sequence of solutions to the problem-posing products. Thus, the collection paper-and-pencil data make the evaluation or processes possible. However, the process of posing one problem cannot be made. Maybe the use of "think-aloud" and the collection of verbal data can help to unpack the process of posing one problem. If that approach is used, however, we suggested the use of more complex the problem-posing tasks than the ones used in this study.

The Role Of Mathematics Knowledge. Our last report brought out differences found in the production of mathematics problems, problems with sufficient data, and, problems that had plausible initial states for subjects with high mathematics knowledge (Leung, 1993). The findings suggested an influence of mathematics knowledge in problem posing. In this study, we found a qualitative difference in the proble-posing process of prospective elementary school teachers. Subjects with high mathematics knowledge systematically manipulated given conditions to make problems and used solutions to prior posed problems as new pieces of information to pose subsequent problems. Subjects with low mathematics knowledge posed problems that might not be solved mathematically and the mathematics problems posed were not necessarily related in structure.

The study provides a cognitive evaluation of the problem-posing process of prospective elementary school teachers on one kind of mathematical problem-posing task. It was a demonstration of the use of information-processing approaches in the evaluation of problem posing. It also suggested a qualitative difference of problem-posing process relating to mathematics knowledge.

Reference

Educational Testing Service (1986). The PPST Guide. Princeton, NJ: Author.

Getzels, J. W. & Jackson, P. W. (1962). *Creativity and intelligence: Exploration with Gifted students.* New York: John Wiley and Sons.

Kilpatrick, J. (1978). Variables and methodologies in research on problem solving. In L. L. Hatfield & D. A. Bradbard (Eds.), *Mathematical Problem Solving* (pp. 7-20). Athens, GA: Georgia Center for the Study of Learning and Teaching Mathematics and the Department of Mathematics Education, University of Georgia.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Hillsdale, NJ: Lawrence Erlbaum Associates.

Leung, S. S. (1993). Mathematical problem posing: The influence of task formats, mathematics knowledge, and creative thinking. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Eds.). *Proceedings of the 17th International conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 33-40), Tsukuba, Japan.

Mamona, J. (1993). On analyzing problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Eds.). *Proceedings of the 17th International conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 41-48), Tsukuba, Japan.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Newell, A., & Simon, H. A. (1972). *Human problem solving*. Englewood Cliffs, NJ: Prentice Hall.

Polya, G. (1954). *Mathematics and plausible reasoning, Vol. 1 & 2.* Princeton, NJ: Princeton University Press.

Silver, E. A. (1993). On mathematical problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Eds.). *Proceedings of the 17th International conference of the International Group for the Psychology of Mathematics Education* (Vol. I. pp. 66-85), Tsukuba, Japan.

Silver, E. A., Leung, S. S., & Cai, J. (Oct., 1991). The marble arrangement problem: Results of an analysis of U.S. students and a comparison with Japanese students. In Jerry P. Becker (Ed.). *Report of U.S. - Japan cross national research on students' problem solving behaviors* (pp. 3-23). Carbondale, IL: Southern Illinois University.

COGNITIVE OBSTACLES IN PRE-ALGEBRA

Liora Linchevski Hebrew University
Nicolas Herscovics Concordia University

The present paper study investigates the pre-algebraic thinking of a class of sixth graders. It confirms the existence of a cognitive gap, uncovered previously in seventh graders, namely, the inability of students to spontaneously operate with or on the unknown. It further investigates the arithmetic skills that can be regarded as being of a pre-algebraic nature and presents five of eight specific cognitive obstacles, identified in the study: 1) over-generalization of the order of operations, 2) failure to perceive cancellation in an arithmetic string, 3) a static view of the use of brackets, 4) detachment of a term from the indicated operation, 5) jumping off from a term with the posterior operation. The three additional cognitive obstacles are: 6) the lack of acceptance of the equal sign as a symbol for decomposition 7) backwards reading of equations, 8) inability to select the appropriate operation for partial sums

Introduction

The past few years have witnessed a growing attention to a new topic in school curriculum, that of pre-algebra. This new topic is, in a way, an answer to teachers' and educators' concern and frustration about the high failure rate in high school algebra. However, without a systematic effort to specify demarcation line between the traditional first course in algebra and pre-algebra, a line that could be specified in terms of the students' cognition, this new topic might either be treated as simply an earlier introduction of algebra or as spreading out over a longer period of time of the standard material. It was in attempt to find some demarcation between the two topics that we undertook an assessment study of the pre-algebraic thinking of seventh graders who were just going to start their first course in algebra (Herscovics & Linchevski, 1994).

In this study, of the seventh graders, we have included a preliminary assessment of some arithmetic skills judged essential for the solution of algebraic equations on one hand, and on the other hand, a systematic evaluation of the solution processes spontaneously used by them to solve first degree equations in one unknown. The results indicated the existence of a cognitive gap between pre-algebra and algebra. The seventh graders did not seem able, even under optimal conditions, to spontaneously operate on or with the unknown (Herscovics & Linchevski, 1994, 1991a). Moreover, both the preliminary assessment and the part on equation solving provided evidence of some unexpected cognitive obstacles, obstacles which were probably rooted in arithmetic and the algebraic context has magnified them and revealed their existence (Herscovics & Linchevski 1991b, Linchevski & Herscovics 1992).

The study with sixth graders, which will be reported in the present paper, was the next step in our investigation. One of our objectives was to evaluate the pre-algebraic potential of even younger students and to see whether the results would confirm the existence of the cognitive gap between pre-algebra and algebra found among the seventh graders. Another objective was to expand our preliminary assessment of pre-algebraic arithmetic skills in order to check whether the cognitive obstacles discovered earlier might exist in a purely arithmetic context and to determine how widespread these obstacles were. In the first part of study we presented the students with 35 questions of arithmetic nature. The second part was based on 28 algebraic equations taken from our Grade 7 assessment study.

In order to include a wide range of abilities, we interviewed a class of 27 sixth graders in a public school. The children were classified as strong (11 students), average (10 students) or weak (6 students), on the basis of their school performance. Their average age was 11:6. We verified that they were not exposed to any algebra in school. Each subject was interviewed individually in two 45 minute sessions. The interviews were semi-standardized in the sense that the interviewer could re-phrase the questions should the original wording seem unclear to the student. An observer with a detailed outline recorded all students' responses.

In this article, due to space problems, we will only report on five out of the eight obstacles that have been identified and will discuss the items which are directly relevant to them. However, since the traces of the obstacles have been spotted not only in the questions aimed directly at them, but sometimes indirectly, in responses to questions which have been prepared to investigate other topics, we include in this paper some data from the other parts.

We will not report systematically on the second part of the assessment which investigates the spontaneous processes used by the students for algebraic equations solving. However, the results obtained in this part confirm the existence of the cognitive gap between pre-algebra and algebra, which was first discovered in the seventh graders (Herscovics & Linchevski, 1994, 1991a). In fact, the procedures used by the sixth graders were identical to those used by the seventh graders beside some differences in the success rate.

Part 1: Arithmetic context

1. Over-generalization of the order of operations: Our students had learned in class the order of operations prior to the interview. They were asked to evaluate the following strings: **1)5+6x10=?, 2)7-3x5=?, 3)8x(5+7)=?, 4)27-5+3=?, 5)24:3x2=?**. The results were somewhat surprising and are shown in table 1.

Table 1: Number of students and choice of first operation.

Choice of 1st operation 2) 17-3x5=?		1) 5+6x10=?
multiplication first addition/ subt. first	9 (33.3%) 18 (66.7%)	14 (51.9%) 13 (41.1%)
Choice of 1st operation 5) 24:3x2=?		4) 27-5+3=?
subt. division first addition, mult. first	21 (77.8%) 6 (22.2%)	18 (66.7%) 9 (33.3%)

For the first string, a majority of students performed the addition first, in spite of their prior instruction on the order of operations. With the second string, this number was somewhat reduced: the majority of the students performed the multiplication first. The reduction could be related to possible detachment of the factor 3 from the preceding minus sigh. These results indicate the same trend we have found in our previous study with seventh graders (Herscovics & Linchevski, 1994). When the order of operations involves subtraction, less students tend to be linear, they mentally split the expression at the minus sign and focus on the obtained part. Question 3 was solved correctly by all students. But, it is with the last two problems that we tested the possibility of over-generalization. The results indicate that a sizable number of children seem to believe that addition takes precedence over subtraction and multiplication over division. Five students over-generalized in both cases. 1 strong, 2 average, and 2 weak.

Impact of challenge. As soon as they were finished with the above evaluations, all the students, whether they had given a correct or an incorrect answer, were challenged on problems 1, 4 and 5 by the question: **Another student worked out the problem differently. He first multiplied 6 by 10 and then added 5** (Or: **He first added 5 and 6 and then multiplied 11 by 10**). **Do you think that he is right, or that you are right, or that you are both right?** A similar question was raised about problems 4 and 5

The responses to the challenge on Q1 showed that the 9 students who had solved string 1 correctly did not change their mind when challenged. However, 15 of those who had evaluated it in the incorrect order, did indeed change their opinion while 3 students remained linear claiming that: "you have to go from left to right". The responses to the challenge of the other strings are remarkably different. Some students with the correct answer, now changed their mind. For the string 27-5-3-?, one changed his mind, but 2 pupils could no longer decide which was correct. Of the 6 students who initially evaluated incorrectly, 3 changed their mind and 3 did not. For the other string, 24 3x2-?, the changes were even stronger. Of the 18 students who evaluated it correctly, only 12 remained immune. Two students changed their mind, 4 could no longer decide. Of the 9 students who were incorrect, only one changed his mind.

2. Failure to perceive cancellation. The perception of terms that can be canceled in an expression plays an important part in equation solving. In gathering like terms and in comparing like terms on either side of an equation, the students have to break away from the tendency to operate sequentially from left to right and need to achieve a more global apprehension of the equation. This is why we included some exercises on cancellation. We presented the students with 6 numerical strings involving only addition and subtraction, each string was presented separately. With each string we asked the students: **Here is a string of operations. Please read it... Without performing the operations, could you tell me what the answer would be?... Can you explain how you figured it out?**. Table 2 shows the problems and the success rate

Table 2: Number of students who canceled additive terms.

String	Frequency	String	Frequency
1) 329+167-167=? (59.3%)	17 (63%)	5) 17+59-59+18 18 -?	both 16
2) 672-256+256=? (40.7%)	14 (51.9%)		none 11
3) 98+214-98=? (48.2%)	15 (55.6%)	6) 237+89-89-92+92=?	both 13
4) 513 -124+100+124 13 (48.2%)	3 (11.1%)		only 89 1 (3.7%) none

Similar questions were raised in the context of strings involving only multiplication and division. The number of students who canceled was almost the same. Comparing the groups of students who canceled in the additive and in the corresponding multiplicative strings, reveals an intersection of 14 students (82.3%).

A comparison between one of the multiplicative strings, 896:28x28-? and Q. 5 on the order of operations (24:3x2=?) provides a form of continuous hypothesis testing. The figures show that among the 13 students who have not perceived the multiplicative cancellation, 10 first multiplied 3 by 2 and then divided 24 by 6. A possible explanation is that in both cases they focused on the multiplication as the first operation to be performed. This represents an intersection of 71.4%.

Another question related to cancellation was **Without using the calculator, just by looking at the numbers and the operations, can you tell which expressions will give the same results?** a)926+167-167=?; b)926-167-167=?; c)926-167+167=?; d)926-(167+167)=?.
If the student stopped after one match, the interviewer asked: **What about the others?**

Table 3: Number of students selecting pairs of strings

Selection of pairs	Frequency		
	Spontaneous	After intervention	Total
(a) and (c)	14	1	15 (55.5%)
(b) and (d)	0	2	2
(c) and (d)	7	1	8 (29.6%)
(a) and (b)	0	2	2

The number of students selecting (a) and (c) above is consistent with the answers to questions 1 and 2 in Table 2. The structure of the strings is identical. If we look at the 10 students who did not perceive the cancellation in 1, six of them are among the students who have not connected (a) and (c). Among the 14 who did not cancel in 2, nine did not connect (a) and (c).

Another 8 students selected (c) and (d) thereby indicating a detachment of the minus sign from the indicated term. The only way they could think that these strings would yield the same answer is by ignoring the indicated subtraction in (c) and focusing on the addition sign. As one of the students said: "C and D are the same, they have the same numbers the same operations and the same order". He was then asked "But in D there are brackets, don't they make any difference?" He said "You are supposed to do first what is in the brackets, and in C I judge what to do by observation". Another student said "It does not look the same but, when you have to <u>solve it</u>, it is the same".

That so few students selected (b) and (d) indicates a very restricted understanding of the use of brackets. Only 2 students showed any awareness of the effect of brackets on the sign of the operation. It might be attributed to the fact that when brackets are firstly introduced, they are presented as simply "do this operation first".

3. A static view of the use of brackets. As shown by their answer to Q. 3 on the order of operations, (8x(5-7)=?), all the students knew the meaning of brackets as "do it first" operation. However, the results to questions which involved brackets in other context indicated a very limited view. In fact only 2 students thought that 926-167-167=? and 926-(167+167)=? yielded the same answer. Surprisingly, 8 students thought that the string with brackets would produce the same answer as 926-167+167=? A third question regarding the use of brackets required the replacement of the second term in 125-67=? by the sum 39+28, and to solve the obtained string. Two students rewrote it as 125-39-28=, while 12 students inserted brackets. Among the 13 who rewrote it as 125-39+28= seven managed to handle it correctly by adding virtual or mental brackets, but 5 students solved it as if it were a new equation. This limited view of the relations between operations, confirms some observations by Booth (1988) and Kieran (1979).

4. Detachment of a term from the indicated operation. In our previous assessment study of seventh graders (Herscovics & Linchevski, 1994), we had found evidence of a detachment of a term from the indicated subtraction. The student "enters" the expression at a certain number, manipulates a part regardless the other parts of the expression and re-insert the partial result into the original expression. They function as if there are invisible brackets around some parts of the expression. In the present assessment, we found that the problem is more general and can be viewed as a detachment of a term from the indicated operation. The issue here is more complex than it may appear. In fact, some detachment is essential if one wishes to follow the conventions regarding the order of operations. For instance, in evaluating 5+6×10+7, the student must temporarily detach the 6 from the indicated addition. This might lead to some over generalizations.

We have found evidence of such detachments throughout this investigation. (27-5+3, was viewed as 27-(5+3), 24÷3×2=, as 24÷(3×2), 926-167+167=, as 926-(167+167), 4+n-2+5+11+3+5, as 4+n-(2+5)+11+3+5). we had, however, prepared three specific string problems that were very likely to induce a high rate of detachment. We specifically asked the students: **Without using the calculator, just by looking at the numbers and the operations, can you show me a quick way to find the answer to this problem? Can you think out loud so that I can follow you?** The first string was 50-10+10+10=? Nearly half of the students, (13, 48.1%), detached the 10s, they first added the three tens, got 50-30 and gave 20 as an answer. The second string was 167-20+10+30=?, 10 students, (37.0%), detached and got 167-60 as an answer. When asked to explain his procedure one of the students said: "I am not sure, but when you first make the additions, it becomes much easier." Another student said: "all the operations are additions and subtractions, being performed at the same time, what's convenient is the basis for my decisions. If they were mixed with multiplication I have to be careful"
Regarding the consistency of the behavior, 10 students detached in both cases, in string 1 and in string 2. If we wish to find a relationship between those students who detached in string 1 and in the order of operations, (27-5+3), or could no longer decide after being challenged, we find that 7 of them, (77.8%), detached in both cases. From this we can infer that a sizable number of students detach consistently. However, these results also show that one might detach in one string but not in another.

5. Jumping off with the posterior operation. In our previous work (Herscovics & Linchevski, 1991b; Linchevski & Herscovics, 1992), we have noticed that some of the students, while grouping like terms in which a distance between the terms is involved, (e.g. 115-n+9+61), tend to focus on the operation following the term ,(e.g. 115-), to ignore the operation preceding the term that is being combined ,(+9), and to subtract the two terms (115-9) instead of adding them (115+9). To further investigate this tendency, we prepared four specific problems. In the first two, students were asked: **In this string of operations, without working it out, can you see terms that could cancel each other?**

1) 217+175-217+175+67=?   2) 217-175+217-175+98=?

Table 4: Perceived cancellations

No. of students who in string 1	Frequency
canceled 217 correctly	19 (70.3%)
canceled 175 incorrectly	13 (48.1%)

No. of students who in string 2	Frequency
canceled 217 incorrectly	15 (55.6%)
canceled 175 incorrectly	13 (48.1%)

To cancel 175 in string 1, the student had to focus on the minus sign following the second term, and to ignore the plus sign preceding the fourth term. The same process had occurred with the second string. It is also interesting to note that two students did not succeed on string 1 because they inserted "mental" brackets around the first two terms and around the third and fourth terms and stated that "the result was 67 because (217+175)-(217+175) will cancel each other"

The last two exercises in this section were: **Here is a string of operations. I would like you to use your calculator, but I want you to work it out in a given order. First I want you to work on the three-digit numbers, and then work on the two-digit numbers.** The two strings were

3) 195+67-117+39=? **4) 217+39-162-17=?**

Table 5: Students handling of strings 3 and 4

No. of students who	Frequency
added : 195+117	10 (37.0%)
subtracted : 67-39	4 (14.8%)
refused to perform exercise	2 (7.4%)
added : 217+162	10 (37.0%)
added : 39+17	4 (14.8%)
refused to perform exercise	2 (7.4%)

When combining the 3-digit numbers in the strings, 10 students added them. When asked to explain "Why did you decide to add those two (three digit) numbers?", one of the students said

" after the 195 there is an addition, so I was not very sure, but after the 195 there is an addition so I thought that I have to add to 195." He then was taken back to the first question, 217-175-217+175+67, in which he had canceled 217, he said "this exercise is completely different from the other one. In this exercise I do in my mind 217-175 to get something, then I take away the 217 to get 175. In the other exercise you asked me to ignore the number in the middle (the two digit) so I have to add"

Two students refused to perform the task stating "But you have to go from left to right. I can't just jump." The same pattern was found in grouping like terms in the context of equations.

Part 2: An overview

The first part of this article dealt with some specific cognitive obstacles in arithmetic. The succeeding part of the evaluation has stressed the fact that these obstacles are of pre-algebraic nature. In the second part of the evaluation we presented the students with linear equations and have tried to trace the spontaneous solution processes used by them prior to any formal instruction in algebra. In most of the cases the students were able to solve the equations successfully using procedures based exclusively on numerical manipulations or numerical substitution. However, when the students failed to handle an equation successfully, we were able to relate it to some of the cognitive obstacles identified in the first part. For instance, the problem of detachment occurred in several instances. For example in solving equation like: 115-_ +9=61, some of the students inserted mental brackets and actually solved the equation 115-(_ +9)=61. For equation like 39- _ -12=74, 6 sixth graders jumped off with the addition following 39 and obtained 39-12. The number of students that were not effected at all, in one context or another, by some of these problems was very small.

The following data will clarify this situation and expose how generalized the following three (out of the eight) problems are: 1) the detachment of a term from the indicated operation; 2) misunderstanding of the order of operations; 3) jumping with the posterior operation.

Fig 1

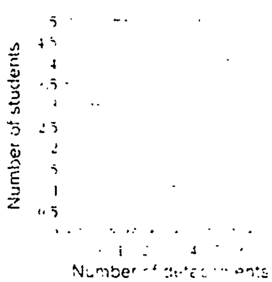


Fig 2

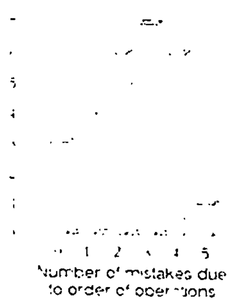
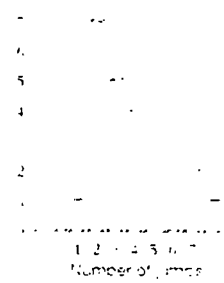


Fig 3



Throughout the Grade 6 assessment there were 82 instances of detachment from the indicated operation. From **Figure 1** we can see that only three students out of the 27 subjects did not face the problem of detachment. Among the 11 students considered strong, the average frequency was 2.73 per student, among the 10 average students, the frequency was 2.50; among the 6 weaker students the frequency was 4.50

In both parts of the assessment there were 66 mistakes related to the order of operations. Again, from **Figure 2** we can see that only three students who did not make even one mistake regarding the order of operations. However, this phenomenon is clearly related to the students' classification: in the group of students judged to be strong the average frequency was only 1.55 per student, for the average students, the frequency was 2.70; for the weaker students, the frequency was 3.67

There were altogether 78 cases of jumps with the posterior operation, see **Figure 3**. Not a single student in our class managed to avoid completely this problem. As far as students' abilities are concerned, the phenomenon did not follow the pattern that might have been expected. The average frequency for the strong students was 3.18 per student, for the average group, it was 2.20, and for the weaker students it was 3.50

Discussion

Since pre-algebra deals with the interface between arithmetic and algebra, as such it concerns itself not only with the initial construction, the genesis. of algebraic concepts but also with the specific arithmetic skills required to handle them. In arithmetic, we can at times bypass the properties and conventions and replace them with an operational approach, while in algebra those properties and conventions prove to be essential. For instance, had we agreed to insert every possible pairs of brackets in each arithmetic string, we could have avoided the convention of the order of operations in most cases. In algebra, when it comes even to a simple equation, like 6+9x__=60 or 6x__-9x__=60, we cannot handle it without the order of operations. The 16 students who failed in solving this equation, in the second part of the assessment, were using an incorrect order of operations.

Our prior studies had identified some cognitive obstacles of a pre-algebraic nature. The results of the present Grade 6 assessment confirm their existence and increased their number to eight The existence of so many cognitive obstacles together with the students' inability to operate spontaneously with or on the unknown (Filloy & Rojano, 1984; Herscovics & Linchevski, 1994), may explain why such a large number of students find it difficult to learn algebra. nor are we likely to have exhausted the topic of cognitive obstacles How to cope with these pedagogically is complicated by the fact that they are so interrelated. For topics presently taught in primary school. such as the order of operations. alternative presentations may help. The rational behind the conventions needs to be explained and discussed so that these rules are viewed as necessary and do not appear arbitrary. The tendency of some students to over-generalize has to be addressed explicitly and this may help to prevent some of these problems. Perhaps the context of the order of operations will prove to be appropriate for a discussion of the detachment of a number from the prec. 'ing operation The introduction of brackets also needs a new type of presentation. It is not enough for the students to know that they indicate the operation that has to be performed first. The operational properties of brackets need to be discussed.

The other cognitive obstacles identified in this paper may be difficult to treat in arithmetic for the lack of contextual motivation. In the context of pre-algebra, questions regarding the cancellation of equal but opposite terms. backwards reading, jumping off with the posterior operation, and partial sums are relevant problems. In any case all these obstacles constitute topics that need to be covered in any course preparing the student for algebra.

References

Booth. L R. (1988) Children's difficulties in beginning algebra In A F Coxford (Ed). *The ideas of algebra. K-12.* National Council of Teachers of Mathematics. Reston Virginia.20-32

Filloy, E & Rojano, T (1984) From an arithmetical thought to an algebraic thought. *Proceeding of PME-NA VI, 51-56.*

Herscovics. N & Linchevski. L (1994) (in press) A cognitive gap between arithmetic and algebra. to be appear in *Educational Studies in Mathematics.*

Herscovics. N & Linchevski. L (1991a) Pre-algebraic thinking Range of equations and informal solution processes used by seventh graders prior to any instruction *Proceeding of PME XV*, Assissi Italy. 173-181

Herscovics. N & Linchevski. L (1991b) Crossing the didactic cut in algebra *Proceeding of PME-NA*, Virginia. V II. 196-202

Linchevski. L & Herscovics. N (1992) "Cancellation within-the -equation" as a solution procedure *Proceeding of PME XVI*, New Hampshire. V I. 265-273

Kieran. C (1979) Children's operational thinking within the context of bracketing and the order of operations. *Proceeding of PME III, England, 128-133.*

Eliciting the meanings for algebra produced by students: knowledge, justification and Semantic Fields

Romulo Campos Lins

Postgraduate Program in Mathematical Education, UNESP, Rio Claro, Brazil[1]

Abstract

For the past six years we have been engaged in developing a theoretical framework which accounts for meaning in mathematics, the Theoretical Model of the Semantic Fields. In this paper we discuss part of a 12 lesson long study with Brazilian sixth graders (11-12 years old), which is based on our theoretical framework. The central objectives of this paper are (a) to argue for the importance of the theoretical construct **Semantic Field** *in the study of pupils'* **knowledge**, *(b) to present and discuss the distinction between* **solution-driven** *activities and* ***justification-driven*** *activities.*

For the past six years we have been engaged in developing a theoretical framework which accounts for meaning in mathematics, in particular in algebra, including a characterisation of *algebraic thinking* (Lins, 1990, 1992a & b, 1993). As a result we have produced *The Theoretical Model of the Semantic Fields* (**TMSF**) (Lins, 1994). We acknowledge that the term *semantic field* has been used by other authors, mainly linguists (see, for example, Grandy, 1987), but also by mathematical educators (see, for example, Boero, 1992). Nevertheless, our conception of a *semantic field* must not be confounded with any of those, as it arises from an epistemological approach which is essentially distinct from the ones supporting those two conceptions (which are, by the way, also distinct from one another).

In this paper we discuss part of a 12-lesson long study with an intact class of Brazilian sixth graders (11-12 years-old). Currently, a fully fledged project is being carried out on the lines of the study here analysed[2]. The central objectives of this paper are (a) to argue for the importance of the theoretical construct *Semantic Field* in the study of pupils' *knowledge*, (b) to present and discuss the distinction between *solution-driven* activities and *justification-driven* activities.

The central research question in the study was the nature of an epistemological obstacle in relation to the "manipulation of the unknown," suggested by Filloy and colleagues (Gallardo & Rojano, 1987). It was our hypothesis, already supported by evidence from an earlier pilot study (unpublished), from an extensive study of pupils' solutions to "algebra problems" and from a historical study of algebra and of algebraic thinking (Lins, 1992a), that those difficulties were directly linked to the ways in which students produce meaning for the equations proposed to them. The teaching experiment, part of which is discussed in this paper

---

[1] Address: Depto de Matemática, Av 24A, 1515, 13500, Rio Claro, SP, Brazil
E-mail: LERCB@BRUNESP (BRFAPESP)

[2] *Producing meaning for algebra: a research and development project*, a collaboration between a British team at the Institute of Education, University of London, led by Rosamund Sutherland, and a Brazilian team at Dept. of Mathematics, UNESP, Rio Claro, led by the author.

aimed at showing that it is possible to develop a teaching approach which avoids difficulties with the "manipulation of the unknown," by producing a working context where that activity may become "senseful," and understood as one way—among others—of producing meaning for equations and their manipulation.

Theoretical Background

The theoretical support for both the development of the activities and the analysis of the results, is drawn from two sources: the *Theoretical Model of the Seman'' Fields* [**TMSF**], and the ideas of Vygotsky, particularly in its influence on the work of V.V. Davydov.

At the heart of the **TMSF** is a particular conception of knowledge: *knowledge* is a pair formed by a *statement-belief*—that is, a *belief* which is stated—together with a *justification* for it[3]; for instance, one might say that in relation to the equation $3x+10=100$ "one may take 10 from each side" (*statement-belief*), with the *justification* that "it is as in a scale-balance." Such *justification* does not, of course, apply in the case of the equation $3x+100=10$, but this does not imply that the *satetement-belief* could not be held by the same subject in relation to the second equation; another *justification* would have to be available, in which case a different *knowledge* would be produced.[4] Such mechanism has been shown to be of great relevance in the study of pupils' understanding of "algebra problems" (Lins, 1992a, 1993).

A second key concept in the **TMSF**, *meaning* is understood as the relationship between the *statement-belief* and the *justification* in a given *knowledge*, the full "being together" of the two elements in a *knowledge*. To say that a piece of mathematics is *meaningful* to a person, is to say that the person possess some *knowledge* about that piece of mathematics. A lack of understanding must, then, be seen as a lack of *meaning*. But if it is immediately possible to relate positively *meaning* and *knowledge*, the mechanism which allows relating the lack of *meaning* to the non-realisation of *knowledge* requires some further elaboration. The third key concept of the **TMSF**, that of *Semantic Field*, provides what is required.

A *Semantic Field* is *a mode of producing meaning*. We can speak, for example, of producing *meaning* for the equation $3x+10=100$ within a *Semantic Field of a scale-balance*, or within the *Semantic Field of algebraic thinking* (see Lins, 1992a), or within a *Semantic Field of whole and parts*. But within the first or the last of those *Semantic Fields*, it is not possible to produce *meaning* for the equation $3x+100=10$. A *Semantic Field* corresponds to possibilities of producing *justifications*, and, thus, of enunciating *statement-beliefs*. The same *statement-belief* may be justified within different *Semantic Fields*, but to each *justification* corresponds different *knowledges*.

---

[3] Although not originally derived from it, our conception bears a similarity with the classical definition of knowledge, $K_p = B_p \cdot p \cdot J_p$ ("justified true belief", see Everson, 1990); a detailed discussion of that similarity is found in Lins (1994). As to any objection in relation to the fact that *knowledge* must have been stated at least once, the reader is referred to Ayer (1986).

[4] Explicitly stated: the *knowledge* constituted by the pair (one may take the same from both sides, it works like a scale-balance) is different from the *knowledge* constituted by the pair (one may take the same from both sides, it is a property of numerical equalities).

A brief example might serve to show in what sense the concept of a *Semantic Field* throws light in the process of *knowledge* production. In relation to the equation $3x+10=100$, a teacher and a student might agree on the *statement-belief* "we can take 10 from each side," although the teacher has a *justification* produced in terms of properties of the equality in relation to the arithmetical operations, while the student has a *justification* produced in terms of a scale-balance; there are *distinct knowledges*. It should not come as surprise—although so many times similar situations do, and, interestingly enough, also for researchers—that when presented with the equation $3x+100=10$, and even being able to deal correctly with negative numbers, the student will say "it doesn't make sense." (see, for example, chapter 4 of Lins, 1992a). The false paradox arises when we mistakenly assume that the student should naturally "apply" to the second equation the *statement-belief* which had been enunciated in relation to the first equation; but *knowledge* is an irreducible composition of a *statement-belief* and a *justification*.

But what the concept of *Semantic Field* also indicates, is that while there is nothing in the equations themselves which can be linked with the production of *meaning*, the same is true of any environment or context, no matter how tempting it might be to say the opposite.[5] In fact, "real objects" are in themselves as "semantically empty" as any x's and y's can be. It is true, however, that *culturally* one situation will probably be more strongly associated with some *Semantic Fields* than with others, as is the case of situations involving money[6].

Although central in the model, the brief discussion of those three concepts—*knowledge*, *meaning* and *Semantic Field*—provides only the elements which are essential in the context of this paper; for a full presentation and discussion of the *Theoretical Model of the Semantic Fields*, the reader is reffered to Lins (1994).

Theoretical support to this study also comes, as we have said, from the work of V.V. Davydov, which is, on its turn, based on ideas from Vygotsky. Davydov has done intensive research on the teaching of mathematics for young children. In one of those experiments (Davydov, 1962) he started from modelling simple situations with whole-part models, and from there moved to exploring the manipulation of quantitative relationships. The use of literal notation was introduced rapidly and without trouble, becoming *a valuable and adequate tool in that context*.

We understand that the importance of those studies is twofold. First, they point out to the ways in which the use of symbols may produce a shift from the solution of problems to the investigation of methods of solution. Second, by starting from the manipulation of whole-part relationships, as a support, and then moving to the manipulation of the expressions themselves, the work of Davydov suggests a fruitful but not fully explored vein. In many respects it may be said that the passage from the "tanks" to the manipulation of expressions is not really different from approaches using "concrete material" or "contextualised situations."

[5] For instance, it is not the case that when dealing with a scale-balance one will necessarily operate within a *Semantic Field of a scale-balance*, some people will, instead, operate within the *Semantic Field of algebraic thinking*

[6] It is hard to believe that even Paul Erdös would set and solve an equation to calculate the change in the market

But there is a distinctive feature in Davydov's approach, namely that the whole-part model is used to *generate* the expressions which are to be manipulated, and not to illustrate the rules of manipulation; what is lacking, however, is the understanding that there are two *modes of producing meaning* in play, and that this situation should be explicitly addressed by teaching

The combined use—in Davydov's work—of symbols interpreted within a familiar *Semantic Field*, within which the *logic of the operations* is sufficiently clear, together with the intention of systematising that *logic of the operations* into principles which would guide the manipulation of the expressions, naturally leads away from the traditional approach of achieving that objective through a generalisation of arithmetic. When Freudenthal (1974) says that "...generality is not always achieved through generalisation," he is in fact pointing out to the need of introducing a kind of activity in which generality is at the starting point, it is not just a target. To those activities we will call *justification-driven* activities, and they will be naturally opposed to *solution-driven* activities.

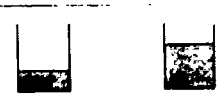
From Davydov's work, then, we borrow those two aspects: (i) operating within a familiar *Semantic Field* as a way of generating *meaningful* quantitative relationships in the form of expressions; and, (ii) the implicit distinction between *solution-driven* and *justification-driven* activities.

The conditions of the study

The study was carried out in 1990, at the Escola de Aplicação, a school set as part of the School of Education of the University of São Paulo. The activities were discussed with the class teacher, in order to guarantee that they would effectively contribute to the already planned teaching, and that they would not be seen by the students as mere "extras." Solving equations and using equations to solve problems were part of the program, and the only required change was in the planned schedule for the lessons. Students were told, however, that those lessons were part of an experimental teaching program. The researcher participated regularly in the lessons, sometimes in the role of a teacher. Students' work has been preserved in photocopies of their notebooks.

Classroom activity

The first activity proposed was based on a diagram given to the students:



With 9 more buckets, the tank on the left will be full; with 5 more buckets, the tank on the right will be full.

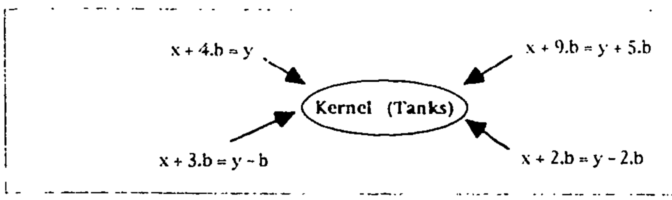
What can we say about this tank situation?

Students were then encouraged to produce, in small groups, expressions which could be shown to correspond to the situation, together with a *justification* for the adequacy of each expression produced. The use of "arithmetical" notation (using the signs for the arithmetical operations) was directly suggested by the teacher, and there was some negotiation as to the letters to be used. The water on the left-hand tank was notated **x**, and the water on the right-hand tank notated **y**, while a bucket was **b**.[7]

Some of the expressions generated, with their justifications, were:

Expressions	***Justifications* for adequacy**
$\mathbf{x} + 9.\mathbf{b} = \mathbf{y} + 5.\mathbf{b}$	"this phrase is correct because the two buckets [sic] will be a whole"
$\mathbf{x} + 4.\mathbf{b} = \mathbf{y}$	"if I add 4 buckets to the tank on the left, they will have the same amount"
$\mathbf{x} + 2.\mathbf{b} = \mathbf{y} - 2.\mathbf{b}$	"x + 2 buckets will fill the tank, with 7 buckets missing [sic]. And in y there are 5 buckets missing, and if we do -2 becomes -7"
$\mathbf{x} + 3.\mathbf{b} = \mathbf{y} - \mathbf{b}$	"6 buckets will be missing on x, and on y 5 are missing; if still another 1 is missing, also 6 will be missing"

As those four examples indicate, the validation of each expression, ie, the production of *justifications*, was done by referring back to a kernel—the "tank situation." The students were operating within a *nucleated Semantic Field of a tanks situation*.

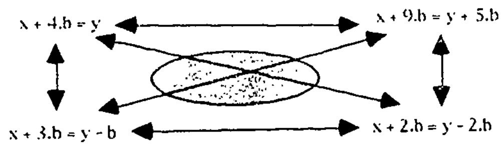


The next step was to propose another approach to the *justification* of the adequacy of new expressions. We asked the students to find a transformation of the expression **x+9b=y+5b** which would lead to the expression **x+4b=y**. The importance of having passed through the first phase—generating the expressions—is that there were now, available to the new activity, a large set of expressions to be manipulated, and those expressions were *meaningful*. The students had no difficulty in producing transformations such as "take 5 buckets from each side."

For each new expression generated from then on, the students were required to produce both a *justification* of its adequacy in relation to the kernel, and another *justification* of its

[7]In Portuguese "buckets" is "baldes"

adequacy from the application of a transformation rule to a previously established expression. Within the new *mode of producing meaning*, expressions are directly linked:



The use of the transformation rules produced expressions which could not be easily made sense of within the *Semantic Field of the tanks*, as, for instance, **2x+8b=2y**, imposing the discussion of the differences between the two *modes of producing meaning*. The distinction was made even sharper when expressions were generated which the students could not be sure whether they made sense at all within the *Semantic Field of the tanks*, as in the case of **x-30b=y–34b**, once one could not be sure of the possibility of taking 30 buckets of water from **x**.

Having established some degree of independence of those specific expressions in relation to the kernel, we could move to the manipulation of expressions which had not been generated within some *nucleated Semantic Field*, as, for example, transforming the expression **3x–4a=2y**. This part of the work was always carried out with a target in mind, for instance, transforming that expression in a way to obtain another expression, of the form **4a=** . . .[8] The fact that the students could correctly deal with this kind of task, suggests that the exit in the previous part of the teaching was not due to the support offered by the "context" of the tanks. Rather, once the direct manipulation of expressions had become a "senseful" activity, not only the technical difficulties did not occur, but, also, the students began to bring into play methods produced in arithmetic (for example, simplifying simple rational expressions).

Discussion and conclusions

The aspect which will be discussed here, of the teaching experiment partly presented in the previous section, is the role played by the theoretical framework in the design of the activities and in the analysis of the results. Any lenghty examination of the learning outcome of the approach we propose has to be preceeded by that discussion; the presentation and discussion of a teaching approach based on the **TMSF** will be found somewhere else in the near future.

The theoretical construct *Semantic Field* was at the heart of the process of designing the activities, pointing out to the need of having pupils to present *justifications* for the correctness

[8]In choosing this format for the activity, we had in mind the introduction of a strongly analytical perception of the expressions (Cf Lins, 1993)

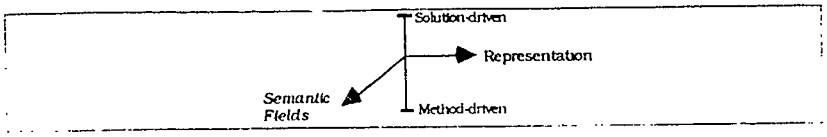
of the expressions (*statement-beliefs*) produced. The importance of those explicit *justifications* is twofold. First, they supported the introduction of the direct manipulation of expressions as one way—among others—of making sense of producing new expressions; in fact, the manipulation of expressions is seen, in the context of the activities proposed, as the production of new, adequate, expressions.[9] Second, in order to focus sharply on the production of *justifications*, we were led to design activities where the possibility of producing particular numerical solutions was denied: instead of "find a solution" activities, we proposed "make sense" activities. Bruner (1988) had already pointed out a possible, and very interesting, paralell between the notion of *given* and *new* tokens in speech—introduced by linguists—and the behaviour of subjects prompted to "think aloud" while solving problems. Bruner observes that those subjects produce a speech (which is likely to be a spoken version of the inner speech) from which the *given* is very much supressed. From the point of view of the **TMSF**, *justifications* certainly belong to the class of the *given*, as they must be accepted before being able to provide an anchor to *new statement-beliefs*. The format adopted in our activities, led the students to deal with both the *new* and with the *given*; as a consequence, they were not working only with solving problems, but also working on producing and enriching—and internalising—new *Semantic Fields*, ie, new *modes of producing meaning*. In Lins (1994) we present a full discussion of the role of *internal* and *external interlocutors* in the process of developing *Semantic Fields* (*knowledge* production).

The fact that our students did not have any substantial difficulties in dealing with literal expressions suggests, in the light of the **TMSF**, that this process is directly linked to the ways in which *meaning* is produced for those expressions. As we had already indicated (Lins, 1993), the "analytical behaviour," dealing with the unknown as if it were "known," is subordinated to particular characteristics of the *Semantic Field* within which the student is operating.

The two key aspects of the dynamics of the teaching approach adopted, are what we call *vertical* and *horizontal developments*. The former consists in the production of new *statement-beliefs* within a given *Semantic Field*, while the latter consists in the reinterpretation of "old" *statement-beliefs* within another *Semantic Field*. *Vertical development* enriches *modes of producing meaning*; *horizontal development* enriches the overall capacity of a *system of knowledge* to produce new *knowledge*, but it also enriches the global *meaning* of *statement-beliefs*.

Based on the theoretical framework, and on the overall results of the teaching experiment, we suggest that the design, conduction and analysis of classroom activity should be considered on a three-component system:

[9] In the process of "properly" solving equations, each transformation is seen as specialised, in the sense that it is almost necessary, although in many teaching approaches one finds the requirement of adding "justifications" to each "step"—eg, "do the same to both sides"—those transformations are dealt with from a very narrow perspective, and as a consequence, the idea of using those transformations to articulate expressions in a way to express more than initially available—for instance, manipulating an expression to show that the number of black tiles on a pattern is always even—is not developed, in the sense of it not being a legitimate strategy



This system should not be seen as a mere "change of basis" in relation to other systems. Although the "concrete-abstract" distinction can be formally interpreted in terms of the three components of our system, such exercise is of no interest. The **TMSF** aims at replacing such traditional polarities with a more flexible and precise framework. We think that research conducted within the framework of the **TMSF** should be concerned with producing a distinct approach to teaching: what to teach, how to teach, rather than with solving learning difficulties which are—more likely than not—produced precisely by the epistemological assumptions underlying those teaching approaches—eg. that there is a "path" from "concrete" to "abstract," and even the assumption that those two categories correspond to qualitatively distinct kinds of *knowledge*.

References

• Ayer, A.J. (1986), *The Problem of Knowledge*; UK, Penguin Books.

• Boero, P. (1992), The crucial role of semantic fields in the development of problem solving skills in the school environment, in J. Ponte (ed.), *Information Technology and Mathematical Problem Solving*, NATO

• Bruner, J. (1988), Vygotsky: a historical and conceptual perspective , in J.V. Wertsch (ed), *Culture, communication and culture: Vygostkian perspectives*; Cambridge University Press

• Davydov, V.V. (1962), An experiment in introducing elements of algebra in elementary school. *Soviet Education*, 5, 27-37

• Everson, S. (1990), *Companions to ancient thought 1: Epistemology*; Cambridge University Press

• Freudenthal, H. (1974), Soviet Research on teaching algebra at the lower grades of the elementary school. *Educational Studies in Mathematics*, 5, 391-412

• Gallardo, A. & Rojano, T. (1987), Common Difficulties in the learning of algebra among children displaying low and medium pre-algebraic proficiency levels, in J.C. Bergeron, N. Herscovocs and C. Kieran, PME XI, pp. 301-307, Montreal.

• Grandy, R. (1987) In defense of semantic fields, in *New Directions in Semantics*, E. LePore, (ed); Orlando, Florida; Academic Press.

• Lins, R.C. (1990), A framework for understanding what algebraic thinking is, in G. Booker, P. Cobb & T.N. de Mendicuti, PME-XIV, Mexico

• Lins, R.C. (1992a), *A framework for understanding what algebraic thinking is*, PhD Thesis, Nottingham University, UK.

• Lins, R.C. (1992b), Algebraic and Non-algebraic algebra, in *Proceedings of PME XVII*, Durham, USA.

• Lins, R.C. (1993), *Analysis and Synthesis*; Papers of the PME Working Group on Algebraic Thinking and Processes; Tsukuba, Japan.

• Lins, R.C. (1994), *The Theoretical Model of the Semantic Fields*, submitted to publication; available from the author.

INVESTIGATION OF A MODEL BASED ON THE NEUTRALIZATION OF OPPOSITES TO TEACH INTEGER ADDITION AND SUBTRACTION

Patricia A. Lytle, UQAM, Montreal, Canada

Assessment studies from various countries have indicated student difficulties with integer operations, in particular with that of integer subtraction. The most commonly used model for integers and their operations, the number line (often used in combination with the traditional subtraction rule) has been inadequate to produce understanding and performance. An alternative model, one based on neutralization of positive and negative quantities, was investigated in this study, and an individualized teaching experiment using this model was conducted with four grade seven students. Results of this teaching experiment were encouraging with respect to the construction of understanding, but not all difficulties were overcome.

INTRODUCTION

Large-scale paper-and-pencil assessment studies are indicators of levels of students performance on various types of tasks. A survey of the integer tasks in 11 such studies is found in Lytle (1992), and the results show that although performance on integer addition items varies from study to study, within each study performance on subtraction items lags behind by about 20%. Two smaller studies involved interviews with students, and findings indicated that success in addition tasks was dependent on reliance on a mental or physical model, while success on subtraction items was related to how well a rule was remembered. These findings were the motivation for investigation into a model for teaching integer operations that might give meaning to subtraction.

MODELS FOR TEACHING INTEGERS

Students develop number sense by constructing a mental model based on the concepts that they know within a physical model, where objects in the mental model (in this case, integers) correspond to objects they have become familiar with (Greeno, 1991). Within this mental model, operations such as addition and subtraction are performed with the mental objects in the same way in which combining and separating have been performed on the physical objects. Models which are used to help students understand integers fall into two main categories: the directed number (number line) model, and the discrete opposites (which can be neutralized) model.

a) **THE NUMBER LINE MODEL**: When the number line is used for addition and subtraction with **natural** numbers, the position of the first number is located, and the direction of movement along the line is determined by the operation to be performed (right for addition, left for subtraction). The distance moved is the magnitude of the second number, and the result of the operation is the position reached. Thus this model's embodiment of number is of both position and distance, while the operations on the numbers are interpreted as directions of movement.

With **integers** the number line takes on added complexity, especially when used for both addition and subtraction, which may be why most authors advocate using the abstract "change the sign and add" rule for subtraction. For addition, the direction of the operation is now determined by the sign of the number to be added (right for positive, left for negative). Since subtraction is the inverse of addition, the subtraction of an integer is in an opposite direction to that of the addition of the same integer (e.g. Chilvers, 1985).

An American survey by Crosswhite et al (1986) revealed that 87% of approximately 500 teachers believed that "use of the number line helped in teaching integers", yet this model has come under criticism by mathematics educators such as Ernest (1985). In addition, in the NAEP 2 survey, Rathmell (1980) compared success for 9 and 13 year olds on addition tasks using either the number line or joining of sets. He found that for both groups of students approximately 40% fewer students correctly used the number line compared to the success rate with combining sets. Kuchemann (1981) recommended abandoning the number line in favor of an annihilation model for integer operations.

b) **THE NEUTRALIZATION MODEL**: The neutralization model uses physical objects, such as different colored bingo chips, to represent quantities of opposite (positive and negative) natures. Physical manipulation of the objects (combining or removing) corresponds to mathematical operations on integers (addition or subtraction). Many educators have explained how this model works, but few have explored it to its full potential. The most concise explanation is given by Grady (1978), while the earliest description of this model appears in Haner (1947).

The model's foundation is the neutralization of equal amounts of opposites, and the concept of equivalence classes which allows any integer to have many physical representations. For example, the integer $^{-}3$ may be represented in its canonical form as ■ ■ ■ or in an equivalent form, such as ■ ■ ■ ■ □, where the combination ■ □ is regarded as a "neutral element" with no value. Addition of integers is modeled by combining the chips that represent the numbers. If the two integers have the same sign, one adds up the total number of chips: thus for example $^{-}9 + {}^{-}4 = {}^{-}13$. When the signs of the integers are different and the chips are combined, neutralization occurs with equal amounts of opposites. Thus for example $^{-}9 + {}^{+}4$ will have 4 sets of neutral elements, leaving $^{-}5$ as the result. Subtraction of integers is modeled by removal of a quantity of chips representing the subtrahend. In subtractions such as $^{-}20 - {}^{-}5$, 5 chips are removed, leaving $^{-}15$ as the result. For the subtraction $^{-}7 - {}^{-}12$, there are not enough chips to take away, so an equivalent representation of $^{-}7$ must be made by adding enough neutral elements so that there will be 12 negatives to remove. The following illustration shows how this is done:

Once the 12 negatives are removed, the result is 5 positives, so $^-7$ - $^-12$ = $^+5$. Mathematically, one asks "how many neutral elements have to be added so that all of the negatives will be removed?", then one sees that while the negatives added are removed, the positives added are what remain. When the signs of the integers are different, as in $^+5$ - $^-6$, one needs to add 6 neutral elements in order to subtract 6 negatives. The chips remaining will be the 5 positives that one started with, plus the 6 positives that came in as part of the neutral element, thus $^+5$ - $^-6$ = $^+11$.

Little research has been reported on the success or failure of the neutralization model, but Rowland (1982) used a variation of this model (using a form of equivalency class notation) with a group of four 11 year olds. He found that while additions were not difficult for the students, the subtraction procedure was not as intuitive. He did not post-test or interview the students. Liebeck (1990) taught two groups of ten grade 3 and 4 students, using the number line for one group and a scores and forfeits representation of the neutralization model with the other. The students were post-tested six weeks after instruction, and the scores and forfeits group scored slightly better than the number line group for additions, but both groups did not do as well on subtraction items, especially when the signs were different.

USE OF THE NEUTRALIZATION MODEL FOR TEACHING INTEGERS: AN EXPERIMENTAL STUDY

a) A PRIORI ANALYSIS: Peled (1991) has made an analysis of levels of procedural knowledge of integers, based on the number line as well as on a quantity representation. Since the focus of this present study was not only on performance, but also on understanding, an a priori analysis was made concerning the concepts inherent in understanding integers both from a mathematical point of view, and from the perspective of the neutralization model, using the Herscovics - Bergeron (1988) model of understanding. This analysis formed the foundation for the teaching experiment, and details can be found in Lytle (1992).

b) THE POPULATION: Two grade 7 classes were given a preliminary pencil-and-paper assessment in which items were designed not only to discover the level of pre-instruction knowledge and performance with integers, but also to investigate knowledge of some of the items inherent in the neutralization model, such as the concept of neutralization itself. The survey revealed that negative number notation was used by 50% of the 53 students on a task such as 2 - 8 = □, and by 82% in a comparable temperature problem. In addition, 50% of the students were successful with addition items involving a whole number and a negative number, and 44% could subtract a whole number from a negative number, while only one student could subtract a negative number from a whole number.

Six students (two strong, two average, two weak) were chosen as possible candidates for the teaching experiment, and were subsequently interviewed individually in a semi-standardized way in order to probe more deeply into their thinking about integers. When given integer operation tasks, none of the students used a consistent solution strategy throughout, and all expressed uncertainty about their solutions. If anything was consistent, it was their tendency to use whole number analogies. For example, one strong student rearranged a task like $3 + {}^{-}10$ to ${}^{-}10 + 3$ because he knew how to add a whole number from any position on the number line. Three of the students expressly stated that they did not know how to add or subtract a negative number since these numbers were lower than zero, and in fact were "like zero" to them. In general, most of the six were successful in operating **on** a negative number, but not **with** one. It seems that students do bring number line notions to integer tasks, but only notions which are in agreement with the natural numbers, and not in conflict with them. When asked to give a meaning for a specific negative number, three of the students used a positional statement (under zero), two mentioned that it represented "something missing", and one reported that it was the result of a subtraction. In general, the students viewed the set of negative numbers as "opposite to" the set of natural numbers.

c) THE TEACHING DESIGN: The teaching experiment consisted of five lessons of approximately 40 minutes each, on consecutive days. Each of the students met separately with the instructor, and all semi-standardized sessions were audiotaped for later analysis. Four of the interviewed students were retained for the study, and they will subsequently be referred to by the following code: S1 and S2 for the strong students, A1 for the average student, and W1 for the weak student.

LESSON ONE began with an exploration of the introductory notions of opposites, neutrality and neutralization in a non-math context. Neutralization was also explored within the context of cancellation of numbers and their operations in a string such as $647 + 299 - 299$. The notion of equivalence was presented in a money (coin) context. The students were then introduced to the concrete materials: the bingo chips, which were of two different colors and marked with either a + or - sign. These were called positive chips and negative chips, with respective values of "one positive" and "one negative". These values were immediately represented by the integers ${}^{+}1$ and ${}^{-}1$, in order to connect the concrete representation to their previous integer notation. The students were then given several chips of the same type, and were asked to write an integer to represent the value of the group. A similar worksheet activity included as the last task a drawing containing 3 positive chips and one negative chip. S1 and S2 spontaneously gave the correct value of $^{+}2$, by using cancellation of opposites. A1 subtracted (3 - 1), while W1 was unable to give a value. Discussion with all 4 students then dealt with opposites (■ ■ ■ is the opposite of ! ! ! ! ! !), neutralization of opposites, the existence of a "neutral element", and equivalence classes of integers. The students were then asked to find the value of groups of chips which contained both types of signs, and the procedure which they spontaneously used was formation of neutral elements by neutralization. They were also asked to draw several equivalent representations of specific integers.

In this first lesson, S1 and S2 had no difficulties with any of the concepts presented, and spontaneously made connections within the model, and from the model to integers. A1 could often manipulate the concrete material, but was slower to make the connection with the abstract number. W1 could not make associations without intervention from the instructor.

LESSON TWO's emphasis was addition, and began with a review of some of the notions from the first day. Written addition tasks were then presented, first with integers of the same sign, then with opposite numbers, and finally with integers of opposite sign. At all times the students were asked to perform the one digit additions with the bingo chips, but were not instructed how to use the chips. All students spontaneously constructed the expected strategies for the first two types of additions, and carried these out mentally for the tasks involving two and three digit integers. For the addition of integers of unlike signs, both S1 and S2 immediately neutralized and did so mentally with the large number tasks. A1 had to be nudged to neutralize in the first task, and then used this procedure for the one digit tasks. The first larger task ($^{-}4$ + $^{+}18$) fell into the category of ones he had been able to solve before instruction (adding a whole number to a negative number using the number line) and he found the answer quickly. However when given $^{+}9$ + $^{-}15$ he was unsure of what to do, and when reminded that he could neutralize, he did so, then said: "Would it always be the right answer if you did the bigger number subtract the smaller one?", indicating his need to generalize. W1 also came up with this rule spontaneously, based on the results of her neutralizations. At the end of the session, the students were given 7 questions of mixed types, and were able to choose the correct solution strategies.

For addition, the physical neutralization model appears to have provided a basis for the students to construct a corresponding mental model for integers. It presented no conflict with their whole number knowledge, but rather was substantiated by it.

LESSON THREE began with a review of equivalent representations of integers, A1 needing reminders of the infinite number of ways to represent an integer. Written subtraction tasks such as $^{-}10$ - $^{-}3$ and $^{-}55$ - $^{-}55$ presented no difficulties for the students, and no one requested to use the chips. Next the students were given a task where the signs were the same, but there were not enough to take away, and they were given the initial amount of chips in order to force them to consider the neutralization model. This produced conflict in all students, S1 stating (for $^{+}3 - {}^{+}8$): "How am I supposed to subtract positive 8? You need the other 5 chips. So this is the wrong way". All students had to be taught to change the representation of the leading number to an equivalent one which would have enough positives in it to allow the removal of the desired amount. S1 and S2 immediately adopted this concept, but S2 often needed intervention to help him decide how many neutrals to add, and S1 often added as many neutrals as the subtrahend. The other two students had difficulty in the actual subtraction, in that after they had used an equivalent representation for the integer, they still viewed the representation in two parts: "neutrals plus the original group", and they needed the instructor's intervention of physically mixing all of the chips to accept that the total amount of chips could be considered to represent the integer. W1 could never decide how many neutrals to add, but she knew that if she added too many the extras would just neutralize, so she often added a random number of neutrals, even after intervention, and this carried over into the tasks which were larger than one digit, and caused confusion for her. When asked to give the result of the subtraction $^{-}4 - {}^{-}6$, S1 remarked "and you're left with positive 2. But we're not doing that, cause these weren't positives", giving indication that although he had learned a procedure for subtraction of chips, he was not able to intuitively be convinced that the result was valid as the result of the written integer problem. On the other hand, A1 and W1 trusted the results that they obtained with this method over the results they would have obtained if left to their own reasoning. In fact, A1, who when pretested had been able to use number line reasoning for starting from a negative position and adding or subtracting a whole number, was able now to apply the concept learned from the neutralization model to the number line: he commented (for $^{-}8 - {}^{-}10$) "smaller negative, and you're subtracting the big negative, so it's gonna have to go above zero, because there won't be place for the 2 left to be below zero".

The next type of subtraction (different signs) was an extension of the above type, and presented the same type of difficulty. Included were tasks in which zero was the initial number. These were followed by a set of a mixture of subtraction types, with which the students still had difficulty deciding how many neutrals to add. The most consistently successful student was the one who made sense of the neutralization model by reference to the number line (A1), and who then seemed to move away again to abstract the notions needed (for $^{+}100 - {}^{+}145$): "There's a hundred. Just forget the 45. There's a hundred take off a hundred, is zero. There's 45 extra, so it's gonna be negative 45".

Non-trivial subtractions (ones in which there is not enough quantity to remove) did not appear to be demystified by the neutralization model. This was the first instance where a non-intuitive, demanding, task was required: to change the representation of the leading integer to one appropriate for the subtraction, and to believe that the new representation was just as legitimate as the canonical one. This procedure appeared to be a "rule without a reason" for the students. The other obstacle was to find meaning for the result: how can the subtraction of two negative numbers yield a positive number?

LESSON FOUR dealt with a subtraction review, and with ordering (using the number line). S1 and A1 were able to solve all subtractions correctly (10 two digit tasks). S2 initially needed intervention to get on track (although evaluated by his teacher as strong, he seemed to be a student who could spontaneously handle the "intuitive" concepts, but who tended to follow a pattern at other times, and here he had lost the memory of the "pattern" of adding neutrals then subtracting). W1 could not handle the integer questions, but when they were scaled down to comparable one digit tasks, was able to use the concrete material to evaluate correctly, but each time needed to be pushed to use the chips, as she constantly tried to make a rule based on one example.

LESSON FIVE began with a mixture of two digit addition and subtraction problems, and here the complexity of looking not only at the signs of the numbers and their magnitude but also at the sign of the operation became too great an obstacle, especially for the weak student.

POSTTEST: In the week following the instruction, the students were given a post-test in the form of a semi-standardized interview. The addition and subtraction tasks were all of a two digit or three digit type. S2 and A2 were able to solve them, while S1 made careless errors, and W1, although she still made errors, was more successful than she had been the week before.

SUMMARY AND CONCLUSIONS

The neutralization model appears to be an intuitive model for the notions of integer opposites and equivalence classes, as well as for integer addition and "trivial" integer subtractions. The whole number notions of number as quantity, addition as combining, and subtraction as removing are built upon rather than ignored. Difficulties emerged with subtractions where there is not enough quantity to remove the subtrahend. It must be noted that the same length of time was devoted to integer subtraction as had been devoted to integer addition, and it is recommended that the more involved subtraction may need to be introduced more slowly, with more time spent at the physical level before larger numbers are encountered. This study, then, could be further refined with more attention paid to the subtraction section. Despite this, there was some evidence that this concrete model could provide students with a mental model which would help them with integer subtractions, certainly for subtractions such as $^{-}82 - {}^{-}27$.

A feature of the model which may cause confusion emerged when students were given a mixture of addition and subtraction tasks. They were sometimes hesitant about whether to "neutralize", or to "add neutrals" and this may be due to the fact that the neutralization procedure used for addition actually calls for **removal** of neutrals (a usual subtraction technique), and that the process of forming an equivalent representation for subtraction calls for the **addition** of neutrals.

Results of this study indicate that the neutralization model is at least a viable alternative for the teaching of integers and of the operations of integer addition and subtraction, since it provides intuitive meaning for all integer additions, and for some subtractions, which the manipulations of the number line cannot do. However, it remains to be seen whether or not all subtractions can be easily understood within the perspective of this model.

ACKNOWLEDGEMENTS

This research was part of a project on pre-algebra at Concordia University under the direction of Dr. Nicolas Herscovics, and was funded by FCAR grant #92ER1032.

REFERENCES

Chilvers, P. (1985). A Consistent Model for Operations on Directed Numbers Mathematics in School, 14(1), 26-28.

Crosswhite, F J , Dossey, J A , Swafford, J O , McKnight, C.C , Cooney, T J , Downs (1986) Second International Mathematics Study. Detailed Report for the United States. Stipes Publishing Company, Illinois, pp 71-71, 115-117.

Grady, M.B.T. (1978). A Manipulative Aid for Adding and Subtracting Integers. Arithmetic Teacher, 26(3), 40

Greeno, J.G. (1991) Number Sense as Situated Knowing in a Conceptual Domain Journal for Research in Mathematics Education, 22(3), 170-218

Haner, W W. (1947) Teaching the Subtraction of Signed Numbers School Science and Mathematics, 47, 656-658

Herscovics, N & Bergeron, J C (1988). An Extended Model of Understanding. Proceedings of Psychology of Mathematics Education - NA 10, Behr, M.J , Lacampagne, C.B & Wheeler, M M , (editors), Northern Illinois University, DeKalb, Illinois, 15-22

Kuchemann, D. (1981) Positive and Negative Numbers published in Children's Understanding of Mathematics 11-16, Hart, K. (editor), pp. 82-87

Liebeck, P. (1990) Scores and Forfeits - An Intuitive Model for Integer Arithmetic. Educational Studies in Mathematics, 21, 221-239

Lytle, P.A. (1992). Use of a Neutralization Model to Develop Understanding of Integers and of the Operations of Integer Addition and Subtraction, Masters Thesis, Concordia University

Peled, I (1991). Levels of Knowledge About Signed Numbers Effects of Age and Ability Proceedings of Fifteenth Annual Psychology of Mathematics Education conference, Vol. 3 Furinghetti, F (editor), Assisi, Italy, 145-152

Rathmell, E.C. (1980). Concepts of the Fundamental Operations Results and Implications from National Assessment Arithmetic Teacher, 28(3), 34-37

Rowland, T (1982) Teaching Directed Numbers An Experiment Mathematics in School, 11(1), 24-27

METALINGUISTIC AWARENESS AND ALGEBRA LEARNING

Mollie MacGregor & Kaye Stacey
University of Melbourne, Australia

Despite general agreement that language proficiency and mathematics learning are related, there is little knowledge of the specific language skills involved. In this paper we report a study of the relationship between students' success in learning algebra and certain aspects of metalinguistic awareness. Data was obtained from tests given to more than 1200 students aged 11 to 15. The results show that although most students could understand and use verbal and numerical information in tables, labels and other familiar formats, many of them made mistakes when consideration of syntactic structure was required and when they had to use algebraic notation. We conclude that metalinguistic awareness of symbols and syntax is an important factor controlling and limiting students' success in learning to use algebraic notation.

The question of whether or how language proficiency is related to learning ability and academic achievement has been debated for many years. There is general agreement that much learning and thinking is possible only through the medium of language. However there have been very few studies of relationships between specific components of linguistic competence (other than vocabulary knowledge) and mathematics learning. In a study of 10-16 year olds who were underachieving in mathematics, White (1985) found that they were less able than their "normal" peers to deal with precise language and to process language normally, although on practical tasks that required deductive reasoning their performance was not significantly different from that of the normal group. Mestre (1988) found that poor performance in algebra by a group of ninth-graders was associated with low scores on a test of verbal ability. According to Cuevas (1984) the learning of mathematics requires particular linguistic skills that some learners, in particular second-language learners, may not have mastered. Cummins (1984) has suggested that there is a certain level of "cognitive language proficiency" that relates strongly to academic achievement. It enables language to be used as an organiser of knowledge and a tool for reasoning. A study of the relationship between two aspects of students' language proficiency and their success in learning the fundamentals of algebraic notation is reported in this paper.

Evidence for association between language proficiency and algebra learning

As a precursor to the main study, data on language proficiency and algebra learning were obtained from a school in a middle-class suburb of Melbourne. A mixed-ability Year 8 class of 25 boys who had not been taught algebra before was observed during three weeks of lessons on algebra given by their mathematics teacher. The boys were then tested on the material taught (i.e., simplifying, expanding and solving equations) using a teacher-made test. Reading comprehension measures (i.e., level of ability to understand ordinary non-mathematical English) for these boys were obtained from the school records, and compared with the algebra test results. The scatterplot shows a linear distribution, with a positive correlation (n = 25, r = .57) between algebra scores and reading comprehension measures. There are three conspicuous outliers to the general trend, and when these are removed from the sample the correlation is high (n = 22, r = .81). This high correlation is surprising because the algebra test contained no verbal material except brief instructions such as "expand" and "solve", and the teaching method had required no reading by the students, who had merely watched demonstrations at the chalkboard and practised symbolic examples. Possibly the teacher believed that by eliminating the reading component from his lessons he was making learning accessible for all students including those with weak English comprehension skills. However almost all the students who had received low scores on the comprehension test learned very little algebra. There are likely to be many factors, both affective and cognitive, that contributed to this outcome. Our hypothesis was that one of these factors was metalinguistic awareness, described below.

Awareness of language structures

Some of the cognitive processes used for generating and interpreting language are automatic, unconscious and spontaneous; others are deliberate, conscious and analytical. The term *metalinguistic awareness* is currently used by researchers in the field of literacy development (see, for example, Hakes, 1980; Tunmer, Pratt & Herriman, 1984) to refer to the linguistic ability that enables a language user to consciously reflect on and analyse spoken or written language. Metalinguistic awareness is involved when the form of an expression is considered rather than its meaning. It enables the language user to reflect on the structural features of language and make choices about how to represent or how to interpret information.

One component of algebraic thinking, according to Lins (1992), is the ability to mentally manipulate abstract objects in accordance with properties of the classes of objects to which they belong. This ability in algebra operates at the same level of abstraction as metalinguistic awareness in ordinary language when words and word strings are treated as instances of variables with general properties (e.g., the word "simplify" could be considered as an instance of "transitive verb" or "word of three syllables" or "word starting with s"). An example in an algebraic context is the classification of $x + 2$ as (a) an instruction to add 2 to x, (b) the result of adding 2 to x, (c) a function of x, or (d) a number greater than x. It is likely that students with low levels of metalinguistic awareness in ordinary language would have difficulty in classifying mathematical objects and hence in interpreting and using algebraic symbolism.

From a list of seven components of metalinguistic awareness in ordinary language (Herriman, 1991) we selected two - *word awareness* and *syntax awareness* - that have mathematical analogues. Translated into the context of algebra, these two components are:

1. Symbol awareness (analogous to *word awareness*)

• knowing that numerals, letters and other mathematical signs can be treated as symbols detached from real-world referents. It follows that symbols can be manipulated in order to rearrange or simplify an algebraic expression, regardless of their original referents.

• knowing that groups of symbols can be used as basic meaning units. For example, $(x+2)$ can be considered as a single quantity for the purposes of algebraic manipulation.

2. Syntax awareness

• recognition of well-formedness in algebraic expressions. For example, "$2x = 10 \Rightarrow x = 5$" is well-formed, whereas "$= 2x = 10 = 5$" is not well-formed.

• the ability to make judgements about how syntactic structure controls meaning and the making of inferences. For example, if $a + b = x$, then $b + a = x$, but if $a - b = x$, then it is not generally true that $b - a = x$.

Development of test items

To test our hypothesis that an association exists between the two components (see above) of students' metalinguistic competence and algebra learning, we prepared test items in three categories to measure (i) symbol and syntax awareness in comprehending written English (ii) ability to select numerical data, choose an operation and calculate; and (iii) interpretation and use of algebraic notation. Examples of items are shown in Figure 1.

Category (i) item

Read the sentence in the box.

There are more than seven words in this sentence.

(i) What is the fifth word? ....(ii) How many letters are in the third word? ....

Category (ii) item

The timetable shows airline flights from Melbourne to Townsville.

Flight	*Days*	*Depart*	*Arrive*	*Stops*	*Meals*
60	MTWTFS-	6:30	11:05	1	B/fast
62	------S	8:15	12:15	1	Snack
35	-----S-	8:30	12:30	1	Snack
38	-----S-	10:00	1:45	0	Lunch
66	MTWTFSS	11:40	3:35	1	Lunch
52	--WTF--	3:00	8:25	2	Dinner
53	M------	3:20	8:45	2	Dinner

(i) How many flights are there on Saturdays? ....

(ii) Which flight takes the least time? .... How long does it take? ....

Category (iii) items

1. Which of the following expressions can be written as $n + n + n + n + n$?

$n + 5$ $\quad$ $n \times 5$ $\quad$ $5n$ $\quad$ n^5 $\quad$ 5^n

2· Sue weighs 1 kg less than Chris. Chris weighs y kg.

Use algebra to write Sue's weight. ....

Easy reading item (See airline timetable above)

What is the arrival time of Flight 60? ....

Figure 1. Examples of test items

We also included an easy reading item requiring no metalinguistic awareness or calculating, in order to identify students who could not understand written English. To construct items in categories (i) and (ii), we used general ideas from the work of Kirsch and Mosenthal (1990), Wickert (1989), and Whimbey and Lochhead (1986). Algebra items (category iii),were based on diagnostic items used in other studies (e.g., Assessment of Performance Unit, 1985; Avila, Garcia & Rojano, 1990; Booth, 1984; Kuchemann, 1981; MacGregor & Stacey, 1993). Students' total scores for the three categories of items are referred to in this paper as (i) *language scores*, (ii) *arithmetic scores* and (iii) *algebra scores.*

Procedure

The test was given to the whole cohort (n = 130) of Year-10 students in a boys' school in a middle class suburb of Melbourne, the same school from which the preliminary data had been obtained. The boys were in their third year of algebra learning. Responses were scored as either right or wrong, and total scores for each student on the three categories shown above were recorded. A similar test was given to a representative sample (n = 1236) of students in 18 schools in year levels 7 to 10.

Results

1. Sample of 130 boys in Year 10. All boys were correct on the easy reading item, indicating basic reading competence. There were errors made on all other items. The arithmetic items were easier than the algebra items, in spite of the fact that the arithmetic items required far more reading and general literacy skills than the algebra items (discussed below). Fig. 2 shows the association between language scores and arithmetic scores. Fig. 3 shows the association between language scores and algebra scores. Note that clusters in the figures indicate that more than one individual obtained a particular score. For example, the three-armed cluster at the top of Fig. 2 indicates that three boys obtained a language score of 65% and an arithmetic score of 100%.

The figures show that none of the students with very low language scores performed well on either the arithmetic items or the algebra items. Fig. 3 shows that only those students who obtained a language score of more than 90% succeeded on all the algebra items. In contrast, as shown in Fig. 2, boys with language scores down to 65% were able to get top arithmetic scores. The different pattern of scatter in the two distributions - Fig. 2 suggesting a general scatter and Fig. 3 showing a lower triangular pattern - is currently being investigated.

The pattern of scatter in Fig. 3 suggests that metalinguistic competence is a limiting factor in the development of algebraic competence, since there is no instance of a student with a low language score achieving a high algebra score. However it is clear that algebra scores cannot be predicted from language scores, because of the considerable number of students high on the language measure but low on algebra.

2. Large sample across four year levels 7 to 10. At each of the four levels, the association between language scores and algebra scores was similar to the pattern shown in Fig. 3. There was no

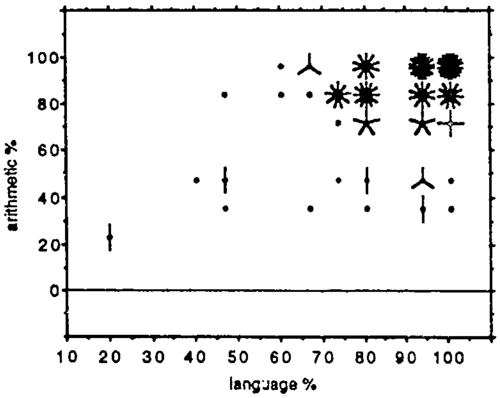


Figure 2. Association between language scores and arithmetic scores for 130 boys in Year 10

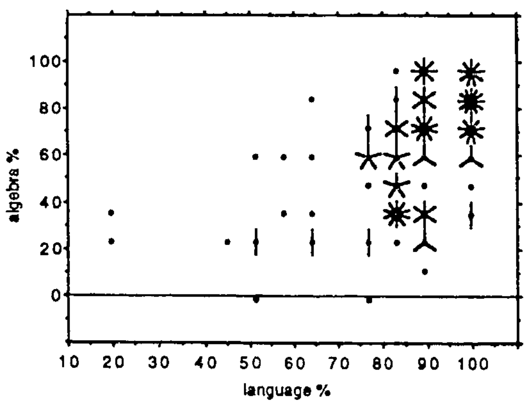


Figure 3 . Association between language scores and algebra scores for 130 boys in Year 10

instance of a student with a low language score and a high algebra score. However there were many students at all year levels with high language scores and low algebra scores (see, for example, Fig. 4). The lower triangular pattern in all language/algebra scatterplots suggests that metalinguistic awareness may operate as a factor that limits students' success in using algebraic notation but does not limit their success on the arithmetic items.

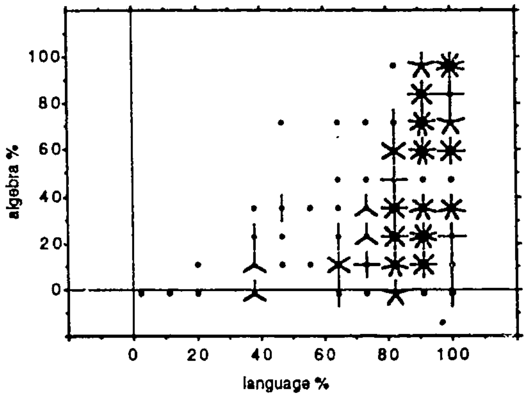


Figure 4. Association between language scores and algebra scores for Year 7 sample (n = 188)

Discussion and conclusion

In contrast to the results of a nationwide survey of adult literacy (Wickert, 1989), we found that almost all students in our samples could select and use verbal and numerical information presented in timetables, charts and labels (Category ii). However in test items requiring awareness of symbols, syntax and word order (Category i), many students made mistakes. Items requiring awareness of syntax in an algebraic context and the ability to distinguish an algebraic symbol from its referent (Category iii) were harder still.

Success on the arithmetic items (Category ii) required choosing an operation and using it for simple calculating, in addition to comprehending the written English and format of the problems. For example, in the Category (ii) item shown in Fig. 1, the time from 10:00 am to 1:45 pm has to be calculated, and there are many decisions to be made in the selection of appropriate information from the rows and columns before the calculating can be done. According to Kirsch & Mosenthal's (1990) investigations of adult literacy, an item such as this one should be relatively difficult because:

(a) the information given in the question does not correspond closely with the information provided in the data, and therefore inferences have to be made about what to do;

(b) prior knowledge may be required (e.g., how to interpret a timetable);

(c) there are many categories of information in the data;

(d) there are many specific pieces of information in the data.

In spite of these potential sources of difficulty and the many opportunities for careless mistakes, approximately 80% of students were correct for all parts of the item. In contrast, only 42% were correct for the apparently very simple algebraic item

Sue weighs 1 kg less than Chris. Chris weighs y kg. Use algebra to write Sue's weight.

Many of the incorrect answers given for this item reveal lack of one or both of the two metalinguistic skills (i.e., *symbol awareness* and *syntax awareness*) that we have selected as necessary for dealing with algebraic symbolism. Nineteen answers (15% of the sample) contained the letters S or C or both, for example, $S = C\text{-}1$, $S\text{-}1 = y$, $C+1 = S$, $1\text{-}C$. As well as revealing syntax difficulties, these expressions indicate the student's belief that algebraic letters are shorthand names (i.e., S stands for *Sue* or for *Sue's weight*), reflecting poorly developed metalinguistic awareness of the distinction between symbols and their referents. Other frequently occurring incorrect answers to this item included $1\text{-}y$ and $1+y$. Both these expressions reveal difficulties in interpreting either the syntax of algebra or the syntax of the given problem. The answer $1\text{-}y$ indicates that the concept of "1 less" was understood correctly from the given problem, but the student did not know how to write "1 less than y" in algebraic symbols. The expression 1+y suggests that the student did not correctly interpret the syntactic structure of the given problem, and decided that the answer (Sue's weight) involved adding 1 instead of subtracting 1. Almost all incorrect answers to the other algebra items in the test could similarly be attributed to lack of symbol and syntax awareness.

As reported earlier in this paper, we found a close correlation between students' English comprehension scores and their scores on a teacher-made test of algebraic procedures. In contrast, in our tests a large proportion of students with high scores on the metalinguistic items obtained low scores on the algebra items. The reason for this discrepancy has not been investigated, but probably reflects the different nature of the algebra items: the teacher-made test required remembering and reproducing rote-learned procedures, whereas most of the items we prepared required students to make choices about the selection and arrangement of algebraic letters, numerals and operation signs in order to produce algebraic expressions.

It is not clear why some students with high language scores were wrong on many algebra items in our tests although they been given the opportunity to learn. One explanation is that these students are not sufficiently aware that the algebraic sign system has its own grammatical rules and conventions that are not intuitively obvious and have to be learned; they continue to use automatic, unconscious language-processing rules (Kaput, 1987; MacGregor, 1991). This in turn may be a result of current approaches to the teaching of language arts and early numeracy, in which children are encouraged to record ideas in unconventional ways that they invent themselves. Students with high metalinguistic awareness may succeed in learning algebra if they are helped to recognize that changes in the order, position or grouping of symbols affect meaning and that the language of algebra has its own set of grammatical rules that are not intuitive or "natural" and have to be learned and practised. However for students with low metalinguistic competence, this approach is unlikely to be effective.

There are many factors that contribute to students' difficulties in learning algebra. Our study shows that one of these factors, not previously reported in the literature, is the student's level of metalinguistic awareness.

References

Assessment of Performance Unit (1985). *A review of monitoring in mathematics 1978 to 1982.* London: HMSO.

Avila, A., Garcia, F., & Rojano, T. (1990). Algebraic syntax errors: a study with secondary school children. In G. Booker, P. Cobb, & T. Mendicati (Eds.), *Proceedings of Fourteenth Conference, International Group for the Psychology of Mathematics Education* (pp. 11-18). Mexico: PME.

Booth, L. R. (1984). *Algebra: children's strategies and errors. A report of the Strategies and Errors in Secondary Mathematics Project.* Windsor, Berks.: NFER-Nelson.

Cuevas, G. J. (1984). Mathematics learning in English as a second language. *Journal for Research in Mathematics Education, 15*(2), 134-144.

Cummins, J. (1984). *Bilingualism and special education: issues in assessment and pedagogy.* Clevedon: Multilingual Matters.

Hakes, D. T. (1980). *The development of metalinguistic abilities in children.* Berlin: Springer-Verlag.

Herriman, M. (1991). Metalinguistic development. *Australian Journal of Reading, 14*(4), 326-338.

Kaput, J. (1987). Towards a theory of symbol use in mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 159-195). Hillsdale: Erlbaum.

Kirsch, I. S., & Mosenthal, P. B. (1990). Exploring document literacy: variables underlying the performance of young adults. *Reading Research Quarterly, XXV*(1), 5-30.

Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London: Murray.

Lins, R. (1992). *A framework for understanding what algebraic thinking is.* Unpublished doctoral thesis, University of Nottingham.

MacGregor, M. (1991). *Making sense of algebra: cognitive processes influencing comprehension.* Geelong: Deakin University.

MacGregor, M. & Stacey, K. (1993). Seeing a pattern and writing a rule. In I. Hirayabashi, N Nohda, K. Shigematsu, & Fou-Lai Lin (Eds.), *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education* (pp. 181-188).

Mestre, J. P. (1988). The role of language comprehension in mathematics and problem solving. In R. R. Cocking & J. P. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics* (pp. 201-220). Hillsdale: Erlbaum.

Tunmer, W. E., Pratt, C. & Herriman, M. L. (1984). *Metalinguistic awareness in children: theory, research and implications.* Berlin: Springer-Verlag.

Whimbey, A., & Lochhead, J. (1986). *Problem solving and comprehension* (4th ed.). Hillsdale: Erlbaum.

White, J. M. (1985). Developmental delay and cognitive processes in mathematics. *Focus on Learning Problems in Mathematics*, 7(2), 11-21.

Wickert, R. (1989). *No single measure. A survey of Australian adult literacy.* Canberra: Commonwealth Department of Employment, Education and Training.

CHILDREN'S DIFFERENT WAYS OF THINKING ABOUT FRACTIONS

Carolyn A. Maher, Amy M. Martino & Robert B. Davis
Rutgers University
New Brunswick, New Jersey USA

A constructivist teaching experiment was carried out with a heterogeneous class of twenty-five 9-10 year old fourth-grade children in order to explore and to study their development of fraction ideas. Problems were presented based on ongoing analysis of children's thinking. All twenty sessions were videotaped with three cameras. The videotapes, children's written work and journals, as well as recorder notes comprise the data. This report illustrates children's retrieval of competing representations of fraction ideas as they discuss the placement of fractions between 0 and 1 on a segment of the number line. It describes, also, the children's effort to reconcile the different representations in an effort to place fractions on a number line.

Theoretical Framework

Freudenthal (1983), in *Didactical Phenomenology of Mathematical Structures*, offers a phenomenological presentation of fractions. In light of the complexity of ideas, he expresses concern with the assumption that "...pupils are supposed to be so advanced as to be satisfied with one approach from reality" (p. 134) and suggests this as a reason why many people never learn fractions. In *Fractions in Realistic Mathematics Education*, Streefland (1991) provides a historical account of the difficulties presented by fractions in mathematics education and demonstrates that the source and application of fractions has been removed from the theory of fractions. Drawing on several years of direct experience working with primary school children, Streefland proposes a theory on the teaching and learning of fractions to illustrate realistic mathematics education. A goal, he suggests, is for "insightful (re)construction" of the system to correspond to the historical learning process.

Behr, Lesh, Post, and Silver (1983) have spent several years studying children's development of rational-number ideas. They indicate that their data have made them sensitive to the issue of concept stability and instability as well as of concept development and suggest that mathematical ideas are present at different levels of sophistication and do not move from ideas that are "understood" to ideas that are "mastered". They indicate that with the development of ideas comes periodic reconceptualization. When embedded in progressively more complex systems, the ideas then may be significantly altered.

Davis, in his 1984 book, *Learning Mathematics*, presents a general theory of selecting among alternative mental representations. He documents many situations in which children select from among alternative mental representations in building solutions to problems.

Clearly, there is no sparcity of documentation of the complexity of fraction ideas and the difficulties children have in building a meaningful understanding of them (See, for example, Behr, Bright & Wachsmuth, 1982; Streefland, 1978; and Maher, Davis & Alston, 1991).

The purpose of this study is to uncover the ideas that children have about fractions when they are encouraged to build their ideas further in an environment where it is easy to express their thinking. The research takes the form of a classroom teaching experiment as described by Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, Perlwitz (1991). In this mode researchers study students' construction of ideas as they attempt to resolve problem situations. Classroom activity is guided by ongoing analysis of videotape data that provide elaboration about student thinking.

Background

This research is a small segment of a longitudinal study of the development of mathematical ideas in children between the ages of 8 and 12 years.[1] The fourth grade students attend a suburban school district in New Jersey where the teachers have been part of a teacher development project in mathematics with Rutgers University that is currently in its fifth year. Mathematics instruction at this school district is intended to create classroom environments that support the learning of mathematics through problem solving. Children are given opportunities to do mathematics in an open environment in which they are encouraged to explore patterns and relationships, make conjectures, and discuss their ideas with others. Students are observed working individually, with a partner, or in a small group, at a personally determined pace and then sharing solutions in whole-class discussions. The classroom environment is designed to encourage thoughtfulness, communication, exploration, and reflection as children are challenged to consider the reasonableness of proposed ideas.

Methods and Procedures

Classroom Setting. In studying the development of mathematical ideas, problematic situations are posed and students are observed building solutions, developing arguments, constructing models, comparing models, discussing their ideas and negotiating their conflicts. The role of the teacher(s) is to facilitate discussion and probe students' thinking through questioning that is related to the student's construction. The teaching style is guided by the students' thinking. Thus, what the students explore is dictated by the questions that they raise. Closure on a topic is deferred until students come to a resolution. New ideas that arise from students trigger further questions which encourage exploration. For this reason, it is not unusual for students to explore a question for an extended period of time that might span weeks, months or in some cases, years (see Martino & Maher, 1991; Maher, Davis & Alston, 1991). Students are encouraged to use resources and tools in their exploration. On days when the teaching experiment is not in progress, the classroom teacher allows the children time to reflect and write about ideas considered in previous sessions.

__Data Source__. Although this research is a continuous phenomenon building upon the prior sessions, our intent in this report is to present a small segment of this teaching experiment. Data for this study in the form of videotape transcripts of classroom lessons and children's written work are from the November 1, 1993 class session involving the placement of fractions on the interval from 0 to 1 on a segment of the number line.

__Procedures for Data Collection.__ In collaboration with the classroom teacher, a 20-session teaching experiment was conducted with a heterogeneous (average to high) class of 25 fourth graders.[2] Each class met from 60 to 80 minutes, on average three times a week beginning in September 1993. All lessons were videotaped with three cameras. Two cameras were used for observing the students building their models and the classroom discussions and interactions. The third camera captured the activity in the front of the room where students presented and discussed their work at the overhead projector. Transcripts of videotapes, the accompanying written work of the students, and researcher notes comprised a __video portfolio__ for each student.

__The Problems__. During the months of September through November of 1993, the students worked on a series of problem tasks using available materials (such as Cuisenaire rods, string, meter sticks, etc.) to explore some basic ideas related to fractions. Students used these activities to explore the concepts of fractional part, equivalent fractions, comparison of fractions, proportion, division of fractions and placement of fractions on a number line. The inter-relationship between ideas evolved naturally. For example, some of the comparison of fraction activities discussion resulted in the production of different models, which gave rise to the discussion of ideas related to ratio and proportion. Thus, one activity frequently led to further questions which were explored by the students.

__The Evolution of the Number Line Activity.__ The placement of fractions on the number line activity was triggered by a class discussion of a comparison problem that had occurred a week earlier. The problem was to share three large candy bars, each containing 10 blocks of chocolate, among the class of 25 students. The regular classroom teacher partitioned the class into three groups, two with eight students and one with nine. The students built three solutions to the problem. For the first two solutions, each of the three groups were given one candy bar. For the groups with eight students, it was determined that each student in that group would get one and one-fourth blocks of chocolate while each student in the group with nine students would receive one and one-ninth blocks of chocolate. Students realized that the people who got one and one-fourth blocks had received more chocolate. The students were then challenged to determine how much larger one-fourth was than one-ninth and spent a few days building models with Cuisenaire rods to compare one-ninth and one-fourth. The students determined that one-fourth was larger than one-ninth by five-thirty sixths and justified their argument using the models that they had built.

At the completion of this activity, another student, Andrew, offered a third solution. He suggested that there be only one group of students, that is, the class of twenty five. He then proposed that the thirty pieces of chocolate be distributed so that each student would receive one and one-fifth blocks of chocolate. (He constructed his mental representation of this solution without building a physical model or making a drawing.) The class became engaged in this idea and this triggered a discussion about the comparison of the three fractions, one-fourth, one-fifth, and one-ninth. An outgrowth of that discussion was the recognition that "the larger the number of people the smaller the piece of chocolate". Students indicated that the larger the number (denominator), the smaller the piece of chocolate. The lesson on comparison had evolved into a discussion of ordering about which the students were both interested and curious. The students' natural interest in ordering these fractions motivated us to see whether the students could order fractions on a number line.

Results

Placing fractions on the number line (November 1, 1993)

The students were given the task of placing the unit fractions 1/2, 1/3, 1/4... 1/10 between the interval 0 and 1 on a number line. Students were also asked to place the number 3/4 on the same interval. After all students had an opportunity to construct a number line, one student, Alan, shared his number line with his classmates (see Figure 1).

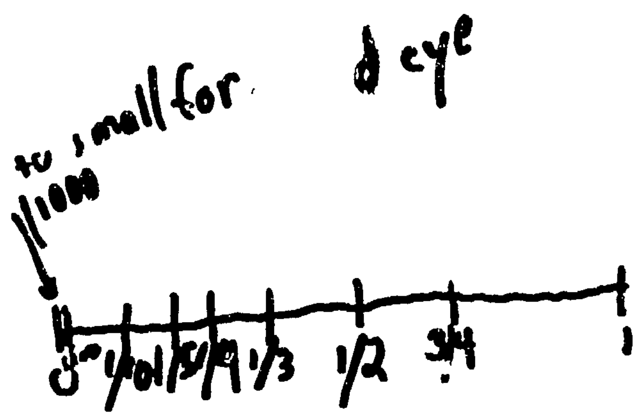


Figure 1. Alan's original number line.

Alan states, "Between zero and one [on his number line] you can divide it into those fractions [see Figure 1]... such as the 3/4 would go there [pointed to a position between 1/2 and 1 on his number line] because you've got 1/3 there and then it would take... [He paused, then began his justification for the placement of 3/4.] ... no... 1/4 there [pointed to the placement of 1/4 on his number line] and you'd need three of those [referred to three segments of length 1/4] to get to that mark [the position of 3/4 on his number line]". Thus, Alan had determined the position for 3/4 by finding the length of 1/4, placing it on his number line and tripling this length to locate the position for 3/4.

In the course of this activity, some children placed the fraction 1/3 in both the 1/3 and 2/3 position on the interval between 0 and 1 (see Figure 2). The children were encouraged by the teacher to discuss this placement of 1/3. Alan, who was focusing on intervals of length 1/3, justified that the 1/3 could go on both 1/3 and 2/3 of the interval between 0 and 1. He explains, "You could put... basically... the 1/3 in any place... in any three places of that number line... because you could have the 1/3's going either way... I mean you could take a third out from there, from there, from there... so you really could put it in three places".

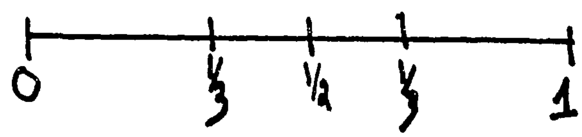


Figure 2. Placement of the fraction 1/3 in both the 1/3 and 2/3 position on the interval between 0 and 1.

Another student, Andrew, challenged Alan about the position we would call two-thirds. In response, Alan built a model with Cuisenaire rods comparing the length of one dark green rod to three red rods (see Figure 3). As Alan placed a mark next to each of the three positions on the interval between zero and one that he would call 1/3 he said, "Here you have the thirds. You could mark the 1/3 for being here, 1/3 for being here, 1/3 for being here".

Figure 3. A reproduction from the videotape of Alan's model built with Cuisenaire rods to show that there are three "1/3 segments" between 0 and 1 on the number line.

Children were then challenged to place the numerals 1/3, 2/3, 3/3 and 0/3 on Alan's model. Alan responded by placing the number 1/3 above his first 1/3 marking; classmates Mark, Danielle and Andrew placed the numbers 2/3, 3/3 and 0/3 on the number line as indicated in Figure 4.

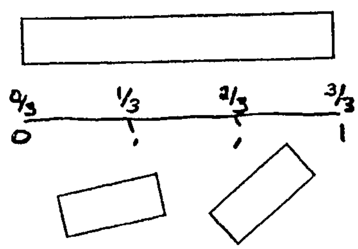


Figure 4. A reproduction from the videotape of students' placement of the fractions 0/3, 1/3, 2/3 and 3/3 on the interval of number line between 0 to 1.

In earlier discussions, students had agreed that there could be many names for the same rational number. For this reason, the teacher again challenged the students to consider whether they could put the numeral 1/3 on the number line in the same position that they had placed 2/3. Alan again considered intervals of length 1/3 and responded, "This could be a third [He referred to the length from 1/3 to 2/3.]... and between there and here [He referred to the length from 2/3 to 3/3.] that would be a third". To justify his placement of the fraction name 1/3 above the interval from 1/3 to 2/3, Alan used the Cuisenaire rods to begin to make a distinction between numbers which define intervals and numbers which define positions on the interval.

Alan: But basically what comes to mind... when you think of fractions... you always think of the first one [He referred to the length from 0 to 1/3 then picks up a red rod.] You could put it here [from 0 to 1/3], here [from 1/3 to 2/3] or here [from 2/3 to 3/3], but you could put 1/3, 2/3, 3/3 in any one of those places [He placed one red rod then two red rods then three red rods along his number line between 0 and 1.]... but you can still go 1/3,2/3... I mean 1/3, that would be 1/3 that would be 1/3 [referred to the three individual thirds between 0 and 1].

Thus, Alan had two different ways of looking at 1/3, the placement of numbers and the length of the segment divided into three pieces, exhibiting a mapping between the two concepts. Alan's explanation encouraged further classroom discussion and interaction.

During this session the class was attentive and contributed to the discussion. Andrew, who was focused on the placement of numerals on the number line and was concerned about preserving additivity of measure, questioned Alan's labeling the middle interval 1/3. As a result of his concern, Andrew presented an accumulated distance argument. He indicated, "I don't think it could really work cause if you just put red in the middle [from 1/3 to 2/3] and call it 1/3, then if you put there on the left side of it 3/3...on the right side of it 2/3... then you'd be reading it 2/3, 1/3, 3/3 [sic] so wherever you put it in your space you always have to start from zero cause you can't go from 1 down to 0. Because that's getting bigger and if you start it like that ... you're then just switching the 0 and 1".

Alan conceded, although still maintaining that one can still regard one-third as one of three segments of equal length and indicated, "Right. But you can still put 1/3 in any of these places, but basically what comes to mind once you think of fractions... you always think of the first one [the length from 0 to 1/3] it could go in any of these". Thus, Alan restated his understanding that since the three red rods (representing the three intervals between 0 and 1) had the same length they had the same fraction value of 1/3. Students continued to explore these ideas about placement of fractions on the number line. They were asked to draw models and write about their models which were shared in the next two sessions. The sharing triggered further reconfiguration of these ideas as children worked to negotiate understanding of alternative representations.

In the construction of his number line, Alan who had built his number line quickly, was asked by the teacher to place unit fractions of 1/100 and 1/1000 on the interval between 0 and 1. In presenting his number line to the class, Alan produced an enlarged version of the small segment of number line between 0 and 1/10 (see Figure 5). He then proceeded to talk about his placement of the fractions 1/100 and 1/1000 on this segment of the number line. He explained, "The 1/2 [position halfway between 0 and 1] you could use as a guideline... now the others... these are the 1/10, the 1/100 and the 1/1000 which I did...now I made a big picture [see Figure 2 for Alan's enlarged view of a segment of his number line] cause you couldn't really see it on that one [referred to the placement of 1/100 and 1/1000 on his original number line]... so that's where the 1/1000 would be [referred to its placement on his new section of number line]. You can't make anything bigger than that cause it would be too small to see".

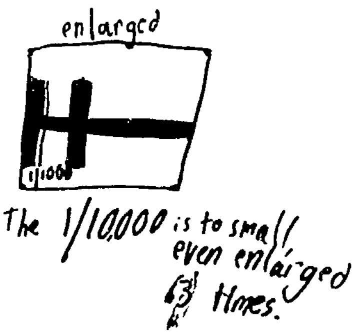


Figure 5. Alan's enlarged segment of his original number line.

Conclusions and Implications

The complexity of building these ideas becomes apparent as we observe students retrieving (or building) different representations and negotiating alternative meanings. Surely these ideas must appear somewhat mysterious to students when each perspective seems equally reasonable. For example, there may be no apparent reason to a child at this point for preserving additivity of measure. Both Alan and Andrew present two different and correct ways of mapping their ideas. Alan considers the segment as divided into three pieces, marking one-third where two-thirds might be expected. In his labeling of the segment, he clearly recognizes one-third of the segment and is not concerned with additivity. Andrew, on the other hand, proposes a cumulative structure, starting at 0 and marking how far each third is from zero.

Alan, interested in the placement of smaller fractions on the interval, extends his understanding. He represents a portion of his line as magnified to indicate the placement of one-hundredth and one-thousandth and indicates the placement of even smaller fractions. He then engages some members of the class in a *lively discussion* in an attempt to convince them that as the fraction gets smaller and smaller, there still is a place for it on the number line. The findings from this analysis are consistent with perspective provided by Freudenthal regarding the complexity of fraction ideas and the multiplicity of approaches for building concepts. This work also supports Streefland's view regarding the importance of insightful reconstruction.

Endnotes

1. This research is supported in part by National Science Foundation #MDR-9053597 to Rutgers, the State University of New Jersey and a grant funded by the New Jersey Department of Higher Education # 93-992002-8001 to develop lesson units. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the above mentioned organizations.

2. We would like to express our gratitude to fourth grade teacher, Joan Phillips, who has been our partner in this long-term study.

References

Behr, M.J., Bright, G.W. & Wachsmuth, L. (1982). Number line representations of fractions. *Proceedings of the Fourth Annual PME-NA*. Athens: GA. pp. 106-112.

Behr, M.J., Lesh, R., Post, T.R. & Silver, E.A. (1983). In R. Lesh and M. Landau (Eds.) *Acquisition of mathematical concepts and processes*. New York: Academic Press Inc, pp. 91-126.

Cobb, P., Wood, T., Yackel, E., Wheatley, G., Trigatti, B. & Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education*, 22(2), pp. 3-29.

Davis, Robert B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* New Jersey: Ablex Publishing Co.

Davis, Robert B., Maher, Carolyn A. & Martino, Amy M. (1992). Using videotapes to study the construction of mathematical knowledge by individual children working in groups. *Journal of Science Education and Technology*. New York: Plenum.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, Holland: D. Reidel Publishing Company.

Maher, C.A., Davis, R.B. & Alston A. (1991). Brian's representation and development of mathematical knowledge: A four year study. *The Journal of Mathematical Behavior*, 10(2), 163-210.

Maher, Carolyn A. & Martino, Amy M. (1991). The construction of mathematical knowledge by individual children working in groups. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth Annual Conference of the International Group for the Psychology of Mathematics Education*. Assisi, Italy.

Streefland, L. (1978). Some observational results concerning the mental constitution of the concept of fraction. Educational Studies in Mathematics, 9, pp. 51-73.

Streefland, L. (1991). *Fractions in realistic mathematics education*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Problem Posing and Hypothetical Reasoning in Geometrical Realm (*)

Nicolina A. MALARA Department of Mathematics - University of Modena (Italy)
Loredana GHERPELLI Middle School "A. Ferrari" Maranello-Modena (Italy)

We expose the guidelines and the main results of a research, realized with small groups of 12/13-year-old pupils, aimed at leading pupils in posing problems on elementary geometrical plane figures, through the construction of their texts. With the research we want to study the real possibilities of the pupils of such age in posing geometrical problems and to obtain information on the effects of cooperative work on this matter. From a general point of view we have realized that problem posing promotes the ability to solve problems under hypotheses (through the individualization and resolution of the possible problems linked to a particular geometrical figure) and metacognition (through the control of the strategies underlying the different situations constructed, the awareness of the fundamental relations among the elements of the geometrical figure studied and the full knowledge of the range of the classical models of problems on them). From a methodological point of view we have highlighted the effectiveness of working in small groups either for the production and solution of problems or for the contribution in overcoming the difficulties of the weaker pupils.

Introduction

What we are going to state here concerns a didactic research aimed at leading the pupils to pose and pose themselves problems through the construction of the texts. This is part of a wide project of educational innovation on the problem started in 1987 and developed into different experimentations, some with other specific aims, but in general with the intention of refining the skills in solving problems and promoting hypothetical reasoning and metacognition (see Malara 1991, 1993 and Malara et alii 1992).

The importance of educating the pupils to formulate problems has been underlined by scholars such as Polya (1962), Sawyer (1964), Freudenthal (1973) and in more recent times Brown & Walter (1983), Kilpatrick (1987), Silver (1993). As the last mentioned has punctuated, the interest in this kind of activity has gradually increased with time up to the point that in the US problem posing has been recommended by NCTM as activity to strengthen for a few years already. Moreover, it is to consider that today this topic appears in the Mathematics :rriculum in culturally advanced countries such as Spain.

We do not intend to linger here on the different aspects of problem posing from a theoretical point of view (problem posing seen as characteristic of a teaching oriented to inquiry, as means of improvement of problem-resolutory skills, as window on mathematical comprehension, as strategy for spotting out or rousing creativity or exceptional ability, etc.) for which we refer to Silver (1993).

Statement of motives

The idea of a research on problem posing is the result of several beliefs of ours which we shall expose.

Above all we believe that many failures in solving problems may depend on the lack of intellectual and emotional involvement of the pupils, whereas taking them into a personal project they may remarkably increase the quality of their performance.

Secondly, we consider this kind of activity interesting because it subverts the usual role of the pupil in front of a problem, allowing him to lead an *underground autonomous* operation of analysis and data elaboration, as well as good control of his reasoning routes. So, as expressed by Kilpatrick (1987, p. 123), *"Problem formulating is an important companion to problem solving"*.

Moreover, we find it an important metacognitive objective to lead the pupils to get aware of the elements cooperating in the construction of a text of a problem: the premises linked to mathematical contents

(*) Work supported by MURST and CNR (contract n. 93.00524.CT01)

required, the familiarity with procedures of resolution, the specific codes of mathematical language, the general linguistic competence. In accordance with other scholars (see for instance Moses et alii 1990), in order to avoid scattering and to focus the pupils' attention on the construction of problems, we consider fundamental that they operate in small mathematical contexts familiar to them, so as to relieve at most any possible difficulty or stress. In particular, it was our intention to study through this research the effective capability -in pupils of such age- of constructing significant problems within the field considered, as well as to investigate on the effects of working in cooperation on such activity.

Our choices

A good field for problem posing is in our opinion the one concerning the measure of classical figures both for its relative delimitation and for its specificity which, besides being an interesting field of experience in itself, allows to eliminate the influences typical of concrete contexts. In our research the attention was drawn onto the following categories of figures: rectangle, triangle (with particular reference to isosceles triangles), rhombus, rectangle trapezium.

The idea on which the activity is based is the following: *the teacher declares to the pupils the measures of certain elements of a given figure*, in the number strictly necessary for constructing a scale model, *and asks them to produce with reference to it one or more texts of problems.*

The reasons for making the pupils operate on a particular model of figure correspond to various needs. First of all we intend to stimulate in the pupils the individuation of more information or data of relational kind relative to the figure in question -and that's why we find it fruitful that they may rely on a faithful graphical representation; by giving convenient values, we then want to reduce to the minimum any difficulty that the pupils may meet on dealing with the numerical values of the data given and we also mean to limit the variability of the latter in order to focus the attention on the procedures (either the pupils' during the phase of analysis and comparison of the performances, or the teacher's for the assessment of the work produced).

The work continues then through the critical revision of what the different groups have worked out, so as to highlight the processes and/or the faults in the construction of the texts.

As in Ellerton (1986), a crucial element of the didactic contract we promote is *to arrange with the pupils* the construction of texts of problems to be presented *as a challenge* to the classmates. We actually believe, together with Silver (1985), that this brings to a considerable emotional involvement of the pupils.

Aims and premises

The main specific didactic objectives of this research can be considered:

- the aware acquisition of the main relations between the different elements of a figure;
- the fact of getting aware of the incidence of mathematical knowledge on the amount of information necessary to solve a problem;
- the critical analysis of texts of problems both from linguistic and mathematical point of view (systematic control of the data inserted, attention to number and compatibility, coherence with the starting situation);
- the recognition of identities of relation and/or analogies and the use of known procedures in more complex situations.

The preliminary activities, tackled right from the beginning of Middle School, concern on one hand linguistic aspects: comprehension of the text of a problem (explicitation of the meaning of terms and sentences, paraphrase) and relative critical analysis (linguistic correctness of the form, possibility of different interpretations, etc.), quite similarly to what Laborde (1990) wished; on the other hand activity of resolution of problems according to our general project. As to geometry specifically, wide attention is given to the definition of the different triangles and quadrilaterals and to their classification; further attention is paid to the pupils' conception of a dynamic vision of the figures by evidentiating, for instance, the relativity of the base and how the height depends on it. Preliminary, too, can be considered the arithmetical activities of approach to the concept of rational number, which facilitate the pupils in expressing the measure of certain segments as fraction of others.

Methodology and structure of the work

The methodology adopted in developing this work is that of small groups (usually 3 units). In accordance to Kilpatrick (1987) we believe that work in groups offers a natural context for the formulation of problems. About this Kilpatrick writes: *"When students work together, they often identify problems that would be missed if they were working alone. A poorly formulated idea brought up by one student can be tossed around the group and reformulated to yeald a fruitful problem. Students participate in a dialogue with others that mirrors the kind of internal dialogue that good problem formulators appear to have with themselves."*

In our experimentations the teacher takes great care in constituting the working groups. Each group collects pupils with knowledge at different levels and with complementary personalities, making sure that there is one pupil who can act as coordinator, too. Still, despite the intentions, not all the groups formed have equal potentiality.

What we present concerns an experimentation carried out with 21 pupils of second grade in approximately 20 hours, namely 2 hours a week, and enacted -in a deeper and more careful way- after a first "test"-experimentation realized in an analogous class with the same teacher (see Gherpelli 1991 and Malara 1993).

The experimentation starts with a preliminary activity concerning the construction of problems on rectangles, in which the teacher, by discussing collectively and "reasoning aloud", highlights the appropriate way of facing the situation, then organizes and coordinates the groups in the composition of the first texts of problems. Already in this initial phase there are some interesting autonomous productions in which data of relational kind appear. The experimentation follows through the phases of:

- construction of texts of problems on isosceles triangle, rhombus, rectangle trapezium;
- critical analysis among the groups of a selection of texts produced;
- recognition of analogies in texts of problems either produced by the groups or taken from textbooks.

In such phases the pupils work without any help from the teacher, who simply observes the course of the cooperative activity and collects their written productions. In order to asses the effect of the activity on each pupil, individual tests are also realized.

In the first phase, in order to simplify the control of the numerical data to be inserted, the use of numerical tables or of pocket calculators is allowed. In agreement to Galletti & alii (1990), in order to induce the pupils to deeper reflection and systematic control on the texts produced, for each problem formulated the solution and its substanciation are requested. The pupils do not receive any information in advance about the category of figures on which they are asked to produce problems, nor are they allowed to turn to the teacher or check in books during the working sessions (about 2 hours without break for each kind of figure). In the following paragraph we shall concentrate on tracing the characteristics of the texts produced by the pupils.

After the phase of composition of texts on the three figures considered, the following step is that of critical analysis of the productions. In this case the teacher hands out to the groups some texts she chooses from among those collected because they offer interesting cues for discussion: the task consists now in verifying their linguistic and mathematical correctness. The pupils read, comment, change/correct them and solve the problems. In this phase the pupils show themselves more careful and rigourous than in the previous one; it has also been remarked that some of the texts are commented and corrected by the authors themselves. In the following paragraphs we shall concentrate in detail on this phase bringing forward some papers produced by the groups which highlight interesting aspects in the pupils' behaviour.

The third phase consists of two different moments of work. First of all the pupils are induced to reflect on the various relations among the elements of the figures used in the construction of those texts which are now patrimony of the class, and to observe the working style of the groups. Then it is asked them to analyse the problems they find in their textbooks or in other books used in the school. This last activity turns out to be easy and pleasant for the pupils. They see now the problem printed in the book as familiar; thanks to the experience gained they realize what are the possible steps followed by the authors on formulating the problems and recognize the criterion through which these are arranged in the sequence second semester of the first year for 15 two-hour meetings. It was team-taught by one of the researchers and another teacher.

Before the Course

Several weeks preceding the course, during the first semester of the program, two questionnaires were administered to all participating teachers. Afterwards, interviews were conducted with six of the teachers. These teachers were chosen according to characteristics such as: teaching experience, experience as math coordinators in their schools, and their responses to the questionnaires.

Questionnaire - Situations (I)

This questionnaire included open-ended questions based on vignettes describing hypothetical classroom situations involving mathematics. Each of eight tasks described a situation in which the teachers had to respond to a student's question or idea and was intended to elicit teacher knowledge. No explanation was given to the teachers about the purpose of the questionnaire. Following are three examples of tasks:

****

Height

A student was asked the following question:

"The height of a 10 year old boy is 1.5 m. What do you think his height will be when he is 20?"

The student answered: "In mathematics it will be 3 m. because 1.5x2 = 3, and in everyday life it will be about 1.80 m."

How would you respond?

Division by 0

A student comes to you and says that s/he checked several division exercises using a calculator. When s/he divided 72:0, 1459:0 or 8:0, the calculator showed "Error." But when s/he divided 0:72, 0:1459 or 0:8, it showed 0. The student asks why it happens.

How would you respond?

Decimal Point

A student was told that 15.24x4.5 = 6858, and was asked to locate the decimal point. The student said that the answer is 6.858 because there are two places after the decimal point in 15.24 and one place after the decimal point in 4.5. Together it makes three places after the point in the answer.

How would you respond?

****

TEACHING SITUATIONS:
ELEMENTARY TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

Zvia Markovits
Oranim School of Education
of the Kibbutz Movement

Ruhama Even
Department of Science Teaching
The Weizmann Institute of Science

This study focuses on elementary school teachers' pedagogical content knowledge. The teachers responded to questionnaires and were interviewed before and after a course that aimed at improving this kind of knowledge. About half of the course meetings were videotaped. This paper reports preliminary results concerning the issue of teacher- versus student-centered responses, consistency of responses and teachers' ways of finding out how students think. We also discuss differences and similarities between this study and a previous study with junior-high teachers.

The recognition that pedagogical content knowledge is an important characteristic of teacher knowledge is growing fast. Shulman (1986) describes this kind of knowledge as knowing the ways of representing and formulating the subject-matter that make it comprehensible to others; understanding what makes the learning of specific topics easy or difficult; and knowing the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning situation.

This study investigates elementary school teachers' pedagogical content knowledge of mathematics and the development of this knowledge. The study has grown out of a previous study which investigated junior-high school teachers' pedagogical content knowledge of mathematical functions (Even & Markovits, 1991, 1993a, 1993b). As part of that study we designed, conducted, and studied three-hour workshops which dealt with students' ways of thinking when they study the function concept, and characteristics of teachers' responses to students' questions or ideas. These short activities could not, of course, change drastically teachers' pedagogical knowledge, but they raised teachers' awareness of these issues. In the present study we concentrate on elementary school teachers' pedagogical content knowledge, this time also focusing on the development of this kind of knowledge as a result of participation in a longer and more comprehensive course for teachers.

The Study

Twenty elementary school teachers participated in the study. These teachers participated in a two-year program for preparing mentors for elementary school teachers. As mentors they are to help teachers from their own school and from several neighboring schools in teaching mathematics. The participants in the program were chosen from a large number of candidates, selection being based on their reputation as successful teachers with the potential to become good mentors. During the two-year program the teachers met every week for one full day in a teacher college. Based on what we learned from the function study, we designed, as part of the program, a course that aimed at improving pedagogical content knowledge. The course, "Classroom Situations", was held in the

96 Gio: Parallelepipeds have ... are nearly the same as a cube, but they have one face longer than the other.
97 Sol: I'd say that parallelepipeds have two by two parallel faces.
98 Teach: A parallelepiped has two by two parallel faces. Well, a cube, has it got two by two parallel faces?
99 Sol: Yes, it has.
100 Teach: Then, the cube is a parallelepiped too, isn't it?
101 Sol: Yes, it is. Chorus: No. No, it is not
102 Sol: I mean, It may be a cube a parallelepiped too, but it is a little bit different.
103 Luc: In my opinion, it is not a parallelepiped, because it has square sides.
104 Teach: What do you mean 'square sides'?
105 Luc: I mean the walls.
106 Teach: You mean the faces.
107 Luc: All of the faces of a cube are equal, on the contrary for parallelepiped they are not.
108 Teach: You say that the faces of a parallelepiped are different.
109 Mat: Thus, all the same the faces are parallel, but a cube is not a parallelepiped. A parallelepiped has the definition of six sides, two by two parallel, but not equal
110 Teach: Did you say "six sides"?
111 Mat: I mean, six faces.
112 Teach: Well, parallelepipeds have six faces, two by two parallel. Is that O.K., for everybody?
113 Fab: I want to say that they are two by two equal [parallel], but it depends on the figure, because that [he refers to the rectangular prism with a square base] has four equal faces and two different, but equal to each other; and the other has two by two equal faces.
114 Teach: Well, here we have a nice collection, this figure has four equal faces and two different, but equal to each other, here there are two by two equal faces, here all of the faces are equal. Our problem is that of finding a good definition for the parallelepiped.
[...]
122 Teach: If all of them are squares it is a cube. Well, *you say that faces can be rectangles or squares, two by two equal and two by two parallel.* 123 Chorus: Yes. Yes.
124 Teach: Well, a parallelepiped is a solid bordered by six faces, which are two by two parallel, two by two equal and of a rectangular shape.
125 Fab: or of a square shape.

As far as geometry is concerned, from the point of view of figural concepts a new harmony between the figural and the conceptual aspect must be achieved, which takes into account the theoretical constraints of formal definitions and openly admits the possible discrepancy between empirical and geometrical concepts. According to our hypothesis, the contributions of different voices in the discussion allowed the conflict to appear and draw towards harmony.
This goal is hard to achieved, but we can say that in approaching this goal a basic role is played by the intervention of the teacher in guiding the discussion and mediating that part of the defining process (from the general to the particular) necessary in order to transform an indefinite description into a 'definition'.

References

Bartolini Bussi M. (1991), Social interaction and mathematical knowledge, *Proceedings of 15th PME Conference* , Assisi, Italy.
Fischbein E. (1963), *Conceptelefigurale*, Ed. Acad. Rep. Pop. Romine.
Fischbein E. (1993), The theory of figural concepts, in *Ed. Stud. in Math.* Vol. 24, n. 2, pp.139-62.
Freudental H (1983), *Didactical Phenomenology of Mathematical Structures*, Reidel.
Leont'ev, A.N. (1964/1976), *Problemi dello sviluppo psichico*, Ed. Riuniti & Ed. Mir.
Mariotti M.A. (1991), Age variant and invariant elements in the solution of unfolding problems, *Proceedings of the 15th PME Conference*, Assisi, Italy.
Mariotti M.A. (1992), Geometrical reasoning as a dialectic between the figural and the conceptual aspect, in *Topologie structurale / Structural topology*, n. 18.
Mariotti, M.A. (1993), The influence of standard images in geometrical reasoning, *Proceedings of the 17th PME Conference*, Tsukuba, Japan.
Poincaré, H. (1968/1902) *La science et l'hypothèse*, Flammarion Paris
Pontecorvo, C. (1989) Social context, semiotic mediation and forms of discourse in constructiong knowledge at school, in Mandl et al. *Learning and Instruction*, Vol. 2.1, Pergamon Press.
Vygotsky L.S. (1962/1934), *Thought and language*, Cambridge MIT Press
Vygotsky L.S.,(1978), *Mind in society: the development of higher psychological processes*, Harvard Univ. Press.
Wertsch J.V. & Stone A. C.(1985), The concept of internalization in Vygotsky's account of the genesis of higher mental functions. In Wertsch J.V., *Culture, communication, and cognition: Vygotskian perspectives*, Cambridge Univ. Press.

Acknowledgements. *The research project is supported by CNR and MURST. I thank Claudia Costa for her precious and compenet collaboration in realizing the teaching experiment, P.Boero and M.G.Bartolini Bussi who read and commented on a draft version of this report.*

Appendix

Excerpt of a collective discussion in a 6th grade

91 Teach: [...]. Well, a cube is a solid limited by six square faces, equal to each other and united. And this [she handles a rectangular prism], what is it? What is it called?
92 Alex: Parallelepiped.
93 Teach: Parallelepiped. And this figure, what is it called? [she handles a rectangular prims, with a square base].
94 Chorus: It is a parallelepiped too.
95 Teach: You remember them from the primary school, don't you? These are parallelepipeds. Can you say why? What are their characteristics?

(102) The class refuses to classify the cube as a parallelepiped, even Solange admits that a cube is a little bit different.
(124 - 128) An agreement is negotiated on the choice of excluding the cube from the class of parallelepiped, and giving it an articulated definition.

The mechanism of the discussion is basic in order to start the dialectic between the general and the particular, it causes the exceptions to emerge, instead of remaining implicit in the general tacit agreement.
A crucial role is played by the teacher, who does not approve any one of the definitions, but focuses on the very process of defining: certain properties are stated, and all the members of the class must have these properties, but also, all objects with these properties must belong to the class. She mediates the mental process requested to verify whether an object belongs to the class and in doing that, she provocatively selects the cube.

The discussion goes on: the teacher (129) proposes a new object, the hexagonal prism, and again raises the problem of classification; this time, pupils start to describe how the solid is 'constructed', taking into account the number and the shape of its faces.
Mattia gives a characterisation which is very interesting from the point of view of generalization, but also from the point of view of the interaction between the figural and the conceptual aspect. His statements refer to the particular object that he is observing, but at the same time they refer to a generic solid of a class. He uses general expressions, such as the assertion (134) "then, on the side there are the faces that are always six", or such as the doubtful utterances (136) "I'm not sure if they are always six, I have to think about it.", and (138) "Yes they are rectangles, ... No, they aren't, I don't know, ... Yes they are, I don't know whether they are rectangles". Mattia mentally moves back and forth from the general and the particular, looking for a characterization without forgetting the particular prism.
The discussion goes on, and it could be interesting to analyse it in detail, but it is not possible here; what I want to stress is the difficulty in accepting the generality of a definition, in which all the figural differences are absorbed. As a confirmation of this fact let us consider one of the formulations given for the prims.
(236) Mattia : So, the top and the bottom are equal and parallel, for some of them they are triangles, for some of them they are other figures, and the lateral faces are rectangles.

6. Conclusions

A classifying task consists of stating an equivalence between similar but different objects, towards a generalization. That means overcoming the particular case and considering this particular case as an instance of a general class, in other terms identifying specific *pertinent* properties, which determine a category.
In the case of geometry, theoretical classifications often resort to structural criteria which are not immediately clear, and certainly are far from those perceptual criteria to which we usually refer in our spontaneous activity of classification. The previous example clearly shows this phenomenon and its consequences, in terms of definitions. Thus, very often theoretical classifications conflict with spontaneous ones, and generalizing requires us to overcome differences corresponding to meaningful perceptual properties. A process of generalization, as it occurs in a theorization, states an equivalence (similarities), overcoming differences; but, as Montaigne said: "It is not similarity that identifies, rather differences", so classifications based on perceptual criteria lead to stress perceptual differences.
The process of generalization requested by a theoretical definition conflicts with the need for differentiating; difficulties arise when theoretical constraints cancel the variety and once and for all, state the equivalence between 'different' things.
It is interesting to note the appearance of particular forms of definitions which state characteristic properties, but accompany them with a collection of cases.

5. Analysis of the excerpt

Let us analyse a short excerpt (see the appendix) of the collective discussion, the aim is to focus on the process of interaction between the figural and the conceptual aspects, in particular defining task.

The discussion achieves the 'definition' of a cube (91), at this point the teacher calls for a characterization of the rectangular prism. Everybody knows the solid, and as they have been thaught at primary school, pupils call it 'parallelepiped'. This is a more general term but pupils use it with the restricted meaning of rectangular prism (in the transcript and in the following we retained the original term parallelepiped).

The first reaction (96) is to refer to the cube and differentiate the new objects according to it, then Solange gives, or maybe recalls, a 'definition' of the parallelepiped (97): "I'd say that a parallelepiped has two by two parallel faces". This is a very general characterisation of a parallelepiped, according to which a cube is a parallelepiped too. But this fact would remain implicit if at this point, the teacher did not intervene. Instead of accepting this definition and stopping the process, she realizes the possibility of conflict between having a certain characteristic and belonging to a certain class, thus she focuses on this conflict.

She makes the deduction, necessary to pass from the general to the particular case: the cube has two by two parallel faces, so it is a parallelepiped. While Solange accepts both the premise and the conclusion, the rest of the pupils agree with the premise, but refuse the conclusion: a cube cannot be a parallelepiped. The evidence of this fact compels Solange to admit that the cube is 'different' (102). This particular case cannot disappear in the general class, and the following interventions aim to differentiate all the particular cases; so, the possible definitions, as they are proposed, sound like a collection of cases respecting figural differences.

The protocol clearly shows the difficulty and at first glance, we have the impression that there was a regression from the 'definition' of Solange (97) to the 'definition' accepted in the following (122)

It is interesting to note the presence of this collection of cases, which conflicts with every criterion of classification; it witnesses the difficulty in reconciling the attitude to differentiating, which originate from observing objects (figural aspect) and the process of generalization which makes differences disappear in the equivalence relationship stated according to specific properties (conceptual aspect).

The harmony between figural and conceptual aspects is not achieved and the discussion shows how the definition proposed by Solange, which the pupils accept "on principle", is submitted to several exceptions.

Apparently, there is no conflict between a definition and the possibility of some exceptions. As Fabio says (113) "I meant that they are parallel (equal) two by two, but it depends on the figure, because this (he refers to a rectangular prism with a square base) has four equal faces and two different, but equal to each other, while the other has two by two equal faces".

As regards the process of defining, consisting of a double process from the particular to the general and viceversa, only one direction is considered: from the particular case some features are abstracted, a parallelepiped has two by two parallel faces; but the opposite direction is neglected: the possession of a certain property does not warrant the cube to be classified as a parallelepiped.

It is important to note that abstracting some features is consistent with maintaining the differences; so, during the discussion all distinctions appear and give origin to a definition with a 'case of cases'.

Let us summarize the evolution of the discussion.

(97) Solange gives a 'definition".

(98) The teacher launches the provocative argument about the cube.

squares and rectangles must be distinguished at least as much as triangles and quadrilaterals; and prisms and rectangles cannot be considered mathematically 'similar', but certainly they can be said "to have the same shape" (Freudental, 1983, p. 228).
On the other hand, geometrical definitions correspond to specific theoretical demands and give origin to the classifications of geometrical objects according to well defined (explicitly stated) specific geometrical properties, which could be said to be *conceptually pertinent*. For instance, the classification of triangles can be carried out in terms of angles' congruence, of sides' congruence ... . Similarly, quadrilaterals can be classified according to the number of axes of symmetry.
In the first case we have fuzzy classifications, often incomplete and characterized by instability, whilst, in the second case, clear and unambiguous rules are stated according to which a certain figure belongs to a category. In this way, geometrical figures participate of a theoretical system
In fact, a definition relates the new object to all the others, in such a way that a chain (*system*) of definitions is built up; this system is an organic and coherent whole. It is just this structure of stated relationships, which constitutes a 'theory' and differentiates between theoretical and spontaneous concepts. "But the *absence of a system* is the cardinal psychological difference distinguishing spontaneous from scientific concepts ... all the peculiarities of child thought stem from the absence of a system in the child's spontaneous concepts - a consequence of undeveloped relations of generality (Vygotsky, 1962, pp. 116)".
Thus, defining holds an important position among school mathematical activities: pupils must *appropriate* (Leont'ev, 1964/76) intellectual objects belonging to a culturally relevant theory, but they need to share their meaning too.

4. The task-situation

Let us describe the situation in the classroom. Some cardboard models of polyhedra are on a desk: a cube, some rectangular prisms, one of them with a square base, a hexagonal prism, a pyramid with a square base ...
The teacher picks up the cube, shows it to the pupils and asks: "What is this?" Everybody agrees that it is a cube and they characterize the cube as "a solid (three-dimensional object), delimited by six equal faces, each face being a square". Then, the teacher reminds them of the goal of the discussion: "Let us try to characterize these objects" (pointing to the polyhedra standing on the desk). She picks up the rectangular prism shows it to them and asks "What is this?". The whole class answers: "A parallelepiped!" At this point the teacher encourages the pupils to go on characterising the object.

As regards the task, the situation simultaneously requires the intervention of the figural level (observing the object as it appears) and the conceptual level (relating the properties which characterise the geometrical figure, embodied by the object). In other words, the task requires an alternate movement from the figural to the conceptual level that is the core of the interaction between the two aspects. Thus, a dialectic process may be expected: the main steps may be summarized as follows:

- observing the object,
- identifying the main features,
- stating the properties according to these characteristics,
- returning to observe the object, checking the definition, and so on.

Obviously, no one step can be considered either purely figural or purely conceptual: the object is observed from the perspective of a specific conceptualization, yet any definition has its origin in the figural features of the object itself. At the same time, the process of defining consists of a double process from the particular to the general and viceversa, from the general to the particular.

interaction between figural and conceptual aspects of geometrical reasoning, particularly in a defining task.

2. Reflections underlining the teaching experiments

A sequence of activities has been proposed: the teaching plan was designed and discussed with the teacher who included the teaching unit in the regular program of her class. The teaching unit consisted of twelve two-hour sessions the schedule foresaw different kinds of activities, among others, collective discussions. The teacher had already experienced collective discussions with her classes, and this was not the first time for these pupils either, even though they couldn't be considered 'experts'; at that moment, the school year had just started and the teacher had introduced them to this new kind of activity.
Discussions were recorded and the transcripts of the tapes analysed.

As the dialectic of figural concepts is concerned, there seem to be two relevant moments in the collective discussion:
- In the course of the argument pupils refer to both figural and conceptual aspects, when they are confronted with a disagreement, the two aspects must interact and, when pupils attempt to find an agreement, the two aspects can finally harmonize.
- In the attempt to convince classmates that their solutions are correct, pupils are compelled to make their reasoning explicit, and to get a conceptual control of the situation. "Interactive conditions enable discourse to be shared and potentially produce a *higher inter-individual functioning*" (Pontecorvo, 1989).

As for mathematics, as a theoretical knowledge, a basic role is played by the process of definition and that of validation. First of all the *objects* which we are dealing with, must be stated and clearly defined, then properties about certain objects will be considered as true only if they are derived from other true statements via argumentations on which there is the agreement of the 'scientific' community.
In the field of elementary geometry, common mathematical activities such as defining and validating constantly involve the interaction between the figural aspect and the conceptual aspect, towards a good harmony.
According to this theoretical reference frame, the role of the teacher becomes relevant. The teacher plays a basic role because she/he has a task of mediation between culture and pupils, between mathematics, as a product of human activities and the new generation of human beings. Therefore, teacher's interventions are planned according to the specific activities, we are convinced that the teacher's main role is that of "guiding pupils' work" (Bartolini Bussi, 1991), and introducing them to mathematics and mathematical activities. The mere role of transfer of knowledge is to be overcome, and a role of mediation promoted.

3. Discussing towards a definition

In the specific case of a *conceptualization discussion* (Bartolini Bussi, 1991, p. 10), aimed to state a 'definition', the confrontation with different points of view leads pupils to overcome the conception of definition as a 'description' of what is seen, towards the conception of definition as a "condition" and to some extent, a "convention".
Certainly, geometrical definitions are not mere conventions in the field of pure arbitrary acts. "La géométrie dérive-t-elle de l'expérience? Une discussion approfondie nous montrera que non. Nous conclurons donc que ses principes ne sont pas que de conventions; mais ces conventions ne sont pas arbitraires, et transportés dans un autre monde (que j'appelle le monde non-euclidien et que je cherche à imaginer), nous aurions été amené à en adopter d'autres.", (Poincaré, 1968/1902), p. 26).
Geometrical definitions seem to originate from a two-fold process. On the one hand, definitions must take into account 'spontaneous' classifications, according to similarities and differences which are *figurally pertinent*. For instance, from the figural point of view

Figural and conceptual aspects in a defining process

Maria Alessandra Mariotti
Department of Mathematics, University of Pisa, Italy

Abstract. *In the reference frame of the theory of "figural concepts" (Fischbein, 1993) a research project on the theme of geometrical reasoning has been designed. Taking into account the results of previous investigations (Mariotti, 1991, 1993), we planned a sequence of classroom activities, that have been carried out in a 6th grade. The general aim is that of observing the interaction between figural and conceptual aspects of geometrical reasoning in a classroom situation. This paper discusses some aspects concerning a defining process in a geometrical context, as it emerges from a collective discussion (Bartolini Bussi, 1991).*

1. Introduction

According to the general reference frame of the theory of 'figural concepts' (Fischbein, 1963, 1993), geometry (in elementary, Euclidean terms) deals with specific mental objects, 'figural concepts', which possess, at the same time both conceptual and figural aspects. These aspects are usually in tension, so that geometrical reasoning is characterized by a dialectic between them.

An experimental research project has been desig..ed with the aim of observing and describing modalities of this process. In order to do that, we decided to move along two different, complementary directions, following two different methods of investigation: a first phase was devoted to the observation of spontaneous performances of the children, during individual interviews. Some results of this first phase have already been presented (Mariotti, 1991, 1992, 1993).

A second part of the experimental design has been devoted to classroom observation, when pupils are engaged in school geometrical activities. A teaching experiment has been carried out with pupils of a 6th grade. According to the theory of figural concepts, the objective of this investigation is to observe the process of interaction between figural and conceptual aspects in geometrical reasoning, as it take place in a social environment (for instance during a collective discussion) and compare the results with what we obsevred during the interviews.

Particularly, we aim to test a general hypothesis about a positive influence of specific education on the development of the dialectical process between figural and conceptual aspects in geometrical reasoning.

The argument that social interaction plays an important role in cognitive development is long-standing in the psychological and educational research literatures. Indeed, several different research studies have produced a sizable body of theoretical analysis and empirical evidence regarding how social interactions may influence children's cognitive development. According to the Vygotskian reference frame, social experience is considered as basic in cognitive development. "Vygotsky argued that there is an inherent relationship between external and internal activities, it is a *genetic* or *developmental* relationship in which the main issue is how external processes are *transformed* to *create* internal processes". (Wertsch & Stone, 1985, p. 163).

As Vygotsky says in the famous quotation: "Every function in the child's cultural development appears twice: first, on the social level, and later on the individual level; first *between* people (*interpsychological*) , and then *inside* the child (*intrapsychological*) (Vygotsky, 1978, p. 57)".

In this reference frame, we consider collective discussions (Bartolini Bussi, 1991), the core of our teaching experiment, as they prove to be a good environment for the process of

Greater communication between us and the teachers might eliminate some of the organisational difficulties our students faced. We might also be able to work more closely with the teachers in providing information for them about our students' findings and aims. We are likely still to face some problems when the philosophy of participating teachers differs markedly from our own.

Despite the difficulties we have experienced in supervision and communication, we remain enthusiastic about the project. We are confident that our teacher education students have made important gains in their skills of diagnosis and design of appropriate teaching activities, along with insight into the construction of mathematical understanding by a child.

References

Ball, D. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics, 8*(1), 40-48.

Cronin, L. L. (1986). *Intended versus implemented curricula: An interpretive study.* Unpublished doctoral dissertation, University of Georgia, Athens.

Cronin-Jones, L.,& Shaw, E. L. (1992). The influence of methods instruction on the beliefs of preservice elementary and secondary science teachers: Preliminary comparative analyses. *School Science and Mathematics, 92*(1), 14-2?.

Denvir, B., & Brown, M. (1986a). Understanding of number concepts in low attaining 7-9 year olds: Part II. The teaching studies. *Educational Studies in Mathematics, 17*, 143-164.

Denvir, B., & Brown, M. (1986b). Understanding of number concepts in low attaining 7-9 year olds: Part I. Development of descriptive framework and diagnostic instrument. *Educational Studies in Mathematics, 17*, 15-36.

Harrison, B., Schroeder, T. L., & Bye, M. P. (1987). As a child learns, so must one teach (elementary school mathematics diagnosis and instruction). In J. C. Bergeron, N. Herscovics, & C. Kieren (Eds.), *Proceedings of the 11th International Conference of the International Group for the Psychology of Mathematics Education* (pp. 318-324). Montreal, Canada: PME.

Karp, K. S. (1991). Elementary school teachers' attitudes toward mathematics: The impact on students' autonomous learning skills. *School Science and Mathematics, 91*(6), 265-270.

Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1987, April). *Teachers' pedagogical content beliefs in mathematics.* Paper presented at the annual meeting of the American Educational Research Association, Washington, DC.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Thornton, C. A., Jones, G. A., & Toohey, M. A. (1983). A multisensory approach to thinking strategies for remedial instruction in basic addition facts. *Journal for Research in Mathematics Education, 14*(3), 198-203.

Wilcox, S. K., Schram, P., Lappan, G., & Lanier, P. (1991). The role of a learning community in changing preservice teachers' knowledge and beliefs about mathematics education. *For the Learning of Mathematics, 11*(3), 31-39.

Zeichner, K., Tabachnick, B., & Densmore, K. (1987). Individual, institutional, and cultural influences on the development of teachers' craft knowledge. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 21-59). Eastbourne, UK: Cassell.

considered they were responsible for some breakthrough in their pupils' understanding of mathematics. It is particularly pleasing that our students seemed to be conscious of the need to provide activities that increased children's confidence as learners of mathematics and were able to report the maintenance of or improvement in risk-taking and confidence.

It is also clear that the breakthroughs reported by our students were in a narrow range of topics, concerned overwhelmingly with place value, basic number facts, and number algorithms. Difficulties with these topics were likely to be quite evident in the pupils' work and readily identified. Most of our students also considered that skills in these topics are essential to further progress in mathematics. Given the brevity of the teaching program, these difficulties could be usefully addressed in ways that led to some obvious changes in the pupils' understanding. This proved satisfying to our students. We are unable to say whether there were other conceptual or procedural difficulties that the students did not identify or identified and did not choose to address in their teaching program. In a future phase of the project, we will need to investigate how our students make their choices.

While a pleasing number of students considered that they had developed useful diagnostic skills, many felt that their diagnostic skills were inadequate, especially at the beginning of the program. Our own view is that the students were generally able to devise appropriate activities and games for use in teaching and that after the first few teaching sessions, most were confident in talking with their pupils and observing the child's actions to determine strengths and weaknesses. Some students experimented with different formats for the presentation of questions requiring standard algorithms for calculation. Nevertheless, we need to provide more assistance to our students in deciding how to start in a way that provides them with useful information but does not overwhelm the child with formal testing.

The data we have available do not allow us to make any conjectures about the long-term effects of the program on the skills and attitudes of participating pupils. In their questionnaire responses, most students reported some improvement in either the understanding and skills or the attitudes of their pupils, and these responses form one type of data. Again, the brevity of the teaching program means that to be effective, the teaching needs to be focused and not to attempt an unrealistic number of objectives.

We have no information about the classroom teachers' views of the project. Some teachers may welcome the presence of our students and see them as providing extra individual help that they are unable to provide to specific pupils in their class, given the demands of their normal program. On the other hand, some teachers may feel that the presence of our students is intrusive. Most teachers would be well aware of the difficulties being experienced by their pupils and would not need our students to identify those difficulties for them. For some teachers, the reports written by our students may seem irrelevant. One of the key reasons for us to have our students write final reports is to have them reflect on the reasons for the choices they have made, and to present a sound, clearly written and concise summary of their diagnoses and teaching strategies. Most students find this a difficult task.

disappointed, but not surprised, by the heavy emphasis on the children's number concepts and skills, given our students' initial attitudes towards mathematics and mathematics teaching.

Our attitude towards the learning of basic facts is that memorisation must only follow an understanding of addition and multiplication attained through reflection by the child on the manipulation of concrete materials then pictorial models, followed by the construction of strategies by the child to relate new facts to known facts, and finally the repeated use of facts in games, activities and challenges. We discourage an emphasis on speed of recall and our students are aware that in Western Australia, automatic recall of basic multiplication facts is not expected of all children before the end of year five. Comments made during the program and in the lesson evaluations written by the students suggested that many of our students found lack of recall of basic facts hindered children in calculations. Some students were advised to concentrate on basic facts by the classroom teachers and felt an obligation to meet these requests. Some students also found it comparatively easy to identify weaknesses in basic fact recall and to address these through the development of strategies and through simple games.

Throughout the program, we emphasised the importance of encouraging children to develop autonomy and positive attitudes towards mathematics and themselves as learners of mathematics. We believe that this can be done best by providing activities in which children achieve success, and which are set within a context of interest to the child. We are also aware that repeated interaction with a single interested adult may in itself be of benefit to some children, regardless of the nature of that interaction. In a question that asked the students to describe any changes in the child's attitudes that occurred over the ten weeks of teaching, 18 students reported that the child had no problems to begin with and maintained the same level of confidence. Another 28 students reported that the child's attitudes improved. A typical comment was "To begin with the child was hesitant to take part/risks, but by the end she was purposefully (sic) making herself do more maths." These responses were gratifying.

A series of questions asked the students about relationships with the parents, the child, the teacher, and the school. Our belief is that the involvement of parents may have a positive effect on the learning of the child, although we acknowledge that sometimes parents are not well-informed about the mathematics the child is learning in class or how best to provide assistance. Some of our students wrote letters to the parents introducing themselves and inviting contact during the program. Twenty-seven students reported little or no contact with the parents, while 22 reported that the family played games provided by the student with the child at home and that this was helpful. Forty-two students said that their relationship with the child was good, and a typical comment was "He has become a real little friend telling me lots of stories and experiences." About half the students described their relationship with the child's teacher as very good, while about half reported limited or no contact.

Discussion

These preliminary data suggest that in some respects the project is achieving pleasing outcomes with respect to our teacher education students. From the students' reports, it is clear that most of them confidence grew, they were able to identify difficulties that were sometimes quite profound and complex. Students identified a feeling of remoteness from the classroom and the teacher. Some of their pupils were tentative about their involvement in the project and this made communication at first quite hard. Some students found it difficult to determine appropriate levels of language to use, making communication with the children quite difficult.

Amongst the positive comments were that students saw the experience as a dose of reality, giving them a rehearsal of the skills they will need to use in their first school appointment. They felt that they were learning to make quick decisions to manage unexpected situations that arose. It was also very gratifying when they could actually see a child "catching on" to something for the first time.

On a rather sombre note, the students felt that they had to please us by being able to report that they had brought about real change in the pupils' mathematics understanding, even though we had specifically said that we did not expect them to be able to make very significant changes over the few weeks of the program. Nevertheless, some of these students reported that they had considered faking their lesson evaluations in order to impress us.

Questionnaires

A simple questionnaire was administered to the students in the final week of semester. There were 51 responses. Questions 1 and 2 asked the students to list skills they had developed during the project and skills they thought were inadequate. Thirty-one thought that they had constructed adequate diagnostic skills but 18 considered that their diagnostic skills were inadequate. Specific skills that were mentioned as developed during the project included the use of appropriate language in explanations to the child (17 students); designing creative activities to cater for specific problems (28 students); planning an appropriate program that was flexible and sequential (18 students); and a greater knowledge of the syllabus (10 students). While many students thought that they had developed skills in designing creative and appropriate activities, 16 students considered that they lacked this skill. A further 18 students said that they found it difficult to know where to start in assessing the child's strengths and weaknesses.

Question 4 asked the students to describe their most exciting moment in working with the child. The responses to this question referred principally to concepts associated with place value and basic facts. A few responses referred to changes in the child's attitudes and confidence. Question 6 asked students to identify the specific areas of mathematics in which they felt they contributed to the child's learning. Not surprisingly, the three most commonly mentioned areas were place value (23 students), basic number facts (25 students) and algorithms (18 students).

Throughout the program, we continually stressed the importance for children of a good understanding of place value as required in calculations, in an understanding of larger numbers and their relationships, and in the development of decimal ideas. We were therefore not surprised that this topic was so frequently mentioned. However, our entire mathematics education program has a heavy emphasis on children's understanding of all strands of our syllabus, space and measurement as well as number, and the integration of these strands in planning activities with children. We were possible teaching techniques in a situation where the responsibility for choice rests with them and where their choices are tested against the realities of the time constraints of the program and the needs of the pupils with whom they are working.

In line with this view, our students are not directed in terms of the topics they should cover in the mediation program. Once they have worked to understand the difficulties their pupils are experiencing, the students choose which topics to work with. They also choose the activities to be used and the contexts within which they will set these activities. The pupils' own classroom teachers may suggest topics to be covered and such suggestions may act either as useful information about where to focus in the program or as a constraint on the program.

Aims of the project

In order to enhance the skills and understandings of our teacher education students, we provide them with in-depth discussion and reflection on issues related to teaching strategies, how children learn mathematics, and the teaching of mathematics to pupils experiencing difficulties. We aim for the students to develop skills of understanding children's understanding and teaching strategies through working in an intensive one-to-one basis with a child in the school environment. We also aim to assist the pupils through having our students help them to construct mathematical skills, understandings and processes, developed increased autonomy, and enhanced beliefs and attitudes.

As the project has developed over the last three years, we have begun to explore whether the project is making a contribution to the skills and understandings of each of the groups involved. Two major research aims direct the collection of data that has begun this year.

1. To identify the factors influencing primary teacher education students' teaching strategies and beliefs about mathematics, how children learn mathematics, and the teaching of mathematics to pupils experiencing difficulties.
2. To identify the effect of participation in the mediation program on the construction of mathematical skills and concepts of primary school pupils and on their attitudes towards mathematics and themselves as learners of mathematics.

The systematic collection of data will continue to be undertaken in future years using various research techniques. Because we are particularly interested in whether any effects are sustained beyond the initial impact, we will seek data over a six month period following the end of the teaching program. We report here some preliminary results concerning the teacher education students' beliefs about their own development of understandings and skills.

Some preliminary results

To gather these data, we asked the students to complete a questionnaire. We also interviewed seven students in depth.

Interviews with students

The students commented that the task of diagnosis was very daunting and they initially found it difficult to identify the child's difficulties. The child was not able to articulate his or her own difficulties. This made it difficult to provide a focus for their program in the early weeks. They found it difficult to remain optimistic and see any prospect of helping the child. However, as their topics in mathematics, teaching strategies, and classroom activities. Each student was allocated a pupil selected by classroom teachers. The pupils were chosen on the basis of their need for extra assistance in mathematics. Pupils experiencing severe clinical problems were not selected.

The students made at least ten visits to the school to work with their pupils. They were required to prepare lesson plans for each session. An essential component of this process was the evaluation of the sessions, which was done in terms of their own development of understanding of the pupil's knowledge, skills and attitudes. This provided the basis for the next session. The students were also required to write detailed reports of their work at the end of the program to provide feedback to the classroom teachers.

The philosophy underlying the project

Within a constructivist philosophy of learning, children can be said not to understand an idea in mathematics when they have not yet assimilated that idea into their existing cognitive schemes. This may occur because their existing schemes are seen as plausible and correct and new information is not seen as more fruitful or useful. New information may in such cases be rejected, or may exist alongside existing ideas for use within school settings, while existing ideas are used in out-of-school contexts. Where a child seems not to understand an important idea, this is seen to be a more complex situation than just a deficit in the child's knowledge, which might be fixed by some simple direct instruction. Rather, the teacher assisting the child needs to ascertain what the child's existing schemes are then re-present the new information in ways that help the child to construct links between new and existing knowledge, assimilating the new information and modifying the existing knowledge.

In practical terms, this translates to some principles that guide our students' work with their pupils: (1) Find out what the child does understand and help the child build on that; (2) The understanding the child already has can be determined best by observing the child as he or she attempts activities, games and problems and talks about the strategies being used; (3) New information must be presented in ways that help the child create links to existing understanding; (4) Concepts and procedures must be presented in ways that will help the child decide they may be more useful and more plausible than existing understanding and procedures; (5) The child must be encouraged to develop his or her own strategies, and to articulate those strategies and share them with others, in this case the student teacher; (6) Activities should be enjoyable, difficult enough to be challenging but not so difficult that the child is unable to achieve some success; (7) Understanding needs to precede attempts to memorise; (8) The construction of new understanding helps determine and is determined by the child's views of the nature of mathematics and of him or herself as a learner of mathematics. These principles are no different from principles that apply to any good program of mathematics teaching.

We hold a similar constructivist view about the development of understanding and skills by our teacher education students. That is, teacher education students must themselves construct skills of understanding children's knowledge constructions and strategies of teaching. This can best be done through involvement in and reflection on activities that require the students to experiment with majority of research into teachers' beliefs has focused on teachers with three or more years of experience. It appears that the beliefs of preservice teachers may be more amenable to change given appropriate experiences and instruction (Cronin-Jones & Shaw, 1992). If we are to help teacher education students re-think their beliefs about mathematics, mathematics learning, and children as learners of mathematics, we need to create situations where these beliefs are faced, tested against the reality of a teaching situation, and reconsidered.

While there is now an extensive literature on how children construct ideas in various topics in mathematics, there is very little reported on studies that aim to assist pupils with difficulties in mathematics to re-construct their conceptual knowledge and skills in line with new constructivist ways of viewing learning. Diagnosis and instruction projects in primary school mathematics have tended to focus more on mathematical content and on error patterns than on the conceptual understandings of the learner. In their review of more than 300 diagnosis and instruction research documents, Harrison, Schroeder and Bye (1987) claim that there are weaknesses inherent in such approaches and contrast these with the strengths and practical value of programs built on insights from genetic epistemology.

Denvir and Brown (1986a, 1986b) report a series of studies into the development of number concepts in seven to nine year old low-attaining pupils. These studies focused on a narrow range of number concepts and identified a framework to describe the order of acquisition of those concepts. Their studies devised, carried out and evaluated teaching activities to extend the children's understanding of those number concepts. In Australia, Thornton, Jones and Toohey (1983) also worked with primary school pupils in remedial mathematics programs. Their study again focused on a very small range of number skills, but the instruction was carried out by teachers who had an additional twelve months of training in diagnosis and remediation beyond their initial teacher education qualification.

The project described in this paper is intended to help teacher education students construct enhanced attitudes and beliefs and better skills in working with primary school pupils experiencing difficulties in mathematics. It is also designed to assist primary school pupils to develop autonomy, independent learning and critical thinking, construct enhanced attitudes and beliefs, and construct new mathematical understanding. Because the word 'remediation' suggests a deficit model of teaching, we use the word 'mediation' to describe our project, which is based within a constructivist framework and sees the teacher's role as assisting the child to construct mathematical understanding for him or herself.

Background

Our mediation project started in 1991 at the request of the local school district which was concerned by the large number of primary pupils having difficulty with mathematics. In 1993, we had 76 final year teacher education students participating in the project. All the students had completed two semester units of mathematics education and were concurrently taking a third semester unit. Throughout these units, there had been many class sessions devoted to discussions of different philosophies of mathematics and views of learning, research on children's understanding of various

TEACHER EDUCATION STUDENTS HELPING PRIMARY PUPILS RE-CONSTRUCT MATHEMATICS

Helen Mansfield, Joy Scott, Yvonne Burgess
Curtin University of Technology, Western Australia

The paper describes a project in which teacher education students work with primary school pupils experiencing difficulties in mathematics. The students must themselves construct new beliefs, attitudes, and understandings about the processes of teaching and learning mathematics as they help the children to develop autonomy and enhanced attitudes, and construct new mathematical understanding. The philosophy underlying the project is based on the belief that teachers can help children with difficuties in mathematics by mediating *in their construction of mathematics rather than by* remediating *their weaknesses. The paper presents some preliminary data that have been obtained about the value of the project as perceived by the teacher education students involved in its implementation.*

Introduction

Children's difficulties in mathematics may stem from many factors, including absence from school at critical times, negative self esteem, lack of valuing of mathematics, the use of inappropriate teaching strategies or materials by their teachers, inappropriate pacing of instruction and so on. Children who experience early difficulties in constructing adequate mathematical ideas are unlikely to be able to re-construct those ideas and skills for themselves when their daily schooling places them in a classroom of about 30 pupils all with claims on the time and skills of the classroom teacher.

Karp (1991) found that the daily experiences of pupils in mathematics classes of teachers with positive attitudes were found to be substantially different from those of pupils in classrooms of teachers with negative attitudes. Her data suggested that teachers with similar professional backgrounds and with pupils of comparable abilities engaged in quite dissimilar instructional activities. Overall, teachers with negative attitudes towards mathematics employed methods that fostered dependency whereas teachers with positive attitudes were found to encourage student initiative and independence.

Our teacher education students generally have deeply rooted ideas about teaching and learning mathematics. Despite lectures and workshop discussions promoting a constructivist philosophy of learning mathematics, they think of mathematics as a fixed body of knowledge that is best learned by memorising facts and rules, and procedures for applying them. They see the role of the teacher as carrying out goals determined by the syllabus, providing demonstrations and examples of tasks to be completed, and checking written work for completeness and accuracy (Wilcox, Schram, Lappan & Lanier, 1991) and conceive of teaching as a matter of technical competence rather than reflection and decision-making based on the mathematical knowledge children are constructing and how they are constructing it.

Researchers have acknowledged the impact of teachers' beliefs on practically every aspect of the teaching/learning process (Cronin, 1986; Peterson, Fennema, Carpenter, & Loef, 1987; Thompson, 1992). Research points to the difficulty in overcoming ingrained notions developed during previous school experiences (Ball, 1988; Zeichner, Tabachnick & Densmore, 1987). The king. In the phase of critical analysis of the texts, the pupils show a strong critical attitude, since they find out easily ambiguities, errors on data and on their compatibility. Several protocols, either of groups or of individual tests, confirm our belief that ambiguous texts or texts with errors constitute the ideal situation for inducing the pupils to discuss possible interpretations, formulate hypotheses and infer implications.

To work on different situations with reference to the same figure shows to the pupils the possibility or impossibility of connecting certain pieces of information and, if compatible brings them to assess the "grade of closeness" (i.e. the number of steps needed for connecting them); which on one hand encourages concatenation of data, essential for developing mathematical speech, and on the other hand produces conceptualisation of the meaning of relation, important for starting abstract thinking.

Moreover, the activity favours metacognition in the pupils because it brings them to hold strong control on the strategies underlying their (or the other's) productions, to be aware of the fundamental relations among the elements of elementary geometrical figures and to master the whole range of classical models of problems. The systematic analysis of the correct texts produced with reference to each figure allows the pupils to sort out that for each figure there is a same minimum number of pieces of information to assign in order to pose a problem on it and that such number is therefore a characteristic of such figure. The inquiry on what happens for figures of a same category (e.g. triangle) but with different peculiar features (e.g. scalene, isosceles, equilateral) carried out by examining simultaneously the data inserted with reference to the figure given, allows the pupils to get aware of the role of the implicit data linked to the quality of the figure itself. Furthermore, the comparison of problems analogous in structure composed before and after the study of Pythagora's proposition, and the observation of the overall number of data inserted in the first and second case bring them to be aware of the economy of information involved in the knowledge of general mathematical facts.

References

Brown S. I. e Walter M.I., 1983, *The art of problem posing*, Franklin Institute Press, Philadelphia

Ellerton N. F., 1986, Children's made up mathematics problems: a new perspective on talented mathematicians, *Educational Studies in Mathematics*, vol. 17, 261-271

Freudenthal H., 1973, *Mathematics as an Educational Task*, Reidel Publishing Company, Dordrecht

Galletti L. et alii, 1990, La strategia dell'"inventar problemi", *L'Insegnamento della Matematica e delle Scienze Integrate.*, vol.13, 201-221

Gherpelli L., 1991, Esperienze di laboratorio con allievi di 11-13 per la costruzione di testi di problemi e l'avvio al ragionamento ipotetico, in D'Amore B. ed.), *La Matematica fra gli 8-15 anni*, Apeiron, Bologna, 112-113

Kilpatrick J., 1987, Problem Formulating: where do good problems come from? in Schoenfeld A.H (ed.), *Cognitive Science and Mathematics Education*, LEA, Hillsdale NJ

Laborde C., Language and Mathematics, in Howson A.G., Kahane J.P. (eds.), *Mathematics and Cognition*, Cambridge University Press, 53-69

Malara N.A., 1991, Improvement of ability to solve problems in pupils aged 11 to 14: some results of a long lasting research, *Proc. CIEAEM 42*, Szczyrk (Poland), 46-60

Malara N.A., 1993, Il problema come mezzo per promuovere il ragionamento ipotetico e la metaconoscenza, *L'Insegnamento della Matematica e delle Scienze Integrate*, vol. 16, 928-954

Malara N.A., Pellegrino C., Iaderosa R., 1992, Avvio ad attività di matematizzazione attraverso problemi, in D'Amore B. e Pellegrino C. (eds), *Convegno per i sessantanni di Francesco Speranza*, Bologna, 111-121

Moses B., Bjork E., Goldemberg E. P., 1990, Beyond problem solving: problem posing, in Cooney J. (ed.), *Teaching & Learning Mathematics in the 1990s*, NCTM Yearbook, 82-91

Polya G., 1962, *Mathematical Discovery*, J. Wiley & Sons

Sawyer W.W.,1964, *Vision in Elementary Mathematics*, Penguin Books

Silver E.A., 1985, Research on Teaching Mathematical Problem Solving: Some Under-represented Themes and Needed Directions, in Silver E.A. (ed.), *Teaching and learning Mathematical Problem Solving*, LEA, Hillsdale, NJ, 247-266

Silver E.A., 1992, On mathematical Problem Posing, *Proc. PME XVIII*, vol. I, 66-85

hypotenuse measures less than the two catheti and it should be the contrary.
Critic: the text has too many data.
Correction of the text: "The measure of the height relative to the different side of an isosceles triangle ABC is 24 cm. Since you know that the area measures 240 cm^2, find the measure of the perimeter.

Please note the improper reference to the Pythagorean proposition which however reveals the pupils' process of thinking in the discovery of the absurd and the imprecision of the expression "the hypotenuse measures less than the two catheti" probably due to the incapacity of formulating correctly into negative the expression "in a rectangle triangle the measure of the hypotenuse is bigger than that of (each of) the two catheti". The absent reference to triangle *AKC* witnesses the pupils' acquired capacity of seeing parts of figures and of autonomously operating on them.

Ambiguous text with redundant data which gives vent to more than one possible solution

In a rhumbus the area measures 600 cm^2 whereas one diagonal differs from the other by 10 cm and it is its 4/3. Calculate the perimeter and the height.

Protocol 4. The pupils assume that one diagonal is 4/3 of the other and give the solution to the problem ignoring the information on the area.Then comment as follows:
The data are coherent, even if too many. You could take the measure of the area and the problem would become: "In a rhumbus the diagonal is 10 cm different from the other and it is 4/3 of the latter. Calculate the perimeter and the height."
You could take off the difference of the diagonals(10 cm), the problem would be then: "In a rhumbus the area measures 600 whereas the diagonal is 4/3 of the other. Calculate the perimeter and the height."
From the point of view of grammar there is an imprecision when it says "one diagonal is 10 cm different from the other and it is the 4/3. This datum (underlined) has no reference.

Non-ambiguous text with redundant data respectively compatible

A rectangle trapezium has the major base of 39 cm, the minor of 32 cm, the height measures 24 cm and the oblique side 25 cm. Find the measure of the diagonals, the area and the perimeter.

Protocol 5. The pupils solve the problem schematically using all the data and then write:
There are too many data in the problem. The height or the minor base or the oblique side or the major base can be taken away from the text so that it comes out right. The correct problem would be: "A rectangle trapezium has the major base of 39 cm, the minor base of 32 cm and the oblique side of 25 cm. Find out....". Or "A rectangle trapezium has the major base of 39 cm, the minor base of 32 cm and the height of 24 cm. Find out ....". Or "rectangle trapezium has the minor base of 32 cm, height of 24 cm and the oblique side 25 cm...".

Non-ambiguous text with incompatible and redundant data

The perimeter of the isosceles triangle ABC measures 72 cm and the different side (third side) is 13/10 of the isometric one; since you know that the unequal side is 1/3 of 60, find the measure of the two sides.

Protocol 6. The pupils work schematically in two ways, neglecting in the first case the piece of information "the different side is 13/10 of the isometric one" in the second case the piece of information on the perimeter and write:

1°way	2°way
60 : 3 = 20 cm 3° side	60 : 3 = 20 cm 3° side
72 - 20 = 52 cm isometric sides	20 : 13 = 1,5
52 : 2 = 26 cm isometric side	1,5 x 10 = 15 cm isometric side

The text is wrong because in the first the isometric side measures 26 cm, in the second measures 15 cm. They made a mistake in writing "the diffentent side is13/10 of the isometric one. They should have written "the different side is10/13 of the isometric one" or "the isometric side is 13/10 of the different one".
Actually: 20 : 10 = 2 cm 2 x 13 = 26 cm (isometric side). If they wanted a single solution they had to put either the fraction and not the perimeter or the perimeter and not the fraction.

Table 2

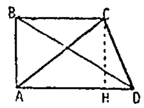
Protocols of critical analysis of the texts

Ambiguous text with equally acceptable interpretations

In a rectangle trapezium the sum of the bases is 71 cm, their difference 7 cm and the diagonal measures 40 cm. Calculate the area.

Protocol 1

(71 - 7) : 2
64 : 2 = 32 = b
32 + 7 = 39 = B



B - b = 7
B + b = 71
d = 40
? A = 852 o 312,4

We don't know which diagonal it is.

If it were the minor
$(40^2 - 32^2) = 1600 - 1024 = 576$
$\sqrt{576} = 24$ *(h)*
A = (b + B) x h : 2
71 x 24 : 2 = 852 (A)

If it were the major
$(40^2 - 39^2) = 1600 - 1571 = 79$
$\sqrt{79} = 8,8$ *(h)*
71 x 8,8 : 2 = 624,8 : 2 = 312,4 (A)

This text is "wrong" because it doesn't specify which is the diagonal corresponding to the measure of 40 cm.

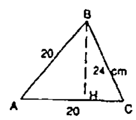
It is to be underlined that on facing this text some other groups choose the first solution declaring that they eliminate the second result because "inconvenient".

Ambiguous text with two possible interpretations, one of which is to reject

The area of the isosceles triangle is 240 cm². Since you know that the height corrispondent to the base measures 24 cm, find the perimeter of the triangle.

Protocol 2. The pupils initially refer to the position of the isosceles triangle in the standard model and directly execute the calculus of the perimeter,then examine the case of the same triangle lying on another side and write:

OTHER WAY OF SOLVING IT



480 : 24 = 20 cm = base and other side

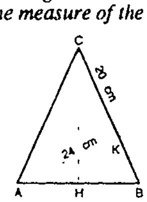
Critic

1. *The preposition "of the" is not appropriate because the isosceles triangle is not unique, there are many, so we shall change it into "of an".*
2. *The text does not specify which of the 3 sides is the base. We have made two conjectures, but the second is impossible because one of the catheti would measure more than the hypotenuse.*

Ambiguous text with redundant data which gives vent to two interpretations, one of which is to reject.

The measure of the two different heights of the isosceles triangle ABC are respectively 24 cm and 18,4 cm; since you know that the area is 240 cm², find the measure of the perimeter.

Protocol 3



- *(240 x 2) : 18,4 = 480 : 18,4 ≅ 26 (cm)*
- *(240 x 2) : 24 = 480 : 24 = 20 (cm)*
- *(26 x 2) + 20 = 52 + 20 = 72 (cm) perimeter*

Solution: we have used all the data of the text, the text does not specify to what side the two heights are relative. But we know that the smallest height is always relative to the bigger side, therefore 18,4 is the height relative to the major side whereas 24 is the height relative to the minor side. Let's suppose that AC is 20: the side AC cannot measure 20 cm because [considered triangle *AKC*] *if we apply Pythagoras we see that in this case the*

continues

document the different typologies of mistake on which they are asked to work, and to observe how the different situations are tackled by them. As it is possible to see, they analyse:

- ambiguous texts with equally acceptable interpretations;
- ambiguous texts with different possible interpretations among which some are to reject;
- ambiguous texts with redundant data which give vent to several interpretations among which sorr are to reject;
- ambiguous texts with redundant data which give vent to several possible situations;
- non-ambiguous texts with redundant data respectively compatible;
- non-ambiguous texts with redundant data respectively not compatible.

Problems with ambiguity of interpretation always give vent to more than one reading and consequent solution in the different cases (see *table 2*, protocols 1&2); problems with redundant data give vent, on cases of non-compatibility, to selection and rejection of data (see *table 2*, protocols 3&6) whereas on cases of compatibility give vent to combination of data and, accordingly, to birth of more than one problem as a reduction from the original problem (see *table 2*, protocols 4&5).

The protocols show how the pupils' attention concentrates on commenting the other's work rather than justifying their own way of operating; this makes us conclude that they don't feel this activity as one of problem solving. Let's not forget that, because of didactic contract, to solve a problem means to these pupils to explicit the reasoning enacted and the difficulties met.

From the metacognitive point of view, a further interesting aspect in this phase concerns the control of the correctness of the data inserted into a text as to those initially assigned with reference to a certain figure. In particular, in some situations the pupils had to deal with the distorting effects deriving from the approximation of the square root of some numerical values, which brought them to a reflection on the limits of approximation and to the awareness of the distinction between a number and its approximate decimal representations.

It is to be underlined that in this phase the pupils have made spare use of graphic representation; this demonstrates their good interiorisation of the models of figures in the knowledge of general mathematical facts.

General results

From a global point of view, on the base of our experience we are able to state that problem posing is a difficult activity at individual level for 11/12-year-old pupils; on the other hand, as team activity -thanks to the "mixture" of different personalities and abilities- not only is it suitable, but also effective, and it turns out to be more fruitful the more all participants are willing to confront and to take each contribution into consideration, whatever the cultural level of the classmate expressing it. Team activity involves all the pupils in the work according to the characteristics of each: for instance, non-creative pupils often act as supervisors.

The observation of the groups reveals how some aspects of the pupils' individual personalities can affect their productions, e.g. pupils who are not willing to share their knowledge with the other compromise the development of the right interaction stream within the group, whereas open pupils who are willing to share their knowledge determine synergic effects in the group.

For brilliant pupils the activity is particularly profitable because having to communicate to the other compells them to explicit intuitions, images, processes which would not arise during individual work. As strong, if not superior, is the benefit to the weak or unconfident pupils, as revealed by the individual tests. It is clear that the quantity of problems tackled, the open relation with classmates with a successful working method, the experience as first actor of the construction of a problematic situation make them aware of the right attitude to have in front of a problem, of how to associate the different data in order to evidentiate their relations, of the incidence of mathematical knowledge on the whole activity. However, for some of these pupils the difficulty of linguistic formulation is confirmed.

It is clear in our experimentations how team activity enhances in the pupils rigour in the use of language: if an idea is expressed roughly or in an incomplete way, it is discussed, analysed, and precised. Clarity of speech becomes to the pupils an essential element for a correct communication of their ideas.

Another strongly stimulating aspect is to work on mistakes, which become elements of intellectual of the contents exposed. In particular, if they don't find in books some "difficult" problem they have constructed, they are mostly satisfied, which gives them a key to assess the skills they have reached. This last activity gives the pupils a new way of approaching the textbook, which is particularly meaningful.

The texts constructed by the pupils

The texts produced refer either to direct or indirect problems. Various groups suggest problems in which the data inserted are expressed as fractions of other, or are anyway of relational kind. Initially, despite the fact that the pupils must also solve the problems they create, there are texts which are ambiguous or with redundant data -compatible or not; in some cases the presence of wrong relational data was noticed (e.g. inverted fraction). As the activity goes forth, the problems composed by each group are more and more correct, and towards the conclusion of the activity, having the pupils acquired the right working manner and a certain methodicalness, they produce a wide variety of problems, ranging from simple to complex. Globally, the problematic situations they suggest cover almost the whole range of problems present in the Italian textbooks. The language in which these texts are composed reflects traditionally accepted codings, even if sometimes it draws away from the standard under the influence of the discussions held in class on definitions and of the work carried out for fostering a dynamic vision of the figures. As appears in the protocols, many pupils can easily see isosceles triangles in non-typical positions and recognize rhumbi as parallelograms. For reasons of space, we may not enclose all the texts produced, however they will be available for the participants to the conference.
From a general point of view, the productions of each group (at least 4 for each kind of figure) show the use of the simplest relations among the elements of a figure; other relations which are more hidden, such as those involving sum and difference of measures of segments, or the expression of a measure as fraction of another are used with awareness only by the groups with stronger potentiality. It is interesting to notice how different groups organize their work by determining -starting from the elements of the figure assigned by the teacher- the measures of all the other elements directly obtainable from them and constituting what they call "the data base", from which they draw the data for contructing the various problems on the figure. Some groups, then, adopt a working method based on shifts through analogy of a series of relations from one kind of figure to another (see *table 1*).

Table 1

Group productions witnessing the use of analogy as a procedure
<u>Problems in which the measure of a segment is given as fraction of that of another</u>
- *A rectangle has the area measuring 240 cm^2. Since you know that the height is 5/12 of the base, find out the perimeter of the rectangle.*
- *In an isosceles triangle ABC, the base AB is 5/6 of the height $\overline{CH}$ relative to it. Since you know that the area of ABC is 240 cm^2, find out the measure of its perimeter.*
- *The area of a rombus is 600 cm^2. Since you know that one diagonal is 3/4 of the other, would you be able to find out the measure of the perimeter?*
<u>Problems in which sum and difference of measures of segments are inserted as data</u>
- *Two sides of a rectangle differ by 14 cm and their sum is 34 cm. Find the area.*
- *In an isosceles triangle ABC the sum of the 3 sides is 72. Since you know that the two equal sides differ from the different by 6 cm, find the measure of the 3 sides.*
- *The sum of the diagonals in a rhombus measures 70 cm. Since you know that between one diagonal and the other there is a difference of 10 cm, find out their measure.*

Pupils' critical analysis of ill-structured problems

As already mentioned, although a solution is requested, there are in the first phase ambiguous texts, with redundant data, either compatible or incompatible. Such texts are precious for the phase of critical analysis since they stimulate hypothetical reasoning and other important activities such as analysis and comparison of resolutory strategies or control of the coherence to the initial situation of the texts made. We present in *table 2* a series of pupils' protocols appeared in this phase of the activity in order to

The tasks were based on student ways of thinking and misconceptions as known from research and personal experience (e.g., Markovits, 1989).

Questionnaire - Beliefs about Teaching and Learning

This questionnaire probed teachers' thoughts about teaching strategies, questioning, skills versus conceptual goals, the process of learning, etc. Following are examples of the items:

- Many times math teachers ask students to explain their answer when it is incorrect. Do you think they should also ask for an explanation when the answer is correct? Explain.
- Do you think a math teacher should tell the student whether her/his answer is correct or should the student make this decision by herself/himself? Explain.

The responses to these items were not analyzed on their face value only but a great deal of attention was given to the teachers' explanations that were given to support their decisions.

Interview - Teacher Responses

The interviews centered on the three situations described above: "Height", "Decimal Point" and "Division by 0". The subjects were presented with responses that were given by other teachers to the same situations, and were asked to react to these responses. Each response highlighted certain characteristics of teacher responses, such as ritual versus meaning orientation, student versus teacher centered, or content knowledge. The following responses were presented for the Decimal Point Situation:

****

1. I will tell the child: You located the decimal point correctly and also explained it correctly.
2. I will ask the child to find two whole numbers that are close to the given numbers and to multiply them. I will then ask him to look at his exercise and the given exercise and to check what is going on.
3. I will tell the child that the multiplication of the whole numbers alone (15 x 4) is 60. So we will have more than 60. That's why the answer should be 68.58. In addition, I will write down the exercise, and ask the child to multiply. The answer will be 68.580 and I will explain that 68.580 equals 68.58.
4. The child does not understand how to multiply decimal numbers. I will give him several exercises and ask him to solve them using the standard algorithm.
5. I will tell the child: You stated a correct rule but your answer is incorrect, because when you multiply 4 and 5 the answer has a 0 at the end. 0 is not shown in your answer, and that's why you made a mistake in locating the decimal point. The answer is 68.580.

****

The Course

At the beginning of the course the teachers talked about the ways they had responded to the questionnaire tasks. They were also presented with the other teachers' responses and analyzed them according to (i) awareness of students' misconceptions (ii) kinds of teacher responses, and (iii) content knowledge. Then the instructors presented some other situations and discussed them as well.

The second part of the course concentrated on better understanding how students learn. Special attention was given to the issue of a conception of learning as construction of knowledge by the student as opposed to a conception of learning as transfer of knowledge from the teacher to the student. Students' misconceptions and their possible sources were discussed, and various ways for dealing with these misconceptions were suggested.

In the third part of the course the teachers looked for relevant situations in their own classrooms, shared them with the other participants and analyzed them.

As part of the final assignment for the course the teachers were to explore students' ways of thinking about mathematical situations and teachers' explanations. To do that, each teacher interviewed a pair of sixth-grade students. They presented three situations to the students (the "Height", "Decimal Point" and "Division by 0" situations) and asked the students to respond. Then they presented the students with teachers' responses to these situations (the same responses that were used in the teachers' interview before the course) and again asked the students to react.

Seven of the course meetings (out of fifteen) were videotaped. Four times during the course the teachers submitted written reflections on their learning from and feelings about the course experiences.

After the Course

A questionnaire was administered to all the teachers immediately after the course ended. After they had responded to the questionnaire, the six teachers who had been interviewed before the course were interviewed again.

Questionnaire - Situations (II)

A questionnaire that included three situations similar to the ones in the first questionnaire was administered to the teachers.

Interview - Other Teacher Responses and Beliefs

The interview had two components. In the first part, the subjects were again presented with responses that were given by other teachers to the situations from the second questionnaire, and were asked to react to these responses. In the second part of the interview the subjects were probed on their responses to the Belief Questionnaire that was administered to them before the course began.

Preliminary Findings

In this paper we report on some preliminary findings concerning the pedagogical content knowledge of the six interviewees. First we focus on their thinking about the "Decimal Point" situation by examining their own responses to this situation as well as their reactions to the other teachers' responses to the same situation. Then we analyze their ways of finding out how students think, as expressed in the interviews which the teachers conducted.

Teacher Responses to the "Decimal Point" Situation

Teachers' Own Responses

The most common response to the "Decimal Point" situation on the first questionnaire was the suggestion to use estimation as a response to the student. Four of the teachers suggested this. (One suggested to use estimation together with the standard multiplication algorithm.) The extensive use of estimation might be explained by the fact that the teachers had a course on estimation and number sense during the first semester--the time when they were administered the questionnaire. Another teacher based her response on the standard multiplication algorithm, showing that actually there are five and not four digits in the final answer. Still another teacher said that the child was right and the answer is correct.

Almost all teachers' responses included some positive statement about the child, something that can make him feel good although his answer was incorrect. For example, "Basically he is right and he remembers nicely how to multiply decimals. But..." This behavior was not in evidence in our previous study on junior-high school teachers (Even & Markovits, 1993b).

Reactions to the Other Teachers' Responses

When reacting to the other teachers' responses presented to them during the first interview, none of the interviewees explicitly mentioned the difference between Responses 2 and 3. That is, between putting the student in the center, suggesting to him activities that have potential for helping him construct a solution by and for himself, and putting the teacher in the center, telling the student how to reach the correct solution. This finding agrees with our findings in the previous study (Even & Markovits, 1993b).

None of the interviewees chose Response 1 nor 4. They rejected Response 1 on the grounds that the student's solution was wrong. They opposed Response 4 which states that the child does not understand how to multiply decimal numbers. The teachers argued that "even though the child did not locate the decimal point correctly, it does not mean that he does not know how to multiply decimals" and, therefore, they claimed, there is no need to ask the student to solve several exercises using the standard algorithm.

Interrelations Between Teachers' Responses and Their Reactions to the Other Teachers' Responses

The interviewees' preferences when asked to choose among the other teachers' responses had similar characteristics to the responses they themselves gave when answering the questionnaire several weeks before the interview took place. This consistency was apparent not only in the mathematical approach they used but also in the way they chose to approach the child. For example, two teachers chose Response 2, which is more student-centered, as the most appropriate among the five responses presented to them. These teachers' responses were also more student-centered than the others:

"I'll ask him how much is 15x4. Since he will get 60 by himself, it will be no problem to get the right answer."

"Pay attention -- is the answer reasonable? Is it reasonable to get to 6 from 4x15?"

In contrast, the two teachers who chose Response 3, which is similar to Response 2 but is more teacher-centered, also presented a more teacher-centered approach in their responses to the questionnaire. For example:

"What is the whole number? 15x4. So, the answer has to be more than 60, because there is more than 15 and more than 4."

One interviewee chose a combination of Responses 2, 3 and 5. The same combination appeared in her own written response, where she started with estimation, then added the idea that ten times hundreds should give thousands. She said she would ask the child to perform the multiplication and locate the point. Finally she explained that 580 thousands and 58 hundreds are the same.

The teacher who missed the whole thing in the written questionnaire and claimed that the student's answer is correct, realized during the interview that she was wrong. She chose Response 5, the only response that can be interpreted as if the student's answer has a correct component in it: The rule is correct but the answer is not.

Teachers' Ways of Finding How Students Think

When asked to interview pairs of students the teachers were told that the objective of this task is to explore students' ways of thinking. To do that they needed to ask students to explain and clarify their thinking. Probing was an important component of the interview procedure. Besides their free, voluntary explanations, the students needed also to be probed to clarify ambiguities.

Some of the teachers used neutral probes such as "Why?", "What do you mean by that?" This kind of probing does not direct the student to any specific direction and at the same time helps the teacher-interviewer to better understand the students' thinking. Even though it was tempting at times to "teach" the students "the material" during the interview, some managed to avoid it. For example, one teacher, whose students did not use estimation for the "Decimal Point" task, did not mention it to them during the interview. In her report she wrote that she had realized that there was a need to teach this to the children and therefore taught it in her class in subsequent lessons.

Other teachers were more occupied either with directing the students to what the teachers thought students' ways of thinking are or with straightforward teaching of the material. The following example illustrates the first approach. One teacher presented the students with the exercise 15.24x4.5=6858 and asked them to locate the decimal point. Then, instead of letting them decide, she turned to one of the students and directed her to use a wrong method:

> *What do you say? Where would you locate the decimal point? Do you remember the rules for decimal fractions? How do we locate the point? What do we do in order to locate the point after we multiply [decimal] fractions?*

The child, of course, replied: "We count the number of digits after the point." The teacher continued in this direction: "So, where would you put it?" The child: "After the 6." Such interviewing does not really help the teacher explore students' ·ays of thinking. Instead, it "helps" the students guess what the teacher has in mind.

Some teachers felt that they had to "teach" the students the correct answer or the "correct" way of thinking during the interview. For example, after the students had decided where to locate the decimal point, the teacher comprehensively discussed with them the difference between the "rule method" and the method of estimation. She discussed the issue from many angles and only then asked the students to react to the teachers' responses. By doing that, the teacher directed the students to what she considered to be the correct choice. This teacher kept correcting mistakes that the students made. She also told the students at times whether she agreed with their answers. While some of the teachers who "talked too much" during the interview referred to this issue in their report, claiming that they had directed the students too often, this teacher wrote that there was a need to direct the students so they would not get confused.

Conclusion

One major difference between this study and our previous study has to do with the attitudes towards the student that the teachers presented. While most of the elementary teachers phrased their responses in ways that would make the children feel good even when their answers were incorrect, the junior-high school teachers never included such a component in their responses. They also never showed any awareness of this issue. The junior-high teachers concentrated only on the correctness of the children's solutions, not taking into consideration their feelings.

The findings of this study agree with the findings of our previous study with regard to teachers' awareness of the issue of teacher- versus student-centered responses of teachers. Before the workshops and the course, both the junior-high and the elementary teachers did not seem to be aware of the distinction between responses that are teacher-centered (i.e., the teacher tries to teach by telling the student how to do things--to "transfer knowledge" whereas the student is expected to stay relatively passive) and responses that are student-oriented (responses that emphasize the construction of knowledge by an active student).

The short activities conducted in the previous study raised teachers' awareness of this issue. However, knowing that making a change in the ways teachers teach is not an easy task, we designed and conducted a much longer and more comprehensive course in the present study. Some of the findings of this study indicate that a one semester course is also not enough for changing teachers' behavior, not even when they deal with two students only (as became apparent from the interviews that some of the teachers conducted with pairs of students at the end of the course). This conclusion is supported by observations made of these teachers' teaching during the second year of the program. A detailed analysis of the data collected in this study should inform us more about the nature and extent of teachers' awareness of the issue.

References

Even, R. & Markovits, Z. (1991). Teachers' pedagogical knowledge: the case of functions. In F. Furinghetti (Ed.), Proceedings of the 15th International Conference for the Psychology of Mathematics Education (Vol. 2, pp. 40-47). Assisi, Italy.

Even, R. & Markovits, Z. (1993a). Teachers' and students' views on student reasoning. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. L. Lin (Eds.), Proceedings of the 17th International Conference for the Psychology of Mathematics Education (Vol. 2, pp. 81-88). Tsukuba, Japan.

Even, R., & Markovits, Z. (1993b). Teachers' pedagogical content knowledge of functions: characterization and applications. Journal of Structural Learning, 12(1), 35-51.

Markovits, Z. (1989). Reactions to the number sense conference. In J. T. Sowder, & B. P. Schappele (Eds.), Establishing foundations for research on number sense and related topics: Report of a conference. San Diego, CA: Center for Research in Mathematics and Science Education, San Diego State University.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher. 15 (2), 4-14.

MATHEMATICAL IMAGES FOR FRACTIONS: HELP OR HINDRANCE?

Susan E.B. Pirie
University of Oxford

Lyndon Martin
University of Oxford

Thomas E. Kieren
University of Alberta

This paper is concerned with the images that learners and teachers have for fractions. We introduce the notion of Cycles of Responsibility and from this perspective focus on the images for rational numbers that teachers and pupils bring to the beginning of work on algebraic fractions. The work of a class of 15 year-olds is observed and analysed and the role of the teacher in enabling the learners to fold back to, or form, appropriate images is considered. Implications for others in the Cycle of Responsibility are tentatively suggested.

Introduction:

We bring together here two aspects of learning, namely the images held by the learner and the images held by those who attempt to influence the learning, and examine their interaction in the process of students coming to understand the concept of fractions. Our work is based within the Pirie - Kieren model for the dynamical growth of mathematical understanding. The model itself has been fully described at previous conferences (PME 15, PME 16), and many of its features elaborated on, there and elsewhere (Pirie and Kieren, 1991; Pirie and Kieren, 1992; Pirie and Kieren, 1994). In this paper it is with the stages labelled 'Image Making' and 'Image Having' that we are concerned. Image Making means the performing of actions by the learner, actual or mental, to get some idea of the concept under consideration. These actions may be prompted by the teacher or another 'outsider', but can equally well be the result of the pupil's own initiatives. Image Having is the level at which the learners actually have some images for the concept and thus they no longer need to rely on the actions that occasioned the understanding and can carry and use the ideas they have constructed. This does not imply however, that their images are complete, appropriate or even sufficient for the work in hand. Many learners develop strong early attachments to particular dominant images and this can seriously hamper later growth of understanding.

Learners' Images

The concept of division provides a vivid illustration of the encapsulation of two completely different images within one mathematical notion. Although either image is sufficiently powerful to allow an understanding in terms of integers, it is in fact essential that students have both images if they are to move to a comprehension of division with fractions. Consider the example: '12 ÷ 3'. This can be interpreted as '12 pizza shared between 3 people' and represented diagrammatically as:

oooo oooo oooo

The answer lies in the number of pizza in each group, namely 'four'.

An alternative interpretation, based on a different image for division, is "How many groups of 3 pizza can be made from 12 pizza?", represented diagrammatically as:

ooo ooo ooo ooo

Here the answer lies in the number of groups.
Either image is powerful enough on its own to enable the learner to function adequately with the concept of division related to whole numbers. Many pupils, therefore, rely on only one of these images.

However problems arise if we consider 'a half divided by 3'. Translating this in terms of the images suggested above we see that 'Half a pizza shared between 3 people' makes sense, whereas 'How many groups of 3 pizza can be made from half a pizza?' leads to the answer 'none' - one can make no groups of 3 pizza from only a half pizza. The second image is of no use here.

Now consider '12 divided by a half' - translating this in terms of the above images we get '12 pizza shared between half a person' - meaningless - whereas 'how many half pizzas can be made from 12 pizza?' is clearly a sensible question. To understand the process of division of fractions, a student needs to have <u>both</u> images. Either image alone is not sufficient.

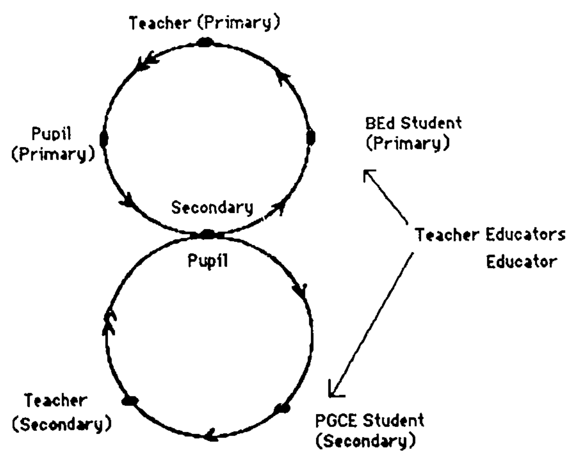
Work concerning fractions has focused predominantly on either the processes used by pupils in working with fractions, or on the ways in which the pupils develop their conceptual understandings or on pupils misconceptions of fractions (e.g. Kerslake, 1986; Hart et al, 1989; Markovits and Sowder, 1991; Cramer and Bezuk, 1991; Peck and Jencks, 1981; Kieren, 1988; Behr et al, 1992).

Kieren (1993) emphasised the importance of images and stated that 'fraction learning...is seen in two kinds of growth: broadening one's image of fractions and hence one's fraction conceptualising capability and deepening, making more formal, and more sophisticated one's understanding actions' and that 'even within a particular fraction environment, building up of the various mechanisms requires folding back to reconstruct one's image of fractions through action in a fraction space.' It is on these personally held images, be they helpful, inadequate or even inappropriate, that learners will base the further growth of their rational number understanding. Fraction learning involves constructing an ever more elaborate, complex, broad and sophisticated fraction world and developing the capacity to function in more complex and sophisticated ways within it. Such an achievement will prove impossible if the foundations laid by the images the learners hold are not adequate to the task. This is not to say that the whole spectrum of images encapsulated in the concept of fractions needs to be created before any progress in working with them can take place, but it is our contention that limited and inaccurate images will inhibit learning and that there will be a constant need to fold back to the level of Image Making to develop the necessary broad range of images.

Kerslake (1986) discussed the different models of fractions that children are familiar with. She noted 'that the problems children have with fractions are due to their restricted view of a fraction' and that this 'makes it difficult to make any sense of addition or of placing a fraction on a number line.' She also highlighted the dominance of the 'part of a whole' model of a fraction in children's thinking and suggested that 'the use of the 'part of a whole' model can inhibit the development of the more general idea of a fraction.' Other images that she suggests are those of fractions as numbers and fractions representing a division process. Hiebert, Wearne & Taber (1991) in looking at fourth graders' construction of decimal fractions talked of the 'partial understandings' constructed by many learners and noted that 'many students did not construct *full and flexible* internal representations' for decimal fractions and that this led to problems with regard to the notion of fractions in terms of continuous rather than discrete environments. In the classroom it is the teacher's responsibility to create an environment in which the learner can construct an understanding of fractions based on appropriate images. Ball (1990) looked at the subject matter knowledge of pre-service mathematics teachers but was more concerned with the generation of representations for processes involving fractions than with the images underlying the subject matter that form the core of any teaching of the subject. The images held by the teacher, however, will have a significant impact on the images constructed by the pupils and on their ability to understand these processes.

Cycles of Responsibility

This leads us to the second of the foci of this paper which we have termed the "Cycles of Responsibility" and this looks at the chain of influences which impinge on the growth of understanding. This chain can be represented by the following diagram:



Circles of Responsibility

If we start with the primary pupils, then these pupils, with their own personal images, constructed from the environment in which they have been learning mathematics and from other, outside influences, become the cohort of secondary pupils. From the secondary pupils, with their potentially changed and enlarged images, are drawn the students who will train to become either primary or secondary teachers - students whose images of mathematics are further refined during the courses they undergo. These in turn become the next generation of teachers - primary and secondary school teachers, who complete the cycles by influencing the pupils in their classes.

We have been working on a project to look at the groups involved within these Cycles of Responsibility as they are affected by and affect the images available, on which an understanding of the concept of rational numbers can be built.

The process of initial data collection in our study involves the use of a six-item, written questionnaire[1]. The questions here were deliberately chosen to be open-ended, to allow for the possibility of a wide range of responses. The participants were not interviewed, as spoken language immediately conveys a more restricted image than written symbolism (Pirie 1992) and can itself conjure up inappropriate images. Take for example the verbalisation of "$\frac{3}{4}$". When associated with the image of fraction as 'part of a whole' this could be rendered "3 divided into 4 (pieces)". With the image of fraction as 'division', on the other hand, "3 (divided) into 4" is a frequent verbalisation of $4 \div 3$ or, written as a faction, $\frac{4}{3}$.

Three groups have been looked at so far :- students training to be secondary mathematics teachers, students training to be primary teachers with mathematics as their specialist subject, and a class of fairly able, 15 year-old pupils. Although at this early stage, from our pilot work, we predict some striking differences between each of the groups involved in the cycle, we wish to concentrate our discussion in this paper on the group of 15 year-olds who were subsequently video recorded over a two week period as they worked on algebraic fractions.

Images for Rational Numbers

Four major images for fractions emerged from the pupils' written responses to the questionnaire and we illustrate these with quotations from these responses:

1. **Division** - e.g. "A quantity - number or sometimes a letter - divided by another quantity."
2. **Part of a whole** - e.g. "A fraction is a section of a whole thing."
3. **Number** - e.g. "It is a number that has not been put into a decimal."
4. **Way of writing** - e.g. "A number over another number" - This image was not among those suggested as necessary by Kerslake (1986), although she does quote one pupil as saying "two

---

[1]The questions were:
i) What are fractions ?
ii) How would you explain $^{3}/_{4}$?
iii) How would you explain $^{15}/_{4}$?
iv) When you want to add fractions, why do you find a common denominator ?
v) When you want to multiply fractions, why do you multiply the tops and bottoms and give $\frac{\text{Product of tops}}{\text{Product of bottoms}}$ as the answer ?
vi) When you want to divide fractions, why do you turn the second one upside down and multiply ?

numbers just put on top of each other". Yet in more sophisticated mathematics, along with the image of a fraction as a quotient number, this image of $\frac{a}{b}$ as simply a way of writing an ordered pair is a powerful, necessary image.

The introduction for these students to the topic of algebraic fractions was made through a deliberately invocative[2] input by the teacher, in which she repeatedly referred them back to the earlier images that she assumed they would have created in working with rational numbers "in primary school". It is interesting to look at the images that she considered useful for them to fold back to, the images that the students themselves verbalised and the images that they and the teacher actually folded back to when working on the algebraic fractions.

The teacher started by reviewing basic rational number work. For each operation, she grounded her explanations firmly in the 'part of a whole' image while saying "...go back to nice simple things first and apply them to algebra". For addition she wrote $\frac{1}{5}+\frac{2}{5}$, and said "if I add the top two numbers together I get three and if I add the bottom two I get ten", then, while drawing a cake, continued "It's pretty obvious that if you have one fifth of a cake and another two fifths of a cake then you have three <u>fifths</u> of a cake. So you mustn't be tempted to think sweepingly 'add them all together' because it doesn't work". Writing $\frac{1}{2}+\frac{1}{4}$ she posed a new problem, again in terms of cakes, and cautioned them "... difficult question, the numbers are different" and when offered the answer "three quarters" she emphasised the point that "later on when you get something like this with algebra and you think I'll just add the top numbers together and add the bottom numbers together ... but if you do that you'd end up with $\frac{2}{6}$ which isn't the same thing. You've got to go back to basics". She then asked the pupil how he had done it. His response was "well I just know it, (pause) well with the cake" and the teacher drew an appropriate 'cake diagram' on the chalkboard. Once they were working with the problem of $\frac{1}{6}+\frac{1}{8}$ however, both pupils and teacher switched to the image of fractions as a 'way of writing' while they talked about "the denominators" (the teacher) and "whatever you times the bottom by, to get whatever you're getting, you times the top by" (pupil). Writing $\frac{4}{(a+1)}+\frac{5}{(x+2)}$ the teacher commented "this one looks a bit more difficult because you've got the algebra bit on the bottom ... but follow the principles of what we've done before". The pupils' suggestions for dealing with this computation all centre on "multiplying the top and the bottom" and no reference is made by anyone to the image of fractions as 'part of a whole'

When working on multiplication, the teacher again initially worked within the fractional image of 'part of a whole' and illustrated $\frac{1}{2}\times\frac{1}{3}$, (verbalised as "one half multiplied by one third") with a drawing of a chocolate bar cut first into three pieces vertically then into two pieces

---

[2]Invocative is used in the specialised sense to imply an intervention that necessitates folding back by the learner. See Pirie and Kieren, 1992.

horizontally, saying "a third of a chocolate bar ... and you want a half of that third ... that will mean you have that much, which if you split these bits into two, you'll have one bit out of six". This teacher had done more than simply think about the basic images that she felt were necessary to the understanding of the manipulation of algebraic fractions, however. She was also aware of the fact that language could influence these images,. She continued at this point by saying "It is important when you multiply, to realise that 'multiplied by' means 'of' ". Why is this apparent digression appropriate? What imag. - might the pupils have for multiplication that could hinder their understanding of fraction multiplic ion? The most common and fundamental image for multiplication is that of repeated addition, repeated accumulation of groups of objects, whereas the teacher here seems to want to evoke an image tied to language.

Consider the problem: 5x3.

This could be thought of as '5 added 3 times: 5+5+5, - an action of addition repeated over time' (one image) or as 'the result of getting 5 (groups) <u>of</u> 3 (things) - ooo ooo ooo ooo ooo - and looking at the end result'- (a second image). Either image is sufficient for a functional understanding of integers.

Now consider the problem: $\frac{1}{2} \times \frac{1}{3}$

From the first image one would verbalise this as 'a half added one third of a time' which makes little sense, whereas saying 'the result of getting half (a group) <u>of</u> one third (of a thing)' allows for the creation of an image that involves looking at the end result and is applicable to both integers and fractions. It is the language here that conveys the image across the embodiments.

Having folded the pupils back to their basic understandings for fraction multiplication, the teacher moved on to consider the process of division. At this point the image for fractions as 'numbers' was spontaneously called upon by both teacher and pupils. They talked of "turning 'it' upside-down and multiplying". The teacher suggested "If you think of easy fractions it perhaps makes it a little bit easier. If you had two multiplied by a half *(writing* $2x\frac{1}{2}$ *on the board)* you know the answer to that is one, obviously, and that is the same as two divided by two *(writing* $2 \div 2$*)*, turning this one upside-down, it is the same thing, you get one. And four multiplied by a half *(writing* $4x\frac{1}{2}$*)* which is two, is exactly the same as four divided by two *(writing* $4 \div 2$*)* which is two". Here the 'half' is being treated as a number that one can multiply by, in exactly the same way as the 'two' is a number that one can divide by. Indeed this image of 'fraction as number' is vital to the inference that she wants the pupils to make, which is that one can similarly equate $2 \div \frac{1}{2}$ with 2x2 and $4 \div \frac{1}{2}$ with 4x2. In other words, the teacher wishes the pupils to use the notion of multiplication and division of numbers as inverses, together with the image of 'a half' as a 'number' and infer that one can therefore divide by a half, the result being the same as multiplication by two.

The Influence of these Images on the Understanding

In all the groups so far investigated by the project looking at fraction images held by the various participants in the Cycle of Responsibility, the 'part of a whole' notion emerged as a strong image for a large number of people. For many of the students at the start of their training to become primary teachers it appears to be their only image. For the pupils in the class discussed above, the initial questionnaire revealed that they, too, held the 'part of a whole' as a common image, and it was certainly seen as a dominant referent during the teacher's early, whole-class work, yet analysis of the video recording of the pupils and teacher throughout the following weeks spent working on the manipulation of algebraic fractions revealed no evidence that *these* fractions were at any point seen in terms of the 'part of a whole' image. Indeed is it hard to envisage how they could be: given $\frac{2x^2}{3y+4}$ what might 'divide a whole one into (3y+4) parts and take $2x^2$ of them' mean? What is clear, however, is that the teacher considered the 'part of a whole' image for "ordinary fractions" to be a useful image to fold back to, in order to enable the pupils to access the images they had (or if necessary make new images) for manipulation of these "ordinary fractions" as a precursor to the successful manipulation of algebraic fractions. She knew the importance of the notion for addition of common denominator and she folded back to its recall through a 'cake' diagram for $\frac{1}{2}+\frac{1}{4}$. From this approach she moved them to viewing fractions as a 'way of writing'. She had prompted folding back to 'part of a whole' imagery to evoke a broader image for rational numbers which was crucial to more sophisticated, algebraic working. What we cannot yet say is how or when the additional broader images are created as we saw no evidence of the teaching of these other images. What is also unclear is why these images are not accessible to the student primary teachers who ostensibly lie at a later stage in the cycle.

One further deduction from the analysis of the questionnaires of the 15 year-old pupils and their subsequent class-work appears to be that it is those with a multiplicity of images for fractions and an ability to move flexibly between these images who are most easily able to grasp an understanding of the new algebraic work. The image of fractions as 'a way of writing' would, in addition, seem to be essential to the ability to cross the 'don't need' boundary into formalising and working symbolically with algebraic fractions. We do not know how vital all these different individual images are for a functional understanding of rational numbers but from the whole collection of questionnaires so far analysed it seems likely that those with only the image of 'part of a whole' find it virtually impossible to understand the processes of multiplication and division of fractions.

References

Ball, D. L. (1990). Prospective Elementary and Secondary Teachers' Understanding of Division. Journal for Research in Mathematics Education, 21(2), 132-144.

Behr, M., Harel, G., Post, T. & Lesh, R. (1992). Rational Numbers, Ratio and Proportion. In D.A. Grouws (Ed.), Handbook of Research on Mathematics Teaching New York: Macmillan Publishing.

Cramer, K., & Bezuk, N. (1991). Multiplication of Fractions: Teaching for Understanding. Arithmetic Teacher, 39(3), 34-37.

Hart, K. M. (Ed.). (1981). Children's Understanding of Mathematics: 11-16. London: John Murray.

Hiebert, J., Wearne, D., & Taber, S. (1991). Fourth graders' gradual construction of decimal fractions during instruction using different physical representations. The Elementary School Journal, 91(4), 321-341.

Kerslake, D. (1986). Fractions: Children's strategies and errors. Windsor: NFER-Nelson.

Kieren, T. E. (1993). The Learning of Fractions: Maturing in a Fraction World. a background paper for a conference on Fraction Learning and Instruction, University of Georgia.

Markovits, Z., & Sowder, J. T. (1991). Students understanding of the relationship between fractions and decimals. Focus on Learning Problems in Mathematics, 13(1).

Peck, D. M., & Jencks, S. M. (1981). Conceptual issues in the teaching and learning of fractions. Journal for Research in Mathematics Education, 12(5), 339-348.

Pirie, S. E. B., & Kieren, T. E. (1991). Folding Back: dynamics in the growth of mathematical understanding. In F. Furinghetti (Ed.), Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education, 3 (pp. 169-176). Assisi: Program Committee of the 15th PME Conference.

Pirie, S. E. B., & Kieren, T. E. (1992). The answer determines the question - interventions and the growth of mathematical understanding. In W. Geeslin & K. Graham (Ed.), Proceedings of the Sixteenth International Conference for the Psychology of Mathematics Education, 2 (pp. 1-8). New Hampshire: Program Commitee of the 16th PME Conference.

Pirie, S. E. B. (1992). Pupil discussion and mathematical understanding: is there any connection? Plenary presentation at the International Congress for Mathematics Education, Quebec.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we represent it? Educational Studies in Mathematics, 26(2).

THE ROLE OF SYMBOLS IN STRUCTURING REASONING: STUDIES ABOUT THE CONCEPT OF AREA[1]

T. NUNES*, P. LIGHT**, J. MASON***, M. ALLERTON*
* Institute of Education, University of London
** Department of Psychology, University of Southampton
*** Department of Mathematics, The Open University

Abstract

The classification of problems in mathematics education research has taken into account only the problem situation and disregarded the means of representation available during problem solving. We wish to argue that systems of representation available in the situation play a structuring role in problem solving because they constitute resources for the subjects' actions and operations. A series of investigations is reported that analyzes the impact of measurement systems on children's reasoning about area. The use of bricks instead of rulers was found not to result in additive misconceptions and significantly enhance children's likelihood of discovering a multiplicative solution captured in the formula number of bricks in a row times number of rows. Theoretical and educational implications are discussed.

The literature on the development of mathematical concepts showed considerable progress through the analysis of the effects of situations on children's problem solving abilities. Problems that have the same formal solution can differ significantly in terms of their difficulty for children. These findings have been interpreted as indicating that problem solving activities are structured by children's understanding of situations (or, in Piagetian terms, by the logic of actions and operations). The theoretical contributions of this analysis are significant. It allows researchers to make more specific hypotheses about the cognitive processing of problems. There are also contributions to mathematics education because this analysis enables teachers to develop instruction plans that take the level of difficulty of problems into account, ensuring that instruction starts from easier problems and eventually covers the whole range of types.

However, it has now become clear that children's success in solving problems is not determined only by the problem situation but also by the symbolic systems used in solving the problem. Nunes (1993), for example, observed that 12-18 year old Brazilian students show significantly different levels of success in solving the same problems with negative numbers when they are asked to solve the problem through either written or oral procedures.

The significance of representations for thinking and reasoning was initially emphasized by Vygotsky and Luria, who proposed that complex intellectual functioning often involves people in activities *that can only be carried out through the mediation of systems of representation. The*

---

[1] This project was supported by the ESRC grant R000233961; we are thankful for their support.

systems of representation, in turn, organize reasoning. For example, we use a numeration system with a base ten to count and write numbers. The base ten, which is a characteristic of the system and not of number itself, becomes a source of organization for the way we calculate and think about numbers.

If representation systems become tools for thought, what happens when the same problem situation can be represented in two ways? We report a series of studies that indicates that the understanding of problems situations is not independent of the way in which the problem is represented. We wish to suggest that systems of signs structure the subject's activity in a problem solving situation and thereby influence the reasoning principles (or the logic of actions) that is developed by the subject in the situation. The studies we will report refer to children's understanding of area, we will first review briefly a difficulty children seem to face in their attempts to deal with the concept of area of relevance to this research and then we will describe the empirical work.

Children's understanding of area.

Previous work has indicated that the concept of area is prone to misconceptions, is difficult to teach, and remains unclear to many students even in the upper-middle school age range. The most common misconception documented in the literature pertains to the relationship between area and perimeter. Vinh Bang and Lunzer (1965) first documented this misconception in a task in which children had to judge whether the area delimited by a perimeter remained the same when the perimeter remained the same but the shape and delimited surface changed. This was accomplished by limiting an area with a string fixed at the corners and gradually changing the surface by moving the tacks which held three of the corners. Children tended to think that the area remained constant despite the changes of surface. More recently, Douady and Perrin Glorian (1989) observed that children treated perimeter and area as interchangeable "measures" of a surface and Hart (1981) obtained further confirmation of these findings with British children.

Teaching children about area has proven rather difficult. Hart (1981) found that, despite having been taught formulae for the area of simple geometrical figures, a large percentage of 12 to 14 year olds could not indicate the area of a figure covered by a grid with squares of 1 cm^2 (26% of 12 year olds, 20% of 13 year olds, and 11% of 14 year olds). She also noted that a significant number of children in these age levels *could only* find the area by counting squares (percentages here are 21%, 20% and 17%, respectively), making no use of multiplicative short-cuts. Dickson (1989), Douady and Perrin Glorian (1989) and Rogalsky also reported only moderate success in their teaching experiments.

It is clear that much more research is still needed on how children come to understand the concept of area. Research has investigated children's *misconceptions* and the *difficulties of teaching* but there is little in terms of positive findings about *how children succeed in understanding the measurement of area*. It would clearly be helpful to know whether the children who succeed in understanding area (and we know that some do so before teaching) do so through diverse routes or whether there is a more certain path to understanding.

The empirical studies

The present studies investigated the hypothesis that children's success in understanding area is not independent of the resources they are given to represent area during problem solving. The measurement tools are part of the situation to be mastered and thus constitute a source of structure for children's actions and strategies.

We designed a series of studies with the aim of investigating: (a) the impact of the tools (rulers vs. bricks) used in measurement on children's understanding of area; (b) the type of conception of area developed by children in interaction with these varied resources; and (c) the relative effectiveness of a teaching session as a function of the tool used in measuring and of the type of conceptual schema (isomorphism vs. product of measures) used by the experimenter in explaining how to calculate the area of a rectangle. The investigation was carried out in three phases described below.

Phase 1

First we wanted to obtain experimental evidence for the impact of tool used in measurement (square centimetre bricks vs ruler) on children's solutions to problems of comparison of areas. Three questions were investigated: (1) do children have a better chance of discovering a multiplicative description of area if they work with rulers or if they work with bricks? (2) do they conceive of area in the same way regardless of the resource used or are the two types of resource related to two different conceptions (isomorphism vs product of measures)? (3) can children transfer what they learned for simple rectangles to solving problems involving other geometrical figures (such as a parallelogram, where there is a need to distinguish between the side and the height of the figure?)?

Method

Subjects: Subjects were 48 pairs of children (16 in each age level from 8 to 11 years, corresponding to Years 4, 5, and 6) randomly selected from three schools in London. The schools cater to both working and middle class families. Ethnic background was varied: approximately 39% of the children were white, 26% African/Afro-Caribbean, 31% Asian and the remaining 4% from diverse backgrounds.

The age range was selected because few children are likely to know the solution to the problems from the outset but it is still possible to anticipate positive effects of instruction. The children's past experience with area as described by their class teachers in an interview was based on the Cambridge Maths scheme. The method involves children in counting the number of squares covered by regular and irregular shapes drawn onto a grid sheet. From this work, children are led to the use of the height x width formula in Year 6.

Procedure: The children, working in same-gender pairs, were presented with a series of tasks that involved the comparison of areas. The task was posed in the context of attempting to decide how to distribute the payment received for a painting job if each child in a pair had painted a wall that resembled one of figures drawn on the paper.

All pairs of children solved two blocks of two problems. The first block involved the comparison of two rectangles (10cm x 4cm and 8cm x 5cm in the first trial and 9cm x 3cm and 8cm x 4cm in the second trial). The second block involved comparing one rectangle with a different figure (a complex figure that can be decomposed into rectangles for one trial and a parallelogram for the other trial). The areas could not be easily compared visually because the wider forms had lower values for their heights.

The pairs of children were randomly assigned to one of two experimental conditions--either having access only to rulers or to 20 bricks, which did not suffice to tessellate any of the figures. They were asked to give one mutually agreed upon answer to the comparison of area questions. At the end of each trial, the children were given feedback by comparing the areas with the help of coloured paper previously cut so as to exactly cover one of the figures (or both, if they were the same) and the parts of which could be arranged over the other figure in order to show whether the areas were the same or different.

A delayed post-test was given to all children individually approximately one month later in order to evaluate the long-term effects of th initial learning experience. In this session, the children were asked to figure out the area of four figures and had at their disposal a ruler and 20 bricks. Two simpler problems (a rectangle and a complex figure that could be analyzed into rectangles) and two more difficult problems (a parallelogram and a triangle) were included. The problems were given in order of difficulty. No feedback was given. All sessions were tape-recorded and the tapes were transcribed for the analysis of problem solving strategies.

<u>Results</u>

An initial analysis of gender effects was carried out in order to test for the need to keep apart the pairs of girls and boys in the subsequent analyses. No significant differences were found. Subsequent analyses do not consider gender.

An one-way ANOVA indicated that the children who worked with bricks were significantly more successful in solving the problem than those who worked with the rulers in the first session ($F = 29.84$; $p < 0.001$).

The results of the delayed post-test were analyzed once using the average for each pair as their joint score in the individual post-test and a second time using each child's score. Similar results were obtained in these two analyses. We report here the second analysis. An one-way ANOVA revealed a significant effect of tool in this delayed post-test ($F = 9.53$; $p < 0.003$). The higher rates of success of children who worked with bricks were observed both in the simpler items (identified below as Block 1) and more complex ones (Block 2) in both sessions. Table 1 shows a summary of these results.

Thus the facilitative effect of the use of bricks in contrast to rulers for reasoning about area was obtained in this experiment. The effect persisted even after one month and could still be observed when the children were solving the problems individually.

Table 1
Mean number of correct responses as a function of tool

	Session 1			Individual post-test		
Tool used	Total	Block 1	Block 2	Total	Block 1	Block 2
Bricks	2.6	1.6	1.0	1.5	1.1	0.4
Ruler	1.2	0.6	0.6	0.7	0.6	0.1

An analysis of children's strategies showed that 39% of the children using bricks solved the problems of comparing rectangles by calculating each area through a multiplication whereas only 21% in the ruler condition did so. This difference in the use of multiplicative solutions persisted in more difficult items. Approximately 30% of the children who had the bricks as measuring tool used multiplicative strategies to solve the problems involving the complex shape and the parallelogram whereas only 13% of the children who had rulers did so. The comparison involving the triangle was very difficult for both groups and success rates were rather low.

An analysis of children's explanations of their reasoning when they multiplied clearly indicated that they were using an isomorphism of measures model when they had the bricks at their disposal. They often made references to number of rows and number of bricks that could be fit in a row. Some of the children (23% of the pairs in the first two rectangles) who had rulers as their tool attempted to use the ruler as if it were an area unit: instead of reading numbers from the rulers, they tried to figure out how many times they could fit the ruler over the figures. This strategy suggests that the ruler was a tool that did not fit well with the way they reasoned about area; they would rather have had area units at their disposal.

In short, this study supports the idea that measurement tools have a structuring role on children's reasoning about area because they are resources in the situation with which children can operate. In a more general way, the results strengthen the idea that systems of signs structure reasoning by becoming part of the logic of actions in a situation.

Phase 2

The aim of this investigation was to probe further into the role of systems of signs by looking at the results of instruction that relies on different tools. Three questions were addressed: (1) can children successfully apply a procedure demonstrated to them for comparing one pair of rectangles to a new pair? and (2) can children successfully adapt the same procedure to solve two transfer tasks?; (3) does the success of adaptation of the learned procedures vary as a function of the measurement tool used in teaching?

Method

Subjects: Subjects were 72 pairs of children randomly selected from the same three schools and from the same year groups as in Experiment 1.

Procedure: The children worked in same gender pairs randomly distributed in equal numbers to one of three instruction groups. The instruction procedures consisted in demonstrating how to calculate the area of a rectangle in one of three ways: a) the experimenter measured the sides of a rectangle with a ruler and told the children that the area was equal to the width times height (the "Ruler Condition", identified below as Condition 1); b) the experimenter built a row of bricks along the height and another one along the width of a rectangle and told the children that the area of the rectangle would be given by multiplying the two values (the "Bricks - product of measures" Condition, identified below as Condition 2); c) the experimenter formed three rows of bricks along the width of the rectangle and showed the children that the number of bricks in the rows was always the same; calculation of the area was demonstrated using the formula number of bricks in a row times number of rows (the "Bricks - isomorphism of measures" Condition, identified below as Condition 3). The first two conditions were similar in their use of the formula height times width in the calculation of the area but differed with respect to tool used. Condition 3 was similar to condition 2 in the use of bricks but differed in the explanation offered for the formula, being clearly connected with an isomorphism of measures explanation.

After the initial instruction, all children solved the same problems as in Study 1. They were interviewed once as a pair and a second time approximately one month later individually. Sessions were tape-recorded and tapes were transcribed for analysis.

Results.

The results are summarized in Table 2, which displays the average scores for each condition by session and block of problems. There were no significant differences across groups in Block 1, where all three groups performed close to ceiling. In other words, children seemed to have little difficulty in repeating the procedure they had been taught regardless of how it had been explained. However, in Block 2, which involved the comparison of a rectangle with a complex shape in the first trial and with a parallelogram in the second trial, a significant effect of teaching procedure was observed ($F = 6.60$; $p = 0.002$). Post-hoc tests indicated a significant difference between Group 3 children, who received the demonstration through the Bricks-isomorphism of measures procedure, and Group 1 children, who were instructed with the ruler; no other comparisons were significant. Thus, although all children could repeat a procedure they were taught immediately after instruction, children instructed through an isomorphism of measures procedure were more likely to devise adequate solutions to new area problems.

Results of the individual post-test showed no significant differences across groups either for the simpler or for the more complex problems.

Table 2
Mean number of correct responses by teaching approach

	Session 1			Individual post-test		
Condition	Total	Block 1	Block 2	Total	Block 1	Block 2
1	2.6	1.9	0.7	0.8	0.7	0.1
2	2.9	2.0	0.9	1.0	0.8	0.2
3	3.0	1.8	1.2	1.0	0.9	0.1

Phase 3.

In this third phase we compared the children from the previous experiments. Children taking part in Experiment 1 were essentially engaged in a "discovery experiment" whereas those participating in Experiment 2 received a demonstration before trying out the problems. We wanted to know whether these differences in their experience significantly affected learning.

Method

A factorial design with two factors--2 (type of tool used) by 2 (type of learning experience in the first section)--was created by pooling the data in the delayed post-test from experiments 1 and 2 (and disregarding the Brick - product of measures conditions). An ANOVA was then carried out with the results of the individual post-test as the dependent variable. In this analysis 190 children were included.

Results

The only significant effect observed was that of the tool used ($F = 7.96$; $p = 0.005$); the comparison between the groups with/without demonstration and the demonstration x tool interaction were not significant. Thus although some of the children had observed a demonstration in the first session, they did not have an advantage over those who did not receive such a demonstration. However, the type of tool used for thinking about area in the first session significantly affected children's success in the delayed post-test.

General conclusions.

This series of studies demonstrates the need to take into account alternative ways of representing the same problem both in theory and practice in mathematics education. From the theoretical viewpoint, it is clear that one cannot analyze the cognitive demands of an area problem without considering the resources available in terms of problem representation to the problem solver.

The studies also show that representation must be taken into account when instruction programmes are designed. They indicate possible reasons for the lack of success of current approaches to teaching children about area. Although children have the opportunity to tessellate figures in order to find their area, they are as a rule only asked to count squares. They have no problem to solve and no need to find a mathematical formula to accomplish this task (see Nunes, Light, and Mason, 1993). After counting squares, a height times width formula is taught. This formula, as we saw, does not fit well with the pattern pupils are likely to discover for themselves. It is possible that by changing these two characteristics of instructional programmes we could make the teaching of area significantly more effective.

References

Dickson, L. (1989) Area of a rectangle. In D. C. Johnson (Ed) Children's Mathematical Frameworks 8-13: A study of classroom teaching. Windsor: NFER Nelson.

Douady, R. & Perrin-Glorian, M. J. (1989) Un processus d'apprentissage du concept d'aire de surface plane. Educational Studies in Mathematics, 20, 387-424.

Nunes, T. (1993). Learning mathematics: Perspectives from everyday life. In R.B. Davis & C.A. Maher (Eds.), *Schools, mathematics, and the world of reality* (pp. 61-78). Needham Heights (MA): Allyn and Bacon.

Nunes, T., Light, P., & Mason, J. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction*, 3, 39-54.

Rogalsky, J. (1983). L'acquisition des notions relatives a la dimensionalite des mesures spatiales (longueurs, surface). Recherches en didactique des mathematiques, 3, 343-396.

Vinh Bang & Lunzer, E. (1965). Conservations spatiales. Etudes d'epistemologie genetique, Vol. 19. Paris: Presses Universitaires de France.

COGNITIVE MODELS OF THE CONCEPT OF ANGLE

José Manuel Matos

Universidade Nova de Lisboa

Idealized cognitive models developed by Lakoff (1987) provide a theoretical framework upon which several features of mathematics learning can be based. This paper describes several cognitive models exhibited by 4th and 5th graders of an elementary school in the United States about the concept of angle, showing the complexity of their mental representations.

Lakoff contends that any situation is represented by a mental space. For example, our immediate reality as we see it, fictional situations, past and future situations, as we see them, hypothetical situations, abstract domains, and others. Mental spaces are structured by cognitive models each of which is a complex whole that may have four structuring principles: a propositional structure (based on a definition), an image-schematic structure (based on mental images), metaphoric applications (based on analogies), and metonymic applications (based on metonymies) (Lakoff, 1987). To accou : for cognitive models in geometry it was suggested previously that a fifth structuring principle should be added, namely a script structure (based on a typical sequence of events) (Matos, 1992, 1992).

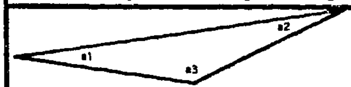
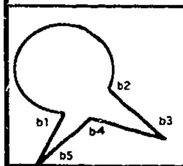
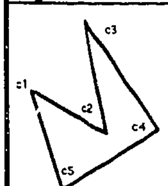
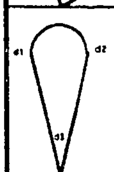
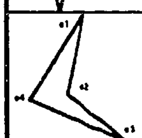
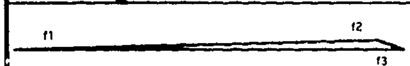
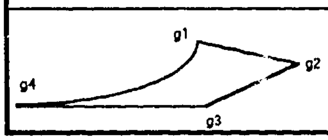
This paper will classify and describe some cognitive models found when investigating the concept of angle exhibited by some 4th and 5th graders of an elementary school in the United States. These models were inferred by the researcher on the basis of a qualitative analysis of semi-structured interviews (Strauss, 1987) of 16 students and a test passed to 57 students. Elements of the model and a set of relations among these elements will be specified for each model.

Images angles

It is plausible to assume that almost all participant students used image-schematic models composed of rich mental images (Johnson, 1987). It is possible to obtain a first grasp of these models by analyzing both the interviews and the answers to the test. Table 1 shows the number of students that identified angles in several configurations. Students did not recognize concave angles (angles on a concave vertex of a configuration) as much as they did with convex angles. Also convex vertices of configurations with curved sides (g1, g4) were recognized as angles, even by fifth graders.

These imagetic models are organized around central elements (cognitive reference points) which can be observed in the interviews. For example, Beth says that to make different angles we could "turn it around and make it like a L, or you can make it a different way... You can put it a different way and then make a angle". Louise said that to make an obtuse angle a friend of her would have to "kind of try to make an A, but not just like an A. (...) [It's an A] but further apart". And a right angle "is kind of like half of a square". And Jessie said that "an angle is just (...) a sort of a V shape". Many times these prototypical images are associated with a preferred orientation of the angles. Most angles drawn by students had one side horizontal. A non-horizontal side was considered "slanted" by Beth.

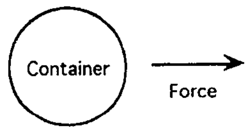
Table 1. Number of students that identified angles in several configurations.

Configurations		Grades G4 (n=33)		G5 (n=24)		Total	
Put an A at the point of each angle in the figures below:			%		%		%
	a1	32	97	22	92	54	95
	a2	32	97	22	92	54	95
	a3	29	88	23	96	52	91
	b1	6	18	4	17	10	18
	b2	6	18	4	17	10	18
	b3	32	97	21	88	53	93
	b4	9	27	13	54	22	39
	b5	32	97	21	88	53	93
	c1	31	94	22	92	53	93
	c2	14	42	16	67	30	30
	c3	31	94	22	92	53	93
	c4	16	48	20	83	36	63
	c5	16	48	20	83	36	63
	d1	5	15	3	13	8	14
	d2	5	15	3	13	8	14
	d3	32	97	24	100	56	98
	e1	32	97	21	88	53	93
	e2	14	42	12	50	26	46
	e3	32	97	21	88	53	93
	e4	24	73	23	96	47	82
	f1	23	70	21	88	44	77
	f2	16	48	16	67	32	56
	f3	27	82	23	96	50	88
	g1	30	91	17	71	47	82
	g2	31	94	24	100	55	96
	g3	23	70	19	79	42	74
	g4	25	76	19	79	44	77

Angles as points

Many students thought that "[angles] must have a point" (Laurie), or "have (...) a point in their end" (Rick). A similar statement is presented by Rick that considered that angles have a "sharp end". James said that an angle is "kind of a point", but distinguished between triangles and angles. The former *have* three points, whereas the later *are* actually points. Turns don't have points because "they are round". Alice says that "the corner point always has an angle".

This is a metaphoric model projected from an image-schematic model characterized by two elements: a container, or part of a container, and a force. The force is a protruding movement to the exterior of the container. In the expression *angles have a point*, an angle is a container that has a part that is projected outwards. This model has an implicit "viewpoint" (Johnson, 1987, p. 36) because the movement is observed from the outside, so we can watch the *sharp ends* of angles.



If angles are thought as pointed objects, it makes sense to use with them language usually associated with pointed objects. Some angles can be "sharper" than others (Laurie, Susan) and some angles "doesn't really have a pointed end" (Mally).

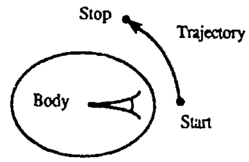
Forces in models may be associated with movements as is instantiated by Rick using a mixture of words and gestures. He says that angles "got pointing ends at the end, they got pointing ends when they come up" and he places his hands vertically as an inverted V, or angles "come out as a point" (Rick) or angles point "across "(Mike).

This model does not exclude angles with curved sides. Some students accepted them but rejected drawings where the lines would not connect. The notion of sharpness helps to understand the structure of this model further. Mally, Susan and Laurie, for example recognize obtuse angles in several contexts. However they mostly refer to acute and right angles. Although they are aware that there are other types of angles, right angles and acute angles seem to stand for the whole set of angles. Their model of angles has a metonymical structure in which these types of angles can stand for all the angles. This metonymical structure seems to be a source for prototype effects because an obtuse angle "doesn't really have a pointed end" (Mally), or is not as sharp as an acute angle (Susan, Laurie). This metaphoric model can also be the source of other metaphoric models. Laurie says that an angle is like "the edge of a pencil, the point of a pencil", an angle "looks like... sort like a thorn on a rosebush" (James), "like the tip of a triangle" (Susan), and like "the pointing top of a triangle" (Mike).

Angle as turning bodies

Students used several ways to refer to the relation between angles and turns: "angles turn", "angles have turns", "angles are turns", "angles and turns are the same", and some of them explicitly identified angles with a rotating line. Most of these usages of the notion of turn underline variations of the model of an angle as a turning body. More specifically, an original image-schematic model of a turning body was projected on the model of angles as turning bodies.

The original model is an image-schematic model composed of a body with an orientation, a trajectory, which have a starting and a stopping point. This model is associated with our bodily experiences of turning, as in *turn around, turn left/right*, and the act of turning so as to face a different direction. These experiences are not associated with an endless movement, because our body rotates on a trajectory which has a starting position and a stopping position.

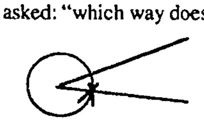


There are certain kinds of turns that we perform more often than other, like *turn around* or *turn left/right* or *turn all the way around*. Those turns are much more used and are good candidates to constitute cognitive reference points. In fact it is difficult to find common words in English to express what, in a mathematized language would be, for example, a *45° turn to your left.*

Angles are thought by some students as metaphoric projections of turning bodies. Ten students said that angles turn. Sometimes they identified angles with turns. Bob, for example, sees "no difference" between angles and turns, Jessie says that "a turn is the same as [an] angle", and Angela states that turns and angles "sort of are alike, because an angle it can turn". Louise said that angles can "turn bigger or wider". James several times compared angles by saying that one turned more than the other. Louise explained that "turn around" is an angle:

> There is different kinds of angles and one of them is turn around. You can take it and [split it into] two parts of a square. It is just one line and the other two connect it. And you can fit a corner of a paper into it. That's how you turn around. (...) Turn around is just the corner of a piece of paper.

Some students associated some gestures with this model. Marie, for example, when asked to show points in the interior of an acute angle asked: "which way does it turn, this way or this way? [traces]":



The metonymic structure of the image-schematic model of a turning body endows the metaphoric model of angles as a turning body with a metonymic structure. We have seen above how Louise describes the special angle she called "turn around". In another example, Jessie, when referring to an obtuse angle, says that "[the obtuse angle] is almost a finished turn". Previously she said that the same obtuse angle "is almost half of a whole turn". The finished turn here seems to be a 180° turn, and the obtuse angle is compared with it. Both the "finished turn" and the "whole turn" are used here as cognitive reference points.

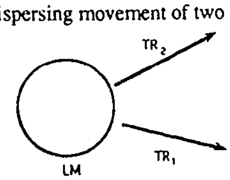
Some students associated angles and turns with a circle. They are usually using the turning body model. Alice, for example, was able to provide an extensive explanation of this association:

> A circle that's bigger than all of the angles because there is a full turn [traces a circle on the table, then makes half a turn with the pencil]. (...) A circle is really an angle because if it's like you started [at] one point [points to the edge of the pencil]. This is the point you started [points to the other endpoint of the pencil]. (...) You started here [points to the edge of the pencil]. (...) You can turn the pencil all the way around [rotates the pencil 180° to the left]. Then you can start there. There would be a angle. Then you keep turning. That would be a angle, that would be a angle [indicates successive angles as she rotates the pencil around an endpoint]. And that would be the end where you started at [ends the rotation of the pencil by moving it to its initial position]. And that would be a circle.

Alice goes one step further and actually uses the turning body model as a way to imagine the sphere as an angle: "in a way it's a angle, because it starts and it turns around [traces a maximum circle of the sphere]. It starts as an angle but then it goes around. So in a way I would say this is a angle". Hill uses this model to reject the possibility of curved sides in an angle saying that the drawing is "not a line that turns". In fact, in this model the sides of an angle are conserved by a rotation, and so they cannot became curved. This model is not useful to interpret angles having curved sides, and no student used this model to explain his/her acceptance of angles with curved sides.

Angle as a source

Angles are thought by some students as sources of trajectories. This is an image-schematic model that comprises a landmark (LM) and two trajectories (TR). Both trajectories are straight, start at the landmark, go on for an indefinite distance, and are indistinct from each other. In this model, the vertex is associated with the landmark and its sides with the two trajectories. It is a dynamic model, because angles are thought as a source of a dispersing movement of two lines.



When asked why she thought there was an angle in a triangle, Alice answered: "really it starts at a point [points to the vertex of the angle] (...) and it has a certain way that it goes [traces the sides of the angle]." Alice is identifying angles with a certain kind of path of which she indicates both the origin and the trajectory. Hill expresses the same idea: "[angles] have a starting point like a ray [traces one side of an angle], and then they go on and on. You can make the angle like a ray". Sometimes both Alice and Hill referred only to some elements of the model. The idea that the vertex is the source of something can be found in Alice. In several instances she mentions that angles "start at a point", or "have a starting point". Hill describes the trajectory as the extension of two lines.

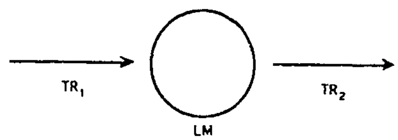
There are certain metaphors that students use that are related with this model. Hill, for example, believes a ray is similar to an angle:

> Because if you have a ray [draws a ray] you have a point [points to the endpoint of the ray] that goes on an on, but they might get it confused for an angle because it has a point like an angle, a vertex, not a vertex because a vertex has... it's where two lines meet, but a point to where a line goes on and on, never stops does not go away, as a, sort like a flashlight.

Alice proposes a different metaphor. She says that an angle "is like, if you're walking straight, if you start from a point on the playground and now you can keep walking for a long time".

Angle as a path

Alice uses a model of angle as a path, which is a metaphoric model of an image-schematic model of a special kind of path. This image-schematic model is composed of two trajectories and a landmark. The landmark is the endpoint of one trajectory and the starting point of another. This is a dynamic model and the switch from one trajectory to the other is performed instantly. It also presupposes that the two trajectories do not have the same direction.



A good example is provided by Alice:

> [An angle] would always have a point [and she gestures].

> You always... it would always have a point, and... A corner, it doesn't always... Sometimes it can curve (and she gestures]

As Alice puts it the easiest way to draw an angle is "just look at how does the lines curve" at the same time that she makes a gesture similar to the one above. She is talking here about a continuous path that comes along a line, and at a certain point makes a sudden curve and continuous along another line. This curve is so sudden that produces a point. At a later time she even stresses this instantaneous change of direction by drawing a small curve around the vertex of an angle. Alice uses the path model to explain why the sides of the angles are straight.

> [It is not an angle] because this curves [traces one "curved" side back and forth] and angles mostly, don't curve. It's like one line goes straight and then it curves like that [draws a semicircle]. But a angle doesn't curve, when it goes [draws a line segment].

In this quotation we can see that the proposed angle is composed of a "line" that "goes straight" but that "curves" at some point. Alice's problem with this is that angles don't "curve" as they go.

Alice's preference for angles as paths leads her to easily associate angles and circles:

A circle is really an angle because if it's like you started [at] one point [places a pencil on the table and points to the edge]. (...) You started here [points to the other end of the pencil]. It is like you can turn the pencil all the way around [rotates the pencil 180° centered at the endpoint opposite to the edge]. Then you can start there, there would be a angle. Then you keep turning. That would be a angle, that would be a angle [successively indicates several intermediate positions of the rotating pencil]. And that would be the end where you started at [ends the rotation of the pencil by moving it to its initial position]. And that would be a circle.

Hill also uses this model as a basis to develop a metaphor for angles. An angle can be seen as a path that people follow along a line. Then they stop at a corner (the vertex) and then they turn and continue in another direction.

A right angle [traces one side of a right angle], when it comes as a vertex toward sort of like a stopping point (...) it makes a turn go up, just like a sidewalk on a corner. People come this way, and then they stop and come this way, and then they have sort of a stopping point [points to the vertex of a right angle] before they turn, or they can come this way [traces the other side of the right angle away from the vertex].

Angles as two connecting lines

In this model angle is thought as two intersecting lines. It is an image-schematic model having three components: two lines and a point. The lines themselves have one endpoint each, and these endpoints coincide with the point of the model.

Jessie used this model extensively as a means to justify the existence of angles. From her statements we can observe that this model is associated with a visual configuration of two intersecting lines and the action of two lines that attempt to connect to each other. When asked why there were angles in a triangle she replied: "right here is connecting the lines, is connecting right there, and they're having a point right here, there and there [points to the vertices]". Later when explaining why she did not identify angles in other triangles she says that "I must have overlooked (...) the lines connecting this and that [points to the vertices]".

In other instances she discussed about angles in terms of connecting lines and even seemed to endow the connecting lines with a will to connect. For example, when explaining what was more difficult to learn about angles she says

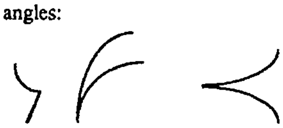
the most difficult thing, is when you try to have (...) parallel lines [draws two parallel lines with a curved portion on one end so that the lines intersect] (...) and get them confused when, you know, the lines that have a line, you know, the lines that keep going, and then the lines that connect when they keep going so they are curved, it connects. I think it's the most difficult thing about learning angles.

She latter repeats this same drawing:

[I am talking about] the ones that connects when they're still going [draws again two parallel lines with a curved portion on one end so that the lines intersect] , like when they start going like that [emphasizes the small curved piece she drew].

This is a type of image schematic model that excludes configurations in which lines do not intersect visibly or that do not contain two distinguishable lines. Jessie does not identify angles in the following configurations:

but does identify the following as angles:



Conclusion

Mathematical concepts were thought for a long time as abstract entities endowed with definitions that would make them impervious to the real world. Moreover, any attempt to use real world entities to interpret mathematical objects was thought as a dangerous contamination weakening the search for abstraction. The models described above present a different picture. They reveal students making use of their experiences, condensed as cognitive models, to interpret a new concept. These previous experiences are very important as tools that we can use to reason mathematically. We have seen, for example, how Alice uses the metaphor of angles as turning bodies to think creatively about the links between angles, circles, and spheres. There are other models that were not presented here (angle as an interior corner, another model of angle as a source, angle as a meeting point, and angle as an opening object). All these models together compose a picture of the diversity and complexity of the mental representations of a mathematical concept and the ways in which we use our imagination as a powerful mathematical tool.

References

Johnson, M. (1987). *The body in the mind. The bodily basis of meaning, imagination, and reason.* Chicago: University of Chicago Press.

Lakoff, G. (1987). *Women, fire, and dangerous things. What categories reveal about the mind.* Chicago: University of Chicago Press.

Matos, J. M. (1991). *Changes in the van Hiele theory to accommodate prototype effects originating from image-schematic, metaphoric, and metonymic models.* Paper presented at the Fifth International Conference on Theory of Mathematics Education (TME-5), Paderno del Grappa, Italy.

Matos, J. M. (1992). Cognitive models in geometry. In J. Ponte, J. F. Matos, J. M. Matos, & D. Fernandes (Eds.), *Mathematical problem solving and information technologies: Research in contexts of practice* (pp. 93-112). Berlin: Springer.

Strauss, A. (1987). *Qualitative analysis for social scientists.* Cambridge, England: University Press.

ON THE UTILIZATION OF NON-STANDARD REPRESENTATIONS IN GEOMETRICAL PROBLEMS*

Ana L. MESQUITA

Universités de Lille and Universidade de Lisboa

This paper presents some results of a study on the utilization of non-standard figures in geometrical problems. Two figures and two statements of problem were used by 7th graders in a French junior high-school. Contrasting results of pupils's answers from questionnaires, it concluded that this kind of representation can foster pupils' high-level geometrical procedures, such as encoding, control and correct utilization of figures and geometrical properties.

This paper is concerned with external representation in geometry and its purpose is to contribute to the understanding of how pupils use representation in geometrical problems. Before presenting our study, we will clarify some theoretical framework and assumptions about representation in geometry.

Theoretical framework

1. Double status of a representation. External representation doesn't have always the same status in geometrical problems. In previous studies, we showed the importance of what we called the **double status** of a representation (Mesquita, 1989, 1992a), which is related to the status of geometrical objects introduced by Husserl (1936/62). In fact, in representing a concept or a situation in geometry, the material trace, or drawing, of these ones can suggest two different possibilities or status:

a) a "finiteness" in the sense of a finite and varied form in its spatio-temporality;

b) a geometrical form in its "ideal objectiveness", detached from the material constraints linked to external representation.

To Husserl, a geometrical object has one of this status. But the status of a represented object is less clear. For instance, the trace of a piece of a straight line can be considered, depending on the situation, the representation of a : a) straight line, b) semi-line, c) segment of a given length, d)

* This paper reports some results from an analytical study on the utilization of external representation in geometry, developped in a cooperative project with C. Janvier, CIRADE/Université du Québec à Montréal.

segment of an arbitrary length or e) vector. It means that a same figure[1] can represent either an abstract geometrical object or concept, either a particular concretization of this one. Depending on the problem, in some cases we are interested in the first situation, in some others, in the second one. Besides, in some cases, one particular characteristics "stands for" all the situations, in others doesn't. In the basis of this ambiguity, is the fact that the only figurative register doesn't enable to distinguish between these two cases. To avoid this situation, conventions are used; but even in this case, in many situations status are not easy to distinguish; it's the case in the distinction of a given or an arbitrary length segment, for instance.

From a mathematical point of view, external representation is considered as a kind of ideal objectiveness, in connexion with the properties underlying it. Mathematicians, as it's known, use a representation as a network of geometrical relationships between elements. From a didactica' point of view, pupils may identify acessory relationships, depending on the constraints of the concretization, and relationships induced by perceptive properties. In consequence, pupils tend to see an external representation as a finiteness, while experts see it as an ideal objectiveness. Therefore, the distinction between these two status may appear ambiguous to pupils (Mesquita, 1989, 1992a,1992b).

It's well-known that this phenomenon is not specific of geometry: the word "number" can suggest a particular number or a generic one. But, contrarily to that happens with representation in geometry, the contribution of symbolic language enable a clarification of the situation: we can use symbols and notations to distinguish them; for instance, in algebra, a letter is in general used to represent an unknown quantity (variable or parameter), whereas ciphers will represent a particular quantity.

2. Modes of representation. Depending on the properties considered, different modes of representation are usually considered: topological, projective, affine, or metrical (Piaget et al., 1947/72, Pallascio et al., 1992). But, in spite of the psychological genesis of the geometrical notions, the universe of pupils in school remains metrical. Therefore, in this paper 'geometry' will refer to metrical geometry, unless a mention in contrary will be made.

3. Nature of representation. Depending on the criteria utilized in the representation, different properties can be considered in the representation. In this sense, the nature of an external representation is not always the same; it determines the type of treatment allowed. Note that the properties that can be extracted from the figure depends on the properties conserved on the representation. In some cases, geometrical relationships utilized to the construction of the

---

1 In this paper, we use the term 'figure' as a synonym of external (materialized on a support, paper or other, by opposition to mental, or internal), and iconical (or figurative, centered on visual image, by opposition to other possible semiotic systems) representation of a concept or a situation in geometry.

represention can be reutilized: we say that the external represention has the nature of an **object.** It's the case, for instance, in the problem: "The following figure is formed by three demi-cercles, each pair of demi-cercles having in common two extremities of diameters. Compare the lengths of a) C, b) C_1 followed by C_2".

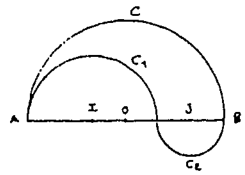


Figure 1

The geometrical relationships used in the construction of the figure may be used in geometrical reasoning. But in some other cases, it's not possible to extract directly geometrical relationships; in these cases, the external represention appears as a kind of topological or projective diagram, from which we can't extract directly geometrical properties. In this case we say the represention has the nature of an **illustration.** For example, in the following situation (where under the assomptions that **1** is an equilateral triangle, **2** is a rectangle, **3**, **4** and the figure formed by **3**, **4**, **5** are squares, we wish to proof the equality of **AC** and **LF**), figure 2 is an illustration:

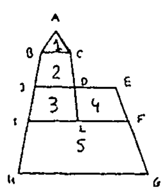


Figure 2

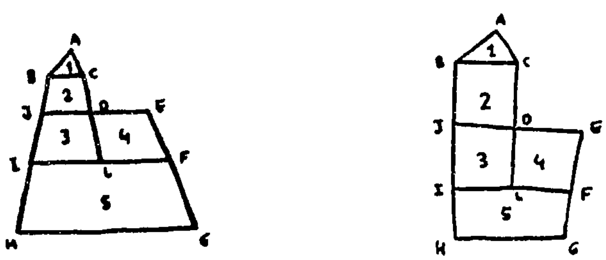
In the case of illustration, any metrical relationship can be extracted from the figure. These differences between the nature - object or illustration- of the external representation are not automatically understood by pupils, which use properties that figure is supposed to have, suggested by its iconical appearance. By this reason, the nature of the external representation may be an obstacle to young pupils' geometrical reasoning: it's possible that there are no coincidence between the nature of the figure associated with the problem and its interpretation by pupils. By other side, textbooks, using almost exclusively objects as external representation, reinforce this trend. Therefore, pupils are used to deal with figures with a nature of "object" and they use easily the properties given by perception.

The distinction between what is admissible or not is connected with the understanding of the nature of the external representation and consequently helps pupils to use external representation as mathematicians do it, i.e., as a network of properties given by the statement. In this sense, working with representations of different natures can help pupils to consider a representation as a translation in another register of the properties given by statements. Encoding, for instance, can contribute to the understanding of the nature and of the limitations of the external representation.

Definition of the problem and methodology

To analyse the utilization of an external representation of a geometrical problem, we proposed a questionnaire to each of the ninety-nine seventh graders, 12-13 years-old, from a French junior high-school. Two different contexts of problem statement and two different representations were considered. In both cases, figures have the nature of an illustration, where projective properties were used. Our main goal is to analyse the procedures of pupils' resolution; in particular, we were interested in contrasting the effects of the different versions of external representations and of statements on the pupils' answers. A second goal concerns the utilization of encoding and of its effect in pupils' procedures: where was encoding more used? and more efective? is encoding associated with correct answers? in which situations?

We have used a geometrical problem concerning properties of basic concepts in geometry -squares, rectangles and triangles- and the transitive property. We used two external representations of a figure - figures 3.a and 3.b, which were an adaptation of a situation studied in Mesquita (1989)- called in the following the strongly (figure 3.a) and the slightly version (figure 3.b), and two statements of the problem, a mathematical one, classroom type, and a "context" problem, where it's supposed to resolve a daily situation.



a b

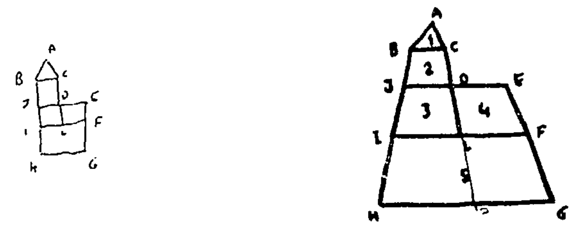
Figure 3

In the four questionnaires resulting of the crossing of the two mentioned criteria, we use a hand-made external representation of the problem. In all the cases, the representation was not congruent with the text, in the sense of Duval (1989a), i.e., the information given by the statement was not concordant with the one given by the figure.

Results

1. On the diversity of treatments called by representation

Use of figural modifications. Fifteen pupils used modifications of the figure. Modifications used are in general optical ones (Duval, 1989b), i.e., resulting from a change of point of view on the figure. Four pupils used mereological transformations (ibid.), i.e., they drew complementary strokes on the figure.These pupils drew another representation, by an optical transformation of the first one, transforming the given representation, non-congruent with the statement, in a new one, congruent, by an alteration of the nature of the proposed figure, using an object instead an illustration. An example of optical modification, done by pupils, is shown in figure 4.a, while figure 4.b shows a mereological modification. These transformations are analogous to the dynami imagery considered by Brown and Presmeg (1993).

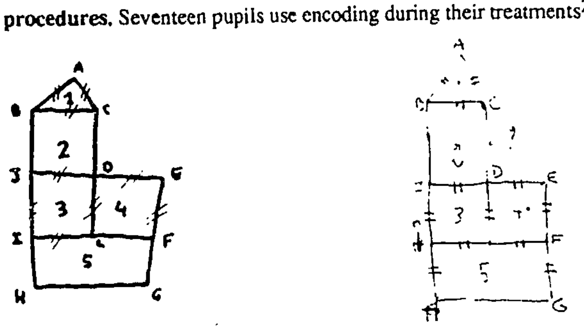


a b

Figure 4

The procedure of transforming a figure seems associated with correct answers. Optical transformations of the external representation are present in 23% of the correct answers.

Use of encoding procedures. Seventeen pupils use encoding during their treatments[2]. In general,



a b

Figure 5

2 Five pupils used encoding and transformations of the figure.

answers associated with encoding procedures are correct ones; more precisely, encoding is present in 30 % of the correct answers. Several types of codes indicating equality were used by pupils: slashes, numbers, colours (figure 5).

Measure and proportionnality treatments. These treatments are present in 17 copies of pupils, which means that 42% of the incorrect answers are due to these treatments. More precisely, to 2% of the pupils, measures are the unique treatment observed; to 15% of the pupils, figure is used exclusively as a support to proportionnality computations.

Utilization of the figure. Analysing the answers of the pupils, we constated that, to 78% of the pupils, properties are used, in an articulation (more or less correct) with the figure; for 22% of the pupils, figure appears to be the exclusive support of reasoning and the properties seem to be ignored by them.

2. Comparison between versions of questionnaires

◊ **"math" vs. "context"**: correct answers are more frequent in "math" copies than in "context" copies (correct answers in both cases represent 69% against 46% of the total answers, resp.);

* encoding procedures : they were used with the same frequency in both cases;
* measure and proportionnality treatments : they were used with the same frequency in both cases.

Note that 58% of the pupils gave correct answers, all modalities considered.

◊ **slightly deformed figure vs. strongly deformed figure**: correct answers are almost equally shared between these two modalities.

* encoding procedures : they were used with the same frequency in both cases;
* measure and proportionnality treatments : they were used with the same frequency in both cases;
* transformation: they were used more frequently in the strongly deformed version: 9 of the fifteen pupils using transformations do it in a strongly deformed version, whereas 6 do it in the slightly deformed version;
* utilization of the properties: correct and systematic utilization of the properties is much more frequent in the strongly deformed version than in the other: 19 pupils made correct and systematic utilization of the properties in the strongly deformed version, while 4 pupils do the same in the slightly deformed version.

We explain this fact as following: in the strongly deformed figure, the effort that is necessary to interpret correctly the nature of illustration of the figure is bigger. In this case, to discover a solution exigs a correct association between properties mentioned by the text and the ones given by the figure. By this fact, the control procedures must be increased. In doing this, pupils discover the solution, understanding why they give that answer. For this reason, it is easier for them to explain their reasoning. Whereas, in the slightly deformed figure, the resolution seems obvious and implicit, control procedures are less important, so the explanation becomes poorer.

Conclusions and discussion

This study shows how pupils with same age treat the external representation in very different ways. The external representation appears as a finiteness, to the two pupils, to which measures are the unique treatment observed. To others, the external representation appears as a proportionnality schema, where ratio is conserved, as a kind of equivalence class. It's the case of the 15 pupils to which figure acts as a support to proportionnality computations. To others, to the 23 pupils to which figure properties are used in a correct articulation with the figure, the external representation appears as a kind of diagram from which only incidence properties can be issued, all the other geometrical properties are issued from the statement. Only in this case it is possible to refer to an ideal objectiveness.

The influence of the context seems to have a slight effect on the performance of pupils, rather than on their utilization of external representations. "Context" copies seem to appear slightly more difficult than "math" copies. However treatments (encoding, modifications, correct utilization of properties) are equally present in both modalities. We explain this in the following way: pupils are used to mathematical problems rather then to context problems. Familiarity to this kind of problems makes pupils aware of the implicite rules used in scholar problems; therefore, they become more performant on them.

This study makes appear the importance of the utilization of external representations with different natures. Figures have a central but hidden part on the learning of geometry and; however, they are not studied by themselves. Also, they are used in a symbolic way, but the rules of this symbolism remain implicit. An effective importance should be given to these subjects in the learning of geometry.

Figures with a nature of illustration seem appropriate to minimize the bias introduced by perception. Moreover, we have seen that the utilization of representation with a nature of illustration can stimulate pupils to use high-level procedures such as control and encoding, which appears to be a tool to understand an external representation as a network of properties. In this situation, encoding and control procedures appear easily and in a natural way. These behaviours can foster pupils to understand geometry and rules of its reasoning.

References

BROWN D. L. & PRESMEG N.C. (1993) Types of imagery used by elementary and secondary school students in mathematical reasoning, *Proceedings of the 17th PME Annual Conference*

(Tsukuba, Japan, July 18-23), vol.II, 137-144.

DUVAL R. (1988a) Ecarts sémantiques et cohérence mathématique, *Annales de Didactique et de Sciences Cognitives*, *1*, 7-25.

DUVAL R. (1988b) Pour une approche cognitive des problèmes de géométrie en termes de congruence, *Annales de Didactique et de Sciences Cognitives*, *1*, 57-74.

HUSSERL E. (1936 /1962) *L'origine de la géométrie*, PUF, Paris.

MESQUITA A.L. (1989a) *L'influence des aspects figuratifs dans l'argumentation des élèves en géométrie : Eléments pour une typologie*, Thèse de Doctorat, Université Louis Pasteur, Strasbourg.

MESQUITA A.L. (1989b) Sur une situation d'éveil à la déduction en géométrie, *Educational Studies in Mathematics*, *20*, 55-77.

MESQUITA A.L. (1992a) Les types d'appréhension en géométrie spatiale: Une étude clinique sur le développement-plan du cube, in *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (edited by W. Geeslin & K. Graham), New Hampshire, USA, August 7-11, vol.II, pp. 104-111.

MESQUITA A.L. (1992b) The types of apprehension in spatial geometry: sketch of a research. Proceedings of the International Symposium on Spatial Representation and Spatiai Skills (Montreal , August 28-30), in *Structural Topology*, *18*, pp.19-30.

PALLASCIO R., ALLAIRE R. & MONGEAU P., Spatial representation and the teaching of geometry, Proceedings of the International Symposium on Spatial Representation and Spatial Skills (Montreal , August 28-30), in *Structural Topology*, *19*, pp. 71-82.

PIAGET J., INHELDER B. (1947/1972) La représentation de l'espace chez l'enfant, PUF, Paris.

Construction of the Limit Concept with a Computer Algebra System

John Monaghan
School of Education
Nottingham University
Nottingham NG7 2RD, UK
tezjm@ten2.nott.ac.uk

Shyashiow Sun
Dept of Electronic Engineering
National Institute of Technology
Taipei, Taiwan (106)
Republic of China

David Tall
Mathematics Education Research Centre
University of Warwick
Coventry CV4 7AL, UK

Abstract *Students encounter many cognitive difficulties with limit ideas: sequences never end; functions do not attain their limits; series do not produce a final answer. Limit is further both a process and an object and students usually focus on the process. Studies investigating these difficulties are noted before presenting some results from a new study that examined the limit conceptions of students who learnt calculus concepts with the aid of a computer algebra system. Differences with traditional approaches emerge in that process problems are suppressed and the limit as an object appears clearer but this brings its own problems. Limit is a deep notion and each approach highlights and suppresses different facets of the concept.*

Conceptual difficulties with limits

The limit concept is known to cause difficulty in learning. A number of research studies reveal that students often conceive of the notion of $\lim_{x\to a} f(x)$ or $\lim_{n\to\infty} a_n$ not as a static concept but as a dynamic process of 'getting close to' a fixed value, often with the implication of 'never reaching' the limit (see summaries in Cornu (1991) and Tall (1992)).

Gray & Tall (1993) considered a wide variety of instances where a symbol can ambiguously represent either a process or a concept. They call this a procept. For instance 3+2 might evoke a process of addition, perhaps by counting on two, or the concept of sum. The symbols $\lim_{x\to a} f(x)$ and $\lim_{n\to\infty} a_n$ both represent either the process of getting close to a specific value, or the value of the limit itself. The limit is therefore an example of a procept. But unlike the procepts of elementary mathematics, which have a built-in algorithm to calculate the specific value of the concept, the limit value does not have a specific universal algorithm that works in all cases and, in some cases, such as $\lim_{n\to\infty}\sum_{k=1}^{n} 1/k^2$, there may be no simple algorithm. (In this instance, the theory of residues in complex integration or a sequence of key strokes on a computer algebra system shows the limit to be $\pi^2/6$.) The circuitous routes by which limits are calculated in the early stages of the theory exacerbate the difficulties students have with the concept. As Cornu (1981) observed: "mathematics no longer reduces to calculations and simple algebraic properties; infinity intervenes and it is shrouded in mystery".

Monaghan (1986) studied the growth of 16/17 year old students' conceptualisations of real number, limit and infinity over one year as the experimental group studied traditional calculus and a matched control group studied other subjects. The students' fundamental concepts of infinity and limits hardly changed in the period concerned. Their notion of real number showed them happy to manipulate whole numbers, fractions and such numbers as $\sqrt{2}$ and π, but they became less secure when attempting to deal with infinite decimals. The latter were regarded as being 'improper' and described as 'infinite numbers'. An expression like $\sqrt{2}=1.414\ldots$ does not say that "$\sqrt{2}$ can be computed exactly as (the limit of) a decimal expansion", rather that "$\sqrt{2}$ can be described to any required accuracy by approximating to a specific number of decimal places". Thus the number line is viewed as consisting of positive and negative whole numbers and fractions and combinations of other expressions including $\sqrt{2}$, π, etc., together with a more peculiar set of 'improper numbers'. In practice, computations can be carried out with finite decimal approximations but this gives a perception that such arithmetic is inherently inaccurate.

To build a precise theory of limits on such a foundation is bound to contain the seeds of cognitive conflict and sow confusion in the students' minds. A traditional approach to the limit in such circumstances is fraught with difficulty, so much so that Davis & Vinner (1986) suggested that there are *unavoidable* obstacles which the student must confront when beginning to study the topic.

Approaching the limit by functional programming

Prior to the arrival of the computer the introduction of the limit concept required the student to have considerable experience of the limit as process, so that the latter is unavoidably embedded deeply in the student's psyche. Computer software can now evaluate many limits, so the possibility arises that it may allow the curriculum to give a more balanced view of limit as concept and process by early focus on the limit as concept with the computer carrying out the process internally.

Li & Tall (1993) investigated an approach using programming in structured BASIC which allows definitions of named functions. This allowed a function to be considered either as a procedure of computation, or as an object whose name could be used in building other functions. The course was largely successful in giving a proceptual view of function as process or object and, in defining a series as a function adding up the terms of a sequence, it was able to help students discriminate between sequences and series. But it was not successful in moving from a view of limit (of a sequence) as a process to a limit as a concept.

This failure to encapsulate limit as a concept in the majority of students was predictable in hindsight. The numerical basis on which it was built was computer arithmetic with numbers stored to approximately 8 significant figures. The experience was therefore built on a foundation of limited numerical accuracy. This was built in to the experience by computing numerical values of sequences s(n) for large values of n and looking for the values stabilising to a given accuracy. The overt message was that, to this given accuracy, from some term $s(N)$ on, the terms stabilised to a fixed value, thus for $n \geq N$, the terms $s(n)$ became indistinguishable. This was used in class to 'motivate' the idea that the greater accuracy required, the greater the value of N beyond which stabilisation would occur, leading to the ε-N definition of limit. This approach should lead to the notion of Cauchy convergence, but after two weeks the students had forgotten most of the discussion and mainly conceived of the limit notion in a dynamic sense.

We conjecture that any computing experience intended to 'motivate' the limit notion by computing limits using approximate arithmetic may be fraught with this underlying problem. As the limit is found by a *process* of computing values of $s(n)$ for larger and larger n, it will implicitly confirm the students' belief that the limit is a *process* not a concept. Dubinsky (1992, 1993) proposes a theory of encapsulation of process as object and uses the language *ISETL* for programming functions as procedures which can then be conceived as objects and used as inputs to other procedures. This computer language is better structured for mathematical purposes than BASIC and is a good environment for conceptualising mathematical thinking in a wide variety of ways. Although it includes rational arithmetic, its fundamental numerical mode appropriate for the calculus is floating point arithmetic. It has no facility for computing the limit as a precise value, so it too is flawed as an environment for the limit concept.

Using a computer algebra system

Given the student's perception of 'proper' numbers, an environment which might prove suitable for exploring the limit concept is a computer algebra system such as *Maple*, *Mathematica* or *Derive*. These allow manipulation of 'proper' numbers such as rational numbers and rational expressions in $\sqrt{2}$, π, e, etc. and do not simplify these expressions to approximate answers unless explicitly instructed to do so. All allow programming to a greater or lesser degree .

Tall (1993) suggests that the computer relieves the learner of the tyranny of having to encapsulate the process before obtaining a sense of the properties of the object. By using software which carries out the process internally, it becomes possible for the learner to explore the properties of the object produced by the process before, at the same time, or after studying the process itself. This new flexibility in curriculum development which gives new possibilities in the order in which the concepts are constructed is called *the principle of selective construction*.

Sun (1993) and Monaghan researched the effects on the limit concept of using the software *Derive* freely in the early stages. This computer algebra system was selected because it is available on a hand-held computer and the facilities are easily available through the use of simple menus. For instance, the sequence of actions to find $\lim_{x\to\infty} \frac{2x+5}{3x+4}$ is shown below. By a routine sequence of key strokes the student can move from the original expression to obtain the value of the limit in the form $\frac{2}{3}$.

1: $\dfrac{2x+5}{3x+4}$

2: $\lim\limits_{x\to\infty} \dfrac{2x+5}{3x+4}$

3: $\dfrac{2}{3}$

```
CALCULUS LIMIT: Point: INF                    From:(Both)Left Right

Enter limit point
User                                          Free:100%          Derive Algebra
```

This is analogous to a child carrying out an arithmetic operation by a sequence of key-strokes. It therefore behaves in a manner familiar to the student and, in one sense, is less "shrouded in mystery" than the traditional dynamic approach to limit. In another sense, however, the internal process by which the computer carries out the process remains mysterious. But, just as someone who knows what a square root *is*, but not how to calculate one, may be satisfied that the square root key gives a satisfactory approximation to $\sqrt{2}$, so the student may give some meaning to the result by other means. For instance, by computing values of the expression for large values of x, or dividing numerator and denominator by x and noting that, as x gets large, so $1/x$ gets small.

Selectively, therefore, the student may focus on the production of the limit object (using the computer) or on the limit process (by a paper and pencil or computer calculation). Therefore the student can see the two complementary facets of the limit procept as concept and process, in whichever order is desired.

The Experiment

The students in the study were able 16/17 year olds at the end of their first year, of two, Further Mathematics Advanced level course. Advanced level mathematics is open to the highest attaining quartile of 16 year olds and covers most of the differential and integral calculus of a single variable. Further Mathematics is taken by able and motivated students within this population.

The experiment was motivated by the access of the first two authors to a group of nine students in a college who had made extensive use of the computer algebra system *Derive* throughout their studies: 50% of lessons in rooms where *Derive* was 'on call' at desktop machines and for two months they were given palmtop computers fitted with *Derive* which they could use at anytime. We shall call these the *Derive* group. The two authors were intrigued as to the possible effects of this exposure on students' limit conceptions and a comparison group was found. Three schools provided 19 students with similar backgrounds who were following an identical curriculum but who had not met a computer algebra system.

One of these schools, which we shall refer to as school A and which accounted for seven students in the comparison group, had students who were very closely matched to the students in the *Derive* group.

A questionnaire was designed to elicit students' conceptions of the limit of a sequence, a function (graphically and algebraically) and of a numeric series. In addition discontinuity was incorporated into one question and the definition of a derivative was examined. The questions drew upon the work of Li (1992) and Monaghan (1986). The *Derive* group were free to use *Derive* on their palmtop computers in the questionnaire. Within two weeks of students completing the questionnaire they were interviewed. 25 of the 28 students were interviewed including all those in the *Derive* group and school A. Interviews lasted for about 20 minutes and were designed to probe reasons behind specific responses.

The Results We report on responses to three of the questions.

Q1 Please find the following limits if they exist. If there is no limit, then write 'no'. Please explain your results.

$$\lim_{x\to\infty} \frac{2x+3}{x+2} \qquad \lim_{x\to 1} f(x), \text{ where } f(x) = \begin{cases} 3 & \text{,if } x = 1 \\ \dfrac{x^2-1}{x-1} & \text{,if } x \neq 1 \end{cases}$$

Eight of the nine *Derive* students used *Derive* to find the first limit and claimed they did not know any other method. The exception was the one student who 'did not like' the computer system. Of the 19 non-*Derive* students 12 divided and then used the concept that $\frac{1}{x}$ approaches 0 as x approaches ∞. Three substituted numbers and four left the question blank.

The second limit with the discontinuity is comparatively difficult and was novel to all students. There was greater diversity in responses. Nevertheless six of the nine *Derive* students simply used *Derive* to find the limit of the major part of the function and ignored the discontinuity. Of the 19 non-*Derive* students, 11 gave diverse answers that considered the discontinuity, six left it blank and only two ignored the discontinuity. Lack of space prevents a full analysis but the point we focus on here is using *Derive* as a button-pushing process that can obscure deeper consideration of the function.

Q2 a Can you add $0.1 + 0.01 + 0.001 + \ldots$ and get an answer? Why?
b Can you add $0.9 + 0.09 + 0.009 + \ldots$ and get an answer? Why?

Although these may appear to many as innocuous questions with easy answers, they cause great conceptual problems to students and to those who have not pursued pure mathematics to any depth. The reason, as documented at length in Monaghan (1986), is that they never end and so you never get an answer – they are always in a state of becoming.

Only two of the 28 students distinguished between the cases, both stating 'no' to the first and 'yes' to the second because $0.\dot{9}$ rounds up to 1. We shall thus illustrate differences between the groups by using the first question. 7/9 of the *Derive* group indicated that there is an answer, three using the formula for the sum of a geometric progression and 4 stating it as $0.\dot{1}$. The general view in the non-*Derive* group was there may be an answer but there were problems. For example school A students gave similar responses to the *Derive* group with 2/7 using the formula for the sum of a geometric progression and 5/7 stating it was $0.\dot{1}$ but all with qualifications that this was only an approximation or that the answer tended to this. Interviews, however, revealed some similar thought behind the *Derive* group's 'object' answers. "First I though 'no' because it just goes on forever and ever. Then I checked it on *Derive*. I did get an answer."

Q3 Please explain the meaning of $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

It was clear from the questionnaire results that only school A and the *Derive* group had properly considered this notation in their mathematics lessons. We thus compare these two groups. All the students in school A gave satisfactory theoretical explanations of the expression but none gave any examples. However, none of the *Derive* group gave theoretical explanations and only two students mentioned the words 'gradient' or 'differentiate'. Four of the *Derive* group gave examples. They replaced f(x) with a polynomial and performed or described the sequence of key strokes that calculated the limit.

Discussion

Different proceptual ideas permeate all three questions. The 'find the limit' questions reveal that *Derive* generates a specific process for computing limits:

- select [Author] and type in the expression
- select [Calculus], then [Limit]
- specify the variable (e.g. **x**)
- specify the limiting value of the variable (e.g. **inf**)
- [Simplify] the result

As mentioned, this is analogous to the processes younger children use in carrying out arithmetic operations on a calculator. This has the advantage that it allows the students to focus on the limit object by suppressing problematic notions of infinity and 'getting closer'. The other side of the coin is that we do not want students to ignore 'closeness' ideas and the fact that seven of the nine *Derive* group students ignored the discontinuity in the second part of the question suggests that this is happening.

The series questions indicates that some students are beginning to see the limit sum as an object. A possible explanation for this is that when summing the series using pencil and paper or, indeed, programming in BASIC such as Li and Tall (1993) did, the mind concentrates on the process of summing, this takes time and time becomes a factor in students' conceptions. However, when a student uses *Derive* to perform the summation, the mind is freed to concentrate on the outcome, the object.

The question concerning the definition of the derivative reveals an 'action schemata' in some of the *Derive* group: to define the derivative from first principles is to produce a sequence of key strokes as outlined above. Why are specific examples used? Is this a case of students seeing generalities only via specific examples? We believe not, but that it is only in the context of a specific example that the key strokes make sense for otherwise the key strokes would merely replicate the notation in the question. Again there are dually positive and negative aspects of this approach in that while an object is produced many of the finer points (discontinuities, stabilization, etc.) are ignored.

Comparing these focuses with those produced in a programming environment reveals some marked differences. The focus on a sequence of key strokes was not apparent in the work of Li and Tall (1993). The object of the process is stronger in the computer algebra environment. In Li and Tall's work the stabilization of a sequence was a focus of student thought but this was not the case in the computer algebra environment.

It appears that the programming approach, with its emphasis on the value of finite terms and closeness, is more like the paper and pencil approach than the computer algebra approach. This implies that the reality of approaches is much more complex than simply computer approaches vs non-computer approaches. Li and Tall (1993) posited three limit paradigms: a dynamic limit paradigm; a functional/numeric computer paradigm; the formal ε-N paradigm. To this we add a fourth: the key stroke computer algebra paradigm.

The different focuses of different paradigms has similarities to Schwarzenberger's (1980) claim that calculus cannot be made easy because the real number line is simultaneously complete, an ordered field, a metric space and a normed metric space. "A certain viewpoint may make certain calculations easy but in other directions it may make things more difficult." Limit is a related deep notion. It is not possible to make it 'easy' but, by using the principle of *selective construction*, however, it should be possible to design curriculum materials that exploits the potential of all of these approaches and so gives an improved cognitive base for a flexible proceptual understanding of limit. The curriculum designer, or the student exploring the new ideas, can select which part of the new notion is to be constructed at a given time – the processes, or the resulting concepts and relationships between them.

For instance, a calculator allows the child to perform arithmetic without the process of counting or the use of the standard algorithms. It is therefore possible for the child to concentrate more on the properties of arithmetic than on the procedure of counting in the early years (Doig, 1993). Likewise it may be possible for the student to develop a more balanced view of the limit dually as process and concept by using a computer algebra system which produces a symbolic limit as a 'proper' numerical expression.

References

Cornu, B. (1992) Limits. In: Tall, D. O. (ed.) *Advanced Mathematical Thinking.* Dordrecht: Kluwer, 153–166.

Cornu, B. (1983) *Apprentissage de la notion de limite: conceptions et obstacles.* Thèse de doctorat, Grenoble, France.

Davis, R.B. & Vinner, S. (1986) The Notion of Limit: Some Seemingly Unavoidable Misconception Stages. *Journal of Mathematical Behaviour*, 5(3), 281-303.

Doig, B. (1993) What do children believe about numbers? Some preliminary answers. *Proceedings of PME 17*, Tsukuba, Japan, II, 57–64.

Dubinsky, E. (1992) Reflective Abstraction. In:Tall, D. O. (ed.) *Advanced Mathematical Thinking.* Dordrecht: Kluwer, 95-123.

Dubinsky, E. (1993) Computers in teaching and learning discrete mathematics and abstract algebra. In: Ferguson, D.L. (ed.) *Advanced Technologies in the Teaching of Mathematics and Science.* Berlin: Springer-Verlag, 525–563.

Gray, E. M. & Tall, D. O. (1993) Success and Failure in Mathematics: The Flexible Meaning of Symbols as Process and Concept. *Mathematics Teaching*, **142**, 6–10.

Li, L. (1992) *Studying Functions and Limits Through Programming.* Unpublished M.Sc. thesis, Warwick University, U.K.

Li, L. & Tall, D. O. (1993) Constructing Different Concept Images of Sequences and Limits by Programming. *Proceedings of PME 17*, Tsukuba, Japan, II, 41–48.

Monaghan, J. D. (1986) *Adolescent's Understanding of Limits and Infinity.* Unpublished Ph.D. thesis, Warwick University, U.K.

Schwarzenberger, R.L.E. (1980) Why Calculus Cannot be Made Easy. *Mathematical Gazette*, **64**, 158-166.

Sun, S. (1993) *Students' Understanding of Limits and the Effect of Computer Algebra Systems.* Unpublished M.Ed. thesis, Leeds University, U.K.

Tall, D. O. (1992) The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof. In: Grouws D. A. (ed.) *Handbook of Research on Mathematics Teaching and Learning.* New York: Macmillan, 495–511.

Tall, D. O. (1993) Computer environments for the learning of mathematics. In: Biehler, Scholtz, Sträßer, Winkelmann (eds.) *Mathematics Didactics as a Scientific Discipline – The State of the Art.* Dordrecht: Kluwer.

Reflecting on prospective mathematics teachers' experiences in reflecting about the nature of mathematics

Candida Moreira
Faculdade de Ciências, Universidade do Porto, Portugal

A dawning awareness of the importance of discussing epistemological issues can be seen in the writings of several authors in the recently published *Handbook of Research on Mathematics Teaching and Learning* (Grows, 1992). What has motivated such an interest in the topic? On the one hand, there is the idea that "we teach [mathematics] in a particular manner because we believe something about the nature of mathematics" (Pateman, 1989, p. 12). Dossey (1992) describes the importance of the topic on a larger scale: "perceptions of the nature and role of mathematics held by our society have a major influence on the development of school mathematics curriculum, instruction, and research" (p. 39).

In my view, "a discussion of the nature of mathematics must come to the foreground in mathematics education" (Dossey, 1992, p. 42), because mathematics teachers must know what *is* the subject they teach, and not only to have domain and pedagogical content knowledge. Moreover, helping mathematics teachers to reflect on the nature of the subject with appropriate reference to the foundations of mathematics and the social and cultural context in which it is elaborated may lead to a context of long-term knowledge acquisition and change, a desirable attitude to be nurtured in teachers.

This study questions whether there is a place for epistemological issues in the content of a *Mathematics Methods* course for prospective secondary mathematics teachers and how should these be implemented in such a course. In considering this question, I devised, in the previous academic year, a *Philosophy of Mathematics and Philosophy of Mathematics Education* unit, which I piloted and implemented with both student-teachers and practising teachers (Moreira, 1993). Here, a refined version of that unit is posited and prospective mathematics teachers' interactions with it are re-examined.

This paper addresses an additional issue on behalf of the teacher. As researchers and teacher educators continue to seek ways to improve the process of becoming a mathematics teacher, more attention ought to be paid to their own action and its effect on teachers. In this respect, I share with Elliot (1993) the aspiration of wishing to enlarge the teacher as a researcher movement to teacher educators themselves. Thus, I focus on the design and implementation of the Mathematics Methods course, of which the *Philosophy of Mathematics and Philosophy of Mathematics Education* unit was an important component, and I look back on my intervention as a teacher educator. Hopefully, this will allow me to see things more clearly and look for ways to do them better in the future.

Background

The study setting

This study began in October 1993 at the "Faculdade de Ciências da Universidade do Porto". In the following two sections, a description is provided of both the course structure/theoretical orientation, and the participants in the study. Here, I will focus on the broader context in which the study was carried out.

In line with decades of tradition at the University, mathematics education has maintained a tenuous position as contributory to the initial preparation of mathematics teachers. This consists of a five years degree, four of which are spent at the University. The fifth and final year is school based: teacher trainees work in a school under the supervision of both a cooperative school mathematics teacher and a University lecturer. The University curriculum is almost exclusively mathematics-centred. Only in their third year, enroll the students in a non-mathematics course for the first time. Fourth year courses should shift from mathematics-centred subjects towards education studies, but that has not been the case. Apart from courses such as *Psychology* and *Curriculum Development*, all the other courses emphasis has been on mathematics itself. Moreover, traditional methodological choices stimulate neither students' learning processes based on continuous reflection nor their autonomy. Accumulating and reproducing mathematical knowledge appears to be their typical way of learning.

The degree has achieved a considerable status in scientific terms, but a very low one in pedagogical quality. It was the recognition of the necessity of introducing a stronger mathematics education dimension into teacher training that led the Department of Pure Mathematics to appoint me in October 1993. I was the first lecturer with a Ph.D. in mathematics education to join it.

Sources of data
In outline, the study methodology and the plan to gather data were simple. Following my assignment to teach the *Mathematics Methods* course, the need was there to design and implement a new programme. Clearly, evaluating such a programme was a must. By orienting my work on action-research principles, I systematically submitted my own action and the actions of my students to scrutiny. Data come from several sources. They included participant observation enriched by field notes taking. In addition, there were the students' written answers to both the structured and non-structured questionnaires administered throughout the course sessions. The students' written assignments and informal conversations constituted other equally important sources of data.

The Course Structure and Theoretical Orientation

The two major goals of the programme were: (1) to develop a critical understanding of both mathematics (from university to school level) and mathematics pedagogy, and (2) to promote professional development by ensuring that the students acquire disposition to question and solve problems within their future careers as teachers. The procedures for this acquisition and development were to include a variety of teaching strategies, from lecturing to discussion, from individual to group work, from work based at the University to a *Preparation for Teaching Practice* component based on schools. Another important aspect of the programme was that it was Activity-oriented (see below).

The course was to meet for three hours twice a week for one academic semester. Following the conventional framework laid by the Mathematics Department policy, each session would fall into two considerably distinct parts -- *theoretical* sessions and *theoretical/practical* ones. The timing and tone of the two different kinds of sessions matched the established policy, but there was enough room for pursuing my own agenda.

I was committed to "breaking the continuity" (Ball, 1990), by having the students to participate actively in their own learning. Thus, in the theoretical sessions, the students would be introduced to the various themes under scrutin, at the heart of which were the *Philosophy of Mathematics* and the *Philosophy of Mathematics Education*, but lecturing was kept to a minimum. The students were encouraged to express their own views, as well as to reflect in their own learning. The spirit I tried to infuse in the theoretical/practical sessions, on the other hand, was inspired in Christiansen and Walther's (1986) *principle of activity*. The focus of each of these sessions was not on a particular topic, but on suggested activities which would serve to inform the various topics dealt with in the theoretical sessions as a whole. The students would work on these activities in small groups, mostly groups of four (which were formed in the first session on the basis of familiarity), and engage in them between the sessions.

A second fundamental principle on which the course was based was that there are valid different points of view that should be met and considered, and that coming to know means integrating, as distinct from aggregating, knowledge. The different groups were expected to show this 'new' kind of knowledge by writing a essay on one aspect of the philosophy of mathematics or philosophy of mathematics education, as part of their formal assessment. The emphasis on gathering information from a range of sources, reading, discussing and reflecting -- something I suspected most of the students had never done -- led me to organise the theoretical/practical sessions in a such a way that these activities could be surfaced and encouraged. For example, during the early sessions, half of the groups were go to the department library (many of the students did not even know that such a library existed). At a later stage, some of the sessions were totally allocated to have the groups to work on their own essays. Moreover, the groups were also to hold tutorials with me.

Half of the theoretical/practical sessions were devoted to activities of mathematical nature. These intended to enable the students to become involved with as many aspects of mathematics as possible, so that they would consider alternative ways of think about mathematics and more valid experience of what it means to do, learn and teach mathematics. A premise on which these activities were based was that the students held appropriate mathematical knowledge content to carry them out. Another premise was that they would deviate from the kind of mathematical experiences the students had had. These activities would constitute a basis for a second group project, another of the students' marked assessments.

The Participants in the Study

There were 48 participants in this study, 39 female and nine male. The great majority of the students were in their early twenties, but some of the students were in their mid-thirties and early forties. A couple of them had had previous experience as teachers of mathematics in schools. For most of the students, it was their enjoyment and interest in mathematics that prompted them to want to become teachers of mathematics. The liking of the teaching profession was their second driving force.

Most of the students expressed the opinion that they were not completely satisfied with the quality of their previous courses at the University. Concurrently, they were also concerned with the limited number of courses which would prepare them for the teaching profession. Another consideration may help to understand the students' dispositions at the beginning of the course: in evaluating their own involvement with previous courses, there was a noticeable contrast in their answers depending on whether these concerned students' traditional roles (attending the sessions, and paying enough attention to them) and less traditional ones (such as reading the suggested literature and hold meeting with the lecturers in order to clarify any doubts).

Further, one may consider the students' expectations about the Mathematics Methods course. Foremost among a set of twelve goals, was the students' desire "to know strategies to motivate pupils to learn mathematics", with over 78% of the respondents strongly endorsing it. The second most desired goal (with almost 60% of the respondents showing strong agreement) dealt with the need for the students "to develop competence in planning and preparing mathematics lessons". In line with existing literature, the students were showing that they were most concerned with the practicalities of teaching. An interesting and unanticipated result arose with the third most endorsed goal -- willingness "to develop habits to reflect upon their own learning and understanding of mathematical ideas". Given that these students were not familiar with the fashionable 'self-reflective' orthodoxy, it is unlikely that they were trying to say the "right" thing. The implicit message may be: "If I can reflect upon my own learning then I may understand how children learn". It may also be interpreted as a sign that the students felt ready to engage in such kind of self-reflection.

The least popular goal (though with the majority of the respondents agreeing with it) was that of reinforcing the treatment of one or two topics of mathematics. Naturally, the students felt that they had already had enough of mathematics. Moreover, the students also tended to perceive relatively little need for knowing both some basic ideas about the nature of mathematics, and the general trends and results of research in mathematics education. In this case, it was probably the students' lack of familiarity with the two themes that was the constrain. One may feel a little reluctant in plugging into something that one does not know at all.

The Students' Development

This section aims to provide a picture of what meant for the students their experience throughout it.

Emerging voices

An atmosphere in which the students' views and judgments were accepted and encouraged was judged crucial and should therefore be nurtured from the very beginning of the course. At this stage, however, it was rather difficult to have the students to take part in such kind of dialogue. Obviously, the formality normally associated with mathematics teaching tended to produce a view that there is no place for negotiation and discussion about the subject. As the course progressed, an increasing number of students were becoming more and more keen in articulating their own ideas and beliefs. Yet by the end of the course, there were still a few students who preferred not to risk their voices.

The students' views about mathematics

Identifying the student-teachers perspectives about mathematics provided the starting point for dealing with the question of the nature of mathematics. The questions 'If you have to explain to somebody what mathematics is about what would you say?', and 'what is the number one' were asked. I had no doubt, that these are complex questions, not given to simple answers. The aim, however, was not so much seeking for an answer, but rather having the students to face an issue that I suspected they have never addressed before. In what follows, I only concentrate on their responses to the first question.

Not unexpectedly, answers of the type *"If I have to explain what mathematics is about, I would have serious difficulties, because, until today, I have never been aware of the importance of this question"* were very frequent. One student, however, had already read a book on the topic:

> Courant and Robbins wrote a book ('What is Mathematics?') in which they sought to answer to this question. What they did was to describe the several branches of mathematics [...]. Now, in my opinion, mathematics is ... (after I wrote 'mathematics is, it took me about five minutes to think what I should write next, and, I realise, nothing has to do with what I was supposed to write, or may be, it has...

Although this student had examined the question before, there was still relatively little work concerned with the reconstruction of the meaning of mathematics. One can also perceive in the student's comments the tension he faced in trying to give an explicit answer to the question.

Another typical answer was that mathematics is a science. A couple of students, however, spoke of mathematics as a language. Another one referred to mathematics as a game: "there is a set of initial rules(axioms), from which different theories are elaborated". For a couple of them, still, mathematics was essentially the mathematics of the school curriculum.

Some interesting considerations are suggested by these data. First, despite considerable difference in the mathematical background of these student-teachers (and also by the teachers in the pilot programme) and that of a group of primary teachers who I interviewed in a previous study (Moreira, 1992), they share essentially the same sort of views of mathematics. Second, the student-teachers' familiarity with a formalist perspective of mathematics, which is present in most of their previous mathematics courses, does not appear to have had a significant impact on their views about the subject. It has been often said that future teachers' teaching styles are most influenced by what they bring with them into teacher training courses. These findings suggest that, similarly, their inputs on teachers' views of mathematics tend also to be limited.

Developing new learning dispositions

A central issue was to assess how the students were coping with new teaching strategies. About half-way throughout the course, I asked them to reflect upon their own learning. A few students felt that they had learnt most from the background reading:

> I learnt by reading and becoming interested in what I was reading. Moreover, I was assessing what I had learnt from what I had read and what I already knew (which was not too much), and the extent to which this knowledge was important.
>
> My learning was essentially based upon the reading of the suggested texts. The sessions serve mainly as an input to suggest and motivate topics for reading and bring to light why to read those texts was important.

These comments show that some students were prepared and even eager to participate in non-familiar teaching approaches.

For most students, however, reading was more problematic, not least because many of the texts were in English. For some, the process merely led to a confirmation of their preference for a traditional way of learning. This can be illustrated most vividly by one of the student's comments: *"I learnt form the information that was transmitted to me, mainly through what was dealt with in the sessions. I learnt little from reading texts, because I am not accustomed to reading"*. And in writing about her difficulties and how these difficulties might be overcome, this student remarked: *"I think the texts are quite complex...; it is quite hard reading them, and this does not stimulate me to read.... It would be necessary to use books of easier access and easy to read, so that they could help me to enjoy reading"*.

There is no doubt, that people are often happiest doing what they are used to doing. But this student's arguments are compelling ones, and should be taken into account. The preceding discussion also tindicates that not all the students benifited in the same way from the same strategies. Problems of learning are complex, and needed is research to explore how relevant variables interact with different teaching approaches. The need to accomodate the student-teachers' needs, abilities and learning styles may be as important as the urge to use new pedagogical approaches with them.

Internalising perspectives about mathematics

I have argued for extending the 'conventional' content of a Mathematics Methods course by adding a more emphasis to make explicit different perspectives of mathematical thought. In the event, five conceptions of mathematics were presented to thes students: platonism, logicism, intuitionism and fallibilism. Which of these perspectives did the students feel closer to their own views?

Arguably, to espouse a particular perspective of thought and commit to it is not an easy task, and as I envisage, the students' conceptions would evolve and he progressively elaborated as their knowledge and understanding of the five perspectives increase. The relationship between choice and commitment is an important element, and the students had not quite worked it. As one student, who expressed her choice of platonism, commented:

> I have found it quite difficult to reflect on issues such as 'what is mathematics' and 'what is mathematics for'... I was not accustomed to reflect on such themes... May be because of that, I could not understand well some of the topics of the perspectives of mathematical thought. But I think that I will be able to overcome this problem if I try hard and read more about these issues.

This direction of further learning leaves, of course, room for modification and adaptation.

The importance of the students' answers to this question extends far beyond their personal preferences. What is a significant is that the students' choices reflected something else than uniformity. Six students (only 29 students were present in the session in which that question was asked) considered that their views were closer to those of fallibilists. Fallibilism had provided them with a vision of mathematics which was simultaneously "more human, funnier, and more in line with today's world". At the other extreme, there were four students who agreed to a platonist view. The adoption of such a view was based upon the conviction that "mathematics is to discover, not to invent". In the words of one of the students, such an "activity of discovery embodies in itself enough beauty to make one think of mathematics as an enjoyable and pleasant experience". Formalism was the perspective that appealed to other two students, and intuitionism appeared to be closer to the views of other two students. Only one student expressed a clear preference for the ideas of the logicists. Of the remaining fourteen students, six did not answer the question, and seven subscribed the view that all the five schools have both aspects which they supported and features that they rejected. Finally, one student preferred to answer in terms of stating the perspectives that he would not endorse, namely formalism and logicism.

At a later stage, an attempt was made to assess the students' understanding of the different perspectives by having them to analyse the discourses of five imaginary mathematicians. Very few students were able to match the five discourses to the five mathematicians successfully, which shows that the topic is not an easy one. An interesting result was that most students were able to identify the fallibilist mathematician. This may be related to the fact throughout the course, and specifically by means of the mathematical activities, much emphasis was placed on promoting a view of mathematics as a tool to solve problems, to make conjectures and to create new mathematics. Fallibilism became, thus, something that was experienced rather than just talked about, and this facilitated internalisation.

The Character of the Students' Activities

The students' favourable response to the course was not so much due to the subject matter, but rather to the opportunity it provided for peer exchange, writing essays and engaging in mathematical activities. This section illuminates some successes, as well as some difficulties experienced by the students in the process.

The essays

The students generally found the task of writing the essays quite demanding. Though some clarification helped the students to sort out certain ambiguities, there were still some considerable difficulties hovering around. "Lack of practice in engaging in such a kind of activity" proved to be troubling for most of them.

Difficulties were not only at the level of competence in writing. What was at stake, too, was the students' views of knowledge. Accustomed to rely on the authority of a single text book or of the notes from the University lecturer, knowledge was clearly seen as totally reliable, objective, and learning was perceived as receiving knowledge. Now, learning was being shaped by their inquiries and this was causing some insecurity. One student specifically noted that he was embarrassed by getting different information from different reading sources.

Other students were unable (or unwilling) to face a variety of sources. One solution was to stick to just one. The following anecdote is illuminating. In writing about various themes on mathematics education, three different groups pursued the same issue -- relevance of mathematics. In all these groups, the way in which they grappled and explored the issue was very similar. Moreover, the examples they included to illustrate their ideas were exactly the same. Not only the students had used the same reference book, a kind of textbook on the didactics of mathematics, but the book had also informed their disposition to address the issue.

The majority of the essays showed that most students were able to recognise and order important ideas, but unable to record them in own words or write some personal thoughts about them. In other words, writing meant, in most cases, copying (or translating) verbatim relevant information selected from one or various sources. I would like to offer two possible explanations for this fact, in addition to the students' lack of familiarity with this kind of activity. First, it is a fact that that most of the students were most concerned with showing factual knowledge about the topic. (Some, however, were able to 'look inside' the activity to pull out a different kind of knowledge acquisition. One comment illustrates this: *"I think that the main objective of the activity was to develop our ability to inquire, select information and critically analyse a specific topic"*).

A second explanation may reside in the fact that within the educational culture of these students there had been little space for interpretation and communication. As Pimm (1987) suggests, "mathematics is not commonly viewed as a discursive subject" (p. 47). Moreover, in mathematics symbols are overwhelmingly confused with mathematics objects themselves, and writing in mathematics classrooms is limited to using a very restricted language according to clear-cut rules, in order to find the right solution of a specific problem. This can determine to a large extent a low fluency in communicative and writing skills.

It was readily apparent that discussion within the groups enabled the students to clarify their ideas and understanding of the theme addressed. Most of them recognised that working in a group was a valuable experience. A few students, however, did mention some difficulties emerging from the need to reconciling ideas of the different elements of a group. Finding the right time for some of the groups to meet and work together between the sessions was not easy either. I detected a further problem . In organizing their work, some groups distributed tasks among its different elements. A bunching effect was the result. This, however, was overcome, at least in part, by having the students revise their work a greater number of times.

Another novel aspect for the students was the tutorial mode of instruction. I had anticipated, and it proved to be true, that without my support and help, and a close monitoring of their work, the groups could have hardly produced any kind of essay. Curiously enough, the students did not verbalise any appraisal to that kind of face-to-face instruction. I wonder how to interpret this. It is possible that given the group situation, the students construed the meaning of their activity only in terms of transactions among themselves. Perhaps, I should be satisfied with such an omission: I was able to "graft" in a non-interfering way.

The mathematical activities

A full list of the titles is included in order to give a flavour of the kind of mathematical activities the students were engaged in: *Spirolaterals, Snooker, The MU-puzzle, The Ideal City, The Game of Life, Pascal's Triangle, The Seals Island, The Sierpinsky Carpet, The Elections in the Green-Yellow Land, The Square Pyramid, and Tessellations.* For the reader, most of these activities may sound rather familiar. For the students, however, they were almost totally unknown.

Initially, all the students were engaged in all these activities. Most of them felt that these activities were particularly important to them:

> The activities that most contributed to my learning were the mathematical activities. I think they are very important since there were topics that I did not know and which deal with mathematics in a way that I have never addressed before.

> I learned topics in a way that I never thought I could learn here at the University, such as, the practical activities... The games with triangles, squares, among others, helped me to see how I could teach mathematics to my pupils, in a new and funny way.

The unfamiliar, enigmatic and game-like aspects of the problems proposed captured the students' involvement and enthusiasm, and also promoted a lot of discussion.

Of particular significance for me, and for most of the students too, was their surprise at the fact that their knowledge of mathematics was, most of the times, insufficient to help them to solve the problems proposed. The following illustrates patterns in the students' approaches to the activities. First of all, it is important to note that whether or not the students could relate the task at hand with some familiar problem appeared to be determinant on how they approached the different activities. For example, in *The Ideal City* activity, a set of related problems in Taxicab Geometry, there was a tendency for the students to use an analytical approach based upon the distance formula given. By way of contrast, the *Tesselations* activity was dealt with in a purely visual manner. Taking a closer look at this activity is worthwhile. In fact, the students experienced difficulties that were totally unexpected to me. For example, they had to struggle hard to find the condition for regular polygons to tessellate. Perhaps even more surprising was to find out that many of them did not know a definition of a regular polygon and/or were not able to retrieve the formula for the sum of the internal angles of a polygon. Rediscovering this formula was not easy either.

Second, the students' actions tended to be uncreative and rather inflexible. The use of diagrams or other tools to organise information in a systematic manner was also absent. *The MU-puzzle Enigma*, an activity adapted from Hofstadter (1980) to illuminate the idea of formal systems and proof, highlights these aspects. In order to find out whether or not a particular theorem hold in the system given, the students' only alternative was following the rules for ever. Years of formalised mathematics, of following procedures, of focusing on answers have yielded inattention to meaning making, creative thinking and solving problems.

Following the initial phase, each group would tackle a specific problem for a considerable period of time (about three weeks). The aim was to encourage personal and inventive approaches, emphasising processes of learning as well as learning products. It was also expected that the students felt encouraged to use the problems given as a starting point for posing their own problems. These may seem quite ambitious aims. What was achieved, however, was not disappointing. Seven of the twelve groups clearly grasped what was intended. They were able to pursue the potential avenues open by the initial problems and release creativity that surprised me and the students themselves.

A few examples are worthwhile. The group engaged in the *Snooker* activity 'discovered' a formula to determine the number of the times the ball hits the billiard walls. This group also extended the original problem in two different directions -- a billiard with two balls and a three dimension version of a billiard. Another group found in the computer a prerequisite to foster their creativity. The activity in which they were engaged in, *The Elections* activity, was extended in several ways by means of a computer program (an unexpected solution for me, given the course context). The most noteable result was, perhaps, the creation of a *Pascal Tetrahedron* by the group for whom *The Pascal Triangle* was the starting point.

Discussion

Before initiating this discussion, I should remind the reader that this study was not about "changing the teacher". Based on my own (personal) experience, I have become increasingly aware of how painful and complex change can be. Research studies, too, have shown that learning experiences for changing teachers' conceptions and practices may enhance their capacity to grow, but may also discourage them (Moreira, 1992). A statement by Pimm (1993) entails my position: "it is dangerous to lose sight of how difficult personal change can be -- and we should not talk lightly or glibly about it, let alone expect or demand it" (p. 31). My intention was to claim that there is a place for an epistemological discussion about mathematics in the context of *Mathematics Methods* course for prospective secondary mathematics teachers. I hoped that this paper has given support to my claim. For the purpose of this final discussion I would like to extend Cobb's (1989) model of the three contexts for the learning and teaching of mathematics -- cognitive, experiential and anthropological -- to the learning and teaching of the nature of mathematics. The experiential, cognitive and anthropological modes were informed by the various experiences the students had during the course. Though access to the topic of the nature of mathematics had not formally been part of their experience thus far, the students held, conscious or unconsciously, beliefs about the subject, as they formerly had not been deprived of access to mathematics itself. Thus, not only did the students heard, read and write about the nature of mathematics, they also reflected on their own responses to dealing with the issue. This was the experiential perspective, contingent on the students' current understandings of mathematics. From writing the essays (and associated reading), and from engaging in the mathematical activities, the students came to understand the nature of mathematics from different perspectives, and this is the realm of the cognitive context. From an anthropological perspective, the social setting within which the students were learning about the topic was most important. It was locatable in both the interactions they had among themselves in the small group work and in their interactions with myself. The emphasis on these three aspects is the crucial aspect of the course.

Not unexpectedly, different students reacted differently to the course. These differences notwithstanding, it seems clear that, for most of the students, the experience was a gratifying one. For a few, the course even appeared to offer a whiff of fresh air that they were longing for. Their investment in terms of time and energy was considerable, and the work they produced was notable. There is no question, too, that all the students had to overcome some hurdles. For some, and I am remembering of one student in particular, the course might have represented a daunting experience. This particular student hardly show any kind of enthusiasm for the course activities, and her involvement and participation both in her group and in the whole class discussions was practically nil. It is possible that in a more traditional kind of programme, her apathy was less evident, but I wonder whether her involvement with and profit from such a programme would be any bigger. There are still many unanswered questions in the area of student-teachers' reactions to "learning teaching".

This paper has identified issues that need attention, and it seems important now to begin to develop ways to help prospective teachers overcome many of the hurdles in examining the question of the nature of mathematics (and mathematics teaching/learning) that were pointed out. Clearly, the topic can be an attractive one for them. All the participants in the study invested considerable time in it, and most importantly, all of them stayed "in the ball game". In addition to come know better what mathematics is about, they seem to have become more autonomous learners and more willing to experience a realm of active and creative learning. In so doing, the students may also have felt committed to a particular kind of mathematics teachers model that they want to become. What kind of teachers they might become is, of course, a different matter.

References

Ball, D. (1990). Breaking with experience in learning to teach mathematics: The role of a pre-service methods course. *For the Learning of Mathematics, 10* (2), 10-16.

Cobb, P. (1989). Experential, cognitive, and anthropological perspectives in mathematics education. *For the Learning of Mathematics, 9* (2), 32-42.

Dossey, J. (1992). The nature of mathematics: Its role and its influence. In *Handbook of research on mathematics teaching and learning: A project of the National Council of Mathematics* (pp. 39-48). New York: Macmillan Publishing Company.

Elliot, J. (1993). Academics and action-research: The training workshop as an exercise in ideological deconstruction. In J. Elliot (Ed.), *Reconstructing teacher education: Teacher development*, (pp. 176-192). London: The Falmer Press.

Grows, D. (Ed.) (1992). *Handbook of research on mathematics teaching and learning: A project of the National Council of Mathematics.* New York: Macmillan Publishing Company.

Hofstadter, D. (1979). *Godel, Escher, Bach: An eternal golden braid.* London: Penguin Books.

Moreira, C. (1992). *Primary teachers' attitudes towards mathematics and mathematics teaching with special reference to a Logo-based in-service course.* Unpublished Ph.D. thesis. Institute of Education, University of London.

Moreira, C. (1993). *A filosofia da matemática como domínio da educação matemática.* Personal communication presented at Simpósio Perspectivas Actuais e Futuras no Ensino das Ciências e da Matemática, Faculdade de Ciências da Universidade de Lisboa.

Pateman, N. (1989). *Teaching mathematics: A tantalising enterprise.* Victoria: Deakin University.

Pimm, D. (1987). *Speaking mathematically: Communications in mathematics classrooms.* London: Routledge.

Pimm, D. (1993). From should to could: Reflections on possibilities of mathematics teacher education. *For the Learning of Mathematics, 13* (2), 27-32.

TEACHERS ASSESSING INVESTIGATIONAL MATHEMATICS:
THE ROLE OF "ALGEBRA"

Candia Morgan

Institute of Education, University of London, U.K.

Recent curriculum and assessment developments in the U.K. have been introduced with the aim of broadening students' mathematical experiences and making assessment procedures capable of assessing a wider range of types of achievement. This paper describes part of a study of the ways in which students and teachers have responded to the use of written reports of investigational activity as an assessment tool and discusses some of the implications of this innovation. Teachers were asked to assess students' reports and to explain the basis on which they were making their assessments. One of the themes arising from analysis of these interviews was the importance of "algebra" for these teachers. The reasons for this importance, however, are ambiguous, illustrating the potential contradictions between teachers' pedagogic role and their role as assessors. Questions are raised about the effects of a curriculum reform that was originally intended to allow students the possibility to be mathematically creative.

When a curriculum reform is introduced, there is no guarantee that the same understanding of the intentions of the reform are shared by all those involved or that its implementation will fulfil the expectations of its initiators. It is recognised in particular that teachers are likely to adapt new materials and teaching methods to fit into their familiar ways of working (Olson, 1980). This paper reports part of a study of the ways in which teachers and students have come to terms with the introduction of examination at 16+ by coursework, in particular by written reports of investigative work in mathematics.[1] Children's perceptions of what makes a "good" written report (Morgan, 1992a) have raised issues about the relationship between their understandings of the requirements of the "investigation" genre and those of the more traditional "school maths" and about the role that may be played by teachers in the construction of these perceptions (Morgan, 1992b). Interviews with teachers have explored the ways in which they read and evaluate students' written reports. One of the themes arising from analysis of these interviews that appears significant for all the teachers interviewed is the presence and nature of "algebra" within a student's work; the role of algebra for these teachers and its implications for their roles as teachers and as examiners will be discussed here.

Background

The introduction in the UK of assessment by investigative work may be traced back to the Cockcroft report (DES, 1982) with its recommendations that all mathematics teaching 'should include opportunities for ... investigational work' (para. 243) and that examinations should include teacher assessment of aspects of mathematical achievement not amenable to timed written papers; this, it was claimed, would 'encourage good classroom practice' (para. 535). These recommendations were institutionalised in 1988 by the introduction of assessment by coursework in the new GCSE examinations at age 16+ (although it was not compulsory in mathematics until 1991). The relationship between curriculum and assessment practices was recognised and it was explicitly stated that methods of assessment 'should not conflict with

[1] These reports are expected to contain evidence of the investigative processes as well as communication of the results, e g. planning, working systematically, hypothesising, evaluating as well as using effective and appropriate means of mathematical communication.

the provision and development of appropriate and worthwhile mathematics courses in schools' (DES, 1985). Indeed, coursework is said to be 'perhaps above all else, an instrument to facilitate and encourage curriculum development in schools' (ULEAC, 1993: 3). It has widely been seen as an agent of curriculum change to shift the emphasis from content towards process and to encourage students 'to create their own mathematics: actively taking part in mathematical thinking rather than passively receiving mathematical thought' (Pirie, 1988: 7). In recent years, similar widening of methods of assessment to validate and encourage a broader mathematics curriculum has also occurred in other countries (e.g. Stenmark, 1991; Stephens & Izard, 1992).

While the expressed aim of examination by coursework may be to value and validate creativity, this does not guarantee that unusual or unexpected student responses will actually be valued (Haylock, 1985; Flener & Reedy, 1990). There is a tension for teachers (who both prepare students to undertake the coursework and assess their written reports) between the aim of encouraging creativity, the need to advise students and prepare them to perform successfully, and the desire to make the final grade appear valid and "objective". One of the ways in which this tension may be resolved is through the transformation of "investigational" or "investigative" work into "investigations" which are perceived as a separate part of the mathematics curriculum and are relatively clearly defined. Many such investigations may be approached using a routine method, described disparagingly by Hewitt (1992) as 'train spotting': generate numerical data, tabulate it, "spot" a pattern in the numbers, describe the pattern. Such an algorithmic approach provides clear guidelines for both teacher and students for doing and assessing the work but does not allow much room for creativity.

The end point or solution of an investigation approached by this method is the description of the number pattern, which may be in procedural or relational terms, in words or using algebraic notation. (Proving that the pattern so described will necessarily continue is normally considered to be beyond any but the most able student and is 'deemed as worthy of a very top A'. (LEAG, 1991: 77)) Both the process of generalising and the use of algebraic notation are clearly of importance in mathematics itself as well as within the criteria for assessing such investigations. It is not surprising, therefore, that algebra emerged as one of the significant themes in interviews with teachers in the current study. What follows will include an examination of the pictures teachers present of their perceptions of the nature of algebra and of the reasons for its importance within the practices of doing and assessing investigational mathematics.

Methodology

The data discussed in this paper is taken from interviews, each approximately one hour long, with six teachers from three different schools. All these teachers had experience with doing coursework with students and assessing their reports, in some cases since the inception of GCSE in 1988. Two of them (Andy and Dan) had known the students whose work was assessed during the interviews and were familiar with the investigational task involved.

Out of a set of students' coursework scripts, three complete reports of work on the investigational task entitled 'Inner Triangles' (LEAG, 1991) were chosen to be assessed by teachers in the interviews. These three texts were chosen to display a number of contrasting features on the basis of a textual analysis using tools derived from Halliday's Systemic/Functional linguistics (Halliday, 1985). They had all originally been awarded similar grades by their teacher and, although taking different routes through the investigation and using different means of communication, had reached broadly comparable conclusions including in all cases a generalisation for the number of 'unit triangles' contained within an isosceles trapezium. It is not claimed that the texts used were typical or representative of the set as a whole or even of the subset of 'Inner Triangle' scripts, although none was remarkably unusual. In addition, extracts containing expressions of generalised solutions to the 'Inner Triangles' problem, were chosen in a similar way from a further three scripts.

The aim of the interviews was to explore the ways in which the teachers read and evaluate such reports of investigations and to identify features of the students' texts which influence their judgements. The form of the interviews was therefore 'discourse-based' (Odell, 1982): the teachers were given the three complete texts and were asked, as experienced assessors, to read them, talking about their judgements as they made them. The teachers were then asked to rank the pieces of work and to justify their rankings. This was a familiar activity for the teachers as they had all been involved in group moderation within their schools. Finally they were similarly asked to read, talk about, and rank the three short extracts. This part of the interview, while a less familiar activity for the teachers involved, focused attention on different ways of expressing generalisations and thus enriched the data on attitudes to the use of algebraic notation. By basing the interviews around the assessment of students' work it was hoped that the teachers would respond from a position within the discourses of coursework and assessment, thus providing a valid picture of their practices. Some of their responses, however, particularly when talking about general principles rather than about the work of individual students, may have been influenced by the interviewee-interviewer relationship and by their beliefs about the expectations of a researcher known to be involved in Higher Education and in-service education of teachers.

The analysis of these interviews aims to identify and explore themes which are of significance to the teachers involved, the features of the students' texts that they pick out for comment and the picture of the nature of coursework that is constructed through their discourse. It was anticipated that the use of algebraic notation would be one of the themes present because of its significance not only within mathematics itself but also in the assessment scheme. The discussion presented in this paper relates primarily to this theme, which illustrates a more general issue of the tensions between teachers' roles as teacher and as examiner.

"Algebra" as a sign of high achievement

It has become an accepted "fact" that, in order to achieve one of the top grades (A - C) for an investigation in GCSE coursework, it is necessary to have used "algebra" (Wolf, 1990) What actually counts as "algebra" is, however, unclear. One examination board, for example, includes in its grade description for a grade C: 'Devises simple formulae when generalising' (ULEAC, 1993: 16), but does not make it explicit whether the formulae should use algebraic notation or might be written in words. Andy appears to feel his own conception of "algebra" to be in conflict with that required by the examination board when evaluating a student's formula expressed in words as '(TOP LENGTH + BOTTOM LENGTH) x SLANT LENGTH = No. OF TRIANGLES':

> A I like the generalisation in words and it's quite clear from the use of brackets that they understand that addition has to be completed before the multiplication takes place. There's no ambiguity with algebra. They could easily change that formula into a shorter algebraic statement. And again that seems perfectly reasonable and straightforward
>
> I Would they get extra marks for writing it as t plus b rather than top length plus bottom length?
>
> A The exam board seems to think so. They often make the point that hasn't got algebraic - well I think that's algebraically quite correct and I see no real difference between that and t plus b times s. Once you start putting brackets in you start making it algebraically correct - I think that's right. And it's quite concise.

Similarly, Dan sees little difference between this formula and one expressed using algebraic symbols and suggests that he might interpret the examination board's criteria (perceived by him to demand such notation) flexibly:

> the person here obviously understands what they've generated and can do that, she's put the formula like that nice and clear. It's a bit of a shame they didn't go on to the bottom line to use algebraic notation. If every other criteria [was fulfilled] and the only thing that was stopping it being a particular grade was the fact that it wasn't expressed algebraically I wouldn't mind. To me, the difference between that and sticking letters in instead is so minimal that it doesn't really matter.

Of another student who did use algebraic notation, however, Dan shows that this does influence the value he puts on a piece of work:

> He got a formula for it I think he's into 'B' country, isn't he, immediately.

Using algebraic notation thus appears not only to be "officially" necessary for the higher grades, but also to be sufficient.

Clearly, however it is defined, algebra is highly valued by these teachers and by the assessment system. Considering the importance within mathematics of algebraic thinking and of symbolic representation, this value would appear to be well justified. A question arises, however, of the nature and purpose of algebra within the context of the "investigation" genre. The primary, and in most cases only, use made of symbolic notation in the set of coursework scripts examined is to express a generalisation. This generalisation is seen by both students and teachers as the end point or solution of the task and is in many cases underlined, presented on a separate page under a heading *FORMULA*, or otherwise signposted as the "answer". While forming such symbolic generalisations is an important mathematical process, outside the school context it would not normally be seen as an end in itself. Symbolisation is not merely a process of translation from one language into another but is the starting point for developing new ways of looking at a problem and for enabling manipulations that may lead to new discoveries and further generalisations. As Mason points out, 'Classification is for the purpose of formulating theorems, not simply to achieve superficial classification' (1987: 77).

In one of the pieces of work read by the teachers in the interview, the student (identified as 'Number 1') had manipulated her formula:

We were able to put this into a formula:-	$ab + ac = d$
This simplified becomes:-	$a(b + c) = d$.

This was commented on positively by several of the teachers:

> I think number 1 is very good, is good the way, you know, they've simplified their algebra as well. (Fiona)

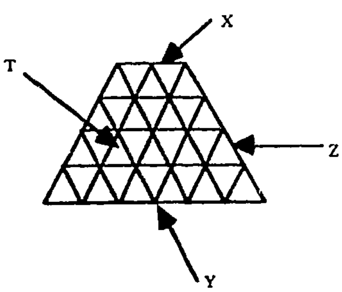
> Number 1 has brought in algebra - factorisation, substitution - that's good. (George)

The teachers' praise of the display of manipulative skill suggests that they are applying criteria from more traditional assessment modes. The manipulation is an end in itself: the student is displaying her skill in performing a routine procedure rather than using that skill to further the mathematical investigation. As an assessment objective, routine manipulation may well be better achieved within more traditional assessment contexts. It seems likely that many of the other students could have successfully performed the above manipulation if specifically asked to do so. They did not, however, choose to do so within the investigative context. We might indeed ask whether it was necessary or appropriate in such a context, particularly as, for this student, it was the end point rather than a tool.

Generalise in words first

An algebraic generalisation, however, is not considered to be adequate without some verbal elaboration. One of the complete texts (by Richard) presented to the teachers consisted essentially of a table of results, a (correct) symbolically expressed formula and a diagram labelled with the variable names used in the formula (see figure 1 below).

Figure 1 (Richard's diagram and formula)



$Z(X + Y) = T$

Most of the teachers took this to be a sign of relatively high achievement or ability (see also Morgan (1992b) for similar teacher evaluations of non-verbal student texts). For example, Fiona praised the algebraic notation as 'quite advanced', while Dan, who had at one time taught Richard, described him as 'quite an intuitive mathematician'. It is interesting to note that both Fiona and Dan "hedged" their praise with the qualifier *quite* as they, like all the teachers interviewed, also criticised Richard for not including more "writing".

The reasons given by the teachers for requiring words as well as algebraic notation are multiple and suggest that teachers occupy a number of (potentially contradictory) positions while reading and assessing coursework which lead them to construct various meanings from the same text.

All the teachers at some point in their interview appeared to be positioned as examiners in a formal, public examination system. One of the public criticisms that the innovation of examination by coursework has had to face is the concern that, because it is not done under controlled conditions, it may not represent the student's own work but may have been copied from another student or even done by a parent. Teachers, in their role as examiners and as professionals, are naturally concerned that their judgements of students should be seen to be valid. They feel the need, therefore, to find evidence within a student's script that the work "belongs" to the student. For example, Dan commented that:

> the formula . . . did sort of appear out of nowhere as though he did have a sly look round somebody's arm or something.

Andy, who actually taught Richard, acknowledged the lack of evidence in his script but was prepared to act as teacher/advocate on his behalf:

> I'm very confident that, although there's no evidence of it, what he's produced is right and he's done it. It definitely wouldn't be a copy. . . . That's interesting, that can only be a teacher's inside knowledge. Somebody marking that cold wouldn't be able to state that.

While satisfied that the formula was Richard's own, Andy nevertheless went on to give a number of other reasons for wanting to see the generalisation expressed in words:

> I would like to see that he can generalise in words first of all. It kind of gives the understanding, I think, putting it into words the patterns which they see. Then I think it underpins the algebra which they produce later.

Initially, Andy claims that he wants to see that Richard 'can generalise in words'; he is looking for evidence of a skill. In the next breath he is suggesting that this would also provide evidence of 'understanding'. The expression that he uses is, however, ambiguous. 'It kind of gives the understanding' to whom? Do the words give evidence to an examiner who needs to know that the student understands, or is the understanding given to the student himself by the process of expressing the generalisation in words. This ambiguity marks a shift in Andy's reading between an examiner role, looking for evidence, and a teacher/advisor role, suggesting ways in which students can be helped. In the final sentence of the extract above, the role of generalisation in words has clearly shifted from being evidence towards assessment of the student to being a pedagogic device for helping the student to gain understanding. At the same time Andy has shifted from referring specifically to Richard to using a non-specific *they*; he is now stating general pedagogic principles. He went on to describe the assessment methods used by the teachers in his school:

> When we moderate, we usually do it as a group in the same room usually working in different corners on the scripts that we're moderating and frequently we ask across the room to the teacher concerned 'where did this come from, is that alright?', and . . . on we go. But there's usually a check of that kind. We try very hard to tell the children generalise in words first of all and we say if you know a pattern, can you tell us about it, tell your friend about it. When they can explain the pattern in words and they write those words down then they're ready to produce the algebra.

Again, he moves from a description of the way the teachers behave as examiners looking for evidence to a description of the pedagogical device they use to help the students to become 'ready to produce the algebra'.

Andy and the other teachers interviewed appear to be using a criterion (unwritten) that any algebraic generalisation appearing in a piece of coursework should be preceded by a verbal statement of the same generalisation. Stated like this, it is a simple matter to judge whether or not a student has fulfilled the criterion. The justification for its use, however, is not so simple but takes a number of forms: it is evidence that the student has the skill to write a verbal generalisation; it shows that the student understands the algebra; it proves that the formula "belongs" to the student; it helps the student to understand the pattern; it prepares the student to 'produce the algebra'. The discourses of assessment and of pedagogy are intertwined here - perhaps an inevitable consequence of the teacher's role.

Advising students to generalise in words as a step towards using algebraic symbols is a common pedagogic strategy. The process is described by Mason as 'a necessary part of the struggle towards meaning along the spiral [of symbolising]' (1987: 80). It seems for these teachers, however, to have been transformed into a prescriptive algorithm for "doing investigations". Regardless of the individual student's actual facility with symbolising, she or he is expected to have gone through the entire 'struggle towards meaning'. As Amy put it, criticising Richard's (correct) symbolic generalisation:

> He needs to explore. There's something needed before he could generalise.

Although Richard is clearly able, at least in the context of this problem, to generalise symbolically without such aids, because he has failed to comply with the conventions of the coursework genre he is judged harshly; the "need" is a requirement of the examination, expressed in the language of pedagogy.

Conclusion and Implications

Analysis of the teachers' discourse reveals both the high status accorded to what they identify as algebra and some ambiguity about what it is and why it is important. One aspect that is completely missing from the interviews is any perception of algebraic symbolisation as a tool for creating further mathematics or for solving problems. For teachers as well as for the students, writing an algebraic formula is the solution. While this may be in part a reflection of the limitations of the 'Inner Triangles' task itself, a picture is presented of symbolising and manipulating symbols as discrete non-contextualised skills which are being tested by the coursework examination. The consequences of this picture for students' attitudes to and further learning of algebra need to be considered.

Both the use of algebra as an indicator of high achievement and the insistence on the expression of generalisations in words as well as in symbols may be seen as symptoms of the difficulty of separating process from product. While one of the aims of the introduction of coursework was to value and assess mathematical processes, the primary means of assessing the process is through examination of the product - the written report. As teachers attempt to help their students to produce high value products, the process of developing a generalisation is distorted, changed from a support for mathematical thinking into a skill to be learnt and assessed.

While reading the students' texts, teachers shifted their positions between reading and speaking as examiners, as teacher/advocates, as teacher/advisors, and in some cases as imaginary naive readers (for further discussion of this, see Morgan, 1993). In describing these multiple and sometimes contradictory positions I am not suggesting that the teachers are confused or incompetent. It is a consequence of the institutional context in which they are expected to fulfil all these roles. The transformation of podagogic advice to use words as an aid to symbolising into an assessment criterion is a product of this context. The need to justify teachers' assessments and give them an appearance of reliability has led to the development of a standardised algorithmic approach to "investigation". This leads me to question the possibility of using such an assessment system to 'encourage good classroom practice' or to encourage creativity and active mathematical thinking in students. Distortions of the original intentions of this attempt at curriculum reform are the result, not of teacher resistance or adaptation, but of the means of implementation, which create the very tensions and contradictions that defeat its purpose.

References

DES, 1982, *Mathematics Counts*, London: HMSO

DES, 1985, *GCSE: The National Criteria - Mathematics*, London: HMSO

Flener, F.O. & Reedy, J., 1990, "Can Teachers Evaluate Problem Solving Ability?", Proceedings of PME XIV Volume I, Oaxtepec: 127-134

Halliday,M.A.K., 1985, *An Introduction to Functional Grammar*, London: Edward Arnold

Haylock,D.W., 1985, 'Conflicts in the assessment and encouragement of mathematical creativity in school children', *International Journal of Mathematical Education in Science and Technology 16(4)*: 547-553

Hewitt,D., 1992, 'Train Spotters' Paradise', *Mathematics Teaching 140*: 6-8

LEAG, 1991, *Mathematics Coursework Tasks and Performance Indicators (1988-1991)*, London East Anglian Group

Mason,J.H., 1987, 'What do Symbols Represent', in C.Janvier (ed.), *Problems of Representation in the Teaching and Learning of Mathematics*, London: Lawrence Erlbaum Associates

Morgan,C., 1992a, 'Written Mathematical Communication: The Child's Perspective', Short Oral Presentation, PME XVI, New Hampshire

Morgan,C., 1992b, 'Written Reports of Mathematical Problem Solving: The Construction of a Discourse', paper presented at ICME7 (Working Group 7, Language and Communication), Quebec

Morgan,C., 1993, 'Teachers Assessing Coursework: Themes and Tensions', Proceedings of British Society for Research in the Learning of Mathematics, Manchester

Odell,L., 1982, 'Writing in a non-academic setting', *Research in the Teaching of English 16(3)*: 201-223

Olson J, 1980, 'Teacher constructs and curriculum change', *Journal of Curriculum Studies 12(1)*:1-11

Pirie,S., 1988, *GCSE Coursework - Mathematics: A Teachers' guide to Organisation and Assessment*, London: Macmillan Educational

Stenmark,J.K. (ed.), 1991, *Mathematics Assessment: Myths, Models, Good Questions and Practical Suggestions*, NCTM

Stephens,M. & Izard,J. (eds), 1992, *Reshaping Assessment Practices: Assessment in the Mathematical Sciences under Challenge*, Proceedings of the First National Conference on Assessment in the Mathematical Sciences (1991: Geelong, Victoria), Australian Council for Educational Research

ULEAC, 1993, *Coursework Assessment 1993: Mathematics A*, University of London Examinations and Assessment Council

Wolf,A., 1990, 'Testing Investigations', in P.Dowling & R.Noss (eds), *Mathematics Versus the National Curriculum*, Basingstoke: Falmer Press

Parental Involvement in mathematics : What teachers think is involved.

Ceri Morgan University of Warwick, UK
Ruth Merttens University of North London,UK.

Abstract *Both in the USA and in Britain it has long been demonstrated that the socio-economic background of the home the largest single factor in determining children's educational attainment. However parents' active support and involvement in children's education can make a major difference. This paper describes IMPACT, an educational initiative in which parents are systematically involved in their children's learning of mathematics through the use of weekly teacher selected take home tasks and how teachers perceive the nature of parental involvement. Starting in 1985 with 12 elementary schools in inner London, the project now runs in over 3000 schools across England, Scotland and Wales. It is the largest project of its kind in Europe. Drawing upon some of the research findings, this paper will describe how the programme works, the percentages of parents who participate and the attitudes of some of the participating teachers.*

The IMPACT Project was started in 1985 as an attempt to mirror the work of the shared reading initiatives in the area of maths. The late 1970's and early 80's had seen the establishment of a substantial body of evidence, gleaned from both research and practice, of the efficacy of involving parents in their children's learning to read through a programme of regular reading at home and sustained dialogue between teacher and parent about the child's progress (Topping & Wolfendale, 1985, Hamilton & Griffiths, 1984). The mechanisms by which this dialogue was maintained usually included small 'reading diaries' completed by parents and children at home and, responsively, by the teacher in class.

IMPACT is an educational initiative in which parents are systematically involved in their children's learning of math through the use of weekly teacher-selected take-home tasks. Started in 1985 with 12 elementary schools in inner London, the project now runs in over 2000 schools across England, Scotland and Wales. It is the largest project of its kind in Europe, and, in addition to the intervention programme, IMPACT supports a flourishing research centre. The IMPACT project is designed to foster greater integration between a child's' mathematical knowledge from outside school and the learning within the school and also to gain from the well documented benefits (educationally) of sharing activities between pupils and parents (Topping and Wolfendale, 1985, Hamilton and Griffiths, 1984).

Impact activities fall into two types. The first type are those that require the child to collect some data. This can then be collated, analysed and represented back in the classroom. An example of a data handling activity is 'Telephone calls'. The second type of impact activity are those that practice or develop a skill. An example of this is the activity called 'Money bags'. These two activities are detailed below :

Telephone Calls Count and record the telephone calls that arrive at your home over a weekend. Who do you think will get most calls ? Design a tally sheet to keep by the phone. Who got the most calls ? Who spent longest on the phone ?

Money bags. How much would you have if you had one of every note and coin that is legal currency ? Ask as many people as you can. 10 seconds to give a guess. Work it out. Who was closest ?

IMPACT is, in a sense, another name for homework in that children take a mathematical task and share it at home. However, there are some important differences between IMPACT and the ways in which homework is traditionally conceived and executed.IMPACT activities are designed to be shared. They require that the child talks to someone, either to play a game, or to collect some data, or to make or do something. Parents are not being required to teach maths. All they are required to do is to support their child's learning, to talk through a task or to act as a resource. Sometimes parents help by supplying skills children have not yet developed. The results of the maths activity which is shared at home are brought back into the classroom and are elaborated by the teacher in the course of the subsequent week's classwork. This is of crucial importance in that it provides parents with a direct means of influencing or commenting upon the classroom curriculum. Traditionally, homework was perceived as very much an additional activity - the class lesson the next day proceeded regardless if one had failed to complete one's homework, except, of course, for any disciplinary action which might ensue. By contrast, the IMPACT task is embedded within the classroom maths curriculum although its actual performance takes place in the context of the home.

A crucial aspect of this process is represented by the IMPACT diary. This is a small diary, divided into sections, in which parents and children comment upon the task undertaken at home. This provides a mechanism whereby parents can and do make assessments of their child in relation to clearly specified mathematical skills. The diaries also enable a sustained parent-teacher dialogue to develop over an extended period of time.

IMPACT has been subject to detailed monitoring and research evaluation from its inception in 1985:

1. A two-year period of intensive monitoring and research in 3 LEAs with 36 schools. This included weekly visits to schools, a large quantity of interviews with teachers and parents, observation of teacher practice, the completion of diaries by children, parents and teachers, and video-taping in a sample of parents' homes.

2. As the scheme has spread to an increasing number of schools across the UK, we have monitored annually the progress of IMPACT in a sample of schools through questionnaires and teacher records.

Early results: We were able to present evidence that IMPACT works successfully over a number of years and with schools in different regions and in widely differing socio economic areas. There is evidence to suggest that children's performance in maths is improved - in some cases quite substantially - and that their attitude to maths becomes more positive. The response rates, in terms of the numbers of parents and children regularly taking part are extremely encouraging with well over 80% of families in a typical infant (grade 1 -4) class responding. Interestingly, the response rate does not appear to be related to the socio-economic class of the parents in the catchment area of the school. However, the teacher emerges as an influential factor here - an enthusiastic teacher can get nearly 100% of the families involved.

Significance: The IMPACT project is the largest and certainly the fastest growing project of its kind in Western Europe. The demand for traditional homework in primary schools within the UK is increasing, due to the pressure on teachers to improve standards of numeracy, and also to the desire on the part of many parents to support their child's learning in what is now perceived as a competitive educational market place. IMPACT provides a radical alternative to traditional homework. It succeeds in raising standards, encourages sustained parent/teacher dialogue and, crucially, is not socially divisive in the fashion of traditional homework.

Qualitative Research findings: In this paper we are concerned to illuminate what teachers perceived as the problems and benefits of involving parents in maths. This automatically raises the question of teacher's perceptions of parents and of parental attitudes to maths in particular, and to young children's learning in general. We looked at why teachers had wanted to start IMPACT in the first place, what they felt about it once it was used as a way of working, and what their feelings were about the long term effects of this type of non-traditional mathematics homework.

All of the teachers who took part in this particular piece of research said that they saw IMPACT as a means of raising the standard of pupils' attainments in maths. They were also keen to improve home-school relations and to bring about a better dialogue between parents and teachers. Most said that having some support had been a major factor in enabling them to start this sustained programme of parental involvement in their school and that this support could come from a variety of places. Colleagues – in their own and other neighbouring schools – were mentioned as one important source, and the Local Education Authority was mentioned as another. Surprisingly the head teacher (principal) was less important than these two.

Once the IMPACT programme was in place the teachers very quickly saw the benefits. In summary:

- children talked a great deal more about their maths, and about mathematical conversations and events which take place in the context of the home,
- children developed a much more positive attitude to their maths, and appeared more confident as a result,
- children were experiencing more of particular (and important) areas of maths –in particular data handling and representation, and practical activities, including games and investigations,
- children – especially infants – were almost all extremely keen to do their IMPACT and, in most cases, it was them who cajoled their parents into doing it rather than the other way round,
- the shared maths at home stimulated work in other areas of the curriculum, especially in science and language,
- the parents became far more involved in the child's everyday routine mathematics work. They also became much more aware of, and realistic about, what their child could and could not yet do.

The problems of running IMPACT included a number of issues surrounding the question of resources, and also a perception of some parental comments as extremely negative and therefore very discouraging. This was a much discussed point, and one which we feel justifies further research.

There is also a background to this initiative that helps to put a gloss on the context it provides. In recent years there has been considerable political and educational pressure on primary schools in the UK to increase the involvement of their parent group in the implementation of the school curriculum in its widest sense. This is apparent in the publications of the requirements on schools to increase the number of parent governors on school governing bodies and in the various documents published by the Department for Education but is especially enshrined in the Education Reform Act of 1988. (HMSO 1988). There had always been some level of involvement but the significant change was that parents had a *formal* right and role to play that may not have been exercised previously. As a result many schools were seen to consider ways of increasing the part parents played in school life in general. This phenomenon was also seen as relevant to what had previously been considered by many teachers as 'their professional domain', i.e. the curriculum content and implementation. Evidence from some commentators (McLeod 1990) illustrate clearly the well defined boundaries of responsibility that participants in the parent - teacher divide see and act upon. Clearly the changing nature of the roles of parents and teachers in schools challenges this boundary and thus challenges the socially constructed norms and behaviours enacted at the interface. It is this change that provides for a tension and readjustment that is only now being recognised as 'problematic'. (Merttens and Vass 1990, Walkerdine 1989).

That all human interaction is a social construct has been well documented (Berger 1968) and so far as the interface between schools and 'their parents' is concerned the process of construction is seen as a dialogue between two interested parties. This can also be seen as problematic and the nature of a dialogue is potentially misleading in this context. There is a growing body of evidence to suggest that the concept of an equal partnership between school and parents is fraught with difficulties. Some of this evidence comes from studies derived from various home - school reading projects (Wells 1986) that have been commonplace in a great many schools in the UK and USA for some years now. Despite this, the rhetoric of current educational orthodoxy is to present parental involvement as both desirable, unproblematic and affording equal status for both parties. Much of this reflects political considerations and not simply educational concerns. For a detailed discussion of some aspects of this see Merttens and V ..., 1990. Yet more evidence to show how schools construct and deal with their parent group in ways designed to maintain the status quo and the school as the powerful factor in any relationship is supported in the earlier work of Deal, 1983 and McDill and Rigsby, 1973, when they discuss the influence of subcultures on the culture and success of a school. However this work tends to accept the parent body as a cohesive group and treats its' influence as unproblematic. There are, however, some more current authors who are beginning to question some of the assumptions about parental opinion and school attitudes towards them in a more systematic fashion. One such work is by Rogoff and Lave who studied how knowledge gained from the home (or any non institutional environment) is often discounted and ignored in school. (Rogoff and Lave, 1984).

The size of the group selected for a structured discussion was 12 teachers and advisors and 48 questionnaires were distributed. 37 of these were returned completed, leaving 11 that received no response. The teachers were invited to a group discussion about the issues of increased parental involvement generally and their participation in a specific project in particular.

Some questions and responses.

What factors made you think you could start involving parents ?

The responses could be grouped in the following way :

A commitment to home school links	33%
Other local schools already doing it	20%
Staff colleagues keen to try something new	40%
Parents seen as responsive	7%

Other reasons stated included :
Support of senior management team.
Good parent/school links already established.

This would suggest that less than 40% of those staff who gave their primary reason as believing they are ready to involve parents in this way actually do so because they believe per se that it would help improve relationships or learning. Several respondents were vague about why they thought they had a commitment to home school links generally although most indicated an awareness of some of the literature documenting the potential success of such a scheme. Teachers were very aware of what other schools around them were doing and what they felt their parent group wanted to see. This was often based, however, on anecdotal evidence from colleagues in other schools who passed on positive comments from their own experiences.

What are the most important factors in deciding to involve parents in this way ? Indicate in order of importance with 1 as the most important and 9 as least important.

The respondents were asked to score the following categories. The total scores allocated to each section were then averaged by dividing by 37. A no response or a negative response was treated as a 9 (least or not important) and a opportunity was available for other replies although only one person actually did this. This was a new headteacher who wanted to develop a fresh initiative in the school that would be seen as her own idea.

The potential mathematical benefits for your pupils	2.3
Need to develop home/school links generally	3.5
Support from your headteacher	4.2
Your perceptions about your colleagues	5.6
Your perceptions about the parent group	5.8
Need to increase practical work in mathematics	6.6

Need to develop homework initiatives	7.3
Need to develop maths curriculum	8.2
Other reason - please specify * see above	8.9

Once again the teachers perceptions about the needs of the parent group generally only came 5th in order of importance. The perceived school needs were ranked as the first (via a need to enhance pupil learning) and second most important factors. Interestingly the mathematical element was not considered as important as the need for peer support. As most of the sample were not head teachers (only 2 were) it was surprising to find that headteacher support was ranked only third. This conflicts with some of the work of Nias who indicates that the support of a powerful sponsor (i.e. in a school this is usually the headteacher) is often the most important factor in initiating or preventing institutional change. (Nias 1981). It is also possible that ambiguous categories could mask different reasons. For example some teachers could have started involving parents because their colleagues are receptive to the process and others because they felt their colleagues ought to be but were not yet. Both of these perceptions could be contained within one response and the categories contain a potential weakness as a result. A further question about support reinforced this perspective.

Could you rank ,in order of importance ,the people whose support was most important to you. 1 as most important.

The responses were scored by adding the totals and averaging by dividing by 37. These were, in order :

The LEA (Advisory teachers)	2.8
Colleagues	3.3
Head teachers	3.8
Project office	4.2
Parent group	4.8
Specific colleague	7.0
Other	7.0

The findings from this question support the view expressed above that teachers do not rate the support of parents as high or essential when considering ways of involving parents more.

Could you detail the most important benefit you hoped to gain from involving parents in this way ?

Help pupils talk about their maths	32%
Explain maths to parents better	20%
Improve the image of maths with parents/pupils	42%
Develop home/school links	6%

Other responses included :
Share information about pupil progress better
Help to develop practical work in maths
Children to see maths outside of classroom
To develop a better understanding of maths generally.

These responses indicate that teachers feel parents 'need' to understand the mathematics their children do in school more significantly. It emphasises the schools needs rather than those of the parents or pupils. This could be evidence of power retention on the part of schools as the teachers clearly define the needs of both pupils and parents. There is also clear evidence, however, that teachers recognise the importance of language in the development of mathematical thinking but also seek to define the pupil needs in their own terms. Various responses were categorised as wanting to improve the image of mathematics generally with more responses emphasising the importance of this in respect of parents than in respect of the pupils. It was surprising that this proved to be the most frequent response despite the well recognised problems that mathematics has in terms of subject image. Teachers casual comments indicated that they felt this to be an important step in motivating pupils and gaining parental support for what they are trying to do.

What potential problems concerned you most about involving parents in this way ?

Ensuring parents understood their role	42%
Schools losing interest eventually	8%
Resources. Do the school have enough ?	32%
Parental response may be a problem	8%
Extra burden on staff colleagues	6%

Other responses included
Schools seeing this as a solution to all of their problems
Parents seeing only number as important in maths
Long term problems for the school.
Raising parental expectations unrealistically

It is clear from these responses that once again teachers feel that their parent group need educating about the process of learning mathematics. They appear to have little confidence in the parent group as a whole to understand, approve or support any initiative without subverting it to a formal, restricted and numoer based curriculum. The issue of resourcing such an initiative is to be expected but 50% expect a 'problem' with the response of parents. About a quarter expec, the school to fail by not keeping the momentum going having raised expectations generally. They anticipate this leading to problems with the parents. These responses are consistent with a model of the school as expert and controlling and the parents as recipients of a perceived wisdom but passive or problematic.

How important was it to have the support of others ?
Please specify who. On a scale of 1-5
1 as the most important
5 as the least important.

86% said that to have some support was essential.
Those indicated as important included the following :

Colleagues	57%
LEA	26%
Advisory staff	14%

This score indicates that not one of the sample said that the support of the parent group was required or important. The single most important group to provide support is seen as colleagues. This is consistent with the work of Nias in her research into peer group support as essential (Nias 1991 ibid.). The next section concerns teachers' opinions about what implications parental involvement has.

What would you describe as the two main benefits as far as the parents themselves are concerned ?

They can see what their children are doing	41%
They are more involved in their child's learning	25%
Informs them of their child's progress	18%
They can see the 'ifficulty of certain concepts	9 %

Other responses included
It structures the help they can give.
Gives parents a specific role
They can see their child's potential.

What would you describe as the two main problems ?

Parents not listening to pupils	25%
Parental lack of confidence with maths	16%
Parents doing the work for the child	16%
Parents not understanding the tasks	12%
Parents not finding time to get involved	12%
Parents/Teachers disagreeing over tasks	12%
Establishing whole school approach	8%
Parents not understanding the maths involved	3%

Other responses included
Teachers coming to know the tasks too well
Parents not seeing the potential

This question attracted a mixture of responses. However it is possible to classify them as teachers seeing parental weakness or inadequacy in 7 out of the 8 most common replies. Only 8% of the replies indicated that one of the main problems could be to do with the school role. 92% of these replies anticipated that the parental role would provide the main problems and usually this reflected their 'perceived' inabilities in mathematics as a subject in one form or another.

Could you say what makes a supportive parent ?

One who gives their child time and attention	68%
Encouraging parent	12%
Works with pupil not for the child	8%
Works with the school	4%
One who sees their role as a partnership	4%
Treats work as a pleasure	4%

No other responses although the most common response was expressed in several differing ways.

Could you say what you think makes a parent who is not supportive ?

One who does not get involved	35%
Negative about child's work	21%
Don't recognise the value of the work	21%
Separates schoolwork from homework	7%
Some negative about school generally	7%
One that does work for the child	4%
Academic parents who over formalise work	4%

Other responses indicated that parents who don't talk with their children are not supportive. Teachers generally felt that openly hostile parents are easier to cope with than the disinterested or uninvolved. Some teachers felt that not getting involved was the same as being negative about their child's work because of the implications carried by the opinion.

Does this kind of involvement alter parental opinions ?

No	76%
Yes	12%
Unsure/ Too early to say	12%

Do you feel that such greater involvement is valued by parents generally ? Detail if possible.

Parents generally like the formal aspects of maths.	42%
Sometimes	36%
No	12%
Yes, they feel involved	12%

There are certain inconsistencies in attitude implied by the responses offered. For example, teachers say they like to see projects of this type as a partnership and that parents welcome the initiatives. However, they do not generally welcome any change in their role in response to parental pressure and do not credit parents with an expertise or knowledge that could serve to balance any partnership. They negate or discount problems as being the responsibility of the parents and yet claim to wish to see greater integration of parental views. Thus teachers retain control over the educational and political agenda. This status quo is illuminated by the overwhelming claim that teachers do not respond to parental pressure and that they do not feel parents change their opinions anyway as a result of enhanced involvement. The attitude is summarised by the view that problems are caused by and thus can only be solved by the parents intervention. This releases the school from much of the responsibility. Indeed many parents are seen as failing to understand the mathematics involved and thus requiring training of some sort and those that do demonstrate an expertise are dismissed as "academic". This betrays a stereotypical view of parental opinions on the part of teachers and confirms the view that teachers feel parents are over formal in their approach to mathematics and generally uninformed about the process of educating children. This view also confirms the perspective of school as expert and powerful and parents as subordinate and passive. Despite claims to the contrary schools do not seek to challenge this orthodoxy in the arena of parental involvement.

Some conclusions.

We have tried to voice some tentative conclusions in the text of this report but some more general and summary conclusions are suggested here. Teachers say they are committed to the principle of increased parental involvement but the reasons for this are mixed. Most of them appear to see its importance from the perspective of the school and not pupil or parent. They also have strong opinions about how far that 'involvement' should permeate what they see as their professional area of expertise. The overwhelming view is of teachers and schools 'educating' parents into accepting what the school has deemed appropriate. This reinforces the inequality of the power relationships within the school- parent divide. I feel there is now evidence that, despite their expressed opinions, teachers have no real desire to see that relationship fundamentally challenged. On the occasions that it is challenged by parental activity the parental viewpoint is negated and then discounted.

Teachers feel the need for support in their work and peers and head teachers are overwhelmingly the primary source of that support. The support of parents is expressed as important but not treated as such. Teachers are also very aware of what they feel they must 'be seen to be doing' in this area and parental involvement schemes or projects are perceived as what their task involves. One source articulated this as " what parents and government expect these days". This view would be represented as being held by a teacher who thinks parental involvement is important but for very different reasons. Teachers opinions underpin their stereotypical view that all parents see maths as hard and threatening and need educating to understand it. This is passed on to their children and thus makes the teachers task that much more difficult. Parents who do not see it as difficult are treated as not typical and thus discounted when they intervene. Parental opinion has been effectively negated through this stance.

Teachers do not have a clear view of what parental involvement is. Clearly it is problematic and clearly teachers accept multiple levels of involvement. Some teachers see parental role within these constraints as supervising homework whilst others accept a higher level of involvement. Very few, despite the claims, welcome any threat to the status quo in which teachers transmit knowledge and expertise and parents receive. Schools have yet to take on board any notion of true partnership, despite the rhetoric. I feel that despite the increase in schemes of this sort, despite the political pressure and despite teachers' claims, parental involvement in educational change is still seen as a threat by schools and is not established or welcomed.

The whole question of 'What teachers think parents think about teaching and learning' seems to us to be at the heart of much of this work in the area of parental involvement – in particular in the field of mathematic .. The questions of teachers orientations to parents, and how they construct what is a 'good' parent are, we feel, crucial to our understanding of how parental partnership programmes can act to improve children's opportunities and attainment in education.

References.

Bastiani, J. 1991, '*Reporting to Parents.*' A survey of the views of parents, teachers and pupils, University of Nottingham, School of Education with Warwickshire County Council. Warks. UK

Berger P, 1968, The Social Construction of Reality, Penguin, London, UK

Deal, T. and Kennedy, A. 1983, Culture and School performance. *Educational Leadership*, 40(5), 14-15.

Education Reform Act 1988. HMSO , London.

Hamilton, D and Griffiths, A. 1984, Parent, Teacher, Child, Methuen, London, UK.

Heath, S.B. , '*Questioning at Home and School : A comparative study.*' In Hammersley, M . (Ed) 1986, Case Studies in Classroom Research, Open University Press, Milton Keynes, UK.

Hughes, M " Parental Involvement. Rhetoric or Reality", in Partnerships in Mathematics, Parents and Schools. Eds Merttens and Vass 1993, Falmer Press, London, UK.

McDill, E and Rigsby, L. 1973, Structure and Process in Secondary Schools. Baltimore, John Hopkins Press, USA.

McLeod, F. 1990, Parental Involvement : The Contemporary Challenge, Falmer Press, London, UK.

Merttens, R and Vass, J 1990, Sharing Maths Cultures, Falmer Press. London, UK.

Merttens, R. and Vass, J. 1993 Partnerships in Maths - Parents and Schools, Falmer Press, London, UK.

Nias, J. 1981, 'Commitment and motivation in primary school teachers', *Educational Review*, 33, pp 181-90.

Rogoff, B and Lave, J, 1984, Everyday Cognition - Its development in social context, Harvard University Press, Cambridge, USA

Ruling the Margins: problematising parental involvement. 1993 Merttens et al. (Eds). University of North London Press, UK.

Topping , K and Wolfendale, S , 1985 Parental Involvement in Reading, Croom Helm, UK.

Walkerdine, V. Ed. Counting Girls Out. 1989, Virago, London, UK.

Wells, G, 1986, The Meaning Makers - a report on the Bristol Language Development Project, Ontario Institute for studies in Education, USA.

Wolfendale,S 1983 Parental Participation in Children's Development and Education, Gordon & Breach, London.

MATHEMATICAL CONCEPT FORMATION BY DEFINITIONS VERSUS EXAMPLES IN ELEMENTARY SCHOOL STUDENTS

Malca Mountwitten – The Israeli Curriculum Center, Ministry of Education

Shlomo Vinner – The Hebrew University, Israel

Abstract

The study examines the cognitive effectiveness of two learning methods for mathematical concept acquisition: a) learning by examples b) learning by definition and description of the concept's general properties. The findings show that a) a certain percentage of students are able to master mathematical concepts after learning them by either one of the methods used. b) There is an interaction between the nature of the concept and the success in acquiring it by the specific learning methods. c) In most cases, there were no significant differences in the performance of different cognitive tasks between the groups using the different methods. d) In all groups, for the 4 concepts examined, the older students' achievements were better than the younger ones'. This age related achievement gap was most evident in the group that learned by examples.

Introduction

Cognitive psychology and philosophy are preoccupied with the question: What is a concept? An important aspect of the discussion pertains to the question of whether or not it is possible to characterize a concept unequivocally by stating its attributes. This question is easily answered concerning mathematical concepts: in mathematics, every concept, which is not primary, has a definition, which states the meaning of the new expression in terms of previous ones. The definition determines the extension of the concept (the set of all the examples which belong to it) and how it relates to other concepts.

From the point of view of the logical aspect of "concept," the situation in mathematics is clear; this is not the case with psychological aspects, namely questions relating to concept acquisition.

Concept acquisition, which is one of the main components of learning, has different aspects: knowing the extension of the concept, knowing the content of the concept (its defining attributes), understanding the name of the concept and so on. According to Klausmeier (1976) there are 4 levels of concept acquisition: the concrete level, the identity level, the classification level, and the formal level.

In this research, learning was tested by the completion of several cognitive tasks concerning the concept:

1. Identifying new examples and non-examples of the concept
2. Constructing new examples of the concept
3. Giving a concept definition
4. Identifying general properties as characteristic or non-characteristic of the concept.

The research in concept acquisition has raised the following questions: What are the different methods of learning concepts? (See, for example: Herron, Agbebi, Cattrell, Sills, 1976) How is the concept represented in the student's mind? (Bruner, Goodnow, Austin, 1956; Rosch 1977; Brooks, 1978, 1987) Is the student's ability to learn a concept related to his present or future developmental level? (Vygotsky, 1934/1962; Mervis, Rosch, 1981; Hershkowitz, Bruckheimer, Vinner 1987) and so on.

Almost all research in mathematical concepts has dealt with geometric concepts. These concepts have a visual aspect that might make learning them from examples much easier. In this study a variety of concepts was examined in order to determine whether or not there is a connection between the nature of the concept and the processes of acquiring it by the students. Unlike other studies whose aim was educational-practical, this one is more basic-cognitive and its aim is to investigate the cognitive effectiveness of different learning methods for the acquisition of different aspects of the concepts learned.

This study examines different aspects of the acquisition processes of mathematical concepts in elementary school students. The students, fifth graders and eighth graders, were asked to learn a concept, independently, in one of two ways:

1) by being presented with a definition of the concept and verbal description of its general properties
2) by being presented with a series of examples and non-examples of the concept

The acquisition processes were examined using the following concepts: ***stepnum*** (*madregon*)* (*stepnum* is a natural number whose digits are consecutive numerals, written in order from greatest to least or from least to greatest), ***seveny*** (sheva'on)* (*seveny* is a natural number, whose digits add up to 7), ***rectagon*** (*metzushar*)* (*rectagon* is a polygon with a right angle), ***arithmetic sequence***.

The first two concepts are artificial-arithmetic concepts, *rectagon* is an artificial-geometric concept; *arithmetic sequence* is a known arithmetic concept.

Methodology

In each one of the four concept experiments, for every learning method, there were 30 fifth graders, (boys and girls) and 30 eighth graders, (boys and girls). All the participants of the same concept experiment were taken from the same school, so that it is reasonable to assume that the differences between them in the outcome of the study are a result of age difference or the specific treatment used and not due to population differences.

Every student learned the concept using one of the methods mentioned above. Special working sheets were developed for each learning method. (see appendices 1, 2.) After the presentation of the concept to the student, an immediate test (see appendix 3) was given followed by a personal interview, during which the student explained the answers given in the test.

Results, Analysis and Discussion

The first research question was whether students are able to acquire a mathematical concept, unknown to them from previous experience, by one of the learning methods used. The number of students who showed mastery in acquiring the above four concepts appears in table 1.

Other related questions were: Is a specific learning method suitable for acquiring some concepts more than others? Is a specific learning method suitable for acquiring some aspects of the concept more than others? Is a specific learning method more adequate for younger students than for older ones?

* The names in brackets are the concepts' Hebrew names.

Table 1: The number (percentage) of eighth graders and fifth graders who mastered the concepts *stepnum, seveny, rectagon, arithmetic sequence,* by the two learning methods.*

learning method / concept	eighth graders		fifth graders	
	examples	definition and general properties	examples	definition and general properties
Stepnum	20 (64%)	13 (42%)	9 (26%)	6 (20%)
Seveny	16 (53%)	17 (57%)	5 (16%)	17 (53%)
Rectagon	13 (43%)	20 (67%)	2 (7%)	7 (23%)
Arithmetic Series	**	**	12 (40%)	12 (40%)

The results in the test (table 1) show the same tendency as the results in the different sub-tests: example identification, example construction etc. These results point to the cognitive effectiveness of different learning methods for concept acquisition.

a. The results suggest that a certain percentage of students will be able to master mathematical concepts, of the same complexity as the concepts examined, after learning them by one of the methods used: by example and by definition and general properties. The percentage of students who mastered the concepts *stepnum, seveny, rectagon, arithmetic sequence* differs from one another but all of the concepts used could be fully acquired by one of the learning methods.

   It is especially interesting to note that even some of the younger students, the fifth graders, mastered the concepts as a result of learning them by definition and general properties (the percentage ranges from 20% in the case of *stepnum* up to 60% in the case of *arithmetic sequence*). This finding is interesting because it contradicts some teachers' intuition that it is impossible to teach mathematical concepts relying on general definitions and statements alone. Also, some studies show that elementary school students cannot learn a concept by definition alone, without added examples (Klausmeier, 1976; Tennyson, 1973).

   According to Vinner (1983), concept acquisition, as expressed by the ability to use a concept, depends on the formation of a *concept image* (The concept image is a set of properties together with the images related to the concept in the person's mind). The concept image might be identical to the concept definition, be part of it or might even contradict it. The role of the definition in the student's concept image formation is not clear: the definition given to the student might create in his mind a "complete" concept image, a partial one or nothing at all. In this study, the definition, together with the statements

* Mastery of the concept means scoring at least 40 out of 46 possible points.
** Eighth graders did not participate in the *arithmetic sequence* experiment because it was familiar to them from school.

about the concept's properties, formed a complete concept image in some of the students' mind, as inferred from their tests.

b. The comparison of the learning methods shows that the fifth graders who learned the concepts by definition and general properties succeeded better than those who learned them by examples. This was particularly evident with the concepts *seveny* a. : *rectagon*. The same phenomenon appeared with the eighth graders in the *seveny* and the *rectagon* experiments but not in the *stepnum* experiment. In the last one, the eighth graders who learned by examples had an advantage over the others. The achievement differences were statistically significant with the tasks of example identification and statements evaluation. These findings lead to the conclusion that there is an interaction between the nature of the concept and the success in acquiring it by one of the learning methods: concepts like *rectagon* are more difficult to acquire by examples because their critical attributes are not prominent. Even though the examples of *rectagon* are drawings which have a prominent visual aspect, the relevant component, the right angle, was not noticed by most of the students. It turned out that finding the defining property of *rectagon*, required an analytical examination of the examples and not merely visual scanning. Many students realized that a rectangle is a *rectagon* but could not justify their answer, which was based only on global perception. These students could not, of course, show mastery of *rectagon*.

   On the other hand, from the examples of *stepnum*, especially from the multiple digit ones, it was much easier to infer the defining attributes.

c. Another question, concerning the comparison of the learning methods was whether there is a difference related to the learning methods used, in performing different cognitive tasks.

   In most cases, there were no significant differences in performing different cognitive tasks between the groups using the different methods. Even though the students who learned the concept by definition had a slight advantage in formulating a concept definition, the general picture was as follows: performing the different tasks was influenced by the student's mathematical ability and by the task complexity and not by the learning method, as could be seen, for example, from table 2. This table gives the number of students who mastered the different tasks with the concept *stepnum* (questions 1,2,4,5 in the test appendix 3).

Table 2: The number (percentage) of eighth graders and fifth graders who mastered the different tasks with the concept *stepnum*, by the two learning methods.

task \ learning method	eighth graders		fifth graders	
	examples 31 s.	definition and general properties 31 s.	examples 34 s.	definition and general properties 30 s.
example identification	27 (87%)	19 (61%)	11 (32%)	13 (43%)
example construction	25 (81%)	19 (61%)	15 (44%)	12 (40%)
definition formulation	5 (16%)	6 (19%)	1 (3%)	4 (13%)
statement evaluation	21 (68%)	12 (39%)	9 (26%)	7 (23%)
complete test	20 (64%)	13 (42%)	9 (26%)	6 (20%)

From this table, it is easy to see that the differences in achievements due to the learning methods, are consistent over the tasks. Those who learned by examples did not have an advantage in example identification; those who learned by definition and general properties did not succeed better in statement evaluation even though they had previous experience with statements during the learning session. On the contrary, among the eighth graders, 68% of those who learned by examples showed mastery in statement evaluation as opposed to only 39% of those who learned by definition and general properties. In other words, one learning method might be more effective than another for the acquisition of the concept as a whole, but not for the acquisition of a specific aspect of the concept.

In Herron et al. (1976), the learning method had a greater effect on performing different tasks with the concept. According to another finding of Herron, the working definition (the one which serves in identification and construction of examples) was more coherent with the formulated definition in students who learned by examples. This finding does have support in our study as well.

d. In all groups, for all concepts, in each subtest, the eighth graders' achievements were better than the fifth graders'. This age related achievement gap was most evident i.· the group that learned by examples (with significant level of .005). This finding is not surprising of course, and agrees with Piaget's developmental theory. According to this theory, some cognitive skills, probably needed for performing the tasks of this study, develop at the age of formal thinking (Sinclair, 1970). Another difference between the fifth graders and the eighth graders demands explanation. In the *stepnum* experiment, the eighth graders who learned the concept by examples did better than those who learned it by definition and general properties. Amongst the fifth graders the picture was reversed, namely there was an advantage in learning by definition and general properties. This finding is surprising: learning by definition and general properties can be considered to be a formal operation task. The above finding is, somehow, inconsistent with Piaget's theory. One possible explanation is that the younger students are not intellectually ready for learning by examples because it requires the ability to analyze, abstract, generalize and draw conclusions. On the other hand, learning by definition and general properties is more direct, although superficial and therefore it is easier for the younger students. In many cases it is difficult to distinguish unequivocally between a correct answer, based on real understanding, and one which is based on partial understanding. Some of the students learned how to "play with words" and give correct answers without real understanding. This behavior, which is pseudo-conceptual (see Vinner, 1993) was found both among the fifth graders and the eighth graders. (The tendency to "play with words" does not decrease with age; on the contrary, it increases slightly.) But, the ability to learn by examples develops significantly with age. The great progress with age in learning by examples makes this method superior for the eighth graders.

As was mentioned earlier, the age related achievement gap was most evident in the group that learned by examples. In other words, the progress with age in learning by examples was greater than in the other learning methods.

This finding can be explained as follows: the fifth graders reach a reasonable verbal understanding which does not improve a lot with age. On the other hand, the **thinking** ability, involved in analysis, generalization and abstraction develops more significantly with age.

Another remark is needed here: It turns out that learning by examples proved to be more meaningful than learning by definition and general properties. (The details can not be given here because of space problems.) Even though learning by definition and general properties proved to be successful, caution must be exercised so as not to fall into the trap which this method could lead to: empty verbalization without understanding (Nesher, Grossman-Mountwitten, 1975).

Appendix 1

Working sheets by examples

Stepnum

1. 12,345 is a *stepnum*
2. 23,456 is a *stepnum*
3. 43,210 is a *stepnum*
4. 76,543 is a *stepnum*

5. Is 2,468 a *stepnum?*
6. No. Is 7,531 a *stepnum?*
7. No. Is 1,234 a *stepnum?*
8. Yes. Is 4,321 a *stepnum?*

32. No. Is 24,315 a *stepnum?*
33. No

Appendix 2

Working sheets by definition and properties

Stepnum

Stepnum is a natural number whose digits are consecutive numerals, written in order from greatest to least or from least to greatest.

1. Are all the digits of a *stepnum* unequal?
2. Yes. In a *stepnum may* two of the digits be equal to each other?
3. No. Can a *stepnum* have 3 digits?

36. Yes. Can a *stepnum* end with 0?
37. Yes.

Appendix 3

Test-*Stepnum*

1. Are these numbers *Stepnums?* Write "yes" or "no" and explain your answer.

a. 8,641
b. 798

j. 456,789

2. Give 3 examples of *stepnum.*
3. Give 3 examples of a number which is not a *stepnum.*
4. Describe what is meant by the word *stepnum.*
5. Write "correct" or "incorrect" and explain.

a. The units digit of every *stepnum* is smaller than its tens digit.

j. There is a three digit *stepnum* whose tens digit is greater than both its units digit and its hundreds digit.

References

Brooks, L., Decentralized control of categorization: the role of prior processing episodes, in: Neisser V. (Ed.) **Concepts and Conceptual Development: Ecological and Intellectual Factors in Categorization,** 1987 Cambridge University Press.

Brooks, L., Non-analytic concept formation and memory for instances. In Rosch, E., Lloyd, B.C., (Eds), **Cognition and Categorization,** Hillsdale, N.J.: Erlbaum, 1978.

Bruner, J.S., Goodnow, J. Austin, G.A., **A Study of Thinking,** N.Y.: Wiley 1956.

Herron, J.D., Agbebi, E., Cattrell, L., Sills, T.W., Concept formations a function of institutional procedure or: What results from ineffective teaching; **Science Education,** 1976, 60 (3), 375-388.

Hershkowitz, R., Bruckheimer, M., Vinner, S., Activities with teachers based on cognitive research, **N.C.T.M. Year Book in Geometry,** 1987.

Klausmeier, H.J., Conceptual Development during the school years. In: Levin, J.R., Allen, V.L., (Eds.), **Cognitive learning in children: Theories and Strategies,** N.Y.: Academic Press, 1976a.

Mervis, C.B., Rosch, E., Categorization of natural objects. **Annual Re w of Psychology,** 1981, 32, 89-115.

Nesher, P., Grossman-Mountwitten, M., Intuitive approach in teaching isometric transformations, **Studies in Education, University of Haifa,** 1975, 6, 89-96 (In Hebrew).

Sinclair, H., Piaget's Theory of Development: The Main Stages, in Rosskope, M.F., Steffe, L.P., Taback, S. (Eds), **Piagetian Cognitive-Development Research and Mathematical Education,** N.C.T.M., Proceedings of a conference conducted at Columbia University, 1970.

Tennyson, R.D. Effect of negative instances in concept acquisition, using a verbal learning task. **Journal of Educational Psychology** 1973, 64,247-260.

Vinner, S., Concept definition, concept image and the notion of function. **International Journal of Mathematics Education Science and Technology,** 1983, 14, (3), 293-305.

Vinner, S., Pseudo-conceptual and pseudo-analytical behaviors in science learning. A paper presented at the international conference on Science Education in Developing Countries, Jerusalem, 1993, January 3-7.

Vygotsky, L.S., **Thought and Language,** (Hanfmann, E., Vakar, G., trans.) Cambridge, Mass.: MIT Press, 1962.

CONSTRUCTING A LANGUAGE FOR TEACHING

Judy Mousley
Deakin University
Geelong, Australia

Peter Sullivan
Australian Catholic University
Christ Campus, Australia

A survey of views 132 teachers and teacher educators, on the elements of quality mathematics teaching, was conducted. Some of the results seem contradictory, pointing to the need for closer examination of terms commonly used in describing mathematics pedagogy. The results of an attempt to explore the use of just one term used frequently by the survey's respondents demonstrated disparate interpretations amongst 180 conference participants. It is concluded that just as teachers need to provide opportunities in mathematics classes for explicit negotiation of meaning, teacher educators need to organise opportunities for active exploration of the language of teaching, in order to establish situated contexts as sources of personal and intersubjective meaning.

Introduction

Cobb, Yackel and Wood (1992) argue convincingly for opportunities for students to elaborate and negotiate different understandings of mathematical processes and terms; and there is evidence that such advice is becoming more widely accepted and practised within the mathematics education community. For instance, the theme of developing meaningful experience-based language was taken up in the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989). Similarly, the Australian Education Council (1990) describes a process of developing and building up mathematical knowledge through describing, questioning, arguing, predicting, justifying, recording, clarifying and "productive sharing of ideas" (p. 13).

> Students should learn to use language as a tool for reflecting on their mathematical experiences and hence for their own mathematical learning. Explaining to oneself and others ... can be a powerful means of working through and clarifying ideas. (pp. 19-20)

However, there does not seem to be the same emphasis on learning about teaching mathematics through exploring its language. In the pre-service and in-service education of mathematics teachers, opportunities for explicit negotiation of meaning for pedagogical terms and phrases are rare. Teachers take on "constructivist" approaches, students engage in "problem solving", learning is "student centred", activities are "non-threatening" and content is "real world". The language bandwagons roll on; but with little elaboration of individual interpretations of such terms, or explication of the assumptions being made about taken-as-shared understandings (Bauersfeld, Krummheuer & Voight, 1988).

Becoming a member of any community involves orientation towards an established but developing set of linguistic behaviours (Steedman, 1991). For the community of mathematics educators, phrases and terms used in describing pedagogy invoke meanings that are usually taken to be shared between teachers, educators, and authors of instructional or theoretical writings. However, as von Glasersfeld (1991) points out,

> Language frequently creates the illusion that ideas, concepts and even whole chunks of knowledge are transported from a speaker to a listener. This illusion is extraordinarily powerful because it springs from the belief that meaning of words and phrases is fixed somewhere outside the users of words and phrases. (p. xiv)

This paper focuses on aspects of the language that teachers and teacher educators used when describing quality mathematics teaching.

Researching quality teaching

Our interest in exploring the ways that people describe mathematics pedagogy has developed further during our current research project. Earlier studies by Mousley and Clements (1990) and Mousley, Sullivan and Clements (1991) had demonstrated that what student teachers reported that they had observed in classrooms seemed to be the antithesis of current theory and of the behaviours espoused in their pre-service teacher education courses. While teachers believed that they were demonstrating quality teaching, features of constructivist approaches, student-centred learning and real world content seemed to be absent from the lessons observed. It was thought possible that the undergraduate students surveyed may have been seeing some high quality teaching, but may not have identified such teaching because of the subtle, sophisticated and fleeting nature of some features of classroom interaction. Alternatively, teachers may be using these terms to describe their teaching, with little evidence in terms of classroom practice.

This led to further discussions on how to structure opportunities for guided observation of quality mathematics teaching in a variety of settings; and of how to make the observation processes analytical, interactive and research-based. It seemed that the first stage of such a project should be to identify the features that quality teaching might include, in order to capture these for use in a multi-media resource base.

Summary of methods

A survey was prepared, seeking the views of mathematics eductors and experienced teachers on the elements of quality teaching. It was thought that a questionnaire would allow collection of the opinions of a sample of experienced teachers and teacher educators in Australia and the USA. After initial trailing in a number of situations, and consequent refinement, the survey was completed by 132 respondents.

Two different sets of survey items were used. The first section consisted of an open-response item:

> ... We want you to imagine a mathematics lesson, at any year level, where the students are learning, for example, to estimate the mass of various objects, or to add fractions, or to record given information as a graph. ... Please write down the most important characteristics which a **quality** mathematics lesson on any of these concepts/skills would usually have.

The second, more structured, part of the instrument used fixed format items. There were 78 pairs of descriptors, clustered under headings arising from the original research: *Teaching environment, Lesson aims, Lesson content, Presentation, Class activities, Questions, Aids, Assessment* and *Closure*. Descriptors were presented as bipolars. For instance, one pair of descriptors (the one we wish to focus on for this paper) was listed in the Content cluster. The item read:

In a (quality) lesson, content

is negotiated <--> is imposed

For each set of bipolars, participants were asked to mark on the line where a teacher ought to aim in order to facilitate quality mathematics lessons. These marks were later converted using a scale from 1 to 7 so that means and standard deviations could be calculated and so the distribution of responses between categories of participants could be described.

For each cluster of such items, participants were also asked to add any of their own descriptors, and then to indicate which of the descriptors in each cluster they considered to be the most important feature/s of quality lessons. Later, they were asked to choose, in order, the 5-10 most important of all of the survey's pairs of descriptors.

Summary of results: Open-response items

Analysis of the results of the open-response section of the survey was undertaken using the qualitative data analysis package *Nudist*. Descriptors used by participants were categorised according to what seemed to be their major function. In summary, six groups of factors emerged as components of the replies. Their general descriptions are:

Building understanding: This is about a recognition of a content to be covered, and of strategies to achieve this end by building on existing knowledge, using materials to explain and clarify concepts, choosing appropriate sequences, helping students to make connections and to form relationships, and knowing the meaning of terms.

Communication: Under this heading were included statements related to opportunities for talking, explaining, describing, listening, asking, clarifying, sharing, writing, reporting, and recording. The emphasis within this category was on expressing and communicating mathematics.

Engagement: This group was about facilitating student involvement in their own learning. It included actively involving the students in their learning, and motivating students to learn, by actions such as using personally relevant material or real world situations, and by seeking to make learning enjoyable.

Problem solving: This category includes activities references to problem-solving activity but also descriptors such as risk taking, challenging, exploring, investigating, thinking, asking, and posing. In general, it was about students using their own conceptions to interpret unfamiliar situations and becoming comfortable with their own ability to do this.

Task orientation (Later re-named "Organisation for learning"): This included a focus by the teacher on specific goals which are made explicit, clear instructions, efficient organisation and some assessment of the achievement of the lessons' aims.

Teacher concern: This is about treating the students as individuals, the creation of environments which support opportunities for success by all students, the development of mutually positive relationships between teacher and students, and about shared goal setting.

While these categories represent a summary of the responses, the groups are different in both the focus and the locus of responsibility. For example, the category *Building understanding* was outstanding in terms of the number of times respondents referred to its features (112 times). In this category, strong inferences of teacher decision, teacher direction, teacher explanations and teacher control were evident. Upon reflection, it seemed that each of the other categories could be considered as a subcategory of *Building understanding*, for each was a vehicle for building mathematical understanding. While other groups, such as *Communication, Organisation for learning* and *Use of materials* may be significant in themselves, the main purpose of the comments included within each group seemed to be aimed primarily at building students' mathematical understandings.

Summary of results: Structured items

Data from the second, structured part of the survey also emphasised the building of mathematical understanding by students. Space does not allow full results to be reported, but the composite picture for a quality lesson developed as a teacher-led facilitation of learning, where the students are engaged in meaningful and interesting activities which encourage them to think for themselves. (More detailed results and discussion are presented in Mousley & Sullivan, 1992; Mousley, Sullivan & Waywood, 1993; Sullivan & Mousley, 1992; Sullivan & Mousley, 1993a, 1993b.)

The following tables present quantitative results for a few of the descriptors from the *Lesson aims* and *Lesson content* clusters.

Table 1:

Some Descriptors Related to *Lesson aims*

The stem for this cluster was "In the lesson ...".

Descriptors			Mean	S.D.
there is no clear purpose	<-------------X>	there is a clear purpose	6.6	0.8
pupils are aware of the aims	<--X----------->	pupils are not aware of aims	2.1	1.3

the aims are negotiated	<-----X------->	the aims are imposed	3.2	1.7

(Scale = 1 to 7. Note that there were other pairs included, not listed above.)

Table 4:
Some descriptors related to *Lesson content*

The stem for this cluster was "The content of the lesson ...".

Descriptors			Mean	S.D.
not linked to pupils' experience	<----------X-->	linked to pupils' experience	6 2	1.2
socially relevant	<---X--------->	not socially relevant	2.8	1.4
included all st's backgrounds	<---X--------->	excluded some st's backgrounds	2.2	1.3
clearly sequenced	<--X---------->	not clearly sequenced	2 2	1.2

With regard to the *Lessons aims*, 92 of the respondents chose "there is a clear purpose" as the most important descriptor of this cluster. In contrast, 19 respondents chose "the pupils were aware of the aims" and only 5 chose "the aims were negotiated". Likewise, while 56 respondents rated "there is a clear purpose" in the 5-10 most important descriptors over the whole survey, descriptors suggesting involvement of students in setting aims gained very little support. In the *Lesson content* section, the descriptor "was linked to real life", was rated within the within the 5-10 most important descriptors overall by only 27 people.

There were some interesting contradictions apparent in this section of the data, and these echoed the way terms and phrases had been used by respondents in the unstructured section of the survey. For instance, both the teachers "having a clear purpose" and the students "negotiating the aims of the lesson" were rated, frequently by the same respondents, as features of quality teaching. One would think that it is not possible to achieve a clear purpose while allowing disparate aims to be negotiated, because a teacher's having a clear purpose for the lesson is somewhat indicative of decisions on its direction being made by the teacher. This apparent contradiction was reinforced by the majority of respondents not thinking it important for students to be aware of a lesson's aims. Central to the view of the development of personal constructs is the belief that knowledge is a product of choice (Gergen, 1985), but it is clear that the majority of respondents did not believe that students should decide what they need to learn. Similarly, thinking of mathematics content as pre-planned and clearly sequenced seems to prevent us acknowledging the constructivist claim that mathematics is actually individual learner's activity (Wheatley, 1991).

In general, the weighting of opinions in both the *Lesson aims* and *Lesson content* clusters implied that mathematics teaching should remain a process of initiating learners by progressing through a series of defined mathematical relationships not necessarily linked overtly to the out-of-school experiences of the children. These responses were somewhat surprising in the light of current rhetoric about mathematics education. The concept of negotiation of lesson aims, content and understandings seems consistent with current theory of mathematical pedagogy, because it allows for different backgrounds of experience and understanding as well as individually meaningful activity in classrooms. However, acceptance of this theoretical stance alongside the expectation of the teachers having clear purposes for lessons provides an interesting challenge for teacher educators. The two positions are not necessarily incompatible, but it is clear that we need to explore how teachers may open up opportunities for more meaningful dialogues with students about teaching and learning. The findings suggest that we also need to examine the ways that words such as "negotiate" are used by teachers and teacher educators, in order to build some common ground for communication about locus of control over mathematics learning.

Further exploration of beliefs about negotiating aims and content

It was decided to use workshops in the 1992 conference of the Mathematical Association of Victoria to investigate further teachers' and educators' beliefs about "negotiation" of lesson aims and content.

About 180 teachers and educators attended the two sessions offered. In order to provided a common basis for discussion, each group was shown the same five-minute classroom episode on videotape (Lovitt & Clarke, 1988). The snippet showed a teacher working with a group of five-year-old children, classifying shapes cut from carpet. We had chosen this video because we believed that it demonstrated a teacher drawing out children's ideas and language, allowing the lesson to flow from the ideas raised by the children, linking teaching points with the experiences and language of the pupils, and engaging herself and the children in a process of negotiating starting points for discussions as well as differing interpretations of mathematical aspects of the shapes used and created. In short, the snippet seemed to illustrate "pedagogical constructivism" (Noddings, 1990, p.15) in action. Our intention was to open up a discussion about the language used to describe such teaching – and in particular the words and phrases used commonly by respondents to the survey outlined above.

After the video snippet had been shown, the following continuum was displayed:

was negotiated <1------------2------------3------------4------------5------------6------------7> was imposed

People were asked to use the scale of 1 to 7 to name the extent to which they thought the teaching demonstrated "negotiation" of aims and content. After participants had made individual decisions, they were asked to discuss their ideas in groups of 3-4, and to come to some consensus within each group. Groups then reported their decisions, and the major reasons for these, as well as any interesting discussion points arising.

It was amazing to have groups of participants categorise the snippet from 1 (completely negotiated) through to 7 (completely imposed). Given our impression of the snippet, we were surprised when there was a fairly even spread of responses along the *negotiated <=> imposed* continuum – and that this happened in both conference sessions. Time did not allow similar testing of other terms commonly used in describing teaching and learning, but the incident brought home to us the need to focus further on language used to describe pedagogy.

Conclusion

We can communicate about the teaching of learning only by assuming that in what we say or write there will be enough compatible meaning for our communications to be intelligible, useful in terms of conveying our understandings, and able to be used as a basis for further reflexive interaction. However, it is clear that it is necessary for people discussing the teaching and learning of mathematics to work at establishing such a basis.

The incident described above certainly suggests that there is a need for:

(a) the creation of time and opportunities for both implicit and explicit negotiation of meaning for the many words and phrases used when any group of professionals or mathematics education students is discussing teaching and learning ;

(b) active exploration of pedagogical situations by groups of education students and educators, aimed at establishing situated contexts as sources of personal and intersubjective meaning, which can then used as a basis for meaningful communication about mathematics pedagogy;

(c) a recognition, in lectures, academic writing and instructional materials that the understandings constructed and knowledge gained by listeners and readers will be a product not of the words used but of their subjective interpretation by individuals;

(d) detailed investigation and explication of current key words, by mathematics education students, teachers and teacher educators; with the intention of examining potentially conflicting aspects of different terms used in conjunction with each other; and

(e) the use of contextual, highly situated examples to elaborate terms commonly used in theoretical communications.

References

Australian Education Council (1990). *National statement on mathematics for Australian schools.* A joint project of the States, Territories and Commonwealth of Australia, December 1990. Canberra· Curriculum Corporation.

Bauersfeld, H. (1988). Interaction, construction and knowledge: Alternative perspectives for mathematics education. In T. Cooney & D. Grouws (Eds.), *Effective mathematics teaching* (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics and Laurence Erlbaum Associates.

Cobb, P. Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education 23* (1) 2-33.

Gergen, K. J. (1985). The social constructionist movement in modern psychology. *American Psychologist. 40* (3), 266-275.

Lovitt, C.,& Clarke, D. (1988). *Primary maths discussion starters.* Videotape: Mathematics Curriculum and Teaching Program. Carlton: Curriculum Development Centre.
Mousley, J. A., & Sullivan, P. (1992). *Quality mathematics teaching: Consensus and contradiction.* Proceedings of the Annual Meeting of the Australian Association for Research in Education (Disc). Melbourne: ACER.
Mousley, J. A., Sullivan, P., & Clements, M. A. (1991, July). *The perceptions which student teachers have of teaching practices of classrooms observed during field experience.* Paper presented at the Annual Conference of the Mathematics Education Lecturers' Association, Perth.
Mousley, J., & Clements, K. (1990). The culture of mathematics classrooms. In K. Clements (Ed.), *Whither mathematics?* (pp. 397-406). Melbourne: Mathematical Association of Victoria.
Mousley, J., Sullivan, P., & Waywood, A. (1993). NUDIST: A qualitative research tool or a mirror of our own pedagogical theories. In W. Atweh, C. Kanes, M. Carss, G. Booker, *Contexts in Mathematics Education* (pp. 415-423). Brisbane: Mathematics Education Research Group of Australasia
National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* NCTM Commission on standards for school mathematics. Reston, VA: NCTM.
Noddings, N. (1990). Constructivism in mathematics education. In *Constructivist views on the teaching and learning of mathematics: Journal for research in mathematics education, Monograph no. 4* (pp. 7-18). Reston, VA: National Council of Teachers of Mathematics.
Steedman, P.H. (1991). There is no more safety in numbers: A new conception of mathematics teaching. In E. von Glasersfeld (Ed.) (1991). *Radical constructivism in mathematics education* (pp. 1-11). Boston: Kluwer.
Sullivan, P., & Mousley, J. (1993b). *Quality mathematics teaching: Describing some key components.* Research report 9: Mathematics Teaching and learning Centre, ACU. Oakleigh: MTLC.
Sullivan, P., & Mousley, J. (1993a). Describing teaching: Categories from qualitative review of teachers educators descriptions of quality teaching. In W. Atweh, C. Kanes, M. Carss, G. Booker, (pp. 523-531), *Contexts in Mathematics Education,*. Brisbane Mathematics Education Research Group of Australasia
von Glasersfeld, E. (Ed.) (1991). *Radical constructivism in mathematics education.* Boston: Kluwer.
Wheatley, G. H. (1991). Constructivist perspectives on science and mathematics learning. *Science Education, 75* (1), 9-21.

YOUNG STUDENTS' FREE COMMENTS AS SOURCES OF INFORMATION ON THEIR LEARNING ENVIRONMENT

Hanlie Murray, Alwyn Olivier and Piet Human
Research Unit for Mathematics Education
University of Stellenbosch, South Africa

The NCTM Professional Standards For Teaching Mathematics stresses the influence the learning environment has on what students learn, as well as the importance of assessing the quality and nature of the learning environment. Present assessment tools such as the analysis of videorecordings of classroom events give incomplete information on this complex issue or may be impossible to use on a wide scale. This study explores the extent to which third grade students' observations of classroom events can reveal their learning environment, and the extent to which these young students can become explicitly aware of the existing social contract.

INTRODUCTION

The NCTM document *Professional Standards For Teaching Mathematics* (1989) expresses the assumption that *what* students learn is fundamentally connected with *how* they learn it. The document describes the nature of the learning environment which is considered desirable for encouraging serious mathematical thinking. In this type of environment, students' ideas and ways of thinking are respected, time is provided for students to become involved with mathematical ideas and problems, and students are expected to function as members of a learning community through discussion and argument. It is stressed that the purpose of such an environment is not simply to make students feel good but to foster mathematical thinking. The teacher's ability to create such a learning environment should be assessed, but the *Standards* emphasizes that the aim of such assessment is teacher improvement and professional development.

The document recommends that in general, evaluation of the quality and nature of the learning environment should be based on multiple sources of information. For this we need easily-administered and safe evaluation instruments. For example, although video-documentation of classroom activities make available a wealth of information, this facility is not accessible to most schools. Some research has been done on the development of instruments to assess students' beliefs, ranging from open-ended questions (e.g. Spangler, 1992) to Likert-type scales for particular beliefs (e.g. Meyer & Fennema, 1988; Kloosterman & Stage, 1992), but any questionnaire and semi-structured interview implies that the researcher has already made decisions about possible variables and their relative effects, thereby both eliminating as-yet-unidentified variables, and making it difficult to assign relative importances to the variables affecting the learning environment.

The learning environment involves much more than students' beliefs. The learning environment can also be called the classroom mathematical culture, "... in which—as in all cultural microcosms—there are (most often tacit) sets of beliefs and values that are perpetuated by the day to day practices and rituals of the cultures" (Schoenfeld, 1988:82). The classroom culture is such a complex, multi-faceted phenomenon that it is quite difficult to describe fully, especially since it is doubtful whether all its variables have yet been identified.

We at the Research Unit for Mathematics Education at this University are confronted with a major challenge: A problem-centered approach to mathematics education is being officially implemented in more than a thousand South African elementary schools on our initiative, and to support this initiative, we need ongoing classroom research and ongoing teacher support and development actions especially as regards the improvement of learning environments.

Some salient characteristics of our problem-centered learning classrooms include:

1. Students are presented with problems that are meaningful and interesting to them, but which they *cannot* solve with ease using routinized procedures or drilled responses.

2. The teacher does not demonstrate a solution method, nor does she steer any activity (e.g. questions or discussion) in a direction that she has previously conceived as desirable, yet she expects *every* student to become involved with the problem and to attempt to solve it. Students' own invented methods are expected and encouraged.

3. It is expected of students to discuss, critique, explain, and when necessary, justify their interpretations and solutions.

The unavoidable paradox in such a situation is that confronting students with difficult problems that they are expected to solve without help from the teacher must necessarily create tension, and tension is not conducive to optimal cognitive functioning. Establishing alternative support systems in these classrooms is therefore essential for the learning process, and takes the form of peer interaction. In a previous paper (Murray, Olivier & Human, 1993), we highlighted two main aspects of peer interaction: firstly, that the type of interaction heavily influences learning outcomes and secondly, that the less teacher *direction* there is in setting up peer interaction groups or in the questions to be discussed, the higher the quality of the discourse among these young students.

For our previous studies, we visited and videotaped the classrooms on a regular basis and conducted interviews with selected students. It was decided to explore the possibility of using students' written free comments as a source of information for the following reasons:

- Students' free comments might illuminate the interplay between the different variables which determine the learning environment and supply the researcher with information above and beyond that which can be gleaned from videotapes, scholastic tests and questionnaires. "... we will need to develop theories and methodologies that encompass cultural and cognitive phenomena, *and the dialectic between them*." (our italics) (Shoenfeld, 1988:83).

- In a very large-scale implementation, there is a great need for cheap, easily-administered data-capturing tools which go beyond scholastic tests and questionnaires.

- We believe that the main thrust in teacher development in the elementary school should be focussed on helping teachers to learn about their students. Students' free comments might make available valuable data for the teacher to reflect on.

THE PRESENT STUDY

Three schools (giving a total of seven third grade classes, average class size: 35, average student age: eight years) were selected. On two occasions during the past school year, the first after about three months, the second at the end of the school year, the third grade students in all the classrooms were presented with a blank sheet of paper and were requested to write whatever they wanted to about their school mathematics. They could use as much time as they wanted to: Some students wrote two lines, others two pages. Most students wrote about half a page, mentioning between three and five aspects. Many added drawings of group situations and/or examples of problems they were currently solving.

The students' comments were classified into the following 13 different categories. Some of these categories deal with the same aspect of the learning environment, e.g. categories 1 and 2 address feelings towards mathematics, yet they have been kept separate because it was found that *what students do not comment on* may also be revealing (cf. Winograd, 1991).

1. Positive feelings about mathematics and/or problem solving (I like math/our math period/problems).
2. Negative feelings about mathematics and/or problem solving, superficially anxiety-free (The problems are too difficult/I don't like problems).
3. Different types of non-problem-solving classroom activities (We play games, measure, ...) and apparatus used in the classroom.
4. Outside influences (My mother helps me with mathematics; mathematics is important for my future).
5. Peer support structures (I like to work with Avril; we talk about our problems).
6. The rules governing social interaction (We help each other; you must not say somebody is wrong, you must discuss it).
7. The reasons for interaction (If we discuss the problem, it becomes easy).
8. Positive teachers' beliefs about students' abilities (She says we can all solve problems).
9. Neutral descriptions of an ability group structure used in classroom organization (There are four groups; I am in the Blue group).
10. Comments on teacher activities that do not suggest her helping with problem solving (She teaches us about time/kilograms).
11. The teacher as the source of knowledge and support during problem-solving sessions (She helps me if I can't do the problem; she explains the method).
12. Competition/comparisons between students (Jack and I always finish first; I wish I was as clever as Benedine).
13. Discomfort and/or anxiety (I feel a bit frightened).

We briefly describe and illustrate the comments from three classrooms. These classrooms are not necessarily representative of a problem-centered implementation; although classroom A is considered to have established a successful inquiry learning environment, classrooms B and C were deliberately chosen to illustrate how much these young students' comments reveal about their teachers' beliefs.

Some of the issues mentioned are of course incompatible with a problem-centered approach (e.g. numbers 11, 12 and 13) and it is therefore valuable to know that students' comments will reveal such practices. In all but one classroom, there were significant differences between the profiles that emerged from the first and from the second sets of comments. There were major differences between the different classrooms' profiles. For example, in some classrooms there was a quite dramatic swing from a negotiation classroom practice towards a transmission practice in the course of the school year, whereas in other classrooms where a culture of negotiation was maintained, students' awareness of the rules and reasons for interaction sharpened and became more explicit. In general, these comments not only provide profiles of the different classrooms, but also help to identify issues for further research.

Classroom A. The first set of comments dealt exclusively with

- Peer support structures (A frequency of 21)
  "I work with Xin Xin" (Bryan)
- Positive feelings about mathematics (8)
  "I like sums and I like mathematics" (Michael)

The comments at the end of the year dealt with:

- Peer support structures (11)
- The rules governing social interaction (21)
  "You must be kind" (Ntaoleng)
  "We don't say its rong we say i dont agree" (Ntsasa)
  "Dont laugh at someone if the sum is wrong" (Simon)
  "We talk about other people's answers" (Ashley)
- The reasons for interaction (11)
  "We are not afraid of difficult work because we work with a friend" (Yannick)
  "We talk about the sums so we learn what to do" (Tsebo)
  "You learn a lot from each other" (Kgalalelo)
- Positive teachers' beliefs (4)
  "Mrs Smith says we can all do the sums" (Andiswa)
  "She leaves us alon and when we have finished the problem we call her" (Carla)

The absence of comments about activities other than problem-solving leads one to expect that the main focus point in this classroom was students working at problems in environments of their choice. Videotapes taken of this classroom reveal that students did in fact engage in normal third grade activities like weighing, measuring, counting in multiples, playing games, etc., but that their problem-solving sessions were periods of intense involvement with the posed problem and with their peers, with the teacher completely in the background or not even present. There are no signs of anxiety or competition evident on the videotapes. It therefore seems safe to assume that the students' comments revealed what struck *them* as the essential elements of their learning environment.

Classroom B. The first set of comments dealt with

- Positive feelings about mathematics. (27)
- Negative feelings about mathematics and/or problem-solving. (11)
- The different (non-problem solving) activities offered. (37)
- Peer support structures. (11)
- Neutral descriptions of an ability group structure. (18)

By the end of the year, the comments dealt with:

- Positive feelings about mathematics. (8)
- Negative feelings about mathematics. (5)
- The different activities offered. (27)
- Peer support structures. (32)
- Neutral descriptions of ability group structure. (21)
- Competition/comparison between students. (13)

  "Peter, Neal and John are the cleverest" (Francis)

  "Our group is the second best group in the class" (Magdel)
- The teacher as the source of knowledge and support during problem-solving sessions. (11)

  "If we cannot finish in time our teacher helps us" (Nick)

  "When we have finished a problem, we go to our teacher and she says if it is right" (Peter)

These comments leave the impression that early in the school year students were aware of *doing mathematics*, although eleven disliked mathematics. By the end of the year, however, the focus had shifted away from involvement in mathematics towards competition and comparison with peers and dependence upon the teacher. Some aspects of the latter classroom profile are interesting.

Firstly, there is a high level of explicit awareness of ability differences, which by itself could be taken as an indicator of hidden anxiety, yet there are no overt comments about anxiety. Secondly, the comments clearly indicate that simply having students discuss and help each other does not at all guarantee the paradigm shift from transmission teaching to negotiation, since in this classroom the teacher clearly functions as source of knowledge and as arbiter. We regard this as a timely reminder (and warning) that changing the superficial interaction patterns away from teacher-student interaction and towards peer interaction does not necessarily imply a change in the classroom culture.

Classroom C. The following issues were raised early in the school year:

- Positive feelings about mathematics. (31)
- Negative feelings about mathematics. (3)
- The different activities offered. (27)
- Peer support structures. (7)
- Teacher activities like helping students to measure or tell the time. (11)

The comments at the end of the year dealt with:

- Positive feelings about mathematics. (30)
- The different activities offered. (18)
- Peer support structures. (5)
- Competition/comparisons between students. (25)
  "I always hope I finish first in my group" (Marinus)
  "If you get your answer right first time you are the winner" (Nina)
  "I wish I could be in the Blue group" (Danita)
- The teacher as a source of knowledge. (4)
- Discomfort and anxiety. (6)
  "I am nervous if I have to explain" (Danita)
  "If you make a mistake they laugh at you" (Marnelle)

In this classroom, although there was definitely social interaction between peers, it must have been of such a nature that it never made any impact on the students. There seems to have been a small swing towards the teacher as source of knowledge, and a slightly bigger swing towards anxiety and discomfort. The major change was in the dramatic growth of competition and comparisons between students, and students' very explicit awareness of this feature of their learning environment.

DISCUSSION

The information contained in students' free comments are of two types: overt and covert.

Overt information is offered when students make direct statements about certain aspects of the classroom culture; for example, explicit descriptions of the rules governing interaction (classroom A) or of the role assumed by the teacher.

Covert information is obtained by studying the aspects which *are* mentioned and those which are *not*, thereby deducing the *main* features of the particular learning environment (those which make the greatest impact on the students).

Certain anomalies which invite further investigation appear when the issues addressed by a particular class are juxtaposed. For example, 30 students out of a class of approximately 35 state that they like mathematics, yet there is a significant count of comments on competition, envy and anxiety from the same class (classroom C). But in a classroom where students have developed a truly effective peer support system (classroom A), in the early part of the year only eight students mentioned that they like mathematics, and nobody bothered to mention it at the end of the year.

There is also an interesting difference between the profiles of classrooms B and C which should be investigated: In classroom B the teacher features strongly as a source of knowledge and support during problem-solving sessions and much less so in classroom C; the anxiety count is absent in classroom B but strongly present in classroom C. Since both these classrooms seem more transmission-type classrooms and not really inquiry-based, does that mean that in a transmission-type classroom a strong teacher presence lessens discomfort and anxiety, or should the link with anxiety rather be sought with the very much higher incidence of competition between students in classroom C?

Although the teacher cannot reflect on the types of questions that arise from the differences between different classrooms, there is, however, sufficient overt information available in students' free comments that can be very useful to the teacher, especially if she has invited students' comments on more than one occasion and notes any changes in the comments that may indicate changes in the classroom mathematical culture.

For ongoing teacher support and development actions by ourselves and by education supervisors, the type of information made available by students' free comments is essential for planning support workshops and for developing reading and teaching materials. Clear indications of how teachers misinterpret and/or disbelieve the message about problem-centered learning they had received in their pre-service or in-service training enable us to concentrate on particular aspects and/or to include sensitization activities with particular aims.

SUMMARY

We find students' free comments a particularly rich source of information on classroom culture and the way in which students experience their learning environment. It is especially difficult when dealing with young children to obtain valid information on how they interpret classroom events: Interviews and videotaping are time-consuming, and questionnaires reflect only the designer's preconceptions. Free comments, on the other hand, capture only what the student is motivated to impart.

The three classrooms chosen for this report were not representative of problem-centered classrooms. Classroom A is a successful implementation, and the students' free comments illustrate the *development* of students' awareness of the social contract under which they operate when they *do* mathematics as opposed to *learn about* mathematics, and illustrate the issues on which these students *do not bother* to comment, or which do not exist to be commented on. Classrooms B and C illustrate to what extent a class of third graders' innocent comments, taken as a whole, reveal the effects of a classroom mathematics culture where the focus has shifted from students doing mathematics towards students trying to please the teacher.

We close with some remarks from a class not reported above, leaving it to the reader to identify the needs in this classroom:

> "I like problems but we don't do enough. I would like to do it every day."
> "The problems are so easy I wish we could do more difficult stuff."
> "Can't we do problems every day but not Saturdays and Sundays."
> "Please Teacher in the first and second terms we did problems every day. Why not now?"

It is an open question whether the above type of information could have been obtained more easily or more quickly in any other way than from the students' own free comments.

REFERENCES

Kloosterman, P. & Stage, F.K. (1992). Measuring Beliefs About Mathematical Problem Solving. *School Science and Mathematics, 92* (3), 109–115.

Meyer, M.R. & Fennema, E. (1988). Girls, Boys and Mathematics. In T.R. Post (Ed.), *Teaching Mathematics in Grades K–8* (pp. 406–425). Boston: Allyn and Bacon, Inc.

Murray, H., Olivier, A. & Human, P. (1993). Voluntary interaction groups for problem-centered learning. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. Lin (Eds.), *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education, Vol II* (pp. II-73 – II-80). Tsukuba, Japan.

National Council of Teachers of Mathematics (1989). *Professional Standards For Teaching Mathematics.* Reston, Virginia.

Schoenfeld, A.H. (1988). Problem Solving in Context(s). In R.I. Charles & E.A. Silver (Eds.), *The Teaching and Assessing of Mathematical Problem Solving, Vol. 3* (pp. 82–92). Reston, Virginia: Lawrence Erlbaum Associates.

Spangler, D.A. (1992). Assessing Students' Beliefs About Mathematics. *Arithmetic Teacher, 40* (3), 148–152.

Winograd, K. (1991). Children's Mathematical Beliefs. *Mathematics Teaching*, no. 137, 33–37.

PATHOLOGICAL CASES OF MATHEMATICAL UNDERSTANDING

Elena Nardi
Research Student
Department of Educational Studies, University of Oxford

ABSTRACT

This paper is based on results from the pilot to an ongoing doctoral study of *the novice mathematician's understanding of advanced mathematical concepts.* In Spring 1993 eight first year mathematics students were observed during tutorials given to pairs of them on Analysis, Topology and Linear Algebra. Ordering the data via coding and categorising followed. Here I present a sample of instances illustrating a few student difficulties regarding some mathematical concepts and mathematical reasoning. The examples concern the concept of function, the process of integration (topical difficulties) and the student's logically deficient use of theorems (logical difficulties). The ongoing study aims at identifying patterns in the pathology of the novice mathematician's constructing of mathematical knowledge.

In order to gain access to the mechanisms of the novice mathematician's personal construction of mathematical knowledge, I had to enter the habitat where it takes place. Opportunity for this is provided by the 'Oxford Way' of doing undergraduate mathematics in the form of individual or pair tutorials given by college tutors. Most of these tutorials are weekly, half to one hour long and cover the full range of topics included in the syllabus. Given a high degree of permissiveness on the part of the tutor - regarding mostly her willingness to replace the traditional monologue with a more conversational teaching style - and of openness on the part of the student - as to how exposed she allows herself to be regarding her mathematical understanding - observing tutorials can prove to be very productive of incidents on mathematical thinking.

On the Focus of This Paper

The study on which this paper is based is the fruit of experience derived from a two-level journey: at one level, it features the evolutionary process of what started as a general research skill exercise on minimally-participant unsystematic observation (2) and now grows into still minimally-participant and relatively tightly focused observation. At another level it illustrates my progressive focusing (7) from a general research interest in mathematical understanding to the study of mathematics undergraduates cognitive structures. In short what now constitutes the main study is the product of a **methodological** and a **thematic** gradual selective process that took place during the pilot study on which this paper is based.

The two processes took place in parallel but not independently. Quite early in the observation sessions the affective aspects of mathematical learning were put aside to allow emphasis on the cognitive aspects. Of course it is impossible to discuss issues on mathematical thinking without considering, for instance, the awe or fear with which certain mathematical topics are confronted by the learners. In the term 'cognitive' here two streams of thought converge: the epistemological -concerning the mathematics discussed in the tutorial- and the psychological -concerning the strictly personal ways in which tutor and students construct mathematical knowledge.

While the tutorials and the observation were still going on, preliminary analysis of the data started. Data consisted of written notes produced during the tutorials. Early in the analysis it became evident that the data could be categorised roughly in two ways: Tutor items (signposted **T-theme**: teaching style, philosophy of mathematics and of mathematics teaching) and Student items (signposted **S-theme**: difficulties, intuition, metamathematical worries, originality etc.). Until the end of the sessions no particular emphasis was given to any of these items. This resulted in the aggregation of material which could be analysed either from a T or an S-perspective. In the meantime another network of items, which I clustered under the name **X-theme**, emerged: the study of the impact that certain observed mathematical, linguistic or other conventions have on the students' mathematical behaviour especially in an exam context.

The transformation of the material into manageable units for analysis took place in the form of the categorisation and coding that followed (8 ch3) : the emergent T, S and X themes were synthesised into three narrative texts from which the clusters (categories) of analytical units emerged. I shall provide examples from one of the themes and some of the categories later. At that point I made the decision to concentrate on the S-theme and in particular on items featuring the students' difficulties (code: S.DIFF.) or in other words on the instances of **pathological understanding**. The use of the term 'pathological' is I believe philosophically correct and compatible with espousing the strong conviction that constructing mathematical knowledge is a distinctly personal mental process (5).

The approach adopted here - that is the study of a commonly-recognised-as-flawed mathematical behaviour of the learner (3)- aspires to establishing associations with the **theory of epistemological obstacles** (1 and 4) and is not necessarily negatively judgmental. Other terms which also apply here but are disquietingly strong - since they stir up the tension between constructivist and absolutist views on learning - are errors, misconceptions etc. .

At the next level of analysis further refinement led to a triplet of categories concerning S.DIFF. :
- **topical difficulties** (S.DIFF.TOP.), that is the cases with reference to particular topics or concepts, e.g. the concept of function,
- **logical difficulties** (S.DIFF.LOG.), that is the cases with reference to the students' use of the tools for mathematical reasoning, e.g. the use of mathematical induction and,
- **symbolic difficulties** (S.DIFF.SYM.), that is the cases with reference to the student's interpretation and manipulation of symbols and language, e.g. the use of absolute value notation $| \, |$.

Apart from the S.DIFF. category other clusters of instances regarding the students' perceptions and knowledge construction were identified, such as instances of the students' remarkable use of intuition, their occasional mathematical originality or even these rare but striking instances of expressing metamathematical worries (S.ORIG.MATH., S.INTUITION, S.METAMATH).

The above triplet (S.DIFF.TOP/LOG/SYM) is not free of methodological constraints. It must be noted that by allocating an instance to one of these categories one attaches a preliminary interpretation to it. That is why, so far, categorising has not gone further than that. The possibility of clustering together, e.g. all the instances concerning function, was dismissed as imposing many restricting labels on the data (8 ch2). From that point onwards each instance was analysed separately. One might react to that by pointing out the danger of failing in identifying possible underlying patterns. The search for such patterns is of course one of the components of this study but at this stage categorising aimed primarily at an ordering of the data.

To illustrate the form taken by the analysis that followed, I present a small sample of instances and my analytical approach to them. All were selected from S.DIFF.TOP and S.DIFF.LOG. In most cases my analysis is associated with results in the field of Advanced Mathematical Thinking in terms both of the mathematical topics and of the psychological processes examined. One of the reasons these particular instances were chosen was that they are characterised by a balance between

- revealing elements of the strictly **personal** way a learner constructs mathematical knowledge (which is a benefit from choosing to do case studies) and
- recognising some previously identified by other researchers '**trends**' (or obstacles) in mathematical understanding.

<u>A Small Sample of Data and Some Analysis</u>

<u>Instance I: on the constant function</u> <u>(from S.DIFF.TOP).</u>

The Setting. Today's tutorial to students L and J is about finding the matrix of a map between two vector spaces. The tutor gives a description of the two vector spaces and then the map. They now have to identify the matrix. The particular example on which they are working now, the second for today, is the map Tf(x)=df/dx (x), where f is a differentiable real function on a given interval. Once they have found the matrix for that map...

Extract I

...the tutor asks them a kind of 'reverse' question.
*TUTOR: What would T be if its matrix were the **O**-matrix?*
Silence. They look confused.
*STUDENT L: (with hesitation)...**maps everything to 0**.*

(Now they try to find the map for Tf(x)=f(x+1). They discuss briefly what it means to apply f(x+1).
STUDENTS L and J: (they have both written down the same) f(x+1)=f(x)+1.
TUTOR: No, no. Put x+1 where you see x.
They then find the matrix.)

Again in the end the tutor asks a 'reverse' question.
TUTOR: What if matrix A were the I? (she means identity)
*STUDENT L: ...**everything would go to 1**.*
TUTOR: (calmly) What?
STUDENT: Oh, sorry ...to itself.

The tutor asks the 'reverse' question probably in order to check out how far the relational understanding, in the R.Skemp sense of the term (11), of maps/matrices and vector spaces has gone, that is whether the students have perceived the 'transformation' role that a map plays when applied to a vector space. The smartness of her question lies in the way she chooses to check that out, that is in reverse order. She then chooses not a general form but the particular cases of A=**O** and A=**I**. Mathematically speaking these matrices are redundantly used and are considered almost trivial but one student remains silent throughout the episode and the other hesitatingly gives the answer expected by the tutor only to the first one. The second answer ('everything goes to 1') can be seen as a consequence of the syllogism 'since A=**O** sends everything to 0, then A=**1** sends everything to 1' which **is** the case when f:R->R, f=constant=1, but **is not** when f stands for the identity function which for f:R->R is f(x)=x. I summarise the above in the following list of points of interest:

- The learner confuses f as a function of one variable and f (or A in this case) as a mapping between two vector spaces. She transfers her reasoning from one to the other with the detrimental

result that in the case of **1**, identity element of a group of functions, is confused with the constant 1. The coincidental correctness of **O** as O led to an inappropriate generalisation for **1** as 1.
Another interpretation might be that the incident reported here is an instance of a more general confusion about the meaning of $f:R^n \to R$. In other occurrences of this kind during these tutorials the tutor tries to resolve the confusion by simply saying that $f(x)=f(x_1, x_2, ..., x_n)=a$ is a real number. The students' usual response is silence.
-Underlying this confusion might also be the attitude of suspicion that students have towards the constant function. As a result of their 'expectation of change' they often do not see constants as functions (12) and therefore they have no cognitive structures readily available to confront constants in mathematical problems. The following extract illustrates this thought:
(It might be worth pointing out at the above as a case of exemplary significance for my research focus: in the episode I can recognise themes from the study of epistemological obstacles regarding the concept of function - for example as in (12)- but seen in the relatively little explored context of vector spaces. This element of enrichment of already existing theories is a substantial component of the ongoing main study.)

Extract II

The same pair of students as in Extract I, L and J, in an Analysis tutorial five weeks later. As usual Student J remains silent. In the course of solving a problem sheet question they have deduced that f, the function they study, is a constant.
TUTOR: And how do you find which constant it is?
STUDENT L: .. let's try a couple of values for x.
TUTOR: And which value is obvious to try?
STUDENT L: ...zero.
They try it.
TUTOR: It is a constant so you only need to evaluate at one point

Probably by bringing in the strategy she is accustomed to use with linear functions (when looking for $f(x)=ax+b$, one has to identify a and b, that is by trying *two* values for x one gets a pair of simultaneous equations with a unique solution (a,b)), student L suggests 'trying out *a couple* of values for x'. Interestingly the incident is repeated in another tutorial the same afternoon.

On f and a change of variable. In between the sub-episodes of the two 'reverse' questions (see in Extract I the piece in square brackets), comes the discussion of the problem of finding the matrix of the mapping $T: V \to V$, where $T f(x)=f(x+1)$ and V consists of continuous functions on an interval.
Now $f(x+a)=f(x)+a$ is true for all f with $f(x)=x+b$, that is for all linear functions with x-coefficient equal to 1 (or in visual terms the change of variable $x \to x+a$ means simply that we 'move' the line $y=x+b$ $|a|$ units parallel to its initial position). One approach could be that again the student has inappropriately generalised from the linear case. In these tutorials quite often the students have displayed a tendency to 'genericity' (6). By that I refer to a particular thinking habit of the students according to which they reflect on certain concepts not in their general, abstract form as seen in the definition but in a concrete, exemplary form.

Another approach could be that the student has attributed to f a series of properties according to a strongly personal interpretation (in a similar vein her answer could have been $f(x+1)=f(x)+f(1)$, which is true for $f(x)=ax$ One can imagine several 'distorted' properties of that kind).Considering my limited exploration of the case -the above is only a speculative analysis of the observations made during the tutorials - I can carry my interpretation of the instance no further. Nevertheless, besides its revealing content regarding obstacles inherent in the understanding of the concept of function, the episode is suggestive about the methodological constraints of observation as a method to access the learner's thinking mechanisms. The plethora of instances of a similar nature prompted me to consider substantiating the data produced in the observation sessions of the main study with interviews in which clarification and further discussion of the observations could be attained. This is the sense in which observing these tutorials paved the way for the form and content of the ongoing main study.

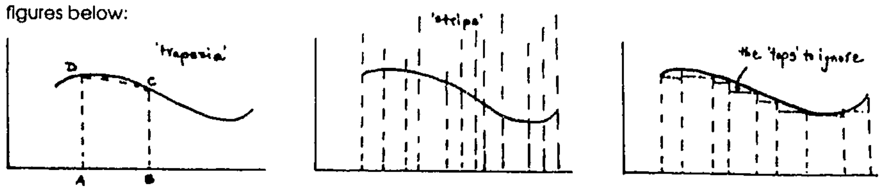
<u>Instance II: A 'fresh' look on the definition of the finite integral</u> (From S.DIFF.TOP)

The setting. In today's tutorials the tutor attempts a general, light introduction to the Lebesgue Integral. For all pairs of students she starts by asking them how they were introduced to integrals at their A-levels. The responses from all of them are exclusively: either (A) 'area under a graph' or (B) 'the reverse of differentiation'. The former is pictorial, the latter procedural. Both constitute aspects of a concept the understanding of which as the limit of a sum is in Orton's words (10) a 'stumbling block'. Here however the episode is about the students' perception of the pictorial aspect of the definition. The tutor then refers to the Fundamental Theorem of Calculus and says that 'we will use (A) to prove (B)'. She then explains the idea of approximating the area under the graph of f with calculating the area of 'simpler' figures. In one of the tutorials Greek Student T suggests using triangles: maybe this is her perception of 'simpler figures'. Actually in Greek Secondary Education, where the syllabus strongly emphasises Euclidean Geometry, calculating the area of a triangle ('base times height over 2' (C)) is a routine task. When the tutor asks her to give the formula for the area of a triangle, Student T stops to think, attempts (C) but soon regrets and withdraws. The tutor then brings the simplicity of rectangles to their attention and goes on to define the integral. In exactly the same fashion, in the tutorial given to Students JN and B later that afternoon, Student JN suggests...

Extract III

STUDENT J: ..trapezia.
The tutor and Student B laugh. Then...
STUDENT B: ...strips maybe.
TUTOR: Yes, what about the top then?
STUDENT J: ...well ignore it...and make it a rectangle.

One interpretation is that the student's image of the situation was similar to the one illustrated in the figures below:



That is Student JN approximates the area under the graph as the sum of the areas of the trapezia such as ABCD, where C is $(a_{k+1}, f(a_{k+1}))$ and D is $(a_k, f(a_k))$. To the others' laughing reaction, Student B changes to strips -and it might be hard to imagine how these 'strips' areas are easier to calculate (see the figure above). So the tutor comes back with what could be credited as a very constructive comment: 'what about the top then?' which points at the underlying difficulty in calculating the areas of these 'strips'. Then Student JN, possibly seeing the 'tops' as dispensable quantities, suggests 'ignore it then...and make it a rectangle' which completes what I saw as an instance of successful mathematical dialectics among three keen interlocutors. That being said probably disqualifies the episode from being classified under S.DIFF.TOP. , since the S.DIFF. category, as mentioned before, is of a 'pathological' nature. Initially the instance had been classified as a S.MATH.ORIG. case which included instances of the students' occasional mathematical originality and intuition. However a closer look at the episode allowed me to identify a few elements of the students' mathematical behaviour which justified classifying the episode as a 'student difficulty'. Such an element is:

- the whole route from trapezia to strips and finally to rectangles seems to reflect a struggle in the students' minds to recall a figure associated in the past with the definition of the finite integral. One might leap to the conclusion here that theirs is an effort to *reconstruct the memory of a definition*, not the definition itself. The tutor seems to expect the latter from them because she believes that they possess the mathematical tools for the reconstruction. One could also argue here that this play with memory is orchestrated by the tutor whose intentions are to elicit the definition from the students' previously established knowledge.

Another possible explanation is that the word 'approximation' evoked in the student's mind the Trapezium Rule for the approximation to an integral as the limit of a sum of trapezia, exact only for linear functions, used in the first years' course on Analytical and Numerical Methods. However as in the other instances, due to <u>the limitations of exploring knowledge constructs merely through observation</u>, my analysis was tied down to the level of hypothetical speculation. In the main study it is aimed that such occurrences are further explored in the interviews that follow observation.

<u>Instance III Theorems stripped of their conditions</u> (from S.DIFF.LOG.)

Theorems in mathematics are of the form 'if...then', that is we can infer the truth of a premise on the condition that certain other premises are true (*Principia Mathematica* B.Russell: what is *implied* by a true premise is true. -my italics). The *mechanics* of this *implication* are illustrated in the *proof* of the theorem. Apparently if one wishes to use the implied premise, one has to assure that in the given context the preliminary premises hold. The above didactically bring in the crucial issue concerning the majority of students' typical mathematical behaviour: I can report rather extensively on how students use the implied premises of theorems without securing the validity of the prerequisite conditions or neglecting some of them. This behaviour could be linked with the more general phenomena of inconsistency regarding students' logical thinking (9) The examples from the observed tutorials include:

- a form of the Mean Value theorem (Differential Calculus),
- the test for the convergence of alternating series (Sequences and Series),
- the ratio test for the convergence of positive series (Sequences and Series) and
- theorems on the existence of an integral of a function (Integral Calculus).

It is worth noting that all the theorems in question have as the implied premise *not a worded statement*, but a *formula* (e.g. in the Mean Value theorem: the formula for the value of the derivative of f on a particular point). Thus generalising about the students' behaviour regarding the use of theorems is in this case a perilous business. A preliminary explanation might be the following: a formula is a condensed, succinct-hence-handy mathematical expression which has a strong appeal for most students. Figuratively speaking, it is like the 'flying sticks' of a magician. When on stage the audience is so fascinated by the ostensible flying of the sticks that the presence of the almost transparent 'invisible' strings that actually hold the sticks and make them move remains unnoticed. Of course the audience know of the strings' existence but the spectacle is so pleasant to the eyes that they prefer to forget about it. Following the path of this metaphor the students have most frequently a good grasp and memory of formulas. They see them as immediate solutions to problems -and often they are. Thus at the sight of a promising formula, they operate in a state of 'cognitive excitement' and ignore the conditions. Let us now take a closer look at the particular examples on which the above theorising is grounded.

In the case of the **Mean Value theorem**, the occasion was the discussion of an exam question. Typical form of these exam questions is to start with what I call the 'recognition theme', that is a question that implies that the student has to recognise, then fully state a well-known theorem and occasionally prove it. The subsequent parts of the question are usually tasks in which the application of the theorem proved in part (i) is needed:

TUTOR: Y, you usually apply the formula of the theorem without looking at the conditions...

STUDENT Y: No, this time I did (shows her draft papers and they laugh).

The reason I include this example is because it is suggestive of this student's problematic handling of theorems in the past. Another thing which I find noteworthy is the relative lightness with which the situation is treated. The reason for the tutor's inalertness might be that these exam questions are 'designed to fit' the undergraduates' up-to-date knowledge. It is thus more likely than not that a mathematical problem, being part (ii) of an exam question with the Mean Value theorem as part (i), is this theorem's application. Therefore in the tutor's perspective very little is at stake when the student neglects the conditions. It is highly unlikely that the student will hit upon contradictions and so the checking out of the conditions is reduced to a sheer formality. What in this case seems to be totally forgotten is the 'power of habit'. A student who cultivates this attitude may escape without any harm this particular or similar situations but soon will find herself in mathematically more open, exploratory situations which call for a scrutiny of the slightest detail and the student will be hardly prepared for that. So what looks now as just one more weary detail necessary to complete the answer in an exam question, might turn out to be a carrier of potential epistemological 'bugs' for the near future.

In the case of the **Alternating Series Test**, 'I know I am missing the conditions', admits one student in her effort to retrieve everything from memory. Despite the almost obvious conditions that would make the Test work, the students do not see them. What escapes them is simply **the necessity of the conditions.** They are merely concerned with (and thus remember) only the 'useful' part of the theorem. Its conditions are sheer paraphernalia. Later on another couple displays a similar attitude with the **Ratio Test for positive series.** They do not seem to know what the role of the ratio is, which features of the series in question it measures and how these features link up with the definition of convergence. Integral calculus appears to be particularly conducive to negligence of the conditions of theorems. With its self-contained integral formulas, students are 'tempted' in a cognitive sense to neglect the conditions under which the formulas hold. The issue brought up here is of a more general appeal. Conditions in theorems play the role of progressive restrictions to the generality of a statement. They are the outcome of an **exclusion process** during which mathematicians verify for which family of mathematical objects this formula holds. One might wonder whether students' behaviour regarding the handling of theorems partly reflects this developmental order, that is whether they behave in such a way because they are somehow at an earlier stage of this exclusion process, that is at the stage of deceptively believing in the generality of the formula. Again the evidence from these instances provided me with an interesting dimension of the students' thinking the further scrutiny of which seems to be worth pursuing in the main study.

(Instead of) An Epilogue

The pilot study was carried out in the Spring of 1993. The following academic year I started the data collection for the main study which is currently going on. The main characteristics of this study which can also be seen as 'aftermath' of the pilot study could be outlined as follows:

- Quantitatively increased material: a wider sample of students is currently being observed. This is expected to add richness and diversity to the data.
- Most students are being interviewed on what appears to be the 'common denominator of trouble' during the tutorials or the core of problematic features in the students mathematical understanding. For instance, the notion of limit (via delta-epsilon definition), the handling of 'iff' statements in Calculus or the concept of a Spanning Set from Vector Analysis emerged as some of the particularly interesting areas to be explored - mainly in terms of the remarkable consistency of the students' difficulties in these topics which implied the existence of underlying patterns - in the first weeks of observation in Autumn 1993. The interviews carried out in December 1993 were designed to illuminate students perceptions on these topics. I thus attempt to go beyond the degree of interpretation to which I was tied down during the pilot study because of lack of sufficient evidence emanating from observation. The tutorials observed during the main study are also being tape-recorded.
- The diversity of topics handled is still high despite my tentative gradual concentration on some of them. The reasons for that are: first, the sometimes unpredictable nature of the mathematics tutorial and second, my openness to the exploration of new unsuspected territories.
- Finally, the main study aspires to investigate students' thinking processes with the dual intention of **verifying** models of these processes created by other researchers and of **modifying** (enriching) these models or even **creating** a new model emerging from the identified patterns of the students' pathology of mathematical understanding. Depending on the nature of the collected material this model can turn out to be either mathematical-topic or psychological-process focused.

References

1. BROUSSEAU G(1983) *Les Obstacles Epistemologiques et les Problemes en Mathematiques* in Recherche en Didactique des Mathematiques 4,2.
2. COHEN L, MANION L (1990) Research Methods in Education ch5, London: Routledge.
3. GAGATSIS A (1992) *Concepts and Methods of Didactics of Mathematics* in GAGATSIS A (ed) Topics on Didactics of Mathematics. ERASMUS: Thessaloniki, Greece.
4. GLAESER G (1984) *A Propos des Obstacles Epistemologiques* in RDM 5,2.
5. GLASERSFELD E von (1983) *Learning as a Constructive Activity* in BERGERON J, HERSCOVICS N (eds) Proceedings of the 5th PME-NA, Montreal, Canada.
6. HAREL G, TALL D (1991) *The General, the Abstract and the Generic in Advanced Mathematics* in For the Learning of Mathematics 11,1.
7. MCINTYRE D, MCLEOD G (1978) *The Characteristics and Uses of Systematic Observation* in MCALEESE R, HAMILTON D Understanding Classroom Life, Windsor:NFER.
8. MILES M B, HUBERMANN A M (1984) Qualitative Analysis: A Sourcebook of New Methods, London, Beverly Hills, New Delhi:SAGE PUBLICATIONS.
9. O'BRIEN T (1973) *Logical Thinking in College Students* in Educational Studies in Mathematics 5,1.
10. ORTON A (1983) *Students' Understanding of Integration* in ESM 14,1.
11. SKEMP R (1979) Intelligence, Learning and Action, London: Wiley.
12. VOLLRATH H J (1986) *Search Strategies as Indications of Functional Thinking* in ESM 17,4.

ACKNOWLEDGEMENT: I would like to warmly thank my supervisor Dr Barbara Jaworski for her meticulous reading of my first draft for this paper.

Slope, Steepness, and School Math

Ricardo Nemirovsky[1]
TERC, Cambridge MA, USA.

We are conducting research and designing tools for the learning of the mathematics involved in the shape of real objects. This paper focuses on a case study of a high school student using a computer-based device, with which one can produce functional graphs of contours and their slope, by tracing real objects. The selected episodes describe her explorations of the slope of a flat board. The analysis details how she used the tool to visualize slope, to enrich ideas and techniques that she had learned in school, and to develop relationships with the everyday experience with tilt and steepness.

1. Introduction

The notion of slope is a meeting point of ideas from a wide range of sources: graph interpretation, the experience of steepness, sense-making of equations, rate of change, and so forth. In American schools slope is introduced in the context of linear equations and graphs. It is commonly identified with the coefficient "m" of linear equations of the form y=mx+b, and with a ratio technique to quantify the tilt of a straight line on a grid, which is known to students as "rise over run."

There are two main strands of research literature on the learning of slope. One focuses on students' understanding of linear behavior and the representation of straight lines. Studies include investigations of students' difficulties with the visual interpretation of linear graphs displayed by computer graphers (Goldenberg, 1988), the complexity inherent in students' learning the correspondence between linear equations and Cartesian graphs (Moschkovich et al., 1993, Confrey, 1993), the reinterpretation of common terms, such as "steepness," that students have to negotiate as they begin to make sense of linear equations (Moschkovich, 1993), and students' use of linear devices, such as winches and springs, in which slope is produced by a mechanism (Greeno, 1988; Meira, in press). Another strand of research (Clement, 1989; Janvier, 1978; McDermott et al., 1987) focuses on students' interpretation of linear and non-linear graphs representing situations, such as the motion of cars or the population of animals. Within this strand, researchers have often emphasized students mistakes like "slope as height" or "graph as a picture."

This paper is a partial report of our research exploring alternative approaches to learning about slope. This work is part of our interest in the use and design of computer-based tools to measure and represent geometrical properties of real

[1]The work reported in this paper has been supported by NSF Grant MDR-8855644. All opinions and analysis expressed herein are those of the author and do not necessarily reflect the views of the funding agency. The author wishes to thank Judah Schwartz for his contribution to the project, and Cliff Konold, Tracy Noble, and Cornelia Tierney for their editorial feedback.

physical objects. The core of the paper is a case study based on a teaching experiment with a high school student, Karen, using a device, the Contour Analyzer. The software can be used to create functional graphs of height vs. distance (horizontal distance) and slope vs. distance on a computer screen, by tracing the surface of an object. Although this paper is based on episodes with Karen, the issues that emerge are relevant to the analysis of all the teaching experiments. Throughout the analysis I highlight three aspects: (1) The relationship between Karen's learning to visualize slope in an object and her inquiries into how the tool can "see" slope, (2) the junctions and disjunctions between her experience in this teaching experiment and what she had learned in school, and (3) the role of a qualitative understanding encompassing the common experience with steepness and the shape of objects.

2. Methodology

This exploratory study involved individual teaching experiments (Cobb & Steffe, 1983) with three high school students. I interviewed the students over three or four sessions during which I posed to them a sequence of problems that incorporated ideas from the students themselves. Between sessions, the videotapes were analyzed to design the subsequent activities. The students used a tool to trace the surface of any object along a certain plane. Figure 1 shows it, the Contour Analyzer, with a flat board resting on a turning table. The bearings of the board can be varied independently of the turning of the table. To create a graph of height vs. distance on the computer screen one moves the cursor by hand over the surface. A software option allows one to generate the graph of slope vs. distance. Two different sets of graphs can be displayed simultaneously.

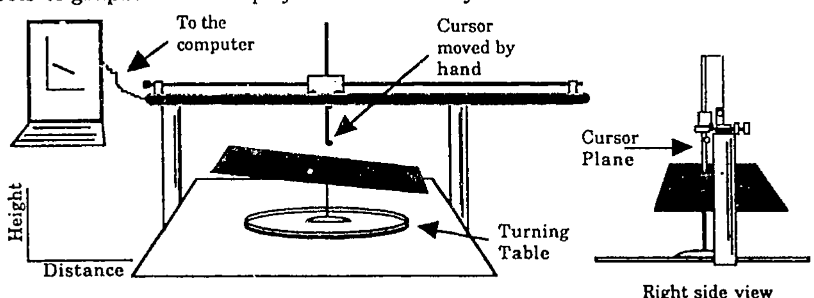


Figure 1

3. Karen

Karen was in the 11th grade. She had taken algebra I, geometry, biology, chemistry and, at the time of the teaching experiment, was taking algebra II and physics. She planned to pursue a career in science, possibly in biology. Karen commented that she was not very good "at theory" and that she did better in science than in math. Her parents were very supportive of her interest in science. She disliked that in math "little mistakes ruin the whole thing." Karen characterized mathematics using the words: "perfect," "rules," "repetition," "nothing ever changes"; in science, on the other hand, there is always something new and unexpected. In science, she said, one needs intuition. "I'm usually interested in what I'm doing," said Karen emphasizing that she had a lot of initiative for science projects. About her teachers Karen said "they think that I'm a very good student, I have a close relationship with them." She liked the school labs because "something unexpected may always happen." During the teaching experiment Karen was active and talkative. She wanted to be useful for the research project, and her engagement never seemed to decline.

4. Commented Excerpts

The selected excerpts took place during the first two sessions and reflect her work with the flat board, before she started to investigate curved surfaces. At the beginning of the teaching experiment I asked Karen to think of the cross sections of a paper cup. Karen cut paper cups with scissors to think about the transformation of the cross sections when an imaginary cutting plane turned. Then I introduced her to the Counter Analyzer using a tilted flat board. As I moved the cursor on the board, Karen commented: "Oh, so you're kind of getting coordinates." Karen easily predicted that the contour graph would be a straight line showing the tilt of the board. This is how she described the contour graph produced by the Contour Analyzer:

Karen: So it's [the computer graph] measuring exactly where it [the cursor] is as if that [the cursor plane] was like a sheet of paper (. . .), and we just pulled it [the sheet of paper] off and put it up here [on the computer screen].

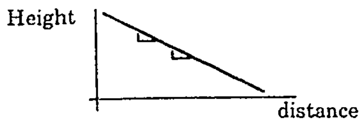
Karen copied the computer graph on a paper sheet (see Graph 1). Then:

Ricardo: So if you would like to indicate somehow the steepness of this contour [Graph 1], how would you talk about the steepness?

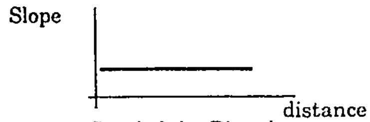
Karen: Steep, you mean as in slope?

Ricardo: Or slope, yes.

Karen: The only steepness I know about is slope from math class. Well, I'd probably set it up on a graph that I can see. And either count it out or I could use, if I could remember the formula, I believe it's rise over run, to figure out slope for a line on a graph. And in that way I guess you could figure out its steepness. I don't know any other methods.



Graph 1, by Karen



Graph 2, by Ricardo

Ricardo: Let's think of a graph of slope versus distance. What do you think? How would the graph look like?

Karen: OK, well, it's [Graph 1] a straight line so the slope isn't going to change. So if it's slope versus distance it again would be a straight line and it would probably be straight across [gesturing a horizontal line], like, what's the word I'm looking for, slope 0, where it's straight horizontal? I believe that's right.

Karen's drew a graph of slope vs. distance with a line on the horizontal axis. To further justify her prediction she drew the little triangles shown in Graph 1 and said:

Karen: If you took a reading here [gestures the upper triangle on Graph 1] it would be the same thing as down here [another triangle]. The slope hasn't changed a bit because it's nice and straight.

Karen thought of the slope function as indicating degree of straightness. She merged this notion with her school-learned definition of "rise over run" that included the visual construction of the small triangles. Then we looked at the slope graph on the computer screen. It displayed a horizontal line but below the axis. Karen was puzzled by the non-zero slope graph:

Ricardo: Then your intuition is that no matter how the plane is

Karen: Tilted, it should have been zero, because the board is not bending that much that I can see. Of course I don't have that. You know. You don't see very, very fine details on the board such as a computer might.

Note how she dealt with the counter evidence by invoking the above-human computer capabilities to "see" details. I asked Karen about the meaning of a positive slope graph:

Ricardo: Let's say that there's something that has the slope like this [drawing Graph 2], that is horizontal but not zero, can you imagine what that could be? Do you understand what I mean?

Karen: Kind of. I've run into this before in class, but it's been a while. It was back at the beginning of the year so I'm a little rusty. All I know is when something is a slope of 2, it's like 2 over 1. So it's like up two for every one over. But I'd have to count that out on a grid. Well, I just thought of something now. One of the reasons it [a slope graph] might be below the zero is because it's going down, and it might be a negative slope. Where if it was going from the bottom off to the upper right, it might be above the zero.

After a failed attempt to use the "rise over run" idea ("I'm a little rusty," she said), Karen experienced an insight regarding the interpretation of the slope sign based on the dichotomy going down/going up. Then Karen recalled her school experience using a motion detector to create graphs of velocity vs. time and feeling perplexed by the sign of the velocity, because

Karen: even if you were still increasing speed it would go the other way. And it's like counting negatives, the numbers go up, and it gives you the illusion of it getting bigger really when it isn't. (. . .)

Ricardo: Do you see a connection between velocity and slope?

Karen: Only the way that it's read on a graph. I don't know if velocity has anything to do with the slope, because if I rolled that [the cursor] very slowly and then I rolled it very quickly it should still give you the same reading. I was just thinking of how you read a graph like that.

Karen recognized a resemblance between velocity and slope in how one has to read their graphs. The connecting experience was that for both of them going "down" to the negative meant "less," even though the moving object may move faster or the board bearing may be steeper. To test her idea about slope sign we tilted the board the "other way" and measured with the Contour Analyzer. As Karen was preparing the experiment she articulated another issue that would reappear several times in our conversation: an observer standing on "the other side" of the board would "see" the opposite sign for its slope because going up from left to right becomes going down:

Karen: Of course if we stand on any other side that would mess up everything. If you think about it we could be reading it from the wrong side. So it depends on, as I said, this [the computer] is probably all calibrated just to that screen [to seeing the board from her angle]. So we have to have that in mind.

Often Karen reflected on how one has to look at the object in order to be consistent with the computer. For example, after noticing that if one turns the computer monitor 180 degrees it does not show the correct tilt of the board, she said:

Karen: You have to be looking straight at the computer on the same side as you're reading. Otherwise you get totally thrown off. That's one of the set backs of a computer. It doesn't understand that.

The device embodied for her a particular way of "seeing" the board; learning to adopt the instrument's perspective (among all the possible ones that were available to her) was for Karen a key aspect in making sense of the computer graphs. Karen's new interpretation of slope sign reopened the question of zero slope. I asked her: "What would that mean? The slope of zero. How to get that one?" Karen envisioned two possibilities for the zero slope: a horizontal or vertical board. She felt more inclined to the latter and tried to use the "rise over run" idea:

Ricardo: Something like this [putting the board vertical] would have a slope of zero?

Karen: I'm trying to remember. I can't remember at the moment. Because the height changes but the distance doesn't. So it would be one over zero. Actually I shouldn't say that. I would be maybe 10 over 0. And whenever you divide a number by zero, I believe it's either zero or the null set. I can't remember which way. I think it's zero divided by something is zero. And a number divided by. A non-zero number divided by zero would be a null set, because I don't believe that one exists or it might be imaginative and I wouldn't know about it, because I haven't learned anything about that yet. But that very well might be what I had in mind earlier. And the closer and closer and closer you get to zero [vertical board] the closer (. . .) they [the slope graphs] should reach toward the zero line.

Reacting to the difficulty of inferring the "rise over run" ratio for the vertical board, Karen developed a limit approach. She avoided the intractable vertical case by postulating a trend in the variation of slope as the board turned. We then measured twice with the Contour Analyzer setting two different tilts for the board. Karen expected that the steeper one would produce a slope graph closer to zero. But the experimental graphs varied in the opposite direction:

Karen: All right. So it's not what I thought. It's just the opposite. It moved further away. Hmmm. So if the plane was more horizontal maybe that would bring it closer to zero.

After Karen measured the slope of a horizontal board, I asked her to describe what happens to the slope as the board turns from the horizontal to the vertical position:

Karen: Um, it increases until you get to the point where it's perfectly straight [vertical] and then maybe it. I don't know how. I want to say jump, but it changes to negative once it's gone past so there must be an explanation for what straight up and down is. I just don't know what it's called. (. . .) When it [the board] hits there [the vertical], it [the slope] suddenly drops and starts the other way.

Note how she characterized the vertical slope not as being infinitely large, but as an abrupt "jump" from positive to negative. Her analysis shows that Karen visualized the vertical and horizontal bearings not as isolated cases, but as landmarks within a continuum along which slope changes.

During the next session we had the board tilted with a positive slope, and I asked Karen what would change if we traced the board moving the cursor left to right or right to left. Karen reasoned that the slope would change sign because it would be equivalent to turning the board 180 degrees or seeing it from "the other side." So we traced the board twice, once from left to right and once from right to left. The slope graphs on the computer screen were identical.

Karen: Actually, it looks the same. This is a smart computer. Good trick. Exactly the same (. . .) The point is they're both positive. So that changes my idea now. The thing is how does it [the computer] know that we just simply moved it [the cursor] the other way? It must have been something that we did.

Striving to figure out how the computer might "ignore" the direction on which the cursor is moved determining the slope, Karen elaborated an alternative scheme:

Karen: Unless this [the computer] somehow knows it always has to move in a certain direction [right to left] to get a negative slope when moving upward. *(. . .) Say you have an x y axis, negative direction [right to left], upward [bottom to top] and that would be a negative slope.*

Throughout the ensuing conversation Karen recalled the school idea of the quadrant of signs and commented:

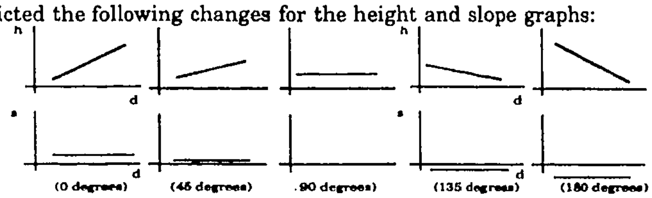
Karen: Because as far as math goes, multiplication and division which would be involved with the slope -- when it's two different signs it will always be negative. When it's the same sign for both it will always be positive. So maybe that's what it's [the computer] reading.

Note how Karen felt compelled to rethink slope sign as she strived to understand how the computer could figure it out.

Turning the round table holding the board, I asked Karen : "Now let's imagine what happens as you rotate this plane." Karen's initial reaction was drastic: "Oh, that throws things. Everything's off." She thought that the slope graph may become curved. She started to imagine the movements of the cursor on the board for the different bearings of the round table:

Karen: So, just a second [observing the board]. (. . .) Just thinking about how I said I think the slope might change [as the board turns]. That's what I'm kind of seeing. I'm now trying to imagine that [the cursor] running across as I'm doing it, because really I'm thinking about it as I'm moving it [the cursor]

By simply moving the cursor of the Contour Analyzer on the board, Karen achieved a sense of what to look at as the round table turned, without measuring. She predicted the following changes for the height and slope graphs:



5. Conclusion

The former episodes illustrate how the Contour Analyzer may become a means to experiment on how to visualize the slope of a flat board. Karen sought to figure out what is that the tool was "seeing" and how it differed from other ways of seeing that were available to her. As Karen learned to visualize the slope of the board she adopted certain points of view (e.g.: looking at the board from a given side), identified conventions (e.g.: going up from left to right is negative), and *imagined* how the cursor would move on the surface (e.g.: her prediction of the changes due to the rotation of the turning table). Karen's struggle to incorporate what she had learned in school, such as the rule "rise over run," shows the limitations of a reductionist view of mathematical concepts, that is, of their ultimate reduction to formal definitions and algorithms; it also indicates the potential richness of learning mathematics as an exploration of the common experience with objects and events. Qualitative analysis was a key aspect of Karen's learning, such as when she reacted to the difficulties on applying the ratio technique to the vertical board by resorting to a limit process ("and the closer, and closer, and closer you get . . . ") and when later she thought of the vertical limit as a sudden jump from positive to negative slope. We expect that future research and tool design along these lines, will create new means for the learning of the mathematics involved in the shape of common objects.

6. References

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics*, 11(1-2), 77-87.

Cobb, P. and L. P. Steffe (1983) The constructivist researcher as a teacher and model builder. *Journal for Research in Mathematics Education* 14 (2):83-94

Confrey, J. (1993) Learning to see children's mathematics: Crucial challenges in constructivist reform. In K. Tobin: *The practice of constructivism in science education*. AAAS Press: Washington DC. 299-321.

Goldenberg, P. (1988) Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representations of functions. *Journal of Mathematical Behavior*, 7, 135-173

Greeno, J. (1988). The situated activities of learning and knowing mathematics. In *Proceedings of the 10th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education*:

Janvier, C. (1978) Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of representations in mathematics learning and problem solving*. (pp. 27-31). Hillsdale, NJ: Lawrence Erlbaum Associates.

McDermott, L., Rosenquist, M. & vanZee, E. (1987) Student difficulties in connecting graphs and physics: Examples from kinematics., *American Journal of Physics*, 55(6), 503-513.

Meira, L. (In press) Making sense of instructional devices: The emergence of transparency in mathematical activity. *Cognition and Instruction*.

Moschkovich, J. (1993). Describing graphs of linear equations during peer discussions: The negotiation and transformation of students' language use. In *Proceedings of the 15th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education*:

Moschkovich, J., Schoenfeld, A., Arcavi, A. (1993) Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.) *Integrating Research on the Graphical Representations of Functions*. Hillsdale, NJ: Erlbaum.

FIVE FINGERS ON ONE HAND AND TEN ON THE OTHER

A case study in learning through interaction

Dagmar Neuman
The University of Göteborg, Department of Education and Educational Research

In a phenomenographic cross-sectional study, 105 7-year-old school starters were interviewed with the goal of understanding how they experienced part-whole relations of numbers within the range 1–10. This study was the starting-point of a two-year teaching experiment with two classes. One child, who had been seen to have hardly any understanding of number in the interview study, took part in interactive, problem-oriented, individual interview lessons twice a week during the two years. The topic of this presentation is the kind of interaction between pupil and teacher through which she began to understand the part-whole relations of number by distinguishing the ordinal and the cardinal meaning of the counting words and by experiencing structural isomorphism between numbers, and how the pupil-teacher interaction was influenced by the earlier cross-sectional study.

Background

In two earlier PME-presentations (1989, 1992), I have described parts of a phenomenographic study (Neuman, 1987), in which 105 children, starting school at the age of 7 in Sweden, were interviewed immediately after school started. The motive for the study was to identify how children experience number and counting before they have received formal instruction in mathematics. It became the starting-point for a two year teaching experiment with two classes from which all pupils had been interviewed (Neuman, 1987). The study showed that certain ways of understanding the part-whole relations within the 'basic numbers' – within the range 1–10 – seemed to result in knowledge of the kind which is usually referred to as 'retrieved number facts'. The teaching experiment aimed to examine the possibility of bringing about such understanding by giving all pupils experiences of these part-whole relationships, through interactive and laborative teaching. This understanding seemed to be formed in a concrete sense when children represented numbers with their fingers in different ways.

About 15% of the school beginners, however, had not yet understood that it is possible to represent numbers concretely. Three pupils did not even seem to understand counting as relating one counting word to each counted object (see Neuman, 1989). One of these pupils was Jenny (J), and she came to me for individual interview lessons twice a week, on average, during the years of the teaching experiment. Our lessons mirrored those going on in the class room but were also directed to more specific problems experienced by J and a few other pupils. One goal of these interview lessons was to acquire a more detailed understanding of the character of interactive learning, taking the preceding phenomenographic cross section study as its starting-point. The interview lessons were tape recorded and transcribed, and their analysis provides the base of this study.

J was the only one of the 105 pupils interviewed who could not answer any question correctly. She knew the counting word sequence up to and including 13, after that came 18 and there the sequence stopped. During one of her first interview lessons she revealed that she actually did not know the sequence after 8. She could not say, for instance, how old her half-sister, Marion, was. J knew that she herself was seven, and that she would be eight next birthday, and also that she would then be as old as Marion now was. But she could not say that Marion was nine without counting from one. J was an 'under stimulated' child. She had a very poor command of Swedish. She did not like to talk, she did not recognise colours, she did not learn the names of her class mates during the whole of the first term, and so on. Even though J's poor language made the interview lessons meagre, of all those who were interviewed she was the most interesting pupil to follow individually because of her almost total lack of understanding of number when she started school.

The teaching experiment

To understand the background for certain elements in the individual lessons with J, something needs to be said about the way in which J was taught in the class room. The teaching experiment her class was involved in set out from knowledge acquired in the earlier interview study, which showed that many school beginners experience numbers as 'continuities', not divided up into units, and that these pupils mix the ordinal and the cardinal meaning of the counting words. Therefore, during the first months, the children played a game related to a story of 'the Long Ago land'. In this land there were no counting words, no digits, no measures, no coins, no schools with a subject called 'maths' on the time table. There was no maths at all. But from time to time problems popped up, demanding different kinds of 'maths'. Then the people in the Long Ago land – the pupils and the teacher – together created the knowledge they needed for the moment.

The game began with problems arising in a series of situations where two servants thought they had not been fairly treated. They had got continuous quantities – 'golden sand', 'oil for their lamps', 'golden bars', 'beautiful silk brocade' and so on – as payment for work they had done for the King and Queen. Since the two quantities always appeared in different forms they could not decide by eye if the King's cashier had been fair or not. 'We'll have to measure', some children suggested. There were no measures, but the children always invented some, and in this way they began to experience that continuous quantities must be divided up into units to be compared. A concept of measuring – an important starting point for development of a conception of number – began to be formed. On one occasion water – oil for the lamps – had to be measured. A plastic glass was chosen as a measure and was emptied over and over again into a bucket. But then the units disappeared and it was impossible to observe whether or not there had been the same number of units in the two vessels. At this point in the story the children felt strongly that they needed number symbols. Some proposed they put up one finger for each unit, others that they should draw one stroke each time a glass was emptied.

Later, when discrete quantities were being measured, it was often impossible for the children to perceive the large number of strokes the cashier had drawn, for instance, one for each sheep paid as tax to the King. Then the number became perceptible by using 'V' as the symbol for the left hand with four fingers 'glued' together, and 'X' as a symbol for two hands with the thumbs crossed.

The reason for introducing Roman numerals – which had been used in their ancient form, once pictures of our hands, where 4 was IIII and 9 was VIIII – was not only that they were part of the story, but also that several children in the earlier study had shown how they used their fingers as if they were Roman numerals: they avoided splitting up the first hand. It was seen that this idea of an 'undivided 5' was responsible for creating the powerful part-whole structure of the 'basic numbers' which would eventually enable answers to word problems to be given spontaneously, without counting. When the fingers were used in this way in a 'building up subtraction', e.g. 2 + _ = 9, or in a 'take away subtraction', e.g. 9 – 7 = _, these subtractions were solved in the same way: the last four of the nine fingers put up initially were split into 2 + 2. Then the answer could easily be read off as 'Seven' or 'Two' respectively, as in VII II. The fingers used in this way lent a structural isomorphism (Freudenthal, 1983) to all part-whole relations of the 'basic numbers' larger than five: the larger part became the first. Some children 'thought with their hands' in this way, illustrating their thoughts with their fingers. Others did not refer to fingers at all, but still illustrated a similar 'biggest first' structure in the more abstract number experiences they expressed.

In the teaching experiment, the Roman numerals were later translated to our modern digits, and the counting words were introduced. The ordinal counting numbers were introduced as well, since several children had used the cardinal words sometimes with an ordinal and sometimes with a cardinal meaning in the interview study; to begin with, the children might say 'number four', 'number five', and so on, if they found the ordinal words difficult.

Jenny's problem: Which counting word denotes the last part of a number?

The first interview lesson with J began at this point, about two months after school started. In this presentation a brief summary will be given of what happened in the seven first lessons, held during the period 28/10 – 22/11. An important theme during the first year of the experiment was to help all children develop the structural isomorphism of the part-whole relations within the basic numbers which some school starters illustrated that they had constructed with the help of their fingers. This theme was also taken up with J from her first individual lesson. I took the experience of 5 plus 5 fingers making 10 fingers altogether as being a suitable starting-point for introducing this theme.

In the first interview lesson (28/10) I therefore ask J how many fingers she has, and she answers: 'Five.' – 'But on both hands?' I add. J has to count her fingers before she can answer this question. I then point to one of J's hands, saying: 'On that one you had ... ?' and J answers directly: 'Five.' After that I point to her other hand asking: 'And there?' J sits q[illegible]tly, moving her lips as if she is counting. Then she answers: 'Ten.' I ask how she has counted, and she says: 'I took six, then it's seven 'n' eight ... ... nine, ten' – 'Yes, then you got how many there are on *both* hands, see?' I say. Then I ask J to sit on the hand to which I pointed when she said 'five,' and add: 'Now let's just count here', pointing to the hand she is not sitting on. But J makes no attempt to count. Finally I prompt her to count with me: 'one, two, three, four, five.'

I began to understand that it might be hard to teach J that 5 + 5 fingers make 10 fingers. Partly she could not say 'nine' or 'ten' without counting from one, and partly she did not understand how to denote the 'last part' of a number – the one where the first object was not related to the word one.

There were fourteen pupils in the interview study who several times used the counting words in the way J was now using them, for denoting the last part of a number; but, they did not do so when they talked about their fingers. For instance, in a guessing game, where they had already counted 9 buttons which were then hidden in two boxes, they might repeatedly guess that there were some number of buttons in one box but still nine in the other one. J had not made guesses like that in the guessing game in the earlier interview study; she had guessed, for instance, that there could be eleven

1011

buttons in one box and nine in the other. Of course, I asked her if she remembered that there were nine buttons hidden in the boxes altogether, and when she answered 'Yes' I went on asking if she thought it was *possible* that eleven buttons could be hidden in one of them. Again she answered: 'Yes.' Her next guess was twelve plus thirteen and the last one thirteen plus eighteen. I asked her again if she remembered how many buttons there were together in the two boxes, and she said 'Nine,' yet made no attempt to change her guesses (Neuman, 1987, pp 97-98). My interpretation of these guesses was that J had not experienced counting as relating each counted object to one counting word, when she thought of the buttons she had counted earlier. If she had done so, no buttons would have been related to the words eleven, twelve, thirteen and eighteen in her guesses.

Now, two months later, however, J seems to experience numbers in the same way as those children who made guesses of the kind 'two in that box and nine in that', which seems to be a more advanced knowledge than that J demonstrated in the interview study. First, it indicated an understanding of counting as relating one counting word to each counted object, since no words outside the number range 1 – 9 appeared in guesses in the guessing game. Second, contrary to Piaget's (1969) interpretation of those answers – that children cannot separate part from whole – to me they seemed to illustrate an early understanding of the whole number as being constituted of its parts, even if the children *could not yet communicate this knowledge in a mathematically correct way.*[1]

The way in which J used the number words to denote the last part of numbers was of a general kind. It did not only concern the fingers and not only the number combination 5|5|10.

When J compares a rod made up of five cubes to one made up of six cubes, for instance, through putting one on the top of the other, and I ask: 'How many more units are there in this rod?' pointing to the single cube that is jutting out in the rod with six cubes, she says: 'Six.' In spite of the fact that I take away this single cube – and J then understands that it is only one cube – she later, when asked about how many more units there are in a rod made of seven cubes than in one made of five, says: 'Six' (under her breath) ... 'Seven!' pointing to the two cubes that are jutting out below the '5-rod. 'Yes, it's the sixth and seventh, is that what you mean?' I say, pointing to the two cubes 'Mmm,' J mumbles. 'And they're together? ... If you count them ... ? How many are there then?' J looks at me uncomprehendingly, and suddenly I become aware of how peculiar my question must seem to her. According to her idea of counting, she *has* already counted these cubes, when she said: 'Six, seven'. I find no words to convey what I want to say to J. Finally *I* count: 'One, two' adding 'the six and the seven, there are two of them.' – 'Mmmm ... ?' J mumbles, just as mystified as before.

We had a 'maths train' in which there were ten dolls, placed in five pairs, five dolls in each of two rows. J talked about these dolls in the same way as she talked about her fingers: there were five dolls in one row, ten dolls in the other and 'ten' dolls also in the two rows together.

The same thing happened when I put out two piles of plastic glasses – five in each pile – or when we divided up a 'rod', which J herself had made of ten unifix cubes, into two similar parts. Every lesson I hid five of the glasses (cubes, dolls, fingers), while I asked J to count those not hidden. And

[1] This interpretation was grounded in the fact that for 'take away' subtractions these children answered with a word related to the last unit thought of backwards, a word different from the last word of the whole (see Neuman, 1989).

every lesson she counted 'One, two, three, four, five,' finally establishing that there were five objects. Yet, as soon as these five objects were included again among the ten constituting the whole, J generally thought that there were five in the first part, ten in the second and ten in the whole.

On two occasions, however, when J had made two rods of five unifix cubes and put them together, I began to hope that she had finally understood how to use the counting words to denote the last part. She said, without hesitation, that there were five in each rod, even when they had been put together. Yet, suddenly I remembered that I should also ask how many cubes there were in the new rod. My hopes were dashed. Without counting J said that there were five cubes in this rod as well. Thus, when ten was divided up into two parts 5 + 10 made 10, since then J first had counted to ten, and related the last cube to the word 'ten'. But when ten was constructed of two fives 5 + 5 made 5, since then the last cube in the last part had been related to the word five, when it had been counted.

Several times I put the palms of J's hands together, illustrating to her that for each finger on one hand there is also one finger on the other hand: five fingers on each hand. This, however, was of no help. J surely knew that her first hand had exactly as many fingers as the second one. What she did not understand was how to use the counting words to communicate this knowledge.

The five-structure constituted by the hands was emphasized in all kinds of number representations larger than five. When, for instance, unifix cubes were used to represent numbers the first five cubes got the same colour, and another colour was used for the cubes from the sixth one and on. When J had constructed such a 'rod' of six cubes, for example, I often asked her first to give me six cubes and the sixth cube, and then to represent six cubes and the sixth cube with her fingers. She was also asked to write the number of cubes and fingers, using the Roman numerals. To begin with she wrote IIIII for five cubes or fingers, but rather soon she began to use the sign for the hand, 'V', for five, and the one for the two hands with the thumbs crossed, 'X', for ten. There was structural isomorphism between the six 'rods', the six fingers and six written in Roman numerals, something Freudenthal (1983) believed to be of importance for an early intuitive perception of number.

I observed, however, that J always chose the first fingers, dolls, cubes and so on in a row when asked to show a number. To help her understand that three can be any three objects, I asked her to take three dolls, and then three other dolls, to put up three fingers, and then another three fingers, also pointing to the difference between 'the finger number 3' – the 3 first fingers – and '3 fingers'. Yet, only after prompts did J sometimes hesitantly show objects other than the first ones.[2]

In the classroom lessons the children had learned to write the ordinal and cardinal numbers in different ways, the ordinal ones as 1a, 2a, 3e and so on (equivalent to 1st, 2nd, 3rd in English). Once J and I gave the dolls paper labels with such ordinal numbers on and put them into a queue, pretending that they were going to enter a skiing competition. This numbering of the dolls was a great complement to the ordinal counting words, which J was still unable to use, not even in the simple form 'number five, number six'. While J looks away I take two dolls, and ask 'How many are there

---

[2] Freudenthal (1983) underlines that children's use of fingers might 'be impeding rather than favouring the constitution of number' (p 56), since it is always the same triple or quadruple of fingers that is lifted, for instance to show the age. This, however, often concerns concrete representations other than fingers, as well, something illustrated even by other pupils than J in the experiment. Therefore the kind of exercises described here seem to be important

here?' – 'Two,' J says. 'Can you write Roman numerals under the dolls to say how many there are?' I go on, putting the dolls on the paper in front of J. She writes II, one stroke for each doll. Then I ask her to translate the Roman numerals to 'our own numerals' and she writes '2' below 'II'. After that I show J the numbers on the dolls' labels, asking: 'Which numbers are they wearing?' 'A seven and a two' J answers. 'But two and seven – the second and the seventh – they're two dolls aren't they?' I go on, pointing first to the dolls and then to the II and the 2 that is written below.

'Yees.' J has taken away the dolls while I am talking and is now making a drawing of them. She writes 7e above the first and 2a above the second doll (fig 1). That gives us an excellent opportunity to talk about 'number two' and 'number seven' being two dolls – one, two, if you count them – in spite of the fact that they are called the second and the seventh when they are in the queue. This is just like J and I being two 'girls' in spite of the fact that we are called Jenny and Dagmar. We take two other dolls several times, and discuss them and write in the same way. J writes the ordinal numbers, but she still goes on to use the words 'two' and 'three', instead of 'number two' and 'number three' – or 'the second' and 'the third' – when she talks about single dolls with certain positions in the queue.

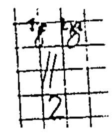


Fig 1

On Nov 18th the first question of the lesson again concerns how many dolls there are in the 'maths train'. After a pause, when J counts under her breath, she answers: 'Ten,' and I go on as usual: 'And how many are there in that row?' – 'Five' (at once). – 'How many in this row, then, J?' After an endlessly long pause J finally says: 'There are five here and, then ... nearly the same.' – 'Only *nearly*?' – 'Yes.' J seems to begin to waver between her two conceptions. Sometimes it seems right to think of the last doll as 'five', sometimes as 'ten', in spite of the fact that there are exactly the same number of dolls in the two rows. 'Maybe,' she seems to think, 'I can say there are almost the same as five in the second row.' However, my astonished exclamation: 'Aren't there *exactly* the same?!' convinces J that this is not a good idea, and she hesitantly answers: 'Mm?' to my question. 'Yes, *exactly* five! *Exactly* the same as there!' I add decidedly. J says nothing. She just looks confused. Some minutes later I ask 'How many fingers have you got on this hand then?' pointing to one of J's hands. 'Five!' the answer comes immediately, as usual. 'How many on the other one then?' Now there is once more an endless pause, before J gives her usual answer to this question: 'Ten'.

In the last lesson to be described (22/11) I put all the dolls pell-mell on the table, and put a piece of paper in front of J on which I write en equals sign. The children have used the equals sign in the classroom from the very first lesson, meaning at first that the two servants have been fairly treated, and J knows that there must always be a similar number of 'units' on each side of this sign. The word 'unit' has also been used in many contexts in the class from the outset. 'J, do you remember how many dolls there were altogether?' I ask, but J can still not remember that. 'It was the same number as you have fingers' I add. 'Five', J says. 'Yes, you've got five fingers on that hand, yes'. I point to one of her hands. 'But there's a lot more than five here ... as many as *all* your fingers ... How many fingers have you got?' I show her all my fingers. J quietly counts her own fingers and answers: 'Ten.' – 'Mmm, and there're ten dolls as well ... There were five there, as many as on one hand, and five here!' I put five fingers into the empty holes where the first row of dolls had sat, and five in the holes of the second row . .'Now let's share out these dolls, you and me, ... so we get just as many each. ... Do you think you can do that?... Share them between us?' J does not seem to understand what 'share them between us' means. She just sits quietly, looking at the dolls. 'We can take it in turns to take one' I say. 'First I'll take one ... ' J just says 'Mmm' uncomprehendingly. How many have I got now?' – 'One.' – 'Now you can take one!' J takes *my* doll. 'Are you going to take *that* one?!' I cry, astonished, but I understand when I see '1a' written on its label. I tell J that this is the doll I have chosen, and that I want to keep it. She has to take another doll. J begins to grub aimlessly in the pile of nine dolls that are left. I tell her that she can take any doll, but in the end I have to turn all the dolls over, so that their numbers cannot be seen, before J will take a doll from the pile. It happens to be the one with '2a' written on it. We establish that we both have the same number of dolls: one doll – one unit – on each side of the equals sign. J writes 'I' below each doll.

After that we both take another doll, and I ask: 'How many have you got now?' – 'Two.' – 'And me ...?' – 'Two.' – 'I took the first and the third - one and three ... And which did you get?', I go on. 'One ...', J hesitates. 'No, you got ... ?' – 'Uhh,' J wriggles. 'Which is this ... it's ... ?' J counts quietly and says: 'Number ten.' Now, for the first time, she uses the early ordinal expression 'number ten' instead of the cardinal word 'ten' that she has used earlier. 'So you got ...?' I go on. 'Number two' J says at once 'You got number ten and number two ... and I got number three and number one ... And there are two of them!' I point first to my dolls and then to hers. We each take one doll more, and 'III' is now written on both sides of the equal sign. Then I say: 'I've got number three and number one and ... ', prompting J to go on. 'A number four', J says. Now she is partly trying to say that there is one doll, and partly that this doll is the fourth, or, as she says 'number four'. 'And you've got ...?' I say. 'Number nine and number two and (J counts) number ten'.

We go on until we have five dolls each. When I then ask which dolls J has, she says 'A number seven, and a number eight, and then a ... ... (counts) ten and two and nine'. 'Ten' is so hard that she forgets to use the ordinal number words when she has arrived at the doll with '10e' on its slip of paper. 'Which have I taken?' – 'A number six ... number five, number three, number one, number four'. Now she is using the ordinal counting words again. – 'Have we got as many dolls each?' I ask, and J answers: 'Yes.' I ask J to write five in another way, pointing to 'IIIII', saying: 'This is five as well, isn't it', showing my hand. J writes 'V' below 'IIIII' on both sides of the equal sign, and I prompt her to put an equal sign between the two V-symbols as well. She does so (fig 2). 'Can you write how many there are with another numeral as well, like those we have on telephones?' I say, and J writes '5' below each 'V'. 'Now you've told me how many we've got in two different kinds of numeral', I add. 'Now ... how many have we got all together?' J counts quietly and says 'Ten'. Then she writes 'X' on her own initiative. 'Yes, there're ten dolls ... five dolls and five dolls make ten dolls. Can you write ten here with the other numerals?' I ask, and J writes '10'.

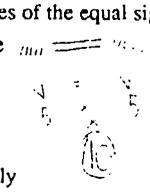


Fig 2

After that I ask the usual question: 'How many fingers have you got on this hand? – 'Five (at once). – 'And on this hand?' Now something unforeseen happens. Almost before I have posed the question J shouts 'Five!' – 'You know it!!' I exclaim, astonished and very happy. The lesson then goes on with some other themes, but before it is finished J has to put the dolls back into their train. Then I say, just to see J's reaction: 'How many are five plus five, then?' To my surprise J immediately begins to count under her breath and answers 'ten', in spite of the fact that I have used the abstract and general expression 'five plus five'. To test if my growing hope is to be dashed once more, I pose the same question in a new context: I ask how many toes J has. She takes the shoe and sock off one foot and begins to count: '(mumbling)... four five ... I know they, that they're ... (mumbling).' – 'How many toes in your other sock, then?' I ask. Once again, J immediately shouts 'Five!!' happily and with great confidence. Now, however, comes the critical question: 'And how many are there on both feet?' J seems to have expected this question, beginning her counting even before I have posed it, since again – happily and almost immediately – she exclaims: 'Ten!!'

Conclusion

The goal of the teaching experiment and the interview lessons was to find out if it was possible, through interactive and laborative teaching, to deliberately form a consciousness of the part-whole relations within the basic numbers which several school starters seemed to have formed already in the school starter study. J had actually acquired one 'number fact', namely 5 + 5 = 10, by the last lesson of the period described here. That, however, was not acquired through drill in number facts. It was anchored in understanding of the counting words and of how part-whole relations of numbers can be communicated through them. And it was a knowledge generalized from experiences of 'structural isomorphism' between fingers, toes, Roman numerals, dolls, cubes, glasses, and also between different numbers of fingers.

My starting point for designing the teaching experiment, and also for communicating with J, was my knowledge of the early conception of number expressed by those children in the school starter study who denoted the last pa[illegible] of the number (experienced by them as a kind of continuity) by the number word related to its las[illegible] unit. The goal was to change this experience to an experience of the part-whole relations of all the 'basic numbers' as similarly structured and related to correct denotations. The way in which J had learned that 5 + 5 = 10 helped her to understand quite quickly that 5 + 3 = 8, 5 + 4 = 9, 6 + 3 = 9, 6 + 4 = 10 and so on,. The result of the teaching experiment was that in the interviews carried out at the end of grade 2 all pupils in the experiment classes – even J – could use this knowledge for addition and subtraction over the 10-border within the number range 1–100, without the help of fingers or any other kind of concrete aid. This result was very different from that of a control group, where there were children in the grade 2 evaluation who, when the guessing game was played, still guessed that there were 9 buttons in one box and 3 in the other.

References:

Freudenthal, H.(1983) *Didactic phenomenology of mathematical structures*. Dordrecht: Reidel

Neuman, D. (1987) *The origin of arithmetic skills. A phenomenographic approach*. Göteborg: Acta Universitatis Gothoburgensis.

Neuman, D. (1989) Early conceptions of subtraction. In Vergnaud, Rogalski, Artique (eds) *Proceedings of PME 13. Psychology of mathematics education*. Paris.

Neuman, D. (1992) The influence of numerical factors in solving simple subtraction problems. In *Proceedings of PME 16. Psychology of mathematics education*. Durham, New Hampshire.

Piaget (1969) *The child's conception of number*. London: Routledge and Kegan Paul.

Acknowledgement: Thanks to Shirley Booth, who has made my English readable

Constructing Meanings for Constructing: An exploratory study with Cabri Géomètre

Richard Noss, Celia Hoyles, Lulu Healy and Reinhardt Hoelzl
Institute of Education, University of London

We further develop our work on the ways in which computational media structure children's mathematical expression and understandings. In this paper, our focus is on dynamic geometry environments. We describe the problem-solving strategies developed by a group of twelve-year-old students during their interactions with Cabri Géomètre. By presenting two case studies, we focus on how and when interactions with the computational objects of Cabri enabled students to make links between their informal intuitions and more formal geometrical representations.

Throughout our investigations of mathematics learning within computational settings, we have been interested in the dialectical relationship between the medium and the mathematics (see Noss and Hoyles 1992). We have described how students frequently construct and articulate mathematical relationships which are *general* within a microworld, yet are interpretable and meaningful only by reference to the tools available: students can construct *situated abstractions*. Additionally, we have described how a computational setting can be exploited by the student to provide *computational scaffolding* onto which they may hang their developing ideas. We have provided examples of both these theoretical notions by reference to students' work in a variety of software settings, and recently we have turned our attention to Dynamical Geometry Environments such as Cabri Géomètre (see Hoyles and Noss 1993). Laborde similarly (1992) recognises that the features provided within different kinds of geometry software (Cabri, Geometric Supposer, Logo) may affect the solution process. Thus, despite the fact that Cabri simulates ruler and compass constructions, it creates a different situation: for example, while paper-and-pencil geometry needs only one type of point, Cabri geometry needs to distinguish three; on a sheet of paper an intersection point is simply marked with a pencil where two geometrical objects intersect, whereas on the Cabri screen it has to be explicitly constructed. In fact, some construction procedures which are totally correct in a paper-and-pencil setting become inadequate in a Cabri environment (see Capponi & Straesser (1992).

In our research, part of which is reported here, we took as our focus geometry-with-Cabri rather than geometry-with-pencil. The aims of the research were three-fold: to explore issues concerning the learning of geometry which are brought to our attention by looking through the window of Cabri; to investigate the problem solving strategies students develop while interacting with Cabri; and probing the ways students constructively generate mathematical ideas within the Cabri setting.

Since Cabri Géometre was introduced at ICME 1988 in Budapest, a wealth of examples has been developed that illustrates the impressive range of school geometry which is amenable to its use (Schumann, 1991). Cabri is often regarded as a microworld for 'guided discovery learning'

(Laborde & Straesser, 1990) in which the concepts associated with geometrical relationships are brought into focus. Other similar software such as Geometer's Sketchpad and The Geometry Inventor have also caught the imagination of mathematics educators, particularly in the US.

In contrast to the abundant illustrations of Cabri's potential, there appears to less published reports which investigate if and how students exploit Cabri to assist them in solving geometry problems or into the nature of students' conceptions of Cabri geometry. There is a need to understand the potentials and the pitfalls of using DGE's, and to examine what students believe they are doing and why. To take one example, one of Cabri's main features, the so-called 'drag mode', allows a drawing to be moved around the screen while preserving the underlying construction. It is this feature which highlights the importance of making an explicit distinction between 'drawing' and 'figure' (Parzysz, 1988; Laborde, 1991; Straesser, 1991). Laborde (1991) points to the fact that it is not yet at all clear how students interpret the drag mode, while Straesser (1992), who analysed students' construction and proofs in the setting of a Cabri square, concludes that although dragging provided a means to mediate between the concepts of drawing and figure, this 'mediation can only be used at the cost of an explicit introduction and analysis, a continuous use of this feature and an — at least implicit — distinction between empirical and theoretical concepts in a construction task' (p 16).

One of the issues we explore in this paper is how students use Cabri to scaffold the move from an empirical to a more theoretical approach.

The setting for the study

We have been conducting case studies of eight 12-year-old students working with Cabri over a period of five sessions, each lasting one and a half hours. The students worked in pairs with us during their mathematics lessons in a room set aside in their school. These students had not been introduced in any formal way to Euclidean Geometry — indeed the newly-imposed 'National Curriculum' in the UK is more or less unencumbered with Geometry altogether. We therefore had both a pedagogical problem and a research opportunity. On the one hand, we could take nothing for granted: while words like 'perpendicular' might be vaguely familiar to some of the students, the notion of pencil-and-ruler construction would be totally alien to all. On the other hand, this lack of background knowledge gave us the chance to study first-hand how students make sense of geometrical ideas with Cabri without bringing to their work a variety of notions derived from experience with the traditional technologies of pencil, ruler and compass.

Introducing Cabri: We decided against giving a quick course in paper and pencil constructions prior to the Cabri work in order that we could probe how students' spontaneous approaches to constructions when they were equipped with Cabri tools. As a way into the Cabri work which would connect with prior experiences, we encouraged the students to construct their own designs — Cabri as a drawing tool!

But how to introduce the notion of dragging in the context of drawing to students who had no idea of geometrical construction? After considerable discussion among ourselves, we hit upon the idea of *messing up* — after a figure was drawn it could be dragged (by anybody, including us) to see if the relationships between the objects within it moved together in a sensible way. For example if a 'face' had been drawn, the eyes and nose would be expected to stay inside the outline of the head! In this way, we hoped that students would begin to conceptualise the distinction between drawing and figure. Although not completely unproblematic, we found messing-up to be a powerful idea. It has given us a language in common with our students, provided a rationale for constructing rather than 'drawing' and afforded us all a mutually acceptable mode of validation for constructions. To our surprise we found that the idea of messing up has spread to other classes and become part of the Cabri culture in the school and beyond (see Healy, Hoelzl, Hoyles and Noss, 1994).

After this introductory phase we offered tasks which increased the opportunities to introduce the main features of Cabri. We now outline two vignettes which illustrate some insights which emerged from our case studies.

Vignette 1: Circle as length carrier

It is fundamental to Euclidean constructions that a circle is seen as a tool for 'carrying' length. Yet it is not at all obvious how to introduce this idea, with or without a computational tool such as Cabri. Our first attempt was to devise some tasks in which students would see length-carrying circles as *useful to them* in the solution of the task.

However, this was not as straightforward as we had hoped. Our first task involved doubling the length of a line segment. Most of the students created a line segment AB and then simply placed a *basic point* by eye which was approximately in the right place, and joined B and C. Students could readily see how this could be messed up, but it was far less obvious what to do about it. In fact, we had to nudge all of them fairly vigorously to consider that circles might be useful, and even then we found that most reverted to eyeballing the solution. A typical response was to draw the circle centre B and radius AB, and then to use 'point on object' to place a point C on the circle at about the correct place and join CB (see Figure 1).

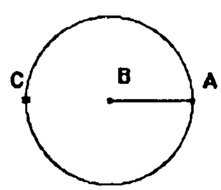


Figure 1: Positioning a point (C) by eye.

How can we explain this behaviour? We suggest three possibilities:

- The students could *only cope with one constraint at a time* — they could fix CB to be the same length as BA, but had lost sight of the second constraint that A, B and C should lie on a straight line;
- The students did not appreciate the need to *construct* a point — we might call this the WYSIWYS phenomenon: *what you see is what you seek;*
- The students had only constructed the circle to please us and had not appropriated the *idea of the circle as a measure of length.*

It is most likely that the complete explanation is some mixture of all three, but we decided to pursue the final possibility in more detail. We devised a further task, in which the idea was to use a circle to maintain a relationship between two lengths in a *dynamic* rather than static way. The students were asked to construct two intersecting lines and a point P on one line (see Figure 2). We then asked them to construct a point Q on the other line so that however we moved P, the distance OQ was always equal to OP.

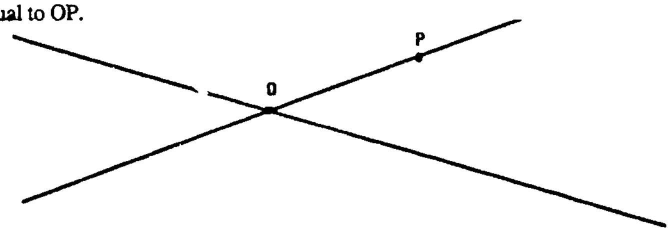


Figure 2: The *moving points* task

We will consider how one pair — Billie and Nora — set about solving the task. Billie and Nora set up the two *basic lines* and proceeded to 'solve' the problem by eye — first by simply placing a basic point on the second line (*WYSIWYS)* and then by restricting it to the line by using 'point on object' *(attending to one constraint).* In both cases when we asked "Does it move with the other point?" they knew that it didn't. The pair became stuck and simply started guessing, adopting a strategy we have frequently observed — random opening of menus and trying out various items in desperation! Eventually they hit on the idea of *symmetrical point* — a construction they had used before and an idea which is at least intuitively familiar to all students of this age. They constructed the reflection of P in the second line, Z (see Figure 3), and discovered that dragging P *did* cause Z to move in unison — one constraint satisfied — but that Z did not (of course) satisfy the second condition, it didn't lie on the second line. Unfortunately they could not see how to exploit the symmetrical point idea as scaffolding towards solution and the pair decided to start again.

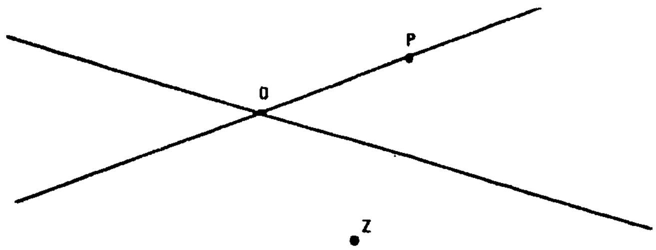


Figure 3: Using a symmetrical point

Once again Billie and Nora used basic lines for their line pair, but this time set them up at right angles (surprisingly easy to do by eye by judging pixels). This time we posed the question in a slightly different way: "Where do you want the point to be?". Both students knew for sure, confidently pointing to the correct place on the second line. What they were less sure of was how to fix its position. They tried a variety of tools from the menus, including parallel lines, basic points, points on objects — anything they could find! Finally, we tried again: "Could you think of a way of making something cross at a point exactly the same distance from the intersection point as the point on object?" And suddenly (somehow) they said "Would it be a circle?" — they tried it, and much to their delight, it worked. They were successful but only as a result of very pointed interventions and it is unclear as to how far they had made sense of the situation for themselves.

We have given considerable thought to try to explain why the students seemed to find it so difficult to entertain *the idea that a construction could be used as a means to an end.* The first and most straightforward possibility — and one supported by our observations — is that students have a natural aversion to constructing objects that they do not envisage in their 'final' construction. This behaviour seems similar to the Logo situation where students 'see' Logo constructions as 'drawings' rather than as, say, compositions of the effects of procedures. Perhaps we have identified the same phenomenon in Cabri? After all, we only have ourselves to blame if we encouraged a 'drawing' perspective in our introductory activities.

Nevertheless, pointing out that it is straightforward to hide unwanted objects did not immediately alter the situation[1], so this explanation cannot tell the whole story. A second and deeper possibility necessitates bearing in mind that the students knew exactly where the object was. They could point to its (approximate) position: that is, for all *practical* purposes, they had fixed its location. The problem is, of course, that practical and mathematical purposes are not the same — an empirical/theoretical distinction. Yet how are the students to know this? How can they be helped to appreciate that a circle (or any other object) needs to be used as a tool? The students are faced with a subtle series of meanings, many or most of which are new. First, they need to shift from seeing a circle as an object to a circle as a tool — especially difficult since the medium to which they had just been introduced focuses their attention on the construction of objects. Second, they need to abstract — at least tacitly — just what role the circle plays in a construction, and we have seen that this is

[1] It did not help that there was an initial (and disastrous) confusion between hiding and deleting objects.

not at all obvious. Last, and by no means least, a crucial way of developing meaning for the notion of construction (perhaps the only way?) is to see geometrical construction as a part of a much bigger picture, in which construction becomes itself an object of study, *and* a tool for understanding geometrical relationships, theorems and abstractions.

Vignette 2: Cabri-Scaffolding

We now turn to events in a later session, in which the students were asked to reflect a flag in a mirror (see Figure 4). Their task was to do this *without using symmetrical points*-in order that they were faced with a more complex construction which invited them to formalise their intuitions of reflection.

Figure 4: The Flag-in-the-mirror task

Helen and Katie first set the mirror line to *look* vertical, and then drew a line segment through one of the points on the flag, P. They then created a basic point, Q, on the other side of the mirror, joined it to P then making PQ *look* horizontal. Q was dragged so it *looked* the same distance from the mirror line as P (see Figure 5). The pair had 'found' the solution 'by eye' and they were perfectly clear that it could be 'messed up'. It was a temporary measure so that they could obtain some feel for what they were looking for: they had something on the screen to guide them; they had exploited the medium to achieve this result (albeit *not* in the way its designers intended); or put another way they had used Cabri as scaffolding in their search for a solution.

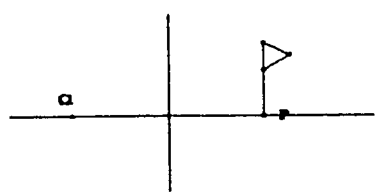


Figure 5: Specifying the problem

Although they were aware that this would not do, it took some exploration with their pseudo-construction before Katie suddenly announced: "We can use circles!". At this, they constructed a circle with centre the mirror intersection point, O, and radius OP, and found the image of P by the intersection of this circle with their line (see Figure 6). They did not bother to measure to check this: they were completely convinced of the correctness of their strategy and immediately started repeating the process for different points on the flag.

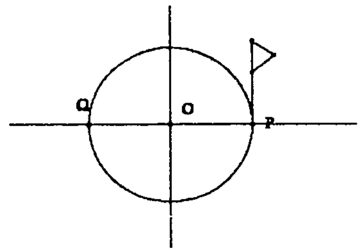


Figure 6: Dealing with one constraint

Helen and Katie had solved the problem *with a constraint removed:* that is, their strategy relied on the mirror being made to look vertical and line from P being drawn as horizontal by eye. So their strategy solved the problem for the distance but did not address, in general, the relationship of a point to the mirror line.

We decided to intervene and asked what would happen if the first line going through the mirror was moved. They both seemed to know that this would "mess up". Once they had noticed this — *and because they had in front of them what they wanted to produce* — they soon came up with the idea of a perpendicular line. Perhaps they were still using the support structure offered by the horizontal/vertical orientation of their initial 'axes' as a way to generalise the relationship that must hold between the lines?

Conclusions

We propose that an important facet which distinguishes the second vignette from the first is the way in which the medium was employed by the students as a form of *computational scaffolding* (Noss & Hoyles, 1992); how each pair used the medium as a support system for exploration and problem solution. In the second vignette, the WYSIWYS strategy provided a sense of the solution, a feeling that it existed, that it was attainable and above all, how it would look when found. All these aspects served as clues to the construction process. Secondly, Helen and Katie used the idea of 'fixing one constraint', as a way to solve a simpler problem — this time one which they were unaware was less general than the original posed by us.

The fact that neither of these strategies were (as far as we can tell) explicit or conscious, underlines an important aspect of the scaffolding idea. It seems that what is important is that scaffolding describes the process by which the informal is linked with the formal, the intuitive with the procedural/symbolic. As in any serious microworld, the core of the Cabri idea is that students are able to interact with computational objects (in this case, points, lines and circles), in order to play with formal objects in an informal way. The crucial element is that students can *articulate* descriptions and connections for themselves, the form and nature of these being mediated by the medium available: this is what distinguishes them from standard tools:

> 'Such programs [*Cabri, Logo etc.*] differ from drawing tools like MacPaint in which the process of construction of the drawing involves only action and does not require a description. In this kind of software, construction tasks are no longer to

produce a drawing but to produce a *description* resulting in a drawing'.(our emphasis).

Laborde C. (in press).

But the construction process and how geometrical objects, particularly the circle, play a part in this process is by no means straightforward, as our first vignette illustrates. Despite using similar strategies, this pair were not able to shift from either WYSIWYS or the fixing of one constraint to solve the problem. To do so required the use of an object as a construction tool — as a means to an end rather than the end itself. The evident difficulties that students had with this notion has made us wonder about geometrical constructions using more traditional tools. Do students constructing equal lengths by arcs of a circle using compasses 'know' they are drawing (parts of) circles? We venture to hypothesise that many would not and simply apply a routine set of procedures (put compass point on A, draw arcs to cut lines AB and AC).

Following from these vignettes, it might be easy to jump to the conclusion that strategies such as WYSIWYS should be discouraged by giving more explicit instruction on the grounds that they obscure the essence of the software and the mathematics to be constructed with it. Our conclusion is the opposite. We need to find ways in which we can exploit software such as Cabri to *negotiate* a problem solution, to find ways to help students build on what seems to them to be a 'natural' approach. That is, after all, the essence of the scaffolding idea: to construct a support mechanism which enables other, more permanent, robust constructions to be built. The findings from this research suggest this is a feasible goal.

References

Baulac I, Bellemain F & Laborde J-M, designers. (1992) Cabri: the Interactive Geometry Notebook. Pacific Grove, CA: Brooks-Cole

Brock, C.F. & Cappo M (Wings) & Arbel A, Rosin M., & Dromi, D.(LOGAL). designers (1992) Geometry Inventor. Scotts Valley, CA: Wings for Learning; Kiryat Shmona, Israel: LOGAL

Capponi, B. & Straesser, R. (1992): Cabri-géomètre in a college classroom. Teaching and learning the cosine function with Cabri-Géomètre. In: Zentralblatt fuer Didaktik der Mathematik 24 (5),165 - 171.

Healy, L., Hoelzl, R., Hoyles, C. & Noss, R. (1994 in press) Messing Up. Micromath 10 [1]

Hoyles, C. & Noss R Out of the Cul-de-Sac Invited Lecture PME-NA, Proceedings of the Fifteenth Annual Meeting, San José State University, Vol 1, 83-90

Jackiw, N. designer.(1990) Geometer's Sketchpad. Berkeley, CA: Key Curriculum Press

Laborde, C. (1991): Zu einer Didaktik des Geometrie-Lernens unter Nutzung des Computers. InStraesser, R. (ed): Intelligente Tutorielle Systeme fuer das Lernen von Geometrie. Bielefeld:IDM, Universitaet Bielefeld (Occasional Paper 124).

Laborde, C. (1992): Solving problems in computer based geometry environments: The influence of the features of the software. In: Zentralblatt fuer Didaktik der Mathematik 24 (4), 128 - 135.

Laborde, C. (in press) The computer as part of the learning environment: the case of geometry. In: Keitel C. & Ruthven K. (eds). Learning from Computers. Springer-Verlag.

Laborde, J.-M. & Straesser R. (1990): Cabri-Géomètre: A microworld of geometry for guided discovery learning. In: Zentralblatt fuer Didaktik der Mathematik 22 (5), 171 -177.

Noss R. & Hoyles C. (1992) Looking back and looking forward. In: Hoyles C. & Noss R. (Eds.)*Learning Mathematics and Logo*. Cambridge: MIT Press.

Parzysz, B. (1988): Knowing vs seeing, Problems of the plane representation of space geometry figures. In: Educational Studies in Mathematics 19 (1), 79 - 92.

Schumann, H. (1991): Schulgeometrisches Konstruieren mit dem Computer. Beitraege zur Didaktik des interaktiven Konstruierens. Stuttgart: Metzler/Teubner.

Straesser, R. (1991): Dessin et Figure. Bielefeld: IDM, Universitaet Bielefeld (Occasional Paper128).

Straesser R. (1992): Students' Constructions and Proofs in a Computer Environment. Bielefeld:IDM, Universitaet Bielefeld (Occasional Paper 134).

SUBDIVISION AND SMALL INFINITIES: ZENO, PARADOXES AND COGNITION

Rafael Núñez E.

Department of Psychology, University of Fribourg, Switzerland*
Institute of Cognitive Studies, University of California at Berkeley

Abstract
From a cognitive viewpoint, the study of subdivision offers an interesting subject matter to address the question of how the mind builds the idea of infinity in the small. 32 students, aged 8, 10, 12 and 14 (high and low intellectual-academic performers), participated in this study, in which a version of one of Zeno's paradoxes was analyzed by means of individual interviews. Results suggest that between ages 10 and 12, a certain intuition of the entailments of subdivision emerges, remaining very labile afterwards such that is very influenced by the context. 66% of the 12- and 14-year-old children said that the process involved in the paradox comes to an end. Less than 25% considered (with deep hesitations) the possibility that the process might continue endlessly. Some epistemological consequences are discussed.

About 2500 years ago in Elea, now southern Italy, a disciple of the philosopher Parmenides, known as Zeno the Eleatic, conceived some paradoxes presumably in order to prove the inconsistency in the Pythagorean ideas of multiplicity and change and to argue in favor of the unity and the permanence of being, which were fundamental principles of his school of thought. Different versions of these paradoxes have come to us, but amazingly, their paradoxical features have always puzzled philosophers and mathematicians, and even today still intrigue us. A shortened version of one of these paradoxes could be stated as follows. Imagine that we are asked to go from a point A to a point B, but we are told to do so by following a rule which says: first go half of the way, then half of what remains, then half of what remains, and so on. Do we ever reach point B? Since each step only covers half of the remaining distance at the previous step, there will always be a short distance to be covered. Therefore we never reach point B. Of course, this is not what our experience tells us when we go from one place to another, whence the paradox rises.

History of mathematics, and studies in mathematical thinking and developmental psychology show that infinity in the small -involved in iterated subdivision- has been much more controversial and elusive than infinity in the large (Núñez, 1993a), and that children begin to understand the idea of infinity in the small much later than infinity in the large (Piaget & Inhelder, 1948; Langford, 1974; Taback, 1975; Núñez, 1993b). Thus, while 8-year-old children easily recognize the endless nature of the counting process, it is not until about the age of 11-12 that they start to consider the indefinite subdivision as a legitimate problem (Núñez, 1993a).

* On leave, supported by the Swiss National Science Foundation (fellowship No 8210-033279).

From a cognitive point of view, the study of paradoxes such as Zeno's offers an interesting subject matter to approach the question of how the mind builds the idea of infinity in the small, contributing thus to the poorly studied domain of infinity (Núñez, 1990). As has been suggested elsewhere (Núñez, 1991), the paradoxical aspects of the arguments propounded by Zeno, as opposed to other situations in which infinity is involved but that don't give rise to paradoxical considerations, could be partially explained in terms of a particularly complex coordination of attributes being iterated indefinitely. In the example given above, if one follows the rule of the problem, there are two attributes that one has to iterate simultaneously, namely, the number of steps one has to make and the distance that each of these steps covers. On the one hand, observe that the type of iteration is different for each attribute, because the number of steps increases whereas the distance covered by the steps decreases. Let's call these two types of iteration, *divergent* and *convergent*, respectively. On the other hand, the nature of the content being iterated is also different for both attributes: the number of steps refers to *cardinality* (number of steps) whereas the distance covered by the steps refers to *space*. Unlike situations in which two attributes of either same type or same nature are indefinitely iterated in a coordinated situation, the coordination that takes place in Zeno's paradox must incorporate heterogeneous components. That is, different type of iterations (*divergent* and *convergent*) and different nature of the content (*cardinality* and *space*), that makes it qualitatively more complex.

In general, any situation involving subdivision, that might lead to conceive the idea of infinity in the small[1], presents this structure: a simultaneous coordination of an increasing number of steps (number of iterations) and a decreasing partial result (in absolute value) to be operated in the next iteration. From a cognitive viewpoint it is interesting then, to study how this complex coordination emerges in mental activity and what forms it takes across different conceptual contexts.

The following presents some qualitative aspects of an invetigation that was inserted in a broader project which intended to study developmental and psycho-cognitive aspects underlying the idea of infinity in a much larger sense.

Method

Procedure: Two variables related to intellectual and academic performance were studied: The former was measured by the Raven test and the latter measured by the marks obtained both in mathematics and French (official language at school). The Raven test was administrated in the classroom during school time. The study was performed by means of individual interviews (50 minutes) and was introduced to the

[1] Of course, this is not the only way that might lead to conceive infinity in the small. In non-standard analysis, for example, infinitesimals are seen as the reciprocals of infinitely big numbers. Nevertheless, from a developmental viewpoint, the idea of subdivision seems to be a more basic one.

students as a study of psychology (mathematics was not mentioned at all). The interviews took place in a room located in the school and were videotaped. The question about Zeno's paradox was included.

Subjects: Thirty-two students from two schools in the city of Fribourg, aged 8, 10, 12 and 14 years old, were interviewed. Each age group (8 subjects) was formed by half low and half high intellectual-academic performers (percentiles 0-33 and 66-100 in each age group respectively, taken from a larger population; N=172). Boys and girls were equally represented.

Material: The problem was presented orally and the question was: "Imagine that we want to go from this side of the table to the other side. First we are told to go half of the way, then we continue with half of what remains, then half of what remains, and so on. Do we ever reach the other side of the table?" Sometimes the subject performing the action was a small insect. The students were allowed to use a variety of materials such as paper, pencils, rulers, etc., which were on the table.

Results

In general, the children's answers could be classified in three main categories, namely, those saying that one reaches the destination, those saying that one does not arrive (either because the process gets stuck, or because it only approaches the destination), and those coming from a hesitating subject defending two different answers. Subjects of all four age groups gave answers of the first two categories. The last category was only observed in the older groups. The following presents the rationale behind these views. No gender differences were found.

8-year-old group:

In this age group, there was a difference between the arguments given by the high and low intellectual-academic performance subgroups, the former showing a more analytical attitude towards the problem. Low performers tended to consider the question as a very trivial one and easily responded with an obvious "we arrive", as if the iteration were to be performed just a couple of times in order to cover the distance.

Based on the idea that in the end one gets stuck due to the smallness of the steps, two children considered the possibility that one might not arrive, and one said that one never arrives. All three were high performers. They vaguely perceived that one might potentially continue for a while making small steps, but in the end they solved this unstable situation by altering the conditions of the problem, either by altering the conditions of the setting (new destination), the of action (i.e., stopping the iteration), or both. An example of the former is the following[2]:

> Ban +(7;9): *If we always go half way we arrive very close, but we are forced to always do halves, always the half.* So? ... (thinks) ... *there is always a small half.* We get closer but we can not

[2] Text in italics corresponds to subjects' utterances. The symbols + and - indicate high or low intellectual-academic performance respectively. The age is also indicated (years; months).

arrive? *We get closer ... but if we go half of the way there is just a step left, and if we go the (other) half, there is just one half left, ... and we go that half and we arrive* (not so convinced). You doubt? ... (thinks and reanalyzes with his fingers near the edge of the table), ... *or maybe there is a small piece left before the arrival and we decide that that's the place where we have to stop.* How is that? *We put the arrival before the place where we want to go, so then we are sure that we arrive.*

10-year-old group:

Like in the previous group, there was a difference between the high and low performers, similar to that described above. 10-year-old children presented more hesitations than the 8-year-old ones and some of them had two different positions, but like the youngest group, sometimes the conflict was solved by altering the conditions of the setting for the last iterations (i.e., new destination).

As it was mentioned in the *Procedure*, other contents were also covered in the interview. Among others, it included a problem described elsewhere (Núñez, 1993b), in which plane geometrical figures were transformed following an iterative process with similar construction (different type of iteration and nature of the content). It is worth noticing that some 10-year-old high performers presented an infinitist position (i.e., acceptance of an endless iteration) in Zeno's paradox but a finitist position in the geometrical problem, and vice-versa. Thus, Euo +(10:1) who was clearly convinced by his arguments in favor of the endless iteration involved in the geometrical problem, did not even hesitate to say that the iteration in the paradox comes to an end:

> Euo +(10;1): So do we arrive to the other side? *Yes, certainly, ... if we make the half, the half, the half, the half, the half, the half, the half, the half, ... afterwards, we arrive, for sure.* I have a friend that says that we can not arrive because there is always a half that you have to make (the Exp. shows the steps on the table and the Subject responds immediately with the following step, as if he wanted to complete the "other half"). ... *and once again, the half, once again, the half, ... we will arrive.* And if we only make it in our imagination? *In our imagination and in physics also* (we arrive).

Ode, +(9,9), on the contrary, maintained a finitist position in the geometrical problem but suggested the endless nature of the iteration in the paradox:

> Ode +(9;9): Does it arrive? ... (examines the "last" steps) ... *always the half?* (the Subject asks). *It's really, ... if it is there it makes the half* (follow the steps on the table) *it will take a long time because afterwards it makes just one step, ... and then the half, and then always the half, ... no, it doesn't arrive. There, for example* (shows the edge of the table) *it is there, and after makes the half, arrives there, and then there, ... no I don't think that I will arrive by always going a half.*

12- and 14-year-old groups:

Unlike the 8-and 10-year-old groups, we didn't find relevant differences between the arguments of high and low intellectual-academic performers in the older groups. The patterns of answers in the 12- and 14-year-old groups were quite similar to each other, so we are going to analyze them as one.

Nearly 2/3 of the 12- and 14-year-old children (high and low performers together) gave answers of the forms "It will take a long time, but we will arrive" or "One will arrive because in the end one will not be able to make the half, it will be too close" (altering the conditions of action). Among these subjects some showed deep hesitations holding antagonic positions, often opposing ideas based on immediate physical considerations (although not necessarily concrete) and ideas based on the fact that the iteration leads to approach the destination (although not necessarily abstract):

> Mal -(14;8): Does it arrive? (thinks, whispers) ... *I'm not going to measure the table, but I think we don't arrive exactly, maybe about a millimeter away. I think that we will arrive, but that we are not going to arrive immediately, ... if we have every time, ... well, I don't know. I think that it will arrive exactly at the point, and I think that it will not arrive exactly. I have two opinions. I don't know whether it will be exactly or at about a millimeter away.*

Others changed their mind when the scale of the problem was amplified:

> Frc -(14;7): Do we arrive? *Oh yes, you're almost there,* ... (thinks), *but it is always smaller and smaller, ... I think that it must arrive.* And if the distance is the one between Switzerland and Sweden? *Oh no, then I don't think so, ... It always remains a short way, .. it is shorter and shorter, afterwards there is a half* (check with the fingers at the edge of the table), *one must make it, but you almost don't advance anymore, it is too tight. ... Uf! It is hard.* So do we arrive? *I don't know I think that for short distances we arrive, but for long distances I don't think so, ... I'm not so sure, but I don't think so.* And what about distances in between? ... (thinks), ... *Yes, we arrive I think, because afterwards there will be short distances.*

Augmenting the distance into a qualitatively much bigger one, seems to put the "last" steps of the iteration further apart, leaving room for further iterations, such that the arrival is not attainable anymore. Qualitatively, this view sees middle distances as belonging to a similar family as the short ones, that is, sharing the same features at the end of the process.

Finally, less than 25% of the 12- and 14-year-old children presented arguments in favor of the idea that one will only approach the destination, but never reach it, although only one subject was truly sure about it. In general, these arguments were characterized: by being loaded with doubts and hesitations; by respecting the conditions of setting and action; and sometimes by the emergence of the idea of infinity, explicitly:

> Joé +(14;4): ... (thinks) ..., at infinity we will never arrive, it goes beyond imagination, but if we see (show the table) we will arrive but in fact we don't arrive, ... I don't know, it surpasses the imagination. If we always make a half of the way that we have to do, it will be difficult to arrive, .. then it will be microscopic, infinitely small.

The only subject who was sure about the impossibility of reaching the destination, despite of the fact that the iteration continues endlessly, showed a pragmatic approach throughout the whole interview. He never engaged himself in extremely abstract and speculative reasoning, although he conceived the objects of the problem as theoretical ones in which the physical constraints were not relevant for the analysis:

> Stn -(14;1): *No, we will never arrive.* Even if we always advance? *Even, because we make only the half We will get close, we will be very close to the edge* (table) *but we will never arrive there. ... there will always be a short way to be done.* Even if we get always closer? *Even.*

He follows the conditions of the problem avoiding any speculative consideration, focusing on the fact that there is always something left no matter how many steps one makes. The position obtained by any iteration is never the final destination and therefore the iteration continues endlessly.

Discussion

Among the variables studied, only age evinced clear-cut differences in the arguments (except for ages 12 and 14). No gender differences were found. The role of intellectual-academic performance seems to have only been significant for the 8-and 10-year-old groups. At these ages the arguments given by the high performers were more analytical and tended to consider the problem as such, whereas those given by low performers tended to consider the problem as a trivial one (reaching the destination after a couple of steps was evident, so that, no further considerations were necessary). The observation that no major intellectual-academic differences were found in the older groups could be partially explained by the fact that, on the one hand, the validity of the Raven test (Intelligence) decreases as age increases due to a ceiling effect, and on the other hand, in higher grades academic performance becomes a more complex phenomenon such that non-cognitive factors other than cognitive play an important role (e.g., early adolescence's changes such as self-esteem, motivation, etc).

In mathematical terms, the version of Zeno's paradox presented here deals with a series of the form $d/2 + d/4 + d/8 + \ldots$ (d=distance). Using this language to describe children's views (which, of course, is not theirs), we could say that we have observed approaches of the following forms:

a) Altering the conditions of action very early: $d/2 + d/4 + \ldots + d/2^k + d/2^k = d$. In general k has a value between 3 and 5. The last two steps are unnoticely considered as equal, leading to an exact arrival. This approach was observed among the low performers aged 8 and 10. The problem then becomes trivial.

b) Altering the conditions of the setting: $d/2 + d/4 + d/8 + \ldots + d/2^k = e$. Suddenly a new destination which defines a new distance $e = \sum_{k=1}^{p} d/2^k$ (where p is a large finite integer) is considered, allowing the arrival at the new destination (in general $e<d$). This approach was observed among some high performers aged 8 and 10, and seems to emerge as a solution to reconcile early intuitions about long lasting *convergent* iterations (subdivision).

c) Altering the conditions of action: $d/2 + d/4 + d/8 + \ldots + d/2^k < d$, where k is a large finite integer. The process gets stuck due to the smallness of the steps and cannot continue. This approach was observed among all four age groups, but k tended to be larger at age 12 and 14 than at age 10.

d) Respecting the conditions of setting and action: $d/2 + d/4 + d/8 + \ldots < d$. The process only approaches the destination and continues endlessly. It was observed in only one subject (14 years old). Nevertheless, nearly 25% of the 12- and 14-year-old subjects also referred to it, although with deep hesitations involving also approach c).

An approach of the form $d/2 + d/4 + d/8 + \ldots = d$, which is "mathematically correct" was never observed.

In order to conceive the paradoxical situation raised by Zeno's problem, one must be able to make the distinction of the elements involved and respect the conditions of the problem (i.e., setting: keeping the original distance to be covered; action: continuing the endless iteration). Approaches a), b), and c) refer to alterations of the conditions of the problem such that they don't give rise to paradoxical situations: Our paradox then, is not children's paradox.

The traditional views in philosophy of mathematics (platonism, formalism, constructivism) consider the existence of mathematical objects (or formulas), in various degrees and forms, as being independent of human beings. My epistemological position is totally different in the sense that mathematics is conceived as totally dependent on human beings: Conceived as emerging in the interaction of biological beings that evolve in their medium, and therefore depending on the very nature of the interwoven process of embodied concepts and social interactions. In other words: No human beings, hence no mathematics. Different biological beings, different mathematics. Mathematics, thus, is a language shaped historically and based on consensus. In Zeno's paradox, respecting the conditions of the problem takes place in this consensus. The consensus in turn, depends, among others, on the biological structure of the subjects (e.g., developing neurobiological structure) participating in the ongoing process of the definition of what the consensus is. In developmental cognitive psychology this issue is essential because the conceptual world that emerges from the cognitive activity of the young children is based on a consensus which is different from ours (because their neurobiology, their language, their embodied cognition is different). Thus, what we call rigor in nowadays mathematics[3] gives rise to different meanings at these ages, and so does the respect of the conditions of Zeno's paradox.

From this point of view, the unnoticed alterations of the conditions of Zeno's paradox by the children (setting and action) takes place in a different domain of consensus which we happens to not share. This becomes evident when the youngest subjects, as determined in their developing neurobiological structure, enact a consensual world clearly different from ours when the meaning of the *convergent* type of iteration is concerned. In fact, before the age of 12 there is a profound and striking ontological difference between infinity in the large (*divergent* type) and infinity in the small (*convergent* type). As a 10–year–old clearly stated,

> Yak +(10;1): If that is the "bigger" infinity, as you say, when you say "smaller" infinity, is it the same thing but in the other sense? *Yes, and there is only one difference. At a certain moment it becomes so small that we can not even know where it is.* So what is the difference then, between the "bigger" infinity and the "smaller" infinity? *At a certain moment, when we are in the smaller infinity, it stops, whereas in the bigger infinity it could continue until, ... infinity.*

there are endless infinities and stopped up infinities, both being infinities! Notice that the latter contradicts our very notion of "In–finity". At 8 years–old, although the idea

[3] Rigor is consensual. History of Calculus shows that rigor had different meaning for the Greeks (Archimedes), the mathematicians of the 17th century (Newton) and 19th century (Weierstrass).

of endless (related to *divergent* type and infinity in the large) is already present in their consensual world, the distinction "infinity in the small" (related to *convergent* type iterations) simply does not exist. It seems that between the ages of 10 and 12, a certain intuition of the iterations of *convergent* type and their entailments emerges, remaining very labile afterwards such that it is very much influenced by the figural and conceptual context. Thus, 10-year-old high performers, and 12- and 14-year-old students gave different answers with different arguments to isomorphic situations in which the context had been changed (e.g., distance to be covered). Contrary to what Piaget has said (1948) and according to our observations, subdivision is not mastered at the age of formal operations.

Finally, I would like to mention that these observations are based in a specific adaptation of Zeno's paradox and the coordination of heterogeneous components (different type and nature). It would be interesting to study similar situations controlling certain contextual factors that might play a role in the cognitive activity due to the labile intuitions of the subjects regarding infinity in the small: (a) Considering qualitatively different distances d (e.g., between the sides of the table, between two places in a building, in a city, in a country, etc); (b) Considering different values for p and q in the series $\sum_{k=1}^{\infty} [p(q-p)^{k-1}/q^k]d$, such that the iteration is not only done by halving (e.g., also 1/10 or 3/4); (c) Presenting the problem in a regressive manner as well (like Zeno's paradox of movement), and not only as a progressive one as it was studied here; (d) Focusing the question not only on the series $\sum_{k=1}^{\infty} d/2^k = d$ ("Do we arrive?") but also on $\lim_{n\to\infty}(d/2^k)=0$ (the elements of the sequence: "What happens with the steps?"). The study of the role played by these factors might reveal more about how our minds build the idea of subdivision and infinity in the small.

References:

Langford, P.E. (1974). Development of concepts of infinity and limit in mathematics. *Archives de Psychologie*, 42 (167-168), 311-322.

Núñez, R. (1990). Infinity in Mathematics as a scientific subject for cognitive psychology. *Proceedings of the 14th International Conference Psychology of Mathematics Education, (Oaxtepec, Mexico)*, Vol. I, pp. 77-84.

Núñez, R. (1991). A 3-dimension conceptual space of transformations for the study of the intuition of infinity in plane geometry. *Proceedings of the 15th International Conference Psychology of Mathematics Education, (Assisi, Italy)*, Vol. III, pp. 109-116.

Núñez, R. (1993a). *En deçà du transfini: Aspects psychocognitifs sous-jacents au concept d'infini en mathémathiques.* Fribourg: University Press.

Núñez, R. (1993b). Big and small infinities: Psycho-cognitive aspects. *Proceedings of the 17th International Conference Psychology of Mathematics Education, (Tsukuba, Japan)*, Vol. II, pp. 121-128.

Piaget, J. & Inhelder, B. (1948). *La représentation de l'espace chez l'enfant.* Paris: P.U.F.

Taback, S. (1975). The child's concept of limit. In Rosskopf, M. F. (Ed.), *Six Piagetian studies in mathematics education.* Teachers College Press.

NATURALLY GENERATED ELEMENTS AND GIVING THEM SENSES: A USAGE OF DIAGRAMS IN PROBLEM SOLVING

Kazuhiko Nunokawa
Joetsu University of Education, Japan

In this paper, we analyze the usefulness of diagrams in mathematical problem solving from the standpoint that diagrams can be changing gradually during solving processes. The aim of this paper is to present one function of diagrams, i.e. generating new elements by integrating information at hand. After considering examples from the previous research, students' solutions will be analyzed in order to find an empirical evidence for this function. The analysis will also suggest that, even if diagrams can show new elements to a solver, the decision about what sense should be given to those elements is left to the solver.

1. Introduction

It is widely believed that drawing diagrams is useful for solving mathematical problems(e.g. Charles et al., 1985; Hembree, 1992). Drawing diagrams appropriate for the solutions is, however, not always easy(Bodner & Goldin,1991; Lopez-Real & Veloo,1993). This suggests that it may not be trivial that diagrams can help the solver who are really stuck in solving challenging problems.

Nunokawa(in press) presents one viewpoint concerning this issue. Taking that viewpoint, a solver gives his/her own structures to the problem situation, and those structures can change during solving processes. We call these structures *solver's structures of the problem situation* (Nunokawa,1993). If we consider diagrams to be outer representations of solvers' structures of the problem situation, we can expect that diagrams can gradually change according to the changes of solvers' structures. In this case, diagrams appearing halfway are not necessarily the exact diagrams needed for the solutions. Such view can be compatible with Davis & Maher's (1993) attention paid to gradual changes in the ways children orgnized or represented the given problem situation.

The purpose of this paper is to show that diagrams can tell new information to the solvers in this changing process. Diagrams can tell some aspects of that information, but other aspects may be left to the solvers. This point will be also presented.

2. Naturally Generated Elements

Larkin & Simon(1987) mention the following point concerning the usefulness of diagrams; Diagrams automatically support a large number of perceptual inferences, which are extremely easy for humans(p.98). They call the data including perceptually obtained information " perceptually enhanced data."

Here we would check this perceptually enhanced data from the point of view mentioned above, from which diagrams can gradually change during solving processes. That is, we would ask how the perceptually enhanced data can emerge. Then we find that, when we put pieces of the information at hand into one diagram one by one, new elements appear naturally and they constitute the perceptually enhanced data. For example, suppose that two lines are intersecting. According to Larkin & Simon(1987), the information that the point of intersection lies on each of the two lines belongs to the perceptually enhanced data. As soon as we draw the second line in the diagram including the first line, we can find that the point of intersection appears naturally in that diagram. In other words, the diagram integrates the elements (i.e. the two lines) and generates new elements (i.e. the point of intersection which lies on each of the two lines).

Diagrams are also used in solving arithmetic or algebraic problems. Take Simon & Stimpson's (1988) example like the following;

> Barbara is 8 years old and Ms. Brown is 38 years old. Their birthdays are on the same day. When will Ms. Brown be three times as old as Barbara? (p.137)

Suppose we have a number line for representing ages and (i) put a mark for Barbara's age on it; (ii) put a mark for Ms.Brown's age; (iii) put a mark for Barbara's final age; (iv) put a mark for Ms.Brown's final age. These actions may correspond to first some steps of drawing a diagram for solving this problem.

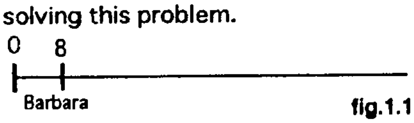


fig.1.1

fig.1.2

fig.1.3

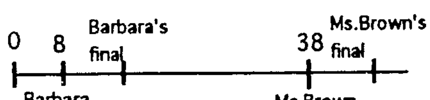


fig.1.4

We should note that, in this example, new elements naturally appear as each element gets represented in one diagram. At the moment when Ms.Brown's age was represented (fig.1.2), a new element, i.e. the difference between Barbara's age and Ms.Brown's age, appeared in that diagram. When Barbara's final age was marked (fig.1.3), a new element, i.e. the difference between Barbara's present age and final one, appeared. When Ms. Brown's final age was marked(fig.1.4), a new element, i.e. the difference between Barbara's final age and Ms.Brown's final age, appeared in addition to the difference between Ms.Brown's present age and final one. In short, when some elements of the problem get represented in one diagram one by one, new elements naturally appear through the diagram's integrating function, as the process of drawing this diagram proceeds.

From the above analysis of two examples, we can assume that diagrams may have a function to integrate some elements in pı oblmes and generate new elements naturally. This implies the possibility that diagrams can tell a solver new information which he/she has not yet known, and can really help the solver. We will check this possibility in the next section.

Diagrams' Function of Generating New Elements

In this section, we would analyze students' solutions to one mathematical problem, in order to find an empirical evidence that diagrams' function mentioned in the previous section can affect solvers' performance.

We take the following "telephone-line problem" ;

> We connect a house to another house by a direct telephone line. We put just one telephone line between each pair of houses. How many phone lines would be there when there are 20 houses?
> (Miwa, 1991, p.82)

It is helpful for finding the total number of lines to attend to a group of lines which are drawn from one house. This way of thinking, however, requires a solver to treat a subtle idea. For example, a line between the first house and the second house belongs to a group of lines drawn from the first house *and* to a group of lines drawn from the second house. This fact introduces the redundancies of lines when a solver tries to find out the total number of the lines by using the number of lines drawn from one house, e.g. by multiplying this number by the number of houses. Thus, the sovler must treat these redundacies in some ways.

These redundancies can naturally appear when more than two groups of lines from each of several starting points are drawn in one diagram(Here, a starting point means a house from which a certain group of lines are drawn). Fig.2 illustrates the case of 5 houses. Imposing a group of lines from one starting point(fig.2.1) on another group(fig.2.2) generates an overlapping of lines, which can suggest a redundancy of lines(fig.2.3). When only one starting point is taken, this overlapping or redundancy cannot appear in the diagram.

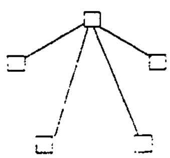


fig.2.1

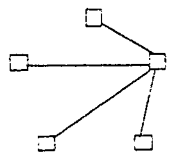


fig.2.2

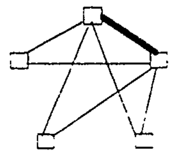


fig.2.3

Therefore, we can assume that the children who took more than two starting points in their diagrams can recognize the redundancies more often than the children who took only one starting point. Analyzing the children's solutions, we will test this hypothesis.

The data analyzed here are the solutions made by the students during the lessons reported in Miwa(1990) and the solutions gathered by two elementary school teachers in Tokyo. The total number of childrens is 372, including 37 fifth-graders, 227 sixth-graders, 41 seventh-graders, and 67 eighth-graders.

Each of the solutions was checked whether it can belong to one of the following four categories;

(a) Solutions having a diagram with only one starting point and showing the idea about the redundancies;
(b) Solutions having a diagram with only one starting point and not showing the idea about the redundancies;
(c) Solutions having a diagram with more than two starting points and showing the idea about the redundancies;
(d) Solutions having a diagram with more than two starting points and not showing the idea about the redundancies.

The criteria concerning the starting points are like the following;

(1) We decide that a solution has a diagram with more than two starting points if those starting points are taken in one diagram. If those starting points are taken in different diagrams, an overlapping cannot occur. Thus, such solutions were excluded taking account of the aim of our analysis.
(2) There are some students who drew a tree diagram. In a tree diagram, a starting point is placed at a very special position, and it might be hard to consider other houses as starting points. Thus, solutions based on a tree diagram were excluded.

A solution was determined as one showing the idea about the redundancies when it satisfied one of the following criteria;

(1) Number expressions include the operations which seem to handle the redundanceis (e.g. dividing by 2, subtracting 20 etc.).
(2) Numbers of lines from one starting point vary depending on which starting point is taken. For example, when drawing lines from one starting point, some students did not draw the lines between this starting point and the previous starting points. In this case, the numbers of lines are decreasing depending on the order of the starting points.
(3) Explanations included in a solution mention the existence of the redundancies.

According to these criteria, the solutions were categorized. Of the 372 solutions, the solutions by 141 students could be identified as one of the four categories mentioned above. Table 1 shows the result of this categorization.

	only one starting point	more than two starting points	total
showing the idea of redundancies	38	56	94
not showing the idea	29	18	47
total	67	74	141

Table 1 The result of the categorization

The computed chi-square value for this table is 5.688 (df=1), which indicates the significant differences ($p< .02$). Table 2 shows the result of the residual analysis for Table 1.

	only one starting point	more than two starting points
showing the idea of redundancies	-2.387*	2.387*
not showing the idea of redundancies	2.387*	-2.387*

Table 2 The result of the residual analysis (* $p< .05$)

These results show that the students who took more than two starting points in their diagrams could show the idea about the redundancies of the lines more often than the students who took only one starting point. This fact suggests that diagrams' function of generating new elements can affect the students' problem solving processes.

4. Solvers' Function of Giving Senses to Elements

Even when solvers recognize the redundancies of the lines, the ways solvers treat those redundancies seem not to be determined fully. In fact, there were several kinds of ways of treatment implemented by the students who took more than two starting points in their diagrams and showed the idea about the redundancies.

Treatment a. Decreasing the numbers of lines from a starting point one by one depending on the order of the starting points(46.6%);

Treatment b. Dividing a tentative total number, which is obtained by multiplying the number of lines from one house by the number of houses, by 2 (26%);

Treatment c. Subtracting 20 from a tentative total number (13.8%);

Treatment d. Finding out the total number by the number expression $18\times19+18$ (6.9%);

Treatment e. Subtracting 19 from a tentative total number (1.7%);

Treatment f. Finding out the total number by the number expression $19\times19+18$ (1.7%);

Treatment g. Subtracting 100 from a tentative total number (1.7%).

In one of the lessons reported in Miwa(1990), most students agreed the merit of the idea like the following; after multiplying the number of lines from one starting point by the number of houses, deviding its result by 2 in order to treat the redundancies. One student was opposed to this, and insisted the idea like the following; decreasing the numbers of lines one by one depending on the order of the starting points and finding out the total number by calculating 19+18+...+2+1. She said " I can understand the idea of 19+18+.... easily. I cannot follow the idea of dividing it by 2 in her solution." However, when she tackled the problem by herself at the beginning of this lesson, first she used the number expression $19\times20\div2$, and wrote under this $\div2$ "dividing because of the redundancies of the lines." Furthermore, in that lesson there was a student who said " I divided it by 2 because one line gets redundant *at each house.*" (Miwa, 1990, p.128, my italic) This student seems to relate the redundancies of the lines to each house, instead of relating them to each line.

These students' behaviors seem to suggest that there were students who were not sure about the following points and felt uneasy in treating the redundancies; To what element these redundancies should be related, what sense should be given to the redundancies, i.e. *how generated elements can be taken into his/her structures of the problem situation.* This unstableness in turn implies that scuh determination may be left to the solvers.

In the following, we will check whether the various ways of treatment mentioned above can be explained by senses which could be given to the redundancies.

Treatment a. If a sense "Redundancies occuring to at each line" is given to the redundancies, this treatment can be derived. Each line belongs to two distinct groups of lines. In order to count each line only once, lines belonging to the previous groups must be excluded. This leads to decreasing the numbers of lines depending on the order of the starting points.

Treatment b. This treatment can be also derived if a sense "Redundancies occuring at each line" is given. Each line will be counted just twice, so a tentative total number must be divided by 2.

Treatment c. This treatment can be derived if a sense "Redundancies occuring at each house" is given to the redundancies. Since there are 20 houses, then 20 must be subtracted from a tentative total number.

Treatment d. This treatment can be derived if a sense "Redundancies occuring at each of the 19 houses except the first one" is given. A student taking this treatment wrote "But there are 19 houses having redundancies." If we remember that the redundancy appears for the first time when the second starting point is taken, this student's determination is somewhat reasonable.

Treatment e. The student taking this treatment wrote the expression $19\times19-19$. We should note that in this expression the number of starting points is considered 19. There was a student writing " All lines can be drawn before I take the 20th house as starting point." Taking account of this, we can assume that the students taking this treatment also attended to the 19 starting points except the last one. This treatment can be derived if a sense "Redundancies occuring at each of the houses except the last one" is given.

Treatment f. This treatment can be derived if a sense "Redundancy occuring at the first house" is given. Subtracting 1 from the number of lines drawn from the first house leads to 18.

Treatment g. There was a student writing "10 lines become redundant at each house." He thought that 20 lines could be drawn from each house, so 10 lines seems a half of them. If we take account of this and remenber that there was a student who thought that all lines could be drawn before the last house would be taken as a starting point, we can assume that the student taking this treatment thought that a half of the lines, i.e. 10 lines, would be redundant at each of the last half of houses, i.e. 10 houses. In other words, a sense "Redundancies occuring to the half of the lines drawn from the half of the houses" is given to the redundanceis.

From the above discussion, each of the ways of treating the redundacies can be explained by senses which could be given to such redundancies. The existence of the various ways of treating the redundancies implies solvers' function of giving them senses.

5. Concluding Remarks

If we analyze diagrams from the standpoint that diagrams can change gradually during solving processes, we can expect diagrams to have a function of generating new elements by intergrating the elements at hand. Indeed, the redundancies can appear as the groups of lines from each of more than two starting points are drawn in one diagram one by one. However, what senses should be given and how new elements should be taken into solver's structures cannot be determined uniquely. This determination may be left to solvers.

The discussion in this paper suggests the possibility that diagrams can help a solver by showing the solver new information which he/she has not yet known. It also direct our attention not to the exact diagrams useful for getting the answers, but to the process of diagrams' changes and the interactions between solvers and diagrams.

Acknowledgment

I would like to thank two teachers, Mr. Masaru Kumakura and Ms. Yuko Mizusawa for gathering a part of data used in this paper.

References

Bonder,B.L. & Goldin,G.A. (1991). Drawing a diagram: Observing a partially-developed heuristic process in college students. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference for the Psychology of Mathematics Education*(vol. I, pp.160-167). Assisi,Italy.

Charles,R. et al. (1985). *Problem-solving experiences in mathematics.* Menlo Park,CA: Addison-Wesley.

Davis,R. & Maher,C.A. (1993). Children's development of methods of proof. In I.Hirabayashi et al.(Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education*(vol. III, pp.105-112). Tsukuba, Japan.

Gick,M.L. (1989). Two functions of diagrams in problem solving by analogy. In H. Mandl & J.R. Levin (Eds.), *Knowledge acquisition from text and pictures*(pp.215-231). Amsterdam: North-Holland.

Hembree,R. (1992). Experiments and relational studies in problem solving: A meta-analysis. *Journal for Research in Mathematics Education, 23,* 242-273.

Larkin,J. & Simon,H. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11,* 188-193.

Lopez-Real,F. & Veloo,P.K. (1993). Children's use of diagrams as a problem-solving strategy. In I.Hirabayashi et al. (Eds.), Proceedings of the 17th International Conference for the Psychology of Mathematics Education(vol. II, pp.169-176). Tsukuba, Japan.

Miwa,T. (1990). *Comparative study of mathematical problem solving lessons in Japan and US.* Technical Report, Institute of Education, University of Tsukuba. (in Japanese)

Miwa,T. (1991). A comparative study on classroom practices of mathematical problem solving between Japan and the US. *Tsukuba Journal of Educational Study in Mathematics, 10,* 81-84.

Nunokawa,K. (1993). Prospective structures in mathematical problem solving. In I.Hirabayashi et al.(Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education* (vol. III, pp.49-56). Tsukuba, Japan.

Nunokawa,K. (in press). Improving diagrams gradually: One approach to the usage of diagrams in problem solving. *For the Learning of Mathematics.*

Simon,M. & Stimpson,V. (1988). Developing algebraic representation using diagrams. In A.F.Coxford & A.P.Shulte (Eds.), *The idea of algebra, K-12.* Reston,VA: National Council of Teachers of Mathematics.

SOCIOCULTURAL MEDIATENESS OF MATHEMATICAL ACTIVITY: ANALYSIS OF "VOICES" IN SEVENTH-GRADE MATHEMATICS CLASSROOM

MINORU OHTANI
Institute of Education, University of Tsukuba, JAPAN

The purpose of this study is to investigate how mathematical activity in a seventh grade classroom is socioculturally mediated. A theoretical framework which is grounded on Vygotskian-Bakhtinian perspectives is taken to the study of sociocultural mediateness in which the concept of "voice" plays a significant role. Transcripts of audio-video recordings, written works as well as field notes during participant observation are analyzed in the light of the framework. Results from the interpretation of the data reveal that the classroom mathematical activity is mediated by monologic voice that ventriloquizes institutional constraints for rationality and homogeneity of formal schooling.

INTRODUCTION

Resent research (Cobb,Wood,Yackel, & McNeal,1992; Lampert,1988; Nickson,1992) have a common and persuasive vision of mathematics classroom as socioculturally mediated milieu. Different classroom cultures mediate different beliefs, attitudes, and values with respect to classroom interaction, and with respect to mathematical activity. In everyday classroom practice, teacher and students coordinate the extent to which they participate in a particular mathematical activity, their role in accomplishing it, and the extent to which they take direct responsibility for accomplishing it.

This article investigates how mathematical activity in a seventh-grade classroom is socioculturally mediated. The investigation focuses on the ways that teacher and students coordinate their purposes and maintain a predictable sense of the ongoing classroom interaction. The following discussion consists of two parts. First part involves an indication of the theoretical framework and the justification for the methodology used. Theoretical framework which is grounded on the Vygotskian-Bakhtinian perspectives is taken to the study of the sociocultural mediateness in which the concept of "voice" plays a significant role. The second part involves description and interpretation of an episode that occurred in the classroom during participant observation. The analysis pays particular attention to the voices depicting teacher's and students' motives before, during and after the lesson.

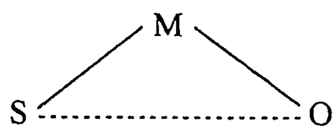
THEORETICAL FRAMEWORK

VYGOTSKIAN PERSPECTIVES

The investigation has been influenced by the writings of Soviet psychologists associated with the cultural-historical or sociocultural school of Vygotskii (Luria, 1974; Vygotskii, 1982a, 1982b, 1983) and neo-Vygotskian works (Cole, 1986, 1992; van der Veer & Valsiner, 1991; Wertsch, 1985, 1991). Central to their formulations is the notion that human beings live in a sociocultural environment transformed by the historically accumulated systems of artifacts (both conceptual and material) of prior generations. The basic goal of a socio-historical approach to mind is to create an account of human mental process that recognize the essential relationship between these processes and their cultural, historical, and institutional settings. Epistemological assumptions inherent in sociocultural school involve "mediation","internalization", and "appropriation."

MEDIATION

A sociocultural approach to mind begins with the assumption that human action typically employs "mediational means" such as tools and language, and that these mediational means shape the action in essential ways (Vygotskii, 1982b). "Individual-acting-with-mediational-means" (Wertsch,1991:12) in particular sociocultural settings is the unit of analysis of the sociocultural approach. Mediateness is represented as a triangle (Cole,1992:11; Vygotskii, 1982a: 104), in which the vertex is a mediating artifact 'M' and remaining points are subject 'S' and object 'O'(see Fig. 1).



(Fig. 1)

In the representation, subject and object are seen indirectly connected by the mediator which is constructed by its participation in the patterned forms of activity embodied in historical, cultural, and institutional settings. Mediators enter into the organization of behavior and act as mental tools and constraints.

INTERNALIZATION

Vygotskii (1983:145) claimed that social dimension of mental functioning is primary in time and in fact and individual dimension is derivative and secondary. For Vygotskii, humans psychological nature represents the aggregate of internalized social relations that have become functions for the individual.

APPROPRIATION

Even though the generic precursor on the intramental functioning can be found in intermental plane, people become to use socioculturally mediated means without recognizing the precursor and history of its ownership. People appropriate socioculturally mediated means and use them as wide reaching scheme of interpretation in order to create and sustain factual character of their environment.

According to Wertsch (1985, 1991), Vygotskii did not examine about the ways in which mediated-mental functioning is shaped by sociocultural contexts. Bakhtin (1981,1986, and disputed work by Volsinov, 1930), however, provide some account of the organization of mediational means in a particular sociocultural setting.

BAKHTINIAN PERSPECTIVE

Bakhtin, contemporary with Vygotskii, sheared basic idea that human communicative practices give rise to mental functioning in the individual. Bakhtin's basic theoretical constructs involve "utterance", "voice", and "dialogism."

UTTERANCE AND VOICE(S)

Unlike many linguists who concern themselves primarily with linguistic form and meaning abstracted from the actual conditions of use, Bakhtin focused on the utterance, "the real unit of communication" (Bakhtin,1986:71).

Speech can exist in reality only in the form of concrete utterance of individual speaking subject. The speaking subject is called "voice" (Bakhtin,1981). Voice refers to more than vocal-auditory signal. It involves much more general phenomenon of "the speaking personality"(ibid:434). An utterance can exist only by being produced by a voice, speaking personality with certain point of view. Bakhtin stressed the idea that utterance is an activity that enacts different values, perspectives, conceptual horizons, intentions, and world views. Thus, voice involves someone who belongs to particular cultural, social, and institutional categories.

Any utterance entails the idea of addressivity. Thus, utterances are inherently associated with many voices. An utterance reflects not only the voice producing it but also the voices to which it is addressed. In the formulation of an utterance a voice address or respond in some way to previous utterances and anticipates the responses of other voices. "[Speaker] expects response, agreement, sympathy, objection, and so forth" (Bakhtin, 1986: 69). Furthermore, Bakhtin did not limit the notion of addressee to only those speakers in the immediate speech situation. Instead, the voice or voices to which an utterance is addressed may be temporally, spatially, and socially distant.

DIALOGISM

Dialogism, the most significant principle that Bakhtin celebrated, is founded on the idea that nothing is ever completed, and there are no ultimate explanations that everyone will accept as exhausting all the possibilities. Bakhtin attempted liberate monologic, oppressive belief system which said that absolute truth is possible.

VYGOTSKII AND BAKHTIN COMPARED

MEDIATIONAL MEANS: SPEECH GENRES

In Bakhtin's view, the production of any utterance entails the innovation of a speech genre, "relatively stable types of utterances" (Bakhtin,1986:60) which is a conventional utterance type characterized primarily in terms of the typical situations of speech communication. Speech genre is a mediational means which is used and privileged by the speaker for the realization of specific social ends.

INTERNALIZATION: INNER DIALOGUE

As Clark and Holquist put it: "Vygotsky's model [of speech acquisition] parallels Bakhtin's model, particularly in its insistence that the self comes from the other" (Clark & Holquist, 1984: 229). The process of mastering patterns of privileging rely heavily on the guidance of others in intermental plane. What begins as external interference on the intermental plane is gradually transformed to a kind of inner-dialogue in an intramental plane.

APPROPRIATION: VENTRILOQUATION

Bakhtin termed the process of producing unique utterances as "ventriloquation" (Bakhtin,1981) whereby one voice speaks through another voice in a speech genre. Even though people have rich repertoire of speech genres, they have no idea that what they are doing. "We speak in divers genres without suspecting that they exists" (Bakhtin,1986:78). Through a process of ventriloquation, a specific type of socioculturally situated voice can be heard alongside the individual voice.

For Vygotskii, socialization or mental development can be profitably viewed in terms of appropriation of mediational means. For Bakhtin, it can be viewed in terms of ventriloquizing through a speech genre appropriate for that particular sociocultural settings. They have come to privilege particular speech genres and registers of these languages and coming to understand the specific contexts in which variants should and should not be used.

AN ILLUSTRATION

To illustrate the Bakhtinian specific constructs of speech genre and ventriloquation, I shall draw on Lampert (1988) and Lakatos (1976).

Lampert (1988) illustrates peculiar form of discourse in the traditional classroom context of mathematical activity. She puts it (ibid; 13):

> *In the classroom, words like "know," "think," "revise," "explain," and "answer" come to have meaning by being associated with particular kinds of activities. Who is responsible for doing the activities associated with these words gets determined in interaction between the teacher and students. ...[and] to be asked by the teacher to revise means "you've done it wrong" and "you've got to do it over" until more of the answers are correct. When the teacher asks a student to think it is often an admonition to be quiet; thinking is not considered to be a process that students--or teacher and students --engage in together.*

The quotation suggests that there is specific speech genre and its registers deemed appropriate for mathematics class.

Lakatos (1976) portrays historical debates within mathematics about a proofs of Euler-Descartes conjecture by constructing a classroom dialogue among a teacher and students. In the dialogue, Lakatos often let the teacher and students, both are fictional characters, ventriloquizes through the voices of mathematicians' own voices, thereby demonstrated how new knowledge develops in the discipline. In the midst of the argument an utterance that belong to a single speaker (either teacher or student), but that actually contains mixed within it many voices, many axiological belief system, conceptual horizon among mathematicians through the last several centuries.

DESCRIPTION OF AN EPISODE

DATA COLLECTION

The episode I shall examine here comes from a seventh-grade classroom (with thirty two students) in a junior high school located at Tsukuba city, Japan.

A yearlong participant observation started April 1993, when the author entered the class as assistant who is preparing for teaching profession. In each lesson, I delivered and gathered worksheet materials, provided audio-visual equipments and had a discussion with students on their requests during individual work sessions.

Every lesson (three times a week) was audio-video taped for later analysis. Utterances which the classroom teacher and students addressed to me are recorded by small audio-recorder put in my inside pocket. These utterances involve their motives, beliefs, situations of definitions. and opinions before, during, and after the lessons. These utterances give the participant observer an opportunity to have access the underrepresented voices in ordinary classroom practice which consists of specific interactional sequences: teacher initiation - student reply - teacher evaluation.

Transcripts of audio-video recordings, written works as well as field notes during participant observation are analyzed in the light of the framework to generate a description of how classroom mathematical activity is socioculturally mediated.

RESULTS

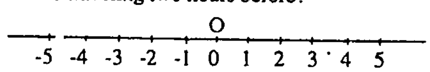
I shall outline an episode that occurred on May 20. The episode is representative of a much larger corps of data. On that day, the teacher planed to introduce multiplication of positive and negative integers. In the teaching staff room, the teacher gave me a worksheet [1] and said that [2]: (In the following excerpt of protocol, the teacher and participant observer are designated by T and PO respectively, and students are designated by their initials.)

[1]

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS
PROBLEM

Mr. T travels at a speed of 4 km an hour from the west to the east. He has just passed point O.

(1) Where he will be traveling two hours later?
(2) Where ha was traveling two hours before?



[2]

T : I made a plan to teach multiplication for today.
PO : I expected.
T : In the introduction, however, student would notice that the worksheet has much to do with multiplication by the title on it. I don't mind even if they notice.
PO : Some days before, when presented a worksheet titled "subtraction of positive and negative numbers", I found that most students formulated the problem using subtraction.
T : Well, ... a worksheet without title would allow them to formulate the task in different ways. I think it would be indeed more interesting.

In the beginning of the lesson, the teacher wrote on the blackboard today's task: "Think about how to do multiplication with positive and negative numbers." Every students received the worksheet and were asked to engage individually. The students engaged the task for fifteen minutes. I observed that fifteen students formulate the problem with multiplication and five students formulate it using addition. The teacher nominated five students to write down their formulation on the blackboard. One of them, students HK formulated the problem (+4)+(+4)=+8 for the question (1), and (-4)+(-4)=-8 for (2) respectively. No sooner had HK returned to his seat, he hesitatingly call my attention to his formulation and said that [3]:

[3]

HK : Ah, Mr. Ohtani, that is my...
PO : Which one is yours?
HK : This one (pointing to his notebook). That white one (pointing to the blackboard).
PO : Which one? Middle one?
HK : Middle? ... No, next one, next to it. Look this (pointing to the title of the worksheet), ..., multiplication. Instead of doing multiplication, I did it by addition.
PO : Is it inappropriate to use addition?
HK : Inappropriate
PO : What?
HK : Yes, inappropriate.
PO : Then, do you get different solution?
HK : No. I think that the answers are the same.
PO : Then, why do you think it inappropriate? You say it is...
HK : Yes. ... 'Cause ...
PO : Can't you solve it by addition?
HK : I can. ... in this case (pointing to the title), however, I should do multiplication. ... Ah (throwing up his hands on his head)... Okay, I don't mind.

At the end of the lesson, the teacher reviewed todays activity and gave summative evaluation [4]:

[4]

T : Today, many of you solved the problem by using multiplication. As I wrote here on this board (pointing to the task), multiplication of positive and negative numbers, ..., what do you mean by multiplication?
S : "joho"[Japanese children use the term "joho" for multiplication in elementary school.]
T : Good to you. The objective of the lesson was to learn "joho". Therefore, I would appreciate if you would formulate the problem using multiplication.

After the lesson, on the way back to teaching staff room, the teacher seemed to be unsatisfactory, saying that [5]:

[5]

T : There were only fourteen [students who used multiplication method]. Together with those who employed another formulations, the total number is less than twenty.
PO : There were indeed variety of solutions among students.
T : Nevertheless, they tend to be mechanically. If students are asked to formalize rules [of multiplication], they would formulate something like "plus and plus make plus" whether they really understood or not.
PO : You say, they tend to do so...

T : Yes. It is the very difficult problem to cope with.

DISCUSSION

Many voices depicting socioculturally mediated means (speech genre) seem to be detected in the present episode. Analysis of the episode shows that classroom mathematical activity is characterized by a monologic suppression by authoritarian voice, rather by a dialogicality of voices.

Teacher's utterances [4] and [5] suggests that he send a strong message that the homogeneity among students activity should be attained, even if another formulation could be used to describe an objects or event more appropriately and usefully in their activity settings. Teacher appeared to be ventriloquizing through speech genres characteristic of the formal schooling. One of its characteristics is a tendency toward certain kind of rationality. This derives in part from the institutional constraints within which formal instruction take place. The formal instruction has to perform a degree of rationalization of what it transmits. A tendency of homogenization created by such rationality reflects the assumption that one must not introduce alternative formulation unless specifically invited to do so. Thus, in place of diversity or heterogeneity, it puts explicit, standardized formulations which are expressly inculcated and conserved so as to be reproduced in virtually identical form by all the students in the succeeding lessons. In order to attain such homogeneity, teacher often use authoritarian directives.

HK's utterances [3] suggest that he was ventriloquizing through voices of the teacher: speech genre of formal schooling. The title of the worksheet becomes directive of imperative which he is expected to follow. The title functioned to regulate his mathematical activity in ways that are appropriate for the classroom setting. By responding to the directive, the student engages in a process sanctioned and regulated by the teacher. The voice of HK can be seen as internalized voice of authorized teacher.

Through months of classroom practice, students would gain adequate facility and flexibility in the patterns of privileging of particular speech genre deemed appropriate in mathematics lessons. In the episode, fifteen students wrote down multiplication such as (+4) + (+4) =+8 for the question (1), (- 4)+ (-4)= -8 for the question (2) respectively. It seems to me that these students appropriate the voices of the previous lessons on addition and subtraction of integers. These voices reflected in the assumption that there is only one true formulation of mathematical task.

The mastery of this privileging pattern of the monologic speech genre of formal schooling was clearly revealed on June 22. On that day the teacher introduce special topic of finding pattern of numerical sequences. In the end of the lesson, the teacher ask students to write their impression of today's lesson?". A student KC wrote that [6]:

[6]

> Unlike ordinal lessons, today's lesson was interesting. Because, I could formulate my answer by myself.

Voice that is temporally and spatially distant become apparent in KC's reply. The voice involves not only positive attitude toward the lesson, but also negative opinion ("unlike ordinal lessons") associated with the way the teacher organize preceding lessons on positive and negative numbers.

CONCLUDING REMARKS

The view I have been outlining in this article suggests that the mediational means of individual mental functioning can be better understood if one goes beyond the individual involved. By focusing on speech genres as mediational means, the author is constantly remind that mediated action is inextricably linked to historical, cultural, and institutional settings.

The investigation itself raised several additional questions. It remains to explore the ways in which such a monologic organization of activity on the intermental plane is mastered, thereby shaping the intramental plane of functioning. Vygotskii-Bakhtin's perspective suggests that what comes to be incorporated into an utterance are voices that were formerly represented explicitly in intermental functioning. Therefore, microgenetic analysis of transition of mental functioning from intermental to intramental plane that would occur over the course of classroom interaction is needed. In addition to exploring the question, it is necessary to show the possibility to create and sustain mathematically appropriate culture in which dialogical organization of activity can occur.

REFERENCES

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays by M. M. Bakhtin.* M. Holquist (Ed.), University of Texas Press.

Bakhtin, M. M. (1986). *Speech genres and other late essays.* University of Texas Press.

Cole, M. (1986). The zone of proximal development: Where culture and cognition create each other. In J. V, Wertsch,(Ed.), *Culture, communication, and cognition: Vygotskian perspectives* (pp.146-161). Cambridge University Press.

Cole, M. (1992). Context, modularity, and the cultural construction of development. In T. Winger & J. Valsiner (Eds.), Children's development within cultural contexts. vol.2, (pp.5-31) Lawrence Erlbaum Associates.

Clark, K. and Holquist, M. (1984). *Mikhail Bakhtin.* Harvard university press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge University Press.

Lampert, M. (1988). The teacher's role in reinventing the meaning of mathematical knowing in the classroom. Research series No. 186. ERIC ED 305 237.

Luria, A. R. (1974). *Ob istoricheskom razvitii poznavatel'nykh processov.* Nauka.

Nickson, M. (1992). The Culture of the mathematics classroom: An unknown quality? In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* NCTM, Macmillan Publishing Company.

Ohtani, M. (1992). Development of activity-oriented approach in Vygotskian school: Social contexts in the zone of proximal development. *Bulletin of Institute of education,* University of Tsukuba, 17(2), 121-135.

Ohtani, M. (1993). Problem solver as a reality constructor: An ethnomethodological analysis of mathematical sense-making activity. In *Proceedings of seventeenth international conference of psychology of mathematics education,* Vol.II, 254-261.

Ohtani, M. (1993). MATHEMATICAL ACTIVITY in social interaction: A Vygotskian perspective. In T, Miwa (Ed.), *Advances in mathematics* education. Toyo-Kan.

Van der Veer, R and Valsiner, J. (1991).*Understanding Vygotsky: A quest for* synthesis. Basil Brackwell, Inc.

Volsinov, V. N. (1930). *Marksizm i filosofija jazyka: Osnovnoe problemy sociologicheskogo metoda v nauke o jazyke.* Priboj. 2nd ed.

Vygotskii, L. S. (1982a). *Sobranie sochinenii, tom pervyi: Voprosy teorii i istorii psikhologii.* Izdatel'stvo pedagogika.

Vygotskii, L. S. (1982b).*Sobranie sochinenii, tom vtoroi: Problemy obshchei pshikhologii.* Izdatel'stvo Pedagogika.

Vygotskii, L. S. (1983). *Sobranie sochinenii, tom tretii: Problemy razvitiya psikhiki.* Izdatel'stvo Pedagogika.

Wertsch, J. V. (1985).*Vygotsky and the social formation of mind.* Harvard University Press.

Wertsch, J. V. (1991). *Voices of the mind: A sociocultural approach to mediated action.* Harvard University Press.

RATIONAL NUMBERS: STRATEGIES AND MISCONCEPTIONS IN SIXTH GRADE STUDENTS

Isolina Oliveira, Instituto de Inovação Educacional, Lisboa
Glória Ramalho, Instituto Superior de Psicologia Aplicada, Lisboa

Abstract

In this study sixth grade children were introduced to some problematic situations in which the rational number concept was involved and presented in different contexts. Individual interviews were carried out in order to undestand children's ways of thinking about rational numbers. Results show the use of different types of strategies and point to specific kinds of misconceptions, which were both found to be different depending on the task that was presented and on its context of presentation. The comparison between "high" and "low" achievers' outcomes leads to the conclusion that much more success is obtained when a task is depicted in a concrete context than when it is introduced within an abstract context, and that this finding is more prominent in the case of "low" achievers.

Introduction

Vergnaud sustains the importance of studying children's conceptions in order to achieve two major objectives: on the one hand, to identify the difficulties and discontinuitites in students reasoning processes while approaching some mathematical concepts; and on the other hand, to determine in which way children's previous conceptions can be used to generate new conceptions.

According to him, a given problematic situation does not, in general, implicate all the properties of a specific concept, and its solution will most likely require a number of distinct constructs. Learning some concept is an enduring process which goes through several phases of interactions and discontinuities. Only amid a vast domain of interwined concepts would it make sense to look for affiliations and ruptures in knowledge (Vergnaud,1990).

The possible different sorts of interpretations of rational numbers and the consequent degree of complexity that they entail is most likely at the origin of the statement, frequently heard among teachers, that this topic, included in sixth grade math portuguese curriculum, is the most difficult of all for children It is, unfortunately, still common to find some teachers, usually the ones who emphasize the main importance of algorithmic procedures, who, contrary to Vergnaud's perspective, advocate that the best way to overpass such difficulties is to persistently exercise on the resolution of many problems of the same sort.

The interest in studying the rational numbers learning process in children can be seen from three different perspectives: 1) from a practical point of view, as the capacity to deal with this concept facilitates the comprehension, and makes it easier to deal with many practical situations; 2) from a psychological perspective, in the sense that such a vast domain propitiates children's mental organization and , thus, enhances their general cognitive development; 3) from a mathematics education point of view as the full comprehension of this concept underlies the further learning of algebraic operations integrated later on in the secondary school curriculum (Behr, Lesh, Post & Silver, 1983).

Children's understanding of the rational number construct has been at the origin of numerous investigations (e.g., Behr et al., 1983; Kieren, 1980). Our intention in this study was to apprehend how children, at this age and in a specific curricular context, think about rational numbers. More specifically, our option was to study the disparities between "high" and "low" achievers, with respect to the strategies employed and to the possible misconceptions involved in the solution of problematic tasks addressing different interpretations of rational numbers.

This presentation will address six of those tasks, whose choice was determined by the revealed degree of difficulty.

The Study

This investigation was organized in two distinct phases. During the first phase, 68 children, attending sixth grade in three different schools in Lisbon, were asked to answer a pencil and paper test whose items regarded the part-whole interpretation of rational numbers. The subjects for the second phase of the study were then selected according to both their performance on this test, and to the math teacher evaluation of these students' degree of achievement in the subject. The sample thus gathered consisted on 15 "high" achievers and 15 "low" achievers.

In the second phase of the study the subjects were individually interviewed and presented with six tasks bearing on the interpretations "quotient", "equivalence" and "addition", in either a concrete or an abstract context.
All the situations were orally presented and the students encouraged to verbalize their strategies during, and after, the solution of each task, to utilize different materials that were available, to draw any sort of diagrams and or register their thoughts in a separate sheet of paper.

The six chosen tasks were the following.
1 You have a pizza that you want to divide evenly among three boys. How much will each boy get to eat?

2. Do you like chocolate? Which one of these would you rather have: 2/3 or 8/12 of a chocolate?

3. In this task, two circular cards were shown, one of them divided into four equal parts, three of which were darkened, and the other one divided into three equal portions with just one marked in the same way.

This (and the first card was pointed out) represents a mushroom pizza and the shaded zone the part that John got to eat yesterday.

This (and the second card was, then, shown) represents a chicken pizza and the shaded zone the part that John got to eat today.

All together, how much did John eat?

4. If you have 1 divided by 3, how can you represent the result?

5. Place one of the symbols (>, <, =) on the dotted line so that a true statement will result:

$$\frac{2}{3} \ldots \frac{8}{12}$$

6. Calculate

$$\frac{3}{4} + \frac{1}{3}$$

The order of task presentation was not exactly the one just displayed since there were more situations whose results are not considered in this presentation.

Results

The strategies put into practice by students were organized by task, by context of the task (concrete, abstract) and by student performance group ("high" achievers = A, "low" achievers = B).

The three following tables display the distribution of students's answers with respect to their correctedness ("C" = Correct, "W" = wrong) and regarding the solution strategies put into practice while solving them, always discriminating "high" from "low" achievers. Table 1 illustrates the answers relative to "quotient" type of tasks, Table 2 refers "addition" and Table 3 "equivalence".

Table 1 - Distribution of strategies put into practice by group of performers, correctedness of the answers and context of task - "Quotient" type of tasks

Strategies	Concrete		Abstract	
	A	B	A	B
	C/W	C/W	C/W	C/W
. Procedure algorithmic (1 3)	-	-	3/0	2/2
. Symbolic representation (1/3)	8/0	3/0	5/0	1/0
. Symbolic representation and procedure algorithmic	-	-	4/1	6/3
. Symbolic and graphical representation	5/0	3/1	2/0	0/1
. Symbolic representation, graphical representation and procedure algorithmic (1.3)	2/0	2/1	-	-
Total	15/0	13/2	14/1	9/6

Table 2 - Distribution of strategies put into practice by group of performers, correctedness of the answers and context of task - "Addition" type of tasks

Strategies	Concrete		Abstract	
	A	B	A	B
	C/W	C/W	C/W	C/W
. Graphical representation and estimation	-	0/1	-	-
. Estimation.	-	2/0	-	-
. Procedure algorithmic (LCD)* and manipulated materials	2/0	6/0	-	-
. Manipulated materials and estimation.	5/0	1/0	-	-
. Procedure algorithmic (LCD).	7/1	3/2	12/0	7/2
. Rule invention/rule transposition.	-	-	0/1	0/5
. Rule invention/rule transposition and procedure algorithmic	-	-	2/0	0/1
Total	14/1	12/3	14/1	7/8

*Least-common-denominator

Table 3 - Distribution of strategies put into practice by group of performers, correctedness of the answers and context of task - "Equivalence" type of tasks

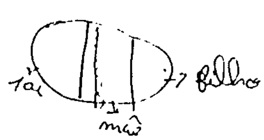
Strategies	Concrete		Abstract	
	A	B	A	B
	C/W	C/W	C/W	C/W
. Procedure algorithmic				
- LCD*	5/0	1/0	3/0	1/0
- decimal form	2/0	-	4/0	-
- equivalent form	-	1/0	2/0	-
. Rule invention/rule transposition	0/2	0/5	0/1	0/4
. Manipulated materials and rule invention/rule transposition	-	0/1	-	0/3
. Graphical representation and rule invention/rule transposition	2/0	1/4	1/2	3/3
. Procedure algorithmic (LCD) and rule invention/rule transposition	4/0	2/0	2/0	1/0
Total	13/2	5/10	12/3	5/10

* Least-common-denominator

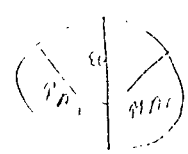
The classification of misconceptions was done in the context of the solution strategy.

Difficulties were found at the level of graphical representation, mainly when the student would not consider an even division, which may have to do with the absence of a full understanding of the part-whole concept, as can be seen in the figures that follow.

3/7

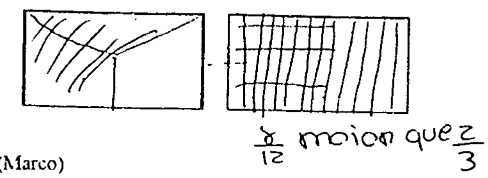


Task 1 (Luis)



Task 1 (Carla)

Task 2 (Susana)



Task 2 (Marco)

Another kind of difficulty has its origin in the incomprehension of the position value of decimal numbers, which happens whenever the child makes use of this sort of representation. Task number four was frequently solved, on the first trial, as "three point three" and sometimes "thirty three", with the student persisting with the answer even when the researcher called his attention.
Finally, another kind of misconception that consistently shows up proceeds from presuming as a rule that applies to rational numbers, one that is true only about integers, as well as some other rules that are just created as the student explain his process of task solving. The examples that follow refer to task number two:

"two thirds is less because the numbers are smaller"
"I prefer eight twelfths because twelve is larger than three and eight larger than two"
"I prefer two thirds because it has less divisions"
"I prefer two thirds because it divides in less parts".

In task number six the imediate answer is, sometimes, four sevenths: the student adds both numerators and both denominators.
We identified other sorts of misconceptions although they seem to us as less interesting from a psychological point of view.

Discussion

In this section we intend to simply elaborate the most salient aspects of the study that was just summarized.

When the answers to the tasks implicating "quotient", "addition" and "equivalence" are considered in this order we can find progressive difficulties, particularly evident in the "low" achievers, which suggests the possibility of a conceptual hierarchy in their understanding

One other aspect that should be emphasized is that the strategy selection done by students while solving tasks that are mathematically identical is not the same whether the context is either concrete or abstract, this being true for both "low" and "high" achievers. In particular, when tasks are presented in a concrete way a better performance is found in the "low" achievers which seems to indicate that a realistic approach in the teaching of fractions allows for a better understanding.
Nevertheless, it is interesting to point out that, for "low" achievers, the kind of context did not have any influence in the success of the tasks implicating "equivalence". In the analysis of their answers the strategy that stands out is the utilization of a rule which they know to be true for integers: apparently, 2/3 and 8/12 are not taken to be numbers, this finding being in agreement with the results of previous investigations (Harrison & Greer, 1993). As a matter of fact, this understanding of fractions as numbers seems to be a real "episthemological obstacle" as defined by Vergnaud (1989, p.38) in the sense that it contradicts the child's previous functional conceptions, it is of an enduring nature, and it facilitates the appearance of earlier notions whenever circumstances are favorable.

References

Behr, M., Lesh, R., Post, T. & Silver, E. (1983). Rational-number concepts. In R. Lesh & M. Laudau (Eds.), Acquisition of Mathematics Concepts and Processes (pp. 92-144). New York: Academic Press.

Behr, M. J., & Post, T. R. (1987). Teaching rational number and decimal concepts. In M. Behr & T. R. Post (Eds.), Teaching Mathematics in grades K-8, (pp. 190-231). Boston: Allyn & Bacon.

Harrison, J. & Greer, B. (1993). Children's understanding of Fractions in Hong Kong and Northern Ireland. Proceedings of the Seventeenth International Conference on the Psychology of Mathematics Education (vol.3, pp.146-153).

Kieren, T. E. (1980). The rational number construct-its elements and mechanisms. In T. E. Kieren (Ed.), Recent Research on number learning (pp. 125-149). Columbus, OH: ERIC/SMEAC.

Kieren, T. E., Nelson, D., Smith, G. (1985). Graphical Algorithms in Partitioning Tasks. The Journal of Mathematical Behavior 4, 25-36.

Vergnaud, G. (1989). Difficultés conceptuelles, erreurs didactiques et vrais obstacles épistémologiques dans l'apprentissage des mathématiques. In N. Bednarz et C. Garnier (Eds.), Construction des Savoirs (pp. 33-40). CIRADE. Agence d'Arc inc.

Vergnaud, G. (1990). La théorie des Champs Conceptuels. Recherches en Didactique des Mathématiques. Vol. 10 N° 23, 133-170.

FIFTH GRADERS' MULTI-DIGIT MULTIPLICATION AND DIVISION STRATEGIES AFTER FIVE YEARS' PROBLEM-CENTERED LEARNING

Hanlie Murray, Alwyn Olivier and Piet Human
Research Unit for Mathematics Education
University of Stellenbosch, South Africa

This paper describes the multiplication and division strategies used by fifth-graders in three schools which have been following a problem-centered approach to mathematics learning for five years. The quality of mathematical thinking and the computational skills exhibited by these students are very encouraging.

INTRODUCTION

The National Research Council document *Everybody Counts* states "More than most school subjects, mathematics offers special opportunities for children to learn the power of thought as distinct from the power of authority" (1989:4). Sadly, these opportunities are seldom realized. We believe that when young students are exposed to transmission-type teaching of mathematics, it encourages the idea that mathematics should be associated with the power of authority rather than with the power of thought, and that the "personal intellectual engagement that creates new understanding" (NRC, 1989:6) does not happen simply because students do not realize that it is expected of them.

A problem-centered learning approach to mathematics teaching (e.g. Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Olivier, Murray & Human, 1990) *expects* students to construct their own knowledge and also attempts to establish individual and social procedures to monitor and improve the nature and quality of those constructions.

Young students enter school with the "power of thought" attitude; to prevent the change to a "power of authority" attitude towards mathematics, teachers in our problem-centered classrooms do not demonstrate solution methods for problems, but expect students to construct their own strategies, and they depend on social procedures, developed during peer collaboration, for error identification and the development of more powerful strategies.

This is possible because young students enter school with a rich repertoire of problem-solving strategies based on sound conceptual knowledge, and they invent their own computational strategies independently of what they are taught. Our approach simply builds on and deliberately encourages these strategies (Murray & Olivier, 1989). This implies that the four standard vertical algorithms for the four basic operations are no longer taught explicitly.

The nature of written algorithms. The essential nature of any non-counting computational algorithm is that it is a set of rules for breaking down or transforming a calculation into a set of easier calculations the answers of which are already known. This process of changing the task to an *equivalent* but easier task involves three distinguishable sub-processes, illustrated here with reference to a procedure to calculate 17 × 28:

- Transformation of the numbers to more convenient numbers e.g.

$$28 = 30 - 2$$

The ability to transform numbers in this way depends on the student's *number concept development*.

- Transformation of the given computational task to a series of easier tasks, e.g.

  $$17 \times 28 = 17 \times 30 - 17 \times 2$$

  The ability to transform the task to an equivalent task depends on the student's awareness of certain *properties of operations* or theorems-in-action (Vergnaud, 1989) (here the distributive property of multiplication over subtraction).

- Calculation of the parts, e.g.

  $$\begin{aligned} 17 \times 28 &= 17 \times 30 - 17 \times 2 \\ &= 510 - 34 \\ &= 476 \end{aligned}$$

  These final calculations are performed at the "automatic" level, i.e. without much thinking, and requires knowledge of some basic number facts.

The process of calculation does not necessarily follow this sequence. The way you initially decompose the numbers depends on your *anticipation*, "looking ahead" to the transformation you intend in phase 2 and the basic facts you are going to use in the final step of the calculation.

Strategies such as these that children in problem-centered learning classrooms use in solving word problems generally indicate that they have a sound intuitive grasp of the properties of operations that underlie their strategies.

Objectives for computation. In our problem-centered learning approach the objectives for computation do not include speed or economy of presentation. Rather, we use computation as a vehicle to achieve other objectives, namely

- understanding of number, operations and properties of operations
- understanding of algorithmic thinking as a process, including executing and explaining given algorithms, comparing and evaluating the efficiency of different algorithms, designing and modifying algorithms for specific tools, using flexible computational procedures (mental computation and estimation)
- using computation as tool for solving problems
- maintaining a positive attitude to mathematics, positive beliefs about mathematics and about learning mathematics, and especially intellectual autonomy.

Our first PME report on division described the different strategies used by first to third grade students in our classrooms (Murray, Olivier & Human, 1991); the second report on division traced the development of division strategies in a third grade class during the course of a school year (Murray, Olivier & Human, 1992). In this report we describe the different strategies that fifth-graders in different schools use for multi-digit multiplication and division. The objectives of the present study were to trace the effect of two further years' practice on students' "power of thought" attitude, to gain insight into the quality and promise their computational strategies hold for further mathematics, and to document the frequencies of the different strategies.

STUDENTS' STRATEGIES

All the fifth-graders in three schools were required to solve the following two problems in written test situations (i.e. individually) at the end of their fifth grade:

> *Miss Schreuder makes tote-bags for students. She uses 27 cm of trimming for each tote-bag. She receives an order for 35 tote-bags. How many centimeters of trimming does she use for this order?*
>
> *There are 784 children at a holiday camp. They are grouped into teams of 16 children each. How many teams are there at this camp?*

The student populations of all three schools are fairly stable, and most of the students have been together for some years, making possible the decided differences in preferred strategies among the three schools. We list below the different strategies used by students to solve these problems and give examples of each strategy.

Strategies for the multiplication-type word problem.

1. Decimal decomposition of each number, leading to the addition of four subproducts.

Helen	20 × 30 = 600	
	20 × 5 = 100	
	7 × 30 = 210	
	7 × 5 = 35	
	600 + 100 + 210 + 35 = 945	

2. Only one of the numbers is decomposed additively into two or more parts. A number of subproducts are calculated and the results added.

Chris	27 × 30 = 810	
	27 × 5 = 135	810 + 135 = 945
Gay	35 × 10 = 350	
	35 × 10 = 350	
	35 × 4 = 140	
	35 × 3 = 105	
	945	

3. One number is kept whole for multiplication with the tens-part of the other number, but decomposed decimally for multiplication with the units-part of the other number.

Liza	20 × 35 = 700	
	7 × 30 = 210	
	7 × 5 = 35	35 + 210 + 700 = 945

4. Repeated doubling of one number, adding the appropriate multiple of the number at the end.

Jon	105 +	105 →	210 + 210 →	420 + 420 →	840 +	105 →	945
	3	3	6	12	24	3	27

5. Changing some of the numbers so that a completely different but more convenient task is executed, and then compensating for the change.

Andy   35 × 10 — 350 × 3 -- 1050
        35 × 3 = 105
        1050 − 105 = 945

6. The standard vertical algorithm for multiplication.

7. Indecipherable strategies.

We have identified only one type of logical error, namely a partial implementation of strategy no. 1, in analogy with a common addition strategy.

Naomi   20 × 30 = 600
         7 × 5 = 35   600 + 35 = 635

Table 1 shows the frequency of use of each strategy for each of the three schools.

Table 1: Multiplication-type problem frequency distribution (%)

Strategies	1	2	3	4	5	6	7	Logical error
School A (n=34)	29	53	9	0	1	2	2	4
School B (n=79)	43	33	8	0	0	1	1	14
School C (n=53)	47	34	6	2	0	11	0	0

Strategies for the division-type word problem.

1. Partitioning of the dividend into known multiples of the divisor, leading to the addition of a number of subquotients.

Alan   640 ÷ 16 = 40
        144 ÷ 16 = 9   40 + 9 = 49

Stefan   320 ÷ 16 = 20
          320 ÷ 16 = 20
          128 ÷ 16 = 8
          16 ÷ 16 = 1   20 + 20 + 8 + 1 = 49

2. A number of separate multiplications of the divisor until the sum of these results is equal to the dividend.

Willem   16 × 20 = 320
          16 × 20 = 320
          16 × 5 = 80
          16 × 3 = 48
          16 × 1 = 16   20 + 20 + 5 + 3 + 1

Gustav   (16 × 20) × 2 = 640
          640 + (16 × 5) = 720
          720 + (16 × 4) = 784   40 + 5 + 4 = 49

3. Counting in the divisor, with an occasional doubling.

Suzanne	16; 32; 48; . . . 784	49

4. Accumulating multiples of the divisor.

Tina	160 + 160 → 320 + 320 → 640 + 32 → 672 + 32 → 704 + 32 → 736 + 32 → 768 + 16 → 784 (20 above 320)	49
Mary	160 + 160 → 320 + 160 → 480 + 160 → 640 + 144 = 784	
	10 10 10 10 9	49
Lynda	320 + 320 → 640 + 64 → 704 + 64 → 768 + 16	
	20 20 40 4 44 4 48 1	49

5. Repeated multiplication or division, adding or subtracting a multiple of the divisor at the end.

Regardt	16 × 10 → 160 × 10 → 1600 ÷ 2 → 800 − 16 → 784	
	10 100 50	49
Susan	16 × 20 → 320 × 2 → 640 + 160 → 800 − 16 = 784	
	20 40 50 − 1	49

6. Repeated doubling of the divisor, adding a multiple of the divisor at the end.

Helga	16	32	64	128	256	512	
	1	2	4	8	16	32	

768 + 16 = 784

48 1 49

7. The dividend is changed to a more convenient number (e.g. a known multiple of the divisor) and the answer is adjusted.

Judy	800 ÷ 16 = 50, 50 − 1 = 49
Michael	16 × 50 = 800, 50 − 1 = 49

8. The standard long division algorithm.

9. Indecipherable strategies.

We have identified only one type of logical error, namely decimal decomposition of the divisor and sometimes the dividend as well, assuming that division is left distributive over addition or subtraction.

Christelle	700 ÷ 10 = 70		
	84 ÷ 6 = 14	14 + 70 = 84	

Table 2 shows the frequency of use of each strategy for each of the three schools.

Table 2: Division-type problem frequency distribution (%)

Strategies	1	2	3	4	5	6	7	8	9	Logical error
School A (n=94)	59	12	6	1	1	1	6	6	6	2
School B (n=79)	21	47	2	0	1	0	2	3	7	17
School C (n=53)	10	32	0	12	8	6	24	8	0	0

DISCUSSION

Students' intellectual autonomy. The fact that these fifth grade students are still using their own invented strategies and not the standard vertical algorithms is an indication that they have maintained the "power of thought" attitude with which they started school over a period of five years, i.e. that the problem-centered classroom culture for mathematics is still reasonably intact.

The presence of the standard algorithms can be traced to new entrants from schools where the algorithms had still been taught, and to parental influence. The seemingly high incidence of the standard algorithms in school C was traced back to an intake of four competent students which in this *small* school counts for 8%. What we find significant is that the use of these algorithms has not spread to any particular degree in any of the schools. Students therefore seem to prefer their own strategies which they use with understanding and with sufficient speed to satisfy the requirements imposed by tests and examinations.

The nature of students' strategies. In the course of a few years, most of these students have developed sophisticated and powerful strategies. In general, the level of number concept of the students as well as their knowledge of multiplication facts is not only quite extensive (note their knowledge of the larger multiples of 16) but also particularly flexible. Their flexibility is shown by the ease with which they construct alternative routes for particular calculations.

The decided differences in preferred strategies among the three schools can be attributed to the effect the problem-centered classroom as a community of mathematics practitioners and talkers has on the thinking of its individual members.

In our previous research reports on the division strategies of students in the first three grades, the different *strategies* differed widely from one another in their economy and the level of abstract thinking employed. This distinction is much less evident among the strategies used by the fifth-graders. With the exception of the quite primitive division strategy of counting in multiples (strategy no. 3), all the strategies are at approximately the same level with respect to use of the properties of numbers and operations. For example, the most useful theorem in action seems to be the distributive property, which pervades all strategies; even the strategies that are based on repeated multiplication or doubling are finally adjusted by applying the distributive property. Even the transformations that the weaker students (or the students with a less well developed number concept or smaller repertoire of known number facts) make in order to simplify matters for themselves are based on an intuitive understanding of the distributive property.

There are, however, great differences *within* each strategy, i.e. with the way in which individual students implement the basic strategy. These differences can be ascribed firstly to an individual student's level of number concept and secondly to the known number facts the student has at his disposal. For example, for division strategy no. 1 (partitioning the dividend), Alan partitioned 784 into 640 and 144, whereas Stefan used 320, 320, 128 and 16.

This is one of the central issues in a successful problem-centered classroom: that students operate at the levels at which *they* feel comfortable. When a student transforms the given task into other equivalent tasks, these equivalent tasks are chosen because the *particular* student finds these tasks more convenient to execute.

It therefore seems that students in problem-centered learning classrooms are aware of the transformations they make and *why* they make them, and that they therefore understand the nature of computational algorithms at a metacognitive level. One can safely say that in a transmission-type teaching environment students do not have this metacognitive perspective on the computational algorithms they are taught, even though some may understand *how* these taught algorithms work.

Viewed from the high school perspective, the strategies indicate a high degree of structural similarity with algebraic multiplication and division.

Logical errors. In a very large-scale implementation of any teaching approach (more than 1000 elementary schools in Southern Africa are implementing this approach) there will not only be found different *interpretations* of the basic principles of the approach, but also *compromises* indicating that the basic principles have not been completely accepted or understood. One of the most common compromises in this implementation arises from some teachers' lack of confidence in their (weaker) students' ability to construct adequate algorithms, resulting in their guiding students towards particular solution strategies. This may pressurize students into trying to use logical structures and/or attempting to deal with number sizes for which they are not conceptually ready.

The occurrence of logical errors among a group of students is a strong indication that the problem-centered learning culture has been compromised, and that students are *not* transforming the given task into tasks that really suit their own level of development.

For example, multiplication strategy no. 1 is the only strategy apart from the standard vertical algorithm which does not allow for differentiation within the equivalent tasks chosen (i.e. there is no choice, the tasks are fixed). Strategy no. 2, on the other hand, allows for very considerable differentiation, both towards much shortened execution (Chris's example) and towards using a number of quite small calculations (Gay does not even multiply by seven, but by four and three). Strategy no. 2, therefore, caters for a wide range of number concept abilities.

The only type of logical error we have identified for multiplication can be described as a partial implementation of strategy no. 1. The high incidence (14%) of this error among school B students, coupled with the high frequency associated with the related strategy no. 1 for the same school, leads one to speculate whether these two phenomena are not related: If some teacher had at some stage guided students towards strategy no. 1, it is possible that a number of students who would have fared much better with strategies of their own choice which allow for more differentiation, are now unwisely attempting to implement strategy no. 1.

The logical error for division also has a high incidence in school B. It therefore seems very likely that the classroom cultures in school B do not allow weaker students complete freedom of choice and make insufficient provision for those social procedures (discussion, support and justification) which identify and clarify errors at an early stage.

Notation. We are very much aware that a classification of strategies which is based solely on students' written expositions may differentiate between strategies which are conceptually identical but which are simply recorded differently. For example, although Alan writes 640 ÷ 16, etc. (strategy no. 1), it is quite possible that he is in fact thinking of 16 × 40, etc., which would be classified as strategy no. 2.

It is not only conceptual and procedural structures that are subject to the influence of the classroom culture; notation is also very much moulded by classroom conventions which dictate acceptable forms of communicating about strategies.

Also, written work serves a number of different purposes: It can function as a computational aid and it aids communication. It can also take the form of reasoned exposition, where the written format no longer matches the original sequence of problem-solving steps, but rather provides a logically coherent justification for the method. For example:

Louis	784 ÷ 160 = 4	*and* 144 *is left*
	144 ÷ 16 = 9	*because* 10 × 16 = 160
		and 160 − 16 = 144
	4 × 10 = 40	40 + 9 = 49

This format of written mathematics could therefore be a quite logical and natural development in a community where mathematics is actively engaged in and where decisions have to be justified.

REFERENCES

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3–29.

Murray, H. & Olivier, A. (1989). A model of understanding two-digit numeration and computation. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education*, Vol. 3 (pp 3–10). Paris, France.

Murray, H., Olivier, A. & Human, P. (1991). Young Children's Division Strategies. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education, Vol. 3* (pp 48–56). Assisi, Italy.

Murray, H., Olivier, A. & Human, P. (1992). The development of young students' division strategies. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth International Conference for the Psychology of Mathematics Education, Vol 2* (pp. 2-152–2-159). Durham, NH.

National Research Council (1989) *Everybody Counts. A Report to the Nation on the Future of Mathematics Education*. Washington, D.C.: National Academy Press.

Olivier, A., Murray, H. & Human, P. (1990). Building on young children's informal mathematical knowledge. In G. Booker, P. Cobb & T.N. Mendicuti (Eds.), *Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Education, Vol. 3* (pp. 297–304). México.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 141–161). Reston, Virginia: National Council of Teachers of Mathematics.

STUDENTS' PERCEPTION AND USE OF PATTERN AND GENERALIZATION

Anthony Orton and Jean Orton
University of Leeds

Four studies in students' perception and use of pattern and generalization are described. One study involved adults and the other three involved pupils aged 9-13 years. Two studies were based on group testing and the others were conducted by means of individual interviews. Analysis of the data has revealed not only different levels of generalizing activity, but also some of the obstacles to successful generalization.

Introduction

Mathematical study involving recognizing, exploring, explaining, continuing, devising and using patterns and attempting to generalize from them has become a significant element of the mathematics curriculum in England According to the DES (1988), "Work in the primary school on number pattern and the relationships between numbers lays the foundation for the subsequent development of algebra" Yet there is still only a limited amount of published evidence as to how successful pupils might be in taking the step which results in them deducing what we can accept as an appropriate algebraic formula from a number pattern. One particularly significant report by Stacey (1989) concerns a collection of studies relating to linear generalizing problems. More recently the use of computers, particularly through spreadsheets, has been suggested as a hopeful approach (Sutherland, 1990; Healy and Sutherland, 1991) Our studies have set out to provide additional evidence concerning patterning abilities and competences, and likely obstacles, in children from nursery school to upper secondary and in adults Here, we report briefly on four such studies, one concerning adults, one involving a group test with 11-12 year old children, and two involving individual interviews with pupils in the 9-13 age range.

Experiment 1: Patterning competences in adults

Mature adults wishing to train as teachers in England may obtain an acceptable qualification in mathematics via an examination provided specifically for them. One legitimate and regular form of question in this examination at the University of Leeds has been concerned with patterning leading to generalization. The format of these questions is limited by the constraints of a written examination, so the amount of data available is less than ideal. Nevertheless, valuable insights have been gained from the data collected over the three years 1991 - 1993. The standard idea in each of the relevant questions from the three years was to provide a sequence of configurations of dots (see Figure 1), and the candidates were asked to predict the numbers of dots (M(r)) for r=5, 10, 50 and n. In each year the nth term involved a term in n^2. There were no cases of candidates successfully answering a particular question without succeeding on all the previous questions, so candidates who provided the answer only for the fifth configuration were classified at Stage 1, candidates who successfully answered both the fifth and the tenth were classified at Stage 2, and so on up to Stage 4 when the nth term was obtained, and this simplified the analysis. There were no major discrepancies between the numbers of candidates who passed the examinations in the three different years.

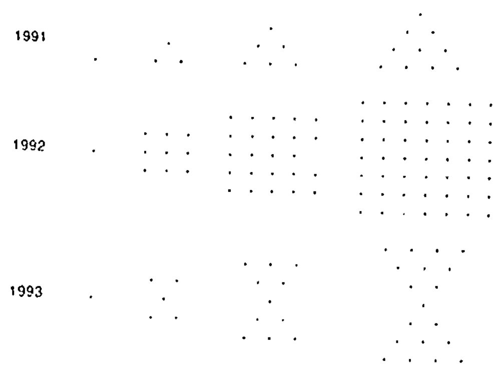


Figure 1

The data indicate (i) that success with the fifth, tenth and fiftieth numbers in the pattern by no means automatically guarantees success on the nth, (ii) that no candidates found the nth term in 1991 yet considerable numbers did in the other two years, and (iii) that the pattern of numbers for 1993 was the easiest. Looking beyond this simple classification, however, other important issues emerge. The use of differencing was ubiquitous as a first attempt to solve these problems, and many candidates persevered with differencing even when attempting to find the fiftieth term. Differencing, of course, normally indicates that the pattern has at least to some extent been understood recursively, but this alone is inadequate for finding the universal rule, in fact it could be said that it sets the candidates off on the wrong track. It was those candidates who were prepared to reject differencing and look for alternative approaches who made progress beyond the tenth term. Thus, success on the fiftieth term was most likely to be achieved by those who had already identified the universal rule. It was particularly obvious in 1991 that, despite the familiarity of triangle numbers, candidates were fixated on differences, and made little progress. In 1992, the numbers and configurations both hinted strongly at square numbers, which perhaps helped, but even greater success was achieved by the 1993 candidates. In this last year, candidates generally began with differencing but, having seemingly failed to identify a meaningful pattern, many ultimately successful candidates then switched to listing squares of r. Most of those who found the formula for the nth term had it available from an early stage and used it to calculate both the 10th and 50th terms. Thus, it appears that not only may such "non-linear" pattern questions not be all equally difficult, but also the most difficult of these three appears to be the one involving the triangle numbers.

In summary, it was clear that mature students, like many children, (i) are first likely to investigate differences, (ii) usually handle the recursive pattern based on differences competently, but (iii) have difficulty in finding an alternative approach which leads them to the universal rule. In a sense, there is a natural obstacle here, in that until students are prepared to discard the recursive pattern, which is often fairly obvious, they are not able to express the universal rule algebraically.

Experiment 2: Large-scale testing with pupils aged 11-12 years

A Group Test was devised as part of a MSc study which was aimed at investigating the appropriateness or otherwise of certain requirements of our National Curriculum and involved a wide range of pattern questions. Subsequently, it was decided to use the test more widely with some 350 pupils aged 11-12 years, partly in order to evaluate the test itself. Space only allows discussion of the results obtained from one particular coherent collection of tasks based on the configuration shown in Figure 2, which it was believed would be unfamiliar to the pupils, and tasks were based on the number sequences contained in the loops. Pupils were first asked to state what they could about the numbers in the loops, and this produced a fascinating variety of responses.

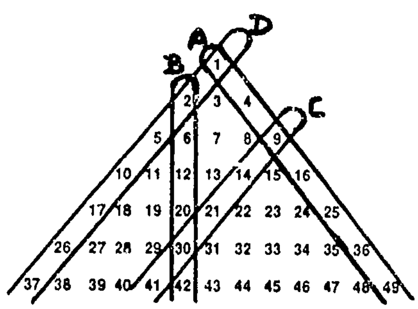


Figure 2

Subsequently, they were asked for algebraic expressions, and they found this very difficult, though about half of the pupils made some attempt. The majority of pupils were able to explain the sequences only in terms of differences between successive terms. Weaker pupils (as judged by their performance on other questions) noticed evens (and sometimes odds), or numbers in the "two times table", and little more. Some of the very weakest had difficulty even with odds and evens. It was expected that Loop A would turn out to be the easiest for pupils. In terms of writing a formula, this was certainly the case but still only four per cent of the pupils wrote n^2 or similar, with a further fifteen per cent writing "square numbers", or "numbers timesed by themselves". In fact, forty-four per cent did not respond.

Taking Loop B as an example for more detailed analysis, 35% of the pupils pointed out that "it goes up 4, 6, 8, 10, 12" and 21% mentioned either "multiples of 2" or "divisible by 2" or "all in the 2 times table". "Even numbers" were mentioned by 34%, of whom nearly half wrote, "Goes up by even numbers", and a further 20% wrote, "Goes up by 2". As regards recognizing the number pattern, very few children recorded a connection with triangle numbers, and hardly any wrote down an appropriate formula. In fact, only four pupils wrote "triangle numbers", and one other wrote "triangle numbers x 2". Four pupils wrote "rectangle numbers", one pupil explaining that it was "one number times a number less, e.g. 4x3=12, 5x4=20", and another explaining "each is a product of consecutive whole numbers". Half of the pupils did not attempt the algebraic formula.

The Group Test was accepted verbatim from its use within the initial MSc study and has turned out to be less than ideal. For example, it was clear that some of the words adopted in the questions were perhaps not appropriate for such young pupils. A further experiment is currently being conducted with a new Group Test which does not use any of the words, 'algebra', 'formula', 'equation', or even 'pattern', and which, because of the nature of the questions, will more readily allow comparison with the data from our other experiments

Experiment 3 : Interviews with low attainers

Individual interviews were used to explore the understanding of number patterns, based on arrangements of matchsticks, with 24 low attainers aged 11-12 years (Root, 1993). The pupils were shown the diagrams of squares and rectangles in Figure 3 For each pattern the pupils were asked how many matches were required for 1 square (or rectangle), for 2, and for 3 and were then asked to predict the number for 5 and for 10.

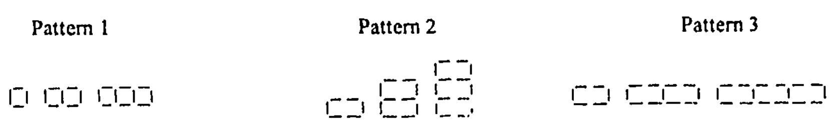


Figure 3

Most pupils used repeated addition, or even counting, and were reluctant to use any multiplication apart from doubling. Further questioning of pupils often led to correct answers for the string of squares but it was rare for this to be carried forward to either of the rectangle patterns. In other words very few pupils came to any appreciation of the linear relationship between the shape number and the number of matches. Just one pupil, consistently and unaided, used an appropriate "formula" (e.g. "timesed 10 by 3 and added one from behind") suggesting she did possess some mathematical insight. Many of the pupils were able to provide correct answers for 1, 2, 3 and 5 shapes, but not 10, through a mixture of stick counting, repeated addition and finger counting.

Over half of the pupils then proceeded to double their answer for 5 to give the answer for 10 When this phenomenon first occurred with Pattern 1, pupils were shown a picture of two lots of five squares put together and questioned about the number of sticks in the middle In this way, many were persuaded to think again and with further questions were led to a correct answer However, this amended approach usually did not carry forward to subsequent questions, when the same errors were committed again Only 5 pupils were able with guidance to generalize the method and achieve correct answers for the rectangles Other pupils were more easily led to consider a formula For example, in response to a pupil who was having difficulty repeatedly adding 3 in Pattern 1.

Interviewer	Look at this diagram (3 squares). How many matches ?
Pupil:	Ten
Interviewer.	But 3 times 3 is not ten so how did we get it?
Pupil	It's add 1 as well

The interviewer was here using pupils' appreciation of the recursive rule (add 3) and leading them to the universal rule (times 3, add 1) In this way, 6 pupils eventually adopted a systematic approach.

Weaknesses in handling large numbers and with number bonds greater than 10 were common, and many pupils were able to do very little mentally. There were also some instances of reversion to 'counting all' instead of 'counting on'. It seemed that the majority of these pupils were a long way from being able to generalize in the extension of number sequences, certainly without considerable further questioning to prompt their thinking. This study raises the whole question of the nature of effective teacher intervention. Further research is clearly needed.

Experiment 4 : Interviews across the ability range

A further study is ongoing and involves pupils having a wider range of age and ability. Individual interviews are being conducted with pupils aged 9 to 15 using the matchstick patterns in Figure 4:

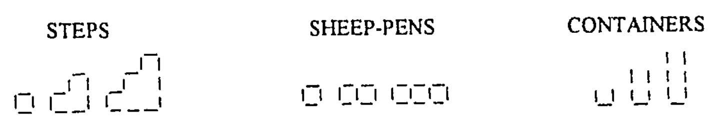


Figure 4

For each pattern, the shapes in Figure 4 are built in front of the children by the interviewer and the first task for the children is to build the next (4th) shape. Pupils are then asked to state how many matches have been used to build each shape and these numbers are recorded within the shape.

The shape number (No 1, No.2, No.3, No.4) is also displayed below each shape. Pupils are then asked to predict, without guidance or hints from the interviewer, the number of matches needed to build the 5th, 20th, 100th and nth shapes and to explain how they have arrived at their answers. A calculator is available for those pupils who want to use it

From the data collected so far (18 pupils) only one pupil (who suffers from the effects of cerebral palsy) has had any difficulty building the 4th shape and all pupils have successfully given the correct number of matches for the 5th shape. The STEPS pattern provided a simple relationship which many pupils have noticed ("It's the 4 times table"). Nearly 70% have given the correct answer for the 20th shape though less than half have managed the 100th shape. Two pupils have given a wrong relationship, i e "Double it". A few of the younger pupils have used repeated addition, sometimes with the help of a calculator, for all the patterns, and there is also evidence of short-cut methods being attempted Using r as the shape number and M(r) as the number of matches, the most common methods which have yielded wrong answers with the SHEEP-PENS and CONTAINERS are:

(a) using the short-cuts M(20)=M(5)x4 and M(100)=M(20)x5;

(b) using difference products

M(20)=3x20 and M(100)=3x100 for SHEEP-PENS

and M(20)=2x20 and M(100)=2x100 for CONTAINERS.

Method (b) is the equivalent of the doubling method, M(10)=2xM(5), employed in the previous study Stacey (1989) refers to (a) and (b) as the Whole-object Method and the Difference Method, respectively. Of course, both of these methods give correct answers for the STEPS pattern, where there is a relationship of direct proportion. It could be argued that the STEPS pattern might draw pupils into adopting either (a) or (b) for the other two patterns. However, both methods were found by Stacey (1989) whose examples of linear patterns did not include direct proportions. Moreover, pupils often mixed the methods:

e.g. "10 pens would be 30 matches" (difference product) ..."so the 20th shape will have 60" (short-cut),

or switched from one method to another:

e.g. "You have to get the answer for 20 steps first and then times it" (short-cut for M(100)), and then later "Times the number by 3 so 100 (pens) is 300" (difference product).

There seemed to be too much information for a few pupils to work with and they revealed confusion between the shape number and the number of matches:

e.g. "20 pens is 60 matches" (difference product) "so 100 pens will be ... you have to add on 40 to 60 to make 100.....".

There were also several examples of pupils' own unique reasoning like the answer that the 20th container would need 23 matches because "Add 1 for all the bases .... that was 20 ... and add 3 for the first". The reluctance of pupils to check their patterns or rules has been noted before (Lee and

Wheeler, 1987; Stacey, 1989). In this study, some pupils noticed that their rules for working out M(20) or M(100) did not fit with the first few shapes but were totally unconcerned.

It was not expected that younger pupils would be able to use the letter n, but one 9 year old managed the explanation "You'd have to times the number of steps by two and then add it again" to describe her generalization, and a 10 year old boy offered "Take the number (shape number for CONTAINERS) times 2 and add 1". Less than 20% of pupils successfully used n (where n4 for STEPS and nx3+. for SHEEP-PENS were accepted as successful, as well as 2n+1 for CONTAINERS). There was little evidence that finding the 20th and 100th terms had helped in the process of generalization. In fact, 2 pupils gave the correct generalization in terms of n after restudying the first few patterns, having previously given erroneous answers for the 20th and 100th terms.

The number of pupils who actually used the matches to help with the process of generalization was disappointing. A few pupils pointed to the 3 matches added on to make a new SHEEP-PEN or the 2 additional matches for CONTAINERS but did not use the end or base match in any attempt at calculation or at devising a formula. The youngest pupil, however, used matches to explain: "You've already got this one (4 matches) so add on 3x19", for M(20), and "3x99+4" for M(100), and this method was generalized using the matches to 2x19+3 and 2x99+3 for CONTAINERS. It had been hoped that the experience of actually handling the matches and building the next shape would help pupils to focus on the matches and make use of the structure of the shapes but, once the numbers had been made explicit, it often appeared that the matches were set aside. Certainly the task of coping with large numbers seemed to prompt a search for a quick method of obtaining an answer and to take the thinking away from the matches.

Conclusions

The individual interviewing, in particular, has revealed generalizing taking place at more than one level. Some pupils were able to use a generalized method for finding the terms of a particular sequence, but their generalizing ability did not enable extension to subsequent sequences. Other pupils revealed a level of generalizing ability which allowed them to proceed beyond the one sequence. As regards the relationship between r and M(r), some pupils had achieved a level of generalization which only allowed them to provide numerical answers, others were able to sum up the relationship using words, and just a few were able to convert this into recognizable algebraic form.

The four studies have also revealed that there are a number of potential obstacles along the road to successful generalization. First, it is clear that arithmetical incompetence prevents progress for some. Secondly, a kind of fixation with a recursive approach can seriously obstruct progress towards the universal rule. Thirdly, a considerable number of students are tempted to use inappropriate methods such as those described above as short-cut and difference product. Finally, we cannot ignore the fact that idiosyncratic methods are adopted by individual students in unpredictable ways.

Bibliography

DES/WO (1988) *Mathematics for Ages 5 to 16*. London: HMSO.

Healy, L. and Sutherland, R. (1991) *Exploring Mathematics with Spreadsheets.* Oxford: Blackwell.

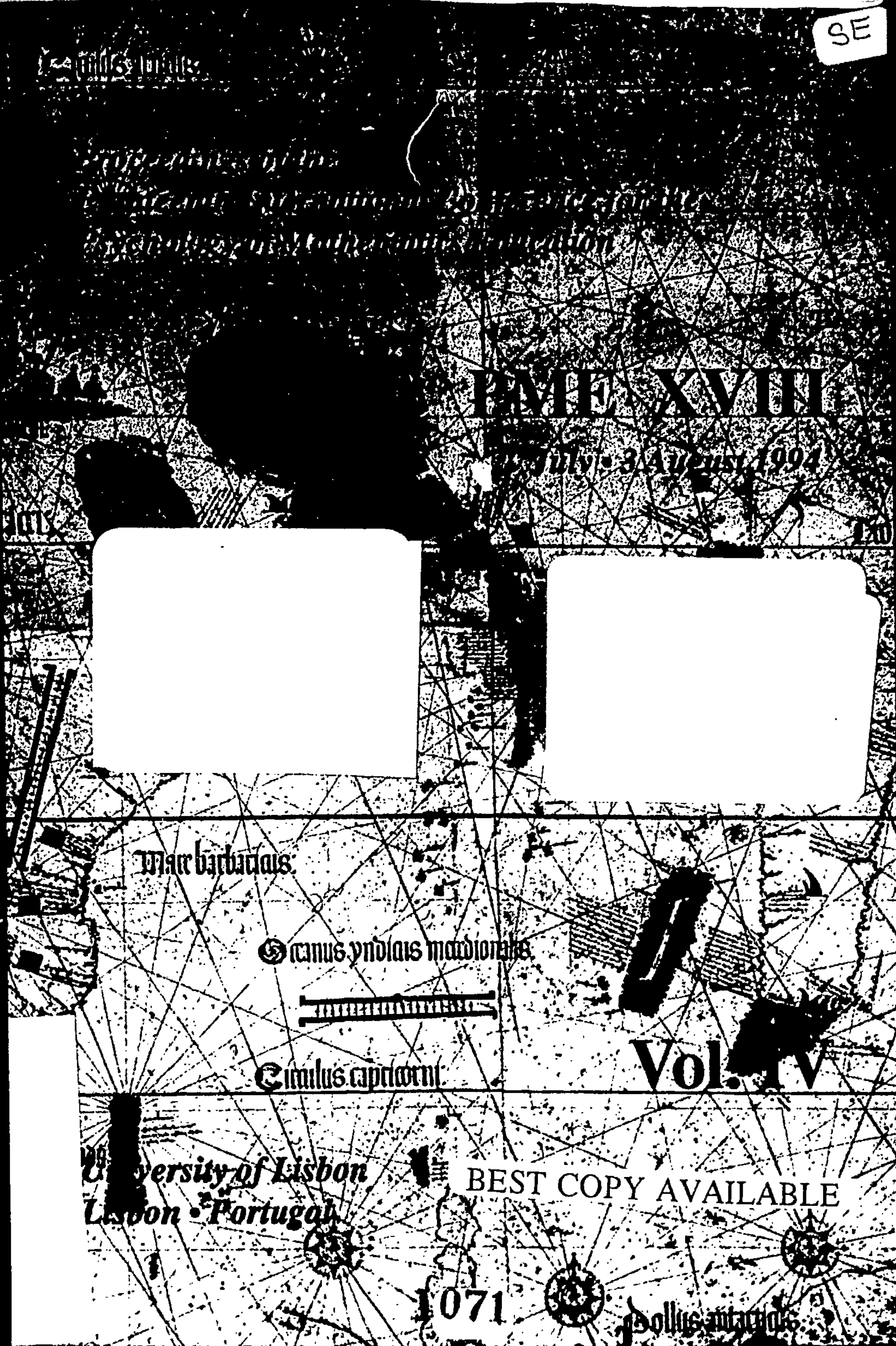
Lee, L. and Wheeler, D. (1987) *Algebraic Thinking in High School Students: Their Conceptions of Generalization and Justification.* Montreal: Concordia University.

Root, P. (1993) Children's Understanding of Number Pattern. University of Leeds: Unpublished MSc dissertation.

Stacey, K. (1989) Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics*, **20**, 147-164.

Sutherland, R. (1990) The changing role of algebra in school mathematics: the potential of computer-based environments. In P.Dowling and R.Noss (eds) *Mathematics versus the National Curriculum.* London: The Falmer Press.

SE



Proceedings of the
Eighteenth International Conference for the
Psychology of Mathematics Education

PME XVIII

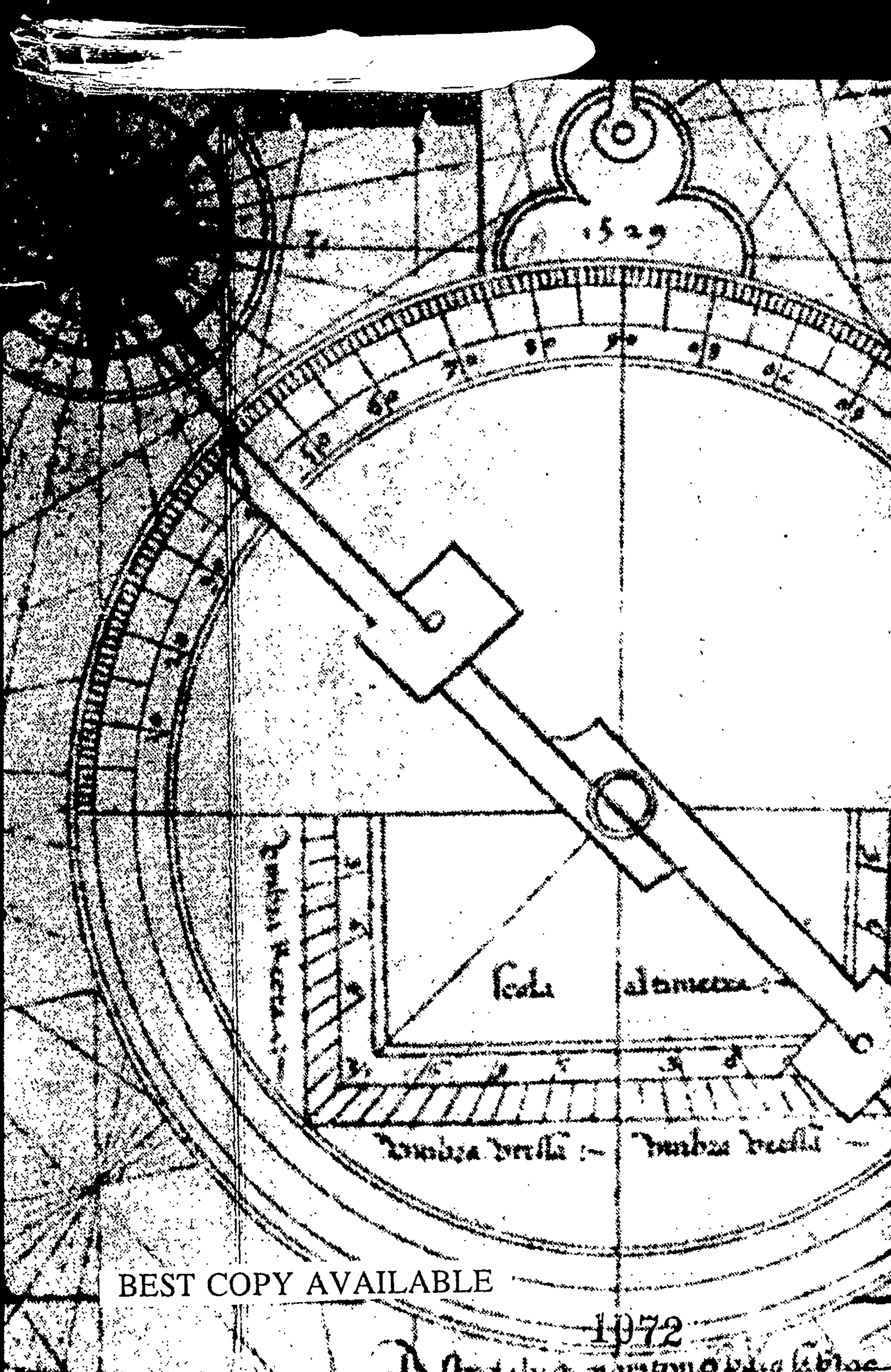
29 July - 3 August 1994

Vol. IV

University of Lisbon
Lisbon • Portugal

BEST COPY AVAILABLE

1071


1529

Proceedings of the Eighteenth International Conference
for the
Psychology of Mathematics Education

PME XVIII

29 July — 3 August 1994

Volume IV

University of Lisbon
Lisbon — Portugal

Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education

Volume IV

Published by the Program Committee of the 18th PME Conference,
Lisbon, Portugal

Editors: João Pedro da Ponte and João Filipe Matos

Cover: Eduardo Veloso

Depósito legal nº 78569 94

ISBN 972-8161-03-4

Printed by GRAFIS, Coop. de Artes Gráficas, CRL

Cover: The background is a reproduction of the *Cantino* chart. This Portuguese map was smuggled from Portugal in 1502, by the spy of the Duke of Ferrara, Alberto Cantino. It is the first world map of the modern times, and it takes into account the findings of the voyage of Vasco da Gama to India.
Cover overleaf: The astrolabe and the quadrant, the most used nautical instruments of the beginning of astronomical navigation, in the 15th century, are represented by two drawings included in the world map of Diogo Pereira, 1529.

Departamento de Educação
Faculdade de Ciências da Universidade de Lisboa
Edifício C1 — Campo Grande
1700 Lisbon — Portugal

Contents of Volume IV

GENDER FACTORS IN AN ADULT SMALL GROUP MATHEMATICAL PROBLEM SOLVING ENVIRONMENT

Vic Parsons and Stephen Lerman
Greenwich University, London, U.K.
South Bank University, London, U.K.

Abstract

In this paper we discuss research aimed at identifying gender factors that may affect problem solving, through a study of a group of twenty seven Open University undergraduate mathematics students. They were undertaking small group mathematical problem-solving activities in the academic year 1993 as part of their first year Foundation Course. The evidence suggests that gender linked preferences are directly relevant to mathematical problem-solving in small groups for the mature students involved in the research study.

1 Introduction

The rationale for using cooperative mathematical problem-solving groups to enhance mathematical learning has been described by Yackel Cobb and Wood (1991) and Dees (1991). The issue of concern here is whether and how, for such adult students, gender issues could influence their mathematical learning. Scott-Hodgetts (1986, 1987) suggested that one important variable involved in mathematical problem-solving strategies was a possible preference by women for a sequential problem solving style, as opposed to a holistic problem solving style. The concept of sequential and holistic problem-solving strategies had initially been suggested by Pask (1976). Given the powerful nature of social conformity to norms within British society such a hypothesis would appear to be theoretically plausible. Evidence of the influence of gender factors within mathematics education are to be found in the work of Fresko and Ben Chaim (1986), Crawford (1986), Joffe and Foxman (1986), Burton (1992) and Hammersley and Wood (1993). The research methodology used in this investigation of preferred problem-solving strategies, and other variables with possible gender links, was primarily ethnographic in nature and based on a pilot study conducted with four students in 1992. An ethnographic research methodology was selected because of the desire to obtain in-depth insights into men and women's reflections on how gender might have affected their mathematical problem-solving investigations.

The twenty seven students in the first part of the main study reported here, were requested to work together in small cooperative groups of three or four on a range of numerical, geometric, abstract and applied mathematical problems, that had been selected on a continuum from closed to open in terms of the types of solutions suggested. The key instruction given to all the students involved in the mathematical investigations was to work cooperatively, explaining to other members of the group the reason for their conjectures, in a manner similar to that outlined by Yackel Cobb and Wood (1991), in order to pool the ideas of all the members working in the group. Members were encouraged to convince the others in the group that their conjectures were valid, and requested to value other members' contributions. A mathematical problem-solving protocol using clarification of the problem, specialising, generalising, conjecturing and convincing was suggested as a problem-solving strategy (Mason 1984).

In addition to the ethnographic data quantitative data was collected by means of questionnaires administered after the ethnographic interviews. Initial conclusions from these data will be presented here. This research study is part of a larger research project aimed at investigating the influence of possible gender effects in such a mathematical problem-solving group context.

2 Analysis Of The Quantitative Data

The quantitiative data collected by means of a questionnaire containing forty questions derived from the pilot study was used to try to substantiate inferences deduced from the ethnographic interviews. The questionnaire was administered immediately after the ethnographic interviews.

2.1 Problem-Solving Style

Table 1 below summarises the perceptions of the students of the problem-solving style they used when tackling the mathematical investigations.

Table 1 Problem-Solving Style

	Men	Women
Sequential	7	11
Neither Sequential or Holistic	1	1
Holistic	7	0

The results seem to bear out Scott-Hodgett's (1986,1987) claim that the men could use both holistic and sequential problem-solving strategies whereas the women exclusively used a sequential mathematical problem-solving style.

2.2 Interactions Between The Students In The Mathematical Problem Solving Groups

Table 2 below summarised the students' perceptions of group dynamics, as measured by who was perceived to dominate group discussions in the mathematical problem-solving investigations.

Table 2 Opinions On Group Domination

	Men			Women			Total		
	Agree	Neutral	Disagree	Agree	Neutral	Disagree	Agree	Neutral	Disagree
Men Tend To Dominate	8	4	3	4	5	3	12	9	6
Women Tend To Dominate	0	4	11	2	5	5	2	9	16

The results suggest that overall the men were perceived as much more likely than women to dominate the group discussions. Further, this was noted to be the case more by the men than by the women students. This suggests that men were more likely to perceive the women as cooperative, than the women themselves. The case study provides some support for the concept that within mathematical problem solving groups women are more likely to be cooperative and men more likely to be competitive. Such a cooperative stance may be a psychological characteristic developed by the desire of women in British society to conform to social norms of cooperative behaviour (Boswell, 1985, Isaacson, 1987).

2.3 Preferred Type Of Mathematical Investigation

Tables 3(a) and 3(b) below indicate the mathematical problems preferred by the students in terms of the nature of the problem and its mathematical content respectively.

The definitions open and closed mathematical problems refers to those mathematical problems which are not well defined and whose solution will depend upon assumptions made (open), or to those mathematical problems which are well defined with convergent solutions (closed).

Table 3(a): Preferred Type Of Mathematical Problem

	Men	Women
Open Types Of Problem	5	2
No Particular Preference For Open Or Closed Types Of Problem	2	1
Closed Types Of Problem	8	9

Table 3(b): Preferred Mathematical Content

	Men	Women
Abstract Mathematics	2	2
No Particular Preference For Abstract Or Applied Mathematics	4	3
Applied Mathematics	9	7

Table 3(a) suggests that women in particular in the problem solving groups did not like tackling the open-ended mathematical problems. Overall, both men and women preferred closed problems to open-ended ones. As adult mature students none of the mathematical problem solvers had experienced mathematical investigations prior to their Open University undergraduate mathematical studies and consequently had not experienced open-ended types of problems before.

Previous research (Pa[illegible]ns, 1990), had indicated that men are more likely to prefer applied mathematics to pure mathematics problems because of their perceived utilitarian value for gaining jobs in the physical sciences and technology where job opportunities for men were likely to arise. Such sex role stereotyping of men into physical sciences and technology occupations and women into the 'caring' professions is well documented. (Gilligan, 1982, Grant, 1983). However, table 3(b) above suggests that women also preferred applied mathematics rather than abstract mathematics problems, with the notable exception of mechanics which was revealed in the ethnographic interviews and is noted in section four.

2.4 Reasons Ascribed For Difficulties Encountered In Mathematical Problem-Solving

Attribution theory which is the causal perception for ascribing difficulties experienced in mathematical problem-solving to internal or external factors (Child, 1986), has significance with regard to mathematical problem solving because of the gender bias (Eisenberg, 1991) and its effect on building self-confidence in mathematical problem-solving.

Table 4 below shows the students' responses when asked whether they perceived internal or external facors as being to blame for difficulties experienced in mathematical problem-solving.

Table 4: Reasons For Difficulties Encountered In Mathematical Problem-Solving

	Men	Women
Blames External Factors	2	1
Don't Know The Reasons	7	3
Blames Their Own Ability	6	8

Both the men and women mathematical problem solvers in the sample were more likely to blame internal rather than external factors for encountering problems when conducting mathematical problem-solving investigations. The women, however, were more certain that this was so, because only twenty five per cent of the women compared to forty seven per cent of the men were uncertain about where to attribute blame, which is significant.

3 Analysis Of Interview Data

The powerful nature of gender affects in small mathematical problem-solving groups can be seen from the following lengthy extracts from interviews with two students, called here Mandy and Peter, as they demonstrate many of the affective issues identified in the literature.

MANDY:-

"Peter seems to whizz ahead of us, he goes off and solves it, and sometimes I feel I'm not getting anywhere. He's done it, and he says to me, 'that is how you should do it,' and I haven't even worked it out for myself. As I said the problem is that Peter is more intelligent than me, he seems to take the lead, and he's whizzing off because he can see something straight away, whereas I have to work through a problem bit by bit.

I'm not happy about Peter. Peter decided the problems we would do. I know he doesn't do it intentionally, but he'll butt in and say, 'oh, the answer's that,' and I can't see how he gets that answer. I like to do things in little bit by bit stages, and when I've come to terms with the first bit, then I can move on to the next bit, then the next bit, I couldn't just look at something as a whole. When we're in the group I feel left behind, and left out of it, and when I do say things I don't think they listen to me, or don't believe what I say. Maybe they think I'm thick or something. We were trying to put our ideas in, but Peter seemed to dominate the group as far as the maths was concerned. The rest of us did give our ideas, but we didn't see the formulas involved until he explained them to us. Peter would either take over, or he'd go off with Gary, and I'd end up doing something with Stephen.

When I'm doing problem-solving by myself, I feel a bit more confident because I know what's going on, but in the group it's more worrying, I feel less confident, I feel intelligence-wise I'm inferior to them. I just don't know enough maths. Maybe it's because I just did arty subjects with my maths at school for 'A' levels. The thing that let me down with my mathematics at school was the mechanics. I just couldn't do applied maths at all, I just had a mental block. Gary's the opposite to me, his favourite subject is applied maths, it's his speciality. All the girls in my school had trouble with the applied maths compared to the pure. Gary says he doesn't like pure because he's accepting rules and things; but with applied he can visualise what's going on. It's true, with pure you're given the rules and you just use them, whereas applied is like physics. Maybe I have problems with applied because as a girl I wasn't encouraged to do practical things."

PETER:-

"I've done electrical engineering and I've covered all this maths before, completely. I was always good at maths at school, but when I went into my degree, I didn't want to do mathematics purely as a sole subject. I went for something applied, that took as much maths as possible namely electrical engineering, where I knew I could get a job.

It's hard for all four of us in the group to work together, because we are at different levels basically. See, Gary and me both did electrical engineering, and we've definitely got a kindred spirit. Steve's got more of an economic background, but Mandy hasn't even done maths to degree level, only to 'A' level. It usually ends up with me and Gary working on the problem, and Mandy and Steve having to do another one. When Steve and Mandy go through a problem together I gave them advice when they got stuck on a problem I'd chosen for them, because I knew I could do it.

I always find it difficult working in groups. I prefer to do it on my own, because I can work out my own structure. If a problem's numerical then I use a sequential approach and work my way through it from the base upwards. If it's an open problem I use a holistic approach; I'll look at the whole problem. I tend to be slower if I work in a group, it usually slows up my train of thought. I've got my own way of thinking, and if I try and work through their logic it usually throws me out of what I'm thinking. I know what I'm doing; I don't build up in stages, but just look for a solution using trial and error for the whole problem. I'm competitive when solving mathematical problems, the mathematics involved in the mathematical investigations is easy, I'm far beyond that in my mathematical knowledge."

Discussion

The analysis of the interview data suggests that organising mathematical problem-solving on the basis of small group working where students are encouraged to work cooperatively does not eliminate power relations within such groups, in particular with a gender bias.

The continuing gender bias within such mathematics problem-solving groups appeared to persistently influence the group of students involved, and it is the intention of this research critically to examine these effects.

Two important gender related variables, namely, problem-solving style and group dynamics appear to influence mathematical problem-solving investigations within such small mathematical problem-solving groups. The extent to which they influence and interact with one another in the student's perceptions of their performance in solving such mathematical investigations will be the aim of further stages in the research. Students' work will also be analyzed for the problem-solving style used in mathematical investigations. Consideration will also be given to the effect of possible tutor intervention.

However at this stage some tentative claims may be made. A first interpretation from the data is that women because of their preferred problem-solving style may not gain the benefits from small group problem-solving sessions that Yackel Cobb and Wood (1991) suggest. It must be noted however that in Yackel Cobb and Wood's research they work with small children and not adults. It would appear in adult groups power relationships in terms of group domination also appear to influence the working within such groups and does not necessarily disappear with tutor intervention. To our knowledge there is no research which shows either that small group problem-solving influences preferred problem-solving styles or that they have any effect on gender-based power relations in the mathematics classroom.

Further interview extracts will be presented at the Eighteenth International Conference of the Psychology of Mathematics Education.

References

Boswell, S L (1985): 'The influence of sex role stereotyping on women's attitudes and achievements in mathematics ' In: Chipman et al (eds) 'Women and mathematics: Balancing the equation'; London: Lawrence Erlbaum Associates

Burton, L (1992): 'Is there a female mathematics or a feminist style of doing mathematics ' Presentation to I.O.W.M.E - Gender and Mathematics Education: International Congress of Mathematics Education (Summer 1992)

Child, D (1986): 'Psychology and the Teacher' (Fourth Edition); London: Holt Educational pp 52-53

Crawford, K (1986): 'Cognitive and Social factors in problem-solving behaviour'; Proceedings of the Tenth Conference for the Psychology of Mathematics Education: London, Vol. 1 pp 415-420

Dees, R L (1991): 'The role of cooperative learning in increasing problem-solving ability in a college remedial course '; Reston, Virginia: Journal for Research in Mathematics Education: Vol. 22; 5: pp 409-421

Eisenberg, T (1991): 'On building self-confidence in mathematics'; Proceedings of the Fifteenth Conference for the Psychology of Mathematics Education: Assissi, Vol. 2 pp 9-16

Fresko, B and Ben Chaim, D (1986): 'Learning environment differences in the mathematics classroom'; Proceedings of the Tenth Conference for the Psychology of Mathematics Education: London, Vol. 1 pp 342-347

Gilligan, C (1982): 'In a different voice: Psychological theory and women's development'; Cambridge, Massachusetts: Harvard University Press

Grant, M (1983): 'Mathematics counts in craft design and technology: But not for some'; London (Goldsmiths College): GAMMA newsletter 4: pp 20-21

Hammersley, M and Woods, P (1993): 'Gender and ethnicity in schools'; London: Routledge: pp 1-8

Isaacson, Z (1987): 'Of course you could be an engineer dear, but wouldn't you rather be a nurse or teacher or secretary' In: Nickson, M and Lerman, S (1993) 'The Social Context of Mathematics Education'; London: South Bank Press

Joffe, L and Foxman, D (1986): 'Factors affecting small group performance in problem-solving'; Proceedings of the Tenth Conference for the Psychology of Mathematics Education; London: Vol. 1 pp 360-362

Mason, J (1984): 'Learning and doing mathematics'; Milton Keynes: The Open University

Parsons, V E (1990): 'Gender and perceptions in mathematical problem-solving' (MSc dissertation); Milton Keynes: The Open University

Pask, G (1976): 'The cybernetics of human learning and performance'; London: Hutchinson

Scott-Hodgetts, R (1986): 'Girls and mathematics: The negative implications of success'; in Burton, L (Ed) 'Girls into maths can go'; London: Holt Rinehart and Winston: pp 68-70

Scott-Hodgetts, R (1987): 'Preferred learning strategies and educational/ mathematical philosophies: An holistic study'; Proceedings of the Eleventh Conference for the Psychology of Mathematics Education: Montreal, Vol. 3, pp 339-345

Woodley, A (1990): 'Equal opportunities: Statistical digest'; Milton Keynes: The Open University

Yackel, E, Cobb, P and Wood, T (1991): 'Small group interactions as a source of learning opportunities in second grade mathematics'; Reston, Virginia: Journal for Research in Mathematics Education: Vol. 22, 5: pp 390-408

ON STUDENTS' CONCEPTIONS OF AXIOMS IN SCHOOL GEOMETRY

Tasos Patronis, University of Patras, Greece

A research on students' epistemology was carried out with a class of 16-years old students, in which the students produced "pragmatic" or "utilitarian" arguments for the justification of axioms used in school geometry.

INTRODUCTION AND THEORETICAL FRAMEWORK

What do students think of mathematical axioms (or postulates) which they learn at school? How do they use them? And how would they justify an axiom if they were going to do so? The present research is an attempt to answer these questions in the context of school geometry.

Euclid's ELEMENTS have remained for a long time a unique paradigm in mathematical thought, in general, and a unique theorization of experience with space and forms, in particular. For these (and possibly other) reasons, Euclid's Geometry has been considered - particularly in school tradition - as an example of a piece of certain knowledge, a logical system which lies on a-priori intuition. But even for Kant, this knowledge was by no means "formal". Euclid's theorems do not follow, according to Kant, by merely logical arguments from the axioms alone, but by a construction or "generation" of geometrical figures, which takes place in "pure intuition" (Friedman, 1985).

Adopting or rejecting a mathematical axiom does not depend on logic and syntax only, but also - and sometimes mainly - on arguments about "reality" and the "uses" of mathematical theory, i.e. on pragmatic arguments. Axioms are assertions (affirmations) that cannot be proved or disproved within the theory; they can only be <u>justified</u> (or not). Habermas (1972, 1976) discusses such assertions among other subjects, of course in a more general perspective. We shall take Habermas' setting as a framework for students' epistemology. Since the justification of axioms usually does not depend on formal proofs but rather on informal arguments, it should be better studied by means of a "dialogical epistemology" such as that developed by Habermas (see Shovsmose, 1990, p. 126). Habermas refers to Toulmin (1964) for an analysis of the structure and use of arguments. By an "argument" (in general) we understand a justification that intends to make someone recognize the validity of an assertion. As stated in the French edition (1987) of Habermas work (p. 309):

> "L'argument est la justification qui doit nous motiver à reconnaître la prétention à la validité d'une assertion, d'un commandement ou d'une évaluation".

By a "pragmatic argument" we mean an argument depending on experience in a particular context. Such arguments cannot be limited into experience only, because they necessarily interpret experience, during the discussion (op. cit. p. 283). As an example of pragmatic argument one can mention any of the arguments, encountered among mathematicians, educators e.t.c., about "usefulness" of mathematical theories; these arguments interpret mathematical activity and in some way predetermine curricula as social constructs.

THE METHOD, THE SUBJECTS AND THE THEME

In agreement with Habermas' epistemological framework on "arguments", we have used participant observation and informal discussion for studying the students' own thinking, conceptions and justifications of mathematical axioms, as developed in an ordinary classroom environment. More specifically our method is the following.

A passage of particular importance is selected from the textbook and analysed epistemologically and didactically. The corresponding lesson is delivered as usually by the teacher in the classroom, while it is observed and recorded. Then a phase of "open" interview follows (in the form of discussion) in two sub-phases: the researcher addresses a number of questions first to the whole class and then, in more depth, to a small group of students. The questions refer to particular points of the preceding teaching-learning process; they do not refer to the mathematical content formally only, but mainly to the meaning and use of definitions, axioms and theorems, to justifications and proofs produced by the teacher or by the students.

The research was carried out with a class of thirty 16 years-old secondary school students at the central part of Athens. The students had been taught some deductive geometry and had heard about Euclid and the possibility of building a Non-Euclidean geometry as well. They had formally covered only the first chapter of the textbook in which the axioms of incidence, order and measure for line segments and angles are introduced, following Pogorelov's axiom system for school geometry.

THE PROCESS

The lesson of the day was to prove that the bisectors of two adjacent angles whose sum is a straight angle are perpendicular to each-other. The teacher told the students to draw two adjacent angles whose sum is a straight angle, and asked them what do they notice about the bisectors of these angles and if they could prove their observation. The students produced a proof as follows: If $X\hat{O}Z$ and $Z\hat{O}X'$ are the given angles, and OM, ON are correspondingly their bisectors then we have, from Fig. 1,

$$X\hat{O}Z + Z\hat{O}X' = 180^\circ$$

which implies

$$\frac{X\hat{O}Z}{2} + \frac{Z\hat{O}X'}{2} = \frac{180^\circ}{2}$$

or, again from Fig. 1, $$M\hat{O}N = 90^\circ$$

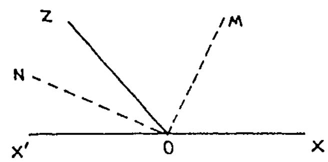


fig. 1

In this proof there is a confusion between angles themselves and their measures, and this was noticed by a student. No reference, however, to the axiom relating angles with real numbers as their measures, was made by any of the students during the lesson, nor during the phase of interview, when the researcher asked the students which axioms did they use in their proof.

The next question of the researcher to the class then was:

- What does an axiom mean for you, and how do we accept an axiom in geometry, according to your opinion?

The main answers were the following:

(An axiom is)

- A definition which looks very simple... it is very simple, by logic, to say it, but very hard to prove it. (Haris)
- A proposition which we consider as true without having a proof for it. We accept it but we cannot justify it by calculations as in algebra. (Viky)
- Something logical, self-evident, obviously true. (Maria)
- A basic proposition which helps us in the solution of problems and exercises. (Two other students)

RESEARCHER (adressing to the last two students:)
Can you give an example?

STUDENTS. In the proof that we gave previously, we have used as an axiom that we can multiply both members of an equality with the same positive number.

There follows a discussion of the researcher with a group of students around Haris:

R. Do axioms and definitions mean the same for you?

HARIS. (Hesitates) Well... not exactly but...

R. Let us take an example. What lines do not call "parallels"?

STUDENTS. Those which never meet, no matter how much they are prolonged. (They clearly mean coplanar lines).

R. So, is this statement an axiom?

STUDENT. Yes.

ANOTHER. No, this can be proved!

HARIS. The proof can be practical only, by measurements; it can never be theoretical, by using formulas and such things... But an argument questioning the axiom can be not only practical, but also theoretical.

R. What do you mean?

HARIS. Our teacher told us that there exist other geometries, in which there can be more that one parallels to a given line from a given point.

R. Do you consider this possible?

STUDENTS. Not at all, we cannot accept this, it is illogical.

HARIS. It depends on what you <u>define</u> as parallel lines. I could call for example these lines "parallels" (he draws two intersecting lines on the table), but then what would it happen?

R. Can we do mathematics like that, with arbitrary rules, like in chess for example?

ANOTHER STUDENT: Then mathematics is not like chess, because the rules of chess are <u>not</u> arbitrary, while this is (he means Haris' example of definition of parallels).

ANALYSIS AND DISCUSSION

In Pogorelov's axiomatic exposition of school geometry the system $\mathbb{R}$ of real numbers is supposed to be known in advance, and axioms are introduced for the measure of straight line segments and the measure of angles, which are considered as mappings into $\mathbb{R}$ (Pogorelov, 1987). The Greek textbook does not always follow the abstract form of proofs in this axiomatic system and sometimes it uses figural representations in the proof arguments. Thus the students often come to confuse an angle with its measure in degrees, as it was the case in the preceding students' proof about the bisectors of two adjacent angles whose sum is a straight angle. The textbook writers and the teachers are aware of the concept of angle measure and of the dependence of proofs on the axioms <u>even when they identify (by abuse of language) an angle with its measure</u>, while the students may not be aware of these things. This was particularly evident in the preceding classroom discussion.

Looking closer at the conceptions of students about axioms, there seems to be a confusion between axioms and definitions (Haris and some of the students around him). Haris and some of his colleagues also speak of the possibility of <u>proving a definition</u>. This probably sounds strange to a mathematics teacher or even a "working mathematician" of our epoch, but it could have a meaning in a <u>semantics</u> for a mathematical theory. In the discussion of logical and methodological problems of science, indeed, such an enterprise does not seem senseless at all. For example, Gorsky (1974) devotes one chapter (Ch. 7, pp. 229-252) to "the applicability of truth-value assessments to definitions", where he discusses Aristotle's view in "Analytica Posteriora" (Book II).

For Haris, Maria and probably some other students, an axiom is "logical" and "simple to say (it) by logic", "self-evident", "obviously true", but "very hard to prove (it)". The same qualities are ascribed to some definitions. Can we prove that those lines which are <u>called</u> "parallels" <u>are</u> parallels in reality? This question has a meaning in the pragmatics of geometrical theories - for example the students would rather have a big problem in accepting Lobachevski's definition of parallels. For many students of the observed classroom the only axioms and definitions that share the above qualities in geometry are those of Euclidean Geometry - the other possibilities seem to them as illogical and arbitrary (more even that the rules of chess!).

While the other students are not very "dialogical" in their assertions about the validity of the definitions and axioms of Euclidean Geometry. Haris offers also an argument. He says that the adoption of a Non-Euclidean axiom about parallels depends on what we define as parallel lines. If we define as "parallels" two straight lines which in reality intersect one-another, then we may adopt an axiom that creates a new geometry. This argument is an indirect pragmatic argument against Non-Euclidean Geometry. Firstly, it clearly depends on the particular field of everyday experience with space and of planar drawings (e.g. on the surface of a table). Secondly, it is not limited into this particular field of experience, but it "projects" this into the heart of the discussion. It seems to say: "It depends on how contrary to common experience you want to be: if you want me to call these lines "parallel", I may call them so, but this is another language game, here we talk of real parallels." This argument is also intended as an indirect justification for Euclidean axioms.

Concerning direct verification (or "proof") of axioms, the students' responses indicate the belief that verifying an axiom does not involve algebraic calculations (Viky) or formulas (Haris), but perhaps only measurements (Haris). Thus verification of axioms can only be "practical" , which excludes verification in terms of a theoretical model - although, according to Haris, there can be a "theoretical argument" (perhaps some new theory) "questioning" the validity of an axiom.

The interview has revealed also another conception of what an axiom in school mathematics is, which at the same time offers a criterion of justification in the eyes of some students. According to this view an axiom is "a basic proposition which helps in the solution of problems and exercises". This is rather a "utilitarian" criterion, and the example offered by the students shows that this conception is closely related to "procedural" understanding of the system of real numbers at school.

REFERENCES

Friedman M., 1985. "Kant's thory of geometry", Philosophical Review Vol. 94; also published (1992) in C.J. Posy (ed) Kant's Philosophy of Mathematics, Kluwer publications.

Gorsky D.P., 1974. Definition: Logico - Methodological Problems, Progress Publishers, Moscow.

Habermas J., 1972. "Théories Relatives à la Vérité", published (1987) in Jürgen Habermas Logique des Sciences Sociales et autres essais, Ed. P.U.F.

Habermas J., 1976. "Signification de la Pragmatique Universelle", published (1987) in J. Habermas Logique des Sciences Sociales et autres essais, Ed. P.U.F.

Pogorelov A., 1987, Geometry, Mir Publishers, Moscow.

Skovsmose O., 1990, "Mathematical Education and Democracy", Educ. Studies in Mathematics 21, 109-128.

Toulmin S., 1964, The Use of Arguments, Cambridge.

TEACHERS PERCEPTIONS OF ALGEBRA

Barbara Pence
San José State University

Throughout a 15 week algebra course, K-8 teachers were asked to describe their views of algebra. The course was a study of functions derived from modelling of real-world situations and investigated through multiple representations in a technology rich environment. Based upon weekly journal questions, exam questions, and student interviews, teacher reflections about concepts of algebra were traced. Student interviews organized and conducted by the teachers enriched the case study of changes in meanings attached to what is algebra and the objects with which one thinks about algebra. Perceptions change across time but the rate and path of these changes vary across teachers. Teachers relate different views of algebra and the objects of algebra.

Introduction

On a recent episode of a television game show, a contestant's correct response to the statement *"The high school subject you have had the least use for"* was *"Algebra."* This response seems simple and direct but it may indeed mean different things to different people. What is school algebra? What are the "big ideas" in school algebra? When we think and talk about algebra, what is meant? Is there a common understanding, a shared language? In a recent conference, sixty eight educators representing all phases of mathematics education were asked to think about algebra for the twenty-first century and address several questions including what is school algebra and what are the "big ideas" of algebra. One of the most revealing fact about the conference is that it failed to produce any answers to the questions. The diversity among the participants brought a multitude of perspectives and visions about the fundamental ideas in algebra.

This paper examines the beliefs of six elementary and middle school teachers as they study a semester of algebra. In addition to the questions of what is school algebra and the key ideas of algebra, questions such as how stable are beliefs about algebra and what modification can be realized as teachers work through a prototype reform curriculum are examined through case studies carried out across 15 weeks. The paper examines the thoughts of the teachers as learners of algebra and also their thoughts as teacher researchers when they interview their own students.

Background Descriptions

The 15 week, 45 hour algebra course was not a traditional symbolic manipulation course. Rather, the focus was on functions and the investigation of families of functions. Each function was introduced through an applied situation and the family of functions were studied as needed in developing the model of the applied situation. The text for the course was Computer Intensive Algebra by Fey and Heid. Throughout the course technology was assumed and used as an exploratory tool. Due to the use of graphing calculators and computers, explorations moved across a variety of representational forms including manipulatives, pictures, tables, graphs, and symbols. The format of the class was also different. Students worked in groups, presented the results of their explorations in written papers and oral reports, and even worked on exams together. Thus both the content and the classroom environment violated each of these experienced teachers' perception of what was and should be "Algebra" or even what should be a "Mathematics Class".

Before beginning on a discussion of qualitative data collected from the teachers during the class, it is important to say a bit about these teachers. Each of the teachers was thoughtful, articulate, and dedicated. All of them had college degrees which meant that they had taken 3 to 4 years of high school mathematics, and had completed at least two college mathematics courses. Of the six teachers who finished the class, three were actually teaching algebra at the middle school level, one was teaching 4th grade, one was a long term substitute for 1st grade and one was a student in the teacher education program. Each teacher was working toward a credential entitled The Elementary Mathematics Specialists Credential designed for teachers who enjoy teaching mathematics and who want to gain more knowledge about mathematics in order to become resource teachers in their schools. The program consists of 7 mathematics courses and 3 education courses including two problem solving courses, courses in algebra and geometry, and introductions to discrete mathematics and calculus.

Data Collection

In order to maintain an ongoing log of the reactions of each of the teachers in the class, a journal was required weekly. Specific questions were posed each week, a total of 30 questions across the course, which elicited responses about content, feelings towards algebra, perceptions of their students reactions to questions, and reactions to the unique structure of the class. Teachers were always invited to add to or delete any questions which they were not ready to respond to. Each teacher kept their own journal and was able to review previous responses at any time. Teacher perceptions were also elicited through in-class discussion questions and exam question. For a final course project, each teacher was asked to interview at least one of their own students.

The focus of this paper will be the teachers perception of what is algebra and what ideas are central to learni g algebra? Ideas about variables, functions, and modelling will be included in the discussion along with general views of what is mathematics.

What is algebra?

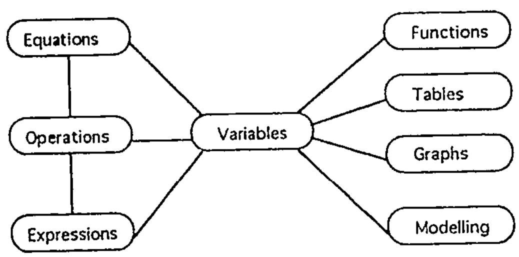
Usiskin's model of conceptions of school algebra and uses of variable is used to provide a framework for analyzing teacher responses. In summary his 4 conceptions of algebra and corresponding uses of variable include: generalized arithmetic - pattern generalizer, means to solve certain problems - unknowns, study of relationships - arguments and parameters, structure -

arbitrary marks on paper. These four categories will be used to structure teacher perceptions and changes in teacher perception of what is algebra. The issue of what is a variable will draw as well on Kuchemann's hierarchy in which he describes the six levels of understanding of variable as letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter used as a generalized number, letter used as a variable for a range of values.

To begin with, the teachers were asked to identify 3 - 5 key idea which make up the course of algebra. Their collective list included sy‹ ..bolic reasoning, equations and inequalities, variables, functions, factoring, exponents, probability. Using a subset of this list combined with ideas of modelling and multiple representations, a set of the eight key words: equation, expression, function, graph, modelling, operations, table, and variables was compiled. Each teacher was asked to develop a concept map for these eight ideas, that is to arrange these ' ,ght ideas into a picture so that those ideas which were related were close to each other and the nodes of the network were connected through arcs, directed if possible.

These concept maps will be the springboard for case study analyses for three teachers. Responses to other questions will be examined as the perception of algebra is developed. Summary of the other work will also be provided.

The following is the map given by Joan.



In this concept map, equations, operations and expressions go together while functions, representations and modelling form a second cluster. Both groups are connected through the central idea of variables. Additional discussions by Joan reflected a perception of algebra as that of solving equations to determine the value of the unknown. The object of algebra then for Joan was a letter which she regarded as a specific but unknown number. Midway through the 15 week class, Joan was asked to react to a Family Circus cartoon statement that *"Reading is easier than math. There are only 26 letters, but millions of numbers."* Even though she had been working with functions for 6 weeks in a technologically rich environment, she stated that" *actually, there are only 10 numerals compared to 26 letters. All it takes is to get the hang of putting them together. There are both hard to spell and pronounce words and hard problems to solve."* Here, her original description of algebra is generalized to her view of mathematics. She sees not only algebra but all of math as working with strings of numbers (numbers, operations, and relations). The digits from 0 to 9 form the objects for her perception of algebra. In no way is she moving to the level of working with variables which assume multiple values.

During the 15 weeks of class, Joan struggled with the idea of function but was constantly in a state of uneasiness. Although she worked with functions and represented them in various forms, her conceptual framework changed little. For Joan, a function was a relationship between variables and numbers. That is, for each representation of the function, she parsed it into individual numbers and unknowns (f(x) meant f times x). In working with tables, she focused on each entry and her graphs were developed as a finite collection of points. When pushed to say more about functions, little was accomplished since she would quote statements from the book.

For the final class project, Joan interviewed an eight grade student. She was given only the general outline that the interview should examine the student's view of algebra. Although she could use any of the questions or probes used during the class, development of the interview protocol was her responsibility. She asked the student what is algebra, what is it used for, and how will it be used after the class of algebra. In the report, Joan described the student's responses to the questions and also related her evaluation of the student responses. Thus, the interview illuminated both what Joan wanted to explore and what she believed was an appropriate answer to the question.

For the question of what is algebra, Joan relates that" *Megan had a fairly good understanding of algebra. She basically thinks algebra is math and it helps people figure out things. It involves using numbers and letters in different operations to solve equations, word problem. polynomials, and fractions."* In this interview she failed to probe the student's first response due to the match with her own definition of algebra. In fact, it agrees with the statement Joan wrote in her final that *"Algebra is the branch of mathematics that uses the operation of arithmetic in dealing with the relations and properties of quantities by the use of symbols and letters, negative numbers as well as ordinary numbers, and equations to solve problems involving a finite number of operations and that a variable is a symbol that represents an unknown. It has the property of being a placeholder."*

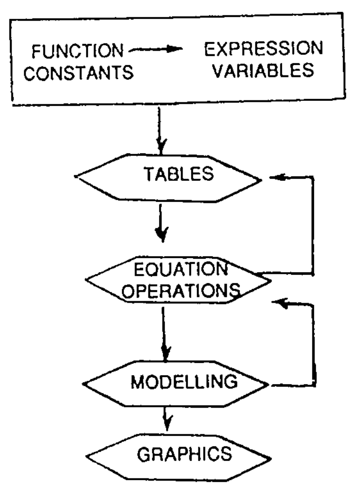
In contrast, another teacher Elisha, whose original responses were very similar to those described by Joan, reflects significant growth. In the beginning, she described variables and in some sense algebra as *letters that are used to represent an unknown number. In language, letters are put together to form words, but in algebra the letters aren't put together to form numbers. I have always found using variables to be very intimidating. I can solve a problem, but when I am given a problem where I have to solve for x, it seems so much harder to do."*

One of her favorite stories in class discussions was when her teacher the previous semester used the letter *n* as a variable. She was shocked and her anxiety level increased when she discovered that there was more than one letter which could be used for a variable. This view of variable was also reflected in her reaction to the cartoon caption. Here she said that it was true for her, *Although letters can be arranged millions of ways, I must have better reading strategies. Reading is consistent, but math isn't..*

By the end of the semester, Elisha's early conceptual conflicts were moving to a resolution. In her student interview, she asked the high school student what she thought about algebra and then probed the student's thoughts regarding variables. Instead of moving to another question when her initial questions about variables yielded variables by examples, that is, a variable is x in the equation $2x = 8$, she continued to probe. She asked about functions and when this question produced no reaction, she asked the student to talk about expressions as compared with equations. Her investigation continued by asking the student about graphing expressions and equations, and describing them in tables. She even took the expression 4x and asked the student to make a table with different input values and corresponding output value and describe how the table entries related to the expression. Pursuing this line of inquiry, she encouraged the student to describe what was meant by a linear relationship and even a quadratic relationship. Elisha's final reflection in the interview report was *"I focused on these questions because I was embarrassed that I didn't know the definitions of these at the beginning of class."*

Elisha modified her perception of algebra from that of a means to solve certain problems to that of a study of relationships. She was able to operate with functions as objects but was unable to move to functions of functions or to explore the structure of the objec`` ^he was using. In her final journal entr;' she stated that *"I have learned more than I thought I would about algebra . . . I understand so much more than I thought I ever would about algebra.* At the end of two pages she was concerned, however, that she was unable to build matrices and understand their meaning.

Two of the teachers began with a concept of algebra which at least included functions. Judith for example gave the following as her concept map:



From the map, it can be seen that functions are not an add-on but a component in her conceptual overview. Several elements, such as the separation of the multiple representations and the restricted interaction between modelling and equations, indicate that functions are not a central object. Rather, equations seem to be the core of her framework. Although this diagram indicates that Judith has some fairly coherent perceptions of algebra, her initial description of algebra included the statement that *"algebra as a pulsating glob that eluded all definition and/or boundaries."*

Even though the material was difficult, Judith made significant progress in understanding the content. In addition to understanding functions,mJudith struggled with the structural view of algebra. This concentration on structure can be seen in her response to the Family Circus cartoon. In reaction to the statement that reading is easier than math because there are only 26 letters, she stated that*"Actually, reading is much more difficult than math because math has a more dependable structure that adheres to predictable rules. However language contains structures that contradict themselves under irregular circumstances. Also reading requires us to read between the lines and interpret text with contextual considerations."*

For the student interview she selected a student from another teacher's prealgebra class. She asked about the meaning of algebra and then probed by asking how algebra was different than arithmetic. She also asked what a variable was, whether a variable could have different values and /hat was meant by a function. One question of special interest to Judith was does your teach ever talk about the structure of mathematics. Although this line of questions produced little, she was happy with the student's comment that he likes structure because it helps to clear up a messy equation. In the final journal entry, Judith described her interest in structure and stated that she was*"both fascinated and relieved to learn that algebra can be reduced to a structure. It dispelled the notion I held that algebra was a pulsating glob that eluded all definition and/or boundaries. It gave me some hope that I'd master it some day."*

In her final, Judith described variables as*"a letter that represents a changing value, i.e. it can represent more than one numeral depending upon the situation. In other words, letters can vary in meaning depending upon the context, so they are not as discrete and exclusive as numerals"* and Algebra as*"higher-order thinking that demands an ability to think abstractly about conditional values. It is imperative that the variables we manipulate are clearly defined so that our solutions make "real-world" sense."* In describing function she stated that*"I began to understand that algebra is not really concerned with integers, equations, variables, constants, etc. Rather, it attempts to model real life and the inherent relationships of life."*

Profiles for each of the teachers' perceptions of algebra could be constructed but due to space restrictions, a summary of the number of teachers in each category both at the beginning and end of the class is given below:

Conceptions of Algebra	Beginning of Class	End of Class
Generalized Arithmetic	2	1
Means to solve problems	4	2
Study of Relationships		2
Structure		1

Thus, across the semester, even though these teachers had taken 4 years of secondary school math, several college math courses and taught algebra themselves, the beliefs about what is algebra changed. These changes were seen in the written responses of the teachers and in addition, the changes were reflected in both the questions which the teachers asked their students and the way they listened to and interpreted the student's responses.

Language of Algebra

A continuing theme which occurred throughout class discussions and teacher reflections was the barriers developed due to both the verbal and written communication. Words and symbols meant different things to each individual.

This paper has examined the concept of algebra for teachers and established, across teachers, the existence of differences in meanings attached to what is algebra and the objects with which one thinks about algebra. Within each individual, these meanings were shown to be consistent across pictures, written reflections, and interactions with students. Changes did occur across time for most but the rate and path of these changes varied from one teacher to another.

Consistencies within individuals and the differences across teachers affected the classroom interactions. Each teacher had to deal with not only the mismatch between their concepts and the concepts of the instructor, but they also had to try to understand the other teachers in the class. In this final quote, Judith talks about this issue as she reflects on her collaboration with Joan on a project. Following the submission of a major project, each teacher was asked to describe where he/she felt comfortable or uncomfortable in talking about mathematics. Judith said*"Often my partner had a very different way of looking at the problem. She was satisfied with isolating single bits of information, finding a rule that fit that situation, then generalizing slowly, testing the rule at different points along the way. I, on the other hand, tended to look at the whole problem or topic. I attempted to generalize first, then isolate a single situation to see if my general idea worked. We seemed to use semantics quite differently. I always needed to reword what my partner said so that it made sense to me. My partner wasn't comfortable with my rewording."*

The conversations between Judith and Joan used the same words to impart different conceptual meanings. Both used the basic terms of algebra: variable, functions and modelling. The overlap of shared meanings was, as shown in this paper, small. Maybe, indeed, it was too small to allow for productive collaboration. How can these diverse meanings attached to the same words be communicated through a common language? As long as the language camouflages these differences, there is no explicit conflict or need for conflict resolution.

References

Begle, E. G. (1979). Critical variables in mathematics education. Washington, D.C.: Mathematics Association of America.

Fey, J and Heid, M. K. (1991). Computer-Intensive Algebra. College Park, MD: The University of Maryland.

Kuchemann, D. (1981). Algebra. In Hart, K.M. (ed). Childrens Understand of MATHEMATICS: 11- 16. London, UK, John Murray.

National Council of Teachers of Mathematics (1993). Algebra for the twenty-first century: Proceedings of the August 1992 conference. Reston, VA: author.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In Coxford, A. and Shulte, A. (eds.). The Ideas of Algebra, K-12. Reston, VA: National Council of Teachers of Mathematics.

TREE GRAPHS: VISUAL AIDS IN CASUAL COMPOUND EVENTS

Angela Pesci, Department of Mathematics, University of Pavia

In this study an analysis of tree graphs as graphical representations which can work as visual aids in the understanding and in the solution of casual compound events is proposed.
It is reported how tree graphs can describe, in figural terms, the conceptual relationships involved and stimulate the use of an adequate calculation procedure.
It is then examined how tree graphs are used by 13-14 year old students to solve two problems with different characteristics.
After a prior study of the two problems mentioned above and a description of the "theoretical background" in the classes where the investigation was carried out, an analysis of the results achieved is proposed and the most significant errors are examined.

1. INTRODUCTION

The idea that knowledge, for each of us, passes necessarily through sensorial experience is certainly not new. The belief that anything which is in our intellect must have also passed through our senses was typical of scholastic philosophy but it can be traced back to Aristotle.
In modern thought this idea has been resumed, starting from J. Locke, and developed in a markedly empiristic sense opening up the perspective in which the current experimental researches can be set.
At present, the sensorial dimension is universally thought to be fundamental in the process of knowledge building.
Also research in mathematics didactics is increasingly highlighting the involvement of perception in the personal building of knowledge. It must be noted that the sensorial component of our knowledge is not recognised as transitory (something which leaves its place to "rational thought") but rather as a stable component of adult knowledge.
This is what can be deducted, for example, from Dörfler's theory of meaning and from his recourse to "concrete carriers" for the image schemata corresponding to concepts (Dörfler 1991, p. 21) or also from Goldin's theory about the five types of internal representations of concepts and strategies, one of which, called "imagistic", is strictly connected with the interpretation of sensorial experience (Goldin 1992, p. 248).
Within the sensorial perception, the visual perception has undoubtedly got a central place: it is sufficient to browse through the latest volumes of the acts of PME or CIEAEM, or through the acts of ICME 6 to realise how the themes connected to visualisation in mathematical knowledge building have been given increasing space.
In relation to this research area the directions are manifold, and they differentiate, for instance, according to the type of mathematical content examined - i.e. geometrical or not - or to the type of visual image production - i.e. by means of a computer or not -.
This study deals with the representation of casual compound events by means of tree graphs as graphical tools. It aims at providing possible answers to the following questions.
"Which graphical representations are particularly effective? In which conceptual context? Which could be the reasons of their effectiveness?" (Mariotti, Pesci, 1992).
These questions, together with others, arise spontaneously when the role of graphical representations in problem solving and in concept building is explored in detail.
In particular, the analysis which is proposed here has the two following objectives:
- a prior study of tree graphs as visual mediators, making explicit the connection between graphical representation and theory;
- an analysis of the use of this graphical tool in two problems with different characteristics and an examination of the most meaningful errors.

2. TREE GRAPH MATHEMATICS

The tree graph as a graphical representation and as a definition, was born from the studies which developed from the 30s onwards and which are usually known as "the graph theory". In the field of probability the tree graph is not used in theoretical classical texts (like Feller, 1968 and Gnedenko, 1979) but it appears when probability is introduced to "non-insiders" (see for instance Kemeny et al., 1968 and Fintacuda, 1983) or in pre-university school (see for instance Bowie, 1968, Cundy, 1968, Engel, 1972 and Varga, 1976). Today, it is one of the most familiar graphical representations, starting in compulsory school, to visualise both combinatorial and probabilistic situations

Tree graphs, in Fischbein terminology, belong to "diagrammatic models" They are " graphical representations of phenomena and relationships amongst them" ... "In the case of diagrams one system, the original, exists in its own right while the other, the diagram, is an artificial construct, intentionally created to model the first" (Fischbein, 1987, p 154)

This type of representation presents important intuitive characteristics Quoting Fischbein again, "Firstly it offers a synoptic, global representation of a structure or a process and this contributes to the globality and the immediacy of understanding Secondly, a diagram is an ideal tool for bridging between a conceptual interpretation and the practical expression of a certain reality" (ibidem)

The use of a diagram though is not automatic. A perfect awareness of the "conversion" principles from one system to another is needed in order to exploit the diagram's potentialities to the full.

In the case of casual compound events with tree graphs as visual aids, in order to fully understand how graphical representations may describe, in figural terms, the conceptual relationships connected to an original reality, it is important to remember some definitions They are essential for interpreting the calculation procedure wh`-h as will be seen, is stimulated by the graphical scheme involved.

We say conditional probability of the event A given the event B, with P(B)>0, and it is indicated with P(A/B), the ratio.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

which is obtained measuring A assuming that B is verified and, therefore, limiting the examination from Ω (total set which is interpreted as the certain event) to B as environment set of the eventualities

Two events A and B are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$. If A and B are independent we obtain that: $P(A/B) = P(A)$

From the definition of conditional probability we obtain

$$P(A \cap B) = P(A/B) \quad P(B)$$

which allows us to calculate more easily the first member, namely the probability that A and B both occur.

If the certain event is divided into disconnected parts E_1, E_2, .., E_n, representing the n possible alternatives, each set A can be seen as the separate union of $A \cap E_1$, $A \cap E_2$, . . , $A \cap E_n$, therefore for the additive property of probability and for the identity seen above we obtain:

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + .. + P(A \cap E_n) = P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2) + . + P(A/E_n) \cdot P(E_n)$$

Therefore the probability of A is the addition of the products of the probabilities of the single alternatives for the corresponding conditional probabilities of A assuming that the given alternative is realised.

This "rule" is effectively illustrated by a tree graph: from a first vertex we draw as many branches as the possible alternatives, from each vertex branches are then drawn which represent the following possible alternatives. Next to each branch the probability of that alternative is marked, namely its conditional probability given the event described at the vertex from which the branch originates.

In order to calculate the probability of an event, the paths on the graph which are relative to the event have to be considered, from the initial vertex down to the end, for each path the product of the various probabilities met has to be calculated and, finally, the contributions obtained have to be added (see the solutions to the problems presented in 4.).

Once more, it has to be underlined that the branches descending from each vertex must represent events which are a partition of the "total" event relative to that vertex. Therefore for each vertex, the possible events being indicated with E_1, E_2, ...E_n as done before (in relation to the vertex considered) and the "total" event with Ω, applying the usual set notation, the former request is translated into the verification of the two conditions:

1) $E_1 \cup E_2 \cup ... \cup E_n = \Omega$ 2) $E_i \cap E_j = \emptyset$ if $i \neq j$

It is, therefore, clear that at each vertex the sum of the probabilities marked on the branches descending from that vertex must be 1.

3. THEORETICAL BACKGROUND IN THE EIGHT INVESTIGATED CLASSES

In this paragraph the essential lines of the didactic proposal on probability presented to all the students from the eight 3rd year Junior Middle School classes (13-14 years old) are described. The students were given two probability problems which will be examined in detail in the following paragraphs. The description will help to outline the students competence and the strategies they used to solve the problems mentioned.
In the eight classes considered, probability had been introduced by the teachers since the first year; the text "Statistics and Probability in the Junior Middle School: a didactic proposal" (Pesci, Reggiani, 1987) had been used.
Due to the students' age (11-14 years), neither axioms nor theorems were given, but everything had to be drawn from examples. In the finite case the definition of probability as the ratio between the number of favourable cases and the number of possible cases comes forth from the analysis of some examples in which one tries to evaluate, the composition of two or more containers being known, if it is more convenient (for a given result) to extract from one container rather then from another.
The book includes many work-cards to be completed by the students who can, in this way, discover the properties by themselves starting from suitable examples and through discussions guided by the teacher.
The propositions relative to the properties introduced in the work-cards are the following (obviously, this is not the form in which they were introduced to the students):

$P(A \cup B) = P(A) + P(B)$ if $A \cap B = \varnothing$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(\varnothing) = 0$
$P(\Omega) = 1$
$P(CA) = 1 - P(A)$

where A,B indicate events, Ω indicates the certain event, $\varnothing$ the impossible event and CA the event opposite to A. The connection with logic is foreseen for the use of the connectives "and", "not", "or".
As far as the study of casual compound events is concerned, neither independent events nor conditional probabilities are dealt with in this didactic proposal, but conditional probabilities are tacitly used with the graphs. More precisely, the situations the students were introduced to (as well as the problems in 4.) can always be regarded as successive extractions from a container with or without putting the balls back to the container. Once an adequate tree graph has been built, the (conditional) probabilities can be associated to the various branches and the rule of "multiplying along the branches" (described in 2.) comes forth spontaneously, in the students, from the concept of "fraction of fraction". For further theoretical details and for the underlying didactic choices see Pesci, Reggiani (1987) mentioned above, which also contains the collection of work-cards presented to the students.

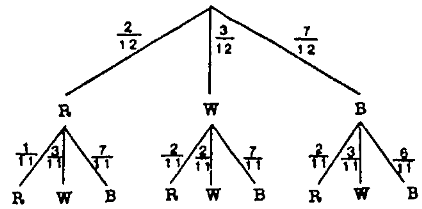
4. THE PROBLEMS AND THE PRIOR ANALYSIS

The text of the two problems is the following:

1. In a sack there are 12 balls: 2 red, 3 white and 7 black. If I draw one ball:
a) What is the probability that it will be black?
b) What is the probability that it will be white or red?
c) Can you find an event with the probability 15/12?
If I draw two balls, without putting the first back in the sack:
d) What is the probability that they are both black?
e) What is the probability that they are of different colours?

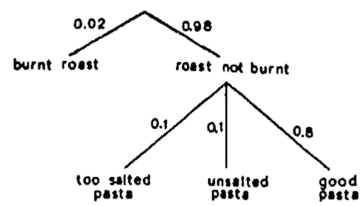
2. In a restaurant the probability that the cook will burn the roast is 0.02, the probability that he will forget to salt the water for the pasta is 0.1, and that he will salt it too much is also 0.1.
What is the probability that the lunch prepared by the cook will turn out well (supposing that he does not make other errors)?

The first problem is fairly easy as far as points a), b), and c), are concerned, whereas points d) and e) are more complicated as they foresee a second extraction.
With the following scheme, the five points for the problem can be immediately answered:



The solutions are the following (the meaning of the symbols being obvious):
a) P(B) = 7/12 b) P(R or W) = 2/12 + 3/12
c) none of the events has probability 15/12 d) P(BB) = 7/12 . 6/11
e) P(different colour) = P(RW) + P(RB) + P(WR) + P(WB) + P(BR) + P(BW) =
= 2/12 3/11 + 2/12 7/11 + 3/12 2/11 + 3/12 7/11 + 7/12 · 2/11 + 7/12 · 3/11 = 41/66 or
P(different colour) = 1- (P(RR) + P(WW) + P(BB)) = 1- (2/12 1/11 + 3/12 2/11 + 7/12 · 6/11) = 41/66

In order to best point out the connection with what was illustrated in 2 , it has to be noted that at the second level of the graph (corresponding to the second extraction), for instance, the event marked R with probability 1/11 is, actually, the event "R given R" and using the symbols introduced in 2, it is P(R/R) = 1/11. Multiplying 2/12 for 1/11 we have P(R) P(R/R) that is P(R¬R) which we indicate with P(RR).
Even the second problem can be schematized with a tree graph as the following:



We have P(meal will turn out well) = 0 98 0 8 = 0.784

There is no need to complete the graph beyond the event "burnt roast" because what concerns us is "meal will turn out well", therefore we will read the result starting from the event "roast not burnt".
It is also possible to build a graph considering the three possible events concerning the cooking of the pasta exchanging the two levels of the previous graph.
It is to be noted that when attributing the probabilities to the various branches, the "level" of the branches is not taken into consideration in our situation the events regarding the roast are supposed not to be influenced at all by those regarding the pasta and vice versa (they are independent events). Therefore the situation is different from the one proposed in the first problem in which, on the contrary, the first level events (first extraction) must be taken into account when attributing the probabilities to the second level branches (second extraction).

From the prior analysis of the two problems, with reference to their solution by means of a tree graph, the following substantial differences emerge:
a) In the first problem, the scheme in the two successive spatial levels reflects the temporal sequence of the events (first and second extraction). This does not happen with the second problem in which the graphical scheme is only a "theoretical" combination of the events involved (those regarding the pasta and those regarding the roast) so much so that the two levels, as noted above, can be exchanged.
In the second problem there is no natural analogy between the tree graph and the problematic situation proposed. The graphical representation (and it happens in other cases, for instance with Venn's diagrams) can

be said to be the description, "in figural terms", of "conceptual relationships which, in turn, are the symbolic processed expression of an original reality" (Fischbein, p.158). Such description reflects the reality represented in a more or less direct way. In the case of the two problems involved, we can say that the representation with the tree graph is more "analogical" than "propositional" in the first problem whereas in the second problem it is the opposite (the terms "analogical" and "propositional" are used in the sense which is peculiar to mental images, see for instance Palmer, 1978).

b) In the tree graph of the first problem, three branches originate from each vertex, namely each time the partitions have the same number of elementary events, and the elementary events are the same at each level (extraction of black, white and red ball) while in the second problem there is a different number of branches from vertexes (the partition events given in the text are two for the roast and three for the pasta) and also the elementary events are different at the different graph levels. Therefore, in the second problem the task is made more complex by a global "difference" between the two levels.

5. SOLUTION AND ERROR ANALYSIS

The test was proposed to 151 students While correcting the problems, calculation errors were not taken into account if the tree graph had been correctly done and if the probabilities requested had been expressed in the right way

5.1 First problem

Ninety-eight students (65%) answered correctly to all of the three questions a), b) and c), whereas only 46 students (30%) did the whole exercise exactly
For the sake of brevity, only the most salient observations, and those strictly connected to the use of tree graphs, are reported here. The solutions relative to questions d) and e), concerning a double extraction without putting the ball back to the container, will be examined.
As to point d), among the 80 correct solutions (53%), there are 78 solutions which, in conformity with the preparation received, used a tree graph. The non-given solutions are 31 (21%) and the wrong ones are 40 (26%)
Among the 40 incorrect solutions, 20 present a tree graph, the other 20 do not
Among the 20 students who make a mistake and do not use a graph, 13 write only P(BB) = 6/11 without any explanation, the other 7 provide other incorrect fractions without any explanations.

As to the 20 incorrect solutions which use a graph, it must be pointed out that they all have a correct graph (three branches at the first level, nine at the second) but the answers are wrong for the following reasons:

- in 15 solutions, wrong values attributed to the branches are multiplied. The most frequent error is due to a reasoning based on the mistaken assumption that before the second extraction the ball had been put back in the container (6 students answer P(BB) = 7/12 · 7/12); moreover, some others consider the total number of balls to still be 12 in the second extraction (4 students answer P(BB) = 7/12 · 6/12). On the other hand, two other students maintain the number of black balls in the second extraction constant (they answer P(BB)=7/12·7/11). The last three students answer with incorrect fractions, difficult to interpret (respectively P(BB) = 7/24 · 6/23, P(BB) = 6/12 · 6/11, P(BB) = 7/12 · 8/11);

- in 3 solutions the probability of the two branches involved are added instead of being multiplied.

- in 2 solutions only the second branch of the BB path is considered and the solution is then P(BB) = 6/11: it is probably the same type of "skipping of steps" seen in the 13 solutions already examined which do not use a graph and provide the same answer.

As to point e) all the 61 correct solutions (40%) point out a proper use of the tree graph The non-given solutions are 48 (32%) and the incorrect ones are 42 (28%): among the incorrect solutions, the tree graph is not used in 5 whereas it is used in 37.

For the purpose of our study it is interesting to examine in detail the errors made by the 37 students (24%) who use the tree graph:
- the attribution of incorrect probabilities to second level branches is the most frequent mistake, after the graph with the nine resulting events has been correctly constructed (14 solutions) This implies that the overall probability, though correctly set out, results incorrect the errors in the attribution of incorrect probabilities are those already considered for point d).

- in 9 solutions there is a tree graph with only two events "black ball" and "non-black ball" in both extractions. obviously with such a partition, if it is possible to answer question d), it is still impossible to determine the event in point e);

- in 7 solutions there is a correct graph and the various branches are attributed correct probabilities but then only some probabilities of the six events concerned are summed.

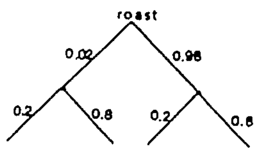
- in 3 solutions the probabilities of the six events considered are calculated separately, but then they are not summed,

- in 3 solutions the tree graph referring to the first extraction is correct but at the second level there are only two events instead of three.

- finally, in 1 solution, although the graph and the probabilities on the branches are correct, there is only the answer P(different colour) = 22/11 without any further details to interpret the error

5.2 Second problem

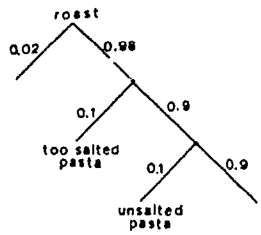
The problem was correctly solved by 19 students (13%), it was not solved by 31 students (20%) and it was incorrectly solved by 101 students (67%) we conclude that the problem resulted very difficult
Among the 19 correct solutions, 2 are obtained without using the tree graph both students write P = (1 - 2/100) (1 - (1/10 + 1/10)), the other 17 are obtained using the tree graph. It must be pointed out that, among these, 8 present the tree graph of the solution described in 4 whereas the other 9 present the following "contracted graph"



in which the two branches referring to "too salted pasta" and "unsalted pasta" are effectively represented by one branch only

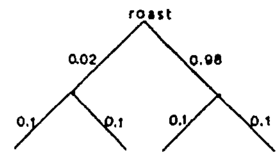
Among the 101 incorrect solutions, 61 do not use a tree graph and 40 use it
In reference to the 61 solutions aforementioned
- the most common error (39) is the following the probabilities given in the text are summed, that is 0 1 + 0 1 + 0.02 = 0.22, the next operation is 1 - 0.22 = 0 78 and then the students conclude that the probability for "meal will turn out well" is 0.78. This is incorrect because probabilities of events which are not disconnected are summed together;
- the same procedure is adopted by another 8 students but they only write 0 1 + 0 02 = 0.12 and they obtain 0.88;
- the remaining 14 incorrect solutions, in which the tree graph is not used, often present apparently "sensible" verbal explanations. Errors are sometimes difficult to interpret in some solutions not all the events are considered, or probabilities of non-disconnected events are summed, or probabilities of independent events are summed instead of being multiplied.

Now the incorrect solution typologies adopted by the 40 students who use a tree graph will be examined
- The most frequent error, found in 12 solutions, is the construction of a three floor graph like the following:



The solution provided by the students in this case is 0 98 0.9 0 9. They did not understand that the events "too salted pasta" or "unsalted pasta" belong to the same partition therefore they must be at the same level.

- Another example of an incorrect graph, made by 7 students is



with the solution P = 0.98 0.1 = 0 098 in this case the partitions at the second level are not complete, in both cases the branch referring to "good pasta" is missing

- Eight students write on three branches at a first level the probabilities given in the text and then 3 students do not continue with the graph and answer P = 1 - (0 02 + 0 1 + 0 1), other 3 students repeat the same branches at the second level and then "combine" the probabilities in a strange way, the last 2 students simulate, at the second level, a sort of extraction decreasing by 1 the denominators and varying the numerators in an inexplicable way

- Four students draw two separate graphs, one for the pasta (three branches for each vertex) and another one for the roast (two branches for each vertex), with two levels for each graph and with the same probabilities of the first level on the second level branches These students probably think of a second extraction and "combine" inexplicably the various probabilities

- Three students draw a tree graph which presents, at each level, both the events referring to the pasta and the events referring to the roast and they combine (with + e ·) the various probabilities unjustifiedly.

- It has to be noted that 2 students draw a correct graph, they attribute the various branches the right probabilities but in one case the exercise remains unfinished, whereas in the second case the student selects an incorrect path.

-The last 4 students present incorrect graphs with different errors from the ones described so far It does not seem useful to give further details regarding these errors but instead to make more general observations.

If we read the data reported here we can observe that among the 120 students who solved the problem, 63 did not use a tree graph and only by two of them the problem was correctly solved, 57 students used a tree graph and 17 of them gave the correct solution.
Notwithstanding the difficulty of translating the problem into a tree graph (already highlighted in the prior analysis in 4. and confirmed by the list of errors) this strategy was shown to be the most effective. It must be observed that, in theory, (see 3.) the students had all the elements needed to give an "abstract" solution, so much so that there are 2 solutions of this kind.

6. CONCLUSION

From our analysis it emerges that the students immediately use tree graphs for solving problems like the first in which the various graph levels represent the event's temporal sequence. As a matter of fact, the larger

number of correct solutions were obtained in the first exercise. Errors were mainly due to incorrect probability attribution to the various branches or to the lack of consideration of the whole paths for the events requested rather than to the drawing of the graph.
The second problem resulted more difficult to the students because the schematization of the given situation in a tree graph does not come as naturally as it does in the first problem (see 4.)
Understanding "the functioning" of tree graphs is not difficult for students, it is more difficult to draw them correctly in order to represent situations which, as in the second problem, cannot be "spontaneously" translated into tree graphs.
In this case the same graph drawing was the cause of many errors and the tree graph did not properly represent the given situation. When the conceptual situation is not thoroughly understood and an inadequate graphical model is built, the student is misled by the automatism induced by the model - i.e. "multiplication along the branches" - and he obtains an incorrect result.
From the analysis presented here we can infer that the tree graph was, in any case, an effective aid even in the solution of the second difficult problem: it can be useful to introduce students to the study of casual compound events. It is clear, though, that the "rules of the game" (properties 1) and 2) in 2.) underlying the drawing of the graph and which make it a very functional aid must be skilfully mastered. Only in this way the graphical aid can become a suitable representation of a problematic situation and it can provide an immediate visual solution.
It is certain that the dialectics between the understanding of the problematic situation and its graphical representation should be played at its best, with mutual advantage of the two moments.

REFERENCES

Bowie P. G. (ed.), 1968, *The School Mathematics Project. Book 4*, Ch. 13, Cambridge University Press

Cundy H. M. (ed.), 1968, *The School Mathematics Project, Advanced Mathematics. Book 2*, Ch.20, Cambridge University Press.

Dörfler W., 1991, Meaning: image schemata and protocols, *Proceedings PME XV*, Assisi, Vol. 1, 17-32

Engel A., 1972, L'insegnamento della probabilità nelle scuole medie, *L'insegnamento della Matematica*, Vol 3 n.2-3-4, 53-110, transl. by C. Sitia from *International Journal of Mathematical Education*, vol 2, n. 3, 1971.

Feller W., 1968, *An Introduction to Probability Theory and Its Applications*, New York, Wiley & Sons, vol.I.

Fischbein E., 1987, *Intuition in Science and Mathematics*, Dordrecht, Reidel.

Gnedenko B., 1979, *Teoria della probabilità*, Roma, Editori Riuniti

Goldin G. A., 1992, On developing a unified model for the psychology of mathematical learning and problem solving, *Proceedings PME XVI*, Durham, Vol. III, 235-261

Kemeny J. G., Snell J. L., Thompson G. L., 1968, *Matematica ed attività umane*, Vol. I, Ch. 4, Feltrinelli, Milano, Translation of Introduction to Finite Mathematics, Prentice- hall, Inc. Englewood Cliffs, N.J 1966.

Mariotti M. A., Pesci A., 1992, Visualization in problem solving and learning, Discussion Group, *Proceedings PME XVI*, Durham, vol I, 22.

Palmer S. E.,1978, Fundamental aspects of cognitive representation, in Rosch E., Lloyd B B., (eds.) *Cognition and Categorization*, L. E. A., Hillsdale.

Pesci A., Reggiani M., 1987, *Statistica e Probabilità nella scuola media inferiore: una proposta didattica*, Quaderno n. 1 Progetto Strategico del C.N.R. Tecnologie e Innovazioni Didattiche.

Pintacuda N., 1983, *Primo corso di Probabilità*, Ch. 2, Muzzio, Padova.

Varga T., 1976, *Giochiamo alla matematica!*, Vol. 2, O. S., Firenze, Translated by Varga T. and Brusa I. from the original edition, Budapest, 1972.

PROSPECTIVE ELEMENTARY TEACHERS' CONCEPTUAL AND PROCEDURAL KNOWLEDGE OF FRACTIONS

George Philippou & Constantinos Christou
University of Cyprus, Department of Education

One of the most widely offered explanations of why students do not learn mathematics is the inadequacy of their teachers' knowledge of mathematics (Fennema & Franke, 1992, p.147).

Prospective elementary teachers' procedural and conceptual knowledge of fractions was investigated through a written instrument. Results indicated that these preservice teachers had a narrow understanding of the ideas underlying the conceptual knowledge of fractions. The findings suggest that prospective teachers need programs for their mathematical preparation that emphasize the connections between procedural and conceptual knowledge of fractions.

In recent years much research has been reported on students' procedural and conceptual understanding of fractions and decimals (Harrison & Greer, 1993; Vergnaud, 1983). Though the basic concepts of fractions are generally introduced from the primary grades, there is convincing evidence that most middle school students do not create appropriate meanings of fractions (Bezak & Bieck, 1993; Kieren, 1988; Ohlsson, 1988). Among the reasons for this insufficient learning, the mathematical nature of fractions and poor instruction seem to be the most widely mentioned. There exist fundamental differences between "whole numbers" and fractions regarding both the symbolic representation and their interpretations, which produce long-term difficulties (Hiebert & Behr, 1988). The one-to-one correspondence between a whole number and its symbol, in fractions becomes one-to-many, whereas the limited number of interpretations of "whole number" is in rational numbers expanded to include at least: the part-whole, the ratio, the quotient, and the operator interpretations (Kieren, 1988). Despite the complicated nature of the rational number concept and related terms which constitute a distinct semantic field (Ohlsson, 1988), they usually receive little attention and are often taught in a meaningless manner. Most teachers proceed to instruct procedural rules before strengthening students' conceptual understanding.

Fennema and Franke (1992) claimed that "the evidence is beginning to accumulate to support the idea that when a teacher has a conceptual understanding of mathematics, it influences classroom instruction in a positive way" (p.151). It is evident that if teachers themselves have difficulties they are not likely to facilitate the construction of meanings for fractions by their students or to recognize related errors the students make (Graeber, Tirosh & Glover, 1989). The insecurity felt by teachers with limited knowledge in this area was expressed by one of them as follows "I had to teach mathematics to the 4th Grade and I didn't know how to multiply fractions" (Fredericou & Folerou, 1991, p.66).

Despite the concern expressed about "proper" instruction in fractions it seems that preservice elementary teachers' knowledge in this area has not been studied enough. The purpose of our research program was to explore in depth preservice elementary teachers' knowledge, mathematical power and attitudes at the point when they enter teacher education, as a means to design mathematical courses suitable to their needs. This particular study focuses on preservice teachers' procedural and conceptual understanding of fractions.

Conceptual Background of the Study

The underpinning of this work is the conceptual and procedural knowledge as it was defined by Hiebert and Wearne (1986). Conceptual knowledge involves building relationships between existing bits of knowledge and it grows through the creation of relationships between existing knowledge and new information that is just entering the system. Procedural knowledge is composed of the formal language of mathematics and the rules, algorithms and procedures needed to solve mathematical tasks. A host of research studies agree that conceptual knowledge is more significant than procedural knowledge (Silver, 1986). However, teachers should be competent in both the conceptual and procedural knowledge in order to teach effectively mathematical ideas.

Mathematical knowledge includes significant, fundamental relationships between conceptual and procedural knowledge. Critical links between conceptual and procedural knowledge would contribute in many ways to the development of a sound knowledge base. According to Hiebert and Wearne (1986), there are three phases where links between conceptual and procedural knowledge take on special importance. In phase 1, a person gives meaning to the numerical and procedural symbols of a mathematical task. The meanings assigned to the symbols may come from the syntax or they may come from connecting symbols with their conceptual referent. Phase 2 involves the procedures selected to solve a problem and emphasizes the links between procedures and their conceptual rationales. Phase 3 involves the solution evaluation and emphasizes the reasonabless of solutions. The three phases of symbol interpretation and algorithms, procedural execution, and solution evaluation provide a scheme that allows the analysis of the existence of relationships between conceptual and procedural knowledge. In this study, we used the basic ideas of the three phases to describe prospectives teachers' conceptual knowledge of fractions.

Purpose of the Study

The work reviewed to this point led to the study's first research question regarding the present state of preservice teachers' conceptual and procedural knowledge of fractions and the degree to which preservice teachers connected their procedural and conceptual knowledge of fractions. Furthermore, the study purported to determine if there were differences between preservice teachers coming from different types of secondary education with or without additional emphasis on mathematics and science. Specifically, the following questions guided this study:

1. What is the present state of preservice teachers' knowledge of fractions?
2. Do preservice teachers differ as regards their procedural and conceptual knowledge of fractions?
3. Do precollege mathematics explain the differences among preservice teachers on their procedural and conceptual understanding of fractions?

Method

Subjects

The subjects were 83 preservice elementary school teachers enrolled in the first semester of their studies at the University of Cyprus. Eighteen of them were graduates from lyceum types where mathematics was not among the subjects of specialization. Twenty one of them took, during their studies in lyceum, both mathematics and physics as their major subjects, while the rest (44) took mathematics and economics.

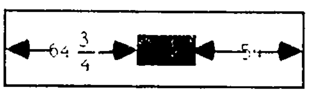
Instrumentation

A test of 30 questions was constructed to assess preservice teachers' knowledge of fractions as well as their ability to connect procedural and conceptual knowledge. Twelve items were designed to assess preservice teachers' procedural knowledge and 18 items to assess their conceptual knowledge of fractions. The items for conceptual knowledge were designed to measure the connection between real-world situations and symbolic computation, the connection between whole number and fraction operations and the solution evaluation. Specifically, five items were designed to assess preservice teachers' ability to connect the symbolic computation with real-world situations (Phase 1). These items provided the numerical expression and asked for posing word problems (Simon, 1993). Seven items were designed to assess preservice teachers' ability to select and apply the appropriate procedures to solve mathematical tasks (Phase 2). These items also involved questions designed to assess the ability to connect whole number operations with fraction operations. Finally, six items were designed to assess the ability of preservice teachers' to evaluate the reasonableness of a solution or an answer given to a problem task. Table 1 provides examples of selected items that clarify the above mentioned categories.

Results

To answer the first question of the study about the present state of preservice teachers' knowledge of fractions, subjects' success rates are summarized in Table 2. The subjects had the greater success on addition and division and the least success on multiplication. Although multiplication is considered as prerequsite to division, the subjects in this study demonstrated quite a different behavior. The written work of the subjects provided some hints that they considered multiplication and division as different operations and could not see the connection between them. Evidence for this result is provided by the subjects' strategy of solving division problems. Most of the subjects (70%) used the mnemonic rule of complex fractions to provide an answer to division exercises and thus tried to remember a different method from that used in solving the exercises on multiplication.

Table 1
Examples of Items Used in the Study

Problems	Knowledge assessed
1. (a) $^1/_5 + ^7/_5$, (b) $16 - ^3/_5$, (c) $^1/_2 \times 4^1/_2$, (d) $^1/_3 \div ^1/_6$ (e) In fraction multiplication the answer is always greater than the factors. **Yes No**	Procedural
2. Write three different story problems that would be solved by dividing 51 by 4 and for which the answers would be respectively: (a) $12^1/4$ (b) 13 (c) 12 (Simon, 1993).	Connection between real-world situations and symbolic computation
3. (a) Find the remainder of the division: $28 \div 3^1/_2$. (b) How far to the left should the picture be moved so that it is centered on the wall? (Cooney, Badger, Wilson, 1993). 	Connection between whole number operations and fraction operations. Selection of the appropriate procedure to solve the problem.
4. The product of the multiplication $6^3/4 \times 4^3/7$ is between: (a) 33 and 40, (b) 29 and 32, (c) 18 and 24, (d) 25 and 28.	Solution Evaluation

The least success of preservice teachers was on the items measuring their ability to connect real-world situations and symbolic computation (25% of success). The majority of the subjects (75%) failed to construct problems that could be represented by a given number expression, indicating that preservice teachers did not possess a broad concept of the operation of fractions to make sense of fractions. Although numbers were kept simple, preservice teachers were not able to connect the symbolic computation with real world situation. This is in agreement with Ball's (cited by Fennema & Franke, 1992) finding that the majority of the preservice elementary teachers were unable to develop an appropriate representation of the operation $1^3/4$ divided by $^1/2$.

Table 2
Summary of Procedural Knowledge Data (expressed as rounded percentages).

Types of Problems	Correct Answer (%)
Interpretation of Symbols	88
Algorithms: Addition	90
Subtraction	88
Multiplication	74
Division	90
Connection Between:	
*real-world and computations (C1)	25
*whole number and fraction operations (C2)	39
Solution Evaluation	56

A large proportion (61%) of preservice teachers could not recognize the features of whole numbers that are similar to fractions and those that are unique to whole numbers. They did not easily extend concepts that applied to fractions, and they overextended features that did not apply to fractions. For example, 96% of preservice teachers failed to give the correct answer to the problem "Niki has 35 cups of sugar. She makes cakes that require 3/8 of a cup each. If she makes as many cakes as she has sugar for, how much sugar will be left over?"

Fifty six per cent of the subjects answered correctly the items on solution evaluation. However, a large again, number of the subjects (44%) were not able to evaluate the reasonableness of the answer. For instance, 44% of preservice teachers failed to answer correctly the problem " The product of the multiplication $6^3/4 \times 4^3/7$ is between: (a) 33 and 40, (b) 29 and 32, (c) 18 and 24, (d) 25 and 28", indicating that these subjects may have weak understanding of the size of fractions.

To answer the second question of the study whether the preservice teachers exhibit the same performance on procedural and conceptual knowledge of fractions, a multivariate analysis of variance (MANOVA) was conducted with procedural and conceptual knowledge as dependent variables. It showed that the preservice teachers' procedural knowledge perfromance (M=0.887, p=.00) was significantly better than their conceptual knowledge (M=0.398, p=.00). This result may indicate that for these preservice teachers the procedural and conceptual understanding of fractions remain as separate entities.

Finally, to answer the third question of the study, a multivariate analysis of variance was conducted with procedural knowledge, and connectness as dependent variables and the type of lyceum from which they graduated as independent variable. Table 3 shows the means and standard deviations of the performance of preservice teachers, grouped according to the type of lyceum from which they graduated, on procedural and conceptual knowledge.

Table 3
The comparison of Performance by Type of Secondary School.

Knowledge	Major	M	S.D.	Significance (p)
Procedural	Greek Literature and Latin	0.88	0.15	.81
	Maths and Physics	0.90	0.12	
	Maths and Economics	0.89	0.12	
C1	Greek Literature and Latin	0.30	0.15	.06
	Maths and Physics	0.33	0.22	
	Maths and Economics	0.31	0.18	
C2	Greek Literature and Latin	0.39	0.15	.07
	Maths and Physics	0.44	0.15	
	Maths and Economics	0.42	0.16	
Solution Evaluation	Greek Literature and Latin	0.51	0.20	.74
	Maths and Physics	0.57	0.29	
	Maths and Economics	0.57	0.28	

As can be seen from Table 3, there were no statistically significant differences among preservice teachers regarding the type of lyceum of graduation. This indicates that precollege mathematics, as taught in Cyprus, may not be a critical variable for explaining the difficulties the preservice teachers encounter in grasping the meaning of fractional concepts. Specifically, it seems that the mathematics taught in lyceums may not help students to conceptually develop fractions

Discussion

This study extends the information available on prospective teachers' knowledge of fractions. It offers evidence as to the usefulness focusing on conceptual and procedural knowledge. The prospective elementary teachers in this study exhibited serious short-comings in their understanding of fractions. They seemed to have appropriate knowledge of the symbols and algorithms associated with fractions, but many important connections seemed to be missing. These findings are consistent with those of Ball (cited by Fennema & Franke,1992; 1990) and Simon (1993) on prospective teachers' inability to develop problem representation of fractions and insufficient knowledge of division.

Although most of the preservice teachers could calculate correctly, they had significant difficulty with the meaning of operations of fractions indicating a narrow understanding of the concepts underlying the procedural knowledge. The prospective teachers' understanding appeared to comprise remembering the rules for specific cases. Evidence for this conclusion is especially clear in the division and multiplication of fractions. Division and multiplication with fractions is rarely taught conceptually in schools (Ball, 1990); most of the prospective teachers probably learned to divide or multiply with fractions without necessarily thinking about what the problems meant. Indeed, most of them could carry out the procedure to produce the correct answer. Yet when they tried to pose word problems that would be solved by a provided numerical expression most of them failed. They also seemed to be unable to connect the symbolic computations with real world contexts (Simon, 1993). This was explicit by the low percentage of the subjects who answered correctly the relevant items of the test. For example, data indicated that subjects did not readily see how the answer to the division 51 divided by 4 would be constrained by different real world situations and that it is possible in some cases to give $12^3/4$ as an answer and in some other cases 13 or 12. It is likely that their previous experience had led them to believe that the answer was determined by the directions given by the teacher or the textbook.

A further demonstration of the lack of connection between procedural and conceptual knowledge is found in the data for problems purported to measure the ability of subjects to connect knowledge of whole numbers with fractions. Subjects were unable to interpret or misinterpreted the remainder of the fractional part of the quotient. This was explicit by their answers to problems requesting the remainder of division of fractions. According to Simon (1993) this is probably an indication that the subjects did not know how to identify the appropriate units. It seems likely that the handling of remainders and fractional quotients in school was a procedural matter which did not help the generalization of the appropriate whole number concepts.

The conceptual weakness demonstrated provides an assessment of precollege mathematics. It seems that lyceums provide students with procedural knowledge which is sparsely connected. It is likely that students in lyceums lack not only certain understandings but also a vision of the kind of understanding that is possible and appropriate in the study of fractions and perhaps in mathematics in general. The present study highligths that relying on what prospective teachers have learned in their precollege mathematics classess is unlikely to provide a sound teacher preparation. Mathematics courses must be refocussed in order to facilitate the making of connections much more than the imparting of additional information.

REFERENCES

Ball, D. (1990). Prospective elementary and secondary teachers' understanding of division. Journal for Research in Mathematics Education, 21, 132-144.

Bezuk, N.S. & Bieck, M. (1993). Current research on rational numbers and common fractions: Summary and implications for teachers. In D.T. Owens (Ed.). Research ideas for the classroom: Middle grades mathematics (pp.118-136). Macmillan. NCTM.

Cooney, T., Badger, E., & Wilson, M. (1993). Assessment, understanding mathematics and distinguishing visions from mirages. In N. Webb & A. Coxford (Eds.), Assessment in the mathematics classroom. Reston, Va: NCTM.

Fennema, E. & Franke, M.L.(1992). Teachers' knowledge and its impact. In D.A. Grows (Ed.). Handbook of research in mathematics teaching and learning (pp. 147-163). Reston, Va: NCTM.

Graeber, A., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. Journal for Research in Mathematics Education, 20, 95-102.

Harrison, J. & Greer, B. (1993). Children's undersatnding of fractions in `Hong Kong and Northen Ireland. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F.Lin (Eds.), Proceedings of the Seventeeth International Conference (vol.III, pp. 146-153). Tsukuba, Japan: University of Tsukuba.

Hiebert, J. & Behr,M. (1988). Capturing the major themes. In J. Hiebert & M. Behr (Eds.). Number concepts and operations in the middle grades .Reston, Va: NCTM.

Hiebert, J. & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In (J. Hiebert (Ed.). Conceptual and procedural knowledge: The case of mathematics (pp. 199-225). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Fredericou, A. & Folerou, F. (1991). Teachers of the elementary school. Athens:Y-Books. (In Greek).

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.). Number concepts and operations in the middle grades (pp.162-181). Reston, Va: NCTM.

Ohlsson, S. (1988). Mathematical meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert & M. Behr (Eds.). Number concepts and operations in the middle grades (pp. 53-92). Reston, Va: NCTM.

Silver, A.E. (1986). Using conceptual and procedural knowledge: A focus on relationships. Conceptual and procedural knowledge: The case of mathematics (pp. 181-198). Hillsdale, NJ: Lawrence Elrbaum Associates, Publishers.

Simon, A.M. (1993). Prospective elementary teachers' knowledge of division. Journal for Research in Mathematics Education, 24, 233-254.

Vergnaud, G. (1983). Multiplicative structures. In R.Lesh & M.Landau (Eds.), Acquisition of mathematics concepts and processes (pp.127-174). London:Academic Press.

ATTENDING TO UNCONSCIOUS ELEMENTS

David Pimm
The Open University

Psychoanalysis is part of psychology. It is not medical psychology in its traditional sense, nor the psychology of pathological processes. It is psychology proper; certainly not all of psychology, but its substratum, possibly its very foundation.
(Sigmund Freud)

Mathematical science ... is the language of unseen relations between things. But to use and apply that language, we must be able fully to appreciate, to feel, to seize, the unseen, the unconscious.
(Ada Lovelace)

Mathematics is a disembodied version of the human personality.
(Donald Winnicott)

This paper is about exploring some of the implications of these quotations.

For much of this century (from at least the time of Piaget and Thorndike), the development of mathematical concepts, particularly early mathematical concepts, has apparently held a fascination for psychologists. Increasingly, this preoccupation has grown to the point where some psychologists seem to consider this to be their natural domain. Moreover, it is on occasion *rationality* itself that is apparently to be studied by means of an examination of thinking and the use of mathematical concepts (see, for example, Sutherland, 1992). Walkerdine (1988, 1990), a psychologist herself by training, has in some sense both continued this tradition while sharply criticising its presumptions.

Relatively recent developments in *social* aspects of cognition, in particular following the rehabilitation in the West of the work of Vygotsky, have led *outwards*, into the social world, into the learning *community* (see, for instance, Lave, 1988). In philosophy of mathematics too, discussion of the isolated, supremely rational, individual, idealised mathematician has given way to a consideration of communities of mathematicians (see, for instance, Kitcher (1983) or Rotman's (1988) fascinating analysis of the language of proof), as well as to wider, cultural embodiments of patterned thought (see, for example, D'Ambrosio, 1991), often unhelpfully[1] termed 'ethnomathematics'.

My intention here is relatively straightforward. I choose to turn my attention in the opposite direction – inwards, not outwards – not because I think the social to be misplaced as a locus of exploration, but because I firmly believe it is not the only one of interest. An indication of this change of direction has been given in Blanchard-Laville (1991, 1992). In an enticing passage, George Spencer Brown (1977, p. xix) writes of his sense of congruence between mathematical and psychoanalytical activity.

> In arriving at proofs, I have often been struck by the apparent alignment of mathematics with psycho-analytic theory. In each discipline we attempt to find out, by a mixture of contemplation, symbolic representation, communion, and communication, what it is we already know. In mathematics, as in other forms of self-analysis, we do not have to go exploring the physical world to find what we are looking for.

So, for now, I turn away from both the social and rationality and towards a brief exploration of individual 'irrationality' in the context of mathematics – and I use that term deliberately, with both its mathematical and illogical connotations resonating. My focus for this topic for discussion will be language. Paul Ricoeur (1970, p. 4) reminds us: "I contend that the psychoanalyst is a leading participant in any general discussion about language". In particular, psychoanalysis may prove insightfully relevant in that important subset of language, the interplay of symbols and meanings. This is probably the richest arena we have for particular discussion about the teaching and learning of mathematics.

One of the most often cited but seldom explored quotations in mathematics education is the following from the mathematician René Thom (1973, p. 202), offered at ICME 2.

> The real problem which confronts mathematics is not that of rigour, but the problem of the development of 'meaning', of the 'existence' of mathematical objects.

His belief that meaning and existence comprise mathematics education's central problem is reflected in the fact that this is the only one of two italicised sentences in his entire talk.[2] What is seldom pointed out is that the central part of his talk, the one which culminates in the above summary sentence, concerns the role of 'unconscious activity'[3] in mathematics, while the etymology of the word 'existence' invokes something standing (or being placed?) *outside*.

Meaning is, on occasion, primarily about reference alone; but can also be about much more. Meaning is often also be about associations of all sorts (including verbal similarities). Young children seem particularly open to the playful aspects of the *sound* similarities of language, and the surprising connections that can sometimes be made through this version of 'moving along the metonymic chain'. In particular, I believe meaning is partly about unaware associations, about subterranean roots than are no longer visible even to oneself, but are nonetheless active and functioning.

Michel Serres (1982, p. 97), writing of the origins of mathematics, claims:

> The history of mathematical sciences, in its global continuity or its sudden fits and starts, slowly resolves the question of origin without ever exhausting it. It is constantly providing an answer to and freeing itself from this question. The tale of inauguration is that

interminable discourse that we have untiringly repeated since our own dawn. What is, in fact, an interminable discourse? That which speaks of an absent object, of an object that absents itself, inaccessibly.

How are we implicated in making ourselves (and others) unaware of certain connections? Are our referential meanings in mathematics absenting themselves, with or without our express permission? What processes may be at work?

What is on the list of 'windows into the unconscious' that Freud came up with? It includes: slips of the tongue, dreams (see Early, 1992 in relation to mathematics), things you know well but forget or are unable to recall (e.g. names of people), and 'faulty achievements' (where I achieve something faultlessly, but it wasn't 'really' what I intended to do). Are there any particularly mathematical ones? Sherry Turkle (1981, p. 247), at the end of her book on psychoanalyst Jacques Lacan ('The French Freud'), writes:

> For Lacan, mathematics is not disembodied knowledge. It is constantly in touch with its roots in the unconscious. This contact has two consequences: first, that mathematical creativity draws on the unconscious, and second, that mathematics repays its debt by giving us a window back to the unconscious. [...] so that doing mathematics, like dreaming, can, if properly understood, give us access to what is normally hidden from us.

One possibility for such direct access may be through working with geometric forms themselves, those strange self-referentialities for which the symbol and the referent are one: 'the symbol for a circle is a circle'. In a book with the evocative title 'The presence of the past', Rupert Sheldrake (1989) talks of 'morphic resonance', an expression which captures exactly what I am concerned with. The power of forms to evoke has been long exploited by sculptors. For instance, sculptor Barbara Hepworth has recalled "All my early memories are of forms and shapes and textures" and her sometime fellow art-school student Henry Moore observed: "There are universal shapes to which everyone is subconsciously conditioned and to which they can respond".

Mathematical analysis: fidelity

> And how reliable can any truth be that is got
> By observing oneself and then just inserting a Not?
> (W. H. Auden)

One general difficulty I have encountered in writing this article is offering convincing descriptions of (let alone accounts *for*) purportedly unconsciously-influenced phenomena: another is the intensely personal nature of much of this material. However, having acknowledged such reservations, I offer an attempted description and attendant discussion arising from videotaped classroom discourse from a lesson which I witnessed.

A secondary teacher (T) started a lesson with a class of thirteen-year-olds on addition and subtraction of negative numbers by invoking the assistance of an image of 'The Linesman'.

> T: I want to remind you of a little cartoon character that we've been looking at over the last few weeks – the Linesman – our little stick man. And he's been helping us do some calculations using positive numbers and the new ones we've been looking at – the negative numbers. Let me show you a picture of the Linesman – here he is.
> [*He puts up an overhead projector slide.*]
> There, right, remember him? And we've drawn various pictures of him. I want to remind you of how *he* was able to help *us* do some calculations about positive and negative numbers.
> Can anyone remember where his number line came from? Where did he get his number line?
> P: His suitcase.
> T: In his suitcase, yes. How long was it? How long was his suitcase? Lorna?
> L: Fidel [*stumbles*], fidelity.
> T: Fidelity? Fidelity?
> [*He speaks with rising tones of surprise and disbelief.*]
> L: No
> [*She laughs, very embarrassedly, completely hiding her face.*]
> T: How long was the Linesman's number line? David?
> D: As long as you want it to be.
> T: As long as you want it to be. OK. We do decide just to take a piece of it, don't we. But if we wanted to take all of his number line, how long would it be. Gary?
> G: Infinity.
> T: Infinity. [To Lorna] Is that the word you're looking for? What does that mean, 'infinity'?
> Pupil: It never ends.
> T: It never ends, it never ends. Right, let's have a look ...

For me, this brief classroom excerpt offers a plausible example of unconscious processes interfering with conscious language. The sexual *double entendres* of the whole dialogue are rampant, as is the wishful thinking: the 'going on forever', the 'being as long as you want'.[4] Although the words 'infinity' and 'fidelity' both have four syllables and the last two of both are the same, why did Lorna did not say '*in*fidelity', surely a far more direct metonymic association with 'infinity'?

Freud writes:

> The subject matter of a repressed image or thought can make its way into consciousness on condition that it is *denied*. Negation is a way of taking account of what is repressed; indeed

it is actually a removal of the repression, though not, of course, an acceptance of what is repressed. It is to be seen how the intellectual function is here distinct from the affective process. The result is a kind of intellectual acceptance of what is repressed, though in all essentials the repression persists. (1955, p. 235)

Lemaire (1986, p. 75) glosses this as: "The repressed signifier is always present in the negation, but, in another sense, it retains the repression though the 'not'."[5]

My account for this incident must obviously remain highly speculative, partly because so little is above ground. I remain struck by the fact that she did not offer 'infidelity'. Assuming part of her attention (unawarely I expect) was taken by the sexual interpretation of the above dialogue, and also perhaps that the charged term 'infidelity' was playing a role in her home life (even suitcases, perhaps), then the above described process of negation could have turned 'infidelity' into the more acceptable 'fidelity' (two negatives making a positive here at least), even though I doubt she has ever heard the parallel neologism 'finity'. Her stumbling over the first uttering of 'fidelity' could indicate either the emotional force of what was at work, or some half-aware realisation that this was not quite right.

Dick Tahta has offered me an alternative account. It concerns the fact that there are strong resonances and connections between a *meaning* invested in 'infinity' and that of 'fidelity', through the notion of 'going on forever'. He offers the thought that there is a strong adolescent investment in fidelity and a corresponding unease about infidelity. Teenage magazines are filled with storied discussions and explorations of this idea.

It is clearly impossible to ascertain what was the case here. But the two accounts given illustrate two general routes for accounting for such occurrences. The first posits a preoccupation that is being kept down, but a connection is made between this concern and the mathematical topic at hand. The preoccupation finds a way to return to consciousness, but is only allowed to do so in a negated form. The links are primarily metonymic: from 'infinity' to 'infidelity' to 'fidelity', which is what finally surfaces. In the second account, the links are primarily semantic, metaphoric. She cannot produce the word 'infinity' for some reason, but the closest word with a strongly related meaning she can get is 'fidelity'.

But unlike with mathematics' excluding, exclusive use of 'or', the one account need not exclude the other. Metaphor and metonymy are offered as *axes* rather than categories by linguist Roman Jakobson, and hence elements of both can be present at the same time.

The resonance of words

> There seems to be no mathematical idea of any importance or profundity that is not mirrored, with an almost uncanny accuracy, in the common use of words.
> (George Spencer Brown)

Mathematicians 'borrow' many everyday words to describe mathematical phenomena of interest. They also use the same words over and over in order to reflect perceived structural or functional similarities, resulting in condensations of experience around terms such as 'normal', 'similar', 'multiplication', 'number'. Curiously, they seem to expect a conscious, overt denial of links to be sufficient so that such connections are no longer heard. (I recall asking in a graduate theoretical course on tensor analysis what the curvature tensor has to do with curvature and receiving the answer "Nothing".)

But such words can and do evoke connections and links, particularly I believe in cases where the mathematical content does not generate powerful images and feelings of its own. The looseness and gap between symbol and referent, regularly exploited for mathematical ends, also permits such slippage to a far greater extent than in other disciplines. 'Circumscribed' is very 'close' to 'circumcised' and the connection is not arbitrary. Teenage girls working on the period of a function can and do make overt connections with menstrual periods. Adolescents can become preoccupied with freedom and constraints upon themselves, and geometry can offer them the possibility of working with the same material as well as the same terms.

Tahta (1993, p. 48) comments:

> Mathematics teachers do not normally expect to make connections between the *relations* of mathematics and the *relationships* of family life. Indeed, many emphatically deny that there is any connection at all; for them, equality of algebraic expressions would be something quite different from, say, equality of esteem.

What do these allusive stories have in common? The first is the resonant quality of language, both in the forms of words and in the topology of those forms, reflected in the belief that nearby symbols have nearby meanings. Thus, metonymic links or 'slips' have semantic consequences by presumed association.

The second commonality raises for me the whole question of 'meaning' in mathematics, in particular some of its unaware contributory components. One reason for teaching mathematics may be so that our students can develop this means of finding out about themselves, in addition to our offering them access to a shared inheritance of mathematical images and ideas, language and symbolism, and the uses for mathematics which humans have so far developed. To the extent that students are enabled to think like mathematicians, this possibility is made available for them. I would like to express this possibility in terms of mathematics deriving from both inner and outer experiences, and meaning as being generated in the overlapping, transitional space between these two powerful and sometimes competing arenas.

References

Blanchard-Laville, C. (1991) 'La dimension du travail psychique dans la formation continue des enseignant(e)s de mathématiques', *Proceedings of the Fifteenth PME Conference*, Assisi, Italy, vol. 1, pp. 152-9.

Blanchard-Laville, C. (1992) 'Applications of psychoanalysis to the in-service training of mathematics teachers', *For the learning of mathematics*, **12**(3), 45-51.

Chevallard, Y. (1990) 'On mathematics education and culture: critical afterthoughts', *Educational Studies in Mathematics*, **21**(1), 3-27.

D'Ambrosio, U. (1991) 'Ethnomathematics and its place in the history and pedagogy of mathematics', in M. Harris (ed.) *Schools, Mathematics and Work*, Falmer, London, pp. 15-5.

Early, R. (1992) 'The alchemy of mathematical experience', *For the learning of mathematics*, **12**(1), 15-20.

Freud, S. (1955) *Complete Works of Sigmund Freud*, Standard Edition, vol. 19, Hogarth Press, London.

Hadamard, J (1954) *The Psychology of Invention in the Mathematical Field*, Dover, New York.

Kitcher, P. (1983) *The Nature of Mathematical Knowledge*, Oxford University Press, New York.

Lave, J. (1988) *Cognition in Practice: Mind, Mathematics and Culture in Everyday Life*, Cambridge University Press, Cambridge.

Lemaire, A. (1986) *Jacques Lacan*, Routledge and Kegan Paul, London.

Pimm, D. (1993b) 'The silence of the body', *For the learning of mathematics*, **13**(1), 35-38.

Ricoeur, P. (1970) *Freud and Philosophy: an Essay on Interpretation*, Yale University Press, New Haven, CT.

Rotman, B. (1988) 'Towards a semiotics of mathematics', *Semiotica*, **72**(1/2), 1-35.

Serres, M. (1982) *Hermes: Literature, Science, Philosophy*, Johns Hopkins Press, London.

Sheldrake, R. (1989) *The Presence of the Past: Morphic Resonance and the Habits of Nature*, Vintage Books, New York, NY.

Spencer Brown, G. (1977) *Laws of Form*, The Julian Press, New York, NY.

Sutherland, S. (1992) *Irrationality: The Enemy Within*, Constable, London.

Tahta, D. (1993) 'Victoire sur les Maths', *For the learning of mathematics*, **13**(1), 47-8. (Tahta also edited this special issue of the journal, on the theme of the psychodynamics of mathematics education.)

Thom, R. (1973) 'Modern mathematics: does it exist?', in A. G. Howson (ed.), *Developments in Mathematical Education*, Cambridge University Press, Cambridge, pp. 194-209.

Turkle, S. (1981) *Psychoanalytic Politics: Freud's French Revolution*, MIT Press, Boston.

Walkerdine, V. (1988) *The Mastery of Reason*, Routledge, London.

Walkerdine, V. (1990) 'Difference, cognition and mathematics education', *For the learning of mathematics*, **10**(3), 51-6.

---

[1] See Chevallard (1990) for a trenchant and penetrating analysis of some of the difficulties inherent in the notion of 'culture' in mathematics education, and in particular his distinction between 'ethnomathematics' and 'protomathematics', the latter offering fertile sources for mathematisation without itself *being* mathematics.

[2] In the other, he criticises the assumption (of mathematical 'modernists') that:

> By making the implicit mechanisms, or techniques, of thought conscious and explicit, one makes these techniques easier (p. 197).

This is the complete reverse of my belief that automation of functioning and the liberation that this can bring results from the making unconscious of conscious control mechanisms. Thom adds (p. 199):

> Certainly, this detachment [of the thinker from his thought] is a necessary step in the process of mathematical advancement: but the inverse operation, which is the reabsorption of the explicit into the implicit, is no less important, no less necessary.

[3] Hadamard (1954) too devotes two chapters to general issues of the unconscious in the context of mathematical creativity.

[4] I have no doubt this occurred completely unawarely on the teacher's part, and certainly was on mine despite being present at the filming. I had remembered the 'fidelity' remark as a complete singularity – and my memory was that she had said 'infidelity'.

[5] There is also a possible link between proof by contradiction and the psychological process of negation (see Pimm, 1993b). Both produce negative statements with the intent of the opposite. In proof by contradiction, the same structure holds. The assertion is made that not-P is true, but the underlying reading that is required is that P is to be believed.

MATHEMATICAL UNDERSTANDING: ALWAYS UNDER CONSTRUCTION

Thomas Kieren
David Reid
University of Alberta

Susan Pirie
Oxford University

In this paper we trace one of the consequences of observing a person's mathematical understanding as a growing, enfolding phenomenon rather than an as an achieved state. Using the dynamical model for the growth of mathematics understanding developed by Pirie and Kieren over the last 6 years, we contrast the patterns of growth in understanding of two university students as they work in two number pattern situations, illustrating the need for a way to observe such growth in process terms. In one situation their growing understanding can be characterised as growing broadly in a connected way out of actions in the situation, while in the second situation, although they had adequate mathematical knowledge, their pattern of growth in understanding was disjoint and sporadic. The patterns and consequences for research and teaching of such differences in understanding growth are discussed.

Background

"Charlene really has a very good understanding of mathematics!" "I want my class to reach an understanding of decimals." Statements like these, which might be made by teachers, are indicative of a view of mathematical understanding as a state or as a particular attainment. Such a view is well supported in the theoretical and research literature (e.g. Skemp, 1987) as well as the professional literature (e.g. van der Walle, 1990) which discuss categories of states and traits of understanding such as instrumental, procedural, conceptual and relational; or even complexes of such states (e.g. Herscovics , 1992; Miller, Malone and Kandl, 1992). Such a view of understanding as a state or complex of states is useful in that it allows us to think of curriculum materials as providing opportunities for students to attain such states or of such states as providing criteria against which to assess students' mathematical understanding.

While realising the value of thinking of understanding as a state, it is the purpose of this paper to demonstrate the value of an alternative way of looking at mathematical understanding. Rather than thinking of understanding in terms of possibly achievable states, we observe mathematical understanding for a person as always being under construction.

Both Sierpinska(1990) and Wittgenstein(1956) see mathematics and its understanding in an active sense. Mathematical understanding for them is a multileveled activity in which a person perceives and overcomes epistemological obstacles and moves on to a higher level. In a series of papers, including several presented at PME meetings (Pirie and Kieren, 1989; Kieren and Pirie, 1992 etc.), we have tried to consider mathematical understanding as an on-going, growing, process by which one responds to the problem of re-organising one's knowledge structures, possibly in the face of epistemological obstacles, by a process of revisiting one's existing understanding. We have termed this process 'folding back' (Kieren and Pirie, 1994). We have developed an evolving dynamical theory for the growth of mathematical understanding of a person with respect to a topic. Our theory entails talking about understanding in terms of a set of embedded modes of knowledge-building activity, both informal and formal. These modes include primitive knowing, image making, image having, property noticing, formalising, observing, structuring and inventising. Modes relevant to this paper will be described as needed and a representation of the embedding model for them is seen in use in Fig. 3 and Fig. 4 below.

It is not, however, the modes alone which define and detail the growth of mathematical understanding. Rather it is the non-linear pathways of students' behaviours, which can be tracked through the modes, which illustrates such dynamical growth most clearly seen in our concept of 'folding back'. At any point in one's knowledge building process one can fold back to a less formal way of understanding a particular topic to gain more experience or to elaborate previous images in order to broaden one's basis of understanding for a topic. We have observed that such folding back occurs not rarely but frequently in the process of understanding and gives the pathways of the growth of understanding, for a person, their non-linear forth and back characteristic. In folding back we argue that one is not experiencing such returned-to actions or images for a topic as if for the 'first time'. On the contrary, more sophisticated language, and concepts that have already developed with respect to the topic, (demonstrated by the previous outward-moving direction of the pathway) are brought to work in the less formal, less sophisticated, more local contexts, in order to broaden the understanding of the topic and create a coherent way of knowing. This revisiting or folding back to inner levels is thus not to be considered as a regression to an earlier state, but as part of the constantly growing understanding which can be represented by the connected pathway drawn on the model diagram as a continuous line.

Of course not all mathematical understanding of all topics grows for a person in this connected manner. In other work we have shown that a person's understanding of a topic can be disjoint (e.g. Pirie and Kieren, 1992; Pirie and Kieren, 1994)and we pursue this theme later in this paper.

Setting the scene

It is the purpose of the rest of this brief paper to show why we think it is useful to think of mathematical understanding not as an achieved state but as a dynamic, changing process. To do this we will consider the work of two university students as they worked on two situations which involved the development of number patterns in a given setting. By using concepts from our model, in particular the ideas of image having, property noticing, folding back and disjoint understanding, we will highlight the pattern of growth of understanding in each situation and demonstrate the ways in which mathematical understanding is always under construction.

The pair of students, Stacey and Kerry, worked together on the following two situations in the given order with a week between, spending between an hour and one and a half hours of class time on each :

1. Arithmagons (from Mason, Burton and Stacey, 1984):
A secret number has been assigned to each corner of this triangle.
On each side is written the sum of the secret numbers at its ends. Find the secret numbers.
Generalise the problem and its solution.

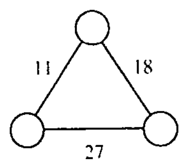


Figure 1

2. The fibonacci sequence begins:

1,1,2,3,5,8,13,21,34,55,89,.....

If you are familiar with the recursive rule defining the sequence write it down. If not, try to discover the rule.

Use the notation F_n to stand for the nth Fibonacci number. For example, F_7 is 13 and F_{10} is 55.

Look for patterns which relate the index n to the Fibonacci number F_n. For example, is there anything special about F_n when n is a multiple of 3, or a multiple of 4, or prime?

Understanding Arithmagons

Like the other pairs in the study Stacey and Kerry used a "guess and test" strategy to find the three corner numbers in the original problem (Fig. 1). The following dialogue occurred during their exploring:

Stacey: *I just want to try something. If you take 27,18 and 11...*[1] *56. Right?...And you have... So you add each of those* [corner numbers] *twice, right?- ...Yeah, you do. You add A,B,C. Then you multiply them by 2. You get this answer* [56]*... Do you know what I mean?*

Kerry: *Sorry. So you add this and multiply by 2; so like the sum of this is 28 .. times 2. And it's 56. Good one! What's it mean? Is that true for them all?*

Stacey: *Yeah.*

[1]The convention used here is that "..." represents a pause in the speech

BEST COPY AVAILABLE

Kerry: *I guess so. It must be. It can't just be a fluke. ...OK I guess we could solve it as a matrix also- ...*

These students were using knowledge that they already had to explore and get an idea or image of this problem space. After they had confirmed their "guess and test" solution, they turned to the question of generalising in a novel way (Fig 2)

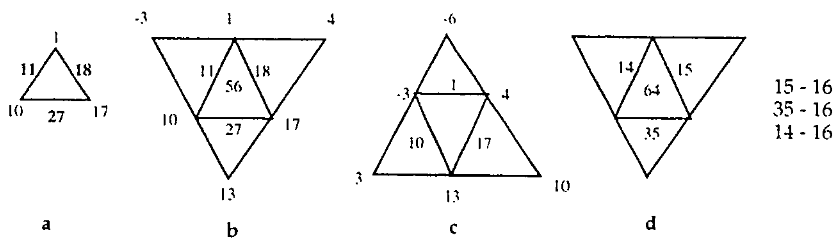


Figure 2

Notice that in going from 2a to 2b these students were developing a related triangle, one which now had the previous solution "vertices" as the given side values. Their solution of 2b included a discussion of whether one could have negative vertex values. After finding the solution set for 2b (-3, 4, 13) using processes which they had used for solving the original problem, they noticed a relationship between the values in the embedded triangle, namely that the vertices were the related side number less 14 (i.e. 27-14=13, 18-14=4, 11-14=-3).

Stacey: (redrawing the -3,4,13 triangle from 2b in a smaller version and omitting the centre, see 2c) *Ooh-... let's keep going.*

Kerry: *Do you want a prediction? This* [each of 1, 10 and 17] *is decreased by 14.*

This prediction failed, but it lead the pair to revisit 2b and see 14 as (11+18+27)/4 and they made a more general prediction about 4 as a key divisor. This led them to try unsuccessfully to use letters to see if they could "derive a formula" which would show the pattern even more generally. As part of this activity the pair made some conjectures about what happened to the sums of the sides of the triangles that

Stacey: *They get smaller and you can have fractions*

as the pattern was continued "outward". They were unable to see why, however.

Stacey: *The sum goes down by half because.......?*

At this point Stacey suggested they try a new problem triangle like the original but with different numbers. After feverishly working on several new triangles they concluded they had a method [see Fig. 2d] which worked for all such problems and were very excited

Kerry: *We've figured out the system! .. now, why the system?*

While their work on the latter question was never successfully resolved, the understanding process did not end there. One week later the students were interviewed.

Interviewer : *Did you happen to work on arithmagons any more?*

Kerry: *Yeah! We bugged any one we knew in the library for hours!*

Not only had they continued to work on triangles, they had also worked productively for several hours on finding "the magic numbers" (like the 4 for triangles) for arithmagons with from four to seven sides and indeed thought that such numbers must exist for any arithmagon.

Figure 3 is a plotting of the pathway of growth of understanding for Stacey and Kerry. They were seen to move repeatedly between working with specific numerical examples - guessing, testing, predicting and recording (image making and image having) (1) - and noticing properties (2) ed to the 'side' and 'vertex' number sums of particular problems. They constantly folded back to work numerically on particular triangles to give themselves insights and a broad foundation of understanding. This eventually led them to what we call the formalising activity of applying a method which they conjectured worked for any triangular arithmagon (3). (*We've figured out the system.*) Interviews revealed that this pattern of growth in understanding of arithmagons continued long after the observed problem solving session. They continued to fold back and added to their experience with numerical examples that were designed for a specific purpose. Previous experience provided a template for new experiences; results from previous experiences served to generate attempts to notice patterns or formal (even if not algebraically expressed) methods. Less formal understandings seemed to unfold into more general or more formal understanding actions; more formal understandings appeared to enfold and indeed access less formal ones. Their personal commitment to the problem was certainly in evidence!

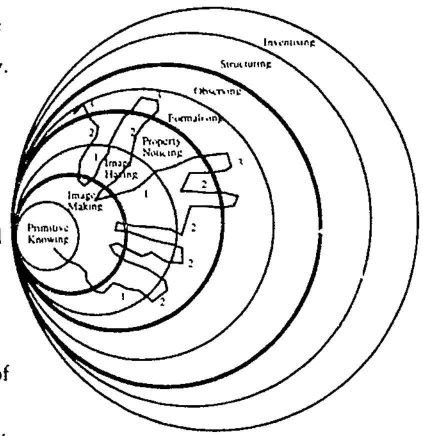


Figure 3

Understanding Fibonacci Sequences

One week after working on arithmagons, Stacey and Kerry worked on the Fibonacci problem given above. At the start of their work Stacey said:

Stacey: *Oh. I know this one. You add the last two to get the next one. 55+89=144.*

Notice that this 'definition' makes no reference to the first terms in the sequence, although the problem states "The fibonacci sequence begins...". Almost immediately Kerry wrote down Stacey's 'definition' using the suggested notation:

Kerry: $F_n + F_{n+1} = F_{n+2}$, *so* $F_n = F_{n+2} - F_{n+1}$. *Neat eh!*

When Kerry used this to derive that F_0 = 1-1=0, one of the observers questioned whether there could be an F_0. Because Stacey's original definition had not taken into account the role of initial terms, the pair saw no problems with extending the Fibonacci sequence in either 'direction': " ...-8,5,-3,2,-1,1,0,1,1,2,3,5,... " Here Stacey and Kerry develop and use a formalised definition (given above) with interesting consequences. They do not try, even when challenged, to test the assumptions underlying their definition.

Following this formal work the pair seemed stuck as to what to do next. They spent the rest of the hour considering the suggestions in the last question posed on the problem sheet and then requested suggestions from one of the observers. In each case they very briefly considered the given set of numbers and made one observation such as:

Kerry: *Prime numbers* [terms with such subscripts] *are all odd.*

and then looked at another question. The one exception to this was Stacey observing that

Stacey: *The multiples of 3* [terms with such subscripts] *are even*

and Kerry formalising this as

Kerry: F_{3k} *is even*

and offering a convincing argument for it. They did not try to consider the challenges in the light of the extent of the Fibonacci sequence nor did they 'play' with the sequence or its elements in the way that they had played with examples of the arithmagons.

The pathway of growth of understanding (Fig. 4) for Kerry and Stacey in this second problem session is strikingly different. We no longer see the same enfolded pattern of understanding growth. Instead the understanding activities and episodes for the most part seemed almost divorced from one another. Stacey opened the problem by saying that she knew this sequence (1) and Kerry immediately stated it in formal terms(2). This led to them folding back(3) to numerical work on the sequence to illustrate the consequence of their formalising. The rest of the understanding activities engaged in by this pair in this setting can be characterised as a series of disjoint lines (4) where the pair would look at the sequence in the light of some query (brief image making and image having activities) and then make some observation about the sequence (property noticing) but make no attempt to link this observation with previous images. One

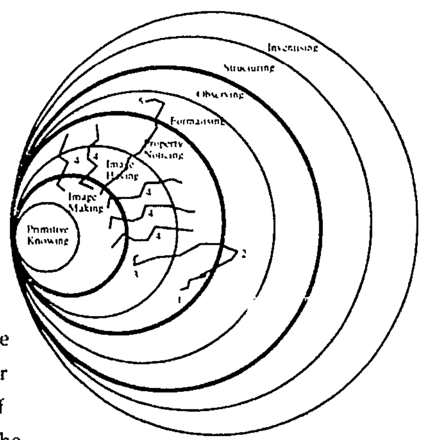


Figure 4

exception in this pattern (5) is when the pair stated and justified a generalisation relating to the terms whose subscripts were multiples of three. Even here the work and the argument were not related to other previous activities but had an isolated character.

Concluding Comments

If mathematical understanding is not thought of as a state or an achievement but as a dynamic process, it is still tempting to assume that the mathematical understanding process would show a pattern of growth that will be roughly the same in all situations. As the foregoing examples show, such an assumption would be problematic. In the arithmagon setting the understanding activity could at any moment be observed as related to previous activity in the domain. Although there were 'dead ends' and although not all previously noticed images or properties were accessed in later work, our dynamical theory allows us to characterise the growing understanding of the pair in this setting as not outwardly uni-directional, but as a continually broadening connected phenomenon always grounded on specific numerical activity (image making). Any property or formal method was seen in relationship both to numerical image making activity and to other previously constructed images or properties. The pair also sensed this connected growth and attempted to keep it going in an unprompted manner.

Such a sense of connected understanding did not carry over into the Fibonacci setting. Perhaps because the pair had a formalised definition for the sequence from the start and the problem sheet suggested areas of exploration, they appeared not to connect each new set of image making activities with previous understanding or work on the problem. Thus they did not generate the numerical experience which might have better allowed them to see their activities as inter-related. There was little drive to elaborate or extend any observation they made. The growth (or lack of it) in understanding could be observed as disjoint. They saw their task as simply to observe the setting in particular suggested areas or directions and did not build up their own image for the setting as they clearly had for arithmagons.

We do not, at this stage, wish to make a value judgement on their two ways of working at the problems, merely to draw attention to the ability of the model to highlight the different pathways in a fashion that would not be possible if one were simply to consider understanding as an acquired state. Although the pair made many mathematically correct, and some incorrect, statements in the face of both settings, such results or achievements do not well characterise their understanding or its growth in either setting. The findings discussed here suggest the necessity of teachers and researchers listening carefully to the on-going mathematical activity of students as a key source of knowledge of student understanding. These findings also illustrate the power of the Pirie-Kieren theory of the growth of mathematical understanding as a tool for observing the phenomenon of personal understanding as always under construction.

References

Herscovics, N. (1992). The construction of conceptual schemes in mathematics. A paper at the Seventh International Congress on Mathematics Education. Quebec: August.

Kieren, T.E. and Pirie, S.E.B. (1992). The Answer Determines the Question: Interventions and the Growth of Mathematical Understanding. In W. Geeslin and K. Graham (Eds.). Proceedings of the Sixteenth Psychology of Mathematics Education Conference. Durham, N.H.: PME. pp. 2-1, 2-8.

Kieren, T.E. and Pirie, S.E.B., (forthcoming) Folding Back: A Recursive Theory of the Growth of Mathematical Understanding, in D. Sawada and M. Galey (eds.), Recursion and Education, New York: Gorden Breech.

Mason, J., Burton, L. and Stacey, K. (1984) Thinking Mathematically. London: Addison-Wesley.

Miller, L. D., J. A. Malone and T. Kandl(1992) A Study of Secondary Mathematics Teacher's Perceptions of the Meaning of Understanding. Paper presented at the Annual Meeting of AERA, San Francisco, April.

Pirie, S.E.B. and Kieren, T.E. (1989). Through the Recursive Eye: Mathematical Understanding as a Dynamic Phenomenon. In: G. Vergnaud (Ed.). Actes de la 13 Conference Internationale, Psychology of Mathematics Education. Paris.

Pirie, S.E.B. and Kieren, T.E. (1992). Creating Constructivist Environments: Constructing Creative Mathematics. In: Educational Studies in Mathematics, 23(5), pp 505-528.

Pirie, S.E.B. and Kieren, T.E. (1994). Growth in Mathematical Understanding: How can we Characterise it and How can we Represent it. In: Educational Studies in Mathematics, 26(3).

Sierpinska, A. (1990) Cognitive Obstacles and Mathematical Understanding, in (M.Quigley,ed.) Proceedings of the Canadian Mathematics Education Study Group Annual Meeting. Vancouver. St. John's,Nfld. : CMESG/GCDEM

Skemp, R. (1987). The Psychology of Learning Mathematics. (Expanded American Edition). Hillsdale, N.J.: Lawrence Erlbaum Associates.

van der Walle, J.A. (1990). Elementary School Mathematics: Teaching Developmentally. White Plains NY: Longman.

von Glasersfeld, E. (1987) Learning as a Constructive Activity, in C. Janvier (ed.) Problems of Representation in the Learning and Teaching of Mathematics, pp. 3-17. Hillsdale, N.J.: Lawrence Erlbaum Associates.

Wittgenstein, L. (1956) Remarks on the Foundations of Mathematics. Oxford: Basil Blackwell.

ACTIVE GRAPHING IN A COMPUTER-RICH ENVIRONMENT

Dave Pratt
Mathematics Education Research Centre
University of Warwick

Abstract

This paper reports on recent results from the Primary Laptop Project in the UK. This project is studying the effects on young children's mathematical learning when they have constant and immediate access to portable computers. The children, aged 8 and 9, have been using graphs and charts to support their classroom experiments. The paper suggests that the computer can support children's active graphing , a process in which children use graphs interactively during (rather than at the completion of) an activity and that active graphing can develop children's interpretative skills. However, the paper also acknowledges the important part played by other factors in the environment.

Introduction

The Primary Laptop Project

This paper reports on some recent results from The Primary Laptop Project in the UK. This project is studying the effects on young children's mathematical learning when they have constant and immediate access to portable computers. The first two preparatory phases introduced the project teachers to the technology and served as a pilot study, allowing the researchers to gain a clearer focus on specific issues.

Pseudo-mathematical use of graphs

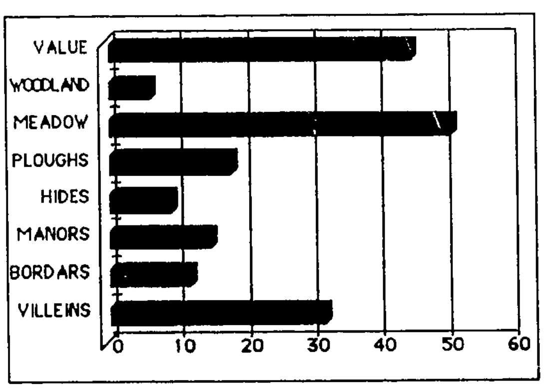


Fig 1 : Sharon's bar chart

The children's use of graphs became one focus during Phase 2. We soon saw children of age 11 working on an historical theme related to a local ruined castle. One group of children were collecting data from Domesday records. Sharon had decided to put the data into a spreadsheet. She included, for example, the area of woodland, the value of the land and the number of ploughs. Sharon decided to used the computer to generate a bar-chart. The bars, being of various units, were not comparable. However, Sharon was pleased with her graph and used it to illustrate her subsequent work.

Children were generating graphs very quickly and reacting to them excitedly. Their reactions seemed to be based on the visual impact that the picture made rather than its relevance to the data on which they were working. It was clear to us, even at this early stage of the project, that the facility to draw graphs and charts easily on the computer carried certain pedagogical difficulties. It was proving problematical for the non-specialist teachers to distinguish between genuine mathematical use of this graphing facility and the sorts of examples above, which we now term *pseudo-mathematical*.

Real use of graphing

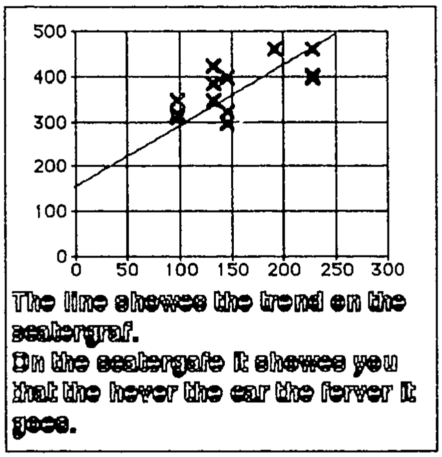


Fig 2 : Rory's scatter graph

Later in this phase we observed younger children having some success with graphs. The Y5 (age 9) children were working on a topic called *Wheels*. As part of this topic, they were investigating how far cars of varying weights would roll. Rory was a boy with considerable interest in the computer but who also had a lot of language difficulties. Rory was encouraged to draw scatter-graphs as his experiment progressed. Finally, he wrote a report by cutting and pasting his graph onto a word-processing document. His brief conclusion shows his understanding of both the graph and his experiment in general.

The classroom environment (Phase 3)

Thus, we now recognised that children's use of graphs would be one important aspect of our research. The project, now one term into its third phase, has been making a detailed study of two primary school classes of age 8 (Y4) and 9 (Y5).

The effect on the children's learning during this phase is, we believe, greatly influenced by the environment within which these children are working. The children were being encouraged by their normal class teachers to work in an increasingly independent way. Two researchers worked as a teacher/ researcher pair for the mathematics and science work, with one taking a teaching role whilst the other observed and took field notes. Both the teachers and the researchers encouraged the children to see the computers as one resource amongst many and the children quickly developed a healthy disrespect for the computers. They were encouraged to use the machines for all of their curriculum work. The children have maintained their eagerness to use the computers both in class and at home over the whole term. In most cases two children share one computer though a few children work in groups of three.

The researchers have worked alongside the teachers to develop teaching plans which were based on a topic approach. During this stage of the Primary Laptop Project, the Y5 children studied a topic on *Outer Space* whilst the Y4 children worked on a topic called *Growing and Shrinking*. In both classes, the mathematical ideas were developed alongside the rest of the curriculum within the framework of the general topic. For example, the Y4 children studied their own growth using their baby records and artefacts; they used Logo to draw families of people of varying heights; they planted and observed the growth of bean plants; they grew mathematical objects such as squares and sequences and they studied the growth of numbers in multiplication tables. Meanwhile the Y5 children developed their own projects on outer space; they used Logo to write computer-based space adventure games; they studied the phases of the moon; they used orreries to observe the motion of the planets and they carried out gravity experiments.

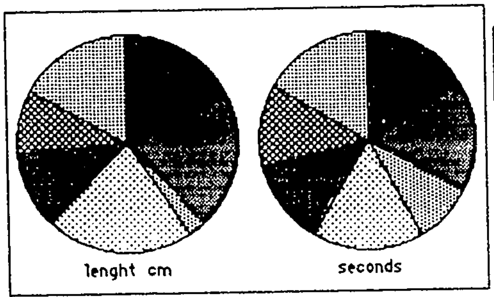
For their Logo work, the children in both classes used LogoWriter. For much of their other work they used ClarisWorks, an integrated package which allowed the children to use graphical, word-processing and data handling facilities.

Children's Difficulties In The Interpretation Of Graphs

We had observed in Phase 2 children using the computer to generate graphs and charts which, though nonsensical in meaning, had been accepted by the children as having value. In Phase 3, we gained more evidence which led to a greater insight into this phenomenon.

Emma (Y5) had been experimenting with the length of a pendulum to find out how the time of one swing was affected. She had collected the data into a spreadsheet and was quickly generating various graphs before choosing a pie chart.

When interviewed she stated that she had chosen this picture because it looked nice.



	A	B	C
1	weight	lenght cm	seconds
2	same weight	39.5	14.41
3		47	14.37
4		6	7.84
5		48	14.69
6		27	11.28
7		21	10.63
8		38	14.31

Fig 3 : Emma's data and pie chart

The tendency for children to interpret given graphs as a picture is a phenomenon which is recognised as a common misconception and has been reported elsewhere (see, for example, Kerslake (1981), Swatton (1994)). However, on many occasions we were led to believe that our children were choosing a graph for its pictorial value rather than for its quality as an interpretative instrument. It would appear that these children were using different criteria from ours in judging the relevance of these graphs. Such examples seemed to suggest that these children had not yet gained access to the criteria which mathematicians would use to assess the value of a graph. The pedagogic question becomes one of how to introduce these children to such criteria.

This is slightly different from the more conventional viewpoint. For example, Bell (1985) has proposed that children need to be made aware of their misconceptions about graphs through a range of teaching strategies which bring the children into conflict with their own ideas. It may be that, for some of our children, it was not yet an issue of misconception but one of perspective or purpose. The teacher's purpose for encouraging the children to use graphs was at variance with that of the children. Nevertheless, some of the children were experiencing success and we were led to ask whether there were aspects of these children's work which was allowing them to assimilate criteria similar to those of the teacher.

Children's Successes - Active Graphing

Rory's surprisingly successful interpretation described earlier gave us hope that we would observe some children giving proper meaning to their graphs. One aspect of Rory's work was the way in which he had been encouraged to draw scatter graphs *during* his experiment. Rory examined his emerging graph for patterns which might give him useful information. He was able to pick out dubious results and check on their validity. He was able to look for gaps in his data and make predictions. Rory's graphing was imbued with a sense of purpose, which went beyond the illustrative, in contrast to that of the children earlier whose graphs we termed *pseudo-mathematical* .

Indeed Rory's understanding had proved to be long term. Six months after Rory had done his work with the toy cars during which time he had very little exposure to the use of computers, we decided to carry out a *thought* experiment with Rory and some of his colleagues. We explained that we wanted to carry out an investigation with another class into how the size of craters on planets such as the moon was affected by the speed and size of the asteroids which hit the planet's surface. There was some discussion about how an experiment could be carried out until the group agreed that heavy balls could be dropped from various heights into a bucket of sand. The group were able to describe how to set up a spreadsheet and get a scattergraph (Rory's fingers could be seen to be moving as he imagined going through this process). The group were then asked to draw out with pencil what the graph would look like.

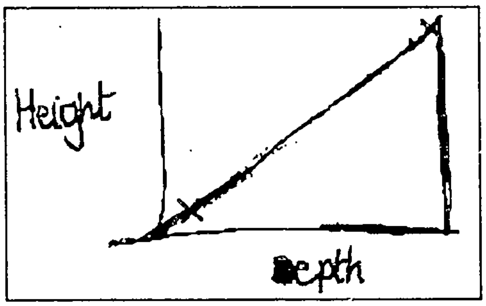


Fig 4 : Rory's hand drawing

Our conjecture was that by using graphing in an interactive way as part of an experiment (a process we now term *active graphing*), the graphing was given a sense of purpose, which was helping some children to gain access to a more sophisticated understanding of graphs. We now have more evidence to support this conjecture.

James and his friends had been collecting data on how the distance travelled by a toy car was affected by the height of the ramp down which it rolled. The researcher encouraged him to draw a scatter-graph of his five results so far and recorded the following field notes:

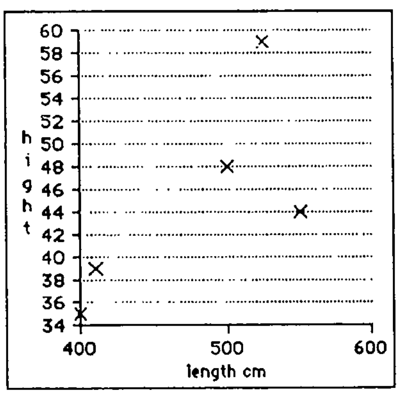


Fig 5 : James's first graph

I got them to try to identify a few points with their experiments, which they all seemed to be able to do. James seemed to have an idea that there was a clear relationship "the higher it is the further it goes", but then said 'Oh no' when he saw that one point didn't fit. They agreed to do it again, though James was concerned that they should do it back in the old place down the corridor to make the test fair.

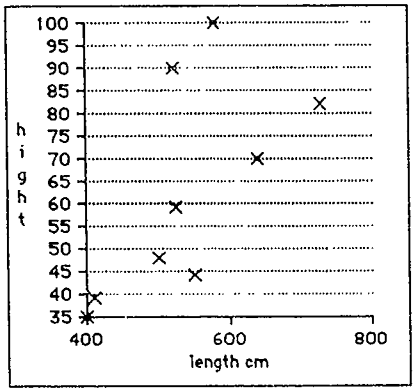


Fig 6 : James's second graph

James seemed to be moving towards a fuller interpretation of the graph. His group went on to collect more data and another graph. James was able to explain that, when the ramp was very steep, the car had hit the ground and slowed up. He was happy that the other results in his scatter graph formed a consistent pattern. It is quite possible that James was basing his conclusions on his first hand experience with the toy cars rather than on the graph. For him, the two experiences were not separate but coincidental, an approach which may help James gain access to a more analytical appreciation of such graphs in the future.

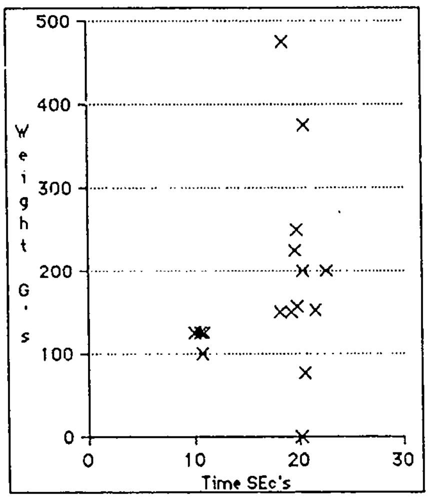


Fig 7 : Joanna's scatter graph

Joanna and her friends had been working on a pendulum experiment looking at how the weight of the bob might affect the time for one swing. She generated the following scatter graph. Joanna was able to observe that some of the crosses did not fit into the general pattern. She went on to explain that her first few results had been measured wrongly - they had timed only half the swing. Indeed the dubious data occupied the first few rows of her spreadsheet. She went on to express very succinctly how the graph showed that the time was the same whatever the weight of the pendulum. This full interpretation seemed to be all the more sophisticated given how difficult it would have been to make this observation directly from the experiment without the aid of the graph.

The success of these children was in direct contrast to other research on children's ability to interpret graphs. Padilla, McKenzie & Shaw (1986) observed that, when children of grades 7 to 12 were given graphs to interpret in pencil and paper tests, only 49% were able to describe the relationship and 26% use a best line of fit. Swatton and Taylor (1994) provided even more depressing results:

> *The description of a relationship proved to be by far the most difficult demand. Not only could very few pupils give a full 'pattern' statement, but almost all pupils stated in response to a short questionnaire attached to each package that they found this aspect to*

be the most difficult........The difficulties which pupils encounter appear to stem from the fact that variable relationships presented symbolically in this way are too abstract for the vast majority of pupils at both 11 and 13.

Indeed, they conclude that a mere 6% of children at age 11 were able to make a statement which gave a proper link between the two variables in question. We might expect our younger children to perform even worse than these figures if the environment's had been similar. It is important therefore to draw together the argument by considering how this environment might have led to some of the children having unexpected levels of success in their efforts to interpret graphs.

Conclusions

We have been struck by two conflicting general observations:

- children will use the facility available on the computer to generate graphs but many of these will be nonsensical. Nevertheless this can be of benefit if children are brought into conflict with their own criteria for judging these graphs.

- some very young children have been able to make accurate and complete interpretations of their graphs. Such events might not have been predicted from previous research except that these children were working in a very different environment.

There are perhaps three main features in the environment which have enabled some children to be successful:

- These children were working in a classroom where the teacher was a mathematics specialist. This had considerable influence on the curriculum planning but also on recognising instances of *pseudo-mathematics*: a first step towards forming a pedagogic strategy which would enable those children to be brought into conflict with their own understanding when appropriate.

- Swatton (1994) found that analysis of different questions, though designed to test the same aspect of graphical understanding, yielded no correlation in their results and concluded:

  Such findings indicate that in order to make an accurate assessment of process skill performance on the strength of just one assessment item or one context is impossible, since each item brings with it its own unique set of 'demands', which are inherent in the very nature of the question itself and the way in which it is presented.

Tall (1989) has argued that learners have a *concept image* which develops as the result of a whole range of experiences and that every individual brings to bear a unique way of understanding a concept. Our children were encouraged to develop strong *concept images* by studying related concepts in a variety of contexts and from different angles. We would contend that the nature of the curriculum planning and the encouragement to explore independently were crucial factors in their success. Similarly, when the children were interpreting the graphs, they were doing so in a context where they themselves had carried out the experiment. They had observed first hand the phenomenon on which they were reporting; they had entered the data into a spreadsheet and seen the phenomenon represented numerically. It is our conjecture that such experiences were influential in helping them to interpret their graphs.

The computer itself was an important part of the environment. We would conjecture that the process of *active graphing* was instrumental in enabling some children to come to a fuller understanding of their experiment, their data and their graph. The graphs that the children generated were judged according to new criteria. Instead of being aesthetically pleasing drawings or pictures, they were imbued with a sense of purpose; the graph could tell them how to proceed with their experiment. Children could discover points which did not seem to make sense according to the trend in the rest of their data. They could see gaps in their data which ought perhaps to be filled by further experimentation and they could predict results from outside their current range of data in preparation for the next experiment. This meaningful interaction with the graph and the data seemed to enable some children to develop a better understanding of their graphs and so eventually be able to give a fuller interpretation of its meaning.

References

Bell A., Brekke G. and Swan M. (1985), Diagnostic Teaching : 4 Graphical Interpretation, Shell Centre for Mathematical Education, University of Nottingham

Kerslake D. (1981), Graphs in Hart K.M. (Ed), *Children's Understanding of Mathematics 11-16*, Murray.

Padilla M.J., McKenzie D.L. and Shaw E.L. (1986), An Examination of the Line Graphing Ability of Students in Grades Seven Through Twelve, *School Science and Mathematics*, **86**

Swatton P. & Taylor R. M. (1994), Pupil Performance In Graphical Tasks, And Its Relationship To The Ability To Handle Variables, *British Educational Research Journal*, **20**

Tall D. (1989), Concept Images, Generic Organizers, Computers and Curriculum Change, *For the Learning of Mathematics*, **8**, **3**, Montreal

CULTURAL MATHEMATICS EDUCATION RESOURCES
IN A GRADUATE COURSE

Norma C. Presmeg
The Florida State University

This paper describes the positive psychological implications for multicultural education of an experimental graduate course, Mathematics Education and Culture, in which the position was taken that multiculture in a mathematics classroom is not a liability but a rich resource to be celebrated and shared. 21 students of diverse cultural backgrounds showed how this can be done, in projects that included the mathematics of the South Korean flag, stick charts of Micronesia, mathematics of American marching bands, baseball, German folk songs, counting in sign language, and diverse cultural games.

With an increasing concern in several countries that education should be fair and equitable for all children (Mathematical Sciences Education Board, 1990), the position has sometimes been adopted by teachers that they are "color blind" in the classroom, i.e., that their teaching is not influenced by the racial or cultural backgrounds of their students. Often this position is held in good conscience, even by whole school districts, as the one most likely to result in equity for learners. But as Nieto (1992, p. 109) points out eloquently, "Equal is not the same". A far more equitable position is to affirm the diversity of students in multicultural classrooms, and to look on this diversity as a potential resource for learning. Otherwise, what happens is that the dominant culture is looked on as the only culture - and hence all others are disadvantaged. For several years I have believed that diversity is an assett rather than a liability in the classroom, but I recently had the opportunity to explore some of the implications of this philosophy for educational practice.

The Course

The 3 semester-hour graduate course, Mathematics Education and Culture, which I taught in Fall, 1993, was an experiment. I did not know in advance which students would be taking the course, but I intended to give these students autonomy to build the course for themselves using the resources of their own cultural backgrounds, based on the premise that *every* student has rich cultural resources to draw upon. 21 students enrolled in the course. Cultures were diverse; "foreign" students came from Tenerife (Spain) and South Korea, but even among American students, cultures ranged from "a Minnesota Jew who is trying to become a Southerner" (as Jenny categorized herself), to Dierdre who wrote,

> "My cultural background is a mixture of ethnicities, a 'masala'. I think it makes me unique. I have African, French, Spanish and Indian in my ethnic background. Not only am I a mix, but the customs and traditions that my family practises is a mixture of the four. When I look in the mirror I see the characteristics that are an amalgamation of the four. I hear in my talk, I see it in my beliefs. New Orleans is where I am from - the city of mixed up people. My family comes in all shades of brown. My race is African American, but my culture is Creole."

A strong theoretical framework for the course was provided by Bishop (1988), which was the only required book, although many other relevant readings were brought in by individual students as well as the instructor. The theme of the course was, 'Mathematics is a cultural product'. The course had theoretical and practical goals, as well as the potential for research in classrooms.

Theoretical component: this included exploration of the meanings of terms such as culture, ethnomathematics, enculturation, acculturation and related constructs as these have been developed in the literature; mathematics of 'exotic' cultures as well as local ones; levels of scale; environmental activities; values.

Practical component: each session the class participated in at least one 'cultural' activity and explored the mathematics of that activity as well as its potential for developing mathematical understanding in schoolchildren.

Research component: participants were required to investigate a theoretical or pedagogical aspect of their choice, which relates to mathematics as a cultural product, to write a paper on their topic, and to organize classroom discussion or an activity relating to this topic. Those students who had the opportunity to try out their activities with schoolchildren were encouraged to do so, and to report on the results.

Assessment was based on two aspects of original research (theoretical paper and an activity or presentation), a journal which students kept each week, and a take-home final which included an analysis of the individual student's research paper and presentation in terms of Bishop's (1988) theoretical framework.

Issues and questions for discussion arose naturally in our reading and activities. For instance, after we read Gerdes (1988) paper and practised some of the sand drawings of the Tchokwe, their was lively discussion of the question, "Whose mathematics is this?" After being introduced to Wendy Millroy's Cape Town carpenters and their mathematics, we found it natural to debate, "If you don't know you're doing mathematics and you don't call it mathematics, is it still mathematics?" We agreed that a broadened definition of mathematics is necessary: Steen's definition of mathematics as "the science of pattern and order" was considered to be broad enough (National Research Council, 1989).

Theoretical Papers and Activities or Presentations

The students who experienced the most difficulty in deciding on a topic for their individual research were the three African Americans in the class. All three of them manifested "race consciousness" (Hall and Allen, 1989) in the sense that they wanted to choose topics that related to and would help African Americans. But the mistaken but still prevalent belief that they lost their culture when they left Africa (Holloway, 1990) caused them some agonizing in their choice of topics. Compromises were made. For instance, Vanessa wrote her theoretical paper on the mathematics of a sharecropper in the United States, but her practical activity was "Egyptian mathematics: multiplication by doubling" - reflecting an African interest. Dierdre reversed the order, writing her theoretical paper on Africa but leading an activity in which students arranged furniture in an (American) entertainment center. For most students, the theoretical paper and the activity were not separated but were different facets of the same (chosen) topic.

What was striking in the projects was the degree of ownership and involvement manifested by the students. The papers and activities were as diverse as their creative authors. Three of the four doctoral students in the course (Phillip, Jenny and In-Gee) all became so involved that they wanted to continue their projects in dissertation research. What follows is a sampling of topics.

Jenny's theoretical paper and activity both involved mathematics education and the various cultures of deaf learners. Her ability to sign fluently at the same time as she spoke gave force to her presentation of the history of signing in France and America. Later, we were asked to work in groups to find a way to represent all the numbers from 1 to 999 using only one hand. After whole-class sharing Jenny showed us the "official" way using American Sign Language. A valuable aspect of the group work was that it stimulated discussion of the properties of our system of numeration, in addition to highlighting the manifold ways in which it was possible to accomplish the task.

Nancy, an experienced Middle School Teacher, investigated gender bias in various mathematics programs in her school, on the basis of the position that stereotyping girls and boys in this way is a cultural phenomenon. Her research revealed several inequities. She presented her findings, and also gave us a bonus activity. Nancy's cultural heritage is German, and after she taught us to sing "O Tannenbaum", we worked in groups to try to represent the song mathematically. This was accomplished in various ways - spatially or numerically or both - including an ingenious likeness of a fir tree in which the branches were lines of the song interspersed with a contrasting theme represented by decorations which twined up the tree to start again at the top.

Denise, who taught English in Micronesia during two years with the Peace Corps, wrote a fascinating paper on the mathematics inherent in traditional navigation between those islands. She had sent away to Pohnpei for a Stick Chart made of bamboo and depicting swells and interference patterns of swells in the Pacific around the Marshall Islands. Her Stick Chart arrived, to her delight, just the day before her presentation. After learning how island navigators use their own bodies to determine 'nodes' in the swells, star constellations (taking the viewpoint that the boat is stationary and everything else moves), birds and phosphorescence, to determine their location and route, we tried to plot a course for ourselves, using these principles, to get from one given island to another.

A mathematical basis was interwoven with cultural elements and their histories, in numerous other activities including Halloween cats made with the tangram puzzle, the quipu of the Incas, the Stomachian puzzle which is more that 2000 years old (attributed to Archimedes), the Chinese abacus, architecture of various cultures, gambling and games of chance and numerous other games. In addition to these the instructor introduced activities relating to mathematics in Scandinavian Yule-baskets, Ndebele house decorations, and various Japanese origami projects.

Outcomes and Implications

In retrospect, the class was successful beyond my expectations. It nevertheless seemed that we had barely got started, merely scraped the surface of something much larger. It is not that 'multicultural mathematics' can be introduced in a Middle School classroom once a week, for instance. It is rather a state of mind, a pedagogical world-view which *constantly* looks to the students for ways to celebrate their cultural uniqueness - as Dierdre felt unique with her four cultures - in the context of learning mathematics. It needs to be done with empathy and sensitivity (Nieto, 1992), and only to the extent that students feel comfortable in sharing, since there is an aspect of vulnerability in opening up what might be misunderstood or misrepresented by outsiders. For instance, in his discussion of the mathematics of the South Korean flag, In-Gee introduced a swastika, describing it as an ancient Buddhist emblem. He became very upset, in fact spent a sleepless night, when it was brought to his attention that this emblem may have very negative connotations for people like his good friend Jenny, who is Jewish. His fears proved to be unfounded, and Jenny and In-Gee are still good friends. But the point is that teachers need to be sensitive to the use of cultural artifacts which may be on the one hand anathema, or on the other, holy objects, to insiders. For instance, mandalas are not just fascinating circular mathematical drawings, they are holy objects of meditation with deep layers of significance, and teachers need to be aware of these connotations. Thus it is the people who belong to a culture who should do the sharing.

The theoretical readings, discussions and analyses were essential elements in our forging of a common classroom culture which was hospitable to multicultural elements. Our constructs changed as we went through the course, since we were not looking for 'final' definitions. For instance, in the beginning Peter was of the opinion that it is possible to have a culture of one, since each person is unique. (Other students did not agree.) By the end of the course there was a consensus that the sharing aspects of culture are paramount, resonating with Stenhouse's (1967, p. 16) definition, "a complex of shared understandings ...".

Perhaps influenced by Jenny's work on deaf cultures, a different change was expressed by Dierdre in her final journal entry, as follows.

> "As a student of this class, my definition of culture has been broadened, from being ethnic based to being a way of living and being. I have played many cultural games and have discussed the relevancy of games in math class. Games are universal to every social group. Many times the rules are simple and the games are fun. I have learned about different number systems, and today, about Egyptian mult./div. I think that it is amazing how they used doubling/halving instead of multiplication tables. This could be an alternative way of teaching multiplication and division. In fact, it shows the relationship of addition/subtraction to multiplication and division. Besides that, I feel that many African American students would feel some sense of pride knowing that this is an African way of calculation."

Peter, of Hispanic background, concluded as follows.

> "The class has made me aware of how much students can bring in from their community, environment and way of living. Classrooms need to make more connections with the student's life and make it part of their real world."

In the sense expressed by Peter, using the cultures of students in the mathematics classroom is a very natural thing to do, in fact it is just one aspect of good pedagogy, practised intuitively and spontaneously by many teachers. However, for this use to be practised more widely and deliberately, the research started in this course needs to be extended to case studies of implementation in schools. I hope to start such research in 1994.

REFERENCES

BISHOP, A.J. (1988). Mathematical Enculturation: A Cultural Perspective on Mathematics Education. Dordrecht: Kluwer Academic Publishers.

GERDES, P. (1988). On possible uses of traditional Angolan sand drawings in the mathematics classroom. Educational Studies in Mathematics, 19(1), 3 - 22.

HALL, M.L. and ALLEN, W.R. (1989). Race consciousness among African-American college students. In G.L. Berry and J.K. Asamen (eds) Black Syudents: Psychosocial Issues and Academic Achievement. Newbury Park, California: Sage Publications.

HOLLOWAY, J.E. (1990). Africanisms in American Culture. Bloomington: Indiana University Press.

MATHEMATICAL SCIENCES EDUCATION BOARD (1990). Making Mathematics Work for Minorities: Framework for a National Action Plan, 1990 - 2000. Washington, DC: National Research Council.

MILLROY, W. (1992). An Ethnographic Study of the Mathematics of a Group of Carpenters. Reston, Virginia: National Council of Teachers of Mathematics, Monograph 5.

NATIONAL RESEARCH COUNCIL (1989). Everybody Counts: A Report to the Nation on the Future of Mathematics Education. Washington, DC: National Academy Press.

NIETO, S. (1992). Affirming Diversity: The Sociopolitical Context of Multicultural Education. New York: Longman.

STENHOUSE, L. (1967). Culture and Education. New York: Weybright and Talley.

Moving through Systems of Mathematical Knowledge: from algebra with a single unknown to algebra with two unknowns.

Luis Radford
Laurentian University
Canada

In the following text, we will concentrate on the didactical problem of going from algebra with a single unknown to algebra with two unknowns. Inspired from epistemology and in genetic psychology, we will develop the concept of system of mathematical knowledge *which serves as a tool of analysis for the experiential data collected from students in the process of learning algebra. Our data is preceded by an historical analysis which sheds some light on the understanding of the noted difficulties experienced by our students.*

§ 1 Introduction: the system of mathematical knowledge

The difficulties of students learning algebra have produced quite a number of studies. Kieran (1991) and Arzarello (1991) approached the problem in terms of a dialectic between procedural and relational thought. Fischbein and Barash (1993), Clement (1982), centered their attention on the significance of the errors made by the students learning algebra. Some other studies approached the problem in terms of the emergence and development of algebraic thought and its relationship with arithmetical thought (Filloy and Rojano 1894; Herscovics and Linchevski, 1992; Bednarz *et al.*, 1992). Herscovics and Linchevski gave special attention to the students' procedures of solving equations prior to a formal instruction in algebra. Filloy and Rojano emphasized the importance of the acquisition of algebraic language and symbolism. Bednarz *et al.* centered their attention on the type of representation that students use when solving a word problem and on the processing of the relationships contained in the word problem. Siding with the research on the development of algebraic thought, we intend to examine certain difficulties experienced by students when going from algebra with a single unknown to algebra with two unknowns in solving word problems. In order to do so, we will introduce, first of all, what we mean by a *mathematical knowledge system*.

General genetic psychology considers that when an individual is placed in a problem-solving situation, s/he will use some concepts and procedures (or methods of resolution) previously acquired, which, through assimilation and adaption become refined, modified, and enriched. These concepts and procedures can further lead to the emergence of newly structured understandings in more complex networks. The construction of a set of procedures, which we will call Σ, depends upon the set of concepts, which we will call Ω. In the same way, the individual's construction of the set Ω depends upon the set Σ. However, the construction of the concepts and the procedures depends upon the set of problems, which we will call Π, that one poses and tries to solve (Bednarz *et al.* 1992). In turn, the set of problems Π that provoked the construction of knowledge sees itself enriched by the possibilities offered by the enlargement of the concepts and procedures: it is thus, that new problems can be envisioned.

From a didactical perspective centered in the cognitive phenomena, the learning of a certain content or mathematical theory τ, can be seen through the cognitive construction (of a trialectic nature) of the Ω, Σ and Π sets[1]. Those sets could be considered as the elemental components of the knowledge associated to the mathematical theory τ. We will refer to them as a *mathematical knowledge system*, and we will represent them by $S\tau$.

Let τ_1 be the part of algebra that we teach in our schools today (based on concepts and methods of solving word problems using a single unknown), and let τ_2 be the algebra of n unknowns ($n \geq 2$), the

[1] An example of this *interaction* can be seen when Lagrange faces the problem $\pi \in \Pi$ of the solution of the equation of the fifth degree, using the methods Σ of the 18th century (based essentially on the elementary operations of algebra). He is obliged to introduce some *new concepts*, like the one of *similar functions* or the one of *reduced equation* (cf Radford, 1990). These concepts open up a whole reflection that runs into the theory of substitutions (cf.vg. Cauchy (1815))

problem that interests us here is that of moving through the knowledge system $S\tau_1$ to the knowledge system $S\tau_2$.

§ 2 Some historical remarks

Historically, the emergence of concepts and methods using two unknowns in the calculations came about very slowly[1]. For a very long time, problem statements of the $p(q_1, q_2, \ldots q_k)$ type, with $1 \leq k \leq 4$, were solved with the help of the simple or double false position (cf. Spiesser, 1982) or algebra with one unknown. The difficulty to conceive two algebraic unknowns on the same level in the calculations can also be detected in certain problem statements of the $p(q_1, q_2)$ type: facing problems of this type, Viète solves the problem by taking, as an unknown, one of the sought after quantities q_i, and once the problem is solved, solves it again. This time taking as the unknown the other sought after quantity (cf. Viète: *Les cinq livres des Zététiques*. Livre I, Zététique VII, for example). One work (probably the first) which gives a treatment of problems using two unknowns is that of the Italian mathematician, Antonio de'Mazzinghi, born circa 1353 (Franci, 1988), where the unknowns are represented by **la cosa** (the thing) and **la quantità** (the quantity). The historical analysis suggests that the emergence of algebra with two unknowns is related to the resolution of certain problem statements of the $p(q_1, q_2)$ type, where one of the givens is the product or the sum of the squares of the sought after quantities (one will find examples in Franci, 1988). However, the introduction of two unknowns did not have as its goal the translation of the problem statement in a new symbolic system, as in present day elementary algebra; the two unknowns were introduced as an heuristic artifice facilitating the resolution of the problem. (In fact, Mazzinghi chose to express the sought after quantities, q_i, as one **cosa** (thing) plus or minus some **quantità** (quantity), i.e., $q_i = x \pm y$, or again $q_i = x \pm \sqrt{y}$). Therefore, the emergence of algebra of two unknowns seems connected to the necessity of solving problems for which the calculations of a single unknown would prove too difficult.

These comments suggest that, from a didactical point of view, the study of the transition between single unknown algebra, τ_1, to two-unknowns algebra, τ_2, could be done through the study of certain problems of the $p(q_1, q_2)$ type of which the resolution could prove more "appropriate" in τ_2. Of the more simple problems, those of the $p(q_1, q_2)$ type that our actual notation allows us to write under the form of the linear equations system: $x + y = d$, $ax + by = c$, seem to be the most appropriate for this study. This type of problem has been historically solved with the help of the simple false position method[2], but also with the help of a single unknown and two unknowns algebraic methods (one will find a "modern" treatment in τ_1 and in τ_2 in Euler's *Elements of Algebra*).

In his *Trattato d'Abaco*, the XV century italian painter and mathematician, Piero Della Francesca, presents a type of problem which interests us, in the following terms:

> «There are two types of cloth, one costs 15 pounds per rod, the other costs 12 pounds per rod. One wants to spend 100 pounds in all and have 7 rods of cloth in all. I ask how many there will be of each type.» (Arrighi (Ed.), 1970, p. 63; the english translation is our own.)

The problem is solved by a false position method (called the double false position) and which consists of giving two values x_0 and x_1 to one of the quantities that one's looking for. Then, one calculates the differences e_0 and e_1 between the answer that one's expecting and the answer that's derived from the value produced by x_0 and x_1, respectively, according to the given conditions in the problem

---

[1] In what follows, we will distinguish between *unknowns of the problem statement* and the *unknowns in the algebraic sense*. Thus, the first three refer to the quantities or sizes $a_1, a_2, \ldots a_k$ that one asks to find in the problem statements, the others refer to the quantities x, y, z, ... upon which the algebraic reasoning and calculation are constructed. If p is a problem where it is a question of finding a certain number k of quantities, we will represent the problem by $p(a_1, a_2, \ldots a_k)$. The term *unknown* will be used throughout this work for the algebraic unknowns and the term *quantities* for the unknowns of the problem statement. In this work we will not deal with the concept of *variable*.

[2] There is a problem of this type in the Babylonian tablet VAT 8391, a tablet that goes back to the first Babylonian dynasty (c. 1900 A.D.); the problem is solved by the simple false position method (cf. Radford, 1993).

statement. A simple calculation on the four numerical values obtained allows us to find the value of the first quantity being sought, and to deduce the value of the second quantity. In later pages, the problem is approached in an algebraic fashion. Let us first look at the double false position method:

> «Suppose that one has 4 rods which cost 15 pounds, these 4 are worth 60 pounds; and 3 rods that cost 12 worth 36 pounds. Add this to 60, that comes out to 96, but you are missing 4 to equal 100. Say for 4 that I write down , I am missing 4. Do the problem once again and say that you had 5 rods of 15, which 5 by 15 equals 75; therefore there are 2 rods which cost 12, so that 2 by 12 equals 24, all together it equals 99 but you want 100 so you are still missing 1. Say: for 5 that I write down, I am missing 1. Now multiply crosswise, 5 by 4 equals 20 and 1 by 4 equals 4; subtract the latter from 20 and you have 16. Subtract 1 from 4, you have 3, which is the divisor. Divide 16 by 3, you get 5 1/3; 5 1/3rods is the cloth of 15 pounds per rod and 1 2/3 rods is the cloth of 12 pounds per rod.» (Arrighi, pp. 63-64)

Even though the problem is stated with two unknowns, as we can see in Della Francesca's solution, the reasoning is essentially derived from one of the unknowns: the choice of the numerical value for the first unknown enables us to deduce the value of the other unknown.

Now let's look at the algebraic solution of Della Francesca.

The unknown is called "the thing" (*la cosa*); the number of things is represented by placing a bar above the numbers of things: thus, $\bar{1}$ or $\overline{15}$, for example, represents 1 thing or 15 things, respectively. In our english translation, we have respected this symbolic code.

> "Say this: let $\bar{1}$ be one part of the 7 rods and the other is 7 minus $\bar{1}$. Multiply $\bar{1}$ by 15 pounds, the cost of the rod, which equals $\overline{15}$ and that is one part; multiply 7 minus $\bar{1}$ by 12, the cost of the other rod, that equals 84 minus $\overline{12}$. Restore the parts, because you want 100, add $\overline{12}$ to 84 minus $\overline{12}$, that equals 84; subtract $\overline{12}$ from $\overline{15}$ and $\bar{3}$ remain. You have $\bar{3}$ and 84 is equal to 100, subtract 84 from 100, 16 remain, divide 16 by the things which are 3, you get 5 1/3: there is the value of the thing and we have put one part $\bar{1}$, therefore it was 5 1/3. Thus, there were 5 1/3 rods of the 15 pounds per rod type. Up to 7 there is 1 2/3 rods of which the price is 12 pounds per rod." (p. 98)

In Della Francesca's algebraic solution, the second quantity sought is expressed in the terms of the first quantity through the given relationship: "7 rods of cloths in all", so here again the reasoning is essentially based on one unknown.

§ 3 The Experimentation

Our goal is to identify and understand some of the difficulties that students encounter in moving through the systems of mathematical knowledge $S\tau_1$ to $S\tau_2$ when solving problems algebraically. Here, we will examine the results of a test that was given to 38 students, 15-16 yrs. old, in the 3rd yr. of a public secondary school in Guatemala City. These students had learned algebra according to the formal arithmetic approach, i.e., the approach in which one introduces symbols as representations of numbers, but which quickly stops on the aspect of formal manipulation of algebraic expressions. These students had seen algebra with one unknown in the 8th grade, and with two unknowns in the 9th grade.

The test was comprised of five problems. There was an "easy" introductory problem which was to give them self-confidence during the test. There were three problems (of which one problem -number 3- was of the same type as the one we analyzed in §2) which could have been solved, *a priori*, in $S\tau_a$ (which, here, designates the system of elementary arithmetic knowledge), $S\tau_1$ or $S\tau_2$.

These are the problems:

Problem 1 (P1):

Maria is five years older than Juan and the sum of their ages equals 25. What are their ages?

Problem 2 (P2):

Pedro is three times older than Juan and the difference of their ages is 20. What are their ages?

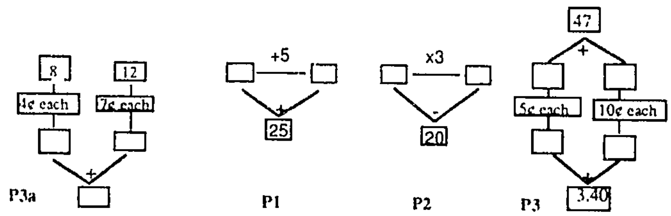
Problem 3 (P3):

Julio has $3.40 made up of nickels and dimes. If he has a total of 47 coins, how many of each denomination does he have?

The three problems (P1), (P2), and (P3) were preceded by an "arithmetic" or a "connected problem" (Bednarz and Janvier, 1993) that we designated as problem P3a, which is, in a way, the arithmetic version of problem (P3). Here's its statement:

Arithmetic Problem 3 (P3a):
Rosa purchased 8 stamps of 4¢ each and 12 stamps of 7¢ each. How much did she spend?

Before going further, it's best to recall that Bednarz *et al* (1992) suggested that a given problem $p(a_1, a_2, ... a_k)$, can be seen as a composition of relations $\rho_1, \rho_2, ..., \rho_n$. For example, in problem P1, there are two relations: one relation called the "sum relation" (which corresponds to "the sum of the ages is 25") and one relation called the "generative relation" ("Maria is five yrs. older than Juan"). In problem P3 (this applies also to Della Francesca's problem), there are 4 relations: two "sum relations" (which correspond to "the number of coins" and "amount of money", which we will designate as ρ_1 and ρ_4, respectively) and two relations called "ratio relations" (the first relation being one that calculates the total money from the number of nickels and the value of the nickels, a relation that we will designate as ρ_2. The second relation being the one that calculates the total money from the number of dimes and the value of the dimes, a relation that we will designate as ρ_3). One can associate a graph with the relations ρ_i called the *semantic or relational structure diagram* (Bednarz *et al.*, 1992). In the case of the preceeding problems, we have the following graphs.



In presenting these problems, we want to see in particular if the students who succeeded (P1) and (P2) in $S\tau_1$ perceived problem (P3) as a problem arising from $S\tau_1$ or $S\tau_2$.

For a student who succeeds (P1) and (P2), what could the difficulties be in solving (P3) in $S\tau_1$? The last question addresses the fact that many secondary school mathematics manuals concentrate most of their introductory chapter algebra problems with the same type of (P1) and (P2) problems (cf. Bednarz and Janvier, 1993; problems that they identified as problems of the "1 sum relation + 1 generative relation" type, according to their relational structure), so that, often, the success of these problems becomes, in practice, the minimal condition for which an individual has developed $S\tau_1$ to a satisfactory degree.

§4 The Results

4.1 The Outcome of the given problems:

The students had an hour at their disposal to complete the test but the required average duration was 40 minutes.The following table represents the outcome of the questions:

Problem	P3a	P1	P2	P3
Correct Answers	32 (84%)	26 (68%)	23 (61%)	11 (29%)

The problems P1, P2 and P3 are of the $p(q_1, q_2)$ type, in keeping with the notations introduced in §2. However, problem P3 is much more difficult than the others. There are several reasons for this: on one hand, we can see in the relational structure diagram that this problem calls upon the concept of ratio or unit cost (the price of each stamp); furthermore, it consists of more relations than the other problems, so that the organization of the data and of the relations is harder to establish. Among the 11 correct responses of this problem, 7 of the students used two unknowns, 3 of the students used one unknown and 1 student solved the problem via an arithmetic procedure. It seems, therefore, that problem P3 is a difficult problem in $S\tau_1$, as the hypothesis we established in §2 suggests.

4.2 The Distribution of the Sample in Algebraic Tendencies

Let $E(S\tau_1)$ be the set of students that succeeded on questions P1 and P2 using algebra with one unknown (τ_1), and that did not try problem P3, or did, but not using τ_2 procedures. Let $E(S\tau_2)$ be the set of students that succeeded P1 and P2 using algebraic procedures, and who tried P3 (without necessarily succeeding) using algebraic procedures with two unknowns (τ_2). Let $E(S\tau_0)$ be the set of students that correctly answered P3a and at least one of the questions P1, P2. Finally, let $E(S\tau^*_1)$ be the subset of students of $E(S\tau_1)$ that correctly answered P1, P2 and P3, and let $E(S\tau^*_2)$ be the subset of students of $E(S\tau_2)$ who correctly answered P1, P2 and P3. [The set $E(S\tau_0)$ can be considered the "frontier" of $E(S\tau_1)$; in this way, $E(S\tau^*_1)$ can be considered the "kernel" of $E(S\tau_1)$ and $E(S\tau^*_2)$ the "kernel" of $E(S\tau_2)$]. Here is the breakdown:

Group	$E(S\tau_0)$	$E(S\tau_1)$	$E(S\tau^*_1)$	$E(S\tau_2)$	$E(S\tau^*_2)$	Other
Size	12(32%)	15(39%)	3	7 (18%)	6	4(11%)

We see then, that the group of students tend to place themselves in the $S\tau_1$ system, even if the student often realizes that his/her procedure of solving problem P3 has failed (cf. examples No. 1, 2 and 3 below).

To better understand this "resistance" to change over to $S\tau_2$ and to better understand the difficulties in the transition from the system of mathematical knowledge $S\tau_1$ to $S\tau_2$, we will present, within the limitations of this article, an analysis of the resolution procedures of problem P3, focusing special attention on the students of $E(S\tau_1)$ and $E(S\tau_2)$ that failed P3:

4.3 Analysis of the Procedures

Example No. 1: (student belonging to the $E(S\tau_1)$ group):
"0.05x = 3.40; 0.10x = 3.40
0.05x + 0.10x = 3.40, 0.15x = 3.40, x = 3.40/0.15, x ="

Example No. 2: (student belonging to the $E(S\tau_1)$ group)[1]:
"x nickels (//) 2x = dimes (//) 0.05x + 0.10x = 3.40 (//) 0.15x = 3.40 (//) 2x + x = 47 (//) 3x = 47 (//) x = 47/3 (//) x= "

Example No. 3: (student belonging to the $E(S\tau_1)$ group):
"x nickels (//) 2x dimes (//) 100 (x/47 + 2x/47) = 340
$100\,\frac{(x+2x)}{47} = 340$ (//) $100\,\frac{(3x)}{47} = 340$ (//) $100\,(3x) = \frac{340 * 47}{3}$
100(x)=53.26 (//) x=0.53 (//) x=53 nickels (//) 2x = 53*2 = 106.
(Afterwards, the student calculates the value of 53 nickels and the value of 106 dimes; he finds $2.65 and $1.06 respectively. He calculates the total: he finds $3.65).

Example No. 4: (student belonging to the $E(S\tau_1)$ group):
"x=number of nickels, 47-x=number of dimes
total of nickels + total of dimes = 3.40 (//)0.05x + 0.1(47-x)= 3.40 (//) 0.05x + (0.1)(47)-0.1x = 340 [...]" (The student answers the problem correctly.)

Example No. 5: (student belonging to the $E(S\tau_2)$ group): "nickels=x. dimes=z. 5x+10z=47"

Example No. 6: (student belonging to the $E(S\tau_2)$ group)
nickels = x
dimes = y 3.40/x + 3.40/y = 47
3.40/x = 47 - 3.40/y

[1] We will use the symbol "(//)" to indicate a change of line in the student's manuscript.

(Afterwards, the students replaces y by 0.10 and making correct calculations, arrives at x = 3.40/13, an answer which he recognizes as being incorrect; he then crosses out the calculations made starting from the substitution of y by 0.10.)

Example No. 7: (student belonging to the $E(S\tau_2)$ group):

"x = nickels, y = dimes

$x + y = 47$	$-5x - 5y = -235$
$5x + 10y = 340$	$\underline{5x + 10y = 340}$
	$5y = 105$ [...]"

(The student has correctly answered the problem.)

Example No. 8: (student belonging to the $E(S\tau_2)$ group):

"x = nickels, y = dimes (//) (I) $x(0.05) + y(0.10) = 3.40$

(II) $x + y = 47$

From (II) one has $y = 47 - x$. One substitutes it into (I) and then one has: $x(0.05) + (47-x)(0.10) = 3.40$ [...]"

(The student continues the calculations and finds the value of x, then, working from (II) he obtains the value of y.)

Comments:

Our results suggest that the students who work from within the mathematical knowledge system $S\tau_1$ (ie. the $E(S\tau_1)$ group students) will concentrate their attention particularly on the sum ρ_4 relation «amount of money» (that's the case of the students in examples 1 and 2), and often they will be incapable to take into account all the ρ_i relations. On the other hand, those that work in $S\tau_2$ seem to have a more global view, and in turn often take into consideration all the ρ_i relations (even if the procedure does not end up solving the problem).

Another difference between the groups $E(S\tau_1)$ and $E(S\tau_2)$, is the way in which the quantities of q_1 (number of nickels) and q_2 (number of dimes) are associated. In fact, as far as the group $E(S\tau_1)$ is concerned, the association of the quantities is made by expressing one of the quantities in terms of the other quantity, by a reasoning that keeps track of the significance of the quantities, be it through the ρ_i relations or other relations. Thus, in the examples 2 and 3, the association of the q_i quantities is made on the basis of the values of the coins, that which allows the construction of the symbolic expressions q_1 and q_2 in $S\tau_1$, ie. the construction of the terms «x» and «2x»: because 10 is 5 doubled, and x is the number of nickels, then 2x must be the number of dimes. In example 4, solved also in $S\tau_1$, he expresses one of the quantities in terms of the other, but, contrary to the case in examples 2 and 3, the association between the q_i is done in terms of the ρ_1 sum relation «number of coins», which permits to **deduce** the expression «47 - x» starting from the expression «x». The student from example 1 (who belongs to the $E(S\tau_1)$ group), shows a less complicated process than the students of the fore-mentioned examples in that he does not try to establish a semantic link between the q_i quantities: the use of the same symbolic representation for the two quantities q_1 and q_2, suffices to come across the equation $0.05x + 0.10x = 3.40$.

On the other hand, in the $E(S\tau_2)$ group, the two sought-after quantities of q_1 and q_2 are introduced by *semantically independent* symbolic representations (one is represented by x, the other by y or z): they no longer have to be *deduced* one from the other, by a reasoning on the ρ_i relations given in the problem statement or eventually from another relation (see examples 5, 6, 7 and 8), as is the case in $S\tau_1$.

In $S\tau_2$, it is the **translation** of the ρ_i relations given in the problem that prevails. The ρ_1 relation allows to arrive at the symbolic expression «x + y = 47». The ρ_2, ρ_3 and ρ_4 relations allow us to arrive at the symbolic expression $x(0.05) + y(0.10) = 3.40$. The attention therefore, is not centered on

the way that one will *express* q_2 in terms of q_1, as in the case of $S\tau_1$, but rather on the translation of the problem statement into a system of equations.
We can see then, that in $S\tau_2$, the processing of the relations is not the same as in $S\tau_1$. The expression «y = 47 - x» deduced in $S\tau_2$, starting from the symbolic representation «x + y = 47» of the ρ_1 relation, is not usually made by semantic deduction (ie. by a reasoning of the significance of the ρ_1 relations within the context of the problem) but by a symbolic deduction (this being, rightly so, one of the strengths of the symbol!). One $E(S\tau_2)$ student, for example, expresses y in terms of x, in writing the equation: $y = (3.40 - 0.05x)/0.10$; this equation would be difficult to justify in a semantic light. It could also happen that in $S\tau_2$ one would do without expressing y in terms of x throughout the problem, as example 7 shows!
However, to better define the difference between the knowledge systems $S\tau_1$ and $S\tau_2$, it is necessary to reflect a moment on this *semantic deduction* that ensues from a reasoning on the ρ_i relations connecting the q_k quantities and that, in $S\tau_1$, allowing the expression of q_2 in terms of q_1. In taking a closer look, we notice that in $S\tau_1$ the student must transform the ρ_1 relation. He must change this «sum relation» into a relation of a completely different nature, that is, a «generative relation» (fig. 1 below) so that the semantic or relational structure of the problem is itself also *completely transformed* (fig. 2 below). It's exactly this impossibility to realize the transformation of relations that often leads certain students (cf. examples 2 and 3) to look for other relations that render it possible to symbolize q_1 and q_2.

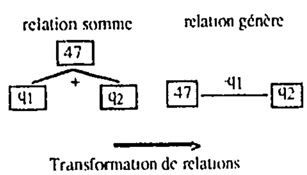


Fig. 1

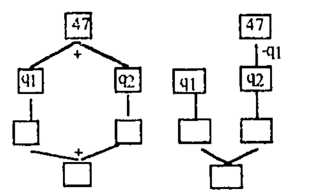


Fig. 2

We see then, that there exists a fundamental difference between $S\tau_1$ and $S\tau_2$ at the level of transformation to which the ρ_1 relation must be submitted. This transformation allows the sum relation «number of coins», to change into a generative type relation. In $S\tau_2$, on the other hand, we do not transform the relation, in such a way that problem P3 stays the same.
If we come back to our historic analysis now, we realize that the transformation of the «sum relation» into a «generative relation» is used in the algebraic process of Piero Della Francesca; but what is even more remarkable is that this transformation is already present, on the arithmetical level, in the double false position method. In fact, in giving an arbitrary value to one of the quantities, say q_1, the sum relation is transformed into a generative relation. (In fact, in giving the value 4 to q_1, Della Francesca deduces that q_2 is equal to 7 - 4 = 3: the «sum relation» becomes the «generative relation»).

§5 Final Remarks:

We have tried to raise certain difficulties experienced by the students in the transition from the mathematical knowledge system $S\tau_1$ to the $S\tau_2$ system. We have seen that there is, in our group of students, a certain "resistance" to change over to $S\tau_2$. Our experimental data suggest that working efficiently within $S\tau_1$ could require some specific skills in the processing of relations in problems such as problem P3, that is, in problems where a semantical transformation of relations is required in order to be able to express one of the quantities in terms of the other quantity. The problems P1 and P2 did not require any transformation of relations, which explains their rate of success in $S\tau_1$. The history of mathematics, in turn, enables us to see that the emergence of the algebraic concepts and methods of problem solving using two unknowns in the calculations came about very slowly. A closer look at the history of algebra allows us to see that the possibility of transforming relations made possible the solving of a wide family of problems of $p(q_1, q_2)$ type. On the other hand, we saw that in $S\tau_2$, the nature of the semantically independent representations for the q_i quantities avoids the transformation of relations, but requires, in return, a conceptualization of the second quantity q_2, as an autonomous algebraic unknown vis-à-vis the first unknown. (Example 5 suggests that this conceptualization has its problems: in fact, it seems that in this student's eyes, x and z have to consolidate their identity while remaining attached to 5¢ and 10¢).

In short, the scope of the concepts in Ω_1 and the methods in Σ_1 of $S\tau_1$ requires, in order to succeed a wide range of problems, some transformation of relations given in the text of the problem. This can be avoided in $S\tau_2$, by extending Ω_1 into Ω_2 by including the second-unknown algebraic concept, which will modify the set Σ_1 into a new set Σ_2. This transition between systems of mathematical knowledge creates new difficulties: in $S\tau_2$, a second equation becomes necessary. Example 6 shows the case of a student who has difficulties obtaining a second equation: so he replaces y by 0.10, which brings the problem from $S\tau_2$ back into $S\tau_1$.

Our results raise some questions: Is there didactical work to be done in arithmetic, to facilitate the success of problems like our problem P3 when students will work in $S\tau_1$? What is the didactical work to be done within $S\tau_1$ to prepare the transition to $S\tau_2$? Specifically, what are the problems that will demand the student to place him/herself in $S\tau_2$? How can the emergence of the modern second-unknown algebraic concept be provoked?

References:

Arrighi, G. (Ed.) (1970). *Piero Della Francesca. Trattato D'Abaco.* Pisa: Domus Galileana.

Arzarello, F. (1991) Procedural and Relational Aspects of Algebraic Thinking. *Proceedings of PME XV*, Assisi, Italy, I, pp. 80-87.

Bednarz, N., Radford, L., Janvier, B. Lepage, A. (1992) Arithmetical and Algebraic Thinking in Problem-Solving. *Proceedings of PME XVI*. New Hampshire, I, pp. 65-72.

Bednarz, N., Janvier, B. (1993) L'algèbre comme outil de résolution de problèmes: filiations et ruptures avec l'arithmétique. Paper presented at the International Colloquium *Approaches to algebra: perspectives for research and teaching*. CIRADE, Université du Québec à Montréal, 12-15 mai 1993.

Clement, J. (1982) Algebra word problem solutions: Thought processes underlying a common misconception. *Journal of Research in Mathematics Education*, 14, pp. 16-30.

Filloy, E., Rojano, T. (1984) From an Arithmetical to an Algebraic Thought. *Proceedings of the sixth annual meeting of PME-NA*, Madison, pp. 51-56.

Fischbein, E. Barash, A. (1993) Algorithmic models and their misuse in solving albebraic problems. *Proceedings of the XVII PME Conference*, Vol 1, pp. 162-172

Franci, R. (1988) Antonio de'Mazzinghi: An Algebraist of the 14th Century. *Historia Mathematica*, 15, pp. 240-249.

Herscovics, N., Linchevski, L. (1992) "Cancellation within-the-equation" as a solution procedure. *Proceedings of PME XVI*. New Hampshire, I, pp. 265-272.

Kieran, C. (1991) A Procedural-structural perspective on algebra research. *Proceedings of PME XV*, Assisi, Italy, 2, pp. 245-253

Radford, L. (1990) Lagrange y el desarrollo conceptual de la teoría de Galois. Cuadernos de Investigación, No. 1, EFPEM, Universidad de San Carlos de Guatemala. 22 pages.

Radford, L. (1993) Le raisonnement algébrique. Une réflexion épistémologique. *Actes du Colloque Élève, École, Société*. CIRADE, Université du Québec à Montréal, pp. 33-45.

Spiesser, M. (1982) Équations du premier degré. IREM de Toulouse, France. 57 pages.

RESULTS FROM PORTUGUESE PARTICIPATION IN THE "SECOND INTERNATIONAL ASSESSMENT OF EDUCATIONAL PROGRESS" - MATHEMATICS

Glória Ramalho, Instituto Superior de Psicologia Aplicada, Lisboa

Abstract

IAEP international study took place between 1989 and 1992. During 1990/1991 a total of 20 countries surveyed Mathematics and Science performance of 13-year-old students and 14 of them also assessed nine-year-olds in the same areas. This presentation addresses the results in mathematics of portuguese students in the context of this study. Results show a low performance of both populations, although there are differences between them, favorable to 13-year-old students. Disparities in achievement are also found in schools in different administrative regions and types of community. Public and private schools show distinct outcomes favorable to private ones in both populations. Children's attitudes toward math appear to be related to their achievement.

Introduction

International comparisons about students' performance in any subject must be cautiously achieved, since there are distinct sorts of differences among countries which may contribute to explain differential outcomes. There are, most likely, dissimilarities on educational systems, on implemented curricula, on emphasis given to specific topics, on student habituation to the testing format, and, generally speaking, there are important cultural differences. In this study, one other source of disparity among participants had to do with the way target populations were defined in each case. For cultural reasons and due to differences in educational systems there were restrictions to the language spoken and to the grades included And the very fact that all participants had limited their assessment to children who were in school, had differential effects on the characteristics of the collected samples, since in some countries this meant a low participation of many age-eligible children. (IAEP, 1992).

It is, however, worth saying that one major concern during this survey's preparation, elaboration and application was to ensure the best uniformity and quality to the project, given the complex characteristics of the international context. Nonetheless, the results have to be interpreted taking into consideration the educational and cultural background in each country.

The purpose of the analysis done over the students' outcomes was threefold: 1) to disclose the relative position of both portuguese populations (nine- and 13-year-old students) in this general assessment process, in spite of all its limitations; 2) to scrutinize some of the possible relationships between students' performance and some characteristics of their personal background, their home and school environment, and lastly 3) to examine possible disparities in the achievement of nine- and 13-year-old portuguese children. This presentation will have to be confined to what we thought were the most relevant aspects of the scrutiny that was completed.

The Survey

The survey was constructed by Educational Testing Service (ETS) with the cooperation of specialists from participating countries, translated and adjusted by technical teams from those countries, and finally administered in selected schools during the school year of 1990/1991 In Portugal the project was sponsered by the Ministry of Education (G.E P) and coordinated by a national representative, who was assisted by a national scientific commitee

This assessment comprised three kinds of instruments. 1) a four-section paper and pencil test addressing five topics (Numbers and operations, Measurement, Geometry, Data analysis, statistics and probability, and Algebra and functions) and implicating three categories of cognitive processing (Conceptual understanding, Procedural knowledge, and Problem solving); 2) a student questionnaire inquiring about both their background, home environment and classroom experiences; 3) a school questionnaire to be answered by school administrators delving on the characteristics of the sampled schools.

The structure of the analysis performed on the results obtained by the portuguese sample relied on a set of research questions which were grounded on the objectives that were listed above. Student achievement is represented, here, by the percentage of correct answers, either in the test all together, or by topic, or by cognitive process assumedly involved In the computation of mean scores weights were used in order to adjust the sample to the target population The results of students who omitted questions at the end part of each section, presumably because they did not reach them, were excluded from the calculations for those questions.

Portuguese students' achievement in Mathematics in the context of IAEP international survey

The mean percentage of correct answers across all countries was 63, with a range of 20 points Portuguese nine-year-old children's average was 55%. lower than the international mean and the lowest among the 14 participating countries (IAEP, 1992)

With regard to thirteen-year-old students the overall average score was 58%, and the total range among countries 52 percentual points. Portuguese students at this age answered correctly to a mean of 49% of the questions, being fifth from the bottom in terms of relative placement among participating countries.

When students' performance is discriminated by cognitive process and by topic we can detect some discrepancies. Figure 1 and Figure 2 illustrate these performances for both nine- and 13-year-old children.

Figure 1 - Average percent correct scores in the test all together and by cognitive process - 9- and 13-year-old students

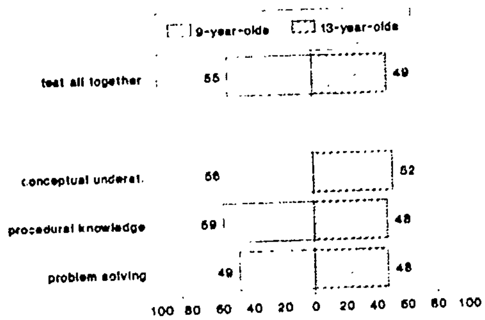


Figure 2 - Average percent correct scores in the test all together and by topic - 9- and 13 year-old students

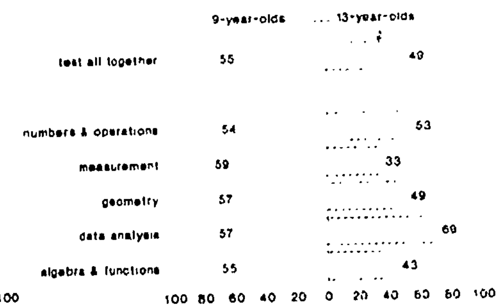
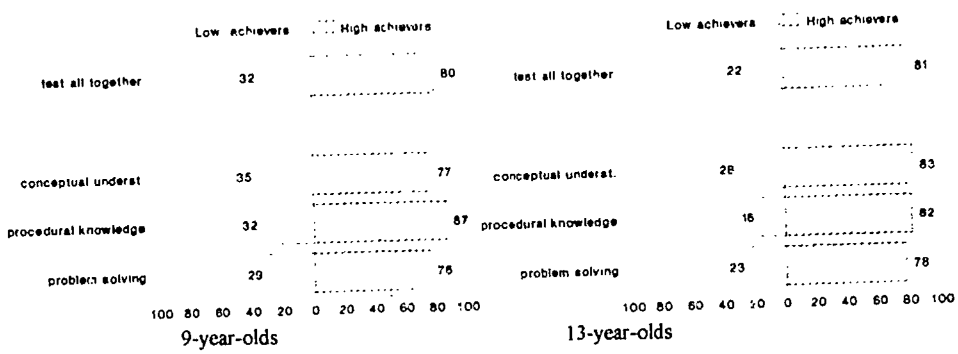


Figure 1 shows us that portuguese children performance depended on the cognitive process involved, although in a different way for each population. Nevertheless, problem solving appeared to present more difficulties in both cases

As can be seen in Figure 2, there is also a differential achievement according to the topic addressed Younger children seem to display some homogeneity in their results, contrary to the eldest whose average scores vary from 33% (measurement) to 69% (data analysis)

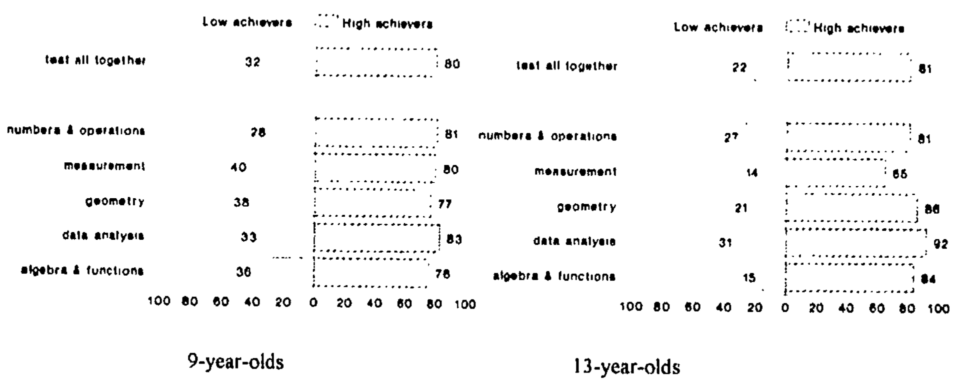
In order to scrutinize the disparities in the performance of "high" and "low" achievers, two groups of students were devised the ones between the 1st and the 10th percentile ("low achievers") and the students above the 90th percentile ("high achievers") Figures 3 and 4 illustrate these groups' outcomes in both age populations.

Figure 3 - Average percent correct scores in the test all together and by cognitive process - "high" and "low" achievers - 9- and 13-year-old students



One striking finding is the range between "high" and "low" achievers' average outcom the oldest population. Nine-year-old "low" achievers seem to be less distant from "high" achievers group of the same age. One other aspect refers to the difficulties found in the first group of children, in both populations, more apparent in questions that implicate either procedural knowledge or problem solving kind of processes. On the contrary, 9-year-old "high" achievers appear to be stronger in procedural knowledge and his older peers perform also well in this sort of questions.

Figure 4 - Average percent correct scores in the test all together and by topic - "high" and "low" achievers - 9- and 13-year-old students



With respect to the topics addressed in the study we can identify some difficulties in "numbers and operations" in the less successful nine-year-olds, although they may be partially due to the predominance of questions relative to this content in the test. "Measurement" appears to be a problematic area for both "high" and "low" group within the oldest population; "algebra and functions" is one other area of concern for the weaker 13-year-old group.

Portugal being a small country, we considered to examine whether or not there were dissimilarities among distinct geographical/administrative regions, and also among different types of community (urban, semi-urban, rural). Results are sketched in Figure 5 and 6.

Figure 5 - Average percent correct scores in the test all together by administrative region - 9- and 13 year-old students

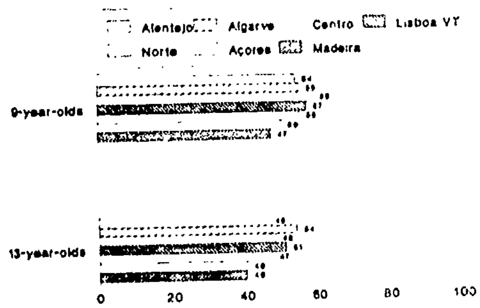
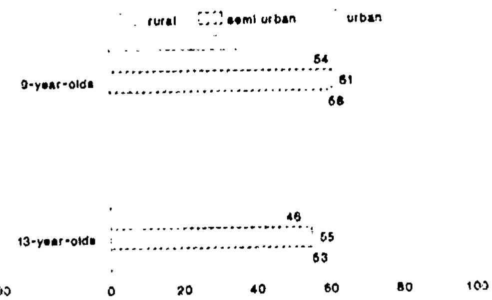


Figure 6 - Average percent correct scores in the test by type of community - 9- and 13-year old students

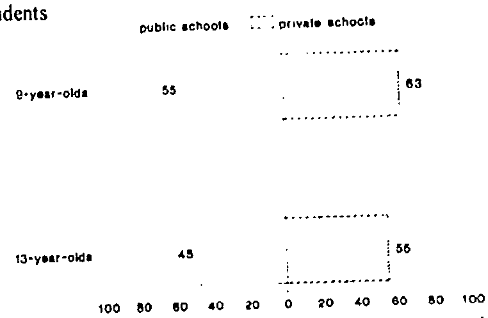


The administrative region where the school is located is found to be related to the students' performance in both age groups. The pattern of this relationship is also pretty much the same across cognitive processes and content areas. We can, therefore, speak about some heterogeneity within the country in terms of the average performance in math.

The country's partition in terms of kinds of community (rural, semi-urban and urban) also exhibits some disparities in the outcomes. Considering either the test all together, or by cognitive processes, or even by content areas, the pattern just repeats itself in both populations: semi-urban schools appear to have better average results, shortly followed by urban schools' and lastly, by rural ones.

The contrast between public and private schools is displayed in Figure 7.

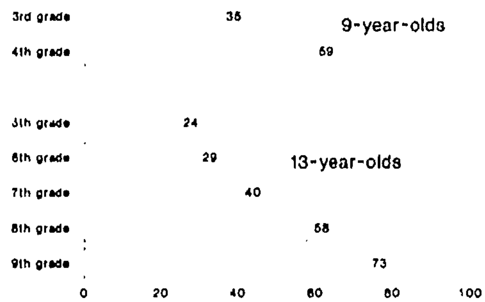
Figure 7 - Average percent correct scores in the test all together in public and private schools - 9- and 13- year-old students



Although the number of children attending private schools is much smaller than the number of youngsters in public ones, the results illustrate a somewhat better achievement of the students in the first kind of schools

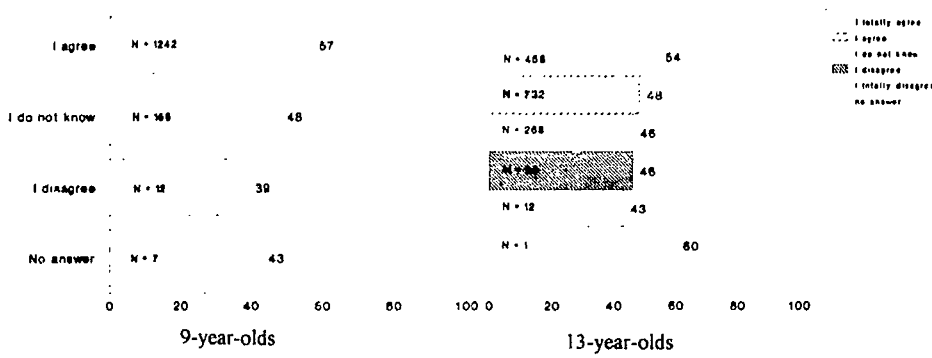
One of the research questions that was thought to be very important had to do with the repetition of grade that was allowed within the portuguese school system at the time of the study. Therefore, the performance of both populations was studied once they were discriminated by grade. The outcomes of this scrutiny are shown in Figure 8.

Figure 8 - Average percent correct scores in the test all together by grade - 9- and 13-year-old students



One interesting finding that results from the observation of the previous figure is that the lower are the grades, (and the age is constant within each population), the worse the outcomes become. And this rule is also true across cognitive processes and topics, and for both age groups, though the differences seem to be magnified in the older students. In summary, students who are kept at lower grades have results far apart from their more advanced peers and, more importantly, reach, in average, very low levels of achievement. It must be noted, however, that the effect of repeating the grade is here, confounded with the fact that the students were exposed to less advanced curriculum, and the test included issues demanding a deeper and wider elaboration, which is not available for children held in the same grade for multiple years.

Figure 9 - Average percent correct scores in the test according to the attitude towards the statement "My parents want me to be good in math" - 9- 13- year-old students



The observation of Figure 9 above indicates that nine-year-olds, as well as 13-year-olds seem to correspond to the perceived expectations of their parents with respect to their achievement in mathematics.

Figure 10 - Average percent correct scores in the test according to the attitude towards the statement "I am good at math" - 9- 13- year-old students

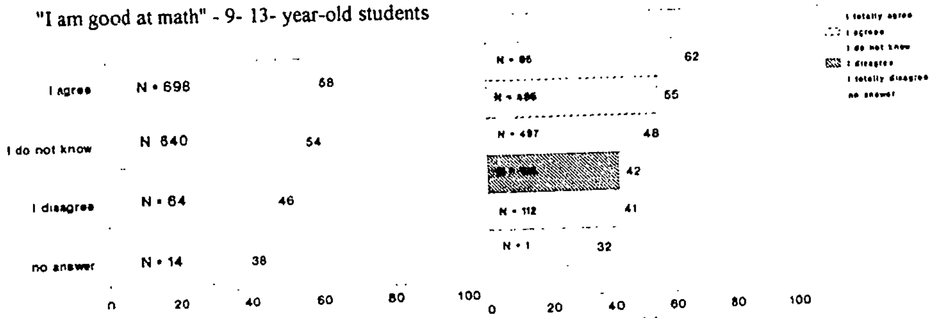


Figure 10 illustrates a strong relationship between students' perceived competence in mathematics and actual achievement in this domain. This relationship apears to be present both in the youngest and in the oldest population

Two home activities were selected to be examined in this presentation, with respect to their potential relationship with student performance in math: Time spent daily watching TV and time in a week spent doing math homework.

Figure 11 - Average percent correct scores in the test and time spent daily watching TV - 9- 13- year-old students

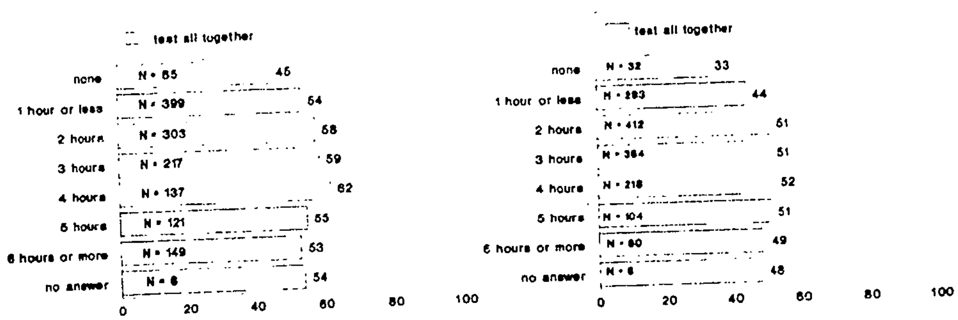
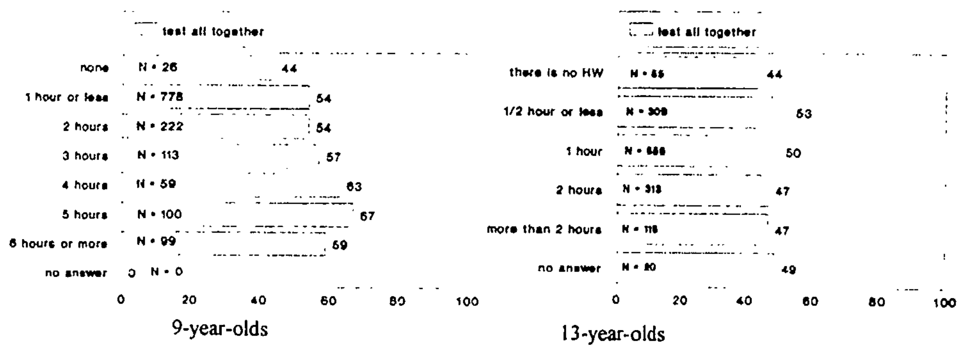


Figure 12 - Average percent correct scores in the test and time spent during a week doing math homework - 9- 13- year-old students



For both age groups there seems to be very uneven distributions of answers regarding these types of activities, and a nonlinear relationship between the regular duration of these activities and the youngsters' outcomes.

Conclusions

Portuguese students displayed a low achievement in average when compared to the students in most of the other participating countries.

Although the country occupies a small area, there appears to be a somewhart large heterogeneity among regions and kinds of community.

Private schools show results that are better than public shools', though the proportion in students attendance to them is very unbalanced.

The fact that a student can repeat a grade does not seem to bring much improvement to his/her degree of success: in both populations, the average scores are worse as the number of (presumed) repetitions increases.

Student attitudes and expectations towards math appear to be related to their performance in this area.

With respect to time watching TV and time spent doing math homework, there seems to exist a non-linear type of relationship between each of these children's home activities and their performance in mathematics.

References

IAEP (1992). *Learning Mathematics.* Princeton, NJ: Educational Testing Service.

Alternative Pathways in the Transition from Arithmetic Thinking to Algebraic Thinking

Ted Redden
University of New England, Armidale. Australia.

Abstract

Some recent discussion at PME and elsewhere has focused on the transition from arithmetic thinking to algebraic thinking among school children. This paper uses data from children's descriptions of number patterns, together with the theoretical framework of the SOLO taxonomy to suggest that there is possibly a number of pathways for children's thinking to take as they make the transition. A model is described that may assist in the mapping of these pathways.

Introduction

In recent years there has been considerable focus on the developmental patterns that represent the transition from arithmetic thinking to algebraic thinking. (Lins, 1992; Redden, 1993; Sutherland, 1992; Mason, 1992; Wagner and Kieran, 1989). In her report to the PME Working Group on Algebraic Processes and Structures, Sutherland posed the questions:

> What shifts in the structure and focus of attention characterise movement from arithmetical to algebraic thinking? What mathematical situations are likely to lead students to encounter the possibility and desirabilities of these shifts?
>
> (Sutherland, 1993)

Mason (1985, 1992) popularised an approach to developing algebraic thinking through the expression of generality in contexts that involve some form of number pattern. This approach has been adopted and developed by others (Pegg and Redden, 1990a; Pegg and Redden, 1990b; NSW Syllabus, 1989; Romburg, 1989; Australian Education Council, 1991). While the approach has had considerable support, the research evidence (Pegg and Redden, 1990; Arzarello, 1991; Redden, 1993; MacGregor and Stacey, 1993) that has been reported on this approach has focused on classification of children's responses rather than the dynamics of child development. This paper asks can a developmental hierarchy be postulated from an analysis of children's responses to questions involving interpretation of number patterns?

Method

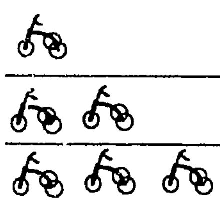
To investigate this question 1435 children aged 10 to 13 were presented with 4 number pattern stimulus items and were asked four questions about each item. The questions were:

(a) What is the next term in the pattern?
(b) Describe a general rule for the pattern in natural language.
(c) Calculate the value of an uncountable term (eg. n=80).
(d) Write their rule in the symbolic notation of mathematics.

The discussion in this paper focuses on the children's natural language description of question (b). Responses to the other questions were at times used to clarify children's intended meaning in question (b). The four stimulus items varied in the complexity of their data and in the context of the number pattern. In figure 1 it can be seen that stimulus item 1 has a simple data structure that involves only one arithmetic operation in the function. The other items involve two arithmetic operations. The data for stimulus item 1,2 and 4 are presented in a "real world" context while stimulus item 3 provides the data in the context of a function table.

Stimulus Item 1

A tricycle manufacturer needs to know how many wheels he needs for different sized orders.



order for 1 tricycle

order for 2 tricycles

order for 3 tricycles

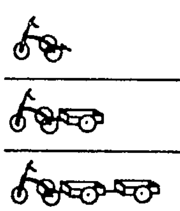
Stimulus Item 3

I have a computer that turns the number in the top row into the number in the bottom row.

□	1	2	3	4		
△	2	5	8	11		

Stimulus Item 2

The same tricycle manufacturer decided to make trailers for his tricycles so that you could make bike trains to carry lots of people.



1 person bike train

2 person bike train

3 person bike train

Stimulus Item 4

Here are some chains of squares built using matches.

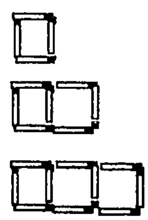


Figure B.1
Stimulus Items

The SOLO taxonomy (Biggs and Collis, 1991) was used as theoretical framework for interpreting the data, since it provided a method of looking across item types and a method of hypothesising hierarchies of development. The taxonomy suggests there are five modes of representation each consisting of three levels of development. The mode of relevance here is the concrete symbolic mode since the children are being asked to use a symbol system to describe a pattern that has a concrete referent. The child may be using the imagery of the ikonic mode as support in the formulation of their responses, however the target mode, in the terms of Biggs and Collis (1991), is the concrete symbolic mode. The three levels in the target mode are unistructural, multistructural and relational. A unistructural response (U) involves the child focusing on one relevant aspect or one piece of data of the stimulus item. Multistructural responses (M) focuses on more than one relevant aspect or data element but fails to integrate then. The sequential nature of these responses differentiate them from a relational response (R) in which an integrated understanding of the relationships between the data is evident.

Results

In interpreting the results in terms of the SOLO taxonomy, there appear to be two hierarchies of growth. One concerns the use of data from the question, and the other concerns the sense of an overview of the data that can be provided in the form of an expression of generality in their pattern description. It would appear that these two concepts are not necessarily developed in parallel. But a child may in fact develop in one or the other independently.

Consider the two dimensions alluded to above. The first dimension has been called the data processing dimension (D). It refers to the number of data elements from the stimulus item that the child can hold in short term memory and successfully process. The second has been called the generality dimension (G) and refers to the child's ability to describe a general relationship that provides an overview of the information that is provided in the stimulus item. In the following, a developmental hierarchy is postulated for each dimension and then their possible interaction is discussed.

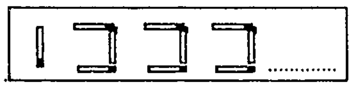
1. Data processing dimension.
Children exhibited the ability to deal with differing number of data elements. By data elements it is meant the number facts provided by the stimulus item. For example in the first stimulus item, there is one element of data and it is that each tricycle has three wheels. In the second and subsequent stimulus items, there are two data elements. The data elements are similar in structure for stimulus items 2 and 4 and consist of the value of the first term of the sequence and the incrementing value. It is argued that a multistructural response is adequate to present the data. In contrast, the data for stimulus item 3 requires two arithmetic operations to be identified for one ordered pair and then tested across the other ordered pairs. This is viewed as requiring a relational understanding of the data.

Some children showed that they can respond to stimulus items that have only one piece of data, but were unable to provide accurate descriptions when more than one piece of data needed to be processed. Other children seized on one piece of data from situations when more than one piece of data was supplied and attempted to generalise from this restricted data set. eg

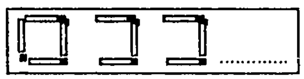
A chain of 80 squares needs 320 matches.

This group of responses could be described, in SOLO terms, as lying in the concrete symbolic mode at the unistructural level (U_D). They have been classified as unistructural responses since they use only one piece of data at a time.

The second group are classified at a higher level than U_D since they were able to process more than one piece of data successfully. Hence they should be classified at concrete symbolic multistructural (M_D) or concrete symbolic relational (R_D). The nature of the stimulus items placed an upper limit on the quality of the response that the children could produce. In stimulus items 2 and 4 the data was presented in such a way as to allow an adequate data set to be able to be produced by a multistructural analysis of the items. Just what these elements of data were depended on the child's perception of the question. In stimulus item 2 some children saw that three wheels were needed for the first person and then two wheels were needed for each remaining person. Others perceived the data as two wheels are needed for each person with one extra wheel needed for the tricycle. As with stimulus item 2, the squares constructed from match sticks in stimulus item 4 were perceived in two different ways.. Some children see the pattern as one match stick with successive additions of three match sticks.



Others see the pattern as four match sticks with successive additions of three match sticks.



These views of the data are analogous with Biggs and Collis' description of multistructural responses as a set of unrelated elements. Children who can adequately describe this data are capable of responding at a multistructural level, of higher, in the concrete symbolic mode. The child is not forced to describe relationships between parts of the data to identify an adequate data set.

On the other hand, in stimulus item 3, such a multistructural view of the data does not lead to an adequate data set. The children who described the data as "the top line goes up by one and the bottom line goes up by three" do not have the necessary overview of the data to describe the relationship between the top line and the bottom line. The data that has to be identified is a set of arithmetical operations that has to be performed on the top line (independent variable) to give the values in the bottom line (dependent variable). Because a common set of elements to describe the relationship has to be found, the identification of an adequate data set for stimulus item three is classified as concrete symbolic relational (R_D).

2. Expressing Generality Dimension.
On coding responses to component (b) of the stimulus items four categories were identified. The first was called Pre-structural and reflect the child's inability/unwillingness to successfully engage the stimulus items. The term pre-structural is borrowed from the first version of the SOLO taxonomy (Biggs and Collis, 1982). It is possible that these children are operating in the ikonic mode and are relying on imagery and intuition but are unable to express their thoughts in the concrete symbolic mode as required by the question. This group have not been included in this analysis.

The first group that experienced some success in responding to component B of the stimulus items were those students who were able to give a specific example rather than a general pattern description. Some examples of these response are:

> *Five tricycles need 15 wheels*
> *Eight squares need 25 matches.*

Since they focus on one example these responses are classified as concrete symbolic unistructural (U_G).

The next group identified were described as successive. The tendency in this group is to focus on outcomes only and to describe how to move from one term in a sequence to the next term. Some examples are:

> *You start with four and ust keep on adding three.*
> *You add one to the top line and three to the bottom.*

This group has been classified as concrete symbolic multistructural (M_G). Under some circumstances there is a tendency to see successive descriptions as a relatively sophisticated iterative function. Such a function, however, implies the existence of a feedback loop to enable the calculation of the next term as in;

$$Y_{n+1}=Y_n+3$$

Such strategies have relevance in many mathematical situations such as Newton's Method for approximating the roots of a function. In many computer settings similar strategies are used for incrementing variable values. Some example are the For-To-Next loop in BASIC programming, recursion in the LOGO environment and spreadsheets. However, these iterative functions are not the focus of the successive category found in this study. The predominant feature of the successive description was the "keep on adding" focus. No description that implied a sophisticated iterative process was identified.

The final group are classified as concrete symbolic relational (R_G) since they have presented an overview of the data by describing a functional relationship between a dependent variable and an independent variable. Some of responses in this category were:

> *You times the number of tricycles by three to get the number of wheels.*
> *The number of squares is multiplied by three and then add one.*

Three categories in each if the data processing dimension and the expressing generality dimension have been identified. The following section synthesises these dimensions into a developmental model.

Towards a Model of Development

Using the above classification of the children's responses into SOLO levels we are in a position to hypothesise a hierarchical model of development for pattern descriptions. Figure 2 is a diagrammatic representation of the model. The model represents children who are capable of responding in the concrete symbolic mode and their earliest entry point to the model is in the top left hand cell (U_DU_G) which represents the unistructural level of both the data processing dimension (U_D) and the expressing generality dimension (U_G).

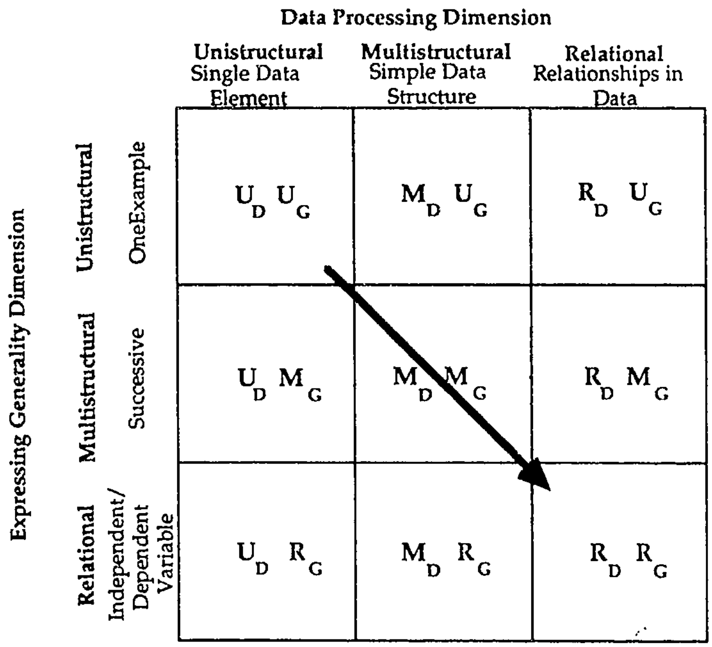


Figure 2
Hypothesised Development Model

In general the desired development path would be represented by the diagonal line in figure 2 This line represents a transition from being unistructural in both dimensions (U_DU_G) to being able to respond at the relational level in both dimensions (R_DR_G). However it is clear from the survey data that the progress exhibited by children is probably not linear, nor restricted to the diagonal cells. It can be seen that there is a possibility of a large number of paths from U_DU_G to R_DR_G. Let us consider some of the development paths that are evident in the survey data. It is not the researchers intension to quantify the likelihood of a student following any of these paths, but rather to provide evidence for their existence. A little later in the discussion some possible reasons for the variability in the paths will be discussed.

1. The path from U_DU_G to R_DU_G. These children demonstrated an ability to identify the appropriate data elements. While their pattern description was low on the generality dimension they were able to provide the correct answer to an uncountable example in component C. Thus they had correctly identified the necessary data set to provide a general description but their response to component B indicated that the concept of expressing generality had not yet developed.

2. Path from U_DU_G to U_DR_G. These children demonstrated an ability to express generality for patterns that had one data element such as stimulus item 1. They were able to write succinct sentences that relate the number of wheels (dependent variable) to the number of triangles (independent variable), but were unable to write such relationships for the more complex data sets of stimulus items two, three and four.

Another set of responses that provided evidence for the existence of this path were those that chose one piece of data from a more complex data set and based all the answers to the components of that stimulus item on this restricted view of the data. An example of this type of response is the group of students who focus on the fact that a square has four sides in stimulus item 4 and then went on to describe the rule as:

Multiply the number of squares by four to get the number of matches.

and then answered component C with:

80x4=320

thus ignoring the fact that only three matches were required for each additional square.

3. Interaction Paths. Thus far in the discussion the two dimensions of growth have been treated as independent of each other. However, there was some evidence of an interaction effect between the two dimensions. This interaction effect was made evident by the variability in response structure across different stimulus items. One common difference was for children to provide a relational description (R_G) to stimulus item 1 and a successive response (M_G) to the more complex data sets of stimulus items 2, 3 and 4. In terms of the model in figure 2 this represents a shift from U_DR_G to M_DM_G. Of course this is not a movement over time since both responses were collected on the same day. A plausible explanation is that the reduction in the level of expression of generality is a consequence of the additional cognitive load brought about by the need to process a more complex data set.

Conclusion

This paper has provided a description of a cognitive map of changes in students responses to number pattern stimulus items. It provides a methodology for explaining the variability in these responses, both between students and between item type.

The SOLO taxonomy has been used as a framework to order the set of responses on both dimensions of the model. The model has been used to predict item difficulty among a number of number pattern questions. These predictions correlated well (rho>0.9) with the hierarchy of actual responses produced by a Rasch analysis of the data. While this support for the model was encouraging, further confirmatory investigation has taken place.

Data from a two year longitudinal study of student responses has been collected. This data will be used to further investigate the validity of the model.

References

Arzarello, F. (1991). Pre-algebraic problem solving. Paper presented at a seminar on problem-solving. Viana do Castelo, Portugal.

Australian Education Council (1991). A National Statement on Mathematics for Australian Schools. Carlton, Curriculum Corporation (Australia).

Biggs, J. B., & Collis, K. F. (1982). Evaluating the Quality of Learning :The SOLO Taxonomy. New York: Academic Press.

Biggs, J. B., & Collis, K. F. (1991). Multimodal learning and the Quality of Intelligent Behaviour. In Rowe, A.H. Intelligence: Reconceptualisation and Measurement. Hillside, New Jersey. Laurence Erlbaum.

Board of Secondary Education. (1989). Mathematics Syllabus: Years 7-8. Board of Secondary Education. Sydney.

Lins, R. (1992). Algebraic and Non-algebraic algebra. In W. Gleesin & K. Graham (Ed.), Sixteenth PME Conference, 2 . Durham: University of New Hampshire.

Macgregor, M. and Stacey, K. (1993). Seeing a pattern and writing a rule. In Hirabayashi, I. (Ed). PME XVII. Tsukuba, Japan.

Mason, J. (1985)) Routes to Roots of Algebra. The Open University Press. Milton Keynes.

Mason, J. (1992). Some Reflections on the Teaching and Learning of Algebra. In R. Sutherland (Ed.), Working Group: Algebraic Processes and Structure. Durham: International Group for the Psychology of Mathematics Education.

Pegg, J. and Redden, E. (1990a). Procedures for, and experiences in, introducing algebra in NSW

Pegg, J. and Redden, E. (1990b). Student experiences in developing algebraic symbolism using number patterns. Paper presented to 13th annual conference of the Mathematical Research Group of Australia. Hobart, Tasmania.

Redden, E. (1993) The transition from Arithmetic to Algebraic Thinking. In Sutherland, R..(Ed) PME Working Group: Algebraic Processes and Structure. PME XVII Tsukuba. Japan.

Romburg, T. (1989) Curriculum and Evaluation Standards. National Council of Teachers of Mathematics. Reston.

Sutherland, R. (1993). PME Working Group: Algebraic Processes and Structure. PME XVII. Tsukuba. Japan.

Wagner, S., & Kieran, C. (1989). An Agenda for Research on the Learning and Teaching of Algebra. In S. Wagner & C. Kieran (Eds.), Research Issues in the Learning and Teaching of Algebra Reston: NCTM.

GENERALIZATION AS A BASIS FOR ALGEBRAIC THINKING: OBSERVATIONS WITH 11-12 YEAR OLD PUPILS

Maria Reggiani
Department of Mathematics - University of Pavia (Italy)

Generalization, intended as the ability to pass from the particular to the general or also the ability to see the general in the particular, represents a fundamental element of algebraic thinking that would otherwise be simply working with symbols. Moreover it is a common experience that often the pure technical aspect prevails in didactic practice and the operative ability sometimes hides the not understanding of the general significance of that which is being done. The bases of algebraic thinking are laid when the properties of the operations between numbers are learnt and one starts to work with symbols in various contexts (arithmetical, geometrical, data processing), but the acquisitions in this field must be considered as a gradual achievement and must be continually consolidated. Our research study proposes to determine the moments of origin of some of the difficulties that consequently cause errors and misunderstandings in working algebraically. This report refers to the level of generalization in 11-12 year old pupils, showing how the ability to grasp the generality of the result may be apparent and trying to understand when it corresponds to real awareness of general relationships.

INTRODUCTION

As is well known, it is customary with the term algebra to indicate a part of mathematics in which various components cohabit from the theories on the solvability of equations to which the name is historically connected, to the manipulation of the symbols in literal expressions, to the study of relations and of structures.

Generalization, intended here in the current sense of the term used in the field of mathematics, that is being able to pass from the particular to the general or being able to see the general in the particular, represents in every case the preliminary and constituent aspects of algebraic thinking that would otherwise be reduced to a set of rules or to a series of formal steps that are carried out mechanically.

Numerous studies, as is well known, both theoretical and empirical have examined the different components of algebraic thinking, they have determined the obstacles in relation to the different age groups and to the contexts in which the algebraic thinking is proposed. In this report we will limit ourselves to quoting those that we intend to refer to.

In recent years there have been several very interesting research studies that have studied the relations between the algebraical language and programming languages, underlining some contributions that the use of particular programming environments can lead to the correct use of the variable and more generally to the training of working algebraically (Capponi-Balacheff 89, Chiappini-Lemut 91, Rojano-Sutherland 93, Ursini 91,...). Some of our research studies on the learning of a programming language with 11-14 year old pupils have highlighted the coexistence of specific difficulties connected to the programming environment with the difficulty connected to the requirement of formalization, that could be revealed in the use of any formal language (therefore also the algebraical language), and with more profound difficulties connected to the conceptualization of the structures involved. These last difficulties could be connected with that of generalization. We refer in particular to the difficulties in the use of the variable (Reggiani 92) and to those in the use of the conditional structure, where it has been noted that the pupils did not know how to detach themselves from the real case in which the alternatives are obviously given one at a

time: therefore they did not manage to consider the alternatives abstractly and therefore as coexistent (Reggiani 91).
Following these observations the attention of our research group moved towards the determination of similar difficulties in the arithmetical-algebraical area. A particular objective was to try and determine some of the stages of awareness of algebraic thinking, considering among the various aspects in particular that of "thinking in general". It has been noted that in didactic practice it is often observed that the acquisition of mechanical procedures is prevalent on the awareness of the formal rules to which they correspond and that the ability to solve a problem translating it into a formula or the mastery of the syntactical rules not always corresponds to the awareness of the generality of what is being done.
In this report we intend to propose some observations made in the scholastic environment regarding 11-12 year old pupils, we tried to make them take the first step towards the gradual achievement of generalization with an activity that takes into consideration, as a particular example of the problem, only the search for a regularity and its motivation. The observations were made by means of analysis of the protocols and class discussions.

THEORETICAL OUTLINE

It is known that the term generalization is used in Psycology and in Philosophy with various meanings.
They can be connected to two fundamental activities of the knowing subject:
a) to express regularities or general properties that are considered included in the "particular";
b) to recognise and to carry out the possibility to pass from a particular to another new object, said to be "general", able to represent the particulars.
In experimental Psycology the question is studied as an "ability" of the subject or also as the development of such ability: sensory elements or mental processes with which the generalization activity is performed are determined.

A brief glance at the history of thought shows us that the problem has been posed since ancient times. Simplifying, one can say that there seems to be the following alternative: is that which generalization attains something which must be discovered or something that must be produced, or even invented? Does it have its own preconstituted objectivity, or is it a construction given by us, in which case it is receiving objectivity from us?
In the first case (Plato, and to a lesser extent Aristotle in ancient times, Frege in the contemporary age) it must be recognised by us, in the second case (the Stoics and recently J.Stuart Mill, B. Russell) it will be elaborated as an image corresponding to particular or "individual" objects.
The point of view of this research study does not exclude the validity of any of the above mentioned meanings, but somehow sets them aside. if one of them were chosen it would arbitrarily impoverish the teaching-learning of algebra, which can not be identified with the possible philosophical or psycological interpretations of its objects.

The term generalization in algebra has a very specific meaning. It coincides with the construction of "variables" that have two complementary aspects: a) expression of a cognitive activity of the subject, b) part of the objective knowledge (Dorfler 91, p.84).
In the context of algebra, generalization reveals itself with the introduction of a symbolic term (algebraical symbol) able to take on numerical expressions or other symbolic expressions of an inferior level.
There are present in literature different possible models to describe such process (let us remember for example those of Dorfler, concerning not only algebra, and Mason).

1176

A phase that is recognised as central is the construction of schematizations of various types (schémas) that can be interpreted as a point of support for the construction of "schèmes" in the Vergnaud sense.

The research study described in this report is limited to studying in an experimental way the "spontaneous" abilities to pass from "particular" to "general" in 11-12 year old pupils, stimulated by a guided procedure, step by step on single concrete examples. It is believed that the possibility to generalize, in this learning phase and at this age, can for many pupils be placed in the "proximal development zone" (Vigotskij), that requires stimulation from an external intervention to be activated. Let us make clear that with the term "spontaneous" here we intend to refer exclusively to the fact that the pupils have not been taught algebra, but we could say that they are in a "prealgebraical" phase

It is thought that generalization is a gradual process, a non continuous acquisition, connected to the knowledge of algebra but not coinciding with it. Also when the pupil shows good mastery of the use of the letters the process can not be considered concluded. In effect it could be a mechanical acquisition as is often shown by class observations also with 14-16 year old pupils, an age in which the Italian scholastic system gives most space to algebra. It is not said that being able to solve a problem by setting out an equation indicates sufficient awareness of algebraic thinking. In the same way indicating confidently an even number with 2n or an uneven one with 2n+1, does not guarantee that it does not mean a repetition of formulae that have nothing to do with a real generality.

The pupils that were taken into consideration, as we will see, are only partially able to see the problem in general, even though they certainly have good mastery of the formulae of the areas of geometrical figure planes, formulae that represent one of the pupils' first contacts with generalization but are not acknowledged as such. Also the correct use of the variable in a programming area is not, in our opinion, a guarantee of acquisition of its meaning in as much as it can simply be seen as the assigning of a name to a memory space to be used and therefore present itself as a concrete and particular element, not abstract and general.

The algebraic game is instead a continual interaction between syntax, that operates at the level of general expression, internal semantics, intended as the comprehension of the meaning of the rules independently from the context and external semantics, that is reference to a specific context and therefore "reparticularization". These aspects of algebraic thinking and of its learning have been studied in great depth by Arzarello et alii, in which the semantics of Frege is integrated with the contribution of linguistics

RESEARCH METHODOLOGY

The research study is part of a vaster one concerning the formation of the bases for algebraic thinking carried out in some first year classes of lower secondary school (11-12 year old pupils), which aims at focusing on the fundamental points of the introduction to algebra, that have been determined in the mastery of conventions and in the ability to distinguish them from the formal properties of the operations, in the ability to pass from verbal writing to symbolic writing and vice versa and in the abil y to generalize. In the frame of this research study a simple problem was given to the pupils th , solution of which did not depend on the data introduced and we tried to guide the pupils to a pc nt where they could free themselves of particular cases and try and find a general explanation. The text of the problem was taken from unpublished material produced by the didactic research group of Genova, with which in that period we exchanged material and compared results. In that work the problem was presented with the same text but with different questions and in a completely different context. This text was chosen and it was decided not to modify it in as much as it appeared significant for reasons that will be listed in the a priori analysis. The original aspect of our work consists of, in our opinion, the subdivision of the questions and the consequent analysis of the results.

Therefore it was not believed important to carry out a comparison with protocols relative to the exercise which inspired us.
The exercise was a written exercise and done individually. The class discussion was conducted by the teacher after having examined the protocols, which provided useful information.
The analysis of the protocols was carried out in part collectively (discussion within the group concerning the significant protocols or the difficulty of attributing them to the established typologies).
After analysing the results guided activities were done in the classes aimed at widening the number of pupils able to grasp the generality in situations of this type and to code in a symbolic way or verbally their acquisitions. The verbalization-symbolization aspect assumed a fundamental role in the research study, as did the use of representations, arrow schemes, tables. It must be remembered that some of the pupils had carried out specific activities on representation proposed by a work group coordinated by A. Pesci (Pesci 93).

THE PROPOSED TEST

Consider the following game:

Think of a number, double it, add 5, subtract the number you thought of, add 2, now subtract the number you thought of again and then multiply the total by 3

Your friend says that whatever number you think of, the result will always be 21.
Try and see if he is right.

a) Start with number 7 and do the necessary calculations. Write the calculations.

b) Now start with number 20 and then, 100 and do the same thing. Write the calculations.

c) Try to understand why you have always got 21 and write your observations (you can use schemes, letters or other symbols to help you).

d) Was your friend right? Why?

A PRIORI ANALYSIS

The simple game that was given to the pupils as an exercise is one of a typology of problem games often used by the pupils spontaneously. The first problem in activities of this type is that of decoding the verbal language that at this age can be done in various ways. The simplest and most spontaneous is the subdivision in steps and then performing the operations one by one as they are requested. If this is done orally or mentally and on concrete cases, there is no trace of the operations performed and therefore it is very difficult to give an explanation of the result obtained even if it is or is not dependent on the initial value. This is why it was suggested the pupils start with single examples and that they write the calculations.
In the specific case the result is independent from the initial number in as much as at first it is doubled and then the initial number is subtracted twice without any other multiplications or divisions being carried out in the meantime. The explanation is not easy and requires adequate formalization or however good ability to generalize.
Points a) and b) were aimed at making the pupils carry out what is required in particular cases.
Point c) wanted to make the pupil reflect on the result obtained and then to try and give an explanation. The use of schemes was suggested thinking of the work previously done in this field in some classes (see research methodology), or that of letters or symbols in order to help the pupils

leave aside the particular cases, for example by means of the writing of an expression or an arrow scheme or a sequence in the form of a flow chart (some pupils were learning a programming language).
Point d) was inserted after a long discussion within the group. Some group members feared that it could hinder, in as much as, in their opinions, it did not add anything new compared to the previous questions and could therefore upset those pupils that having already answered in general terms point c) would think that further explanations should be given.
Nevertheless it was inserted with the aim of taking the pupils back to the original question "Your friend says that WHATEVER number you think of, the result will always be 21. Try and see if he is right." It is in the "whatever" that the generality of the statement lies and therefore, after the fragmentation that is typical of this type of activity, it is our opinion that it is necessary to bring the pupils back to reflecting on the "whatever", so as to discover first of all, if someone does not believe in the generality of the result, but is anchored to the concreteness of his particular cases and then, to lead who is satisfied with a general observation (we will see that most behave in this way) to find the causes.

ANALYSIS OF THE ANSWERS

The exercise was given to 105 pupils belonging to five first year group classes of lower secondary school. One of these classes at that time had experience of working with a computer and three had carried out work on the use of arrow schemes in problem solving, particularly in multiplicative inverse problems. From the analysis of the protocols and taking into consideration the theoretical outline and the a priori analysis, the following typologies were determined for the answers to points c) and d) that were considered collectively:
I) correct answer with complete and general reasoning explanation of the type "I always obtain the same result because I multiply the initial number by 2 and then I subtract it twice"
II) Correct answer with a justification that did not exactly grasp the reason why the result was always the same but tried to generalize. Often it involved a form of verbalization that was a little different from that proposed by the text, even if it probably was significantly for the pupil who wrote it, in as much as it was, at least in part, an interpretation.
An example:
"I always obtain 21 because I multiply by 2, then I add 5, then I subtract the number, I add 2, then I subtract the number again and get 7 that I then multiply by 3."
Another type of justification can be represented by the following:
"I always obtain the same number because if I multiply, add, subtract and add again to different numbers always the same numbers the result does not change."
This explanation seems weaker than the previous one in as much as it is incorrect, it is true only in the sense that the same relation with the initial number is maintained but not that you always obtain the same number, but from the point of view that interests us, it does not seem inferior, in as much as the pupil has tried to say what he did attempting to detach himself from the example.
III) Correct answer with a justification that is not a real explanation but only expressing explicitly an observation: "You always get 7 which multiplied by 3 is 21"
IV) Correct answer with an incorrect justification and often confused as follows procedures seen in other contexts. We found for example attempts to resort to even and uneven numbers or distinguishing between the behaviour of numbers that "finish with zero" from others or phrases such as "at a certain point you get a number similar to the initial number."
V) Correct answer without any justification.
VI) Doubted or did not believe the statement, "I always got 21 but I do not know if I will always get it", " I always get it with whole numbers but I do not know if I will with decimal numbers"
VII) No answer to points c) and d).

The classification proposed here does not intend to list the answers according to merit, also because in our opinion a list in order of merit would be restrictive compared to the complexity of elements involved in an activity of this type. Moreover the answer typologies mentioned above do not take into consideration the use of representations or of symbols that also have an important role in constructing the answer. Nevertheless at least at certain points (for example among types I, II, III) it is believed that they correspond to levels of awareness that can be posed in the same order.
It appears more difficult to us, to compare for example the answers of type VI with the others, that could be due to an inability to detach oneself from the example or, on the contrary, to an awareness of the caution necessary with generalization. The same applies for the pupils who did not answer. In these cases the class discussion is fundamental, although the results of the discussion can not alter this analysis in as much as the situation in which the pupil is placed in the discussion is different.
Points c) and d) have been considered here as a whole.
With a more detailed analysis it can be observed that in most cases the answer is given in point c). At point d) about half of the pupils say that they have already replied, a part repeat what has already been said generally more completely, some, who previously only used symbols, take advantage of the occasion to verbalize, a small part (about 10 pupils in all) pass from an answer that could be classified as type II or III to one of type I.
Our investigation is of a qualitative type, nevertheless it is interesting the quantitative subdivision of the answer typologies:

I	II	III	IV	V	VI	VII
23	15	33	15	7	9	3

The numeric data show that the majority of pupils tried to supply a reason which in most cases is apparently correct (I - II - III).
In reality though, the case indicated with III was not an explanation but only , as previously mentioned, a statement and did not necessarily indicate awareness of generalization.
It can be thought that some pupils placed in II understood that there must be a connection, of a general type, between the initial number, the operations and the result even if they did not know how to use the instruments that they possessed (from symbols, to graphs, to properties of operations, to language) to try to identify this connection by means of common language signs or formalized language. They were, however, in a phase very close to phase I.
For what regards points a) and b) the correctness of the sequence of the operations, that is a correct decoding of the text, was obtained in more than 90% of cases. At least 40% of the pupils used "equal" in a procedural way (about the pupils' difficulties in the correct use of equal sign, see for example Sfard 91,92), as follows:
$7 \cdot 2=14+5=19-7=12+2=$......
This obviously being incorrect, it was also observed that a good number of pupils that used the equal sign in this way managed to supply correct explanations. This can be interpreted observing that this use allows the sequence of operations to be visualized and to grasp therefore such sequence or procedure as a general fact, that is utilizable in any case not yet considered.

Finally we examined and classified the possible symbolic and representative aids used in the answer at point c). In particular we studied the use of arrow schemes, the use of representations comparable to algorithm codings (flow charts or other), the use of numerical expressions.
It was noted, as expected, the wide use of schematizations in the classes where this had already been used in other contexts. Flow charts favoured the production of the correct answer. In some cases, though, they were given as an explanation without being decoded and this highlighted that their meaning, for the students, was purely instrumental.
As clarified in the discussion after, they thought that the scheme, communicable with the computer and that always supplies the same result, was a sufficient explanation. Not always then, does

mastery of a programming language mean having understood the prevalence, from the point of view of the concepts of construction of algorithms in comparison with their calculation.
The students that worked with arrow schemes used them in as much as they had been indirectly invited to do so by point c).
Instruments in general favoured the visualization of the procedure; in some cases though, this did not help with generalization, in as much as here it was not asked for example to use a scheme as an instrument to set out correctly an inverse problem, but to observe it and study it so as to discover the regularities in its construction.

THE SUBSEQUENT STEPS OF THE RESEARCH STUDY

The analysis of the protocols and the clarifications obtained during the class discussion regarding certain points, led us to continue the work with the same pupils trying to act in the margin that separated them from generalization and that seemed very limited in many pupils.
Analysing some of the difficulties that emerged from the analysis of the protocols and in particular starting from the advantage many had from the previously mentioned use of the "equal" sign, we thought of supplying them with the "table" instrument to visualize the sequence of operations done and so to favour the passage to general.
One of the exercises proposed is the following:

Consider the following game:
Think of a number, multiply it by 3, add 7, subtract the number you thought of, add 4, subtract 11.

a) Show in a table the single orders and find the results for at least three different numbers of your choice.
b) Examine the final results obtained.
What did you notice?
c) Is it right to say that by knowing the final result it is always possible to discover the initial number ? Why?

The construction of the table can be done in various steps passing through a phase where the verbal form of the text is subdivided into stages before its symbolization.
The problem here is more difficult in as much as it involves establishing a relation between the final result and the initial value. We will not examine this test in detail here but we will limit ourselves to making some observations regarding the comparison between the results obtained in this test and that which is the object of this report both done by the same pupils.
Almost all the pupils that supplied an explanation of type I in the test examined in the previous paragraphs, confirmed the result, while the others did not know how to establish the relation or they explained it incorrectly or only partially. Some of those who had supplied explanations of type II or III passed to level I, while the others did not know how to establish the relation or how to explain it. Classification variations were noted for all the other typologies even if an overall improvement was observed.
Some considerations can be made regarding these results:

- the table instrument, as observed in other research studies regarding for example the spread sheet, for reasons already mentioned, aids in passing from the particular case to the general.
- the achievement of generalization can not be considered a stable acquisition. It is possible to hypothesize that for some pupils generalization comes and goes alternately.
- it can be difficult to find the relations required by exercises of this type if the formal rules have not been mastered, and on the other hand, mastery of these does not guarantee awareness. Therefore the roles of written verbalization and class discussion are of fundamental importance.

CONCLUSIONS

From the work examined in our opinion the following observations emerge:

- 11-12 year old pupils not educated to think algebraically, nor trained to use formal properties, are able to "sense the general" starting from particular cases.
- some are able to attempt an explanation by means of above all naive verbalization and symbolization and not always, as has been seen, formally correct.
- a part of this latter group are able to reach a spontaneous awareness of the generality discovered even if the level of generalization reached can not be considered a stable achievement.

Moreover it is known that the complicating of requirements and the mechanization of procedures often casts a shadow on the achievements concerning meanings previously acquired. A possible development for this research study is that of studying the obstacles found on the path towards mastering algebraic thinking in the moments in which the use of formalism is most massive and therefore the risk of fractures on the syntax-semantics axis is strongest.

REFERENCES

Arzarello F., Bazzini L., Chiappini G.P. 1992: Intensional semantics as a tool to analyse algebraic thinking, *Proceedings of Workshop on Algebraic Learning*, Torino (in press)

Arzarello F., Bazzini L., Chiappini G.P. 1994: *L'algebra come strumento di pensiero. Analisi teorica e considerazioni didattiche*, Progetto Strategico C.N.R.: Tecnologie e innovazioni didattiche, n 6, printed in Italy.

Capponi B., Balacheff N. 1989: Tableur et calcul algebrique,*Educ Studies in Mathematics*, vol 20, pp. 179-210

Chiappini G.P., Lemut E. 1991: Construction and interpretation of algebraic models. *Proc. of PME XV*, Assisi, Italy, vol.I, pp 199-207

Dorfler W. 1991: Forms and Means of Generalization in Mathematics, in *Mathematical Knowledge: its growth trough teaching*, A.J. Bishop et al. (eds.), Kluer Academic Publishers, Printed in Netherlands, pp. 63-68

Mason J. et alii 1985: *Roots to Roots of Algebra*, Center for Mathematics Education. The Open University, Great Britain.

Pesci A. 1993 : Visualization in Mathematics and Graphical Mediators: an experience with 11-12 year old pupils, in R. Sutherland (ed.) *Exploiting Mental Imagery with Computers in Mathematics Education*, NATO ASI Series F., in press.

Reggiani M. 1991: Computer activities in mathematical problem solving with 11-14 year old students: the conditional structure learning, *Proc. of PME XV*, Assisi, Italy, vol.III, pp.199-206

Reggiani M. 1992: Learning Difficulties in Mathematical Problem Solving with Computers. Some observations in 11-14 year old students, *Proc. WG 17 - ICME 7*, Quebec, in press.

Sfard A. 1991: On the dual nature of mathematical conceptions: reflections on processess and objects as different sides of the same coin, *Educ. Studies in Mathematics*, vol.22, n.1, pp 1-36.

Sfard A. 1992: Equations and Inequalities: processes without objects?*Proc of PME XVI*, New Hampshire, USA, vol.III, p.136.

Rojano T., Sutherland R. 1993: Towards an Algebraic Approach: the Role of Spreadsheets, *Proc. of PME XVII, University of Tsukuba*, Tsukuba, Ibaraki, Japan, vol.I, pp.138 -145.

Ursini S.,1991. First steps in generalization processes in algebra.*Proc PME XV*, Assisi, Italy, vol.III,316-323.

Vergnaud G. 1985: Concepts et schèmes dans une théorie operatoire de la representation, *Psychologie française*,30, pp.245-253

Vigotskij L.S. 1978, *Mind in Society: the Development of Higher Psychological Processess*, Cambridge, Harvard University Press

MEASURING PRE-SERVICE TEACHERS ATTITUDES TO MATHEMATICS: FURTHER DEVELOMENTS OF A QUESTIONNAIRE

Joe Relich and Jenni Way
University of Western Sydney, Nepean

This paper reports on the further development of an instrument designed to measure self-concept in mathematics and the attitude of pre-service teachers toward the teaching of mathematics which was presented at PME 17. It presents the results of the questionnaire's application to teacher education students across various programs at the University of Western Sydney, Nepean. The findings of the study revealed that the students' attitude towards teaching mathematics improved as they progressed through the programs, but that their self-concept in mathematics remained unchanged by the experience. In addition there were clear and significant differences in attitude to teaching mathematics and self-concept in mathematics based on gender and program of study. Males compared to females were found to display consistently higher levels of self-concept and more positive attitudes. Secondary students also displayed significantly higher scores on both constructs than did primary students who in turn scored consistently higher than early childhood students.

Attitudes and Pre-service Teachers

> "An important aim of mathematics education is to develop in students positive attitudes towards mathematics and their involvement in it......The notion of having a positive attitude towards mathematics encompasses both liking mathematics and feeling good about one's own capacity to deal with situations in which mathematics is involved." (Australian Education Council, 1991, p.31)

This extract from A National Statement on Mathematics for Australian Schools reflects the widespread belief among educators in the importance of the development of positive attitudes in the teaching and learning of mathematics.

Although definitions of attitude vary, they generally include the ideas that attitudes are learnt, manifest themselves in one's response to the object or situation concerned, and can be evaluated as being either positive or negative (Leder, 1992). One such definition states...." Attitude is a mental and neural state of readiness, organised through experience, exerting a directive or dynamic influence upon the individual's response to all objects and situations with which it is related " (Allport in Kulm, 1980, p.356). When exploring the attitudes of pre-service teachers toward mathematics it is necessary to not only to consider their attitude towards the subject itself, but also their attitude regarding the teaching of mathematics.

The attitudes of pre-service teachers are of particular importance because of their potential influence on pupils. Although the research evidence is certainly not conclusive it has been sufficient to suggest that positive teacher attitudes contribute to the formation of positive pupil attitudes (Aiken, 1976; Sullivan, 1987). There is an additional concern that female teachers, who constitute the majority of primary school teachers, may be perpetuating negative attitudes with the girls in their classes (Kelly & Tomhave, 1985; Relich, Conroy, & Webber, 1991). Some studies have indicated that teacher attitudes towards a subject and the teaching of that subject influence the instructional techniques they employ and that these, in turn, may have an effect on pupil attitudes (Carpenter & Lubinski, 1990).

One argument presented in support of the need for positive attitudes is that such attitudes can enhance achievement in mathematics at primary, secondary and tertiary level (Dungan & Thurlow, 1989). Most studies on the relationship between attitude and achievement have revealed a low but significant correlation (Aiken, 1976; Kulm, 1980). However, the nature and direction of this relationship is yet to be unravelled (Kulm, 1980; Suydam, 1984).

Unfortunately large proportions of pre-service teachers have been found to hold negative attitudes towards mathematics and mathematics teaching (Becker, 1986; Sullivan, 1987). Some studies have revealed links between attitude to mathematics, the choice or avoidance of mathematical studies, self concept and attitudes towards the teaching of mathematics. It appears that students (both male and female) with low self-concepts in mathematics are less likely to pursue mathematical studies. Not surprisingly then, research has shown that most pre-service teachers who exhibit negative attitude towards mathematics have not chosen to study mathematics in their final years of high school (Aiken, 1976; Relich, Conroy, & Webber, 1991; Sullivan, 1987).

The potential of teacher training courses to change the negative attitudes of pre-service teachers towards mathematics must also be considered. Sullivan (1987) found that almost half of the students entering a teacher training course possessed negative attitudes towards mathematics. He states..."The course improved their attitudes overall, but those who started with negative attitudes still had the most negative attitudes at the end." (Sullivan, 1987, p.1). He concluded that if these initial attitudes are so significant, teacher education courses may need to establish entry criteria based on the mathematics background of the applicants.

Several studies suggest that beginning teachers are especially concerned about methodology as well as content and therefore training courses should cater for this (Aiken, 1976; Blunden & De La Rue, 1990; Mansfield, 1981; Sovchik et al., 1981; Watson, 1987). Relatively high correlations have been found between mathematical achievement, the enjoyment of mathematics and the perception of mathematics being useful, which also carries implications for the design of teacher education courses (Watson, 1987). If the attitudes of pre-service teachers are to be improved, there first needs to be a reliable instrument with which to measure levels of attitudes and perhaps to identify groups of students with special needs (Aiken, 1976; Nisbet, 1991; Watson, 1987).

Measuring Attitudes

In recent years, researchers of 'attitude' have acknowledged that attitude appears to be a multi dimensional construct and therefore requires a multi dimensional measure (Leder, 1992). However, there is some dispute as to which components of attitude, such as anxiety, enjoyment, self-concept and belief in the usefulness or value of mathematics, provide the best indicators of attitude. One component that has received much attention is that of 'mathematics anxiety'. There is little doubt as to the existence of 'mathematics anxiety' and a number of instruments for measuring levels of anxiety have been developed and implemented (Richard & Suinn, 1972). However, Sovchik, Meconi and Steiner (1981) suggest that the construct 'mathematics anxiety' is not as well defined and measurable as assumed by some mathematics researchers. There is some doubt as to whether anxiety is in fact a separate construct. It may just be a reflection of some deeper attitude (Wood, 1988). For example, Bandura (1982) points out a person may judge that a particular action will result in a particular outcome (expectancy judgement), but may doubt his/her ability to successfully perform the action (efficacy judgement), which in turn may be related to existing anxiety or the production of anxiety. Anxiety's relationships to factors such as enjoyment, general attitude towards mathematics and performance are unclear. Wigfield and Meece (1988) concluded that mathematics anxiety and perception of mathematics ability are conceptually distinct, though related, constructs. Others argue that self-concept is a better measure of how people feel about themselves as mathematicians and as teachers of mathematics, and that self-concept has an influence on the formation of attitudes (Relich, Conroy, & Webber, 1991; Relich & Way, 1992). Studies have also found a consistently high positive relationship between self-concept and mathematics achievement (Hackett & Benz, 1989).

Instruments designed for determining the attitudes of pre-service teachers towards mathematics and the teaching of mathematics have been scarce. Nisbet's (1991) instrument consisted of the Fennema and Sherman (1976) 'Mathematics Attitude Scales' plus his own parallel scales constructed to cover the 'Attitudes to Teaching Mathematics' aspect. Neither of these scales included items regarding self-concept. Relich and Way (1992) supplemented Nisbet's instrument with self-concept items developed by Marsh (1988) to form a large composite questionnaire, referred to as the "Teaching Mathematics Questionnaire". The results of this study (Relich & Way, 1992) supported the contention that attitude towards mathematics and the teaching of mathematics can best be defined through feelings about the teaching of mathematics and through self-concept of one's abilities to cope with mathematical tasks. Anxiety did not emerge as a separate variable, but was highly correlated with attitude and therefore subsumed into the attitudinal profiles.

The major purpose of this research was to further validate the relevance of this questionnaire in accurately gauging attitude towards mathematics among pre-service teachers, and to determine how attitude differs based on characteristics such as program of study, year of study, level of previous study of mathematics in high school and tertiary institutions, age and gender. The questionnaire focuses on "attitude to teaching mathematics" as measured by a composite scale (see Relich & Way, 1992) and "self-concept in mathematics". Through a cross-sectional analysis of student data at various levels of study, we can determine whether attitude and self-concept change as pre-service teachers progress through their course and whether changes in attitude are concomitant with changes in self-concept.

METHOD

Subjects

The questionnaire "Teaching Mathematics" developed by Relich and Way (1992) was used to gather information from all students in Education at a NSW university. A total of 564 students participated across a variety of programs with a substantial majority (85%) being female. The subjects represented all the years of study within the programs and were asked to participate on a voluntary basis in completing the forms. They represent approximately 70.5% of the population of students within the Faculty enrolled in these programs. Over 81% studied mathematics in high school at the 2 unit level or less. Those who undertook study at the three or four unit level were represented in all the programs but constituted the majority (67%) of secondary students.

Instrumentation

The "Teaching Mathematics Questionnaire" consists of 20 items, 11 related to specific attitudes on the teaching of mathematics and nine mathematical self-concept items. The former scale is more a measure of "feelings" about the act of teaching and or performing mathematical tasks whereas the latter relates to personal perspective of oneself as a mathematician. The development of this instrument, an adaptation and amalgamation of various currently available instruments, is discussed in detail in Relich and Way (1992, 1993). The 20 items required Likert scale responses on an eight point continuum from definitely false to definitely true. For those statements which referred to practice teaching experiences, participants had an option of indicating that the question was not applicable. The purpose of this category was twofold. First to cater for students in their first year who may not have had such an experience and second to gauge what proportion of students may not view themselves as teachers of mathematics.

A test of reliability using Cronbach alpha suggests that the two sub-scales are highly reliable with coefficients of .92 for the attitude scale and .88 for the self-concept scale.

Procedure

The administration of the questionnaire took place during normal lecture periods when a majority of students within a particular year and program were all expected to attend at the same time. Students were briefly instructed about the purpose of the questionnaire and given information about how to complete it on a voluntary basis. They were however encouraged to participate because the researchers wish to conduct a longitudinal study. While no student can be directly identified in order to ensure anonymity, each student was able to provide an identification number known only to the participant. No more than ten to fifteen minutes of class time were required to complete the task.

Regression analysis was used to examine the results and to determine what set of independent variables were the best predictors of attitude to the teaching of mathematics and self-concept in mathematics. We also used t-tests and oneway analysis of variance to determine if there were significant differences in response patterns based on gender, program and year of study variables. These techniques were used in preference to factorial analysis of variance because the "not applicable to me" response reduced much of the available data from the early childhood and first year students across all programs resulting in either empty cells or cells with very small numbers.

RESULTS

An initial correlation analysis indicates that there are some significant relationships among attitude, self-concept and the set of independent variables. In particular attitude and self-concept are significantly ($r=.62$, $p<.001$) correlated and share over 36% variance. Attitude ($r=.12$, $p<.005$) and self-concept ($r=.10$, $p<.05$) are significantly related to gender, indicating that males are significantly more likely to display positive attitudes to teaching mathematics and register higher self-concept in mathematics than females. There is a similar positive and significant relationship between level of study of high school mathematics and attitude ($r=.15$, $p<.001$) and self-concept ($r=.32$, $p<.001$) suggesting that higher levels of study at this level are associated with more positive attitudes and higher self-concepts. Interestingly, it is significantly more likely that males study high school mathematics at a higher level than females ($r=.09$, $p<.05$). There is also an interesting relationship between age and the two dependent variables. While older students are significantly ($r=.15$, $p<.001$) more likely to have positive attitudes, younger students display significantly ($r=-.10$, $p<.05$) higher levels of self-concept. On an encouraging note, there is a positive and significant ($r=.35$, $p<.001$) relationship between the year of study and attitude, with those in the latter years of study exhibiting more positive attitudes. For self-concept, however, no such relationship exists suggesting little change on this construct as students progress through their programs of study.

Stepwise regression analysis was used to determine which set of independent variables were the best predictors of attitude and self-concept. Gender, level of study of high school mathematics and year of study all contributed significantly (multiple $R=.30$, $p<.001$) to the prediction of attitude to teaching mathematics. For self-concept, however, level of study of high school mathematics provided the most significant ($r=.34$, $p<.001$) relationship which was not significantly improved by the inclusion of any other variable.

Significant gender differences in attitude ($t=3.94$, $p<.001$) and self-concept ($t=3.68$, $p<.001$) were verified through the use of t-tests. In addition significantly ($p<.05$) more positive attitudes and higher self-concepts were found among secondary (m=77.8, s.d.=8.5; m=43.5, s.d.=8.2) students when compared to primary (m=69.2, s.d.=13.0; m=36.7, s.d.=9.8) students who in turn had significantly ($p<.05$) more positive attitudes than early childhood (m=59.1, s.d.=10.3; m=33.4, s.d.=8.2) students. A further oneway analysis based on the year of study confirmed the significant ($p<.05$) attitudinal changes from the second ($m=67.0$, $s.d.=13.9$) year of study to the third ($m=70.4$, $s.d.=11.3$) year of study. No similar changes occurred for self-concept profiles.

These results strongly suggest that while attitude to the teaching of mathematics may be ameliorated through participation in a teacher training programs which focus on the affective state of the student, self-concept seems far more resistant to change. Those who enrol in teacher training programs with high levels of study in mathematics at the high school level, generally tend to have more positive attitudes and better self-concepts. This type of student profile is also much more likely to be exhibited by the males compared to females.

DISCUSSION

The application of this questionnaire to such a large sample and the consequent findings further validates its relevance for teacher trainers in mathematics. It was clear from the results that attitude and self-concept while highly related are separate constructs. Although there were clear attitudinal changes towards the teaching of mathematics occurring in the latter years of study, it is clear that self-concept is stable and not easily malleable. The evidence suggests that pre-service teachers do change their attitudes about their ability to teach mathematics but unfortunately do not perceive themselves as better mathematicians as a consequence. This raises the question as to whether such changes are enduring and even whether such changes are superficial and inconsequential to their overall ability to deal effectively with mathematics lessons. Perhaps the other side of the argument is that as educators we should be grateful for any positive changes that we can engender and that expecting to change self-concept is not a realistic goal. Nevertheless, there is something inherently attractive to a teacher in attempting to change deeply entrenched views about personal ability and the possible consequent positive effect that this may have on the enthusiasm of a future teacher for mathematics.

All the evidence continues to point to differences in attitude and self-perceptions in mathematical ability between males and females. Our evidence is clear that these differences are the baggage that students bring with them from their study at the high school level. Those who studied mathematics at higher levels are more likely to be advantaged and invariably such students are more likely to be male than female. Given the high proportion of females to males, especially at the primary and early childhood levels, we need to continue to seek ways in which to encourage girls to attempt higher levels of study in mathematics. This of course assumes that the level of study has an effect on subsequent levels of self-concept but this is not entirely clear. It is also possible that those with greater self-concept seek out the more challenging mathematics subjects. Whatever the explanation may be, these are issues which we need to continue to study and address if we hope to deliver quality programs which will result in teachers at all levels who are positive not only about their ability teach mathematics but also about their own levels of skill and ability as mathematicians.

REFERENCES

Aiken, L. R. (1976). Update on attitudes and other affective variables in learning mathematics. *Review of Educational Research*, 46, 293-311.

Australian Education Council and Curriculum Corporation. (1991). *A National Statement on Mathematics for Australian Schools*. Carlton, Victoria. Curriculum Corporation.

Bandura, A. (1982). Self-efficacy mechanisms in human agency. *American Psychologist*, 37,122-147.

Becker, J. R. (1986). Mathematics attitudes of elementary education majors. *Arithmetic Teacher*, 1, 50-51.

Blunden, R., & De La Rue, P. (1990). A survey of teacher concerns across one year of teacher training. *Journal of Teaching Practice*, 10, 1-16.

Carpenter, T., & Lubinski, C. (1990). Teachers' Attributions and beliefs about girls, boys, and mathematics. *Educational Studies in Mathematics*, 21, 55-69.

Fennema, E., & Sherman, J. (1976). Fennema-Sherman mathematics attitude scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Psychological Documents* (Ms No. 1225). Washington, DC: American Psychology Association.

Hackett, G., & Benz, N. E. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research in Mathematics Education*, 20, 261-273.

Kulm, G. (1980). Research on mathematics attitude. In R. J. Shumway (Ed.), *Research in Mathematics Education* (pp. 356-387). Reston, V. A.: National Council of Teachers of Mathematics.

Leder, G. C. (1992). Measuring attitudes to mathematics. *Fifteenth Annual Conference of the Mathematics Association of Australasia: Conference Proceedings*.. Richmond, N.S.W.

Mansfield, H. (1981). A report on pre-service mathematics education for primary school teachers in Australia. *Australian Mathematics Teachers, 36*, 12-15.

Marsh, H. W. (1988). *Self description questionnaire: A theoretical and empirical basis for the measurement of multiple dimensions of pre-adolescent self-concept: A test manual and research monograph*. The Psychological Corporation, San Antonio, Texas.

Marsh, H. W. (1990). A multi dimensional, hierarchical model of self-concept: Theoretical and empirical justification. *Educational Psychology Review*, 2, 77-173.

Nisbet, S. (1991). A new instrument to measure pre-service primary teachers' attitudes to teaching mathematics. *Mathematics Education Research Journal*, 3, 34-56.

Relich, J., Conroy, J., & Webber, V. (1991). Gender differences in self-concept: Male and female mathematics teachers as role models. A Report for the Department of School Education.

Relich, J.D., & Way, J. (1993, August). Attitudes to teaching mathematics: The development of an attitudes questionnaire. Proceedings of the Seventeenth Annual Conference of the International Group for the Psychology of Mathematics Education. The University of Tsukuba, Japan.

Relich, J., & Way, J. (1992). Pre-service primary teachers' attitudes to teaching mathematics: A reappraisal of a recently developed instrument. *Fifteenth Annual Conference of the Mathematics Association of Australasia: Conference Proceedings.* Richmond, N.S.W.

Sullivan, P. (1987). The impact of a pre-service mathematics education on beginning primary teachers. *Research in Mathematics Education in Australia,* August, 1-9.

Sovchik, R., Meconi, L. J., & Steiner, E. (1981). Mathematics anxiety of pre-service elementary mathematics methods students. *School Science and Mathematics,* 81, 643-648.

Suydam, M. N. (1984) Attitudes toward mathematics. *Arithmetic Teacher,* 32(3), 8-11

Watson, J. (1987). The attitudes of pre-service primary teaches toward mathematics: Some observations. *Research in Mathematics Education in Australia,* August, 48-56.

Wigfield, A., & Meece, J. L. (1988). Math anxiety in elementary and secondary school students. *Journal of Educational Psychology,* 80, 210-216.

Wood, E. F. (1988). Math anxiety and elementary teachers: What does research tell us ? *For the Learning of Mathematics,* 8(1), 8-13.

CHILDREN'S SYMBOLIZING OF THEIR MATHEMATICAL CONSTRUCTIONS

Anne Reynolds
University of Oklahoma

Grayson H. Wheatley
Florida State University

Abstract

This paper reports on fourth/fifth grade students use of paper, pencils and other materials as they constructed mathematical relationships and solved various non-routine tasks. Participants in the study used these materials to aid them in symbolizing their mathematical constructions and in reformulating their schemes through reflection. It is suggested that students benefit from the opportunity to externalize and elaborate their own meaningful constructions in this way, rather than have the symbolizations of others imposed on them

Background

Reynolds (1993) found that students made rich use of imaging in constructing meaning for mathematical tasks (see also Reynolds and Wheatley, 1992). In contrast to what frequently happens in mathematics classrooms, where students are routinely required to symbolize mathematical relationships in a teacher prescribed manner, these students developed and elaborated their own meaningful ways of symbolizing their mathematical constructions. Dorfler (1991) suggests that an important component of mathematics education is for the learner to be encouraged to make external her/his internal images. Wheatley (1992) argues that learning occurs when the action is accompanied by reflection and suggests that reflection is encouraged when the learner is confronted with the need to take her/his own mathematical actions as an object of thought.

This paper reports on children's use of paper, pencils and other materials, in symbolizing their thoughts and in reformulating their schemes through reflection. It will be argued that these materials helped students to symbolize their mathematical constructions. Further, as these children used their symbolizations reflexively they elaborated their schemes.

Research setting

The research investigated children's mathematical activity in solving various tasks. Four fourth/fifth grade students, Kristin, Elaine, Tracy and Neal, participated in this one year study. Biweekly individual problem solving sessions lasting approximately one hour (which were video-recorded for further analysis) together with field notes taken in the participants' classrooms constituted the data. Tasks which were potentially problematic (Wheatley, 1991) were presented in these sessions. Participants were encouraged to describe the ways in which they were constructing meaning for the various tasks. Because it often happened that when a participant was struggling to make sense of a task s/he was "in the action" and not necessarily able to reflect on her/his actions (Schon, 1983), each participant also watched the videotape of the problem solving session immediately afterwards so that s/he could further explain and clarify her/his thinking. This segment was also video-recorded for further analysis by the researchers.

Analysis of participants' use of materials

During the sessions paper, pencils, rulers, a calculator and other materials as appropriate were available. Participants were encouraged to use pencil and paper in particular only if they needed them. This was somewhat different from the classroom setting where pencil and paper are the normal tools by which a student records her/his work with a task. Because the researchers wanted to focus on the participants' mental activity, the interviewer made it clear from the beginning that the camcorder was recording their words and so we did not need to have a written record of how they had dealt with a task. This was further demonstrated as participants sat and watched the tape of the various problem solving episodes during the session.

It must be remembered that a participant's 'normal' (classroom) way of doing mathematics was with pencil and paper. Thus there may have been times when they used them in this research setting simply because that was something to which they had grown accustomed. However there were numerous occasions when a participant did not use these materials to record their activity or did indicate their deliberate choice of them. It appeared that each participant constructed this research setting as one where their use was by choice rather than because that was what was expected of them. Thus, it is important to examine the ways in which such materials were used by a participant. They seemed to serve a particular purpose

In analyzing participants' use of paper, pencils and other materials we formulated two assertions

- *participants used paper, pencils and other materials to symbolize and explain the patterns and relationships they had constructed;*

- *as they used these symbolizations reflexively they elaborated their schemes*

Each of these assertion will be discussed in the following sections. Tasks referred to are shown in the Appendix

Participants used paper, pencils and other materials to symbolize and explain the patterns and relationships they had constructed Kristin expressed this idea many times during the sessions She frequently reached for pencil and paper in order to record her thoughts, explaining "It helps" The figure she drew was, as she said on several occasions, what was in her mind It was not always possible for Kristin to focus on all the various aspects of her thinking without the aid of pencil and paper, they were like tools, an extension of her mind On several occasions she expressed frustration at how her thoughts were "crowding up" in her mind saying "It's too much again" She described a process of attempting to keep various activities going on simultaneously as she sought a solution to the task This was most poignantly explained when Kristin reflected on her imaging of a 6x6x6 cube

> I tried very hard but it wouldn't fit in It's six by six! I had so many, well, I don't know I couldn't fit it, 'cause I had to look at the insides too. I had to have one picture of the whole thing here (indicating the right side of her head), and there, um, the inside of it (indicating the left side of her head), and there the face of it (indicating the base of her head), and I couldn't fit it all in. I couldn't think about all of it at the same time.

Thus it was that she described an activity in which she had developed three somewhat independent though related components, each expressing an idea that was important in her construction of a solution The problem for Kristin was that all these components needed to be kept in her mind *at the same time* so that she could somehow integrate them into one image and devise a solution. Pencil and paper were thus useful tools by which she could record her mental activity and thus have room in her mind

These symbolizations flowed from Kristin's mental activity as she indicated on many occasions in words such as: "It helps, because it is what I was thinking about in my head." She expressed her frustration with being unable to mentally "see" the bottom of the Rubik's cube in the following way: "I can't see them very well, that was the problem and I can't draw them any better than I could see them!" Kristin expressed clearly the idea that what counted was her mental activity. She must mentally construct meaning for a task, only then could she express those constructions using diagrams

In the session involving the Tigers and Cages task Tracy explained her pencil and paper activity in the following way: "Sometimes when I'm thinking in my head I get mixed up so I just have to write it on paper." She was expressing a need to record her thoughts. She drew rectangles for the four cages and placed the various numbers in these rectangles as she attempted to find a solution. This was a particularly difficult task for Tracy. At the end of a considerable period of time she had just one solution. Her drawing enabled her to hold first the number 5 and then the number 1 constant in the first two cages so that she could concentrate on constructing combinations of other numbers that together with 5 and 1 would add to 15.

In the next session Tracy felt more at ease with the given task. She had several metonymies (Lakoff, 1987) to draw on in devising a solution. Initially she decided: "I don't have to write anything down," indicative of the broader imaging base she felt comfortable about manipulating mentally. However, she began to construct greater complexity in the task as she elaborated her solution for she subsequently said. "I can draw it." She decided she needed to augment her mental activity in this way.

An actual Rubik's cube was available on a number of occasions as participants solved the various cubes tasks. When the Rubik's cube task was first introduced to each participant this Rubik's cube was not in view. Each participant was encouraged to construct a solution based on their prior experiences with a Rubik's cube and, when necessary, the interviewer's descriptions of it as being made from smaller cubes with three such cubes on an edge. Each participant did this with the aid of drawings. However each participant's drawing was quite different from the others

Neal attempted to draw each small cube in 3-D. Tracy drew three columns of three squares, one beside the other. Both Elaine and Kristin constructed an image of the cube without drawing anything. They began to use pencil and paper when they attempted to identify the number of cubes with just two sides showing. Both had identified the relevant cubes to be counted as the middle cube on an edge. Kristin drew a 3x3 grid to symbolize her construction. She had decided that all she needed to do was to determine the relevant number of cubes on one face of the Rubik's cube and multiply that number by six (because a cube had six faces). Her drawing clearly symbolized her construction at that stage. Elaine on the other hand drew six squares connected end to end, a different symbolization.

Something interesting happened when the Rubik's cube was given to Tracy, Kristin and Elaine to aid them in thinking about the task. Tracy was given the cube early on because it became apparent that she was giving little meaning to the question about the cubes with just two sides showing. When given the cube she struggled for a short time but soon identified the appropriate cubes to count by focusing on the "colors" of the small cubes. Thus she determined that she needed to count the middle cube on each edge: "You'd get them from this part right here (touching the middle cube on an edge)." She anticipated that there might be a problem in organizing her counting scheme to account for all these cubes: "It's going to be hard for I may overcount." Though she tried several different ways of organizing this counting activity, she was not able to image the cube in a viable way that allowed her to systematically count the relevant cubes. The Rubik's cube had helped to organize her thinking such that she could identify the appropriate cubes to count. However, even with the cube in front of her she was unable to devise an effective system for counting the cubes.

Kristin was given the Rubik's cube after she had determined the solution to be 24 cubes (4x6) She was not able, even when given the Rubik's cube, to solve the problem of coordinating her counting activity to allow for the problem of the middle cube on an edge being shared with an adjacent face Her imaging of the cube did not aid her in sectioning the actual Rubik's cube in some way such that she could develop a viable system.

Elaine did not see the problem of double counting of cubes shared by adjacent faces when she drew her symbolization of six squares. She calculated 24 small cubes with two faces showing (4 on each face and 6 faces in all). When she was given the Rubik's cube she immediately identified her difficulty and very quickly sectioned the cube into three layers, thus organizing her count of these cubes with two sides showing. She counted four such cubes in each layer of nine cubes (twelve cubes). She re-examined her initial drawing and immediately identified how it had been used inappropriately: she was able to reconceptualize her drawing to see the way faces were being shared.

Elaine quickly and efficiently developed a procedure for counting the cubes with two sides showing because she imaged the Rubik's cube as being in layers. Kristin and Tracy were not able to do so, even though they had the Rubik's cube there to aid them. What counted were their mental constructions Without a method for counting the cubes both their pencil and paper activity and the availability of the cube itself were not helpful. Cobb, Yackel and Wood (1992) have stressed that there is no mathematics in the materials as such; mathematics is a human activity. They have underscored the need for students to build their own mental models as means of constructing meaning for mathematical ideas. This was an idea that was expressed by Kristin on several occasions when referring to her drawings: "It helps, because it was what I was thinking about in my mind." These materials did not help when participants had not constructed the relationships mentally. They were meaningful only when participants had clearly elaborated their constructions of the various relationships. Imagery is personal to the learner as s/he attempts to make sense of her/his world in mathematical terms (Bishop, 1989).

As they used these symbolizations reflexively they elaborated their schemes: On occasion Neal explained that he did not have a well developed image in his mind. His images were often constructed as he reflected on his activity. Pencil and paper were tools that he used to help him record and refine his thoughts. One powerful example of this was in his solution of the Fence Posts task. Initially he drew a square with two fence posts on each corner (one for each side). Upon reflection, he realized that he would only need one post in each corner ("One will do for both"). Thus he erased one fence post from each corner. His diagrams helped him elaborate his constructions.

On many occasions Tracy initially constructed little meaning for a task. Meaning was elaborated as she recorded her ideas on paper and then reflected on them ("Sometimes when I'm thinking in my head I get mixed up so I just have to write it on paper"). In the Mapping task she initially used seventeen sheets of paper to draw the route from home to school. At the end of the session she examined her work: "It's long!" and decided: "I'll just have to make the roads shorter." In the following session she did just that and reduced her map to three sheets of paper. A month later she indicated "I'm going to see if I can get it on one sheet," which she did. By first drawing and then reflecting on her drawing Tracy was able to identify the more salient features of her image and modify that image. She did not just make the routes shorter. Tracy needed to express her constructions through her drawings so that she could reflect on those constructions and further elaborate them

Elaine frequently drew sketchy diagrams that expressed relationships as she was thinking of them. When she began the 3 by 5 Card task, she initially drew a rectangle of no particular size saying: "This isn't exactly twelve by thirty, but this is the twelve side, this is the thirty side." Upon reflection she decided: "Oh, I'd better make it exact," and she proceeded to use a ruler to help her draw a rectangle 12 cm by 30 cm. She adjusted her drawings as necessary to reflect her interpretation of the task. In our final session together Elaine expressed what each participant had alluded to on many occasions:

> Well, instead of like having a picture in your head because, you know, it's not like you can turn around and look at your brain or something; turn your eyes around or something; but I mean, you can picture things in your head and, but it's easier to look at things when they're on paper.

Participants' diagrams were symbolizations that aided them in reflecting on their constructions and elaborating their schemes.

Conclusion

It has been argued in this paper that participants found pencil and paper and other materials useful as they attempted to construct meaning for mathematical tasks. They used these materials in a twofold manner: to symbolize their constructions and as they reflexively elaborated their schemes.

These symbolizations were different for each participant. The data suggests that it is important that students be encouraged to express mathematical relationships in their own meaningful ways rather than have particular symbolizations imposed on them.

References

Bishop, A. (1989). Review of research on visualization in mathematics education. Focus on learning problems in mathematics: Visualization and mathematics education,11(1)7-16.

Cobb, P., Yackel, E. & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. Journal for Research in Mathematics Education 23(1) 2-33.

Dorfler, W. (1991). Meaning: image schemata and protocols. in Proceedings of the Fifteenth Annual Meeting of the International Group for the Psychology of Mathematics Education, 1 17-32. Assisi.

Lakoff, G. (1987). Women, fire and dangerous things: What categories reveal about the mind. Chicago: The University of Chicago Press.

Reynolds, A. (1993). Imaging in children's mathematical activity. Unpublished doctoral dissertation. Florida State University, Tallahassee, FL.

Reynolds, A. & Wheatley, G. H. (1992). The elaboration of images in the process of mathematics meaning making. in Proceedings of the Sixteenth Annual Conference of the International Group for the Psychology of Mathematics Education, 2 242-249.

Schon, D. (1983). Educating the reflective practitioner: How professionals think in action. New York: Basic Books.

Wheatley, G. H. (1991). Constructivist perspectives on mathematics and science learning Science Education 75(1) 9-21.

Wheatley, G. (1992). The role of reflection in mathematics learning. Educational Studies in Mathematics, 23 529-541.

Wheatley, G. H. & Reynolds, A. (1993). The construction and coordination of units in mathematics. Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA, April.

Appendix

1. There are 15 tigers and 4 cages. How many ways can the tigers be put in the cages so that no two cages have the same number of tigers? (Tigers and Cages task)
2. I want to fence in a square. How many fence posts will I need? (Fence Posts task)
3. In a Rubik's Cube, how many small cubes have just two sides showing? (Rubik's Cube task)
4. There are 216 small cubes arranged in a 6 by 6 by 6 large cube. One layer of small cubes is removed from each face of the large cube. How many cubes are removed? (6X6X6 Cube task)
5. How many 3 by 5 cards are needed to cover a sheet of paper 12 by 15? (3 by 5 Card task)
6. Draw a map that could be used by someone to get from your house to the school. (Mapping task)

The research in this paper was supported in part by the National Science Foundation under grant No. MDR 885-0560. The opinions expressed do not necessarily reflect the views of the funding agency.

TWO-STEP ADDITION PROBLEMS WITH DUPLICATED SEMANTIC STRUCTURE.

Rico, Luis. Departamento de Didáctica de la Matemática.
Castro , Enrique. Departamento de Didáctica de la Matemática.
González, Evaristo. Departamento de Pedagogía.
Castro, Encarnación. Departamento de Didáctica de la Matemática,
Universidad de Granada. SPAIN

ABSTRACT: *This paper reports on a study made of 540 schoolchildren between 9 and 11 years old, engaged in two-stage addition problem solving tasks. Of the 64 different types of two-stage addition problems which can be identified by taking all possible pairs from semantic categories and crossing them with pairs of arithmetical operations. 16 problems were chosen for this study: those in which the semantic structure is the same in both stages of the solution. A multivariant analysis with repeated measurements (MANOVA) was used in two intersubject factors and the interactions shown were analysed. Furthermore, the indices of difficulty and discrimination for each of the problems were determined.*

I. INTRODUCTION.

At the end of the 70s and beginning of the 80s, researchers working on simple arithmetic word problems with addition structures obtained similar semantic categories, among them Vergnaud, Greeno and Heller, Carpenter and Moser, and Nesher. It is at this time when research into arithmetic problems divided into two large fields: addition structure problems and multiplication structure problems.

The semantic categories for addition problems, which represent alternative structures of quantitative information, were initially established by Heller and Greeno (1979) and, with successive improvements, are still with us today.

Fuson (1992-b) lists 22 structurally different addition problems, for which she bears in mind the four alternative semantic structures: Combine, Change, Compare and Equalize; two types of relation - increase or decrease - for the last three categories, or static and dynamic for the first category; and three possibilities for the unknown element in the structure of relations which the problem's statement establishes (only two in the Change structure).

A considerable number of research projects have tried to establish the level of difficulty of the addition structure problems in terms of the semantic categories and of the position of the unknown element in the framework of implied relations. The establishing of the knowledge structures employed to solve different types of problems according to the aforementioned classification still leaves a number of unanswered questions today (Fuson, 1991 and 1992).

Our project is placed within the general framework described previously: we study arithmetic problems of addition, classified according to the semantic structure four categories mentioned above. The studies into addition problems have centered on problems whose solution requires a single operation; these problems are called one-step problems and constitute the simplest category of addition problems.

Interest in the study of two-step arithmetic problems, which are those whose solution requires two consecutive arithmetic operations, has appeared recently (Shallin, 1985; Nesher, 1991). Two-step arithmetic problems of addition are those problems whose solutions involve only addition and subtraction and, in all cases, two of these operations are necessary. The study of arithmetic problems of addition is the aim of this research project.

We are going to consider problems in which two elements appear first and with which we must operate to obtain another number. This new number must be used in an operation with the third element to achieve the solution:

---

ordered elements of the problem: a, b, c
order of operations to reach the solution:
a*b ------> d
c*d ------> solution

This study concentrates on those problems in which the semantic structure of the two operations is the same, thus obtaining 16 possible cases that we name *two-step problems with duplicated semantic structure*

I.1. Research Goals.

The following goals were proposed:

1. To compare the performance of schoolchildren in the 4th, 5th and 6th year of primary education (9-11 year olds) in relation to two-stage addition problems of arithmetic.

2. To determine if there are any differences of difficulty between the pairs of semantic structures and the operation sequences, as well as to study the interactions between these two factors.

I.2. Hypotheses to be Verified.

H_{01}. *There are no significant differences due to the "course" factor..*

H_{02}. *There are no significant differences due to the "semantic structure" factor.*

H_{03}. *There are no significant differences due to the "operation type" factor.*

H_{04}. *There is no significant interactive effect between the variables "course" x "semantic structure".*

H_{05}. *There is no significant interactive effect between the variables "course" x "operation type".*

H06. *There is no significant interactive effect between the variables "semantic structure" x "operation type".*

H07. *There is no triple interaction effect between the variables "course" x "operation type" x "semantic structure".*

II. METHOD.

II.1. Characteristics of the Sample.

The sample consists of a total of 540 pupils from six primary schools in the Province of Granada (Spain), five of which lie in the city itself and its surrounding districts, while the other is in a rural area. Three of the schools are state-run, while the other three are state-approved private institutions.

In each school three groups were chosen, from the 4th, 5th and 6th year respectively. The number of pupils in each group was as follows:

164. 4th year pupils.
175. 5th year pupils.
201. 6th year pupils.

Teachers were not previously made aware of the contents of the tests, nor did the children receive any type of specific instruction beforehand regarding the tasks proposed for the research project.

II.2. Instruments Used.

The 16 possible cases were classified into two groups of eight problems each.

The characteristics of these two tests were as follows:

Test A. This test included those problems in which the semantic structure of the two operations is the same, that is, the pairs (Ch, Ch); (Co, Co); (Cp, Cp); (Eq, Eq), and the two operations are also identical: (+, +) and (-, -). The resulting 8 problems were determined as being of duplicated structure.

Test B. As with the previous group, the problems in this test also have the same semantic structure in the two operations, but here the second operation is different from the first: (+, -) and (-, +). 8 solutions also result from this group, forming symmetrical pairs with respect to the operations.

Since the items are dichotomyc, the reliability as internal consistency of the test was calculated with the Kuder-Richardson's Kr-20 coefficient. The resultant reliability index was 0.86.

	Combine-Combine	Change-Change	Compare-Compare	Equalize-Equalize
(+,+)	Peter has 17 white wooden marbles inside a box, 8 red wooden marbles and 22 glass marbles. How many marbles has Peter got inside the box? [Co-Co (+,+)]	Peter had 23 video-games. On his birthday he bought 8 and his friend Joe gave him 15 to. How many video-games has Peter got now [Ch-Ch (+,+)]	This year, Peter had watched 23 films. Joe had watched 8 more films than Peter and Mary 6 films than Joe. How many films has Mary watched? [Cp-Cp (+,+)]	Peter has got 25 pesetas and needs 9 pesetas more to have as much money as Joe. For Mary to have as much money as Joe, she needs 17 pesetas more. How much money has Mary? [Eq-Eq (+,+)]
(-,-)	Kathy has 48 marbles inside a box. 22 of them are glass marbles and the rest are wooden marbles. Of the wooden marbles 9 are red and the rest white. How many white marbles are inside the box? [Co-Co (-,-)]	Kathy had 46 video-games and it broke 15 of them. Of the ones she has left, she gave 8 to Susan. How many video-games does Kathy have now? [Ch-Ch (-,-)]	This year, Susan has watched 48 films. Mary has watched 8 less films than Susan and Kathy 11 less films than Mary. How many films has Kathy watched? [Cp-Cp (-,-)]	Mary has got 51 pesetas and if she spends 17 ptas. she will have the same amount of money as Susan. If Susan spends 9 pesetas she will have the same amount as Kathy. How much money has Kathy got? [Eq-Eq (-,-)]

Problems of the Primary session (Test A)

II.3. Application Procedures.

The tests were given to the children by members of the research team in two sessions, with a maximum interval of one week between sessions, approximately half-way through the first term of the 1993-94 academic year. The tests were all given on different days, leaving at least one day between tests, and also avoiding a weekend break falling between tests.

The tasks were carried out in the children's usual classrooms, using existing groups/classes. The tests were applied in a large group format, the pupils having a maximum time of 30 minutes per session in which to solve their problems in silence.

Before the tests began each of the groups was given a series of verbal instructions to orient the problem-solving tasks. These included:

> * *Solve the problems in the order they are on the question paper. When you finish one problem go on to the next, and so on, until you finish.*
> * *Make all your notes and calculations in the space underneath each question. Don't just write in the answers.*
> * *Don't look at anyone else's work and keep silent so that everybody has a chance to do their best and solve the problems properly.*

The tests were marked by members of the research team, once agreement had been reached over the marking criteria to be followed. The missing subjets of the sample was under 1% of the total number of pupils, this corresponding to those pupils who only completed one of the tests as a result of their absence for one of the two sessions. Each of the answers to the problems was given a value of 1 or 0, according to whether its solution was correct or incorrect. Solutions were considered correct when they showed that the pupil had opted for the appropriate operations in order to arrive at a successful solution to the task. Calculation errors were ignored. In this paper only the correct solutions are analysed, research currently underway concentrates on an in-depth study of of the typology of errors according to empirically defined category frameworks.

A statistical design of repeated measurements has been applied to the matrix of data obtained from the 540 pupils' scores, with the year factor as an intersubject factor, considered as an independent variable with three levels of definition: 4th, 5th and 6th; and two intersubject factors, i.e. the operation type factor and the duplicated semantic structure factor, with four levels of definition, respectively: (+, +), (-, -), (+, -), (-,+) and Ca-Ca, Cp-Cp, Ig-Ig, Co-Co.

The statistical technique used for analysis of the data was a maultivariant analysis with repeated measurements in the two intrasubject factors (MANOVA).

The performance of each course, and the indices of difficulty for each problem according to the operation type and semantic structure variables appear in the following tables:

Structure Operations	Ch-Ch	Eq-Eq	Co-Co	Cp-Cp	Total
(+,+)	.890	.726	.927	.750	.82
(-,-)	.524	.598	.591	.732	.61
(+,-)	.811	.659	.610	.707	.70
(-,+)	.805	.634	.524	.659	.66
Total	.76	.65	.66	.71	

Table 1. Mean punctuation by course 4th

Structure Operations	Ch-Ch	Eq-Eq	Co-Co	Cp-Cp	Total
(+,+)	.937	.880	.977	.863	.91
(-,-)	.691	.789	.789	.886	.79
(+,-)	.903	.834	.789	.834	.84
(-,+)	.920	.840	.771	.851	.85
Total	.86	.84	.83	.86	

Table 2. Mean punctuation by course 5th

Structure Operations	Ch-Ch	Eq-Eq	Co-Co	Cp-Cp	Total
(+,+)	.965	.915	.990	.945	.95
(-,-)	.826	.920	.881	.965	.90
(+,-)	.955	.886	.896	.910	.91
(-,+)	.955	.896	.876	.945	.92
Total	.93	.91	.91	.94	

Table 3. Mean punctuation by course 6th

Structure Operations	Ch-Ch	Eq-Eq	Co-Co	Cp-Cp	Total
(+,+)	.993	.846	.967	.859	.90
(-,-)	.691	.780	.763	.869	.78
(+,-)	.894	.800	.774	.824	.82
(-,+)	.898	.798	.735	.828	.81
Total	.87	.81	.81	.85	

Table 4. Global Mean punctuation by 4th, 5th & 6th

Results of the contrast of hypotheses according to the variables considered are given in the following table:

	F	p (α=0.01)	Level of signification
H_{o1}	55.96	p<0.01	Differences by course
H_{o2}	14.49	p<0.01	Differences by semantic structure
H_{o3}	53.95	p<0.01	Differences by operations
H_{o4}	3.26	p<0.01	Differences by course x semantic structure
H_{o5}	7.42	p<0.01	Differences by course x operacion
H_{o6}	31.17	p<0.01	Differences by operatios x semantic structure
H_{o7}	2.14	p<0.01	Differences by course x operations x structure

Table 5. Hypothesis & Levels of signification by course, semantic structure and operations

The global interaction between operation type and semantic structure variables appear in the following figure:

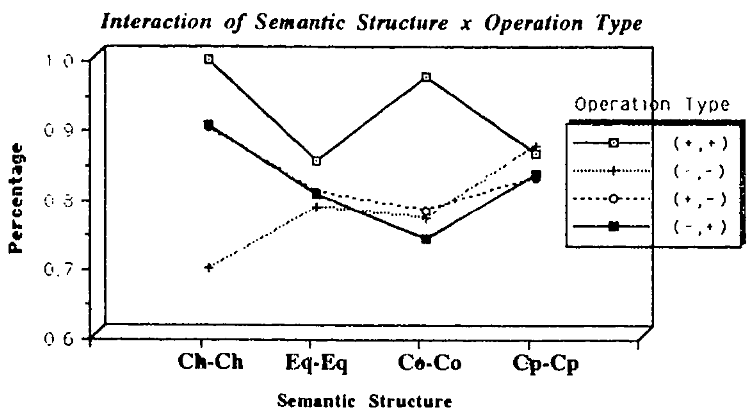


Figure 1. Percentages of appropiate solution for different operations types for de four semantics structures.

IV. CONCLUSIONS.

Globally, following analysis of the variance of the total mark for the 16 items of the two two-stage, duplicated semantic structure addition tests, significant differences between school years were found (F=57.813; p=0.0). Subsequently multiple comparisons between year pairs were carried out according to Scheffe's method at 5% significance level, and significant differences between the school years were found: between 4th and 5th, 4th and 6th, and 5th and 6th.

Analysis of the variance in the two intrasubject factors (semantic structure and operation type) showed significant differences in the following cases: due to the "semantic structure" factor ($F=14.49$, $p=0.000$), due to the "operation type" fcator ($F=53.95$, $p=0.000$) and due to the mutual interaction between both factors ($F=3.26$, $p=0.000$). Finally, significant differences were also found due to the triple interaction of the factors "year" x "semantic structure" x "operation type" ($F=2.14$, $p= 0.003$).

Global compari· ›ns mad: for the variable "semantic structure" showed the existence of significant diffemeces ‹ ·y between the following pairs: Ch-Ch with Eq-Eq, and Ch-Ch with Co-Co.

For the variable "operation type", significant differences were obtained for with the following pairs, by year:

4th year: (+,+) with (-,-); (+,+) with (+,-) and (+,+) with (-,+).
5th year: (+,+) with (-,-) and (+,+) with (+,-).
6th year: (+,+) with (-,-).

V. BLIOGRAFIA.

Carpenter, T.P. y Moser, J.M. (1982). The Development of Addition and Subtraction problem-solving skills , en T.P. Carpenter, J.M. Moser y T.A. Romberg (eds), *Addition and Subtraction: A cognitive perspective*, pp. 9-24. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Carpenter, T.P. y Moser, J.M. (1983). The Acquisition of Addition and Subtraction concepts, en R. Lesh y M. Landau (eds), *Adquisition of Mathematics Concepts and <Processes*, pp. 7-44. Orlando, Florida: Academic Press.

Castro, E.; Rico, L. & Gil, F. (1992). *Enfoques de Investigación en Problemas Verbales Aritméticos Aditivos* . Enseñanza de las Ciencias, Vol 10, nº 3.

Fuson, C. (1992-a) *Research on Learning and Teaching Addition and Subtraction of whole numbers*, en Leinhadt,G; Putnam, R & Hattrup, R (edts) Analysis of Arithmetic for Mathematics Teaching. LEA: Hillsdale, N.J.

Fuson, C. (1992-b) *Research on Whole Number Addition and Subtraction*, en Grouws, D.A. (edt.) Handbook of Research on Mathematics Teaching and Learning. McMillan Publishing Comp.: New York

Goldin, G. A. & McClintock, C.E. (Edts) (1980) *Task Variables in Mathematical problem Solving*. The Franklin Institute Press: Philadelphia, Pensilvania.

Gutiérrez, J.; Morcillo, N; Rico, L; Castro, E; Castro, E; Fernández, F; González, E; Pérez, A; Segovia, I; Tortosa, A; Valenzuela, J. (1993). *Problemas aditivos de dos etapas con igual operación y estructura semántica duplicada. Estudio preliminar en 5º de Primaria.* Actas VI JAEM, Badajoz (en prensa).

Heller, J.I. & Greeno, J.G. (1979) *Information Processing analysis of mathematical problem solving*, en Lesh, R; Mierkiewicz & Kantowski, M. (edts) Applied Mathematical Problem Solving. ERIC/SMEAC: Columbus, Ohio.

Kilpatrick, J. (1978) *Variables and Methodologies in Research on Problem Solving*, en Hatfield, L.L. & Bradabard D.A.(edts.) Mathematical Problem Solving: papers from a research workshop. ERIC/SMEAC: Columbus, Ohio.

Kilpatrick, J. (1992) *A History of Research in Mathematics Education*, en Grouws, D.A. (edt.) Handbook of Research on Mathematics Teaching and Learning. McMillan Publishing Comp.: New York

Nesher, P. (1982). *Levels of description in the analysis of addition and subtraction word*

problems, en T.P. Carpenter, J.M. Moser y T.A. Romberg (eds), Addition and Subtraction: A cognitive perspective, pp. 25-38. Hillsdale, New Jersey: Lawrence Erlbaum Associates.
Nesher, P. (1991). *Two-Steps problems, Research finding*, en Furinghetti, F. (ed.) Proceedings Fifteenth PME Conference, Vol. III, pp. 65-71. Assissi, Italia.
Morcillo, N.; Castro, E.; Rico, L.; Castro, E.; Fernández, F.; González, E.; Gutiérrez, J.; Pérez, A.; Segovia, I.; Serrano, M.; Tortosa, A.; Valenzuela, J (1993). *Dificultad debida al orden de operaciones en Problemas Aditivos de Dos Etapas con estructura semántica duplicada . Estudio preliminar en 5º de Primaria.* (en prensa) Actas VII Jornadas Anadaluzas de Educación Matemática "Thales". Sevilla.
Puig, L y Cerdán, F. (1988). *Problemas Aritméticos escolares.* Madrid: Editorial Síntesis
Rico, L. et al. (1985). *Investigación "Granada-Mats"* . Instituto de Ciencias de la Educación, Universidad de Granada. Granada.
Rico, L. et al. (1988). *Didáctica activa para la resolución de problemas.* Granada: Departamento Didáctica de la Matemática de la Universidad de Granada.
Shallin, V.L. y Bee, N. V. (1985). *Structural Differences between two-step word problems*, presentado en el Meeting de la American Educational Research Association.

This project has received financial support from the *Consejería de Educación y Ciencia de la Junta de Andalucía* (BOJA núm. 131, 19 de diciembre de 1992), with the tittle:***"Diagnóstico de Procedimientos y Evaluación de Destrezas Terminales para la Resolución de Problemas Aritméticos en el Tercer Ciclo de la Educación Primaria Obligatoria (E.P.O.)"***

HOW TEACHERS DEAL WITH THEIR STUDENTS' CONCEPTION OF ALGEBRAIC EXPRESSIONS AS INCOMPLETE

Naomi Robinson
Department of Science Teaching
Weizmann Institute of Science

Ruhama Even
Department of Science Teaching
Weizmann Institute of Science

Dina Tirosh
School of Education
Tel Aviv University

This study investigates seventh-grade teachers' awareness of, and teaching approaches to, a well-documented cognitive obstacle in learning algebra, namely, the incomplete nature of algebraic expressions. Two novice and two experienced teachers participated in this study. Three types of data were collected: 1) lesson plans, 2) observations, and 3) post-lesson interviews. The analysis showed that while experienced teachers were aware of the existence of the difficulty and its possible sources, novice teachers attributed difficulties to other sources, such as notations. The teachers differed in their approach to this difficulty. One experienced teacher used a metacognitive approach and suggested using various ways, while the others used technical methods. In conclusion we raise issues for further research, including the stability of the incompleteness difficulty across various approaches to teaching algebra.

Every teacher encounters situations in which s/he has to decide about the presentation of the subject matter to his/her students either through his/her initiative or as a response to a student's comment. In making such decisions, obviously the teacher's own knowledge of the subject matter plays a major part. Not less important is his/her acquaintance and understanding of students' ways of thinking.

This last component of teachers' knowledge necessitates a body of knowledge of common student conceptions. In addition to teachers' own experiences, such knowledge has been generated mainly in the last two decades of extensive cognitive research on student learning, which has yielded much useful data on student conceptions and thinking in mathematics. It has been shown, especially with reference to algebra, that students often make sense of the subject matter in their own way which is not always isomorphic with the structure of the subject matter or the instruction.

One of the well documented cognitive obstacles in learning algebra has to do with the "incomplete nature" of algebraic expressions. It was found that many students tend to "simplify" expressions such as 3m+4 to 7m or 7. Researchers have suggested several possible sources for this behavior, most of which are related to the fact that while the result of any real numerical expression can be a number without any operational signs (e.g., 15+4-2 is 17), a possible "result" of simplifying an algebraic expression could still include operational signs (e.g., 15+4-2a is 19-2a). Booth (1988). Kieran (1992), Küchemann (1981) and Matz (1980) suggest that the tendency to over-simplify algebraic expressions has to do with students' expectations that the "behavior" of algebraic expressions be similar to that of arithmetic expressions (e.g., 15+4-2a=19-2a=17a). That is, they somehow believe, that the final answer should be "well formed single-termed". Similarly, Collis (1975) suggests that students face cognitive difficulty in "accepting lack of closure." Davis (1975) and Kieran (1991) illuminate another possible source for this difficulty. These researchers noted that in arithmetic, symbols such as + and = are typically interpreted only in terms of actions to be performed, so that + means to actually perform the addition operation and = means to write down the answer. Thus, an expression such as 3n+2m is often interpreted as an instructional (or procedural) statement, which states that 3n is to be added to 2m. Students often tend to "add" such expressions getting answers such as 5nm.

In this paper we examine teachers' awareness of the possibility that their students hold a conception of algebraic expressions as incomplete. We also analyze their teaching from this point of view and hypothesize about their students' learning.

Method

Participants

This study is part of a larger project that examines various differences between experienced and novice teachers with respect to teaching equivalent algebraic expressions. In this paper we refer to four seventh-grade teachers who participated in this study, two were novices with one or two years experience, and the other two had more than 10 years of experience, and a reputation of being excellent teachers. All four subjects used a traditional approach to the teaching of algebra with an emphasis on formal language, procedures and algorithms. Their lessons were teacher-centered with no emphasis on students' investigations.

Data Collection

The collected data are related to the three or four initial lessons on the same teaching segment: equivalent algebraic expressions (open-phrases). Three types of data were collected: 1) lesson plans--each teacher was asked to submit a lesson plan before each lesson, either in writing or in audio-recording; 2) observations--all lessons were observed by one of the researchers; field notes were taken during observations and were supplemented by audio-taped records, and 3) post-lesson interviews--semi-structured interviews after each lesson. All interviews were audio-taped and transcribed.

The Incomplete Nature of Algebraic Expressions: Teacher Awareness and Approaches

In this section we analyze excerpts taken from the four teachers' lessons that illustrate these teachers' awareness of possible student conceptions of algebraic expressions as incomplete. These episodes illuminate different approaches to dealing with this topic. Also included are hypothesized views of students who tend to "complete" algebraic expressions, about the experiences they encountered during the lessons.

Batia

Batia is an experienced teacher. In the ways she plans and teaches her lessons and from her interviews it is obvious that she is aware of the common difficulty students have regarding the incomplete nature of algebraic expressions. For example, in her written lesson plan she wrote: "I expect difficulties in problematic cases [such as] $2x+3=5x$." Also, in an interview, when asked about difficulties that students commonly encounter when studying algebraic expressions, she mentioned, among others, the tendency to add 2a+3 and get 5a. When the interviewer probed her for the reasons for this difficulty, she stated: "They need to get an answer, it does not seem finished to them."

Being aware of this student difficulty, she includes in her lessons various activities and several strategies to expose students' thinking about this issue, to help them reflect on their solutions, and develop a view of an algebraic expression as a possible complete answer. The following episodes illustrate her approach.

Episodes

Episode 1. The teacher writes 3+4x on the board and asks the class what can be done with this expression. One student says: "It should remain as it is". Although this was a correct answer, probably the one that most teachers would expect at this point, the teacher continues to probe for other reactions that students may come up with. Another student says: "No, this is not true, 3+4x is equal to (3+4)x." The teacher then suggests one way to check this by substituting x=5, i.e., $3+4*5\neq(3+4)*5$. She continues to present similar expressions for students to work with.

Episode 2. In another lesson, the expression 3+4x was mentioned again. The teacher asks: "What is 3+4x?" One student claims: "7x". The teacher, aware of the problematic incomplete nature of algebraic expressions, purposely asks: "How about 7?"

Student: Maybe?!

Teacher: Well, let's see again. 3+4x. What is the operation between 4 and x.

Student: Multiplication.

*Teacher: So, first we have to determine what 4*x could be. Can we know that?*

Student: No!

Teacher: So, can I first add the numbers?

Student: No! OK, I got it.

In this episode, the teacher tackles this issue differently. This time she does not use substitution but builds on students' knowledge of the order of operations.

Episode 3. In another lesson the class discusses the expression $x^2 + 3x$. One student asks if this could be $4x^2$. The teacher repeats the question: "Could it be $4x^2$?" in a neutral tone, not implying the correct answer. Another student correctly replies: "No". The teacher then asks a "reverse" question: "In order to get $4x^2$, what should be there?" One student answers: "There should have been $3x^2$ and not 3x". The first student further insists: "Why can't I add them? I still don't understand." The teacher repeats the question. The second student continues to explain: "We don't know what $3x^2$ and 3x are. Therefore, one cannot add them. We should first multiply." The first student seems satisfied with the explanation.

A student's view

A student in Batia's class who believes that an expression such as 3x+4 is incomplete, can by said to experience the following:

- The teacher understands what the student's difficulty is and in her response to the student focuses on his/her way of thinking.
- The teacher offers the student opportunities to express his/her ways of thinking about the issue even when it's wrong, and discusses them respectfully.
- The teacher provides various presentations of the subject matter that highlight different aspects of and angles on the issue (e.g., substitution, order of operation, reversibility).
- The teacher frequently returns to this issue on different occasions, with relation to different types of expressions (e.g., linear, quadratic).

Gilah

Gilah is also an experienced teacher. She is aware of students' tendency to "simplify" expressions such as 3x+4 to 7x as a result of their conception of the "incomplete nature" of algebraic expressions. She explains: "Students tend to do it as simple as possible. They tend to 'finish' it [the expression]." In her opinion, the incomplete nature of algebraic expressions is the main obstacle in teaching how to simplify algebraic expressions. Therefore, she planned a comprehensive introductory activity, devoted to getting the students acquainted with the notions of like and unlike terms. She keeps referring to these notions during instruction. The following episode is a typical example of her approach. It took place after the students had worked for an extensive period on like and unlike terms.

Episode

During a lesson the class arrives at the expression 4x+6. One student suggests that this equals 10x. Another student objects: "4x and 6 are unlike terms because 4x is a number with an x and 6 is only a number." The teacher pushes further: "So what?" The student replies confidently: "The answer stays 4x+6."

A student's view

A student in Gilah's class, who believes that an expression such as 3x+4 is incomplete experiences the following:

- The teacher understands what the student's difficulty is and considers it important and worth taking care of.
- The teacher offers the student opportunities to express his/her way of thinking about the issue even when it's wrong, and to discuss them with respect.
- The teacher spends time and effort on teaching and directing the student towards the use of one specific method: like terms.

- The teacher frequently returns to this issue on different occasions, with relation to different types of expressions (e.g., linear, quadratic).

Drora

Drora is a novice teacher in her second year of teaching. From the ways she plans and teaches her lessons and from her interviews it seems that she is aware of various difficulties related to working with algebraic expressions such as 1x=x, x*5=5x, -1x=-x. Yet she does not mention the difficulty of the incomplete nature of algebraic expressions. Drora, like Gilah, uses the notion of like terms. However, she uses this notion without helping the students in getting the feeling of what it means. Therefore, when she mentions "like terms", the students do not seem to have the same understanding of the meaning of this as she does. The following episodes demonstrate her way of teaching.

Episodes

Episode 1. The class discusses the simplification of the expression 7x-5-x+12. The teacher asks: "What are the like terms that we can add?" One student answers: "7x and 5." (Apparently, the student's conception of like terms is different from that of the teacher.) The teacher replies by stating that: "7x and -x, isn't it? 5 does not have an x nor does 12." She immediately writes the expression 7x-x-5+12 on the board and continues with the idea of like terms, even though this notion was shown to be unclear and problematic for the students. She asks: "What are the like terms? What can I use here?" and immediately answers her own question: "I have 7x and -x." Later, when a student asks why do we add 7x and -x, meaning, why did we leave 5 out (although the teacher keeps using the notion of like terms, several students do not seem to understand it), Drora explains: "Because I add only numbers that have an x, don't I? I add only like terms, terms that have an x, and 5 doesn't have one." (Notice that 7x and $5x^2$, for example, are "numbers that have an x" and, therefore, are like terms according to Drora's explanations.)

Episode 2. While simplifying the expression 16m-9m+2m+3-11 the class reaches the expression: m*8-8. The teacher, attempting to get to the conventional way of writing expressions: 8m-8, asks: "And what is this?" A student, apparently feeling that the expression is incomplete, answers: "Zero." The teacher, probably thinking that the problem is rooted in not remembering the order of operations, asks: "Can we do m times 8 minus 8?" The student replies: "Yes". The teacher goes back to the idea of like terms and states: "No, because one has a variable and one does not." (again, not covering cases such as x and x^2, or a and ab).

Episode 3. A student, while working at the board, writes: 5t-3t+t+2 =11t. The teacher, ignoring his answer, uses the distributive property to get 3t from 5t-3t+t. Then asks: "Can we add 3t and 2? Can we add a numeral with an algebraic expression?" She immediately answers her own question: "No! 3t+2 is the final expression." She continues, attempting to use everyday life situations to demonstrate the idea of like terms:

What is 3t? 3 times t. Is it a n''mber? No! It's an algebraic expression. And what is 2? 2 is a constant number. We can add 3t only to the same thing. If I have 3 pears and 2 apples, can one say that I have 5 pears? We are left with 3 pears and 2 apples. 3t is an expression and 2 is a number. So we cannot add them.

In this episode the teacher makes an inappropriate analogy between 3t and 3 pears, using letters as objects. For further discussion on the problematics of this approach see Küchamann (1981) and Pimm (1987).

A student's view

A student in Drora's class, who believes that an expression such as 3x+4 is incomplete, is thus likely to experience the following:

- The teacher attributes the student's difficulty to reasons such as not understanding the meaning of coefficients of variables, difficulties in the correct order of operations, and obstacles related to the use of the distributive property. As a result the teacher does not focus in her response on the student's way of thinking.
- The teacher does not give students opportunities to express and explain their ways of thinking. When an incorrect response is given, the teacher often states that this is wrong, and provides the correct answer .
- Most of the teacher's presentations of the subject matter concentrates on showing the "correct way of doing things" by using only one method of like terms. Although she keeps using the notion of like terms, the teacher neither defines nor practices what is meant by that.

Benny

Benny is a novice teacher in his first year of teaching. In his lesson plans and during his interviews he did not explicitly mention the difficulty associated with the incomplete nature of algebraic expressions. Yet, he mentioned that adding numbers and algebraic expressions needs to be addressed in teaching and stated the need to provide students with a rule of "adding numbers separately and adding letters separately". This rule is problematic because according to it 5m+2 can be equal to 7+m or 7m or another variation. The following episode shows what happens in his class when he tries to apply his plan.

Episode

Benny writes the expression 3m+2+2m on the board and asks: "To what is this equal?". He immediately states the previously mentioned rule: "Add the numbers separately and add the letters separately". Then he suggests to color the "numbers": 3m+**2**+2m, and writes 5m+2. A student asks: "And what now?" Another student suggests: "7m". The teacher (a bit surprised by this answer) states: "No! 5m+2≠7m." And he repeats the rule again: "The rule is: add the numbers separately and add the letters separately." Then he gives the students another example and colors the [free] numbers: 4a+**5**-2a+**7**. The teacher emphasizes the rule by dictating it to the students and asking them to repeat it loudly. The rest of the lesson is devoted to further work on similar exercises. The students continue to experience difficulties. For example, towards the end of the lesson, a student asks: "Why does not 10+2b equal 12b?" The teacher realizes at this point of the lesson that even though he gave the students a seemingly clear rule, they still don't know how to work with it. He chooses not to address the question.

In his reflection on the lesson Benny expresses his dissatisfaction and frustration from his way of teaching this material. He explains that he felt that there is a difficulty but he does not understand its sources. He adds that giving a rule without explanation was problematic and starts to construct alternative explanations.

A student's view

A student in Benny's class, who believes that an expression such as 3x+4 is incomplete, may well experience the following:

- The teacher feels that the student experiences difficulties but does not understand their sources and therefore does not focus on the student's ways of thinking.
- The teacher keeps repeating a rule which fits the student's way of thinking about the incomplete nature of algebraic expressions, but the use of this rule leads to inappropriate answers.
- The teacher is attentive to students' difficulties and is looking for alternative ways of presenting the material. It seems likely that the teacher will come up with more helpful ways of teaching.
- The teacher provides opportunities for the students to express their ways of thinking even when they are wrong. Yet, when an incorrect response is given, the teacher often states that this is wrong, and provides the correct answer.

Conclusion

The incomplete nature of algebraic expressions is considered one of the main cognitive obstacles in learning to simplify algebraic expressions. Initial investigations show that this difficulty is not limited to the first stages of learning algebra only. At higher grade levels students show a similar tendency, e.g., they feel reluctant to accept 3+2i as a "complete" number. Therefore, it is important for teachers to be aware of this cognitive obstacle and to address it throughout the years of teaching algebra.

Our study shows differences between novice and experienced teachers in terms of awareness and approaches to dealing with the "incomplete nature" of algebraic expressions. The experienced teachers were aware of both the existence of the difficulty and its possible sources. The novice teachers, however, felt that there are difficulties associated with work with numbers and variables but attributed them to other sources, such as notations.

The teachers largely differed in their ways of approaching this difficulty. Batia, one of the experienced teachers, used a meta-cognitive approach and suggested the use of various ways. The other experienced teacher, Gilah, based her teaching on one, rather technical method, which proved to eliminate a great deal of the students' difficulty. Drora, a novice teacher, used the same approach but without preparing a common ground for communication with her students. Benny, who used a rule-based approach, had the same problem of communication with his students.

These initial findings raise several issues for further research.

- Is awareness of students' difficulties related to the incomplete nature of algebraic expressions a natural consequence of teaching algebraic expressions?
- Can we make shortcuts to this "practical wisdom" by teaching preservice teachers about the existence and sources of students' tendency to "complete" algebraic expressions?
- To what extent is the incompleteness difficulty embedded in the traditional approach to teaching algebra? Will other approaches, such as the function-based approach, eliminate this obstacle?

References

Booth, L. R. (1988). Children's difficulties in beginning algebra. The ideas of Algebra, K-12, Yearbook of NCTM (pp. 20-32). Reston, VA: NCTM.

Collis, K. F. (1975). The development of formal reasoning. Newcastle, Australia: University of Newcastle.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. Journal of Children's Mathematical Behavior, 1(3), 7-35.

Kieran, C. (1991). Cognitive processes involved in learning school algebra. In: P. Nesher and J. Kilpatrick (Eds.), Mathematics and Cognition (pp. 96-12). Cambridge University Press.

Kieran, C. (1992). The learning and teaching of school algebra. In: D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 390-419). NY: McMillan Publishing Company.

Küchemann, D. E. (1981). Algebra. In: K. M. Hart (Ed.), Children's understanding of mathematics (pp. 102-119). London: John Murray.

Matz, M. (1980). Towards a computational theory of algebraic competence. Journal of Mathematical Behavior, 3, 93-166.

Pimm, D. (1987). Pupils' written mathematical records. In: D. Pimm (Ed.), Speaking mathematically. London and New York: Routledge & Kegan Paul Ltd.

THE TEACHING OF LINEAR ALGEBRA IN FIRST YEAR OF FRENCH SCIENCE UNIVERSITY : EPISTEMOLOGICAL DIFFICULTIES, USE OF THE "META LEVER", LONG-TERM ORGANIZATION.

Jean-Luc Dorier (DidaTech, Grenoble), Aline Robert, Jacqueline Robinet (Didirem, Paris), Marc Rogalski (Didirem, Paris - Lille)

Abstract : *We present an epistemological and didactical analysis of students' difficulties in linear algebra as taught in first year of French science university. We deduce from it hypotheses for research work. We present an engineering for this teaching based on these hypotheses. This engineering takes the form of a long-term project (over one year) and uses, to a great extent, the "meta lever" and the changes of setting and points of view. The evaluation currently in progress of the project draws out certain methodological difficulties encountered in the validation of the type of hypotheses which have been made.*

Résumé. *Nous faisons une analyse épistémologique et didactique des difficultés des étudiants avec l'algèbre linéaire enseignée en première année d'université. Nous en déduisons des hypothèses de recherche. Nous présentons une ingénierie pour cet enseignement, qui prend en compte ces hypothèses. Cette ingénierie se présente sous la forme d'un projet long (une année), et utilise de façon importante le levier du "méta" et la pratique des changements de cadres et de points de vue. L'évaluation en cours du projet met en évidence certaines des difficultés méthodologiques pour valider le type d'hypothèses que nous avons faites.*

A worrying state of affairs

It is quite a well-established fact that the teaching of linear algebra in first year of French science university does not work well. Students' learning of concepts and methods of linear algebra are poor. Yet the results of final exams may very well hide this fact, as students are often only asked to use techniques, and do not really have to bring the concepts themselves into play (Dorier 1990). Thus traditional teaching remains mainly "bourbakist".

However, a number of attempts to transform the teaching of linear algebra have taken place in the past ten years or more recently, at the CNAM[1] or at Lille university, as well as in other countries (Harel 1986, 1989).

It is only since 1985 that a more profound reflection or a didactical analysis has started on the teaching and learning of linear algebra (Robinet1986, Robert & Robinet 1989, Rogalski 1990, Dorier 1990,1991,1992, Dorier & al. 1993).

A didactical analysis : locating the central question

This state of affairs gives rise to didactical interrogations: where do these difficulties originate in: the teaching, the concepts themselves, the students ? In which way can the teaching of the concepts of linear algebra improve ? We have drawn out several hypotheses on the causes of these difficulties, and presented some proposals for modification of the teaching. Some of these hypotheses are admitted (and already partially supported), others are, on the contrary, objects of our research work, and may be of a more general type.

***First hypothesis** (admitted): one of the reason for students' difficulties comes from the specific epistemological nature of the concepts of linear algebra, this should be taken into account in the teaching.

---

[1] University-type establishment specializing in adult education.

The analyses of J. Robinet and then of J-L Dorier underline the specific nature of the elementary concepts of linear algebra (vector-space, linear operators, image, kernel, basis, dependence, dimension, rank, etc.). The historical study shows that these concepts took their formal shape only after numerous uses of linear methods in specific contexts without much unification. These concepts, in their present form, are the cause and results of unification and generalization. Their relevance only appears a posteriori, as they first renewed in an economical manner the solving of old problems and only afterwards allowed new approaches.

We think that this epistemological dimension plays a part in the difficulties encountered by students, mainly because they only have access to the final phase of the historical process : definition of the concept and systematic use in the solving of problems. Yet the problems they are asked to solve may often be solved with specific tools and methods, more familiar, which do not necessarily imply the use of the new concepts. In this case, the simplification and improvement induced, in the solving of the problems, by a change of point of view (for what we will call the algebraist point of view) cannot be foreseen by the students. They may only do this because they are asked to; this is an effect of the teaching contract.

On a more theoretical basis, this implies that these problems, which can be solved in various settings, cannot yet be chosen as "good" problems in the perspective of the tool/object dialectic as defined by Douady (1986), as the concepts to be taught are not indispensable for the solving. The only problems which would necessitate an absolute use of the formal concepts of linear algebra are all too complicated for our students (they involve non countable infinite dimension vector-space)...

In other words, for the students, the concepts of linear algebra are above all objects before they can be used as tools; students are therefore deprived of the long progression, which brought mathematicians to express these concepts, step by step. Consequently, as a first hypothesis, we state the necessity of the awareness of this epistemological specificity and the difficulties it implies as well its explicit taking into account in the building of teaching sequences.

***Second hypothesis** (admitted, supported by several, even quite old, works): <u>the lack of too many previous elements of knowledge may be an insuperable handicap</u>.

More precisely, in Dorier (1990), following Robert (1985), the authors have pointed out the difficulties in the learning of algebra, of students having elements of knowledge which initially are not well enough distributed in different settings (geometrical, analytical, logical, formal settings mainly). These analyses of students' practice also revealed <u>the lack of ability to change settings and points of view</u>.

One of the consequences we drew from this, is that a preliminary work to the introduction of the concepts of linear algebra is necessary, to give the students sufficient previous knowledge in various settings.

***Third hypothesis**: considering the two first hypotheses, the teaching of linear algebra needs to be built on several interwoven and long-term strategies, using the "meta lever" as well as the change of settings (including within mathematics) and of points of view, in order to obtain substantial modifications for a sufficient number of students.

By long-term strategy (Robert 1992), we mean a type of teaching which cannot be divided up. The long-term is vital because the mathematical preparation and the changes in the didactical contracts have to operate over a long enough period to be efficient for the students, in particular regarding the evaluation. Finally the long-term is necessary to take into account the non linearity of the teaching due to the use of change of points of view, which implies working on a subject more than once.

By "meta lever" (Robert 1993), we mean that the teacher will use information on: the knowledge of the students about their own knowledge in mathematics or about their way of learning; or about mathematics (organization, different uses in other fields, differentiation between general and particular fields, etc.); or about mathematics as they are taught (for instance the use of methods, the different types of questioning in mathematics, how to organize a reflection on concepts, etc.). Moreover, we claim that this type of information given by the teacher must be followed or accompanied by activities involving its pertinent use by the students.

By change of settings or points of view (Douady 1986), we mean that the teaching is organized in such a way that the course and the exercises use translations of the same concept or question from one setting into another (from formal to numerical, from numerical to geometrical, etc.) or lead the students to a change of point of view on a notion (for instance seeing the linear equation $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$ as the n-tuple $(a_{11}, a_{12}, \dots, a_{1n})$, in order to go from the use of linear combinations of equations to the notion of rank of a set of n-tuples, etc.).

This hypothesis is the main goal of our work: it led M. Rogalski to the building of a complete teaching project (Rogalski 1991), which has been tested in Lille with several colleagues, and the evaluation of which is being carried out by us. This is completed by more general research we are working on: the use of the "meta lever" in the teaching, the long-term projects and the role of metacognitive representations of the students concerning their own learning.

A teaching project on which we want to try this hypothesis

1) Choices of strategy as they result from our didactical analysis

(a) One way of taking into account the specific epistemological nature of the concepts is to introduce, at a favorable and precise time of the teaching, some activities which induce a reflection on a "meta" level. This should clarify the specificity of the epistemological nature of these concepts (what is, in general, a linear equation ? What is the relation between the rank of vectors and the rank of equations ? etc.). On the other hand, it should create activities in which the students reflect on certain mathematical concepts (role of the axioms of vector-space, the two points of view used to represent a subspace of R^n : equations and parameters, etc.). Finally, it should give them indicators or methods which can help to solve problems (which methods can I use to show the identity between two subspaces ? to determine the image and the kernel of an operator ? etc.).

(b) As a complement, it would be useful to set up a fairly long preliminary phase preceding the actual teaching of elementary concepts of linear algebra, which would prepare the students in order to understand, through "meta" activities and presentations, the unifying role of these concepts. This means that various problems should be solved at a preliminary stage and then unified with use of vector-space theory, in a way that points out the simplification provided by the above solving.

(c) As far as possible, the changes of settings and points of view are put forward, even if it is not along with a tool/object dialectic. This is especially the case in the beginning with geometrical and algebraical settings which allow the extension of the double point of view equation/parameters to R^n, and also with algebraic and other settings to put forward the "intra-mathematical modeling", when formal linear algebra is introduced.

(d) Finally, we chose to give the concept of rank a central position in our teaching. Indeed historical and epistemological works showed the importance of this concept as well as the difficulties encountered by students with this concept. This led us to give the theory of linear equations a privileged situation in the preliminary phase, as rank can be introduced quite naturally in this field.

It is important to see that our project does not induce a "smooth progression" : R^2, R^3, R^n, general linear algebra. Even if the didactical difficulty increases, certainly, with the dimension and the abstraction, we think that the main difficulty is qualitative rather than quantitative. Therefore, although the organization we chose aims at putting forward similarities and changes of settings and points of view, it also aims at provoking a reflection on what is at stake in the new theories and at giving access to methodological processes which allow a better understanding of the concepts. Thus a paradox might arise : to obtain better understanding, we develop more demanding strategies, which is likely to induce, at least temporarily and apparently, worse scores during the evaluation.

2) **the overall organization of the teaching**

(a) **First step.** The development of preliminary phases along with the beginning of the teaching of linear algebra, and organization of the changes of settings in order to make different points of view converge.

At the beginning of the course, we set up the "circuit éléctrique" activity (Legrand 1990), which aims at the understanding by the students of the rules of mathematical reasoning. We also givethe basic language of set-theory, and some complements about space geometry.

Concerning linear algebra, we introduce during the first course Gauss' method for elimination in the solving of systems of m linear equations with n unknowns, by using R^n and its linear structure, as the set of reference in which the solutions must be found. This straight away gives a powerful tool for solving problems in algebra as well as in plane or space geometry. Moreover, this introduces what will be, for us, one of the central questions in linear algebra: the solving of systems of linear equations, from which the concepts of subspaces of R^n, of rank, of double point of view equations/parameters will be drawn. The basic questions, which we try to induce in the reflection of our students and which will justify the introduction of the concepts of linear algebra are:

* do I have too many, just the right number, not enough equations ?

* How many parameters do I need to describe the set of solutions of a system of linear equations ?

We continue the teaching with Cartesian geometry in R^3, with this double aspect equations/parameters as a central question.

(b) Second step. This starts by an explicit presentation to the students of all the common questions that can be asked about the preceding phase. This also implies the change of point of view consisting in seeing an equation as a n-tuple. Then we define linear independence and rank and we show the invariance of rank using the reversibility of certain linear combinations (as done in the exchange theorem). This is followed by the description, with parameters, of subspaces of R^n, initially defined by their equations, and vice versa. Then, we explore the notions of dimension and bases of subspaces. In this phase, some results on the systems of linear equations, which were only conjectures in the first phase, are proved. The fact that the ranks of the lines and the columns of an array are the same is also proved.

The proofs of fundamental results are made with use of abstract formulation, avoiding the use of coordinates. This is justified to the students by a systematic use of algebraic thinking (on which we make our students reflect during preliminary activities involving elements of group theory illustrated on some concrete finite groups). Another justification we use is the search for generality in order to make the transition with further developments in abstract linear algebra easier.

Finally, we give some elements of methodology concerning the solving of problems about subspaces in R^n : inclusion, equality, intersection, parameters and equations of subspaces.

During this phase we also apply linear methods in other fields : polynomials, linear recurrent series, linear differential equations.

(c) Third step. Here we teach abstract linear algebra (axiomatic theory of finite dimensional vector-spaces, linear operators) and its application as a modeling setting within mathematics. At the same time, we draw out several methods and point out types of problems as references. These problems were solved during the previous phases, but they are now solved again in the setting of formal algebra. These problems have to be rich enough so that their generalization make the use of the formal concepts almost compulsory (one can give 15 conditions for an interpolation, instead of the usual 3 or 4, for instance), the contract is to model the problem in linear algebra terms. One of our most favored problem is the search of a modeling by the general linear equation "T(u) = v". The interaction with the rest of the course of mathematics is quite important in this phase. This creates a constraint which specific to long-term projects.

(4) Fourth step. This phase is more technical and also shorter. It presents the matrices, the techniques associated with the change of bases and the inversion of square matrices. Of course, the tool "matrix", introduced in relation to linear operator, is then used in several problems in relation to various other fields of mathematics. But matrix calculus is not an important goal for us.

How can we use this experimental teaching in order to validate our hypothesis ?

What must be validated can be listed as follows :

1) Evaluations of the results

Essentially, we have to know whether the engineering is viable, in other words if the fact that we demand more from the students does not induce an unbearable number of failures.

2) Our specific hypothesis

(a) The role of the "meta lever"

We have to find some indicators that students use "meta" themselves: What is the students' understanding of the nature of the concepts introduced ? What kind of questioning can they have about a given mathematical situation ? More precisely, do they use geometrical representations, or do they check the coherence between the number of parameters and the number of equations, or between the dimension and the maximum number of independent vectors, etc. ? This led us to give the students situations in which the first goal is a form of questioning or checking.

(b) The changes of settings and points of view.

We must know if the notion itself of "change of setting" has been understood, in which way it is available or even just summonable? Is it used in problems involving a modeling in terms of linear algebra, and how ? Are the changes of points of view helpful to obtain better understanding ? Do they not induce complications, at least for a certain time ?

(c) The teaching of methods

We must evaluate its impact, especially by giving problems which have to be preceded by a reflection on the different types of methods than can be used. We also have to find ways of making students write or speak about this preliminary reflection.

Difficulties encountered in this type of evaluation, temporary results

For a precise report of the difficulties, we refer to Dorier and al. (1993), her, we will just give a short overview, listed in three points:

1) For an engineering concerning long-term teaching, it is difficult to choose the time suitable for its evaluation, as many interferences may occur due to students' own organization of their time and work, in a way that cannot be kept under control. Thus, phenomena of maturing, depending on students' level of implication (variable along the year) are difficult to take into account in the evaluation of the engineering.

2) It is quite difficult to trace indicators of students' reflection at a "meta" level, especially in their writing: indeed in the usual teaching, the contract is to give the answer, never the means which led to it.

3) Several background disturbances often hide the phenomena we want to study. These disturbances are mainly:

* the fact that students have or not already worked on a problem of the type used in the evaluation;

* the variability in what is done during the tutoring;

* the sudden change in the type of students in France, due to a recent democratization of university teaching and changes in secondary teaching; these two phenomena make the sociological implications sometimes more relevant than any didactical explanation.

Therefore, our next goal is to find a better means of access to the didactical variables we want to study, as it is difficult in the present state to come to entirely reliable conclusions.

Let us give a few partial results anyhow:

- the overall experiment is satisfactory: the engineering does not induce massive failures at the final exam; but although this result reassure us, it is of little didactical significance.
- it seems that there is an appropriation by a majority of students of the double point of view equations/parameters on the subspaces of R^n;
- the notion of rank of a set of vectors and the methods of finding it also seem to have been understood;
- for the moment, the chances of autonomous questioning from the students are poor;
- the capacity for spontaneous modeling of a problem in the setting of linear algebra also seems adequate only for a minority;
- as anticipated, some confusions between equations and vectors occur for a while for about half of the students;
- finally, it also seems that it is possible to make the students involve themselves in situations on a "meta" level, in which the main goal is epistemological as much as mathematical (Dorier 1992).

Perhaps the key to a better engineering is in a more extensive use of this type of activity.

REFERENCES

DORIER J.L. : 1990a, Contribution à l'étude de l'enseignement à l'université des premiers concepts d'algèbre linéaire. Approches historique et didactique, Thèse de doctorat de l'Université J.Fourier-Grenoble 1.

DORIER J.L. : 1990b, Analyse historique de l'émergence des concepts élémentaires de l'algèbre linéaire, Cahier Didirem n°7 IREM de Paris 7.

DORIER J.L. : 1990c, Continuous analysis of one year of science students'work in linear algebra, in first year of French University, in the proceedings of the XIVth annual of the PME-Mexico.

DORIER J.L. : 1991, Sur l'enseignement des concepts élémentaires d'algèbre linéaire à l'université, in Recherches en Didactique des Mathématiques, Vol. 11/2.3, Ed. La pensée sauvage, Grenoble.

DORIER J.L. : 1992, Illustrer l'aspect unificateur et généralisateur de l'algèbre linéaire, Cahier Didirem n°14 IREM de Paris 7.

DORIER J.L., ROBERT A., ROBINET J. et ROGALSKI M. : L'enseignement de l'algèbre linéaire en DEUG première année, essai d'évaluation d'une ingénierie longue et questions, Colloque de l'ARDM, à paraître dans Recherches en Didactique des Mathématiques.

DOUADY R. : 1986, Jeu de cadres et dialectique outil/objet, in Recherches en Didactique des Mathématiques, Vol. 7/2, Ed. La pensée sauvage, Grenoble.

HAREL G. : 1986, A comparison between two approaches to embodying mathematical models in the abstract system of linear algebra. Proceedings of the 8-th annual conference of PME, NA Michigan (pp 127-132), Ed G. Lappan, (Michigan 1986).

HAREL G. : 1989a, Learning and teaching linear algebra, For the learning of mathematics 11, 139-148.

HAREL G. : 1989b, Applying the principle of multiple embodiments in teaching linear algebra, School scinece and mathematics 89, 49-57.

LEGRAND. M. : 1990, "Circuit", ou les règles du débat mathématiques, dans: Enseigner autrement les mathématiques en DEUG A première année, brochure de la Commission Inter-IREM Université, IREM de Lyon, 1990.

ROBERT A. et ROBINET J. : 1989, Quelques résultats sur l'enseignement de l'algèbre linéaire, Cahier de didactique des Mathématiques n°53 IREM de Paris 7.

ROBERT A. : 1987, De quelques spécificités de l'enseignement des mathématiques dans l'enseignement post-obligatoire, Cahier de didactique des Mathématiques n°7 IREM de Paris 7.

ROBERT A. : 1988, Rapport enseignement/apprentissage (début de l'analyse sur R). Fascicule I, Analyse d'une section de DEUG A première année (les connaissances antérieures et leur apprentissage), Cahier de didactique des Mathématiques n°18-1 IREM de Paris 7.

ROBERT A. : 1992, Projets longs et ingénieries pour l'enseignement universitaire : questions de problématique et de méthodologie. Un exemple : un enseignement annuel de licence en formation continue, in Recherches en Didactique des Mathématiques, Vol. 12/2.3, Ed. La pensée sauvage, Grenoble.

ROBERT A. : 1993a, Analyse du discours non strictement mathématique accompagnant les cours de mathématiques dans l'enseignement post-obligatoire, à paraître in Educational Studies in Mathematics.

ROBERT A. : 1993b, Prise en compte du "méta" en didactique des Mathématiques, Cahier de DIDIREM n° 21, IREM de Paris 7.

ROBINET J. : 1986, Esquisse d'une genèse des notions d'algèbre linéaire enseignées en DEUG, Cahier de didactique des Mathématiques n°29 IREM de Paris 7.

ROGALSKI M. : Pourquoi un tel échec de l'enseignement de l'algèbre linéaire ? dans : Enseigner autrement les mathématiques en DEUG A première année, brochure de la Commission Inter-IREM Université, IREM de Lyon, 1990.

ROGALSKI M. : 1991, Un enseignement d'algèbre linéaire en DEUG A première année, Cahier Didirem n°11 IREM de Paris 7.

INSTITUTIONALIZATION AS A KEY FUNCTION IN THE TEACHING OF MATHEMATICS

ROUCHIER André
IUFM Orléans-Tours
CREHSTO University of Orléans

+++++++++++++++++++++++++

In the course of this presentation we will examine some of the difficulties that arise through the relationships between teaching and learning mathematics. There is a fundamental tension between these two poles, each of them should be, in a certain sense , antagonistic of the other. Difficulties from the pupil's point of view, difficulties from the teacher's point of view. Extending the scope of analysis should be made through two complementary theoretical approaches, the classical theory of situations form one part, the anthropological approach form the other to focus on the main process, institutionalization.

It is not necessary to insist about the "reality" of the fundamental tension between teaching and learning in the field of mathematical thinking, knowlegde and practice. From the "naive" point of view of the teacher : "I teached them this point and the didn't understand, they didn't learn at all !" to more organised analysis that underline the basic fact of no-symetrical position of the teacher and the learner. The first has to teach and don't have significative means to be sure about what "they" are learning The second has to learn and try to look for what is asked through various indices that are not completely in the terms of knowledge. For instance some of these indices can be related with the didactical contract. Our purpose in this paper is to develop a theoretical model of the relations between the two faces of teaching, the cognitive one for one side which is deeply related to mathematic as a knowledge, the social one for the other for which this knowlegde exist and is "knowable" through special social configurations, institutions. In the first two parts we will present briefly the two main theories in this field, before developing our proper approach about facts and phenomenas linked with institutionalization in mathematical teaching.

1- Institutionalization in the scope of the theory of situations:

The first elements of the theory of didactical situations were proposed by G. Brousseau in the middle of the sixties through various oral interventions and conferences. It is up to this moment the more profound attempt to grasp and describe an intellectual system of analysis and setting up the inscription in action of the teaching intentionality with all its consequences. Farther on the idea of process of abstraction proposed by Diénes, it is the totality of the interactions with an organized environment (including or not effective human partnership) to product (to give rise to) mathematical knowledge. Dimensions of action, formulation and validation are identified and their specific models described in terms of systems. The production of mathematical knowledge is not posible out of particular conditions, cultural and institution nal ones as we will see later. Is is not possible out of situational conditions in terms of mathematics (problems) and in terms of interactions (personal and social) with the other actors of the teaching system. It is well known that these interactions are to be taken as a whole between three sub-systems that cannot be taken into account each of them for itself: teacher, student and knowledge. The variety of situations in which the student has to take by himself the responsibilty of developing the specific knowledge required without any direct help from the teacher are of great theoretical and practical interest, they are called a-didactical situations. Through paradigmatics exemples extremely fruitful for research and significative for teaching a very rich know-how of contruction and analysis of such situations was developed in the course of the late twenty years. The mathematical game "Qui dira vingt ?" (Brousseau, 1978) is one of them, other ones, among a great number, are presented and analysed in Arsac, Balacheff, Mante (1992).

The dimension of effective teaching which is a process fundamentaly intentional, purposeful and goal-directed cannot be reached without a modification of the relations with the knowlegde, tying together the utilitarist face of mathematical knowlegde and the social and cultural one. Knowing involve two aspects, knowing for oneself from one part, a-didactical through a direct confrontation to the problem, knowing in the same way other people do. The institutional dimension of mathematical knowledge which is in the scope of the curriculum, the program and the classroom prescriptions is to be introduced in the model at this level as another moment in the process of mathematization: institutionalization.

The theory of situations was at the beginning of the main stream of the french research during the late twenty years as it was expressed in Laborde (1989). We quoted yet the setting up of situations as a fundamental tool for research under the term of didactical engineering. At the same time, the study of the conditions by which a student enters in a situation and the problem it carries on (devolution of the problem) gave rise to the identification of the didactical contract (Brousseau, 1980), "funnel effect" in the terms of Bauersfeld (1980) when he worked on "the hidden dimension of th so-called reality of the class-room". The difference between the constraints of working on a theorisation of didactical phenomenas and a naturalistic point of view can be particularly underlined in the case of these two concepts.

2- Anthropological models of didactical systems:

This kind of model was particularly developed by Yves Chevallard (Chevallard Yves. 1992). As for the other models and theories. the purpose was to give elements for description and analysis of facts and phenomenas related to didactical events. As a such, a project of this kind is not completely new. A great part of our work, in the field of research is devoted to identify, describe and analyse phenomenas. Nevertheless the setting up of good models or fruitful theories is not a trivial task. We had previously to deal with another theory, theory of situations, which not so elementary.

The point of view of anthropological approach tries to be at the same time broader and more formal. When I want to propose a model, I have, as a first stage, to consider and define primitive terms and relations in which what I want to explain has to be projected. In the case of anthropological theory, we dispose of three bricks: objects, persons, institutions. Objects are the very basic material, in the sense that persons and institutions can also be considered as objects, they can be in position of being objects. In fact an object cannot exist "per se", by itself, separately of the fundamental relation that constitute it as an object: the relation "to be known as an object (O)" by a person or by an institution. This is the fundamental and very fruitful point. It corresponds to an enlargement of the classical theoretical schemas about cognitive facts, especially in mathematical education Mathematical objects, that are in the very core of our interest, are objects among other objects such as "teacher", "school", "knowledge", etc.. with which we be obliged to deal with in an unified approach. As the same time this approach offers the opportunity of specifying, in the sense that we should have the need, inside the model, to clear the relations between different categories of objects (in the naive sense of the terme,I mean, out of the theory itself) that intervene in the definition and the structuration of didactical systems In fact the frame is more general in the sense that it permits to work with systems in which the basic relation is of the kind X knows O, O is an object of knowledge for X (X being a person or an institution) So the power of the theory lies in two main facts. The first one is the following. Considering a given institution X, we can distinguish the objects that exist for X. these are objects for which exist an institutionnal relation $R_I(O)$. Briefly speaking,mathematical objects appear in various institutions in which can be defined institutional relations (institutional ways of knowing these objects), mainly three families of institutions institutions in which mathematics are produced, institutions in which they are used and institutions in which they are teached So mathemetical objects don't exist "per se" independently of the various institutionnal relations by which they are tied. The second important fact is connected to the relations between persons and institutions Institutions being the places of knowledge about subject one have to be a "subject of the institution" to be in relation with the object , i e to know O from the point of view of $R_I(O)$ At least the person. to become such a subject has to modify his relation R(x,O), has to learn. Amongst the institutions through which O can be known some of them have the special property of carrying an intention of modification of R(x,O). With this purpose they define (it can be define) a generic position, the position of being a student in I for O. and a relation $R_I(s,O)$. institutionnal relation to O for a student. These institutions are didactial institutions with respect to O

This short "*aperçu*" is too brief to have a deep insight into this theory and I suggest for those who are interested in to consult Chevallard (1992) for further reading. Nevertheless from the point of view which is ours in this communication it permits to justify the interest to the process of institutionalization as a such. The passage from R(x,O) (eventualy void) to $R_I(s,O)$ for aperson x or the emergence of a R(x,O) conform to the former is governed by the institution in which it takes place. Functionaly to know O in the way is has to be in I is the product of an institutionalization process

3-Institutionalization, Cognition and Knowledge Transposition:

Entering institutionnalisation in the scope of research on mathematics education and didactics of mathematics seems to be a subject of great interest we are going to see that institutionalization has a crucial role in teaching ans learning being at the same time an "invisible" function, naturalised and incorporated in the everyday school activities This particular invisibility is an obstacle in the study of the kind of phenomenas that seem to be in the scope of institutionalization. During a long time, in spite of early questionings coming from some sociologists, the student was the only "character" in charge of learning

One can use two modalities in the analysis of the didactical system. Following the first one consist in examining and defining "a priori" its functions to traduce them in terms of constraints and necessity. The second one should be a naturalist one, looking for, by observation of class-room situations, categories of events that can be linked to institutionalization. We don't adopt this approach for the moment.

Two basic facts are to be at the principle of an "a priori" questioning: accomodation of the subject to the situations and the knowledge necessary to treat them, accommodation of the subject to the forms of knowledge required by the institution, the one that define the knowledge to be teached and to be learnt. These two components will be permanently interrelated through complex mechanisms in which what is directed by the situation and the subject (a-didactical) and what is directed by the knwoledge and the teacher (didactical) intersect. For the student, the passage from the a-didactical side where he is in charge of his relation with the proposed knowledge to the didactical side where he is under the control of the teacher, is the central part of what we name institutionalization. In its plain generality this function consists in allowing the student to establish with the knowledge a quasi-direct relation, out of the mediation of another person, namely the teacher It seems necessary to study systems and didactical organisations through which this function is realised. In the realization of this function, cognitive aspects are involved but they are not only purely cognitive. In fact, effects of social constraints are to be taken into account. Labelling some of them didactical contract or funnel effect underlines the deep relation they have with the intentionality to teach and to learn, at least to satisfy certain aspects of the teacher's dem^ -d.

The first research on this domain was done on the basis of the theory of situation. This choice had two main consequences. The first one was pu. the emphasis on the a-didactical, didactical axis. It was also, with the help of a new material for analysis, the didactical biography, the way followed by Mercier (1992). The second one was to focus our attention on the realisation of situations for institutionalization In other terms: What should be the caracteristics of these situations? Are there environments for institutionalization ? What are the roles that have to be taken by the student and the teacher along the process? These questions are very complex. The fact is that institutionalization can take place in varions general settings. Puting apart, for the moment, that setting in which the knowlegde-aim is directly communicated to the student, we are going to try to develop another model:

In this model we need to take into account the interplay between two situations. We don't forget anyway that these situations have at the same time two caracteristics: mathematical problem are under study and there are social exchanges. The first situation is the basic a-didactical situation, S_A, in which a certain variety of problem was to be solved. Solving this problem a new kind of knowledge or new properties of a previously learned one. The second situation is an instituting situation, S_I. Its purpose is to act upon the previous one to let the "actors" identify the elements of the cognitive experience in S_A that are fundamental to constitute the knowledge-aim. What is done in S_I has to give a signification, related to the knowledge, to the events in S_A. In this later situation, the students and the teacher have to take into account two kinds of necessities: an intrinsic necessity which lies in the (mathematical) situation in the sense that it does fit the aim, in terms of knowledge, an intentionnal necessity which is tied to the didactical purpose. The first is the epistemological caracteristic of the situation. The second prepares the cultural side of the knowledge. In fact S_A presents some caracteristics of an a-didactical situation but, in general, is not completely a-didactical. In particular students know, from their previous experience that they are working to learn something so that they try to identify what is asked in the didactical contract and not through their propoer activity.

In terms of institutions, that can be seen in our pragmatic model, as a group of persons involved in the resolution of a certain variety of problems and the use of certain kinds of solutions (social practices[1]), the two situations S_A and S_I refer to institutions of two levels. The first level is more or less connected with action and with pragmatic knowledge; knowing how to do without having the obligation to know how it works. The second level is connected with other kind of practices, concerned in the reasons of the success of the methods of the preceding level. Let us quote there an historical example: the difference of practices between the egyptian architect to measure right angles and the theorem of Pythagore and its demonstration; they refer to our two levels. Institutionalization is a change of institution (problems, methods or set of practices, knowledge) and the inscription of one's practice in a new kind of relation, defined by the practices shared by the members of the institution. S_I is a place for social practices of the second level and it appears at least for the observer as a kind of institution. In fact S_I is a place in which the teacher and the student have to negociate about S_A and the knowledge-aim. The student had a special relation to S_A, a certain kind of knowledge which is not well known by the teacher. On the contrary, the later has a clear relation to the knowledge-aim but a relatively poor information about the student experience of S_A. Their common problem is a problem of coincidence in their relation to the knowledge.

This model, in spite of being relatively abstract, is supported by some considerations among which we can identify the following ones. The classical socratic method is an extreme form of institutionalization, directly connected with a theory of knowledge, already inscribed in the world of ideas, already present in each of us so that its actualization is possible through the way of reminiscence. With the perspective of transformation of one's experience, the general process is reflexivity, directly if it is asked to the student as a product, indirectly if the student is invited to compare his solution to others solutions. With a more active teacher's participation which is the case in most of the didactical situations, he can be involved in a reformulation of the student's past, in the translation of a social practice to another social practice, in the disclosure of elements of his didactical intentionnality.

[1] Let us name practice(namely mathematical practice) every actualization made by somebody in the course of the resolution of a mathematical problem or a problem the solution of which requires some manifestation of a mathematical knowledge. This kind of practices are concerned by algorithms, methods, formulations, validations. In other words, a mathematical practice is certified by behaviors that are socialy certified to be those of the practice in refering to a cultural universe of conformity.

Concluding remarks:

We cannot pretend that it is completely original to consider "mathematical learning ... as both a process of individual construction and as a process of acculturation into the mathematical meanings and practices of wider society" (Cobb P. and all, 1992; Eisenhart, 1988). Nevertheless our main problem now, from a theoretical point of view as well as from a practical one, is to describe operational approaches of this problem from the point of view of teaching and teaching devices. Our general problematics try to ccodinate the two major settings in which the mathematical activity of the student takes place as well as its mathematical practice takes its meaning. From one side, each mathematical situation and problem requires understanding, previous knowledge and a recourse to other kinds of knowlegde which are not exactly inscribed in what he know. In general cases these late requirements are not very important. On the contrary learning new mathematical objects involve new ways of knowing and new mathematical contents. On the other side, mathematical object are complex and are to be known through a rather long process, didactical process, goal oriented, that can be analysed in the way the general process of entering in an institution is analysed. This kind of analysis is not on the student's side but more on the teacher's side and the special setting of knowledge which is required to make this knowlegde teachable.

References

ARSAC, G., BALACHEFF, N., MANTE, M. : 1992, "Teacher's role and reproductibility of didactical situations", Educational Studies in Mathematics **23**(1): 5-29.

BISHOP, A.: 1988, "Mathematical Enculturation, A cultural perspective on Mathematics Education", Kluwer Academic Publishers.

BAUERSFELD, H.: 1980, "Hidden Dimensions in the so-called reality of a mathematical classroom",Educational Studies in Mathematics, **11**, 23-41.

BROUSSEAU, G.: 1978, "Étude locale des processus d'acquisation en situations scolaires", in IREM de Bordeaux,*Enseignement Élémentaire des Mathématiques*, **10**.

BROUSSEAU, G.: 1980, "Les Échecs Électifs en Mathématique", Revue de Laryngologie, Othologie, Rhinologie, **3-4**, 107-131.

CHEVALLARD, Y.: 1992, "Concepts fondamentaux de la didactique: pespectives aportées par une approche anthropologique", Recherches en Didactique des Mathématiques **12**(1), 73-112.

COBB, P., YACKEL, E., WOOD, T.: 1992: "Interaction and learning in mathematical classroom situations", Educational Studies in Mathematics **23**(1): 99-122.

DOUGLAS, M.: 1989: Ainsi pensent les institutions. Editions Usher.(French translation of "How Institutions think", 1986 by Syracuse University Press) .

EISENHART. M. A.. 1988. "The ethnographic research tradition and mathematics education research'. Journal for Research in mathematics Education **19**. 99-114

LABORDE. C 1989. "Audacity and reason: French research in mathematics education". For the Learning of Mathematics **9**(3). 31-36.

MERCIER. A. (1992). "L'Élève et les contraintes temporelles de l'enseignement . un cas en calcul algébrique". Thèse de doctorat Université de Bordeaux

ROUCHIER. A.· 1991. "Etude de la conceptualisation dans le système didactique en mathématiques et en informatique élémentaire: proportionnalité. structures itérativo-récursives. institutionnalisation". Thèse de Doctorat d'État Université d'Orléans.

THE ROLE OF GRAPHIC AND ALGEBRAIC REPRESENTATIONS IN THE RECOGNITION OF FUNCTIONS BY SECONDARY SCHOOL PUPILS.

L.Ruiz Higueras; J.L. Rodríguez Fernández (University of Jaén)
C. Batanero and J.D. Godino (University of Granada).

SUMMARY.

The concept of function is one of the most important mathematical concepts, due to its unifying and modelizing nature and since it appears on all school curricula from the Primary school years up till the Secondary school. However, it is a complex concept since it contains a multiplicity of representative registers and it generates very different levels of abstraction. Due to these characteristics, in our research work, we have concentrated on differentiating between the different conceptions that the pupils may associate with it. This paper is part of further research into the concept of function, in which we present an analysis of the answers to 14 questions regarding the discrimination of functions based on algebraic expressions and graphic representations. Using a multivariate analysis of the group of answers, we deduce factors which indicate the influence of the task variables in the items regarding the pupils' arguments. Consequently we characterize different local conceptions about the concept of functions.

INTRODUCTION.

Over the last few years research in the field of Didactics of Mathematics has been marked by the efforts made to identify pupils' conceptions, through the regularity to be found in their different work, to distinguish those that may constitute obstacles for acquiring other knowledge and to clarify the conditions in which these conceptions, which are sometimes highly resilient, may be modified. In a great number of recent papers, such as those by Confrey (1990), Bednarz and Garnier (1989), Brousseau (1983), Vergnaud (1990), Artigue (1990), Viennot, Joshua, etc there is one fundamental idea: analyzing the development of the pupils' conceptions and the factors which determine that development.

> *"The term conception is used in Didactics of mathematics for the purpose of establishing a distinction between the mathematical object which is unique and the varied meanings which may be associated therewith by our pupils...... The uniformity of the definitions and exercises proposed in mathematics text books hides the wealth and complexity of the conceptions which may be associated therewith" (Artigue, 1984, p.77).*

In reference to the notion of function the first papers to be found are those by Piaget, Grize, Szeminska (1968) in the framework of genetic epistemology. Later we find those by Thomas (1975) and Maryanskii (1979) which, although they do not deal directly with pupils' conceptions, they implicitly refer to some aspects relating thereto. Following these, various papers have been written, amongst which we would highlight those by Janvier (1981), Dreyfus and Eisemberg (1982), Vinner, Tall and Dreyfus (1983, 1989), Markovits, Eylon and Bruckeimer (1986), Sfard (1989, 1991, 1992), Sierpinska (1989, 1992), Dubinsky, Hawks and Nichols (1989), Dubinsky and Harel (1992), Tall and Bakar (1992). This diversity of research into the notion of function is described in the survey made by Leinhardt and colls. (1991); amongst the conclusions which he gives us we highlight those which characterize the learning stated by the pupils: a wish for regularity in the function's behaviour (a strong tendency towards linearity), an excessively precise vision of the function (as against the problem as a whole) and an incorrect interpretation of the properties which may be abstracted from the graphic displays of functions.

In this paper we study the arguments that the secondary school pupils use (aged between 16 and 18) when discriminating functions based on algebraic expressions or graphic displays. Starting from a multivariate analysis of the group of answers to the different items, we deduce factors which indicate the influence of the task variables in the items on the pupils' arguments. Consequently we characterize local conceptions about the concept of function.

EXPERIMENTAL STUDY.

The work which we present is part of a larger study regarding the concept of function, which includes an epistemological study, an analysis of the teaching and a study about conceptions, using a questionnaire made up of 25 items which was given to 323 pupils (Ruiz Higueras, 1993). The full questionnaire included three sections: a) study of the definitions which, about the concept of function, the pupils supply and of the examples they propose; b) discrimination of functions based on their graphic and algebraic displays; and c) use of functions in solving problems about modelling.

The results which we present in this paper correspond to the analysis of the answers to the 14 questions about the discrimination of functions, which appear as an Annex. When drawing them up we chose to use open questions instead of multiple choice questions, so as to study the range of possible arguments that the pupils are capable of expressing when justifying their answers. By means of these arguments the pupils will reveal different conceptions about the notion of function.

The task variables included in these questions were as follows:

-V1: Presentation setting for the item; which could be graphic or algebraic.
- V2: Type of correspondence presented: application/non-application.
- V3: Continuity: Continuous/discontinuous in the whole domain.
- V4: Derivability: Derivable/non-derivable in the whole domain.
- V5: Unique algebraic expression or different in parts of the domain.

In each one of the questions we have considered two dependent variables: the answer to the item (correct or incorrect) and the argument used as a justification. We have used the following categories for these arguments:

(a) Application (coded as APLIC in Table 2): There are pupils who justify whether a given graph represents or not a function by considering the latter as a single valued correspondence. ("It does not involve a function since at one given moment it is in several places at the same time" (graph GA in the Annex). When trying to justify whether an algebraic expression is a function or not, some pupils also use the criterion of it being univocal ("If it is a function, for each value we give for x, it would correspond to a value of y" (expression AE).

(b) Iconic-ideogram elements (ICON): we have assigned this category when the pupil bases his recognition of the function, not on an analysis of its characteristics, but because it is a prototype example, used frequently in the teaching he has received. Lacasta (1992) states that "on numerous occasions, both the school books and the teachers in their classes treat graphs of functions as ideograms: a conventional symbol for recognizing shapes in an ostensive manner" (p.8). One example would be the following: "If it is a function, it is a circumference" - graph GF). For the algebraic expressions the pupils also base their arguments solely on the "shape" they have, i.e., they recognize or do not recognize it as a model familiar to them ("If it is a function by parts" (expression AA).

(c) Algebraic expression (EALG): In this section we include those arguments which justify that a given graph represents or not a function, based exclusively on the fact that it is possible to give an algebraic expression for it. ("If it is a function, it is the function $y = 8$" (graph GA).

In the case of algebraic expression, solely the fact of its existence may be enough to justify it in order to determine whether or not it is a function ("If it is a function, there is a mathematical formula" (expression AC).

(d) Domain-Image (DOMINIO): There are some pupils who consider the graph of a function as a kind of abacus, they can give values to the x and, therefore, by means of the graph, they can obtain the corresponding value of y. ("Yes, it is a function, because we can take points from the x and find others for the y, and so we would achieve this graph" (graph GC).
In this category we also classify those answers that, when referring to the algebraic expressions, the pupils justify the existence of a function by the possibility of giving values to the x and obtaining values of the y; i.e., they see in the algebraic expression a certain algorithmic procedure. ("It is not a function, because I don't know where I'm going to give the values to the x" (expression AB).

(e) Continuity-Discontinuity (CONT): Justifications based on the characteristics of continuity or discontinuity. ("It is not a function, a function cannot have sudden jerks" (graph GE).

(f) Increase-decrease (CREC): Arguments based on the increase or decrease in the curve shown. ("It is not a function because it increase and decreases in an irregular manner, it looks more like an earthquake reading" (graph GG).

(g) Others (OTROS): We include in this section those sporadic and strange answers which it is not possible to classify in any of the previous sections.

RESULTS AND DISCUSSION.

Answers to the items.

Table 1: Percentages of correct answers, according to the pupil's age.

	ITEMS													
AGE	AA	AB	AC	AD	AE	AF	AG	GA	GB	GC	GD	GE	GF	GG
16	87.1	47.5	41.7	51.1	54.7	71.2	66.9	20.1	87.8	40.3	67.6	86.3	18.7	25.2
17	88.4	44.2	29.1	34.9	53.5	72.1	73.3	15.1	90.7	37.2	52.3	89.5	48.8	39.5
18	92.9	60.2	62.2	53.1	46.9	79.6	72.4	18.4	98.0	43.9	84.7	100	52.0	38.8

As may be seen on the Table, generally speaking, age does not influence the pupils' answers. We may say that the education received by the pupils does not determine significant changes as regards the identification of functions.

Analysis of the arguments.

Once the arguments were classified, we carried out an analysis of correspondence (Greenacre, 1984) in table 2, which crosses the different arguments used by the pupils in each of the 14 questions presented in order to recognize whether or not they are functions.

Table 2: Frequencies of arguments in the different questions.

	ITEMS														
ARGUMENTS	AA	AB	AC	AD	AE	AF	AG	GA	GB	GC	GD	GE	GF	GG	TOTAL
APLIC	11	5	4	4	6	5	7	30	4	3	1	0	9	5	94
ICON	138	163	161	141	45	77	38	98	112	86	90	130	118	64	1461
EALG	0	0	0	0	0	0	0	4	91	15	26	36	15	8	195
DOMINIO	33	5	16	13	39	45	66	13	15	8	13	8	7	8	289
CONT	8	3	0	0	3	0	1	0	9	2	5	32	3	6	72
CRECI	0	0	0	0	0	0	0	2	7	16	9	9	3	14	60
GRAFICO	36	35	17	22	14	20	10	0	0	0	0	0	0	0	154
OTROS	6	25	3	4	9	9	7	0	5	1	2	0	0	6	77
TOTAL	232	236	201	184	116	156	129	147	243	131	146	215	155	111	2402

A highly significant Chi-square value was achieved (1413.722; 91 d.f. (degree of freedom) and three main factors were observed, amongst which they explain 80.40% of the inertia. All the arguments and items used presented a good quality of representation on the axes of projection obtained. We shall interpret these factors below.

FIRST FACTOR (47.25% of the inertia): "Presentation chart for the item".

On this first axis clear opposition was shown as regards the coordinates on the axis between the items expressed in algebraic form and those in a graphic form, due to the fact that the arguments, used by the pupils to determine in each one of them whether it involves or not a mathematical function, are clearly opposed. So the "graph" argument ("it is a function because we can display it on a graph") is used exclusively when algebraic expressions are referred to; the "domain" argument ("it is a function because we can give values to the x and we obtain the value of the y" or "it is not a function because since we don't have x we cannot give it any values") is used much more abundantly to refer to algebraic expressions. On the contrary, the "algebraic expression" argument is used by the pupils exclusively when referring to the functions expressed graphically ("it is a function because its equation is y = 1/x"); the same may be said about the arguments of "increase" and "continuity", they are only used if there are graphs.

Consequently, we interpret that the importance of this factor is indicative of the existence, amongst a large part of the pupils of a pre-conceived conception due to the strong association between formula and graph: "Every formula may be displayed on a graph", reciprocally, "Every graph comes from the values given to a formula". Evidently this conception would have a very limited field of validity, even in this same test, since in expressions like AE or GG it would not work.

The arguments about "increasing" (CRECI) or "continuity" (CONT) are only included in arguments which involve trying to justify the existence of a function by means of describing its graph, however, they are not used for algebraic expressions such as AA, AD, AE or AF. This would indicate that these properties are considered to be graphic characteristics and not analytical ones. We interpret that the origin of this could be due to the fact that the pupils' education reflects the influence of the "ostensive" nature whereby the teachers use graphs in their teaching to present highly formalized concepts such as lateral limits, continuity, growth, derivability, etc. The pupils do not respond to the question of whether or not a function is involved, but rather they limit themselves to "classifying" the graph.

SECOND FACTOR: (21.1% of the inertia) "Comparison of arguments relating to the domain and ideographic arguments on the algebraic chart".

This factor separates the algebraic items into two groups and it hardly has any influence from the graphic items: on the one hand, items AG and AE appear; on the other hand items AD, AC and AB do so. The first group of items appears to be associated to the domain argument and the second one to the ideographic argument, which are opposed to each other. In fact, items AG and AE present a percentage of items relating to the domain which is quite higher than the average and a percentage of ideographic arguments quite lower than the average; the opposite happens with items AD, AC and AB. This explains to us that there are algebraic expressions which the pupils identify or not as a function simply bearing in mind the "shape" they have: they are prototype examples that are easily recognized by the pupils.

This kind of ideographic argument is for us indicative of another highly restricted conception of the function, since an algebraic expression shall be considered as a function only under certain "syntactic" conditions and a graph only under strict limitations regarding its configuration. Its domain of validity is also very limited since functions such as AB, AD, AE, GC, GD or GG would be excluded. We believe that its origin is also in the teaching habits, due to the usual repertoire of graphs and functions that appear in mathematics teaching. The presentation that both the maths books and the teachers make of the notion of function in all their generality contrasts with the limitations of the field in which the examples and exercises are normally chosen.

As we described above, the arguments which we have called DOMAIN indicate that only the algebraic expressions which make it possible to give values to the x and to obtain the corresponding values for y may be considered to be functions. For us, they indicate a conception of the function as a certain procedure of algorithmic calculus (input-output). Its domain of validity is also very limited since it excludes functions like AB or AE. The pupils show this when referring to the functions expressed algebraically. We may say that they are induced by a deeply rooted practice in teaching, as is the drawing of "tables" based on algebraic expressions.

THIRD FACTOR: (12.1% of the inertia) "Idea of application"

This factor is marked almost exclusively by the idea of application and it opposes item GA to the rest. In this item, the argument of application was used very frequently, which, in general, was not used in the others. The lesser weight of the third factor shows that the pupils first looked for other arguments (ideographs, algebraic expression, graphs, etc..) before checking whether or not it involved an application. For them, the function had to meet other prior requisites before this one in order to be considered as such. We may also point out that the GA graph is very peculiar and strange, due to its shape (it does not usually appear in text books) and for the pupils it was very effective and economical to use the "application" argument. Nevertheless, item GF would also be included in these very same circumstances, but because it involved a circumference, a very familiar curve for them, they do not use this argument, but rather the ideograph argument ("it is a circumference, therefore it is a function") and this is sufficient for them to consider it to be a function, they do not need to prove whether or not it is an application.

This application argument is also indicative of a conception of the function as an application, although generally this was observed very little in our sample, in spite of the fact that this is the definition of a function which appears in the text books which these pupils used.

Although the variable of the academic year involved has been included as a supplementary variable, no influence therefrom was obtained in any of the axes. Consequently we observe a permanence of the arguments used throughout the different academic years, which might be a sign of the permanence of conceptions, independently of the teaching received.

REFERENCIAS

ARTIGUE, M. (1984). *Contribution à l'étude de la reproductibilité des situations didactiques. Divers travaux de mathématiques et de didactique des mathématiques.* Thése de Doctorat d'Etat. Université Paris VII.

BEDNARD, N., GARNIER, C. (Eds) (1989). *Construction des savoirs: Obstacles & Conflicts.* Otawa. Canada: Agence d'Arc.

BROUSSEAU, G. (1983). Les obstacles épistémologiques et les problèmes en Mathematiques. *Recherche en didactique des Mathematiques, 4*, 2, 164-198.

CORNEJO, J. M. (1988) *Técnicas de investigación social: El análisis de correspondencias (Teoría y práctica).* Barcelona: Promociones y Publicaciones Universitarias.

GRENACRE, M. J. (1984). *Theory and applications of correspondence analysis.* London: Academic Press.

DREYFUS, T. EISEMBERG, T. (1982). Intuitive functional concepts: a baseline study on intuitions. *Journal for research in Mathematics Education*, 13, 5, 360-380.

DUBINSKY, E., HAWKS,J., NICHOLS, D. (1989). *Development of the process conception of function by pre-service teachers in a discrete mathematics course.* Proceedings of the 13th Annual Conference of the International Group for the Psychology of Mathematics Education, 291-298.

DUBINSKY, E., HAREL, G. (1992) *The concept of function: Aspects of Epistemology and Pedagogy.* Mathematical Association of America. USA

JANVIER, C. (1981) Use of situations in Mathematics Education. *Educational Studies in Mathematics*, 12, 113-122.

JANVIER, C., CHARBONNEAU, L., RENE DE COTRET, S. (1989). Obstacles épistemologiques à la notion de variables: perspectives historiques, en Bednarz, N., Garnier, C. (ed), *Constructions des savoirs: Obstacles et conflits,* Quebec, Canada: Cirade, Agence d'Arc.

LACASTA, E. (1992) *Génesis y desarrollo de un tema de investigacion en Didáctica de las Matemáticas.* Ponencia presentada en el Seminario Interuniversitario de Investigación en Didáctica de la Matemática. Madrid

LEINHARD, G., ZALASVKY, O. y STEIN, M. K. (1990) . Functions graphs, and graphing: Task, Learning, and Teaching. *Review of Educational Reseach,* 60, 1, 1-64.

MARCOVITS, Z., EYLON, B., BRUCKEIMER, M. (1986). Functions today and yerterday. *For the learning of Mathematics*, 6, 2, 18-28.

MARNYANSKII, I. (1979). *Psychological characterics of pupils' assimilation of the concept of function.* En Kilpatrick, J., Wirszup, I., Begle, e.G., Wilson, J.W. (Eds.) (1979). Soviet Studies in Psychology of Learning and Teaching Mathematics. Vol. XII. Stanford: NCTM

PIAGET, J., GRIZE, J.B., SZEMINSKA, A. (1968) *Epistemologia et psichologie de la fonction.* Presses Universitaires de France.

RUIZ HIGUERAS, L. (1993). *Concepciones de los alumnos de secundaria sobre la noción de función : análisis epistemológico y didáctico.* Tesis Doctoral. Universidad de Granada

SIERPINSKA, A. (1989). *On 15-17 years old students' conceptions of functions, iteration of functions and atrative points.* Warsaw, Poland: Institute of Mathematics, Polish Academy of Sciences, Preprint, 454.

SIERPINSKA, A. (1992). *On understanding the notion of function. En G. Harel, E. Dubinsky (Eds), The concept of function. Aspects of Epistemology and Pedagogy* (p. 25-58). USA: Mathematical Association of America.

SFARD, A. (1989). *Transition from operational to structural conception: The notion of function revisited.* Proceeding of the 13th Annual International Conference for Psychology of Mathema tics Education, 151-158.

SFARD, A. (1991). On dual nature of mathematical conceptions: reflections on processes and objets as different sides of the same coin. *Educational Studies in Mathematics,* 22, p. 1-3.

TALL, D., BAKAR, P. (1992) . Students'' mental prototypes for functions and graphs. *International Journal Mathematics Education Science and Technology,* 23, 1, 39-50.

VINNER, S. , TALL, D. (1981). Concept image and concept definition in Mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151- 169.

VINNER, S. (1983). Concept definition, concept image and the notion of function. *The International Journal of Mathematics Education in Science and Technology,* 14, 293 - 305.

VINNER, S. DREYFUS, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education,* 20, 4, 356-366.

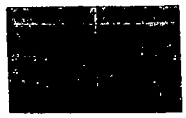
VINNER, S. (1990). Inconsistencies: Their causes and function in learning mathematics. *Focus on learning problems in mathematics,* 12, 3 & 4, p. 85-98.

ANNEXE

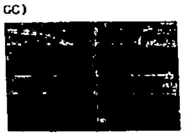
Te presentamos a continuacion varias figuras. Debes decir, para cada una de ellas, si se trata o no de la representacion gráfica de una funcion. Explica detalladamente tus respuestas

GA)

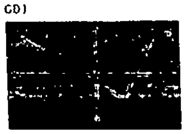
GB)



GC)



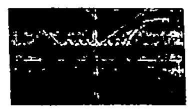
GD)



GE)

GF)

GG)



Te presentamos a continuacion varias expresiones algebraicas. Debes decir, para cada una de ellas, si se trata o no, de una funcion. Explica detalladamente tus respuestas.

AA)

$$f(x) = \begin{cases} 4x + 3, & x \in R,\ x < 2 \\ 0, & x \in R,\ 2 < x < 3 \\ 2, & x \in R,\ x > 3 \end{cases}$$

AB)

$y = 4$

AC)

$x^2 + y^2 = 1$

AD)

$x.y = 5$

AE)

$$f(x) = \begin{cases} 1 & \text{para todo x racional} \\ 0 & \text{para todo x irracional} \end{cases}$$

AF)

$$y = \frac{-3}{x}$$

AG)

$$f(x) = \sqrt[4]{4x+1} \quad \forall x \in R$$

PUPILS' VIEWS OF CALCULATORS AND CALCULATION

Kenneth Ruthven
University of Cambridge

Abstract: *This questionnaire study examines pupils' representations of number work, calculators and calculation, and their dispositions towards them, as they transfer from primary to secondary education. Two distinct dimensions of attitude and belief emerge, associated with pupils' preference for not using or using a calculator: first, their degree of enjoyment of number work and of confidence in alternative modes of calculation; second, their degree of scepticism about the legitimacy and beneficence of the calculator and of confidence in the calculator mode of calculation. Marked gender and school differences are found on the second dimension.*

Introduction

The focus of this paper is on pupils' attitudes to, and beliefs about, number work, calculators and calculation at an important transition point in their education: between primary and secondary school. The concepts of attitude and belief are diffuse, and this has provoked some criticism of them as theoretical constructs. Nonetheless, their persistence reflects their resonance with our commonsense construction of human thought and behaviour; in particular, the way in which representation of experience and disposition towards it appear to combine in our mental activity.

Equally, attitude and belief can be said to have both individual and social facets: in particular, individuation of attitude and belief within a community can be thought of as differential positioning within some shared template. The particular focus of this study was on developing a simple model of such a template. For this purpose, a standard questionnaire methodology was adopted, involving a substantial pupil cohort.

The design and implementation of the study

The pupils taking part in the study had just entered their first year of secondary education in two 11-16 comprehensive schools serving village communities in the Cambridge area. In most cases, the pupils had previously attended local primary schools from which the normal pattern of transfer was to these secondary schools.

The questionnaire consisted of items in Likert format presenting pupils with a sequence of statements to which they were asked to respond on the following five-point scale: ✓✓, described as meaning '"yes!": you strongly agree' (and scored as 2); ✓, described as '"yes..ish": you agree but not strongly' (scored 1); -, described as '"er..um..?": you are undecided or neutral' (scored 0); ✗, described as '"no..ish": you disagree, but not strongly' (scored -1); and ✗✗, described as '"no!": you strongly disagree' (scored -2). The items fell into two groups. The first focused on comparative judgements of three modes of calculation: mental, written and calculator. The second was concerned with more general judgements about working with numbers and calculators. Items from the two groups were alternated in the questionnaire; within each group, similar items were distanced to avoid sequencing effects; and two potentially emotive items were placed at the end of the questionnaire.

Pupils completed the questionnaire in September 1993, within the first fortnight of joining their secondary school. They did so in class under the supervision of their mathematics teachers. Some teachers reported instances of pupils requiring help in identifying which response to choose when they disagreed with a statement that could be construed as a negative. Of the 340 pupils on roll in the two schools, questionnaires were received from 327, a response rate in excess of 96%.

Comparison of calculation modes

The first group of items was designed to elicit comparative ratings of the modes of calculation in terms of three criteria: difficulty, incidence of mistakes, and time taken. The items and the patterns of response to them are shown in Table 1. There is a consistent pattern in which pupils rate the calculator mode relatively favourably on each criterion and the mental mode relatively unfavourably, with the written mode rated intermediate on difficulty and unreliability. Statistically, all these comparisons are highly significant (beyond the .000 level on a one-tailed Wilcoxon test). In the only exception to

Table 1: Items comparing modes of calculation

Item	Response percentages ✓✓	✓	-	✗	✗✗
It is more difficult to do number problems with a calculator	4	10	19	30	37
It is more difficult to do number problems by a written method	6	19	30	29	16
It is more difficult to do number problems in your head	23	34	20	18	6
When you do number problems with a calculator, you make more mistakes	5	12	25	30	29
When you do number problems by a written method, you make more mistakes	4	16	34	27	19
When you do number problems in your head, you make more mistakes	11	37	22	23	7
Doing number problems with a calculator takes more time	6	12	14	35	34
Doing number problems by a written method takes more time	23	38	23	13	4
Doing number problems in your head takes more time	23	39	20	11	7

this pattern, the written mode was rated close to the mental mode in terms of time demand.

Equally, the patterns of response to a particular mode are similar across different criteria, suggesting that relatively undifferentiated judgements are being made. For each mode, the statistical association between ratings on different criteria is sufficiently high (correlation typically around 0.3, significant beyond the .000 level on a one-tailed Spearman test) to make it reasonable to aggregate ratings for further analysis by averaging scores for difficulty, unreliability and time demands to form the compound constructs *CalculatorHasDrawbacks*, *WrittenHasDrawbacks* and *MentalHasDrawbacks*.

Views of number work and number problems

The second group of items was intended to elicit judgements about more general aspects of working with numbers and calculators. The majority were designed as paired items of opposite polarity. The items and the patterns of response to them are shown in Table 2. Analysis suggests a degree of acquiescence, although some pupils may reasonably have interpreted statements as not wholly opposed, or encountered the difficulties mentioned earlier over disagreement with negative statements. Nonetheless, with the exception of one pair which has been excluded from this analysis, statistical association is sufficiently high (correlation typically around -0.25, significant beyond the .000 level on a one-tailed Spearman test) to justify the formation of compound constructs by averaging scores over the pair, after reversing those

Table 2: Items on number work and calculators

Item	Response percentages				
	✓✓	✓	-	✗	✗✗
I enjoy experimenting with numbers	38	36	13	6	7
I dislike working with numbers	7	9	13	30	42
There are several different ways of doing every number problem	31	51	12	5	1
There is one proper way of doing every number problem	15	14	29	22	20
I prefer number problems when I have to think them out	19	34	24	14	10
I prefer number problems when I have already been shown what to do	36	31	14	11	9
You have to understand a number problem to do it with a calculator	22	34	22	14	9
You can do a number problem using a calculator without understanding it	22	33	13	19	14
You can learn new things about numbers from using a calculator	16	26	28	17	13
If you use a calculator you learn less about numbers	29	32	16	15	9
I prefer not to use a calculator to do number problems	16	16	30	22	17
Using a calculator to do number problems is a kind of cheating	19	24	25	17	16

on the negative item. Indeed, interpretation of the tabulated data will be supported in the text by reporting of the proportion of pupils responding consistently positively or negatively over a pair of items.

For the first pair of items, corresponding to the compound construct *LikeNumberWork*, such information confirms the indication of the tabulated responses that the majority of pupils have a positive attitude to number work: 60% agreed with the positive statement and disagreed with the negative; whereas only 5% showed the reverse pattern.

The results from the next two pairs of items suggest some qualifications to Schoenfeld's (1992, p.359) general characterisation of pupils' beliefs about mathematics: that there is only one correct way to solve any problem, usually the rule that the teacher has most recently demonstrated; that pupils cannot expect to understand mathematics, but must memorise it and apply what they have learned mechanically. Responses to the pair of items corresponding to the construct *SeveralWaysToDo* suggest that many pupils take a flexible view of method. The strongly supported positive statement that there are several different ways of doing every number problem was the first on the questionnaire, appearing prior to any mention of different modes of calculation. Even on the more pointed negative statement that there is one proper way to do every number problem, disagreement outstrips agreement. Combined, 38% of pupils agreed with the positive statement and disagreed with the negative; only 3% showed the reverse pattern. On the pair of items corresponding to the construct *LikeToThinkOut*, the evidence is less clear cut: 29% of pupils agreed that they prefer problems both when they have to think them out and when they have already been shown how to do them. The combination of these two positions is not necessarily inconsistent; it could be interpreted as expressing approval for structured intellectual challenge. Overall, 15% took the rather stronger position of agreeing with the first item and disagreeing with the second, with 19% showing the reverse pattern. Nonetheless, even these findings should serve to qualify the stereotype of the intellectually passive pupil.

Views of the calculator

The last two item pairs focus more specifically on use of the calculator. A study by the APU (Foxman et al., 1991, p. 3-17) found that nearly 30% of 11-year-olds considered that use of calculators was harmful because "they stop you using your brains" or "prevent you learning all sorts of sums". The present study distinguished between the constructs *UnderstandWithCalculator* and

LearnFromCalculator (and this differentiation is vindicated by the relatively weak correlations observed between their respective items). Overall, there is more scepticism about the position that use of the calculator promotes learning about numbers (with only 14% taking a consistently positive position, 22% consistently negative) compared to the view that using a calculator to do a number problem requires understanding (where 24% took a consistently positive position, 15% consistently negative). In effect, this indicates greater acceptance of the calculator as a procedural tool than as a learning aid. In this respect, pupils' views coincide with the practice of teachers as observed by HMI (1991, p.18; 1992, p.15) who found little recognition of the role of calculators in promoting the development of number concepts and skills.

The final single entries in Table 2 are the two potentially emotive items already referred to. The more neutral statement, expressing a preference for not using a calculator, was supported by one third of pupils. This is similar to the proportion of 11-year-olds reported in the APU study as disapproving the use of calculators or preferring not to use them. The more impassioned statement (concluding the questionnaire) which labelled use of the calculator as a kind of cheating was supported by over 40% of the pupils, whereas such a view was expressed more spontaneously by less than 10% in the APU study. This can be interpreted either as an expression of simple disapproval or as one of intellectual fastidiousness; a valuation of mental competence and mental activity. Hedren (1985, p. 176) has noted that many mathematically confident pupils enjoy mastering calculations themselves instead of being dependent on a machine; similarly, Ruthven (1992, pp. 92-93) found pupils reluctant to use a calculator on the grounds that by doing so they 'lost control' of their mathematics. In view of the broad and even spread of responses to these two items, they were retained as separate constructs for further analysis as *PreferNoCalculator* and *CalculatorIsCheating*.

The influence of gender and primary school

Gender differences in attitude to both mathematics and technology have been widely reported. Consequently, this is an area which merits particular examination. School differences also need to be considered as they may be acting as an otherwise hidden influence on gender differences. Although school differences are also of interest in their own right, in this study there is no way of distinguishing between an effect which is due to differences in the communities from which schools draw their pupils, and one due to pupils' experience within the school. School effects, then, must be interpreted with corresponding caution, as indicating plausible conjectures about the influence of schools rather than firm conclusions. Possible gender and school differences were explored through a two-way analysis of variance on each of the constructs (with small primary schools transferring low numbers of pupils being consolidated into a single category). In fact, no gender-school interactions approaching significance were found in the analysis. The findings are summarised in Table 3.

Although the APU (Foxman et al., 1991, p.5.2) reports that, at age 11, boys as a group are more favourably disposed towards mathematics than girls, such an effect was not found here. The focus of *LikeNumberWork* is, of course, more narrowly specific, and the same APU study (p.2.3) reports number as an area in which there are minimal gender differences in attainment at age 11. On the two constructs relating to views of number problems, the stronger and mildly significant gender difference is on the construct *LikeToThinkOut* although the trend on *SeveralWaysToDo* is in the same direction. These differences are consistent with those suggested by other work on gender differences in attitudes to autonomy and independence (McLeod, 1992, p.586).

On the constructs relating to modes of calculation, the only evidence of a gender effect is in relation to use of the calculator. Boys' and girls' ratings of both the written and mental modes are extremely close. But while boys and girls both rate the drawbacks of the calculator mode as markedly lower than either of the other modes, the boys' rating is significantly more favourable. On the four remaining constructs relating to the calculator, there are no significant gender differences, although the trends are all towards more positive ratings of the calculator by boys. Here, however, there is strong evidence of a school effect, with three of the four items revealing some degree of significance in the differences by primary school.

Table 3: Construct ratings: overall, and by gender and school

Construct	Overall statistics		by Gender		by School		Significance	
	mean	st-dev	Female mean	Male mean	Lowest mean	Highest mean	Gender effect	School effect
CalculatorHasDrawbacks	-0.78	0.85	-0.66	-0.86	-1.14	-0.40	.055	
WrittenHasDrawbacks	-0.01	0.79	-0.02	-0.01	-0.44	0.18		
MentalHasDrawbacks	0.47	0.83	0.45	0.48	0.20	0.86		
LikeNumberWork	0.91	1.00	0.92	0.90	0.54	1.26		
SeveralWaysToDo	0.62	0.83	0.58	0.66	0.38	0.75		
LikeToThinkOut	-0.17	1.00	-0.28	-0.07	-0.57	0.15	.085	.048
UnderstandWithCalculator	0.09	1.03	0.08	0.10	-0.32	0.41		
LearnFromCalculator	-0.22	1.02	-0.29	-0.15	-0.70	0.21		.048
PreferNoCalculator	-0.09	1.30	-0.05	-0.12	-0.75	0.50		.088
CalculatorIsCheating	0.13	1.33	0.17	0.09	-0.57	0.83		.031

A structural analysis of views

The goal of this study was to model a simple framework within which individual differentiation of attitude and belief can be understood as taking place. This was pursued through a factor analysis of the constructs with the results reported in Table 4. This two-factor model captures a good proportion of variation; beyond this point, the increased explanatory power of adding further factors drops markedly. One way of making sense of the structure is to see it as offering two independent dimensions relating to pupils' preference for using a calculator or not doing so.

The first dimension links preference for not using a calculator to a liking for number work and thinking out number problems, to relative confidence in the alternative written and mental modes of calculation, and to a view that number problems can be tackled in different ways. Correspondingly, preference for using a calculator is linked to a lack of enjoyment of number work and thinking number problems out, to lack of confidence in alternative modes of calculation, and to a view that there is one proper way to tackle each number problem. At heart, then, this dimension relates to pupils' general enjoyment and confidence in the different elements of number work: the more so, the stronger the preference to dispense with the calculator (thus increasing the challenge of the work); the less so, the stronger the preference for using the calculator (thus making the work achievable). This suggests how decisions about use of the calculator enable pupils to control the challenge posed by number work, and match it to their degree of enjoyment and confidence in such work. This is consistent with the report by Shuard et al. (1991, p. 24) that most children decide for themselves that they do not need or want to be dependent on their calculators for all calculation.

Table 4: Factorial structure of views

Loadings accounting for more than 10% of construct variance

Construct Factor	I	II
LikeToThinkOut	.66	
LikeNumberWork	.66	
MentalHasDrawbacks	-.63	
WrittenHasDrawbacks	-.47	
SeveralWaysToDo	.45	
CalculatorIsCheating		.72
LearnFromCalculator		-.69
PreferNoCalculator	.49	.53
CalculatorHasDrawbacks		.42
UnderstandWithCalculator		-.33
Percentage of total variance explained	20.6	15.7

The second dimension links preference for not using a calculator to doubts about its legitimacy and about the facility, reliability and speed of the calculator mode of calculation, and to beliefs about the negative influence of the calculator on understanding and learning number. Correspondingly, preference for using the calculator is linked to acceptance of its legitimacy and of its facility, reliability and speed. This dimension, then, is one of general attitude to the calculator: the more positive the score, the greater the degree of scepticism about the the calculator and the preference against use; the more negative the score, the greater the acceptance and preference for use.

No gender or school difference emerges on the first factor: in particular, the means for boys and girls are extremely close. On the second factor, however, both gender and school effects are significant (at the .051 and .023 levels respectively). In particular, the male mean is markedly lower (0.23 of a standard deviation) than the female. This seems to be a consolidation of the consistent trends for boys to be more favourably disposed towards the calculator on individual constructs. In sum, then, important gender and school differences arise in attitudes and beliefs about the calculator.

References

FOXMAN, D., RUDDOCK, G., McCALLUM, I., and SCHAGEN, I. (1991) *APU Mathematics Monitoring (Phase 2)*. Slough: National Foundation for Educational Research.

HEDREN, R. (1985) 'The Hand-Held Calculator at the Intermediate Level', *Educational Studies in Mathematics*. 16, 163-179.

HMI (1991) *Mathematics Key Stages 1 and 3: A Report by HM Inspectorate on the First Year, 1989-90*. London: HMSO.

HMI (1992) *Mathematics Key Stages 1,2 and 3: A Report by HM Inspectorate on the Second Year, 1990-91*. London: HMSO.

McLEOD, D. (1992) 'Research on Affect in Mathematics Education: A Reconceptualization'. In GROUWS, D. (Ed) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan

RUTHVEN, K. (1992) 'Personal Technology and Classroom Change: A British Perspective'. In FEY, J. and HIRSCH, C. (Eds) *Calculators in Mathematics Education*. Reston VA: National Council of Teachers of Mathematics.

SCHOENFELD, A. (1992) 'Learning to Think Mathematically: Problem Solving, Metacognition and Sense Making in Mathematics'. In GROUWS, D. (Ed) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan.

SHUARD, H., WALSH, A., GOODWIN, J., and WORCESTER, V. (1991) *Calculators, Children and Mathematics*. London: Simon & Schuster.

LEARNING ABOUT TEACHING AND LEARNING: A DIALOGUE WITH TEACHERS

Adalira Sáenz-Ludlow, Marcela Perlwitz
Purdue University

Abstract: *This paper describes our collaborative efforts with five teachers to rethink mathematics instruction in elementary schools during a summer math camp. Our activities included the analysis of videotaped mathematics lessons, working in the solution of arithmetical tasks, and reflecting on our role as teachers. Throughout the camp, teachers changed the focus of their attention and reflection from the teaching-learning activities documented in the videotapes to a restrospective analysis of their own teaching practice.*

CONCEPTUAL FRAMEWORK

Calls to reform mathematics education in the United States (Mathematics Framework for California Public Schools, 1985; Everybody Counts, 1989; NCTM Standards, 1991; MAA Call for Change, 1991) have brought increasing attention to the issue of teacher preparation and professional development. On the one hand, teachers are seen as key elements in transforming mathematics education. On the other hand, it is recognized that teachers are the product of the very system they are expected to change (Schifter, 1993; Bauersfeld, 1980; Ball, 1989; Lampert, 1991; Simon, 1993). The NCTM Standards for Teaching Mathematics (1991) certainly acknowledges the complexity of the task ahead:

> The kind of teaching envisioned in these standards is significantly different from what many teachers themselves have experienced as students in mathematics classes. Because teachers need to develop this kind of teaching practice, appropriate and ongoing professional development is crucial. Good instructional and assessment materials and the latitude to use them flexibly are also keys to the process of change. (p. 3)

This suggests the need for teacher professional development programs that take a holistic approach to the teaching and learning of mathematics within the classroom (Bauersfeld, 1980).

> The process of teaching and learning mathematics can be viewed most aptly as a highly complex human interaction in an institutionalized setting (Bauersfeld, 1980; p. 35).

Thus, among other things teachers need opportunities to

- become actively involved in doing mathematics individually and in cooperation with others while reflecting on their knowledge;
- reflect on pedagogical and cognitive implications of mathematical tasks posed to children;
- reflect on children's interpretation of mathematical tasks and their mathematical activity;
- develop instructional materials that facilitate children's formation of mathematical concepts while developing their thinking strategies;
- develop adequate forms of assessing children's understanding and progress;
- reflect on their own teaching practices as interactively related to students' learning.

Few opportunities, however, arise for teachers to distance themselves from their own teaching practice in order to engage in the solution of mathematical tasks, the observation of children's mathematical activity, and the reflection on their own practice.

The work here described was an exploratory approach to teacher professional development in which we attempted to incorporate as many elements of classroom life as possible so we could come a little bit closer to the teachers' world. Our basic assumption is that teachers and researchers view classroom life through the lenses of their experiences that are ultimately shaped by their domain of activity. Thus, as Bauersfeld (1988) puts it, "in comparison with teachers, perhaps researchers do have 'other' knowledge about classroom realities, rather than 'better' knowledge." Hence, our common desire to improve children's mathematical learning can constitute the basis for our collaboration and team work. In this paper we describe our joint activity with teachers during a summer math camp in which we shared pedagogical concerns regarding elementary school mathematics, explored instructional tasks intended for elementary school children, analyzed unedited videotaped lessons, reflected on children's and teachers' solutions, participated in small-group and whole-group discussion, and reflected on the importance of social interaction and the role of mathematical argumentation in the development of mathematical concepts. We also experienced, first hand, classroom life as a community of discourse, and discussed the need to reflect on our own teaching practice, individually and in collaboration with our colleagues.

PURPOSES, METHODS, AND PROCEDURES

The intent of the study was two-fold. On the one hand, we wanted to investigate how children's solutions of mathematical tasks could influence teachers' awareness of their own mathematical activity and that of the children. On the other hand, it was intended to share and analyze a teaching practice that supported children's inquiry. To this effect, a math camp was designed and conducted during the Summer 1993. The main feature of this camp was that it provided the participant teachers and students the opportunity to solve non-standard mathematical tasks individually and in collaboration with others. These tasks had the potential to support strategic thinking, mental calculations, counting, conceptualization of units, place value, and number sense as some of the interwoven strands essential in the development of quantitative reasoning.

The camp was conducted with 6 children and 5 teachers who participated on a voluntary basis. Neither the children nor the teachers, up to this point, had been exposed to a constructivist approach to teaching and learning. The teachers had between eighteen and twenty-three years of teaching experience each and were searching for alternative ways of teaching mathematics to their students.

The camp included two consecutive sessions: the first one was conducted with students and the second with teachers. The children's camp lasted 4 weeks, and we met 2 hours a day. Teachers' camp lasted 12 days over a 4-week period. We met with teachers 4 hours per day. We were the instructors and gave both groups the same set of tasks to provide teachers with opportunities to compare and reflect on their own solutions and mathematical activity as well as those of the children. Children were divided into two small groups and emphasis was placed on justification and explanation of solutions to mathematical tasks. With teachers, we used two instructional strategies: small-group interaction followed by instructor-led whole-group discussion. Each of the teachers' sessions was broken into two 2-hour periods. After teachers' solutions were discussed, a videotape of the children's solutions to corresponding tasks were discussed and analyzed. Teachers were requested to keep journals of their reflections on students' and their own solutions; on children's collaborative efforts to communicate their solutions; on children's mathematical thinking and reasoning power; and on other issues the teachers deemed important.

Children's sessions were videotaped and we kept field-notes of teachers' solutions and the issues discussed with them.

ANALYSIS

In the course of the summer camp, the participating teachers changed the object of their reflections. At first, they focused their attention on the teaching and learning activities presented in the videotapes. Then, they reflected on their own mathematical activity and their interactive participation with other members of the group. Finally, they made their own practice the focus of their reflections. In what follows we present excerpts from teachers' journals to illustrate their reactions to the videotaped lessons.

Throughout the first week, the group conversation evolved from issues concerning the teacher's actions to children's thinking strategies to the nature of the instructional tasks. That week, the teachers decided to write about the interaction and mathematical activity of the observed group of children.

> While viewing the video I noticed the teacher's role as being a facilitator rather than instructor. The teacher did not "teach" a lesson so that the group could restate answers as in traditional lessons. The teacher posed a problem for the group to work on, then encouraged different ways to solve it....Children shared their solutions and the teacher encouraged discussion by asking questions or restating their explanations to make sure that they were communicated (Lynn, July 7).

> In the video, a problem was selected which could challenge the students to think about numbers in a variety of ways....You did not take on the "traditional role" as the lecturer, leader, or instructor. Instead, you helped the students generate number relationships. I noticed that your role included physical, emotional, and cognitive ways (Megan, July 7).

> The value of the activity in the video was that it presented a problem (not an exercise) for the students to solve using the understanding that they currently had about numbers. It also provided the opportunity for the students to verbalize their thinking. In doing so, the students helped each other because they used language that made sense to them....The students referred to other students' strategies and used them later in solving problems. The verbalization brought the students a sense of accomplishment (Mary, July 8).

While teachers centered their reflections and discussions on the changes they perceived in the children's activity, we observed the teachers' interaction and noticed gradual changes. For instance, at first, teachers did not feel comfortable sharing their thinking and solutions with others in small-group or whole-group discussions. By the end of that week the teachers seemed more relaxed and willing to share their ideas. At that point we suggested that they reflect on their interaction. The following is a sample of their views.

> This class has been changing ever since the first meeting....We have a variety of experiences that we bring to the classroom. This helps to enrich all of us....The more we interact the easier it is to interact. This is facilitated by the instructor who allows free interaction. Each of us feel that we can contribute something valuable to the class. No one feels like they are being judged when expressing

concerns or ideas....The mathematical activities have been stimulating and eye opening (Helen, July 14).

> On the first day of our class, you had us do some mental calculations to get us started. I can remember feeling very nervous and reluctant to share because I only knew one perscn in our group. After four days of working together as a group, I can see changes in our group. I feel more comfortable sharing answers and methods. I even feel that I can risk making mistakes....All of the members seem to be sharing, verbalizing, and asking questions freely (Megan, July 14).

> The social interaction of our group has become more relaxed and questioning of others' solutions. These changes may be a result of a mutual respect of each learner's thoughts and explanations, combined with a safety factor that no one will be negative about somebody else's explanations or thoughts (Lynn, July 14).

These excerpts reveal how the teachers themselves experienced the constitution of a new set of expectations for classroom participation. As instructors we supported a risk-free environment where the participating teachers could interact with others and collaborate in the solution of problems. Our interventions and interaction with them were similar to those with the children. As we pointed out earlier neither the teacher nor the students had previously experienced solving problems in collaboration with others. As we observed the videotaped lessons and the teachers' interaction, we noticed a parallel between the evolution of the interaction within each of the groups. In both cases there was a need to establish an atmosphere of mutual respect and trust in order to have a productive activity. The videotaped episodes provided the context for a discussion of some of the difficulties one might encounter while attempting to establish an inquiry approach to mathematics instruction.

With regard to the teachers' mathematical activity it should be noted that, from the beginning, the teachers did not hesitate to use materials, make graphic representations, or use what they thought to be more elementary solutions. However, they were at first unaware of the variety of solutions and strategies that one could use to solve simple arithmetic tasks. By the second week, we incorporated ourselves to the small groups and actively participated in the solutions and discussions. As our interaction evolved, teachers began looking for alternative solutions and attempted to make sense and use of each other's strategies.

> As we strive to understand each other's thoughts while sharing, we attempt to see how and when we can try the processes to see if they will work for us (Lynn, July 14).

> [Today] each of us had one (occasionally more than one) process we used to solve the problems. More solution ideas were learned as we shared our processes in our small group, and even more in our large group discussion. Some processes by others made sense and could be used to solve problems in the future, however, other processes did not make sense (yet) and could not be used. During our large group discussion I also found comparisons between problems (Mary, July 8).

> Another interesting thing is happening in our group, and that is, one of the instructors takes on the role of one of the students in the video. This creates a "problem situation" for our group and the instructor. This is very valuable because the instructor can model for us how to handle those situations. This is a better way to learn about class management rather than discussing fictional situations. (Helen, July 14)

So far, we have illustrated the evolution of our dialogue with teachers with regard to children's activity and interaction, and those of the teachers themselves. We must point out that although we wanted to provide teachers with an opportunity to reflect on their own practice, we did not expect them to do it overtly. However, during our conversations they began sharing with the group what they considered to be limitations in their teaching approach.

> The summer camp has been interesting. I have found many "gaps" in my teaching method. Although I provided a lot of hands-on experiences in my classes, I did not provide students enough time to think on their own and to discuss their strategies for solving problems. I had the children verbalize *my* [emphasis added] strategies for addition and subtraction and did not provide opportunity for the students to construct their own. I also underestimated what children can do when given challenging problems (Megan, July 22).

> This camp has helped to encourage me in what I have half-heartedly begun in my classroom. We are doing a lot more of the concrete experiences and hands on activities but I see that one of the key elements that has been missing in my class has been taking the time to listen. Listening to their explanations and allowing them to actually teach others their own methods will enhance what I am trying to do (Helen, July 22).

Solving, explaining, and discussing arithmetical tasks was a type of activity that teachers had not experienced either as students or teachers. Generating, individually or collectively, different solutions for arithmetical tasks provided teachers with a context to reflect on their own mathematical activity and, with it, that of their students. Likewise, reflecting on the interaction between the instructor and the children seems to have facilitated the emergence of teachers' introspective analysis of their teaching.

CONCLUSIONS

The analysis of the teachers' reflections suggests that they became aware of the dynamic and evolving nature of classroom social interaction and its influence on the development of arithmetical concepts. As children anticipated solutions, verbalized explanations, interpreted and acknowledged the solution of others, teachers realized the generative and constructive power of their intelligence.

In a way, the children's sessions modeled an interaction that served us well to initiate our interaction with the teachers. This interaction evolved as we established a new set of expectations based on mutual respect for our different kinds of knowledge of classroom realities. Teachers brought with them a wealth of practical knowledge, the product of their years of classroom experience. We brought insights into ways of approaching teaching from a socio-constructivist perspective. From this viewpoint, social interaction supports communication that enhances reflection on one's own activity to facilitate abstraction and thus cognition. The consensual interaction between teachers and researchers generated an environment where non-traditional tasks were solved, argued and defended; children's videotaped sessions analyzed; and reflection encouraged and expected.

The summer camp provided researchers and teachers with an opportunity to work in collaboration and to experience the initiation of a socio-cognitive approach to teacher professional development. This is not to say that the summer camp experiences were enough for teachers to transform their practice. We are fully aware that, at most, the camp represented the beginning of a long-term process of cooperation with teachers to support the evolution of their classroom practice. We concur with Gattegno (1970) that "it is not the duty of the teachers to convince themselves of the particular advantages of any given new approach" but that of the advocates of new proposals and researchers to bridge the gap "between themselves and an audience that in fact come to exist because teachers seek progress and are prepared to look for anything the promises improvement" (p. 51).

REFERENCES

Ball, D. L. (1988). Unlearning to teach mathematics. Issue Paper 88-1. National Center for Research in Teacher Preparation. Michigan State University. East Lansing, Michigan.

Bauersfeld, H. (1980). Hidden Dimensions in the so-called reality of a mathematics classroom. Educational Studies in Mathematics, 11, 23-41.

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In T. Cooney & D. Grows (Eds.), Effective mathematics teaching (pp. 27-46). Reston: National Council of Teachers of Mathematics and Lawrence Erlbaum.

California State Department of Education. (1985). Mathematics Framework for California Public Schools: Kindergarten through Grade Twelve.

Gattegno, C. (1970). What we owe children: The subordination of teaching to learning. New York: Outerbridge & Dienstfrey.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer. American Educational Research Journal, 27 (1), 29-63.

Mathematical Association of America Committee on the Mathematical Education of Teachers. (1991). A Call for Change: Recommendations for the Mathematical Preparation of Teachers of Mathematics.

National Council of Teachers of Mathematics, Reston. (1991). Professional Standards for Teaching Mathematics.

Schifter, D. & Fosnot, C. T. (1990). Reconstructing mathematics education: Stories of teachers meeting the challenge of reform. New York : Teachers College Press.

Simon, M. (1993). Focus on children' mathematical learning in classrooms: Impact and issues. In T. Wood, P. Cobb, E. Yackel, D. Dillon (Eds.), Rethinking elementary school mathematics: Insights and issues (99-107). Reston : National Council of Teachers of Mathematics.

Mathematical Science Education Board and National Research Council. (1989). Everybody Counts: A Report to the Nation on the Future of Mathematics Education. National Academic Press

---

This investigation is part of a larger research project funded by The National Science Foundation (RE-9155734). The results and views stated here are solely the responsibility of the authors.

STUDENTS' UNDERSTANDING THE IDEA OF CONDITIONAL PROBABILITY

Ana María Ojeda Salazar
Departamento de Matemática Educativa
CINVESTAV del IPN, PNFAPM.
México

In this work the answers of students aged 16-22 years given to questions concerning the idea of conditional probability are analysed. The general performance they showed to support their answers depended on the instructional environment to which they were exposed. It was confirmed also that it is more difficult to answer questions in which the conditioning event is supposed to happen later than the conditioned event, than when the conditioning event happens either simultaneously or before the conditioned event. Additionally, there was a confusion between conjunction of events and conditioned events.

This paper is concerned with a part of a wider research on pupils and students' understanding of fundamental ideas of probability before the university level (see Ojeda, 1990). One of the ideas the study dealt with was conditional probability, which was investigated with Mexican preparatory students aged 16-22 who underwent two instructional conditions.

Prior research about understanding of conditional probability has emphasized the difficulty subjects have to assess conditional probabilities for which Bayes' theorem is required for a formal approach. That is, when for the expression of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0),$$

at least the term $P(B)$ and often also $P(A \cap B)$ cannot be computed directly, but they have to be expressed by means of the probabilities of other events, and the property of the addition of probabilities and the multiplication rule (i.e. $P(A \cap B) = P(A)P(B|A)$) have to be used.

To make explicit what follows, we will refer to B in the expression above as the conditioning event and to A as the conditioned event.

Kahneman & Tversky's work (1982) highlighted the issue of the bigger difficulty subjects have to assess conditional probability when the conditioning event is perceived as a possible effect of the conditioned event (perceived as a possible cause), than when the conditioning event is perceived as a possible cause of the conditioned event (perceived as a possible effect). Using these authors terminology, *diagnostic reasoning* is required in the first case, and *causal reasoning* is needed in the second.

The difficulty that Bayes' theorem entails has been investigated by Shafer (1982). He has pointed out that whereas in traditional instruction distinction is not made between simultaneous events and timed events when applying the expression of conditional probability, Bayes used his nowadays well known theorem for subsequent events (see Proposition 5, p. 1077, in Shafer's paper). That is, the main idea in Bayes' theorem is that of using information to assess the likelihood of past events, i.e. the idea of *inverse* probability.

For the case of subsequent events, an explanation of failure when subjects have to assess conditional probabilities when the conditioning event is posterior to the conditioned event was given by Bentz & Borovcnick (1985). According to these authors, when the conditioned event is supposed to have occurred before the conditioning event, the former is not considered as a possible cause of the latter and the condition is not recognized. As a consequence, the most frequent answer given corresponds to unconditional probabilities instead of conditional probabilities.

Falk, in her works on insights, difficulties, uncertainty and conditional probability (1986, 1989), points out that one's understanding of a situation depends on how one considers that the information has been obtained, and calls attention to what one considers as relevant in the information given. In particular, Falk highlights the ambiguity often present in real life problems, where the randomness is not clearly stated and the conditioning event can be either considered as irrelevant or difficult to identify.

Parzysz (1990) has highlighted the richness that tree diagrams offer to the instruction on conditional probability. According to his proposition, the mathematical entities involved in the situation to be modelled (sample space, events, probabilities and conditional probabilities, partitions, the total probability theorem, the multiplication rule) are linked to the notational system through the tree diagram.

These kinds of representations could play the role of the "material forms" needed to generate mathematical knowledge, as Kaput (1991) claims. Moreover, according to this author, to the extent to which the operant features of two particular notations are perceptual and cognitively available to a subject, their physical representation is also available to him or her (in the present case, the identification of events, their probabilities and their relations among them on the tree diagram).

Most of the instructional environments to which students are exposed, what is called here traditional environments, work as if knowledge could be transmitted by the teacher (who exposes the theory and solves problems in front of the class) to the students (who take notes of what the teacher says or writes on the blackboard). Students' beliefs and representations, are among the elements often neglected in this kind of environment, the same as some aspects which were important in the emergence of the concepts that they have to learn.

This work was interested in knowing if a traditional instructional environment enable preparatory students to overcome the bigger difficulty to answer questions which require diagnostic reasoning than to answer questions for which causal reasoning is needed, as reported by prior research. In addition, we were interested in investigating students' understanding of conditional probability after they were exposed to another instructional environment where their activity and opinions were important, and a specially designed didactical guide was used. Among other elements, in that guide Parzysz' proposition was used and the information about errors and strategies that students in a traditional instructional environment make when answering questions about conditional probability was taken into account..

The questions. Four problems were posed to the students: two involving asymmetrically distributed populations, and two involving symmetrically distributed populations. In the last case, one of the problems was posed through a random variable.

For each problem, four questions were asked, each corresponding to the items specified in the former section: for item a) diagnostic reasoning was needed; for item b) causal reasoning was required; item c) asked about the intersection of the events involved in items a) and b); item d) concerned with the conditioning event in item a). As an example, we quote next the first problem (referred to an asymmetrical distributed population) and the fourth problem (symmetrical population, random variable).

1. A cloth bag holds two equally sized counters. One of the counters has its two faces red and the other has one red face and the other is blue. Without looking, a man draws one counter and shows only one of its faces.

a) The face shown is red. Which of the two counters is more probable to have been the one drawn?

b) If the different coloured faces counter was drawn, which colour do you think is more probable to be the one shown?

c) What is more probable, drawing the different coloured faces counter and that the red face be shown, or drawing the different coloured faces counter and that the blue face be shown?

d) What is more probable, having a red face shown or having a blue face shown?

4. After a raffle, the winner has the chance of getting a reward as follows: he/she randomly chooses two cards, one after the other, from four numbered cards, 1, 2, 3, 4. The number of the chosen cards are added and, if the sum is even, the winner is given that sum in million of pesos.

a) If the second card chosen is even, what is more probable, that a reward is given or that it is not?

b) If the first card chosen is odd, what is more probable, that the sum is given or that it is not?

c) What is more probable, that one even card is chosen and the sum be given to the winner, or that one even card is chosen and the sum be not given?

d) What is more probable, the winner being given the sum or being not?

The population and the methodology used. A questionnaire was designed (in four forms) to gather the information. As shown above, open questions were posed and a mathematical justification was required for each answer. To answer the questionnaire, only pencil and paper work was required, and a maximum of $1\frac{1}{2}$ hours was allowed for the answering.

The information was obtained from two samples of preparatory students aged 16-22 years from two standard Mexican schools at this level. These subjects were introduced to Probability (which was compulsory for them according to the usual mathematics programme of study) for the first time. The programme of study used for this instruction is also the usual standard programme at this level (e.g., see Garfield & Ahlgren, 1988). The instructors of the students were three experienced mathematics lecturers at this level who voluntarily accepted to participate in this study.

The first sample of students, with 149 subjects, were presented the idea of conditional probability in a traditional instructional environment. For this part of the research, the lecturers were just asked to give the questionnaires to their students once the instruction on this idea had finished.

The lecturers were asked to use the didactical guide with a second sample which had 106 students, who were introduced to the idea of conditional probability in an activity based instructional environment. At the end of the instruction, the students were given the questionnaire. As quoted above, the tree diagram played an important role in the guide.

Results. Table 1 below shows the percentages of correct answers given by each sample of students.

Table 1.
Percentages of success

Sample	1st	2nd	1st	2nd	1st	2nd	1st	2nd
Item	a		b		c		d	
Problem								
1	26	31	48	50	13	10	44	41
2	26	25	36	42	13	25	11	17
3	46	35	59	56	15	13	54	60
4	38	39	46	42	10	5	26	28

1st sample: 149 students; 2nd sample: 106 students

In general, the scores from the students in the second sample were similar to the scores from the first sample students. The difference between the numbers of students who answered correctly all items a) (diagnostic reasoning) and of subjects who gave a correct answer to all items b) (causal reasoning), obtained from each sample, was significant ($\chi^2 = 3.03$ for the first sample, $\chi^2 = 5.99$ for the second, $\alpha = .05$, one-tailed test). That is, questions requiring diagnostic reasoning were more difficult to answer for the two samples than the questions requiring causal reasoning. This means that the bigger difficulty students had in answering the questions requiring diagnostic reasoning than the questions requiring causal reasoning could not be overcome with the use of the didactical guide in the circumstances described. Nor could the difficulty of answering the questions concerning intersection of events be overcome. In particular, these questions had the lowest scores of success in the two samples.

Generally, problem 3 (asymmetrical population) was the one for which the best results were obtained. Problem 2, which concerned a symmetrical population, was the most difficult. It should be noted that, although problem 4 was problem 2 posed using random variable (which could be thought of to involve an additional difficulty for students), the former had better scores of success than problem 2.

An important difference between the two samples of students arose in the form of the justifications they gave. Most of the students in the traditional instructional environment (73.8%, 110 learners) gave justifications for the answers they advanced using daily language, without applying the formulae which describe the concept of conditional probability or Bayes' theorem. On the contrary, in the second sample, about 70% (76 learners) of the students did mathematize the problems (in the sense that mathematization is explained by Lange (1987, p. 43); that is, students interpreted the problems into mathematical terms.

The difference between the numbers of the students exhibiting mathematizations in each sample is highly significant:

	Mathematization		
	Yes	No	
1st sample	39	110	149
2nd sample	76	30	106
	115	140	255

The value of χ^2 obtained is 50.02 ($\alpha < .001$). The second sample students used the formulae and the methods taught in instruction. Even six students explicitly applied Bayes' theorem. Tree diagrams and Venn diagrams were among the representations that the second sample students more used. That is, since the lecturers for the two samples were the same, giving the same course in the same schools, to students of the same ages regularly registered in the preparatory third year, that difference in performance is attributed to the different instructional environment, in which the guide had a part. Therefore, the use of the didactical guide in the second environment seems to have resulted in the difference in performance with regard to the students' performance in the first sample (with traditional instruction).

There was a decrease in the recurrence of arguments appealing just to chance in the second sample with respect to the first sample students: 18.8% and 6.6%, respectively.

Among the students in both samples who used daily language in their justifications for their answers to items a) (diagnostic reasoning), changing the sense of the conditioning was a common procedure to justify a correct answer to item a). That is, the order of the two events in the expression for conditional probability above is inverted: B is considered as the conditioned event and A as the conditioning event and students argued about the conditional probability for this case (that is, using causal reasoning).

Problem 4 (which involved a random variable) promoted the most the listing of the sample space.

The major drawback the students of the two samples had was item c), as the Table 1 above shows. This question asked about intersection of events. Most of the subjects interpreted the question as referred to conditioned events.

The difference in the performances shown by the two groups of preparatory students in this study points to the need for further research on the role that the representation of mathematical entities plays in students' understanding of the concepts represented.

The provision of some means of organization would lead to filling the gap between having a first approach to fundamental ideas, and mastering those ideas. This kind of means, as extra linguistics representations, would be the agents allowing students the "transition" to higher levels in mathematics. Those representations would provide them with the opportunity to use mathematical notation, and to relate it to the mathematical entities involved in problems. Later on, teaching success will imply the detachment of intermediate resources, such as tree diagrams. However, time is needed to attain this aim.

REFERENCES

Bentz & Borovcnik, M. G.: 1985, On >Representativeness< -A Fundamental Statistical Strategy. Theory, Research & Practice in Mathematical Education (Bell, A.; Low, B., and Kilpatrick, J., Eds.), Shell Centre for Mathematical Education, University of Nottingham, U. K.

Falk, R.: 1986, Conditional probabilities: Insights and Difficulties. Second International Conference on the Teaching of Statistics, pp. 992-997.

Falk, R.: 1989, Inference under Uncertainty via Conditional Probabilities. Studies in MAthematics Education, Vol. 7, The Teaching of Statistics (Robert Morris, ed.). UNESCO, pp. 175-184.

Garfield, J.; Ahlgren, A.: 1988, Difficulties in Learning Basic Concepts in Probability and Statistics: Implications for Research. Journal for Research in Mathematics Education, Vol. 19, No. 1, pp. 44-63.

Kahneman, D., Tversky, A.: 1982, Causal Schemas in Judgments under Uncertainty. In Judgment under Uncertainty: Heuristics and Biases (Kahneman, Slovic & Tversky, eds.). Cambridge University Press. pp. 117-128.

Kaput, J. J.: 1991, Notations and Representations as Mediators of Constructive Processes. Radical Constructivism in Mathematics Education, E. von Glasersfeld (ed.), Kluwer Academic Publishers, Netherlands, pp. 53-74.

Lange JZN, J. de: 1987, Mathematics, Insight and Meaning. Vakgroep Onderzoek Wiskundeonderwijs en Onderwijscomputercentrum, Rijks-universiteit Utrecht, pp. 38-76.

Ojeda, A. M.: 1990, Some Considerations on the Learning of Probability. Ages 10-11 and 14-16 Years. Proceedings of the XIV Conference of the International Group for the Psychology of Mathematics Education. Oaxtepec, Mor. Mexico.

Parzysz, B.: 1990, Un Outil Sous-estimé: l'Arbre Probabiliste. APMEP, 69 annee, 372, France, pp. 47-54.

Shafer, G.: 1982, Bayes' Two Arguments for the Rule of Conditioning. The Annals of Statistics, Vol. 10, No. 4, pp. 1075-1089.

AN ANALYSIS OF THE DEVELOPMENT OF THE NOTION OF SIMILARITY IN CONFLUENCE: MULTIPLYING STRUCTURES; SPATIAL PROPERTIES AND MECHANISMS OF LOGIC AND FORMAL FRAMEWORKS

Lucilia Bechara Sanchez
Associação Universitária
Interamericana- Brasil

Anna Maria Pessoa de Carvalho
Universidade de São Paulo
Brasil

January 13, 1994

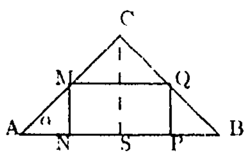
Abstract

This paper deals with some aspects of the acquisition of the notion of similarity of plane figures by students between 11 and 16 years of age from three schools located in São Paulo. The paper uses constructivist assumptions and emphasizes the teaching/learning relations, the development of cognitive structures and the learning of specific contents. The results describe the following construction processes : a. operations with numbers, from multiplication to ratios; b. special relations from the syncretic view to the distinction of variables and their interdependence; c. the logic of the classification vis-a-vis the classificatory criteria; d. the logic of the propositions from acknowledgment of the true values to the implicational relations and determination of the necessary and sufficient conditions. Given the psychopedagogical implications, the paper discusses applications aimed at optimizing the function school plays in promoting learning and developing cognitive structures.

This study is about how students of ages 11 to 16 use their knowledge of numbers, geometry and propositions in solving problems related to enlarging and reducing geometric figures and how they use their logical structure in solving such problems. More specifically, our objective was to investigate the following questions: a) What numerical operation do they use? When and how do the notions of multiplicative compensation and of proportion evolve? b) How do they use geometric knowledge and properties within the conceptionnel field of similarity and how do they relate spatial variables? c) How do they use the logical operations of classification as well as the logic of propositions and implications?

The theoretical basis of this paper derives from researches of Piaget and the School of Geneva relating to classifications, multiplicative structures, proportionality and logical implications. The choice of this study is due to: a) the confluence of the numerical, geometrical and logical knowledge. b) the theoretical and practical complexity; c) the pedagogical implications and psychopedagogical applications.

We decided to conduct a classical type of interactive clinical interview with 26 selected students from 5th to 9th grade coming from three different schools in São Paulo, including public and private, as well as experimental and traditional schools. The research material consisted of a set of eleven signboards, each showing a different drawing on graph paper made of an isosceles triangle and a rectangle forming the familiar picture of "a little house" (see dimension of each one in the table below).



classes	CS x AB x MN	NP x MN x CS
A	3 × 6 × 1.5	3 × 1.5 × 3
	6 × 12 × 3	6 × 3 × 6
	4 × 8 × 2	4 × 2 × 4
	2 × 4 × 1	2 × 1 × 2
B	8 × 12 × 4	6 × 4 × 8
	4 × 6 × 2	3 × 2 × 4
	12 × 18 × 6	9 × 6 × 12
C	7.5 × 10 × 3	6 × 3 × 7.5
	5 × 6.5 × 2	4 × 2 × 5
	2.5 × 3.5 × 1	2 × 1 × 2.5
	10 × 13.5 × 4	8 × 4 × 10

The interviews were focused on three major tasks: 1.To identify similar "houses". 2.To classify "houses" putting together those that are equivalent. 3.To draw new "houses" belonging to those categories.

Analysis and categorization of the student's answers

The following factors were considered in analysing the answers: **1.**The notion of similarity; **2.** The use of the scale factor or equality of ratio between homologous sides (internal ratio) ; **3.** The use of the ratio between distances (external ratio); **4.** Dissociation, combination and interdependence of spatial variables; **5.** Formation and characterization of the equivalence classes.

Factor 1 - The notion of similarity

The first uestion asked in the interview was: What do you mean by enlargement or reduction? The answers were catagorized as follows:

A - The student says that to enlarge is to increase all sides, all distances, the length, the width, etc.

P - The student uses the word "proportion", "proportional" or "proportionality".

R - The student uses the word "reproduction".

F - The student makes an association with the idea of the shape being maintained.

M - The student uses a scale multiplying factor to characterize the enlargement or reduction.

L - The student uses the expressions "equal" or "the same"

Tab' 1

students	grade	A	P	R	F	M	I
4	5th	1	1	0	1	0	2
7	6th	0	1	0	4	2	1
4	7th	2	1	1	1	0	0
8	8th	0	2	0	6	1	2
3	9th	1	1	1	0	1	1
26	Σ	4	6	2	12	4	6

Factor 2 - The use of the scale factor (internal ratio)

Most of the students interviewed used the scale factor **K** to identify similar figures. Those who did not use the scale factor correctly gave two types of replies:

Ad - Additive Compensation: the student adds (or subtracts) the same measure to (or from) all dimensions to enlarge (or reduce).

Oc - Other Compensations: noticing that the figure becomes deformed (that is, it becomes "shorter" or "taller") after adding (or subtracting) the same measure to (or from) all or some dimensions, the student concludes that to larger (or smaller) distances, larger (or lesser) distances should be added (or subtracted), yet does nor know which operation must be performed, and consequently makes approximate calculations and notices the resulting deformation.

Those who used the internal ratio correctly are of five types:

MI - When K is a whole number.

MF - When K is a fraction and the multiplication appears in the form of a fraction of a fraction.

Af - When K is a fractional number and the student does what we call an addition or a subtraction of the fractional part. For example, if $y = \frac{3}{2}x$, the student concludes: $y = x + \frac{1}{2}x$; or if $y = \frac{2}{3}$, the student concludes: $y = x - \frac{1}{3}x$.

S - The student makes ratios with equal sides of the figures and uses simplification to ascertain the equality of the ratios.

P - The student uses $\frac{a}{b} = \frac{c}{d}$ and cross multiplication $a \bullet d = b \bullet c$ for calculating one of the elements of the proportion.

Table 2								
students	grade	Ad	Oc	MI	MF	Af	S	P
4	5th	0	0	2	1	1	0	0
7	6th	0	3	5	2	2	0	0
4	7th	1	1	3	0	1	0	0
8	8th	1	0	6	3	3	1	1
3	9th	0	0	2	0	1	1	0
26	Σ	2	4	18	6	8	2	1

Factor 3 - The use of the ratio between distances in the same figure (external ratio)

Each class of "little houses" can be characterized by two ratios: class A by ratios $\frac{MP}{MN} = 2$ and $\frac{CS}{MN} = 1$, class B by ratios $\frac{MP}{MN} = \frac{3}{2}$ and $\frac{MP}{MN} = 1$ and class C by ratios $\frac{NP}{VU} = 2$ and $\frac{CS}{MN} = \frac{5}{4}$. We found five types of replies among students who made use of the external ratio:

N - The student just begins to perceive that an external ratio has to be maintained.

Rd1 - The student uses the external ratio as a support.

Rd2 - The student uses the external ratio in addition to the scale factor.

Rd3 - The student uses the external ratio as a fundamental factor (sometimes with the addition of a scale coefficient).

RA - The student relates the angle with the ratio between sides of the right-angle triangle (trigonometric ratios).

Table 3						
students	grade	N	Rd1	Rd2	Rd3	RA
4	5th	1	2	0	0	0
7	6th	1	0	0	0	0
4	7th	1	0	1	0	0
8	8th	0	0	0	2	1
3	9th	0	0	1	1	1
26	Σ	3	2	2	3	2

Factor 4 - Dissociation,combination and interdependence of spatial variables

To solve the problem of identifying similar figures, the students need to dissociate the distances and angles, and establish an interdependence among them. We identified three stages in this process.

Stage I - Use of intuition to maintain the form

This stage is characterized by syncretism. Here the students use insufficient variables and complement the lacking data by intuiting the shape.

Substage Ia - The student uses intuition to maintain the shape of the triangle: "more tilted", "more bent", "more upright" and the shape of the rectangle: "taller", "wider", etc.

Substage Ib - The student uses only the angle ACB and sometimes by also maintaining intuitively the shape of the rectangle: "higher", "narrower", "longer", etc.

Substage Ic - The student uses the internal or the external ratio between the base and the height of the rectangle and supplements the solution with the shape of the triangle: "higher", "more tilted", etc.

Stage II - Use of the necessary data

This stage is characterized by analysis. Here the students randomly take into account two, three and sometimes four variables needed to identify similar figures and do it through direct observation. We found students who established causal relations through correspondence rather than by implication. They neither introduced necessary relations nor inferred any new relations.

Substage IIa - Whenever the variables chosen were NP, MN and α, some students replaced the opening of α by the height CS or by the distance AN, thus recognizing, in practice, the interdependence between the angle α and the measurements of CS or AN.

Substage IIb - The variables chosen were NP, MN and AC (or CS or even AN), with no apparent relation among them. Whenever the variable used was AC, the student resorted to CR or CS, recognizing, in practice, a correspondence between these variables.

Substage IIc - The variables chosen were NP, MN and AN (or CS or AB), establishing some interdependence among them.

Stage III - Establishing necessary and sufficient conditions

The caracteristics of students in stage III is their ability to combine variables and make connections between them using implications and identifying necessary and sufficient conditions.

Substage IIIa - The student deals with the possibilities of combining variables and establishes implicational relations among them.

Substage IIIb - The student uses the range of possibilities for choosing variables, and establishes some necessary and sufficient conditions. Therefore the student selects variables, establishes equivalences and implications, and excludes the variables in excess. It is necessary to make a distinction between those who establish necessary conditions taken from their immediate experience and those who do it by deductive hypothetical reasoning, although in relation with a practical conformation.

Substage IIIc - The student selects the necessary and sufficient variables, justifies that choice and explains why other variables were excluded, foregoing the confirmation of the findings based on practice. Such a student develops a deductive hypothetical reasoning.

Table 4

students	grade	Ia	Ib	Ic	IIa	IIb	IIc	IIIa	IIIb	IIIc
4	5th	0	2	1	1	1	1	0	0	0
7	6th	2	0	2	2	2	1	0	0	0
1	7th	1	0	2	2	2	1	0	0	0
8	8th	0	0	0	1	3	4	3	2	0
3	9th	0	0	0	0	0	1	1	1	1
26	Σ	3	2	5	6	8	8	4	3	1

The results show that the apparent simplicity of the defintions based on necessary and sufficient conditions should not deceive us. The line of reasoning used, starting from the students understanding of the concept of correspondence and their establishing of an interdependence among variables to their introducing the necessary and sufficient conditions is a long continuous and complex route.

Factor 5 - Formation and characterization of the equivalence classes

A very important aspect examined was what criteria were used for classifying the "houses" and how these criteria evolved to logical and concrete structures.

In our research we found 11 and 12 year old students who could not classify the figures because they did not recognise similarity as an equivalence relation. In this case we identified three stages:

Stage S1 - The student establishes relationships by successive approximations, with no operational criteria.

Stage S2 - By means of successive approximations, the student progresses towards forming classes, however, with no notion of grouping. He succeeds, by means of succesive and retroactive corrections, in reorganizing previous positions and using the entire classification. However, he is not really making a class extension, since he advances from one element to the next without taking the whole into account. Driven by yet crude anticipations and retractions, the student at this stage follows three paths: **a)** he forms pairs of three-element groupings, and using yet other means, such as transitivity, combines small groupings into larger sets. **b)** he chooses a representative and adds elements to it by successive approximations. **c)** he starts with large groups using anticipatory and generic criteria without characterizing the classes and then improves and subdivides the groups.

Stage S3 - The operational classification is typical of this stage. The subject only starts acting based on anticipatory frameworks and shows eagerness to modify his criteria or add his initial groupings to broader ones. The simplest forms of generalization consist in associating

the old with the new. The higher forms of generalization consist of a retroactive process by which new elements lead to a reorganisation of the whole, modifying even previous concepts and knowledge. The most balanced type of reorganization consists in not destroying the previous structure but adding as much of it as possible to the new structure.
We identified seven criteria used in forming the pairs and the classes.

Discoursive Criteria:

F - The student maintains the shape to approximate elements or characterize classes.

P - The student uses proportionality to integrate new elements or characterize classes.

T - The student applies transitivity after making successive approximations between pairs using the scale factor or maintaining the angle.

R - The student chooses one of the figures as representative and then adds others by successive approximations

Operational Criteria:

A - The student uses the angle ACB of the "roof" or α of the inclination.

K - The student uses the scale factor K to approximate similar figures. The scale factor does not characterize the class, it only makes it possible to approximate two figures. By using successive approximations and other logical recourses such as transitivity, the student manages to make the classification.

Rd - The student elects to maintain the ratio between distances to recognise similar figures, and succeeds in finding general characterization for each class.

Table 5

students	grade	S1	S2	S3	F	P	T	R	A	K	Rd
4	5th	1	3	0	2	0	3	1	2	2	2
7	6th	2	5	0	4	2	2	3	5	4	0
4	7th	1	3	0	3	1	2	0	4	3	0
8	8th	0	7	1	2	1	4	3	4	5	2
3	9th	0	2	1	0	2	1	1	0	3	1
26	Σ	4	20	2	11	6	12	8	15	17	5

Conclusion and discussion

We were able to see how local solutions precede general and canonical solutions. This finding indicates that it is possible to work with students in this age bracket in exploring specific properties related to the study of similarity. This, however, implies a new theoretical-methodological concept based on constructivist presuppositions.

We would like to stress two types of learning: cultural (or spontaneous) and scientific (or monitored) which result in the development of cognitive structures.

The first one is based on concrete experiences. The second one is necessary when the student lacks experience-based knowledge. The two types are not contradictory. Quite to the contrary, they are complementary.

A careful look into the solution-generating processes is required in examining the teaching methodology further, concentrated on:

a) the relations between learning and development, contents and structures, as well their discontinuity and apparent contradictions;

b) the dynamics of the pedagogical interferences which involve the relation between spontaneous and scientific contents;

c) the formalization and their complex construction process,

d) the necessity of integrating the learning of geometry to the learning of other subjects.

BIBLIOGRAPHY

FREUDENTHAL, H. *Didatical Phenomenology Mathematical Structures.* Ed. Library D. Reidel, Publish. Company.Deordrech, Boston, Lancaster, 1983.

FUSON, K. *An analysis of Research Needs in Projective, Affine and Similarity Geometries, Including an Evaluation of Piaget's Results in these Areas.* In: Recent Research concerning the Development of Spatial and Geometric Concepts.Published by R. Lesh & D. Mierkiewics Publishers. ERIC/SME AC, Columbics, Ohio, 1978.

HART, K. M. *Ratio: Children's Strategies and Errors.* NFER-Nelson Publishing Company Ltd, London, 1984.

KARPLUS, R.; PULOS, S. & STAGE, E. K. *Proportional Reasoning of Early Adolescents.* In: Acquisitions of Math Concepts and Processes. Richard Less and Marsha Landauy. Academic Press, New York, 1983.

LAPPAN, G.; FITZGERALD, W.; WINTER, M.J. & PHILLIPS, E. *Similarity and Equivalent Fractions.* Addison-Wesley, New York, 1986.

PIAGET, J. *L'Equilibration des Structures Cognitives.* PUF, Paris, 1975.

PIAGET, J. & INHELDER, B. *La Genèse des Structures Logiques Elémentaires.* Delachaux et Niestlé, Neufchâtel, Suisse, 1959.

PIAGET, J. & INHELDER, B. *De la Logique de l'Enfant à la Logique de l'Adolescent.* PUF, Paris, 1970.

STUDENTS' APPROACHES TO SOLVE THREE PROBLEMS THAT INVOLVE VARIOUS METHODS OF SOLUTION

Manuel Santos T
CINVESTAV, Mexico

Students deal with different types of mathematical problems during instruction. Although getting the solutions of problems could be seen as the ultimate goal while interacting with problems, it is also important to examine the quality of the diverse strategies or methods that students use when they are asked to work on some problems. This paper analyses the work shown by some secondary school students who worked on task-based interviews that included three problems with various forms of solutions. Results show that students experience difficulties in thinking of more than one way to solve the problems. As a consequence, teachers should encourage their students to deal with this type of problems during the learning of mathematics.

Introduction

Problem solving plays an important role in the learning of mathematics. Indeed, the interest in problem solving has made significant contributions to the understanding of how people solve problems. For example, Schoenfeld (1988) pointed out that mathematical resources, cognitive and metacognitive strategies, and beliefs are aspects that characterize the problem solving process. Schoenfeld indicated that in order to engage the students in problem solving activities, they should be encouraged to think mathematically. That is, problem solving is a way of thinking in which students have the opportunity to behave as an actual mathematician while dealing with mathematical tasks or problems. For example, students could make conjectures, design their own problems, discuss and defend their ideas or solutions. An important issue here is the type of tasks or problems that could help students develop this way of thinking. Schoenfeld uses, what he calls, nonroutine problems to explore and categorize the students mathematical process. Here, it is important to study to what extent the students are able to use different methods to solve problems. Therefore, the purpose of this study is to examine the work shown by students of grade 10 when they were asked to work on problems that involve various methods of solution.

Conceptual Framework

If one accepts that the process of developing mathematics is related to the process of learning this discipline, then it becomes important to examine what mathematical aspects are present in the students interaction with some mathematical tasks. Lampert (1990) indicated that "Choosing and using 'good problems' and instituing appropriate means of classroom communication can be thought of as two of the tasks that teachers need to do to teach mathematics" (p. 125). That is, the problems should have the potential to lead the students to make connections with other mathematical ideas and to evaluate or discuss the quality of the strategies that appear during the solution process. In tnis context, it was important to select problems in which the students have the opportunity to deal with various methods of solution. The problems were analysed in detail before that they were given to the students. Here, there was interest to identify some anticipated solutions to each problem and the content involved to solve them. This work provided grounds to suggest a frame of analysis of the students work. The components of the frame include aspects related to the understanding of the problem, and the presence of both cognitive and metacognitive strategies. Here metacognition involves information about the students own process, that is to what extent they are able to express their own thinking; the students monitoring or self-regulation process; and the students ideas about mathematics that influence their problem solving process.

Methods and Procedures

Thirty-five grade ten students from two secondary schools were interviewed. All the students were volunteers. The interviews lasted from 20 to 45 minutes. The students were asked to think aloud while solving the problems. The interviews were audiotaped. The interviewer occasionally asked the stu nts some clarification questions or provided some hints that could be usefu solving the problems. Using a record sheet, the interviewer also took notes on main steps that the students followed during the interviews. The problems used for the interviews were:

1. A farmer has some pigs and some chickens. He finds that together these animals have 19 heads and 60 legs. How many pigs and how many chickens does he have? (farmer problem).

2. Could you find two whole numbers **a** and **b** whose product is one million and neither **a** nor **b** includes a zero in its representation? Are there any other such pairs? Why or why not? (one million problem).

3. A textbook is opened at random. The product of the numbers of the facing pages is 3,192. To what pages is the book opened? (pages problem).

Working with the Problems

The analysis of particular solutions was very important when categorizing the work done by the students. Ideas associated with the different qualities of the solutions were identified while discussing the possible solutions. These ideas were contrasted with the work shown by the students. That is, the work done to the problems before the interviews helped clarify the potential of the problems and organize the process used by the students.

Problems	Methods of possible Solution	Strategies	Content
The farmer problem	•Pictorial •Guess and Test •Correspondence •Algebraic	Use of diagrams or actual pictures, trial and error, organized list or table, comparing, and equations	Basic operation with integers. Equation of first degree or system of two equations.
The one million problem	•Trial divisors •Prime factor •Simpler problem	use of a list or table, thinking of a simpler problem (10, 100, or 1000).	multiplication and division of integers, factoring, exponents, and prime numbers
The page problem	•Guess and test •Factorisation •Square root •Algebraic	Estimation, trial and error, symbolic representation, and equation.	Consecutive numbers, multiplication of integers, factorisation, meaning of square root, quadratic equation

Results from the Interviews

For the farmer problem the most common approach used by the students was to represent the problem algebraically. It was observed that the students spent almost no time attempting to make sense of the information given in the problem and immediately began to write down the data. Some students introduced different variables for representing the chickens, pigs, heads, and legs. This type of representation was confusing for the students, but after they reexamined the problem, in general, they were able to solve it. The interviewer then asked the students whether or not there was another way to solve this problem. They recognized that there was another way, "the long way" in which they could use guessing and testing. When the students were asked to try this method, it was observed that they were not systematic, and often they even tried extreme cases, such as 19 chickens or 10 pigs. Not one of the students used a table or a diagr m for solving this problem and they relied on the use of a calculator to che. even small numbers. One student added 19 and 60 and divided the result by 2 and 4 obtaining 34.5 and 17.2, respectively; then the student claimed that there were 34.5 chickens and 17 pigs.

It was observed that the students did not spend time trying to understand the problem. They read the problem and immediately started to work on some calculations without identifying the main data of the problem and their relationships. The difficulties that they experienced while trying to use the information often pushed them to re-read the problem and to think of a more organized strategy. It was clear that the students associated the information of the problems with some operations but they lacked the basis to select a plan. The metacognitive aspect related to the use of the operations was not present at the students work. That is, the students were fluently while operating with the numbers but they did not relate the meaning of the operation to the information naturally. It was also observed that it was difficult for the students to realize that some of their work did not match the information or conditions of the problem. That is, they have not developed some sort of strategies that could help them to challenge their own work. An example of the students' work on the farmer problem is presented next:

She started to read it. She immediately divided 60 by 4 and 19 by 2 and wrote 15 and 9	The interviewer asked her to explain the answer.
She mentioned that there were 15 pigs and 9 chickers	Here, the interviewer suggested that she could check the information of the problem and the results.

The student then realized that she was not taking into account the information. She then said that there were only 19 animals and perhaps she could have 10 chickens and 9 pigs. However, when she counted the number of legs she found that this answer did not fit the conditions. Here she then worked using trial and error by subtracting and adding a unit but she took 19 as fixed number.

At this time she felt confident and started to work very quickly.

For the one million problem, the students experienced difficulty accepting the existence of such factors. They represented the problem algebraically as **a** x **b** = 1,000,000 and then they isolated a = 1,000,000/ **b** . They assigned several values to **b** and with the use of a calculator they found values for a. After several trials, some mentioned that such factors did not exist. None of the students was able to make progress in this problem alone. When the interviewer suggested trying smaller numbers than 1,000,000, then some of the students were able to find the pattern by trying 10 = 2 x 5, 100 = 4 x 25 and so on. Only one student decided to factor 1,000,000 and found that there were sequences of twos and fives; however, he was not able to see on his own that he had found the desired factors. For the second part of the problem, no one (on his or her own) was able to explain why there were no other pairs. Two students mentioned that because for 10 and 100 it was not possible to find other such representations, then the same could be applied to 1,000,000. When the interviewer asked them how they were sure that for 100 there were no other pairs, they listed all the pairs whose products give 100 instead of checking the prime factors. It was observed that the students tried to solve this problem by selecting some concrete numbers. These numbers did not work and they thought that the problem did not have a solution. This showed that the students experienced serious difficulty for recognizing the consistency of plausibility of the information of the problem. That is, it was not clear for the students that checking for three or four examples does not mean that the numbers could not exist.

Some samples of the students' work with comments is presented next:

Rob read the one million problem and started to try some numbers with the use of the calculator. He try 963 x 56 = 53928; 99987 x 89 = 9799804; 9999 x 89 = 889911; and 99999 x 89 = 899911. Rob then suggested to work with 10 and he wrote: 10 = 5 x 2; Although he hesitated to continue he then wrote 100 = 25 x 4; 1000 = 125 x 8; 10000 = 625 x 16. Rob at this point mentioned that the number of zeros was the exponent of 5 and 2. Hence, he wrote 1, 000, 000 = $5^6 \times 2^6$.	Rob mentioned that it was going to be difficult to get the number by this method but that at that moment he could not think of something else. The interviewer then suggested to consider smaller number than one million.
Rob using a hand calculator found the two numbers. He mentioned that the problem was not easy until he found the pattern.	The interviewer then asked to explain why those numbers did not involve zeros in their representations. Rob was asked about the next part of the problem
Rob mentioned that he was sure that there was not other pair because other representation involves zeros.	His response to this part was based on observing that the numbers 10, 100, ...,had only 2 and 5 as factors.

For the pages problem, the students represented the problem algebraically; however, they experienced difficulty solving the quadratic equation[2] involved in the representation. It was interesting to observe that some students wanted to isolate the variable by dividing the two sides of the equation by the variable, that is, from $x^2 + x = 3{,}192$ to $x + 1 = 3{,}192/x$ without realizing that they were dealing with a quadratic equation. When the interviewer asked the students it they could use other methods, they responded "probably by using the long way" (guess and test). When they were trying this guess and test approach, some of the students were using guesses in which the numbers were not consecutive even when originally they had represented the problem correctly. None of the students used the square root of 3 192 as the approximation of the page numbers. In general, they found products of consecutive numbers larger and smaller than 3 192 and explored all the possibilities between these numbers. It seemed that it was difficult for the students to think of another method of solution even when they had not shown much progress. They stuck to one approach for some time and gave

[2] Although the students were familiar with quadratic equation, the grade ten curriculum only includes particular ways of finding the roots such as factoring.

up easily. Santos-Trigo (1990) showed that students often expect to be told what method to use to solve problems

Discussion of the Results and Recommendations

There was indication that some students experienced difficulties in identifying the key information of the problems and they were not organized in presenting the information that could be used to solve the problem. They believed that the most efficient way for solving the problems was the algebraic approach even when they often struggled in representing the information. They used guess and test as the last resort and failed to use simpler problems as a means to solve the problems.

The students failed to monitor their processes while working on the problems; they often used information that was not consistent with ideas that they had used previously. For example, for the pages problem they recognized that the numbers had to be consecutive, but when they were searching for the numbers, they often tried any number.

The results showed that the use of these type of problems could provide information about the difficulties that students experience while trying to solve the problems. It may be suggested that asking these problems to the students could be an important activity to implement regularly in the classroom. The students must engage in open discussion with their classmates (small groups, pairs) and have the opportunity to explain what they do while solving the problems. Schoenfeld (1988) suggested that a *microcosm of mathematical practice* in which the students have the opportunity to express and defend their ideas, to speculate, to guess, and to reflect on possible approaches for understanding mathematical ideas or solving mathematical problems could be the initial point for attaining this goal. The students engaged in such practice will be exposed to an intellectually demanding environment that should eventually improve their ways of understanding mathematics.

Conclusions

The results of the present study may have direct implications for mathematical instruction. For example, the difficulties shown by the students while trying to understand the statement of the problems suggest that teachers should encourage their students to discuss the main ideas of the problem. That is, class activities should include examples in which students realize the importance of understanding the sense of the problem before going further. Another aspect that teachers should pay attention during instruction is to emphasize the need to explore various ways to solve problems. As well, the need to characterize the qualities of different methods of solution. For example, it is important that students recognize that a problem often could be solved using trial and error (for example) in a better way than using algebra. In addition, teachers should discuss during their instruction problems which can be solve by different methods (geometrical, algebraic, or numerical ones). The idea here is that teachers use these problems to explore the processes used by the students while trying to solve them. As Easly (1977) stated "teachers would have to understand rather well the process of cognitive development and listen to and observe children carefully so to grasp with reasonable accuracy what kind of mental operations they are bringing to bear on a given task" (p. 21).

Students were consistent in trying to find a rule of formula that they could use to solve the problem. This suggest that teachers should discuss regularly examples in which these means could not be useful to get the solution of the problem. That is, to show the students that applying a formula or algorithm is not the only way to find the solution of a problem.

Finally, this study illustrates the importance of discussing the problems among colleagues. As a result, diverse methods or strategies could be identified. This discussion should also be promoted among students as a part of instruction. As well, students should select or design their own problems to be discussed during the development of the class. They may contribute to develop a view of mathematics that reflects the proper activities of the discipline.

References

Easly, J.A. (1977). *On clinical studies in mathematics education.* Columbus, OH: Ohio State University, Information Reference Center for Science, Mathematics, and Environmental Education.

Lampert, M. (1990). Connecting mathematical teaching and learning. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 121-152). State University of New York Press. New York.

Schoenfeld, A. (1988). Mathematics, technology, and higher order thinking. In R. Nickerson & P. Zodhiates (Eds.), *Technology in Education: Looking toward 2020*, (pp. 67-96). Hillsdale, NJ: Lawrence Erlbaum Associates.

AN ANALYSIS OF TEACHER CANDIDATES' REFLECTIONS ABOUT THEIR UNDERSTANDING OF RATIONAL NUMBERS

Vânia Maria Pereira dos Santos
Universidade Federal do Rio de Janeiro

Abstract

This paper addresses the insights of observing and interviewing teacher candidates enrolled in a newly developed mathematics content course at Indiana University. The display of some excerpts of the narratives of two respondents show the contrasting views of mathematics and rational numbers that these students acquired throughout the semester of the course. Some implications for mathematics teacher education are discussed.

The goal of this paper is to discuss the insights of examining two teacher candidates' reasoning and arguments about mathematics and rational numbers. Two reasons motivated this work. First, the fact that elementary school teachers are responsible for introducing and exploring fraction ideas with children. Secondly, the awareness that mathematics educators can learn about teacher candidates' thinking processes when analyzing their reasoning in both written and verbal form. There is evidence that school children have difficulty in understanding the basic ideas of fraction, and that many times they are able to perform computations with fractions without real meaning (Behr, Lesh, Post & Silver, 1983; Hart, 1981; Hasemann, 1981). Studies have shown that many misconceptions about fractions held by students are also prevalent among intermediate school teachers (Post, Harel, Behr & Lesh, 1988). Related studies with prospective teachers (Ball, 1988; Bezuk, 1989; Civil, 1989; Santos, 1991) confirmed the need of further investigation.

This study is part of a research project which was designed to investigate teacher candidates' metacognitive awareness of their knowledge of mathematics (in particular fractions), their implicit theories of learning and teaching of mathematics, and their beliefs about mathematics and its pedagogy in the context of a newly mathematics course specially designed for them. Information from studies of the learning of mathematics and research on teaching as well as documents arguing for changes in mathematics education which are consonant with constructivist principles helped to develop this innovative course for preservice teachers (Connolly & Vilardi, 1989; Davis, Maher & Noddings, 1990; Lester, 1988; National Research Council [NRC], 1989; National Council of Teachers of Mathematics [NCTM], 1989, 1991; Novak & Gowin, 1984; Santos, 1991 ; Schroeder & Lester, 1989). This course is entitled Mathematics For Elementary Teachers Via Problem Solving (T104) and has as its main emphases: a) the use of a problem-solving approach to learn mathematics; b) the frequent use of small group work; c) the use of written activities; d) the use of concept mapping; and e) the use of alternative methods of assessment (For a complete discussion about the conceptual framework that guided the course development and its emphases see Santos & Kroll, 1992; Santos, 1993).

Data Source

The research project involved class observations of 26 preservice teachers (PSTs) (24 females and 2 males) during the Spring semester of 1991 while exploring mathematics in a context of group problem solving. The bulk of artifacts examined in this study was classroom observations, interviews with 16 PSTs (14 females and 2 males), and document analysis. From January to May 1991, this T104 class was observed for fifteen weeks (three days a week for two hours each day). Only eight respondents from the 16 PSTs who were interviewed completed four interviews (semi-structured interviews of one hour). Document analysis of students' work, written reflections, questionnaires, interviews and member checking with respondents were used to compose respondents' narratives about mathematics, teaching and rational numbers. In this research project, I followed the principles to establish and maintain trustworthiness outlined by qualitative studies (Lincoln & Guba, 1985) as well as engaged in a simultaneous process of collecting and analyzing data through coding and categorizing.

In this paper, I examine some excerpts from the narrative of two PSTs concerning their views of mathematics and their understanding of rational numbers. The information to weave the stories was provided from the questionnaires, interviews, written reflections and member checking with the respondents.[1]

Investigating Teacher Candidates' View of Mathematics and Rational Numbers

In an attempt to capture the richness of data, the respondents' stories were told in multiple voices (i.e., respondents tell what they shared with me during the study. See Van Maanen, 1988).

MaryAnne's Thoughts About Mathematics

I think my reflections were "more reflective" toward the end of the semester. At that point, I was really writing about how I felt regarding math, and I was doing this more than I did at the beginning. I think that written reflections helped a little bit because they can give you a chance to sit and think about what you've been doing in mathematics. Often, I'll start writing about something and then I'll discover something while I'm writing. I'll discover that I've been doing something wrong or that there is a better way of doing it. To me math is just a subject in school. I feel like a lot of the basics are used in everyday life, but not too much beyond that. I guess I don't look at things in a very mathematical way. I feel like am able to learn mat, but it might take me a little bit longer than others. I think it's important for me as an elementary teacher to have knowledge of basic mathematics but beyond that it doesn't matter to me. I have no interest in complex and complicated mathematics beyond 10th grade.

In all of my previous math classes, I had to work individually. The group work is something new that I'm experiencing in T104. The type of problems we do in T104 are also different from my previous math classes. In my other math classes, we got problems after problems and the goal was to

[1]The first questionnaire addressed questions about students' family and school backgrounds, and previous experience with mathematics. At the end of the semester, students filled a second questionnaire that focused on their reactions to the T104 instructional approach and my dissertation study The first two interviews concentrated on students' school background, decision to become an elementary teacher, previous teaching experience, good and bad school memories, good and bad mathematics memories, reactions to the T104 instructional approach, self-image as mathematics learners, and self-confidence as mathematics learners. The third interview addressed fraction ideas such as fraction estimation sense and mental computation with fractions, fraction concepts while solving mathematics problems that dealt with the four basic operations (+, -, x, ÷), and explanation of problem solutions via concept maps. Respondents checked rough drafts of their stories with me.

find the answer. In T104 it's more important how do you go about to get an answer, than just getting the answer. The assessment is also different. In other math classes I had only to mark an answer in a multiple choice test and all by myself. Here in T104, the instructor also looks at the work and not only at the answer. And in T104 we have tests with individual and group parts. T104 is very different from my other math classes. Before the teacher would stand in front of the class and give definitions and then we would work. Here in T104, it's more hands on the entire time, we are the ones who have to figure things out. My ability to do mathematics is improving after T104. I would say that along a continuum line that my ability is close to neutral coming toward strong, almost 60% of the way. In T104, I feel more confident that I can solve problems and do math activities.

MaryAnne's Understanding of Rational Numbers

I think I can understand rational numbers better now. We have to do a lot of things ourselves, like finding out the fractions with Cuisenaire rods, solving the Condominium Problem (married men and women), comparing fractions without using the rules of finding common denominator, etc.[2] Before T104, fractions were always portions of a whole. I'm learning more now. I think I'm well prepared for the final exam, I feel very confident that I'll do well. Now, I can solve different problems with fractions, and I learned how to think in different ways to solve a problem. When you asked me to prove that the sum of the angles of a right triangle was 180°, I didn't prove it for you, but I think I can handle these problems about fraction you are asking me to do. Let's see, I have 3 pizzas to share among 5 people, and you want to know how much of the pizzas each person will receive. I can divide each pizza in five equal pieces and I can give 1/5 from each pizza to each person. Then each person will receive 3/5 of a whole pizza at the end of this sharing (see figure 1).

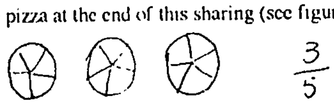


Figure 1. MaryAnne's solution.

Well, you want that I think if there are other ways of solving the same problem. I guess I can think about giving half of a pizza to each person like this. Now, I still have this half of a pizza that I can divide again in fifths. But 1/5 of 1/2 is 1/10 (see figure 2 with this solution).

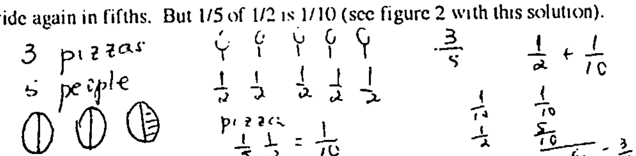


Figure 2. MaryAnne's second solution.

---

[2]**Find all the fractions:** Prove using Cuisenaire Rods, that the yellow rod is half as long as the orange. This relationship can be written as 1/2o = y. Find other pairs of halves and record them. Then do the same for thirds, fourths, fifths, and so on, up to tenths. Record your findings. Then write the expressions in the other way, for instance 2y = 1o. Find at least 4 pairs for each fractional relationship.

Condominium Problem: In an adult condominium complex, 2/3 of the men are married to 3/5 of the women. What part of the residents are married?

Thus, each person receives at the end 1/2 of a pizza and 1/10 of a pizza.

And 1/2 + 1/10 = 5/10 + 1/10 = 6/10. That's the same 6/10 = 3/5. Of course, but I'm surprised that I could do it. Speaking with you has made me more confident in my math abilities. I also have discovered many things that I wasn't aware of while talking. I think I've gotten many ideas for teaching from this experience with the T104 class and the interviews.

Alexis Helga's Thoughts About Mathematics

As long as I can remember, I have always wanted to be a teacher. I love kids and really enjoy giving students something to better their education. I see myself teaching in the future with a lot of hands on work and using manipulatives to help kids figure out what they are doing. I like to use the idea of group work. But I will instruct my students, because I don't like the idea of having to figure out everything on my own. I think the teacher is in the class to instruct us, to show us ways to solve problems. This current semester, I'm enrolled in T104, in E343, the math methods class, and in M201, the corresponding field experience class. Two weeks ago, I taught fractions to 4th grade kids as part of my early field experience M201. I was surprised that students did not understand the work with fractions, and they were still working on some problems when I left after the 45 minutes that I was there guiding their work. For me, we have just to know what are the procedures to operate with fractions and to use the procedures.

In my previous math classes, students had to work periodically after the instructor presented a lecture. Here in T104, I have to figure things out my own, there is no guidance before we start the work. I don't know what to do most of the time, and I think it's a waste of time to have students struggling to learn alone. Why doesn't the instructor instruct and explain things first? I'd much rather have lectures. The only thing I enjoy from T104 is the group work. Maybe in my future teaching, I would use small groups in my class but after I instruct kids.

I think that the best way for me to learn is applying the knowledge I am taught. I need someone to discover the knowledge and show it to me first. If I understood what the teacher showed and if I can check why it works then I can move on to applying this knowledge in exercises But first, I need to understand what I have to do. The best thing for me is to have the teacher really instructing us. I think T104 does not fit with my personal learning style. I feel lost in this class. I need more lectures first, and maybe someday I can discover something on my own. I really don't think that writing reflections or doing concept maps help much. These things did not help me. They are just more work. I don't like all these new things we're doing here. I only enjoy the group effort. This was really a plus.

Alexis Helga's Understanding of Rational Numbers

This last unit in T104 was more like a review to me. Operations with fractions are more clear, and I discover that when you multiply some fractions you get smaller numbers. Our conversation about the concept of fraction is interesting , but I have to keep track of so many details when working with fractions. It's kind of hard. If I had to share 3 chocolate bars between 5 persons, I guess I'll divide the chocolate bars in fifths, and each person would receive 3/15 or 1/5 of the total 15 pieces of chocolate [She was confused here with the meaning of 3/15 and 3/5, and I had to keep probing her to explain her ideas.]

When I think in chocolate bars or pizzas, after I divide each pizza in fifths, I discover that I have 15 pieces of pizza, so to share these pieces among 5 persons, I would give 1/5 of 15 pieces to each one. And this means that each person receives 3 pieces of pizza. It's hard for me to see that I can say 3/5 of a pizza and that this is the same as these 3 pieces of pizza out of fifteen pieces of pizza. I don't know if I can think about other ways to share 3 pizzas among 5 persons. I'm so used to think in dividing the whole by the number that I think is the denominator (i.e., in this problem I think about dividing in five pieces each pizza) that I don't know other ways (see figure 3)

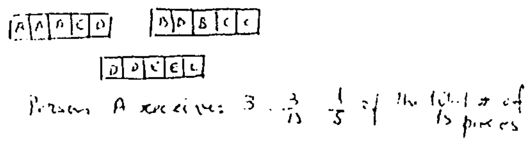


Figure 3. Alexis Helga's solution to the problem of 3 pizzas.

Oh, maybe this can be an idea to start dividing each pizza in half and see how I can share the pizzas among the five persons [I probed her asking to justify if it's possible or not to divide each pizza in halves to solve this problem.]. Your idea works at the beginning but now I had to figure out how to share this last half of the pizza among the five persons. I guess I'll have to divide this last half in five pieces again. And this is 1/5 of 1/2, it's 2/7. No, it's not, wait a minute, I have to do 1/5 x 2/1 = 2/5. No, it's wrong. I don't know, it's too confusing (see figure 4).

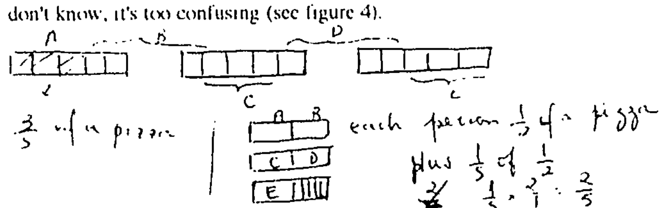


Figure 4. Alexis Helga's second solution to the problem of 3 pizzas.

From the time I began offering her ideas of different ways to share the 3 pizzas among the 5 persons, the interview really stopped and I began trying all possible ways to help Alexis Helga reach some conclusions. I had already perceived that she was not knowledgeable in fractions and other mathematics topics, but she was reluctant to admit her lack of knowledge. At least in this interview context, she began asking for help because she could finally realize that she didn't know how to compute 1/5 of 1/2.

Alexis Helga tried to add numerators and denominators, she tried to work out as if it was a division of fractions, and after a long discussion, while modeling with paper folding she could see that

1/5 of 1/2 was 1/10. We discussed again the pizza problem, and when she tried to discover the final amount of pizza that each of the 5 persons would receive, she said "Each person receives 1/2 of a pizza plus 1/10. Oh, this is 1/2 + 1/10 = 2/12 = 1/6 of a pizza." She perceived she was wrong because in her first solution she discovered (after our conversation) that the answer was 3/5 of a pizza or 3/15 of 15 pieces of pizza or 1/5 of 15 pieces of pizza. Then, I probed her last answer (1/2 + 1/10 = 2/12 = 1/6 of a pizza), and she said "I forgot that I had to find equivalent fractions to add Well, 1/2 is the same as 5/10, so 1/2 + 1/10 = 5/10 + 1/10 = 6/10, and 6/10 = 3/5." (See her computations on figure 5.)

$\frac{1}{2}$ of a pizza + $\frac{1}{10}$ of a pizza

$\frac{1}{2} + \frac{1}{10} = \frac{2}{12}\ \frac{1}{6}$ of a pizza

$\frac{5}{10}\quad \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$

Figure 5. Alexis Helga's computations.

Discussion and Implications for Mathematics Education

These two respondents had contrasting attitudes toward the novelties implemented in T104. It seemed to me that their motivation, interest, and willingness to do well in mathematics classes had a lot to do with how their expectations of teaching and learning matched with their instructor's conceptions of teaching and learning MaryAnne experienced situations of change in her attitudes toward the class throughout the semester, but she was more open to the pedagogical innovations of T104. Alexis Helga's expectations about mathematics teaching and learning did not match with the instructor's conceptions of teaching and learning and since the beginning she had a hard time in the class.

Even though not all of the T104 students ended the course with a complete understanding of why certain principles work or were able to explain what is behind the rule (as it happens in the case of dividing fractions, that you repeat the first fraction and multiply by the inverse of the second fraction), many acquired a more mature vision and were aware of what they understood and knew in mathematics. This new vision of themselves and the development of this metacognitive knowledge as learners of mathematics is very important for their future teaching practices.

Teachers need to become aware of what they are teaching, what they know, what they do not know, and become aware that there are places to search for explanations. The real benefit of T104 for MaryAnne and many others in the class was to expose them to basic mathematics concepts in a problem-solving context in which they were allowed to question their understanding and started to think more critically about procedures used in mathematics and the reasons why certain procedures work. Before this T104 class, MaryAnne and others could not articulate their uncertainty or discover if they really knew or not the mathematics concepts. This was because they learned these concepts in traditional mathematics courses where students only have to solve problems, use procedures, and find a final answer to show they can perform. Very rarely are students asked to think about or discuss their ideas or verbalize their mathematics understanding or lack of it in verbal, written or pictorial forms (e.g., concept maps).

From our first interview and our other interactions, Alexis Helga always commented about how unhappy she was with this class. I met her again during the Fall 1991 (August to December 1991) as a math tutor. I had offered to tutor all of the 16 PSTs who have participated in the research project in a few interviews or during the entire study. When Alexis Helga was enrolled in M118- Finite Mathematics during the Fall 1991, she asked if would tutor her because she was having a lot of trouble in that class. We met a couple of times to go over homework assignments and to help her develop study habits so that she could work on her own later. It was only in the middle of our tutoring sessions, that Alexis Helga realized that she would need to understand the concepts behind addition, multiplication, and division of fractions. One day, she told me "OK, I think you've convinced me that I have to understand how to operate with fractions because I'm always getting confused on what rule I should use. Let's make a deal, you help me do these homework assignments with matrices involving all these strange numbers, and I'll let you discuss with me the ideas behind these operations."

It seems to em that teacher candidates like Alexis Helga, who do not possess a strong mathematics background and who are not willing to engage in the challenges and novelties proposed by a course such as T104 need to be considered when we propose changes in teacher education. Unfortunately, the only thing I can ascertain is that we still let many Alexis Helgas go out with a teaching license without the minimum mathematical knowledge they need to teach and this is a huge and complex problem for all of us interested in mathematics teacher education. I learned from Alexis Helga that our innovations in T104 work only if the student is willing to change. Students need more time to rethink their views about learning and teaching before changes can take place. I believe that students similar to Alexis Helga need to bring their weaknesses at a conscious level. They need to experience classroom environments that model this kind of strengths and weaknesses in mathematics. They need to become aware of what they do not know and they need to receive help to overcome their difficulties. In conclusion, we (mathematics educators) need to conduct other studies and analyze in more depth how and when change takes place in preservice teachers' conceptions and understanding of mathematics.

References

Ball, D. L. (1988). Knowledge and reasoning in mathematics pedagogy: Examining what prospective teachers bring to teacher education. (Doctoral dissertation, Michigan State University, 1988). *Dissertation Abstracts International, 50*, 416A. (University Microfilms No. 8900008).

Bezuk, N. (1989). Preservice elementary teachers' understanding of proportional reasoning missing value problems. In C. A. Maher, G. A. Goldin, & R. B. Davis (Eds.), *Proceedings of the Eleventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 212-218). New Brunswick, NJ: The State University of New Jersey Rutgers.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-126). New York, NY: Academic Press.

Civil, M. (1989). Prospective elementary teachers' conceptions about the teaching and learning of mathematics in the context of working with ratios. In C. A. Maher, G. A. Goldin, & R. B. Davis (Eds.), *Proceedings of the Eleventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 289-295). New Brunswick, NJ: The State University of New Jersey Rutgers.
Connolly, P., & Vilardi, T. (Eds.) (1989). *Writing to learn mathematics and science*. New York, NY: Teachers College Press.
Davis, R. B., Maher, C. A., & Noddings, N. (Eds.) (1990). *Constructivist views on the teaching and learning of mathematics*. (Monograph Number 4). Reston, VA: National Council of Teachers of Mathematics.
Lester, F. K., Jr. (1988). Reflections about mathematical problem-solving research. In R. I. Charles & E. A. Silver (Eds.), *Research agenda for mathematics education: The teaching and assessing of mathematical problem solving* (pp. 115-124). Reston, VA: National Council of Teachers of Mathematics.
Hart K. M. (1981). Fractions. In K. M. Hart (Ed), *Children's understanding of mathematics: 11 16*. London, England: John Murray.
Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics, 12*(1), 71-87.
Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Newbury Park, CA: Sage Publications.
National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author
National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, D. C.: National Academy Press.
Novak, J. D., & Gowin, D. B. (1984). *Learning how to learn*. New York, NY: Cambridge University Press.
Post, T. R., Harel, G., Behr, M. J., & Lesh, R. (1988). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 194-219). Madison, WI: Wisconsin Center for Educational Research, University of Wisconsin-Madison.
Santos, V. M. (1991). *Writing influences prospective elementary teachers' knowledge of mathematics*. Paper presented at the VIII InterAmerican Conference of Mathematics Education, Miami, Florida.
Santos, V. M. (1993). *Metacognitive awareness of prospective elementary teachers in a mathematics content course and a look at their knowledge, beliefs and metacognitive awareness about fractions*. Unpublished doctoral dissertation, Indiana University.
Santos, V. M., & Kroll, D. L. (1992). Empowering prospective elementary teachers through social interaction, reflection and communication. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth International Conference for the Psychology of Mathematics Education, Vol. 2* (pp. 282-289). Durham, NH: University of New Hampshire.
Schroeder, T. L., Lester, F. K., Jr. (1989). Developing understanding in mathematics via problem solving. In P. A. Trafton & A. P. Shulte (Eds.), *New directions for elementary school mathematics* (pp. 31-42). Reston, VA: National Council of Teachers of Mathematics.

SCHOOL CHILDREN VERSUS STREET SELLERS' USE OF THE COMMUTATIVE LAW FOR SOLVING MULTIPLICATION PROBLEMS

Analúcia Schliemann, Cláudia Araujo, Maria Ângela Cassundé, Suzana Macedo & Lenice Nicéas[1]

Universidade Federal de Pernambuco, Recife, Brazil

We analyze whether or not use of the commutative law for solving multiplication problems of the isomorphism of measures type requires school instruction or if this mathematical property can also be understood by children with little schooling who use repeated additions instead of multiplication when they solve problems as street sellers. Subjects were first- to third-grade school children and street vendors with irregular school attendance who had received none or very little school instruction on multiplication. Results of the two studies described suggest that understanding of the commutative law seems to be closely related to school instruction on multiplication, while use of repeated additions to solve isomorphism of measures problems seems to postpone this understanding.

There is nowadays agreement upon the fact that mathematical knowledge can be developed outside of school classrooms and independently of school instruction. The prevalent view is that mathematical knowledge may develop in social contexts, through cultural practices. This does not mean that nothing is left for school to teach. Schools, as an important cultural setting in children's lives, play an important role in developing sound mathematical knowledge through the variety of situations that can be set up to provide opportunity of learning new and more powerful procedures and of understanding new aspects of mathematical contents which does not easily appear in other contexts. Understanding of properties and principles involved in multiplication, such as commutativity, may be one of the areas where school instruction may play an important role while out of school experience tends o postpone its appearance.

We have documented in our previous studies (T. Carraher, D. Carraher & Schliemann, 1982, 1985, 1987; Nunes, Schliemann & D. Carraher, 1993; Schliemann & Acioly, 1989; Schliemann & D. Carraher, 1992; Schliemann & Magalhães, 1990) that children and adults with low levels of schooling usually solve multiplication problems through repeated additions, instead of using multiplication. A common mathematical problem young street sellers solve at work consists in, given the price of one item, calculate the price of a certain amount of the items. Street sellers with little school experience work out solutions to this type of problem through the successive addition of the price of one item, as many times as the number of items to be sold. This strategy preserves throughout the necessary computations clear reference to physical quantities, a typical feature of computation procedures in real life situations where number is rarely conceived without referents. In this case, when computing the price of many items, given the price of one item, what makes sense is to add the number that refer to price as many times as the number denoting how many items are to be sold. It may seem inappropriate for a child to consider such operation on quantities as commutative and to add the number of items as many times as the price of each one to find the total price.

As pointed out by Vergnaud (1988), even when multiplication is used, it may be not obvious for young children that problems of the isomorphism of measures type such as "John wants to buy 5 balls; each ball costs 3 dollars; how much he will have to pay?" can be solved either by multiplying 3 times 5 or by multiplying 5 times 3. At a purely numerical level, multiplicand and multiplier can be inversed without consequences on the results At the level of the measures being dealt with by the subject who solves a problem, commutativity may not seem obvious and "certain pedagogical precautions are needed to make the child accept this commutativity" (Vergnaud, 1981, p. 122). Empirical work by Nunes & Bryant (1992) shows that 9 to 10 year old schooled children do not easily accept commutativity when they try to solve multiplication problems of

[1] This study was sponsored by grants to the first author from CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazil.

this type. One of the few studies focusing on use of commutativity among non-schooled subjects (Petitto & Ginsburg, 1982) demonstrates that illiterate adults do not use the commutative law to solve multiplication problems. After solving through repeated additions a problem like 100 times 6, their illiterate African subjects did not use the previous result for solving the problem 6 times 100. Instead they tried to work out an answer by successively adding 6 one hundred of times.

It is mainly in schools that children learn to operate on numbers devoid of referents. In schools, when multiplication tables are taught children have many opportunities to realize that the same result is obtained if you multiply 5 times 3 or 3 times 5. Moreover, they are explicitly taught that commutativity is one of the properties of the multiplication operation and receive intensive training on algorithms entailing multiplicative relationships. This training is often decontextualized, with emphasis on numerical computation rather than on quantities, with numbers being used without any reference to physical quantities. Despite the well documented lack of sense of many procedures learned by students in schools, this type of training may help school children to deal with a broader range of principles than street vendors with low levels of schooling are accustomed to. This may accelerate understanding of commutativity among school subjects while street vendors with less school experience would not discover and use commutativity even when this property would provide easier solutions for some problems, specially when they are using repeated additions.

How and when an understanding of commutativity in multiplication emerges? Is it the result of school instruction on multiplication? Would procedures such as use of repeated additions to solve multiplication problems postpone children's understanding of commutativity in multiplication? Or do they recognize commutativity between the number that is repeatedly added and the number of times an addend is taken to solve a problem? In the two studies reported below we analyze the emergence of use of the commutative law to solve multiplication problems among school children and among street sellers in Brazil. Our analysis aims at determining whether or not understanding of the commutative law for multiplication requires school instruction or if this mathematical property can also be understood by children with little schooling who use repeated additions instead of multiplication to solve problems as street sellers.

STUDY 1

Subjects were 72 first- to third-grade Brazilian children, in the age range from 6 to 9, and 44 Brazilian street vendors, aged 9 to 12, with irregular school attendance and who had received none or very little school instruction on multiplication. We had to opt for two samples with wide differences in their age range because, while middle class children in Brazil start receiving school instruction on how to solve multiplication problems when they are eight years old, street sellers, who belong to the poorest sectors of the population, enter school much later and are usually not working before the age of nine. Despite the fact that street sellers in the sample had attended school for one to four years, they did not receive regular instruction on multiplication due to the poor level of instruction in most Brazilian public schools.

Subjects were individually interviewed and asked to solve, talking aloud, the following series of verbal problems where, given the price of one chocolate, in terms of number of 'cruzeiros', the Brazilian monetary currency, they have to compute the price of a certain amount of chocolates stated in the problem as follows:

(a) A boy wants to buy chocolates. Each chocolate costs 5 cruzeiros. He wants to buy 3. How much money does he need to buy 3 chocolates?
(b) Another boy wants to buy a type of chocolate that costs 3 cruzeiros each. He wants to buy 5 chocolates. How much money does he need to buy 5 chocolates?
(c) Still another boy wanted to buy 4 chocolates. The chocolates he wanted costs 6 cruzeiros each. How much he will have to pay?
(d) And another boy wanted 6 chocolates. Each chocolate costs 4 cruzeiros. How much he will have to pay?

After attempting to solve each one of the four problems subjects were asked to explain their procedures if these were not made clear during solution. Before solving problems *b* and *d* subjects were asked to try to answer these without doing any computation. If failure to provide a response based on the previous answer occurred, they were asked to proceed with computations to find out an answer.

In problems *a* and *c*, the numbers denoting the price of each item is larger than the one denoting the quantity of items to be bought. For problems *b* and *d* the inverse occurs. In order to control for effects due to the order numbers appear in the problems, for the pair of problems *a* and *b*, in problem *a* the larger number is the first number given while in problem *b* the opposite occurs; for the pair *c* and *d*, it is in problem *d* that the larger number is stated first, while in problem *c* the reverse occurs. Because problems were given in the above fixed order, a direct way of demonstrating understanding of commutativity would be to find the result for problems *b* and *d* by mere repetition of the results of problems *a* and *c*, respectively, without any further computation, since the answers for the two problems in a pair is the product of the same two numbers. On the basis of results obtained by Nunes & Bryant (1992), among schooled subjects, we did not expect such an explicit use of commutativity to occur very often. There is, however, an indirect way to access understanding of the commutativity law as, in Vergnaud's (1981) terms, a theorem in action: A child who considers multiplicatic problems of this sort as non-commutative or asymmetrical should always chose the number denoting the price of items as multiplicand or, if repeated additions is the procedure chosen, as the addend to be repeated. A child who conceives multiplication as commutative, should always choose the larger number as multiplicand or as the addend in the repeated addition procedure, since this order represents a more economical procedure.

Results

Table 1 shows the percentage of correct answers for each group in each problem. Problem *a* was the easiest of all for all groups and only first graders in the school children group had some difficult in finding the correct answer for it. Problem *c* presented the same trend for schooled children but proved to be fairly difficult for street vendors. Problems *b* and *d* were solved by nearly all schooled subjects in third grade, presented some difficulties for second graders, and were very difficult for first graders. The majority of younger street sellers failed to solve problem *b* while the majority of the older ones could solve it. In problem *d*, however, even older street sellers failed in most cases in finding a solution for it.

Table 1: Percentage of correct answers for each problem in Study 1

Group	Problem *a*	Problem *b*	Problem *c*	Problem *d*
School subjects:				
First graders (N=24)	.58	.33	25	37
Second graders (N=24)	.96	.79	.96	.67
Third graders (N=24)	.96	.88	.96	.88
Street vendors:				
First and second graders (N=22)	.91	.32	.50	.27
Third and fourth graders (N=22)	.95	.68	.59	.36

The procedures or justifications for solving each problem were classified in terms of type of arithmetical operation used to achieve or to explain a result. For problems *b* and *d*, a fifth category was added to include cases in which the subject did not computed an answer but, instead, explicitly appealed to the commutative law stating that results for problems *b* or *d* must be the same as the result obtained for the corresponding previous problems. Correct or wrong answers given through non-identified operations or procedures were included the "others" category. Tables 2 and 3 show the distribution of subjects in each grade group, according to type of answers given for problems *a* and *c* (where price is the referent for the larger number) and for problems *b* and *d* (where price is the referent for the smaller number).

Table 2: Frequency of each type of procedure for school subjects in each type of problem

Grade	Problem	Explicit use of commu-tativity	Operation performed to solve problems: Multiplication of larger x smaller	Multiplication of smaller x larger	Additions of larger number	Additions of smaller number	Non-identifi-ed & others
1st	Larger number denotes price (*a* and *c*)	-	0	0	20	0	28
	Smaller number denotes price (*b* and *d*)	1	0	0	0	24	23
2nd	Larger number denotes price (*a* and *c*)	-	11	11	20	4	2
	Smaller number denotes price (*b* and *d*)	7	9	7	5	15	5
3rd	Larger number denotes price (*a* and *c*)	-	19	20	7	0	2
	Smaller number denotes price (*b* and *d*)	16	6	20	2	2	2

Table 3: Frequency of each type of procedure for street vendors in each type of problem

Grade	Problem	Explicit use of commu-tativity	Operation performed to solve problems: Multiplication of larger x smaller	Multiplication of smaller x larger	Additions of larger number	Additions of smaller number	Non-identifi-ed & others
1st & 2nd	Larger number denotes price (*a* and *c*)	-	0	1	30	1	12
	Smaller number denotes price (*b* and *d*)	0	0	0	1	32	11
3rd & 4th	Larger number denotes price (*a* and *c*)	-	3	8	24	1	8
	Smaller number denotes price (*b* and *d*)	6	5	2	2	17	12

Explicit reference to commutativity, or use of the results for one problem as the answer for the following problem, on the basis of the commutative law occurred for nine school children and three street sellers in problem *b*, and for 15 school children and three street sellers in problem *d*. For children who did not explicitly refer to commutativity, understanding of this property was inferred through use, for each pair of problems (*a/b* and *c/d*), of either number as multiplicand, when multiplication was used, or as addend, when addition was the operation chosen by the subject.

Three interesting clear developmental trends appear in the results for school children. Firstly number of errors, "don't know" answers, or unidentified strategies decreases and nearly disappears for all problems when children reach third grade. Second, successive addition, which is the exclusive computational procedure used by first graders gives place to multiplication procedures among the older children. Finally, while children who use successive addition to solve the problems clearly prefer to take as addend the number denoting the amount of cruzeiros, those using multiplication use either number in the problem as multiplicand, denoting an implicit understanding of commutativity in multiplication. Data for street sellers also show a developmental trend, switching from the exclusive use of successive additions to the use of multiplication to solve the problems. This trend however occurs at a later age and is less accentuated, with a majority of older street sellers still exclusively using repeated additions. The general tendency concerning an implicit understanding of commutativity when multiplication is used is also observed while, when successive addition was performed, street sellers, as was the case for school children, only accepted to use as an addend the number denoting the quantity of 'cruzeiros' stated in the problem.

The association between type of arithmetical operation used to solve the problems and presence or absence of understanding of the commutativity law, either explicit or implicit, can be better appreciated by relating the strategies used for each pair of problems. When multiplication was used to solve problems *a* or *c*, in 59 % of the cases for schooled subjects and in 67 % of the cases for street sellers, the answer to problem *b* or *d* was achieved though explicit or implicit use of the commutative law. The same occurred in only 25% and 7 % of the cases, respectively, when addition was the operation used to solve the problems.

Discussion

The above results suggest that when children learn about multiplication in schools they also learn about the commutative law. Street seller's experience on solving problems about prices of various items through repeated additions does not seem to be enough to promote the emergence of understanding of the commutative law. It is possible, however, that street vendors did not demonstrate an understanding of commutativity in the solution of the problems they were given in this study, choosing to perform successive additions in the least economical way, because the difference between the two numbers in the problem was not large enough to really matter. Therefore, the choice of the largest number as the successive addend did not represent a great economy in the steps towards solution. With larger differences, such as that between the numbers in the study by Petitto & Ginsburg (1982), different results may appear. Another aspect that remains to be analyzed concerns school children use of the commutative law: Do these children use of commutativity as the result of understanding a logical rule, or is it just an empirical finding for specific number pairs they have been practicing well at school? Memorization of multiplication tables provide them with a wide range of opportunities to match the results of the multiplication of the same pair of numbers in different orders. If this is the case, even school children who use multiplication to solve the problems may refuse to profit from the commutative law when they have to solve problems involving pairs of larger numbers, for which they have not memorized the results of multiplication. In order to better analyze these issues, a second study, where numbers in the problems differ by a larger margin was designed and run with similar groups of school children and street sellers.

STUDY 2

A total of 72 first- to third-grade children, aged from 6 to 9, and 43 street vendors, aged 9 to 13, with irregular school attendance, who had received none or very little school instruction on multiplication participated in study 2.

As in study 1, each subject was asked to solve, talking aloud, four verbal problems where, given the price of one chocolate, in terms of number of 'cruzeiros', they have to compute the price of a certain amount of chocolates. Each one of the problems now included one large number, as follows:

(e) A boy wants to buy chocolates. Each chocolate costs 50 cruzeiros. He wants to buy 3. How much money does he need to buy 3 chocolates?
(f) Another boy wants to buy a type of chocolate that costs 3 cruzeiros each. He wants to buy 50 chocolates. How much money does he need to buy 50 chocolates?
(g) Still another boy wanted to buy 4 chocolates. The chocolates he wanted costs 60 cruzeiros each. How much he will have to pay?
(h) And another boy wanted 60 chocolates. Each chocolate costs 4 cruzeiros. How much he will have to pay?

Problems were given in the above fixed order and subjects were asked to explain their procedures. As in study 1, before solving problems *f* and *h*, they were asked to try to answer these without doing any computation.

Results

Percentage of correct responses for each problem, in each group is shown in table 4. For problem *e* the results closely follow the pattern obtained for problem *a* in study 1. The other problems however, were more difficult than the corresponding problems used before, especially for the younger children.

Table 4: Percentage of correct answers in Study 2

Group	Problem *e*	Problem *f*	Problem *g*	Problem *h*
School subjects:				
First graders (N=24)	.58	0	.08	0
Second graders (N=24)	.92	.46	.33	.17
Third graders (N=24)	.96	.79	.63	.58
Street vendors:				
First and second graders (N=23)	.96	0	.52	.04
Third and fourth graders (N=20)	.85	.60	.65	.50

Tables 5 and 6 show the procedures used by school children and by street sellers, respectively.

Table 5: Frequency of each type of procedure for school subjects in each type of problem

Grade	Problem	Explicit use of commutativity	Operation performed to solve problems: Multiplication of larger x smaller				Non-identified & others
			Multiplication of: larger x smaller	Multiplication of: smaller x larger	Additions of: larger number	Additions of: smaller number	
1st	Larger number denotes price (*e* and *g*)	-	0	0	30	0	18
	Smaller number denotes price (*f* and *h*)	0	0	1	1	15	31
2nd	Larger number denotes price (*e* and *g*)	-	6	9	28	0	5
	Smaller number denotes price (*f* and *h*)	4	11	7	4	7	15
3rd	Larger number denotes price (*e* and *g*)	-	20	21	6	0	1
	Smaller number denotes price (*f* and *h*)	7	15	23	0	1	2

Table 6: Frequency of each type of procedure for street sellers in each type of problem

Grade	Problem	Explicit use of commutativity	Operation performed to solve problems				Non-identified & others
			Multiplication of: larger x smaller	Multiplication of: smaller x larger	Additions of: larger number	Additions of: smaller number	
1st & 2nd	Larger number denotes price (*e* and *g*)	-	0	1	32	0	13
	Smaller number denotes price (*f* and *h*)	0	1	2	3	4	36
3rd & 4th	Larger number denotes price (*e* and *g*)	-	7	7	18	3	6
	Smaller number denotes price (*f* and *h*)	3	11	8	4	5	9

Explicit reference to the commutative law was even rarer in this second study than in the first: three school children and two street sellers explicitly referred to the commutative law or used the result obtained for the previous problem as the result for problem *f* and eight school children and one street seller did the same when solving problem *h*.

As before, the number of errors, "don't know" answers, or unidentified strategies decreases with age among schooled subjects and nearly disappears for all problems when they reach third grade. Successive addition was, with one exception, the only computational procedure used by first graders. As before, successive additions gives place to multiplication among the older children. Again, children who use successive addition to solve the problems prefer to take as addend the number denoting the amount of cruzeiros while those solving the problems through multiplication use either number as multiplicand.

Data for street sellers also show a developmental trend, switching from the nearly exclusive use of successive additions to use of multiplication. A large proportion of older street sellers, however, still prefer to use repeated additions. When the larger number denoted the amount of cruzeiros the preferred strategy among street sellers was the repeated addition of this amount. When the amount of cruzeiros was represented by the smaller number, a sharp increase in non-identified strategies occurred among younger street sellers in this second study. Only a few subjects tried to reach a result via successive additions of the smaller number (in all cases, giving up after a few steps). Most subjects seemed completely lost when faced with such problems and tried, instead, to reach solutions through non-sense computations.

The association between arithmetical operation used to solve a problem and explicit or implicit use of the commutativity law can also be seen in the following data: For schooled subjects, when multiplication was used for solving the first problem in a pair (*e* or *g*), explicit or implicit use of the commutative law was present in 71% of the solutions to the second problem (*f* or *h*). This percentage dropped to 17% when addition was the operation used to solve the problems. For street sellers the corresponding percentages were 60% and 26 %.

Discussion

In this second study, for both samples, there was a slight increase in the number of responses denoting, either implicitly or explicitly, an understanding of the commutative law. However, in the case of street sellers, problems *f* and *h* pose real difficulty for the children who only use repeated additions. Instead of using the commutative law, these children seemed to be at a complete loss when faced with problems where, according to their usual strategy, they should add the amount of cruzeiros stated in the problem for an unusual large number of times. A similar phenomenon has been documented before by Schliemann & Carraher (1992) in their analysis of street sellers strategies for solving proportionality problems with unusual quantities. School children use the commutative law to solve the problems, even when the numbers they deal with are not part of the multiplication tables they so often rehearse at school. It seems therefore that, when they use commutativity they are not just repeating memorized facts but they are demonstrating a real understanding of a mathematical property.

CONCLUSIONS

Do failure to use commutativity for solving problems about price of items indicates a general lack of understanding of the commutative law or can children who refuse to use commutativity in this context apply it to other contexts? Vergnaud (1988) suggests that failure to accept commutativity would not apply for product of measures problems and Nunes & Bryant (1992) show for schooled subjects that product of measures problems are more easily seen as commutative than are isomorphism of measures problems. Whether or not this would also hold for street sellers or school children who have not yet learned about multiplication and work with repeated additions remains a question to be answered by further research.

As suggested by the data described above, understanding of the commutative law for solving isomorphism of measures problems seems to be closely related to school instruction on multiplication while use of repeated additions to solve multiplication problems tends to postpone this understanding. The everyday practice as street seller where numbers most often are used to refer to physical quantities has the advantage of preserving the meaning of the operations performed throughout the series of computational steps until a final answer is reached. This, as stressed by Nunes, Schliemann & Carraher (1993), is one of the strengths of mathematical procedures used to solve real life problems. And in the street sellers practice a clear reference to physical quantities really matters in order to ensure correct computations of prices to be paid. However, if we want children to further advance in exploring the properties of the number system and operations, explicit instruction and carefully designed situations seems to be necessary if schools want to play their role in widening children's understanding of mathematics.

REFERENCES

Carraher, T.N., Carraher, D.W. & Schliemann, A.D. (1982). Na vida, dez; na escola, zero: os contextos culturais da aprendizagem da matemática. Cadernos de Pesquisa, 42, 79-86.

Carraher, T.N., Carraher, D.W. & Schliemann, A.D. (1985). Mathematics in the streets and in schools. British Journal of Developmental Psychology, 3, 21-29.

Carraher, T.N., Carraher, D.W. & Schliemann, A.D. (1987). Written and oral mathematics. Journal for Research on Mathematics Education, 18(2), 83-97.

Nunes, T. & Bryant, P. (1992). Commutativity in multiplication. Proceedings of the XV International Conference for he Psychology of Mathematics Education, vol. III. Durham, NH, p. 133.

Nunes, T., Schliemann, A.D. & Carraher, D.W. (1993). Street Mathematics and School Mathematics. New York, Cambridge University Press.

Petitto, A. & Ginsburg, H. (1982). Mental arithmetic in Africa and America: Strategies, principles and explanations. International Journal of Psychology, 17, 81-102.

Schliemann, A.D. & Acioly, N.M. (1989) Mathematical knowledge developed at work: the contributions of practice versus the contribution of schooling. Cognition and Instruction, 6(3), 185-221.

Schliemann, A.D. & Carraher, D.W. (1991). Proportional reasoning in and out of school. Em P. Light & G. Butterworth (Orgs.) Context and Cognition. Hemel Hempstead, Harvester-Wheatsheaf (in press).

Schliemann, A.D. & Magalhães, V.P. (1990). Proportional reasoning: from shopping, to kitchens, laboratories, and, hopefully, schools. Proceedings of the XIV PME Conference. Oaxtepec, Mexico.

Vergnaud, G. (1981). L'Enfant, la Mathématique et la Réalité. Bern, Editions Peter Lang.

Vergnaud, G. (1988) Multiplicative structures. Em J. Hiebert & M. Behr (Eds.) Number Concepts and Operations in the Middle Grades (vol. 2). LEA/NCTM.

A TASK-BASED INTERVIEW ASSESSMENT OF PROBLEM SOLVING, MATHEMATICAL REASONING, COMMUNICATION, AND CONNECTIONS

Thomas L. Schroeder, State University of New York at Buffalo, USA

The original purpose of this study was to provide a qualitative assessment of students' problem solving focusing on the nature of their thinking, their problem-solving strategies and heuristics, the mathematical approaches they selected, and the ways they monitored their progress. However, the interview task discussed here also has potential for the assessment of students' communication, reasoning, and connections, themes from the NCTM Standards. Data include interviewers' field notes and students' written responses to a non-routine problem presented orally in 15 interviews with Grade 10 students. The problem can be and was solved using several different strategies. The students' selection and use of strategies is related to the notion of mathematical connections; their successes and difficulties explaining why the solution they had found is unique is related to mathematical communication and reasoning.

In the NCTM Curriculum Standards (1989) four themes, mathematics as problem solving, mathematics as communication, mathematics as reasoning, and mathematical connections, are highlighted, appearing as the first four standards in each grade level section of the document. The work presented and discussed here is part of a small-scale qualitative evaluation of students' mathematical problem solving (Schroeder, 1992). However, mathematical communication, mathematical reasoning, and mathematical connections were quite prominent in the students' performances, and they are the focus of the analysis in this paper. The two main goals of the study were (1) to develop a series of non-routine problems that would be challenging yet accessible, that would demand planning and reflection, that would permit a number of different methods of solution, and that would embody a variety of familiar mathematical processes and operations, but not in obvious, routine ways; and (2) to use those problems in interviews with individuals and pairs of students to describe and assess the mathematical processes and operations they apply and the problem solving plans and strategies they adopt.

Interview Task

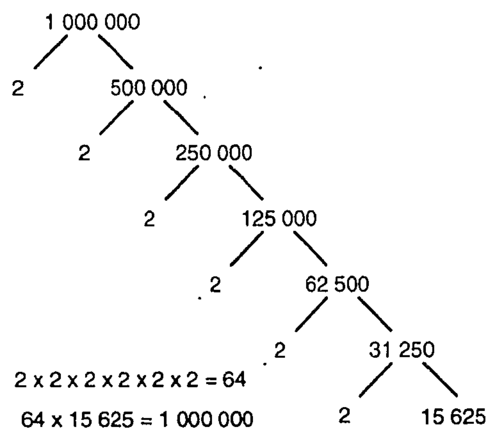
One of the study's seven problems developed for Grade 10 students was the following: "Can you find two whole numbers *a* and *b* whose product is one million and neither *a* nor *b* includes a zero in its representation?" To students who found the required pair of numbers, a follow-up task was posed: "Are there any other such pairs? Why or why not?" Although it does not say so in the statement of the problem, this problem is essentially about factorization and about the roles of prime numbers in a factorization. A fact that can be applied to advantage in this problem is the fact that every number that is a multiple of 10 ends with a zero, as well as the converse of this principle, that every whole number that ends with a zero is a multiple of 10. For students in Grade 10, these mathematical ideas should be quite familiar, even though the problem as stated is an unfamiliar, non-routine one.

In preparing to use this problem in interviews with students, consideration was given to the variety of ways in which students might approach and solve the problem. Indeed, an important reason for selecting the problem was that it could be solved in a number of different ways. One way of beginning to work on this problem is with a trial divisors approach. To find two whole numbers whose product is 1 000 000, one might try dividing 1 000 000 by various numbers hoping that the divisor and the quotient would both be whole numbers not containing a zero. The table below shows some possibilities. Notice that even though the guesses have been made systematically with the factors of 1 000 000 in order of increasing size, it does not look as if a solution is on the horizon.

Divisor	2	4	5	8	10	20	25	...
Quotient	500 000	250 000	200 000	125 000	100 000	50 000	40 000	...
Zeros?	Yes	Yes	Yes	Yes	Yes	Yes	Yes	...

Another systematic way of looking for the two whole numbers is with prime factorization. If instead of taking each factor in order, the problem solver takes out prime factors, each prime being divided out as many times as possible before moving on to the next prime, a solution is reached in just six steps. The figure below shows two ways of presenting prime factorization, one using repeated division, the other a factor tree. In both cases, twos have been divided out as many times as possible, but the on the right hand side 15 625 remains to be decomposed. Both show that six factors of two can be divided out, giving 64 x 15 625 as the two required numbers.

2) 1000000
2) 500000
2) 250000
2) 125000
2) 62500
2) 31250
5) 15625
5) 3125
5) 625
5) 125
5) 25
5



2 x 2 x 2 x 2 x 2 x 2 = 64

64 x 15 625 = 1 000 000

Another variation on the prime factorization theme would be to factor 1 000 000 into tens, and then try to rearrange and collect factors to make two factors each containing no zeros. This might look like the work shown below. The numbers within brackets in the top line all end in zero, and any product formed with them will also end in at least one zero. This means that twos and fives cannot be in the same factor, so the twos have to be put into one factor and the fives into the other, as the bottom line shows. This reasoning also answers the second question of the problem by showing that any rearrangement of the prime factors that puts both a two and a five into the same factor will not work, since that factor will end with a zero.

$$1\,000\,000 = (2 \times 5) \times (2 \times 5) \times (2 \times 5) \times (2 \times 5) \times (2 \times 5) \times (2 \times 5)$$

$$1\,000\,000 = (2 \times 2 \times 2 \times 2 \times 2 \times 2) \times (5 \times 5 \times 5 \times 5 \times 5 \times 5)$$

$$1\,000\,000 = (64) \times (15\,625)$$

A different approach to the problem, and one that seems appropriate in view of the very large number involved, is to solve a simpler problem. To avoid the difficulties of working with a number composed of so many zeros, the student may take a simpler case, say 100, and try to see if there are two numbers neither containing a zero that have a product of 100. It is not too difficult to find the two numbers in this case, 4 and 25. To see how this solution might help in finding the solution to the original problem, the problem solver can solve a series of simpler cases, such as the ones given in the table below, and try to detect a pattern in the solutions.

Product	10	100	1000	10 000	...
Factors containing no zeros	2 x 5	4 x 25	8 x 125	16 x 625	...

In each case, one of the two numbers is a power of two (which is fairly obvious), and the other is a power of 5 (which is somewhat less obvious). Extending the pattern suggests that the way to get one million as the product of two whole numbers containing no zeros is to use $2^6 \times 5^6$, or 64 x 15 625.

Earlier, when we discussed factoring 1 000 000 into powers of 10 and rearranging the twos and fives in those factors, we showed why the solution 64 x 15 625 is the only one possible. That reasoning, using the fact that having both a two and a five in either one of the factors would cause that factor to end in at least one zero, gives a reason why this can be the only answer. Of course, some students may want to or attempt to use brute force to show that all other pairs of numbers whose product is one million have a zero somewhere in their representation. Unfortunately, there are 49 different factors of one million (25 pairs having a product of one million) and 13 of the 49 individual factors do not contain a zero. However, only in the case of 64 x 15 625 do both factors contain no zero. These facts suggest that a student will not be able to be sure that all possibilities have been accounted for, unless the logic involving prime factorization and rearrangement of the prime factors is used.

Procedures

Sixteen volunteers (8 females and 8 males) took part in interviews based on this problem; 14 were interviewed individually, the other two (both males) were interviewed together. The students were provided with calculators (scientific or 5-function) to use if they wished. In the introduction to the interviews students were informed that the activity was not a test and that their performance would not affect their standing in their school mathematics course. They were told that the interviewer was more interested in how they went about solving the problem than in whether they got the correct answer, and that they could ask questions of the interviewer at any time. Students were urged to think aloud as they solved the problem so that the interviewer could understand how they worked on the problem, and they were told that the interviewer might ask them questions for clarification or give them hints or help if they wished.

As the students worked on the problem, the interviewer observed them closely and made field notes. At the conclusion of each interview the interviewer completed a record sheet designed to be a convenient and standardized way of summarizing and reporting students' work on the problem. The sections of the record sheet headed "Understanding," "Strategy Selection," and "Monitoring" reflect Polya's (1945) description of the phases of problem solving and the importance of metacognitive activity in problem solving; they list a number of anticipated features of students' work which can be checked off as appropriate, and they provide spaces in which to describe students' work. In the final section headed "Overview" the interviewer is to indicate whether student(s) solved the problem essentially on their own, or solved the problem with needed help from the interviewer, or did not solve the problem even with help from the interviewer. The amounts of time spent reaching a solution to each part of the problem, finding different methods of solving the problem, e[illegible]ding it or looking back, and are also to be noted.

Results

In the discussion which follows, the unit of analysis is the interview, of which there were 15. Because of the small numbers, results are not reported separately for males and females nor for students interviewed individually or in pairs. The overall results showed that in five interviews (33%) the students solved the problem on their own, and that in nine interviews (60%) the students solved the problem with help from the interviewer; in only one interview did the student fail to solve the problem, because of lack of time to complete the interview. The total length of time spent in each completed interview varied from 8 to 40 minutes with a median of 23 and a mean of 24, and the time taken to reach a solution for the first part of the problem ranged from 8 to 34 minutes with a median of 17 and a mean of 19. Interviews in which the students solved the problem on their own averaged about the same in overall length as interviews in which students solved the problem with help, but the students who solved the problem on their own tended to spend less time reaching a solution than those who received help. The additional time spent by the students who solved the first part of the problem on their own was spent mainly on the second part of the problem and on discussing other methods of solving the first part. The median time spent finding a solution to the first part of the problem for students who solved it on their own was 16 minutes, as opposed to 21 minutes for students who received help; the means were 14 minutes and 21 minutes respectively. All these time measures include about three minutes spent exchanging introductions, recording facts such as names and birthdates, summarizing interview procedures, and obtaining students' consent to participate. Although these gross measures give a sense of the extent of the interviews and an idea of how well the students performed, they were not the focus of the analysis; qualities of the students' work were the main concerns.

In most of the interviews the students seemed to comprehend the problem readily, but in four cases (27%) the students initially confused the term "product" with "sum." In one of these cases the student almost immediately caught his error; having written $a + b = 1\ 000\ 000$, he quickly said "No, product ... ," and changed it to $a \times b = 1\ 000\ 000$. In the other three cases the students also wrote $a + b = 1\ 000\ 000$, and the interviewer intervened only after allowing them to work for some time with that equation (up to three minutes). The questions "What is the problem asking you to find?" and "Why are you adding?" were sufficient to cause the students to recognize that multiplication is required.

In the majority of the interviews the students' initial approach to the problem involved some sort of trial and error behavior. Most often a calculator was used, and in only a few cases were some of the guesses recorded on paper. In some instances students guessed both factors and tested to see whether their product was the desired number, one million. In other cases the students guessed only one factor and divided one million by that number to see whether the other factor was also a whole number containing no zeros. In most cases the students realized that guessing was unlikely to lead them to a solution, and they abandoned it after a short while, but a handful of students persisted with guessing for several minutes, up to 8 minutes in the longest case. Not all of the guessing was "wild guessing." One student made a 9 x 9 table showing the last digit (i.e. the ones' digit) of the product for all the possible choices of last digit for the factors (excluding 0). Noticing in the table that numbers ending in 5 multiplied by numbers ending in 8 yield products ending in 0, the student guessed numbers such as 898 x 565 and found their products using a calculator. This student recorded only a few of his guesses on paper; none of them involved factors ending in 2, 4, or 6, which also give products ending in 0. Another student tried to use other patterns she had recalled from prior experiences playing with numbers. Her guesses (e.g., 99 998 x 89, 99 999 x 89, etc.) are reminiscent of other interesting patterns containing products with repeated digits such as 12 345 679 x 9 = 111 111 111. Other students were noticed making guesses using multi-digit numbers containing repeated fives (e.g., 1 000 000 ÷ 555, etc.).

In the five interviews (33%) where the students solved the first part of the problem without help from the interviewer, the students used either factorization to primes or solving a simpler problem. All three students who used factorization to primes carried out repeated division by twos then by fives in a format similar to the one shown at the left in the earlier diagram, and all three completed the factorization as shown on the left, rather than stopping when the quotient no longer contained a zero, as shown on the right. In two of these three interviews the students reviewed their work and noticed that 15 625 was the first quotient that did not contain a zero. Taking the product of the divisors used up to that point led them quite easily to the other factor, 64, which also did not contain a zero. In the third interview, which incidentally involved the pair of students working together, the students took the six twos and six fives they had found in the prime factorization and explored various ways of rearranging these 12 numbers to form two factors of one million. They used a calculator, but they were not systematic and made no written records, trying many combinations and seeming not to realize that if they put a two and a five together in the same factor that factor would end in zero. Eventually, almost 20 minutes into the interview, they found the correct way to group the 12 prime factors; one might say they stumbled upon the answer which was there all the time.

The other two students who solved the problem on their own did so using the heuristic of solving a simpler problem, in fact a series of simpler problems. The problems they solved were to find pairs factors containing no zeros for 10, for 100, and for 1000, but they did not necessarily tackle them in that order. However, when they did consider their solutions in order, they noticed that one of the factors kept on doubling while the other was repeatedly multiplied by five. Extending that pattern led them with little difficulty to 64 and 15 625.

The students who solved the problem with help from the interviewer were given hints in the form of questions. In one case the interviewer asked, "Would the idea of prime numbers be helpful in this problem?" Interestingly, in this instance the student was not able to make use of the suggestion, even though it can be considered to be a very broad hint, almost a "giveaway." In the other interviews the interviewer asked, "Why don't you try using [or working with] some smaller numbers?" This hint, although somewhat directive, is much less specific. It was given with the expectation that students might think of factors of one million, perhaps even prime factors, as the "smaller numbers." Alternatively, they might think of products like one million but smaller than it, since one million was the number given in the problem. If this were the case, the question might serve as a hint to use the heuristic of solving a simpler problem. In actuality, the students who were given the hint to try smaller numbers all interpreted it to mean that they should try to solve the problem for numbers smaller than one million, and they all did so using 10 or 100 at first. A few of the students had relatively little difficulty taking this hint and running with it, but the majority of them were guided or prompted with further questions, such as "Do you see any relationships between the problems you have solved?"

Detecting and applying patterns seemed to be a source of difficulty for several of the nine students (60% of the interviews) who used the heuristic of solving a simpler problem with help from the interviewer. In two interviews the students had solved the problem for 10 and for 100, and then tried to extend the pattern on the basis that 10 squared was 100, and likewise for each factor of 10 and 100. This led from 2 x 5 and 4 x 25 to 16 x 625, and from there to 256 x 390625, which has a product greater than a million and a factor containing a zero. Thus, the strategy of solving a simpler problem did not lead these students to a solution for the given problem, because the pattern they saw in the two problems they had solved, when extended, did not include the given problem. A few other students were tempted to see additive patterns in the series of problems, rather than multiplicative ones, and this, too, led to difficulties.

The second part of the problem asked whether or not there exist any other pairs of factors containing no zeros, and also for a reason why or why not. None of the students seemed interested in tackling this part, and almost half of them (40%) responded "I don't know" when the interviewer asked these questions directly. In another 33% of the interviews the students replied that they didn't *think* there were any other solutions. All of these students had found their solution to the first part of the problem by solving simpler problems and looking for patterns in those solutions, so they tended to look for additional solutions among their solutions to the simpler problems, as well. Typical of the responses of these students was the statement "I can't find any [others] of those numbers that would make 100, .. so I'm guessing that [because] for 10 you can't and [for] 100 you can't and you can't [have another pair] for a million." The pair of students who were interviewed together and who in the first part of the problem used prime factorization and extensive trial and error to allocate the prime factors to form two factors of one million, were more successful than any other students in explaining why there is only one solution to the problem. Although they had not previously recognized or used the fact that a two and a five cannot be found in the same factor because that factor would then end in a zero, when they addressed the second part of the problem they explained that if you put a two and a five together you would have 10, and when you multiply by 10 you get a zero on the end. Two other students who thought that there were no additional solutions but could not explain why there were not, asked the interviewer to tell them why. In these interviews the interviewer conducted a Socratic dialogue leading the students to produce the prime factorization of one million, pointing out that the solution was essentially $(2^6) \times (5^6)$, and asking what would happen if one of the twos was moved over to the factor containing all the fives. Although these students could not produce the reasoning needed for the second part of the problem, they did not seem to have any difficulty following the argument that they were contributing to by their answers to the interviewer's questions. To see how well these students had appropriated the reasoning for the second part, it would have been useful to ask them to explain it orally or in writing to another student. Unfortunately, this additional procedure had not been planned in advance, and there might not be enough time available in the interviews for it.

Discussion

One important finding of this study is the amount of time that the students spent working on the problem. By comparison with multiple-choice test items, which students are expected to answer at the rate of about one (or more) per minute, or constructed-response items, which take on the order of about five minutes, this task was quite time consuming, and there is a question whether the time required is justified by the information obtained. The amount of time that the students spent reflects not only the fact that the problem was a difficult and unfamiliar one for them; it is also a measure of their perseverance with the task and their willingness to try different approaches and reflect on their work and monitor their progress.

Students solved the first part of this problem without help from the interviewer in only 33% of the interviews. One way of interpreting this result is to say that the problem was rather difficult for them, but an analysis of the students' performance, including their reactions to the hints and questions provided by the interviewers, suggests that for the most part they were not lacking in the conceptual or procedural mathematics background required by the problem, for example understanding of relationships between multiplication, division, and factorization and skills needed to carry out these processes; knowledge of the fact that multiples of 10 end in zero; etc. The obstacles they encountered had to do with difficulties marshaling these resources and failing to monitor their progress. Moreover, most of the students seemed unfamiliar with the problem-solving strategy of solving a simpler problem, although when this was suggested to them by a quite indirect hint they were able to identify and solve at least two relevant and useful simpler problems (i.e. finding pairs of factors containing no zeros for 10 and for 100). Some of the students experienced difficulties detecting and exploiting patterns in the simpler problems that they solved.

The students' performance on this problem illustrates once again the point that mathematical connections are relatively difficult for students. "It seems safe to say that many mathematical connections are not obvious to most students. Substantial amounts of time are required for students to ponder about them. Even when fairly broad hints are given, students may not catch on quickly" (Schroeder, 1993, p. 11). The general idea of mathematical connections is probably valued by most mathematics teachers and researchers, but it often seems to be taken for granted. This research shows that the notion of mathematical connects should remain problematic for both researchers and teachers.

The results of the interviews with students suggest that this problem is a very appropriate one for students in Grade 10, and probably also for younger students since its mathematics content demands are typically met several grades earlier. Either as an assessment tool or as a learning activity this problem provides rich opportunities for students to demonstrate and/or develop their abilities to reason mathematically and communicate mathematically using arguments concerning mathematical necessity and the consequences of various hypothetical choices. For these reasons, this problem should be considered a worthwhile mathematical task. However, as I have argued elsewhere, "Further work to develop additional appropriate tasks for assessment and for instruction [concerning mathematical connections and reasoning] would [still!] be warranted" (Schroeder, 1993, p. 11).

References

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Schroeder, T. L. (1992). *Qualities of problem-solving: The report of an assessment project*. Unpublished report. British Columbia Ministry of Education, Victoria, BC.

Schroeder, T. L. (1993, March). *Mathematical connections: Two cases from an evaluation of students' mathematical problem solving*. Paper presented at the NCTM/SIG/RME Research Presession, Seattle, WA.

GLOBAL THINKING "BETWEEN AND WITHIN" FUNCTION REPRESENTATIONS IN A DYNAMIC INTERACTIVE MEDIUM.

Zipporah Resnick
Science Teaching Department
Weizmann Institute
Rehovot, Israel

Baruch B. Schwarz
School of Education
Hebrew University
Jerusalem, Israel

Rina Hershkowitz
Science Teaching Department
Weizmann Institute
Rehovot, Israel

This paper describes some cognitive development of students exposed to dynamic interactive media about functions. After a few sessions, ninth grade students worked in groups of four on a problem (the Fence Problem Situation). One week later, individuals were asked to reflect upon their problem solving process by answering questions pertaining to the Fence Problem Situation (in a worksheet format). We analyze the actions undertaken by the groups especially the conditions under which students decided to turn to another representation, and to keep on working within a particular representation. The answers produced by the individuals served to analyze the mental representations formed through the problem solving sessions. Finally we attempt to study links between those mental representations and the particular actions undertaken during the problem solving session.

1. Introduction

In the following we describe some cognitive aspects of the process of learning the function topic in a dynamic interactive medium. This work is part of a project whose main goal is to develop and implement a multi-media based mathematics curriculum. The project has three components:
i) the development of a set of activities in different mathematical topics; ii) research on learning processes and on the role of the teacher and students within a dynamic media environment;
iii) implementation at different levels.
At present, three pilot schools participate in the project; the members of the research team, some of whom are teachers, negotiate and redefine the curriculum and implement it in the classroom.

One of the main topics is *functions*. Various kinds of activities are integrated and embodied in different media: work with graphical calculators, with software about functions, worksheets, and verbal interactions (between teacher and students, and in small groups). Most of the activities are open-ended problem situations in which the student is encouraged to make his/her own decision about which representation/s to use, when and how to link them together and in which medium (e.g. paper and pencil, graphical calculator, combinations of both of them, discussion in teams, etc....).

The computer tools enable dynamic linkage between the three representations (algebraic, graphical, and numerical), and actions within each of them (e.g., drawing, scaling, plotting, numerical and algebraic computing). The linkage between the different representation systems is asymetric, in the sense that in order to draw a graph, one must pass through the algebraic representation whereas an algebraic rule cannot be defined by a graph (except for linear functions).

The general approach in the design of the activities is to reduce to a minimum the formal introduction of new topics. Students are exposed to problem situations, and encouraged to use their own intuition. Most teacher interventions are aimed to foster their function sense through the integration of information from different media.

In the following we describe several kinds of reasoning around a problem situation (called the *Fence Problem* described below) given to ninth grade girls (n=36) in one of the pilot schools.

2. Methodology

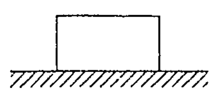
The Fence Problem was given after four sessions of activities about functions with graphical calculators (TI-81). The girls worked in nine groups of four, with the graphical calculators available. We observed the work in the classroom and then collected written group reports and analyzed them, taking into consideration the group's computer actions. In the second stage, the girls worked individually at home without graphical calculaters on a worksheet also based on the Fence Problem. Its design was subsequent to the analysis of the group reports and invited each individual to reflect upon her own mental model constructed during the first stage. We collected these individual report and they were also analyzed. In the following we analyze samples of cognitive activity from both the team and the individual reports.

This methodology is a variant of the recall technique used to evaluate comprehension in discourse processing (Kintsch, 1986) and in arithmetical word-problems (e.g., Mayer, 1982; Cummins, Kintsch, Reusser, & Weimer, 1988; Schwarz & Nathan, 1993). It is based on the assumption that when solving a problem (or reading a text), the solver constructs a representation of all the events, objects, or components of the problem (or of the text). Kintsch calls this mental construction the *problem model*.. When recalling a text or a problem long enough after reading or solving it, text based representations (in reading) and computations (in problem solving) are generally erased from the memory and readers or solvers tend to recall or reconstruct the problem model (see Kintsch, 1983). Consequently, researchers can analyze the problem model by using a recall technique, and infer the understanding of the problem (or of the text).

The homework we gave one week after the problem-solving session required the students to reflect upon the whole solution process and to continue this process.

The Fence Problem Situation

Oranim School bought a 30 m. long wire fence with which to enclose a rectangular garden vegetable lot. The lot is bounded on one side by the school wall, so that the fence is required for three sides only.



a. Find four possible dimensions for the lot, and the corresponding areas.
b. What are the dimensions of the lot with the biggest area?
c. One of the dimensions is 11m. What is the area of the lot? Can you find another lot with the same area? If yes find its dimensions; if not, explain.
d. How many lots are there with the following areas: 80m^2, 150m^2?
e. What is the domain appropriate to the problem situation?

3. The Problem-Solving Session

The justifications in the group reports included evidence of actions undertaken within representations and about decisions which led to linkages between representations, as illustrated by the following three examples.

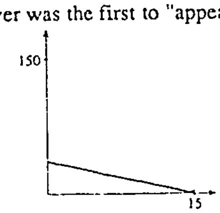
A. From past experience with the activities, it was clear to all groups that:

i) In order to find the dimensions of the lot with the biggest area (question **b** above), they needed to draw the graph of the area function.

ii) In order to draw the graph with their graphical calculators, they had first to enter an algebraic formula.

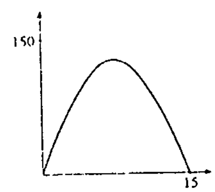
Many groups organized their numerical examples in response to question **a** as a table with three entries: one side, second side, area. Then, they generalized the numerical values in algebraic form: X, 30-2X, X(30-2X). In a few groups they explained very carefully how they proceeded from numbers to algebra, step by step.

Three groups were "in a hurry" to find a formula (the key for drawing a graph), and inserted 30-2X or (30-X)/2 (the formula for the second side) whichever was the first to "appear", and were surprised by the result. For example:



We drew the graph on the graphical calculator, using the formula 30-2x, with a range of 0-15 for x, and a range of 0-150 for y. It did not seem OK because we had to find the area and the formula fitted the second side.

So we understood that the formula was wrong and we decided to replace it by (30-2x)x [there were two arrows pointing to x and to 30-2x and labeled first side and second side] *because this is the area formula. We are looking for the "highest"* [the girls meant the highest on the graph] area.



We got graph that seems to us a more correct one.
Note: we did not change the range only the formula.

The first graph the girls got did not **make** sense because it did not fit the model they had constructed in the numerical representation (question **a**). So they went back to the algebraic representation, corrected it and got what they called later "the correct graph".

B. The area graph in the graphic calculator seems to have a "flat top". So in order to find the largest area, girls did some actions on the graphical presentation - they **changed the** scales in order to magnify the relevant part of the graph, and/or moved between the representations in order to **attune** the information between them. For example:

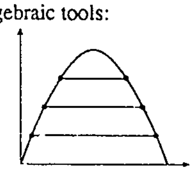
When we drew the right graph, we got a straight line on its top, so we "reduced" [zoom on the neighborhood of the maximum] *the graph (the space of the y-values) in order to find the maximal area of the lot.*
We went up on the graph, and we reached the highest point on it. We read 7.49 on the x-axis and we substituted this value for x, in the equation, so we got:
*(30-2*7.49)*7.49 = 112.49.*
We tried this also on the graph and we reduced the x-axis, and then the y-axis, so we reached the highest point, and it yielded exactly the same as with the formula.

C. In question **d**, some groups gave global reports lacking accuracy, within the graphical representation:

Because the graph goes up and then down it is obvious that there are lots with the same area. [and they drew

Some were more specific, and tried to tackle this problem with algebraic tools:

*We tried to solve (30-2x)*x=80, but we couldn't do it, so we turned to the graph. When the area is 80m^2, then the side can be 11.526 and can be also 3.4736. There are two lots whose area is 80m^2.* [and they drew



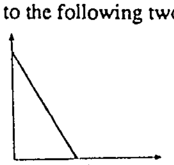
In order **to avoid an impasse** (their inability to solve the equation), the girls moved from one representation to the other.

We can conclude that the passage from one representation to another was induced by three needs: i) to make sense of the problem to be solved (first example); ii) **to attune** the information between representations (second example); iii) **to avoid an impasse** within a particular representation (third example).

4. Reconstruction of the problem model with the homework activities

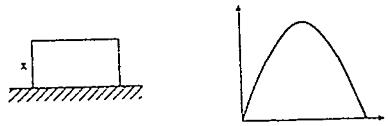
In this second stage the girls were given a worksheet to be solved individually at home, and were asked to justify their answers. Their justifications were based on their *problem model,* which could be characterized by the *mental representations* students evoked to answer the questions in the worksheet. We found some properties of those *representations* by analyzing their written justifications, as illustrated by the discussion of their responses to the following two questions:

Question 1. Some students *drew the straight line as the area graph. Most of these students said that this graph does not fit the area . Explain why. Give as many reasons as you can.*

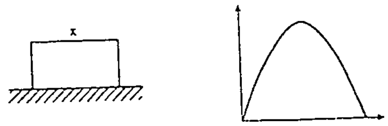


As mentioned above three groups drew this graph on their way to the right graph. In this question we wanted to learn more about "the place" of this graph in the individual problem model.

Question 2. *Some students chose x as the length of the side perpendicular to the wall.The formula for the area is then (30-2x)*x and the graph looks like this:*



*Other students chose x for the length of the side parallel to the wall. The formula for the area is then (30-x)/2*x and the graph looks like this:*



If we draw the two graphs in the same coordinate system, do they overlap? Give as many reasons as possible.

One of the justifications given by girls was discrete in nature. For example:

In the area graph when x=0, then the area is 0 and in the linear graph, when x=0 the area is not 0.

Such an approach is natural and simple. However, only few students used it, and among those that did, more often than else, it was intended as a check that an assumption was correct, rather than to construct a justification on it. For example, in response to the first question:

From the graph ,the largest area can be $30m^2$ and this we can show to be nonsense. For example: we found that when one side is 7, the area is 112, meaning that there is an area bigger than 30 m^2.

In general, students almost always relied on the work they did one week before to articulate their answers (a phenomenon which justified our methodology).

The second kind of justification was very **global**. It uncovered a **constraints and affordance aspect**[1]. It was diagnosed on the graphical representation, because the two questions we asked were graph-centered. In such justifications, the graph is obtained as a series of constraints dictated by the problem data . These constraints limit the general form of the graph, delineate (or attune) through which points the graph must pass, or where it increases. For example, for the first question:

We know that if a side is zero, there is no area (that is, it's not included in the domain); also if the side is 15, there is no area (it's not included in the domain). So I know that between 0 and 15, there must be an arc 0∩15. Because for 1 the area increases, and so on 2, it increases until it must go down in order to get to 15 (where there is no area). And on this graph, it always goes down.

---

[1]This component was recognized by Greeno, Engle, Kerr, and Moore (1993).

down in order to get to 15 (where there is no area). And on this graph, it always goes down.

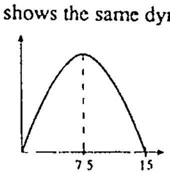
In another kind of justification, the **dynamic-variational** aspect of functions was present. Here, a function is grasped as describing a continuous phenomenon whose change for each value is considered relatively to the change obtained for contiguous values. The student compares these changes using various tools, such as rate of change, or infinitesimal modification of the pre-image in the algebraic or "real world" representation. An example of this kind of explanation for the first question is:

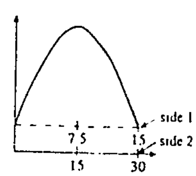
According to the table, the formula and previous examples, we can tell that this is not the correct graph because the y-axis is used to represent the area and I already know that it is not reasonable that a straight line goes down, I mean, that the differences between the areas go down at a constant rate.

Another more sophisticated example is:
The graph does not seem to me very reasonable, because the results do not seem reasonable. The graph cannot be a diagonal line but a curved one because there is one side and another one, and as a result the area increases and drops; for example, 1 28, 5 20-here the jumps are substantial- 14.5 1-here the decrease is very slow.
[Here, the student notices that the rate of change itself is not constant]

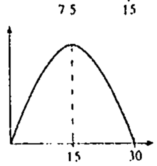
The following justification for the second question shows the same dynamic-variational aspect.

Yes. The graph is the same. The second side is with jumps of two, whereas the jumps for the first side are of one. So the graph of the side 1 is like that,





The graph for the second side is in fact the same, the jumps on the x-axis being of two.



[the student shows a table with the values of the first and the second side].

In another kind of justification (very common in the problem solving), students made up their mind about the problem by **integrating information** from different representations. The following excerpt shows this integration aspect for the answer to the first problem.

According to the given graph, the longer the length of the side, the smaller the area, and that can't be. According to the table, first the area increases, until one reaches the maximal area, when the one side is 7.5 and the other is 15 and this is half the total length of the fence: from there, the second side decreases as the first side increases and then the area begins to go down until it reaches zero

Of course, more then one of the aspects reviewed above is often present in a student's explanation. For example, in the second discrete example there is also a constraint and affordance aspect (when the student asserts that there is an area greater than $30m^2$) in addition to the discrete aspect.

Again, a student might use global considerations to decide that the two graphs have the same general form (constraint-affordance aspect), and discrete considerations to decide that the (same) maximum is obtained for different x-values. Thus for example:

No. The graphs will not fall one on the other because for each set, the position of the x is different. For example, when we use the formula (30-x)/2 and we put 15, we obtain 7.5, but because the second formula is 30-2x, the pre-image x is in a different place.

Students' justifications reveal different global aspects of functions which were used during the problem-solving sessions and made visible by the homework reports. These aspects indicate that the students developed rich *mental representations* for solving problems about functions.

5. Conclusion: some considerations about learning processes with dynamic interactive media

Dynamic interactive media for the learning the function concept differ from other environments in the new status they confer to their *external representations*. Graphical and tabular representations are traditionally *display representations* in which it is possible to discover properties by solely interpreting them. In dynamic interactive media, they become *action representations*, similar to the algebraic representation (Kaput, 1992). The fact that it is possible to make links between representations, and to follow up the parallelism of actions within representations is at the root of student learning (Kaput, 1992; Schwarz & Dreyfus, in press). Our research indicates that students acquire rich and varied kinds of *mental representations* of functions in a very short period of time.

A very general aspect of functions which students evidenced when solving the Fence problem and during their homework, is that information collected from several representations signify the same entity - the function. For this reason passage to another representation is often intended to refine (attune) information which is not accurate enough or available in a particular representation. It addition it is clear to them that in this passage the entity properties will remain, because properties of functions are invariant across representations. For example, when they can't solve $(30-2x)x = 80$ they turned to the graphical representation to look for the unknown x value.

Another aspect students used in their work is that functions are quickly grasped as dynamic-variational processes. As seen in one of the examples above, students considered rate of change in a global way, in contrast to the "en bloc" static view of graphs or formulae.

Functions were also perceived as entities which satisfy global constraints according to the problem to which they refer. The student's problem model of a problem-situation is partly built as a system of constraints (for example, the fact that the dimensions of the lot are between 0 and 15), and affordance (for example, the fact that the area function will first increase from zero, then decrease, until it reaches zero for y=15).

The analysis of the problem solving session gives some clues about the origins of the formation of such rich *mental representations*. We saw that moving from one representation to another is triggered by several causes: the learner seeks to make sense of the problem, to attune the information, and to circumvernt impasses. We believe that, for example, "moving to another representation to make sense of a problem" causes, in turn, students to integrate representational information. Similarly, it seems reasonable that grasping the dynamic-variational aspect of function may stem in part from the fact that "searching for a maximum by algebraic computation often leads to an impasse". The program of the current research is to better characterize the links between the acquisition of these mental representations and the decisions of the learner during problem solving. From the preliminary results reported here, it is clear that the learner *constructs* very rich representations by interacting with dynamic media, and that their use is highly flexible in problem-solving activities.

Bibliography

Cummins, D., Kintsch, W., Reusser, K., & Weimer, R. (1988). The role of understanding in solving word problems. *Cognitive Psychology*, 20, 439-462.

Greeno, J. G., Engle, R. A., Kerr, L. K., & Moore, J. L. (1993). Understanding symbols: situativity-theory analysis of constructing mathematical meaning. In *Proceedings of the Fifteenth Annual Meeting of the Cognitive Science Society*, 504-509, Boulder, CO.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, 515-556, NCTM.

Kintsch, W. (1986). Learning from text. *Cognition and Instruction*, 3,

Mayer, R. E., (1982). Memory for algebra story problems. *Journal of Educational Psychology*, 74, 199-216.

Schwarz, B. B. & Dreyfus, T. (in press). Measuring integration of information with a multi-representational software. *Interactive Learning Environments*.

Schwarz, B. B. & Nathan, M. J. (1993). Assessing conceptual understanding of arithmetic structure and language. In *Proceedings of the Fifteenth Annual Meeting of the Cognitive Science Society*, 912-917, Boulder, CO.

MATHEMATICAL PROOF AS A NEW DISCOURSE.

AN ETHNOGRAPHIC INQUIRY IN A JAPANESE MATHEMATICS CLASSROOM

Yasuhiro Sekiguchi

University of Yamaguchi, Japan

The study investigated the practices of teaching and learning of mathematical proof in a mathematics classroom. The data was collected in geometry lessons of an eighth grade mathematics class in Japan, using ethnographic methods. Preliminary analysis of the data indicates that basic processes underlying the instruction of mathematical proof is an introduction of a whole new mathematical discourse. As pedagogical device, "fill-in-the-blanks" proof was frequently used by the teacher. It functioned as "scaffolding" tool to guide leaning of writing proof, as well as instructional management device. Social nature of learning of proof, and problematic aspects of "scaffolding" practice in the classroom are discussed.

Mathematical proof has taken important part in mathematics curriculum, and many efforts have been done on its teaching and learning. Though the concepts of proof underlying those efforts were often that of formal proof, students need more experience in which they reason in less formal mathematical systems as recent van Hiele theory indicates (van Hiele, 1986). Furthermore, a stress on formal proof may distort the dynamic processes of mathematical inquiry (Hanna, 1983). Recent studies in the philosophy and history of mathematics have proposed a sociological approach to mathematics (Bloor, 1976; Lakatos, 1976; Tymoczko, 1985, Wittgenstein, 1978). They stress the importance of informal mathematics in mathematical inquiry in socially interactive situations.

In addition, though many studies on learning of proof were conducted in the past, they were not necessarily sensitive to the actual classroom contexts. Only few studies (e.g., Balacheff, 1991; Lampert, 1988; Moore, 1991; Schoenfeld, 1988; Tinto, 1990) have been conducted in the actual classroom. The present study was an exploratory attempt to understand the practice of teaching and learning of proof in the actual classroom, focusing on its informal and social nature.

The author conducted an ethnographic study on mathematical proof in 1989-1990 in the United States and reported in Sekiguchi (1991, 1992). The present paper is based on a new study that took place in a culturally different setting, in Japanese junior high school classroom, during 1992-1993, in order to further understanding of the processes underlying the instruction of proof.

Description of Study

The study used an ethnographic approach. It enabled the investigator to access data on the actual practice of proof in the classroom, the context where the proof was situated, and the participants' perspectives. Because proof is first introduced to the students in the eighth grade (the second year of junior high school) in Japan, I selected one eighth-grade mathematics class at a junior high school in southwest of Ibaraki prefecture in Japan. The participants were the teacher and the students (36 students) in the class. The class was observed throughout the second half of the school year (October 1992-March 1993). Interviews of the participants and collection of their writings were often conducted to supplement the observational data. The lessons were audio- and video-recorded. The interviews were usually audio-recorded. During the period of data collection those records were used to write field notes, perform informal analyses of the data, and choose a focus for the next data collection. The records of lessons were also occasionally used in an interview to help an interviewee recall an incident that had occurred in class. After the data collection ended, the records were partially transcribed and used for qualitative analysis (LeCompte & Preissle, 1993) The analysis has not yet been fully completed. The present paper is written based on the preliminary analysis of the data taken mainly from introductory lessons on proof.

Results

The concept of proof is first introduced to the students in the geometry lessons of second year mathematics of junior high school in Japan Though the students did certain kinds of justification and explanation in previous mathematics lessons, proof is not discussed there.

<u>Explaining Why?</u>

After finishing the chapter on application of linear functions in 12 October 1992, the teacher began to deal with geometry. The class discussed the basic properties of vertical angles, parallel lines, angles of polygons, the conditions for congruence of triangles nearly two months. During that period, the terms "proof," "definition," "theorem" were not used (the terms "axiom" or "postulate" were never used throughout the year). The basic properties of figures were introduced to the class after working on examples The teacher encouraged the students' discovery of those properties. At the same time, he encouraged them to "explain" properties of figures using what the students had already learned. Inductive and empirical reasoning using ruler and protractor came to be discouraged in explaining. He frequently asked the students "why?" and requested to give "a reason."

For example, the first topic was vertical angle. The teacher drew two intersecting lines on the board, and asked the students how many angles there were around the intersection (Figure 1).

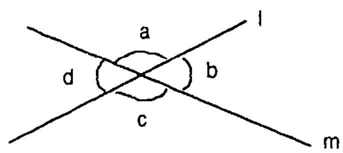


Figure 1.

Students pointed out the 13 angles: a, b, c, d, a+b, b+c, c+d, d+a, a+b+c, b+c+d, c+d+a, d+a+b, a+b+c+d. The teacher asked them which angles were equal each other. A student suggested that angles a and c, and angles b and d are equal respectively. Another student pointed out that a+b and b+c were equal. After noting a+b and b+c both were equal to 180 degrees, the teacher asked the students to work on a problem:

Using $\angle a + \angle b = \angle b + \angle c$, let's explain why $\angle a = \angle c$.

Students found that by subtracting $\angle b$ from both sides they got $\angle a = \angle c$. The teacher then concluded that "when two lines intersect, the angles facing each other at the intersection are equal." He told that those angles were called vertical angles, and reformulated the above conclusion, "Vertical angles are equal."

<u>Introducing "Proof"</u>

In 9 December 1992, the term "proof" was introduced to the class. The topic of the lesson was how to use the conditions for congruence of triangles. The teacher first presented the problem (see Figure 2):

"In the figure, when AM = CM and BM = DM, explain that AB = CD" (the point M is at the intersection of segments AD and BC). He asked the students to solve it using conditions for congruence of triangles.

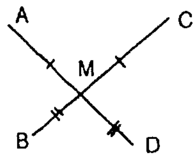


Figure 2

After the teacher connected AB and CD by segments, students noted that ∠AMB and ∠CMD were equal because they were vertical angles. The class then found that Δ ABM and Δ CDM were congruent because "two pairs of sides and their included angles are respectively equal," concluding that AB = CD.

After that discussion the teacher wrote an explanation on the board and asked the students to copy it and fill in the blank by themselves:

In Δ ABM and Δ CDM

AM = CM . . . (1)

BM = DM . . . (2)

Since vertical angles are equal,

∠AMB = ∠CMD . . . (3)

From (1), (2), and (3), because __________ are respectively equal,

Δ ABM ≡ Δ CDM

Since corresponding sides of congruent triangles are equal, AB = CD.

T (the teacher): "Later, we're going to study 'proof.' I want you to remember the writing style like this."

He then presented another problem that asked to explain using conditions for congruence of triangles (Figure 3): "In the figure, when AB = DB and AC = DC, explain that ∠ABC = ∠DBC."

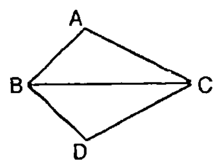


Figure 3.

Students were stuck after writing down "AB = DB . . . (1)" and "AC = DC . . . (2)." The teacher went to the board and asked the class what other parts are equal in the figure.

S1: "∠A = ∠D."

T: "This and this are equal. Well, they appear to be equal. Uh, what would you do if you're asked why so? They are equal, equal, but, if you're asked why they're equal, uhm. S2, what do you think?"

S2: "BC."

T: "Okay? BC and BC are equal, aren't they? Obvious, isn't it? Because they are common. They are sticking together. Uh, we write it like this."

Since they are common, BC = BC.

Using "three pairs of sides are respectively equal," the class finished the problem.

The teacher then introduced to the class the term "proof" as arguing orderly the reasons why a statement is true by relying on what have already been accepted as true, pointing out that the explanations they got in the above problems were proofs (Note: Proof in 8th grade mathematics is written in paragraph form in Japan). The teacher further introduced the terms "Assumption (or Hypothesis)" and "Conclusion," and reformulated proof as deducing a conclusion from assumptions.

The above lesson indicates several important themes of learning proof, which were recurring in later lessons. First, proof is considered "explanation." Second, statements mentioned in proof must be backed by "reasons" even if the statements are actually true. Appearance is not enough to support a statement. Third, writing style is important. The statements used in a proof should be written down (and numbered) whether they are "obvious" or not.

Viewing a Statement as Conditional

The class began to work on more proof problems. The students experienced serious difficulties in identifying Assumptions and Conclusion from a given statement. The teacher often stressed in class the importance of correct understanding of Assumptions and Conclusion in proof: "Being able to solve a proof problem depends on whether you have solid understanding of the Assumptions and Conclusion. Basing on what, and what to prove? If you're confusing them, the proof will make no sense to you at all." When a statement is written in "if-then" form, the students did not have much troubles. But, when it is not in "if-then" form, they became uncertain. Viewing a mathematical statement as conditional seemed not to be trivial for the students. For example, consider a statement "the areas of two congruent triangles are equal." Students tended to identify the Assumptions with the subject (e.g., "the areas of two congruent triangles"), and the Conclusion with the predicate (e.g., "[are] equal"). (They conceived that Assumptions should be phrased using words contained in the statement, and the assumptions and the conclusion should be taken from different parts of the statement.) Also, they sometimes took Assumptions as "reason" for Conclusion. For instance, when asked to write Assumptions and Conclusion of a statement "an angle of equilateral triangle is 60°," some students wrote "the sum of the angles of equilateral triangle is 180°" as Assumption, and "one of the angles is 60°" as Conclusion. They added to Assumptions new information ("the sum of the angles of triangle is 180°") used to prove the statement.

In addition, when proof was involved in, finding of Assumptions and Conclusion got another complication. When proving a general statement like "the base angles of isosceles triangle are equal," it needed to be reformulated in terms of the symbols labeling the given figure. To make the reformulation, definitions of main terms used in the statement needed to be recalled. For example, in the above statement, the definition of isosceles triangle was needed in order to state "AB = AC" (in the given isosceles triangle ABC) in Assumptions. But some students did not memorize definitions well. Also, when a proof was using an auxiliary line, students tended to put a property of the line into Assumptions because it was used as reason in proof.

Using and Writing Reasons

The problems the students had worked this year before geometry were mostly of algebraic nature They were "problems to find" (Polya, 1973) and involved computation mostly In algebraic problems the students just needed to show their calculations In geometry, they suddenly faced "problems to explain" and "problems to prove," and required to tell and write "reasons" with words When they could not find appropriate "reasons" for a statement, the teacher told them, "We can't *say* that," and did not accept it into a proof The students had to learn a new way of discussing and writing mathematics, a new discourse of mathematics

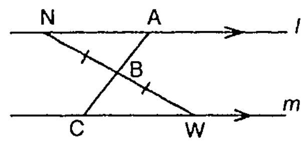
They needed to know the kinds of "reasons" and ways to phase them that were allowed in proof The students came to learn that the appearance of figures and the measuring results were not acceptable in explanation, and get accustomed to using "properties of figures" in explanation When proof was introduced, the students first learned to use Assumptions and then the conditions for congruence of triangles as reasons Later, some proof required to draw an auxiliary line. The students had to learn where to write about the auxiliary line and how to phrase a property of the line as reason in proof. As they learned several theorems, they had often troubles not only on how to apply them but also how to write them in proof They also seem not to be accustomed to remembering precise wordings of theorems, rather different from remembering algebraic formulas.

Finding and Writing Proof

There were two major goals when dealing with a proof problem. Finding and writing a proof. When presenting a proof problem, the teacher usually explained it drawing figures at the beginning If the problem had been assigned as homework, he sometimes asked one student to put his or her proof on the board and the teacher explained it. In other cases, the teacher had discussion with the students about how to prove the problem, using figures and jotting down informal memos on the board. The discussion may stop when they got strategies to find a proof, or may continue until finding an outline of proof. Then, the teacher asked the students to write a proof by themselves on their notebooks, and helping them individually. After a while, he put "fill-in-the-blanks proof" (see Figure 4) on the board as a hint or guide of writing a proof, saying often "your proof doesn't have to be exactly like this." After seeing many students finishing, the teacher filled in the blanks together with the students, and ended the problem.

"Fill-in-the-blanks proof" (let me abbreviate it as FBP), a well-known pedagogical device, is a form constructed based on a complete proof, simply by replacing several parts of the proof with blanks. It was used frequently in teaching of proof, appearing in the textbook, lessons, "minitests" (5-minute review test before lesson), and term exams. The teacher's uses of FBP in lesson seemed to be contingent on his perception of students' difficulties. In the beginning of teaching proof, the teacher put FBP on the board soon after the class finished discussing how to find a proof, and asked the students to fill it in by themselves. But, later lessons, he tended not to put FBP until students made substantial efforts to write proofs by themselves. Then, sometimes the teacher asked the students to write a whole proof by themselves: "You should be able to do now without fill-in-the-blanks proof."

In the figure, when l // m and NB = WB, prove NA = WC



<Assumptions>

<Conclusion>

<Proof> In Δ NBA and

From the assumptions, . (1)

From the assumptions, l // m, and because of of parallel lines,

$\angle$ BNA . . . . (2)

And, because are equal, $\angle$ NBA = . (3)

From (1), (2), and (3), since are respectively equal,

Δ NBA ≅

Since the corresponding sides of congruent triangles are equal, NA = WC.

(11 December 1992)

Figure 4. Fill-in-the-blanks proof

However, students' difficulty was not the only factor for the use of FBP. Time constraint of lesson was also important one. After explaining a problem and discussing Assumptions and Conclusion, the teacher sometimes told the class: "Since we don't have much time, we've got to use 'fill-in-the-blanks'." Then he put a FBP on the board, and asked the students to write a proof using it. The teacher seemed to feel a tension between giving the students ample opportunity of writing a proof by themselves and "getting through" the lesson.

In proof problems in minitests and term exams, FBP were pervasively used. The teacher believed that writing a complete proof by themselves would be too difficult for the students, especially, in limited time like in test. Actually, filling out FBP was much easier for the students. They told me in interviews, "Fill-in-the-blanks is easy, . . . because it helps me to go to a [right] direction." In addition, the use of FBP in proof problems reduced a great burden of grading from the teacher because it limited proof writing uniquely and the teacher needed to check only the answers in the blanks.

Discussion

This study is an attempt to analyze the current practices of teaching and learning mathematical proof and identify its underlying processes. The data was collected in geometry lessons of an eighth grade mathematics class in Japan, using ethnographic methods. Analysis indicated that basic processes underlying the instruction of mathematical proof is an introduction of a whole new mathematical discourse. As pedagogical device, "fill-in-the-blanks" proof was found to be frequently used by the teacher. It seemed to be functioning as a supporting tool to guide leaning of writing proof, as well as instructional management device.

The metaphor "mathematics as language" has come to be recently used in mathematics education community. Viewing learning mathematics as acquisition of language or developing communication ability seems to provide useful implications to mathematics education (NCTM, 1989; Pimm, 1987). The present study indicated that learning of proof involves an introduction of a new mathematical language system, qualitatively different from previous systems: It required for the students to follow and internalize new ways of interpreting, arguing, and writing mathematics. The difficulties that the students experienced seemed to be not only of cognitive nature but also of social nature because the students had to accept values of the new system and follow new rules of communicating mathematics in the classroom. Further classroom research is necessary to understand these social processes in learning of proof.

The FBP found in this study seemed to be used as "scaffolding" device by the teacher to aid the students' learning of writing proof. The "scaffold" is a metaphor, first used by Bruner and his colleagues, in order to capture the adult's (or teacher's) supporting role in child's learning. In that model, the teacher "provides a supportive tool for the learner, which extends his or her skills, thereby allowing the learner successfully to accomplish a task not otherwise possible" (Greenfield, 1984, p. 118). And, as the learner masters the task, the teacher gradually withdraws the intervention, and release the full responsibility to the learner (Wood, 1980, p. 282).

But, the teacher's use of FBP seemed to indicate problematic aspects of "scaffold" in the classroom instruction also. Ideally speaking, the scaffold should be constructed and used contingent on individual student However, the teacher in this study had to handle 36 students, with varying abilities, at the same time in limited period (This situation is normal in Japan). Though he tried to help the students individually when circling among them, some of them often had difficulties in filling out FBP Also, the teacher appeared to have overused FBP because of instructional management purposes. This seemed to have led some students to rely on FBP too much, and fail to develop the ability of writing a full proof. We need to further explore effective ways of "scaffolding" of learning of proof in the classroom (cf. Cazden, 1988, pp. 107-110).

References

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. J. Bishop, S. Mellin-Olsen, & J. V. Dormolen (Eds.), Mathematical knowledge: Its growth through teaching (pp. 175-192). Dordrecht, The Netherlands: Kluwer Academic.

Bloor, D. (1976). Knowledge and social imagery. London: Routledge & Kegan Paul.

Cazden, C B. (1988). Classroom discourse: The language of teaching and learning. Portsmouth, NH: Heineman

Greenfield, P. M. (1984) A theory of the teacher in the learning activities of everyday life In B. Rogoff & J. Lave (Eds.), Everyday cognition: Its development in social context (pp. 117-138). Cambridge, MA: Harvard University Press.

Hanna, G (1983) Rigorous proof in mathematics education. Toronto, Canada: Ontario Institute for Studies in Education.

Lakatos, I. (1976). Proofs and refutations. Cambridge, England Cambridge University Press.

Lampert, M (1988). The teacher's role in reinventing the meaning of mathematical knowing in the classroom In M. J. Behr, C. B. Lacampagne, & M. M. Wheeler (Eds.), Proceedings in the 10th annual meeting of North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 433-480). DeKalb: Northern Illinois University, Department of Mathematical Science.

LeCompte, M D., & Preissle, J. (1993). Ethnography and qualitative design in educational research (2nd ed). San Diego, CA: Academic Press.

Moore, R C. (1991) College students' difficulties in learning to do mathematical proofs. Dissertation Abstracts International, 51, 3354A. (University Microfilms No. 91-07, 213)

NCTM. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: NCTM.

Pimm, D. (1987). Speaking mathematically: Communication in mathematics classrooms. London: Routledge & Kegan Paul.

Polya, G (1973). How to solve it (2nd ed.). Princeton, NJ: Princeton University Press.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. Educational Psychologist, 23, 145-166

Sekiguchi, Y. (1991). An investigation on proofs and refutations in the mathematics classroom. Dissertation Abstracts International, 52, 835A-836A. (University Microfilms No. 9124336)

Sekiguchi, Y (1992). Social dimensions of proof in presentation: From an ethnographic inquiry in a high school geometry classroom. In W Geeslin and K. Graham (Eds.), Proceedings of the Sixteenth Annual Conference of the International Group for the Psychology of Mathematics Education, Vol. 2 (pp. 314-321). Durham, NH

Tinto, P P (1990). Students' views on learning proof in high school geometry: An analytic-inductive approach Dissertation Abstracts International, 51, 1149A

Tymoczko, T. (Ed.) (1985). New directions in the philosophy of mathematics. Boston: Birkhäuser

van Hiele, P M. (1986). Structure and insight: A theory of mathematics education. Orlando, FL: Academic Press.

Wittgenstein, L (1978) Remarks on the foundations of mathematics (2nd ed.) (G. H. von Wright, R. Rhees, & G E. M. Anscombe, Eds G. E. M. Anscombe, Trans.). Cambridge, MA: MIT Press.

Wood, D J (1980) Teaching the young child: Some relationships between social interaction, language, and thought. In D R Olson (Ed.), The social foundations of language and thought (pp. 280-296). New York: W W. Norton.

RESPONSES TO VIDEO IN INITIAL TEACHER EDUCATION

Michelle Selinger, The Open University, UK

The different reactions to video of two lesson extracts illustrating the practice of two experienced mathematics teachers was collected from three distinct groups, tutors from initial teacher education institutions, students in initial training and recently qualified practising teachers There were substantive differences in the responses from the practising teachers compared to the other two groups The implications this has for the current movement in the UK towards a greater emphasis on school-based teacher training reinforce many of the current concerns held by those involved in teacher education.

BACKGROUND

There is currently a strong government-initiated push to school-based teacher training in the UK in which practising teachers will take a lead role in training students to teach their specialist subject (CATE 1992). Although moves towards more direct involvement of schools in initial training are welcomed by many involved in teacher education, there is concern that students' experience will be limited by the localised experience they will receive from working closely with one teacher. The advantages of a higher education-based traditional teacher education course in which students spend more time reviewing a wide range of practice and discussing the relative merits of different ways of working will be lost (e.g. NAHT 1993, SCETT 1992). There is also concern that experienced teachers will be drawn away from teaching pupils to teaching students. However many mentors in school are often teachers with limited teaching experience although they are often supported by more senior members of staff. It is this use of recently qualified teachers in initial teacher education to which this discussion is addressed. The comparative responses to video of the classroom practice of two experienced mathematics teachers between representative groups of recently qualified teachers, tutors in initial teacher education and a group of students in initial training was analysed.

VIDEO IN THE OPEN UNIVERSITY PGCE

Within the framework of increased emphasis on school-based work, the Open University has launched a part-time, distance taught Postgraduate Certificate of Education (PGCE) funded by the Department for Education (DFE). The first 1300 students started this 18 month course in February 1994; of these approximately 180 are secondary mathematics students and 450 are primary students. In line with new teacher training criteria (DFE 1992) students spend a total of 18 weeks in school. This is less than is required at present for students on full-time one-year courses in which the minimum requirement of 24 weeks in school represents two-thirds of the PGCE course.

The Open University course is multimedia using print based material, audio and video tape, computer disk based materials and computer conferencing. In addition students attend tutorials and day schools at regular intervals throughout the course. The video element comprises three video cassettes each linked to the three stages of the course and they contain some generic as well as subject specific materials. The subject specific element was filmed according to broad criteria so that students from different curriculum areas would be working on similar aspects of teaching. In the second stage video each subject domain looks at the development of one or two lessons. Each video extract is accompanied by print-based tasks which invite the reader to analyse some part of the lesson they are viewing and perhaps to look a -elated material like the teacher's lesson plan, examples of pupils' work or extracts from the departmental schemes of work. Viglietta (1992) highlights the importance of written material to support the video extracts and found that 'where teachers were exposed only to the video this had little effect on teacher's attitudes and behaviour'.

In the second stage video for the secondary mathematics students two lessons are shown each giving extracts from the beginning, middle and end of each lesson. Students have access to a lesson plan for each extract. The first teacher is working on the 'function game' (see Hewitt, 1990) with a year 8 class (12-13 year old pupils) in a top set. The pupils and teacher are working in silence and all the action is indicated by the teacher or a child passing a pen to another and encouraging them to write the answer that fits the function as shown below.

2	→	4
3	→	9
4	→	16
10	→	100
n	→	

Pupils then work in fours making up their own functions, each pair in a four is setting enough mappings for the other pair to find the function. The teacher walks around the class and we see her talking to two different pairs of pupils about their work.

In the second video a mixed ability class in year 8 are describing to their teacher what they know about angle. This lesson is very unusual in style, employing a strong constructivist approach and working from the pupils and what they can say about what they know and understand by the word 'angle'. The teacher is trying to find out if these pupils perceive angle as static or as a measure of turn and he does this by prolonged questioning of one individual, then another avoiding any verbal or facial clues as to their correctness. He asks each pupil, "What does the word 'angle' mean to you?"

In making the selection of teachers and lessons to film, consideration was taken of the limited amount of video time available. I wanted to s..ow what I considered to be good practice even though many aspects of the lessons could possibly appear unattainable to students or newly qualified teachers; in both extracts the teachers required their pupils to be very quiet in order to teach to their plan. Yet at the same time I wanted to show what they might aspire to, and to illustrate several aspects of teaching I considered to be important. I was showing teachers attempting to gain the best possible idea about what pupils know rather than making assumptions about their knowledge, as well as highlighting questioning techniques that displayed an understanding of the time necessary for pupils to construct verbalisations of their thoughts.

THE RESPONSE OF PGCE STUDENTS

In order to test out the extent to which I had achieved this I showed the video to a group of students in their first term of a conventional one year PGCE course. They were asked to discuss what they thought video ought to show if it was going to be useful to them and then, secondly, to respond to the two video extracts described above. One of their responses to what they wanted video to show was:

> I'd be quite interested to see their lesson plan. It would make it easier to have a framework from which we can judge what they are trying to achieve and if they're achieving it.

This confirmed the view that related print materials were important. Other responses indicated concerns about general teacher/pupil relationships rather than teaching of content:

> It would be useful to see how the same class reacted to different teachers.
>
> I'd like to see a teacher follow a class to see what happens. Do they have different styles when they get to know a class?
>
> That first lesson with children when they come in, perhaps in year 7 or the first encounter with a teacher, how does the teacher carry out the lesson?

The Centre for Mathematics Education at the Open University have developed a style for working with video (Jaworski, undated) Observers are asked to watch an extract ; at the end of the viewing they are asked to replay the video in their minds for a minute or so without speaking. They are then invited to reconstruct what they saw with another person and to come to some shared agreement of the scenario. The emphasis is placed on an 'account of' the action rather than 'accounting for' it. It has been found that by giving an account of the extract, the temptation to judge the teacher's actions is reduced and attention is focused on other aspects of the sequence. In the 'accounting for' judgements are made but with a shared picture of the sequence which can provide additional insights into why the teacher might have responded in a particular way.

After following this rubric, the students' reactions to the the first sequence revealed much of their experience of schools as well as giving them some new approaches to teaching. Many of their responses also moved focus from general areas of teacher/pupils relationships to concerns about the way in which the teacher worked with individual pupils on their learning:

> She didn't really give her much time you know, one question after another. It takes time to remember.
>
> She didn't say, "Does everyone understand?", so you didn't know if there were any pupils who didn't understand.

Although general issues still concerned some:

> You just see this thing where you see the children sitting in their seats and quiet. How did she initiate that?

When asked to describe *one* aspect of teaching that had emerged from the video that the students might use in their own teaching, they showed the hoped for concern about questioning techniques:

> Don't ask a question if you aren't prepared to wait for an answer.
>
> Not to ask totally leading questions but give the pupil a chance to think things through for themselves.

They also focused on certain teaching points:

> I picked up the idea of letting the class set each other's homework.
>
> Getting children to explain concepts and patterns in words instead of going straight to symbols.

> A maths lesson can be very practical, it doesn't have to be based on a text book.

Others noticed aspects of classroom organisation:

> A good reminder of how effective group work can be in motivating pupils and making lessons more interesting.

The second video extract produced more diverse and fierce reaction:

> I felt so uncomfortable watching that lesson. If I'd been there I would have thought, "Please don't pick on me", - not that I didn't know about the angles.

> I did think that it was a good technique and it was really successful because it gets all the pupils thinking all the time.

Some were also concerned about the lack of reassurance the teacher gave pupils and asked questions like:

> I would like to see the same teacher deal with a pupil who obviously didn't have anything to add. What does he do?

As in the previous extracts there were comments about pupil learning:

> I think the lesson provoked more thought outside the lesson as well.

> It's good that he didn't get them to draw on the board They actually articulated. He kept pushing until they had a firmer understanding.

When asked for one thing they had learnt from the second extract some focused on the way pupils learnt concepts:

> Rather than just have a definition of what angle is, have a real concept of what it is. This concept is personal because they come to it themselves rather than just being given it by the teacher.

> Finding out pupils understanding of something before they start looking at a topic on it.

Others focused on aspects of teaching:

> Pupils can be given opportunity to express their own understanding of the subject and to clarify their explanation.

> Effective teaching is thoughtful and probing questions - allowing the children to dictate and shape the lesson.

> I've learned how it can be necessary to give no feedback in order to allow pupils to be totally honest about what they know.

RESPONSE TO VIDEO OF TUTORS AND TEACHERS

In order to test the findings against the responses of those who are most closely involved with teacher education, tutors in higher education institutions and teacher-mentors in schools, the same video extracts were shown and the same questions were asked of two representative groups. The tutors taught a range of initial training courses, not just PGCE and not all taught predominantly secondary students. The teacher group, with the exception of one, were in the first two years of teaching but they were all expecting to work with students in initial training.

The tutors wanted to use video to show what they considered to be excellent teaching and also practice that they were keen to promote but which they rarely saw in schools. They wanted video to illustrate a range of practice since students often gained experience of only two schools during their training. Their responses to the video extracts appeared to identify the same issues as the students, the very short time the teacher in the first video gave for pupils to answer questions, the style of the lesson, and the attempt to develop an open questioning style. In the second video extract, the tutors also reacted strongly. They emphasised the role of questioning to elicit what children knew and understood through articulation. One tutor saw the sequence as 'highlighting the difference between *teaching* mathematics and children *learning* mathematics'. As expected, their responses were more analytical but the issues were mainly the same issues identified by the students themselves.

The teachers' responses however were more varied. The most experienced teacher being the only one to respond in a similar way as the tutors. The other teachers wanted video to highlight the complexity of teaching and to show the end of the day, starts of lessons as well as 'good teaching'. The video extracts were rejected because they 'appeared too sanitised'. They were concerned about the context of the lesson in terms of the school and the catchment area and whether their own year 8 pupils were able to respond in the same ways as illustrated in the video extracts. They also felt the classes were 'unreal' because the pupils were very well behaved and they considered this might put off students who were unlikely to be able to teach in this way immediately and who might set their expectations too high as a result of watching the video. They thought it was very difficult for pupils to 'deconstruct their ideas' although some agreed that working on thought processes was 'a really good idea'. The main concerns of this group of teachers were rooted firmly in their current practice and they were unable to see the merits of the video in illustrating practice one might aspire to or practice in which they might find an alternative approach they could use in their own teaching. They appeared to see the extracts as an entirety and were unable to unpick some of the teaching points that both students and tutors picked up on.

DISCUSSION

Berliner (1994) identified 5 stages of teacher expertise moving from *novice* to *expert*. Tutors in higher education institutions are often experienced teachers who have moved into initial training and can therefore be classed as *expert*. However in Berliner's classification experts are *arational* in that they have 'an intuitive grasp of the situation and a nonanalytic and nondeliberative sense of the appropriate response to be made' (p110). However they are able to bring 'deliberate analytic processes' to bear when things do not seemingly work as planned. In observing other teachers or students working with pupils, these analytical frameworks can be brought into play; they are able to analyse and reflect on the practice they observe. However teachers in the early stages of their careers are developing through the stages and according to Berliner's classification are most likely at the stage of *advanced beginner* in which they are unable to accept 'personal responsibility for classroom instruction' since they are still unable to wilfully make the results of their actions take place. This explains their responses to the video as 'unrealistic' and 'unattainable'. Hoyle's distinction of *restricted* and *extended* professionals also supports this view (Protherough and Atkinson, 1991):

> Restricted professionals are those who see their task as bounded by the classroom walls; they describe themselves as working at the sharp end of the chalk face and make a hard distinction between practice and theory. Extended professionals, without devaluing the work of the classroom, realize that it cannot be separated from wider ideological issues (p. 48)

Tutors and experienced teachers acting as extended professionals will point to the view that training to teach should involve a combination of theory and practice, whereas recently qualified teachers are still in need of ideas that will 'work' in their particular situation and are not yet in a position to be ready to consider the more theoretical aspects of teaching.

The students in this study did not respond as novice teachers as they had not yet had an opportunity to teach and were still unaware of the complexity of the task; they were still open to a range of views of teaching and the basis of their experience was mainly from the viewpoint of a pupil remembering lessons that ran smoothly. Weinstein (1989) reports on students' 'unrealistic optimism' or 'idealism' at the start of their training course. Their reaction to the video therefore could be based on their own experience as pupils as well as their early perceptions of themselves as teachers.

The implications of the evidence presented here for moving to a greater emphasis on school-based teacher training confirm some of the concerns, particularly where inexperienced teachers are used as mentors. Experienced teachers have mastered many of the concerns of their recently qualified colleagues and are able to work on aspects of their practice which rely on an understanding of the way pupils respond in various situations. They have absorbed the ethos of the school and know what to expect from different groups of pupils and they have attained a higher stage of teacher expertise than recently qualified staff. They are more able to reflect on their own practice and the practice of students and are better placed to advise them on developing practice. The use of video between students and tutors or students and *experienced* teacher mentors can possibly enhance student understanding of aspects of teaching, and, because it is detached from any shared experience of all parties involved, it can view the teaching process critically and analytically without the fear of personal criticism.

REFERENCES

Berliner, D. (1994) 'Teacher expertise' in Moon B. and Shelton-Mayes A. (eds) *Teaching and learning in the secondary school*, London, Routledge.

CATE (1992) *Guidance on partnerships between HEIs and schools, accreditation criteria and institutional development planning for ITT*, London, Council for Accreditation of Teacher Education (CATE).

DFE (1992) *Initial teacher training (secondary phase)* Circular no 9/92. London, Department for Education (DFE).

Hewitt, D. (ed) (1990) *Developing Algebra*, Bristol, Resources for Learning Development Unit.

Jaworski, B. (undated) *Using classroom videotape to develop your teaching*, informal publication, Milton Keynes, Centre for Mathematics Education, The Open University.

NAHT (1993) *School-based Initial Teacher Training, Briefing paper*, Haywards Heath, National Association of Head Teachers.

Protherough, R. and Atkinson, J. (1991) *The making of English teachers*, Milton Keynes, Open University Press

SCETT (1992)Responses to the DES consultation document on the reform of initial teacher training, *SCETT News* 21, Autumn.

Viglietta, L. (1992) Making a science education video: a teacher trainer's perspective *Journal of Educational Television*, 18, 2-3, pp. 117-127.

Weinstein,C.S. (1989) Teacher education students' preconceptions of teaching, *Journal of Teacher Education* 40(2), pp. 53-60

A PERSPECTIVE ON FRACTALS FOR THE CLASSROOM

Fernando Sereno

Polytechnic Institute of Oporto
School of Education

ABSTRACT This paper refers to a fractal geometry Logo microworld tested in two case-studies involving seventeen year-old students in a London comprehensive school. Data analysis has provided insight into the ways in which the interdependence between symbolic and visual modes created by the Logo microworld and the strategic planned mediation of the teacher can generate a context that stimulates understanding how randomness creates deterministic shapes (Sierpinski gaskets)

ABSTRAKT: Dieser Essay spricht von einer "Fractal" Geometrie Logo Mikrowelt, die in zwei Studien als Beispiele gestet wurden, mit siebzehnjährigen Studenten in einem London Gymnasium. Data Analyse hat eine Anschauung über die wechselseitige Abhangigkeit zwischen Symbolischen und sichtbaren Arten von der Logo Microwelt und von der Strategischen, geplanten Vermittlung des Lehrers uns gegeben. Die beiden Arten konnen einem Zusammenshang schaffen um zu verstehen, wie Zufalligkeit kann bestimmte Gestalte erschaffen (Sierpinsiki Gestalte)

A New Language

Fractal geometry is considered "first and foremost as a new 'language' used to describe the complex forms found in nature (Peitgen H-O, Jurgens H., Saupe D., Maletsky E. Perciante T., Yunker L.,1992)". These authors argue that the elements of this language, instead of the basic visible forms such as lines, circles and spheres of the 'traditional language' that is used in the Euclidean and non-Euclidean geometries, are "primitive transformations, and its words are primitive algorithms (Peitgen &als,1992,vol.I-p.256)". Moreover, Mandelbrot (1992) strongly advocates that whenever computer graphics are available we should make them, "not a tool to be called only if needed, but a constant and integral part of [the] process of thinking". And this process of thinking started around the mid-seventies, with the beginnings of the computer era, although a number of shapes now counted as fractals, as the Sierpinski gasket for example, have been known for a long time.

It was around the mid-eighties that appeared increasing claims that programming should be seen as a mathematical activity like other branches of applied mathematics in physics and engineering. One of them came from C.A.R. Hoare, from the Oxford University Computing Laboratory, who argued that as a general philosophical and moral principle computer programs should be seen "as mathematical expressions, describing with unprecedented precision and in the most minute detail, the behaviour, intended or unintended, of the computer on which they are executed". Furthermore, programming would require "the determined and meticulous application of traditional methods of mathematical understanding, calculation and proof, [and] not recognising the absolute need to use mathematics as the basis for a programming discipline could have a number of notorious consequences (Hoare CAR,1986)".

About the same time Donald E Knuth, from the University of California at Stanford, presented a different perspective. He claimed that computer programs should be "fun to write", well-written computer programs should be fun to read, and they could also do a useful work, one of the life's greatest sources of satisfaction. Moreover, it will be also relishing to know that a program will be a pleasure for other people to read, as for oneself Programming, argues Professor Knuth, is best regarded as a process of creating 'works of literature', which are meant to be read Additionally, all major problems associated with computer programming, issues of reliability, portability, learnability, maintainability, and efficiency could be ameliorated when programs and their dialogues with users become more literate (Knuth DE, 1984.1992,p.ix)

The issue of programming in the school mathematics curriculum began to be raised in the beginnings of the eighties as the constructivist perspective to mathematical learning was gaining more acceptance as "a reorganising activity, where activity is interpreted broadly to include conceptual activity or thought, or when children are given the opportunity to interact with each other and the teacher, or they can verbalize their thinking, explain or justify their solutions, and ask for clarifications (Koehler MS, Grouws DA,1992,p.118÷119)". These researchers share a belief that (1) "new mathematics is brought about through "a process of 'conscious guessing' about relationships among quantities and shapes, with proof following a 'zig-zag' path starting from conjectures and moving to the examination of premises through the use of counter examples or 'refutations'", (2) "mathematics knowledge is constructed as a joint venture from teacher to student (Koehler MS, Grouws DA,1992, p 122, Lampert M.,1990)".

Seymour Papert, who proposed the word '*mathetics*' to mean the 'art of learning', is proposing more recently the word '*constructionism*' for a concept of education that he claims must be built on the assumption that (1) "children will do best by finding ("fishing") for themselves the specific knowledge they need", (2) "organized or informal education can help most by making sure [children] are supported morally, psychologically, materially and intellectually in their efforts". *Bricolage*, another neologism he proposes in the educational field, comes from his criticism to the tendency of our intellectual culture being traditionally "so dominated by the identification of good thinking with abstract thinking", by the belief that "the achievement of balance requires constantly being on the lookout for ways to re-evaluate the concrete", by the warning that 'the right way' of programming, for example, "may express and insidious form of abstractness that may not be recognized as such by those who use it". Our traditional epistemology, according to Papert, is based on the chaining of propositions therefore closed linked to the medium of 'text', and he suggests that this linking may change and become more closely bound to *bricolage* and concrete thinking, which always have existed but were more marginalized until the emergence of more dynamic media as we moved into the computer age (Papert, 1992,p.139,146,156)"

The following is my attempt to summarise these perspectives. To the new *'fractal's language'* view programs and the computer can enable students to achieve 'general representations' in a subject mater that Cantor, Peano and Poincaré were sometimes reluctant. In the perspective of some applied mathematicians, e.g. Hoare and Knuth, who for the first time recognized how problems raised by programming could lead to a so great complexity, there was a need to devise structured approaches and computer-aided-software-engineering environments, and they claimed either for more mathematical or more literary methods for the software job. The *construtivists*, and/or the *consructionists*, (and/or the *sociological* or the *epistemological*) adepts suggested either that the teacher should adjust activities to the student, or that the student style of organizing his work should be *bricolage*, 'negotial' rather than planned in advance, 'heterarchial', as McCulloch called, rather than hierarchial.

In the next section I revisit part of a LOGO microworld for fractal geometry that I developed in 1991, at the London Institute of Education, as a part of a MSc thesis under the supervision of Richard Noss, and suggest a role I see for computer technology that is in connection with all these perspectives outlined above .

An Episode Taken from 'The Chaos Game' Case-Study

The design process of the 'chaos game' activity has benefited from the extensive research on Logo mathematics conducted by Celia Hoyles and Richard Noss (Hoyles C& Noss R,1992) Others alternative approaches using the programmable graphing calculator, instead of the computer, were at the same time being published in USA by Evan Maletsky & als , and Heinz-Otto Peitgen & als (Peitgen&als., 1991).

The 'chaos game' activity was tested in a London comprehensive school that followed the SMILE (School Mathematics Independent Learning Experience) curriculum, where the students were used to work at their own individual level and to perform tasks arranged in topics and levels of difficulty. Two A-level students, 17 years-of-age, strongly motivated to computing, were participants in an experiment involving magnetic record of the student-teacher-computer interactions, and data analysis. This activity was the last one of a package of a total of seven activities designed for the fractal geometry microworld. Its pedagogical components comprised also the teacher's agenda and a semi-structured inquiry to guide how to facilitate students extending the range of the abstractions situated in the randomness of the process and the creation of a deterministic Sierpinski shape. The technical components of this microworld comprised an *Apple Macintosh* computer, the *Paradigm Object Logo* programming language, a set of pre-built tools, and a graphical user interface composed of three object-windows, for graphics, data/listener and editing (Sereno F,1993).

In this activity the student was asked to discover what would happen to the space between three points A and B and C supposing that: (1) a turtle (starting equidistantly from these points), chooses to look at one point, A, B or C, according to the result of tossing a die, (2) the turtle goes forward one half of the distance between the present position and the chosen point, and it plots a big dot (3) this process continues over and over.

The way a student may think about randomness, experiment with three pre-built tools, which constitute a very simple translation to Object-Logo of the Barnsley's chaos game algorithm (Barnsley M.F,1988,BartonR,1990,Peitgen&als,1992), change and write procedures can be illustrated by the following excerpt.

(🔊 - recorded dialogue, ✍ - typing; 💻 displayed at the computer screen)

🔊

JAMES I wonder if it will fill up the all triangle?

FERNANDO: ...Do you think it will?

J: we can still never reach the point B. Yeah, in a triangle I can't reach the point B from going half way from there to there. I can never reach B. So... you can't reach the point B, and you can't reach the point A, and you can't reach the point C.

F: [pointing to the vertices A, B and C] Don't think in these points but about the space inside.

J: Yes, I will imagine then How would you get the edge of the line [the side of the triangle] So you can do all the edges of the line, that's no problem. I would imagine you could probably fill up. Yes, I would imagine you could probably fill up the triangle with points . You could fill up the edges in the triangle upon the points A, B, C. I would imagine.

F. Why don't you do some trials ?

J: [typing]

✍

? ThreePoints

? LookAtThreePoints random 3

[... James wanted to use the output of 'random 3' instead of the built-in input a, b or c '... The procedure was edited in the computer

💻 (See figure 1 a)

```
To LookAtThreePoints :input
if Or :input "A :input "a [lookAtOneHalf -70 100]
if Or :input "B :input "b [lookAtOneHalf 103 0]
if Or :input "C :input "c [lookAtOneHalf -70 -100]
end
```

✍ and changed to this

```
To LookAtThreePoints :input
if :input 0 [lookAtOneHalf -70 100]
if :input 1 [lookAtOneHalf 103 0]
if :input 2 [lookAtOneHalf -70 -100]
end
```

? LOOKATTHREEPOINTS redefined.

(3)

F: I think you understand this procedure...

J: It's fine ... Is it possible to run this program many times ?

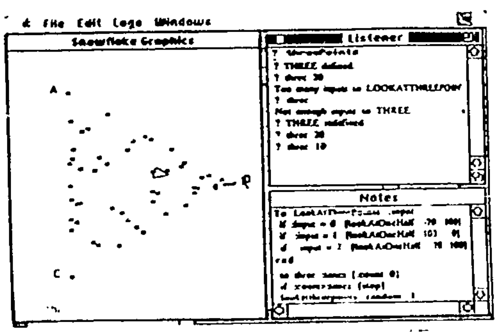
F: If you can write a procedure to do it.

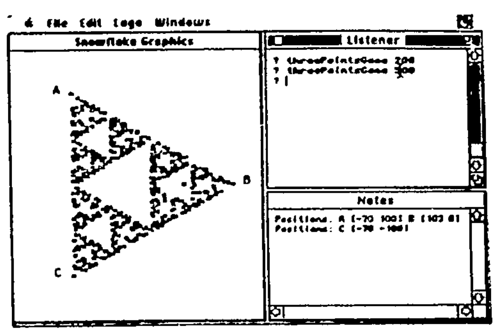
J: ... Ha, probably yes. This one here. Yeah, it's good. [typing]

and

```
To Three :times [:count 0]
if :count = :times [stop]
lookAtThreePoints random 3
(three :times :count + 1)
end
? THREE defined.
? three 20
? three 10
```

I wish to point out three observations about this excerpt taken from my dissertation (Sereno F,1991). Firstly, my mathematical agenda had not an explicit purpose to evaluate if the student could alone change the procedure '*LookAtThreePoints*' to allow the input *Random 3*, but I was only interested in observing if he would raise this issue, as he really did, and if he was able to understand the effect of each change that was introduced. Secondly, my view of the role of the microworld was one of expecting that the student would be able to construct by himself either the procedure *LookAtOneHalf*, that allows the turtle to do displacements of one half the distance between two points, or the procedure *LookAtThreePoints*, that allows random displacements toward points A, B or C. Nevertheless, in the time available I could not observe conveniently this issue. Coming now to the last observation, the criteria I used to design the microworld tools were: (1) they should allow the student to do the task without a need to be guided in every step, (2) the student-user should feel encouragement in developing his own paths towards the construction of orbits of a single point under iteration, (3) the student-user should be stimulated to play the role of an apprentice of the art of microworlds designing and, at the same time, being its user. Put another way, the microworld should act as an attractor for fostering programming and induce a formalisation of mathematical constructs and logico-mathematical reasoning.





Concluding Remarks

From the analysis of the data in two case studies it can be said that the 'microworld for fractal geometry' can help mathematics teachers:

- To pose problems and "ask for clarification in order to engage students in mathematical discourse (Koehler M S, Grouws D A,1992;Lampert M,1990)"; For example, why the orbit s(t) of the 'turtle' under the iteration of a collection of three different displacements, in a random order, has an attractor ?
- To illustrate "the unity and integrity of the discipline (MSEB & NCR,1990,p.48)", or to provide topics for reasoning which can interest some students, or "to encourage them to pursue a study more in-depth (DES,1989,p.6)". This seems to be of potential value in the terms of the development of technological interests, software engineering at the first place, as for example, if we want to know ways of achieving programs better explained and therefore better programs.

Additionally, the feedback received at the Oporto School of Education from students of an Initial Teachers Training course who used a MS-DOS *Logowriter* version of the 'microworld for fractal geometry' to illustrate an algebra and geometry course, in 1993, makes me believe that it had an important role in order "to find the language and symbols that students and teachers can use to enable them to talk about the same mathematics content (Koehler M S, Grouws D A,1992; Lampert M,1990)". For example, it could make sense to some motivated students statements like the following that were taken from (Barnsley M,1988, Edgar,1990): "The geometric distribution of the points s(t) in the plane is a proper subset S of R^2 such that every point of s(t) is close to some point of S for t large enough"; or questions like "How can be proved that "*ind S < dim S*" (the small inductive dimension of S is smaller than the Hausdorff dimension of S), and therefore the set S is a fractal?".

The interest to learn geometry in a computer environment and do research by stressing the computer-student interaction to create learning situations that facilitate the acquisition of visual skills, specific geometrical concepts, and other thinking processes, has improved in the last few years. Despite this common interest there remain two very important and complex questions. The first one, raised by Hershkowitz R (1990), is the problem of knowing "what the effect of programming in a "geometrical language", like Logo, has on concept formation and vice versa", has illuminating clues in the Efraim Fischbain (1993) '*Theory of Figural Concepts*', as he points out to understand how images and concepts cooperate or sometimes conflict in the cognitive activity of the student. The second one, raised by Philip J. Davis (1993), is the question of knowing if the visual aspects of mathematical intuition and reasoning, '*visual theorems*', can "be considered as really mathematics", which seems to be an open question to me

The aim of this paper is to present a perspective about the connections that can be created between the discovery of how randomness can generate textures that students can organize coherently in a certain way to convey high information content, and the activity of programming, by doing literary 'bricolage', or producing mathematical theorems in computers' environments that, as diSessa (1993) pointed out, satisfy the criteria of good learnability, comprehensibility, general expressiveness, usefulness and collaboration.

Acknowledgements

A short-duration (JNICT) research grant and Polytechnic Institute of Oporto Chairperson Professor Luis Soares support. The opinions expressed here are those of the author and not necessarily of the Institutions. I thank Richard Noss for his enthusiasm for our work. I owe improvements in this paper's readability and the abstract in German to Luisa Félix.

References

Barnsley M F, 1988, *Fractals Everywhere*, San Diego, Academic Press Inc.

Barton R, 1990, *Chaos and Fractals*, in *Mathematics Teacher*, October 1990, Reston, NCTM.

Davis P J, *Visual Theorems*, in *Educational Studies in Mathematics* :24:333-344.

DES,1989, (Department of Education and Science) *Mathematics from 5 to 16*, 2d. editition, Curriculum Maters 3, London, HMI.

DiSessa A, 1993, *Collaborating via Boxer*, in Kynigos C & als (eds), 1993, *Proceedings of the 4th European Logo Conference* (suplement: p.s35-s44), Athens, Doukas School SA.

Edgar G A, 1990, *Measure, Topology and Fractal Geometry*, New York, Springer-Verlag.

Fischbeim E, 1993, *The Theory of Figural Concepts*, in *Educational Studies in Mathematics* :24.139-162

Koehler M S, Grouws D A, 1992, *Mathematics Teaching Practices and Their Effects*, in Goruws D A (Ed.), 1992, *Handbook of Research on Mathematics Teaching and Learning*, NCTM, New York, MacMillan Publishing Company.

Lampert M, 1990, When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching *American Educational Research Journal*, 27(1), 29-63.

Hershkowitz R, 1990, *Psychological Aspects of Learning Geometry*, in Nesher P & Kilpatrick J, 1990, *Mathematics and Cognition*, Cambridge, CUP.

Hoare, C A R, 1986, *Mathematics of Programming*, Conference on the occasion of BYTE's 10th Anniversary celebration, Byte, August 1986, Peterborough.

Hoyles C & Noss R (Eds.), 1992, *Learning Mathematics and Logo*, The MIT Press.

Knuth D E, 1992, «*Literate Programming*», Center for the Study of Language and Information, Stanford, Lecture Notes Number 27.

Mandelbrot B, 1992, *Fractals and the Rebirth of Experimental Mathematics*, in Peitgen & als,1992, *Fractals for the Classroom*, NCTM & Springer-Verlag

MSEB & NCR,1990, (Mathematical Sciences Education Board and National Research Council) *Reshaping School Mathematics, A Philosophy and Framework for Curriculum*, Washington, National Academay Press.

Papert S, 1992, *The Children's Machine, Basic Books.*

Peitgen H-O, Jürgens H, Saupe D, Maletsky E, Perciante T, Yunker L, 1991, *Fractals for the Classroom (Strategic Activities - Volume One)*, NCTM & Springer-Verlag

Peitgen H-O, Jürgens H, Saupe D, Maletsky E, Perciante T, Yunker L, 1992, *Fractals for the Classroom (Part one, Introduction to Fractals and Chaos)*, NCTM & Springer-Verlag

Sereno F, 1991, *A Microworld for Fractal Geometry*, Unpublished MSc thesis, Institute of Education, University of London.

Sereno F, 1993, *Fractals and School Mathematics*, in Kynigos C & als (eds), 1993, *Proceedings of the 4th European Logo Conference* (suplement: p s35-s44), Athens, Doukas School SA

THE TALE OF TWO STUDENTS: THE INTERPRETER AND THE DOER

Anna Sfard and Liora Linchevski
The Hebrew University

While carrying out an experimental study in which several sixth graders were introduced to algebra, we noticed a quite striking diversity not only in the students' understanding of the subject, but also in their attitude toward the issue of understanding. We decided to observe closely two children, Shir and Dana, who obviously differed in their aims and expectations. This difference had a distinct impact on the way their knowledge developed. The two students displayed disparate learning behaviors and different approaches to the issue of explanation, and they progressed at a different pace. Shir's example has shown that a constant attempt to understand may have its price: a struggle for meaning may mean postponement of an "automatic" mode of dealing with algebra and a slowdown in the acquisition of skills.

The story we are going to tell in this paper is a story with a moral. The lesson to be learned concerns the impact of students' attitudes toward the issue of understanding mathematics on the outcomes of instruction. This moral was a serendipitous finding of an experiment in teaching beginning algebra. At the following pages we will focus on this special by-product of our algebra study rather than on its principal aims.

Let us begin with just a few indispensable details about the teaching experiment. Teachers and researchers often complain that student's understanding of algebra is purely instrumental: the children can "go through the motions" but they cannot explain the things they are doing. The findings of our previous studies forcefully confirmed these complaints (see [4], [5]) and brought some insight into the nature and the sources of the problem. In keeping with the implications of our analysis, we designed an experimental teaching sequence which was expected to bring the beginners at a deeper, more flexible understanding of algebra. It was an introductory material on linear equations aimed at seventh-grade children (age 12-13) who, by the time we met them for the first time, were already acquainted with the notion of algebraic expression but not with the concept of equation. Each of the six children who participated in the experiment had five individual one-hour long teaching sessions devoted to the basics of equations solving. After a month another meeting took place during which the participant of the experiment was asked to help in explaining the subject to an "uninitiated" peer.

To shed some light on the way in which we tried to foster student's understanding of algebra we must first say a few words on the meaning of the term "understanding" in this particular context. The simplest, most obvious way to define the word is to say that to understand algebra means to be able to justify algebraic procedures. At the school level, the only method to provide the reasons for the operations performed on equations is to ground the formal transformations in the numerical computations which they symbolize and generalize. For instance, the transition from, say, $3x+7 = 2x-5$ to $3x = 2x-12$ can only be explained by saying that whatever number is substituted instead of x, the first equality holds if and only if the other holds, and this means that subtracting 7 from both sides of the equation does not alter its solution. In other words, in order to apply algebraic processes in a meaningful way, one must be able to relate them to the properties of numbers.

The teaching sequence we designed and used in our experiment aimed at raising student's awareness of the connections between algebraic procedures and numerical calculations. All along the way we kept interpreting algebraic expressions as prescriptions for computational processes which may have infinitely many inputs and outputs. In the semi-structured dialogue between the teacher and the learner, requests for explanations and justifications accompanied every step and every utterance made by the student. In this way we tired to prevent the children from becoming addicts of an automatic algebraic mode -- from slipping into the habit of performing symbol manipulations without ever thinking about their meaning or justification. We wanted the children to always keep their eyes on the links with the underlying arithmetical rules, or at least to be able to make an appropriate reference whenever necessary.

The heroes of this paper are a boy and a girl whose names are Snir (S) and Dana (D). In what follows, we present a number of snapshots from their interactions with the teacher (T). It is not our aim in this paper to give a full account of the teaching experiment or of its results. We shall only report on the special phenomena that attracted observers' attention and stirred their curiosity. Some telling differences in the behavior of the two children have been spotted. These differences indicated a disparity between Snir's and Dana's attitudes toward the issue of understanding. Obviously, the students differed in their aims and expectations, and this fact had a considerable impact on the way their knowledge developed.

Stage 1: Grappling in the dark

During the first session of our experiment, an interview was conducted the aim of which was to examine student's knowledge of equations prior to instruction. The difference between Snir and Dana started to show up already at this early stage. It was a subtle disparity between the types of explanations provided by the children rather than a difference in their knowledge of the subject that attracted our attention.

Both Snir and Dana said they did not learn equations before. So far, they have never seen anything like $7+x=23$, even though they did deal in the past with such expressions as $7+?=23$. In spite of this, and in full accordance with findings by other researchers ([1], [2]) the children could deal with 'arithmetical' equations (equations of the form $ax+b=c$) in an intuitive manner. They solved these simple problems by 'undoing' the computational process, namely by reversing the operations in the formula on the left and by applying them in the reverse order to the number on the right. This is the way Snir tackled the equation $7x+157=248$:

> Here, I have to find a number so that 7 times this number plus 157 is 248. First, 248 minus 157 is 91. Now, the number times seven... 91 divided by 7 is 13. The number is 13.

The difference between Dana and Snir began to show for the first time when the children were asked to justify their solutions. Snir, when requested by the teacher to 'convince' her that his solution of the equation $7x+157=248$ was correct, exclaimed "Can't you see? 7 multiplied by 13 is 91. 91 plus 157 equals 248!". This response was clearly based on a spontaneous interpretation of the expression $7x+157$ as a computational process that, if applied in reverse to the number on the right (248) would produce the value of the unknown number x.

In contrast, Dana's first answer to the question why she acted the way she did was "I don't know". Her next responses were tautological: she claimed that she did the only thing that could be done ("I think this is the way and there is no other") or she simply repeated the procedure. Sometimes, when pressed for a more informative answer, she would recite the rules: "It says that 3 times x equals 141... when we have a multiplication, we have to divide". These rules of reversal were obviously taught to her in the past, when the class dealt with problems of the type $7+?=23$.

Quite predictably, none of the children got through the 'didactic cut' (see [1]): neither Snir, nor Dana could cope with non-arithmetical equation in which the unknown appeared in both component formulae ($ax+b=cx+d$) and where the technique of undoing did not work anymore. There was much to learn about the children's thinking from the ways they handled the problem they could not solve.

When Dana was faced with the equation $15x+12=8x+47$, she seemed baffled and did not know what to do. And then, after a few moments of silence, she said:

> There are two exercises here and they are equal one to the other. I do 15+12=27, and then 47+8=55. We can see here that they are not equal... I don't know what to do.

Dana, who just a moment ago successfully coped with an arithmetical equation, was now confused and could not interpret the new equation or even to grasp the purpose of the task. A strong urge to do something forced her into the only type of calculation she could perform here: adding the numbers that appeared in the formulae while ignoring the x. In fact, we had doubts whether Dana felt a real need to interpret or explain, and we could sense that her principal aim was to do something.

Snir's behavior was quite different. Here is a sample from the dialogue between him and the teacher:

> S One must find something ... that when I multiply 15 by a number and I add 12 to it, it will be equal to eight times a number, and I add 47 to it. Let's start with something simple. Here, times 3 and times 1 [writes 3 over x on the left-hand side and 1 over x on the right-hand side]. No, it's not that ... I don't know
>
> T What are you looking for when you are solving the equation?
>
> S: We must find two things. Something that when I multiply it or divide it ... doesn't matter.. will be equal to the other thing. One must find the way of making them equal

Obviously, Snir had his understanding of the problem: he looked for the inputs for which the outputs of the procedures represented by the two sides of the equation would be equal. His interpretation was not entirely correct since he did not require the 'equalizing inputs' to be the same. In addition, he had no knowledge of solving techniques, the fact of which he was aware ("The only way I know is trying the numbers.. It's a mess and takes time") but which nevertheless did not stop him from the attempts to find the solution. In spite of all these shortcomings, Snir did act in a meaningful and consistent way. The main point to be made here is that, unlike in Dana's case, Snir's actions were guided by his interpretation of the problem, and were not just a mere implementation of a randomly chosen algorithm.

<u>Stage 2: Finding the way</u>

During the following three sessions the students learned the concept of equation. In an attempt to make the learning as meaningful as possible, we decided to use student's spontaneous knowledge of arithmetical equations as a point of departure and to gradually generalize it in such a way that it would become applicable to all kinds of linear equations. Thus, we had to translate the technique of undoing used by the students in the case of arithmetical equations into the algebraic idea of 'the same operation on both sides of an equation'.

At the first stage, only arithmetical equations were tackled. The students had to look carefully into the procedures they performed. Gradually, they were encouraged to translate their verbal description of the solution (like the one by Snir, quoted above) into the algebraical representation. The next step was to discuss the relationship between the different equations appearing in the solution sequence and to justify in a new structural way the operations that led from one of them to another. Our last move was an application of the 'permissible operations' to non-arithmetical equations. To sustain student's awareness of symbol-numbers connections, the concepts of equivalence of equations and of permissible operations was recalled and discussed all along the way.

We were surprised to find out that of our two students, Dana was much swifter in making the transition to the algebraic mode. Indeed, she adopted the new method without hesitation: after just one experience with algebraic solution like the one presented above, she was ready to apply the technique to all the problems with which she was presented later. She never went back to the technique of undoing. In contrast, Snir clearly gravitated toward the old procedure. During the first session he had to be constantly prompted to use the algebraic method, and at the beginning of the next session the whole idea had to be re-discussed before he agreed to use it (compare our findings with those in [3]).

As far, however, as student's explanations were concerned, Snir was still the one who had more to say. Like in the opening interview, he had no difficulty with justifying his solutions. In contrast, Dana had to be given much hints in order to say, eventually, that the correctness of the result she has found so easily could be proved by substitution into the original equation.

The concept of the equivalence of equations seemed to be quite clear to Snir from the very beginning. For example, when asked whether two equations that just have been solved, are equivalent he said "Yes, they have the same solution". To Dana, all this did not seem so obvious. She had no answers to our questions. When asked by the teacher whether the equations 2x+1=11 and 3x-2=19 were equivalent, her responsse was correct, but when requested to give an explanation, she had a difficulty: "They are not equivalent because we did not add,...we did not multiply...". We guessed she was trying to say that there was no operation which could transform one equation into the other.

It seemed to us that Snir's temporary resistance to the new technique stemmed from his inability to fully justify it. As long as the undoing was more meaningful to him than the algebraic method, he would rather choose the former. By the same token, the relative easiness with which Dana 'switched allegiance' from the technique of undoing to the algebraic method stemmed from her strong wish to actually do something on one hand, and from her relative indifference to the issue of meaning on the other hand. Or, in a slightly different language: since she did not expect her understanding to become much different from what it was already at this early stage, she could see no reason why the use of the efficient algorithm should be postponed.

Stage 3: Being there and guiding the others

The fifth session was devoted to a review of the whole subject of solving equations. This time, neither Snir nor Dana had any difficulties with the algebraic technique. Both were also able to verify their solutions. However. when asked why $7x+157=248$ and $7x=91$ were equivalent, they did not give the same type of answer: while Snir stressed the fact that the two equations had the same solution ("There is the same solution... the operation that was done on both sides did not change it"), Dana concentrated on the operation itself: "They are equivalent because when we subtract 157 from both sides we get an equivalent equation." When asked what it meant, she responded: "The equations are equal".

The children were then requested to justify the permissible operations. This question left Dana speechless. She could produce no answer in spite of the teacher's insistence and encouragement. Snir, as usual, was prepared to give an exhaustive explanation:

> These operations do not change the solution ... I do the same operation on both sides, and I substitute the same number on both sides ... if I substitute and compute the sides of the [new] equation, I get a different results [than in the old equation], but it is still the same structure ... the same ... how shall I put it? so the solution is the same solution ... [Snir had obviously some difficulty with finding the right words, but what he seemed to be saying was that the permissible operation did not change the equality relation between the sides of the equation.]

Obviously, Snir's first need was to have a consistent vision of the situation, whereas Dana was mainly interested in doing things. This message about Dana's attitude was brought with particular force in an exchange she had with her "uninitiated" friend Zohar during the sixth meeting. The later had a difficulty with the equation $112=12x+28$ and was told by the teacher to try to subtract 28 from both sides.

Z How would I know that I should subtract 2[illegible] and not 12x?
D. [cannot find the explanation, remains silent for a few seconds] You can't subtract 12x. How would you do it? You don't have any x here [points to the the formula on the left]. How would you subtract? You can subtract 28
T: Is the subtraction of 12x forbidden? Is it against the rules?
D: It is forbidden, we wouldn't get the result

These utterances by Dana came to us as a surprise since by that time she was already in a slightly more thoughtful mode and was able to give more satisfactory answers to the questions on permissible operations ("We are permitted to add, subtract, multiply both sides of an equation... because this doesn't change the solution"). Dana's urge to do things and obtain a result was evidently strong enough to make her oblivious to all the previous explanations. The permissible operation was one thing when discussed theoretically, and something quite different when put into practice. In the latter case, the only criterion that counted was the criterion of effectiveness: permissible operation was an operation which would result in a simpler formula.

Discussion and the morals

From our brief report it seems quite clear that Dana and Snir came to our experiment with quite different aims and expectations as far as their learning and understanding of mathematics was concerned. For Dana, the activity of equations solving had initially only one dimension: that of doing. In her eyes, the means were also the ends and the actions she performed were undertaken for their own sake. Often, Dana seemed unable to understand the meaning of the request 'explain' the same way the teacher did. Her single-minded preoccupation with doing and manipulating persisted in spite of the instructor's constant effort to entice her into a more reflective mode. It was only toward the end of the experiment that Dana's attitude showed first signs of a change.

For Snir, in contrast, there were always two dimensions in the problems he tackled: the dimension of means, namely of procedures to be performed, and the dimension of ends -- of the purpose for which the procedure was employed. The purpose was something that had to be defined with no reference to the procedure. Snir recognized the independent existence of a universe of mathematical entities, such as numbers and algebraic expressions, and whatever he did was a derivative of his knowledge of these entities. From the first moment he seemed to be in a pursuit of links between the manipulations and the properties of the objects on which these manipulations have been performed. Thus, unlike Dana who, having nothing to lose, switched to mechanical manipulations without qualms the moment she saw the algorithm, Snir rejected the automatic mode as long as the reasons for the 'rules of the game' did not seem sufficiently clear to him. We decided to call Snir an <u>interpreter</u>, while Dana was named a <u>doer</u>.

Teachers and researchers are often bitterly disappointed to find out that even most reasonable and carefully implemented didactic ideas would not bring much change. In particular, they are frustrated by their inability to significantly improve students' understanding of mathematics. It seems, however, that the meaningfulness (or should we say meaninglessness?) of the learning is, to great extent, a function of student's expectations and aims: true interpreters will struggle for meaning whether we help them or not, whereas the doers will always rush to do things rather than to think about them. The problem with the doers stems not so much from the fact that they are not able to find meaning, as from their lack of the urge to look for it. In a sense, they do not even bother about what it means to understand mathematics. All this may be not entirely new to an experienced teacher, but in our study we had a rare opportunity to look at the phenomenon through the magnifying glass of a fully videotaped individual exchange between the student and the teacher.

To sum up, the main moral of Snir and Dana's story seems to be this: whatever route we choose while teaching a certain subject, we must be prepared for the possibility that some students will not be able to benefit from our efforts. Our success will depend, among others, on our ability to turn doers into interpreters.

[1] Filloy,E. & Rojano,T (1989) Solving equations the transition from arithmetic to algebra <u>For the learning of mathematics</u>, 9(2), 19-25

[2] Herscovics, N and Linchevski, L 1991, 'Pre-algebraic thinking range of equations and informal solutions used by seventh graders prior to any instruction', in F Furinghetti (Ed), <u>Proceedings of PME 15</u>, vol. 2, 173-180, Assisi, Italy

[3] Herscovics, N and Linchevski, L 1993, 'The cognitive gap between arithmetics and algebra' To appear in <u>Educational Studies in Mathematics</u>

[4] Linchevski, L., & Sfard, A. (1991). Rules without reasons as processes without objects - The case of equations and inequalities In F. Furinghetti (Ed), <u>Proceedings of PME 15</u> (pp 317-324), Assisi, Italy

[5] Sfard, A & Linchevski, L (1993) The gains and the pitfalls of reification the case of algebra. To appear in <u>Educational Studies in mathematics</u> (special issue in theories of learning and teaching mathematics)

IS INFINITY A WHOLE NUMBER?

Gilli Shama & Nitsa Movshovitz-Hadar
Technion - Israel Institute of Technology
Haifa, ISRAEL 32000

Abstract

This paper presents preliminary results from a naturalistic study of the way people perceive periodicity. Periodic phenomena are infinite but they are presented by finite means. The results point that there is a close tie between the infinite extendibility of periodic phenomena and its presentation by a whole number of periods. This may indicate that many perceive infinity as a whole number.

The Problem.

Periodic series and periodic functions, like any periodic phenomenon[1], are defined on an unbounded domain. Consequently, their values occur periodically, each an infinite number of times. However, finite and bounded means, such as time and space available, limit the *actual presentation* of periodic phenomenon. Obviously, this finite nature of the presentation does not imply necessarily a presentation by a whole number of periods. Is it indeed that obvious? In other words:

- Is not *mental representation* of periodicity also limited to a whole number of periods?
- Do many perceive infinity as a whole number?

This paper describes a study that seeks answers for these two questions. It is a part of an on-going research focusing on the concept of periodicity.

---

[1]A series $\{a_i\}$ is ***periodic*** if there exists an integer k such that $a_n=a_{n+k}$ for all n. We will use the term ***period*** to describe any set of k consecutive elements in the series.
A real function f is ***periodic*** on a domain D if f isn't constant on D and there exists a real number t such that for each x in D x+t is also in D and $f(x)=f(x+t)$. For a complete definition of a complex periodic function see Van Nostrand Comp. 1977, p. 654. We will use the term ***period*** to describe a half open segment with length t in the domain. t is the ***length of the period.***
Examples for periodic phenomena can be found in Berry, Norcliffe, and Humble (1989).

Background.

The literature includes many studies, empirical and theoretical, about *teaching* the concept of periodicity (Dreyfus & Eisenberg, 1980, 1982, 1983; Eylon & Rosenfeld, 1990; Fraser, 1968; Hersberg & Talsma, 1990; Hirsch & Novak, 1987; Hornsby, 1990; Hughes & Freier, 1991; Jacobs, 1975; Lichtenberg, 1978; Lovell, 1982; McGinty & Mutch, 1982; Olson & Olson, 1988; Sconyers, 1986; Seybel, 1979; Sherard, 1982; Wagner, 1982). However, a thorough literature survey did not yield any report on research about how people perceive the concept of periodicity. Related studies about the way people perceive the concept of *infinity* are described by Tirosh (1991).

Method.

The lack of background, as described above, yielded a need for an exploratory approach to the problem. This need lead quite naturally to the selection of qualitative methods (Glaser & Strauss, 1967; Romberg, 1992). We employed unobtrusive classroom observations, personal interviews with students of various ages and with their teachers, as well as an analysis of existing teaching materials. This paper presents protocol analysis of personal interviews with seven 11-graders (4 females, 3 males), all taking mathematics as an advanced topic. They were each interviewed right after their class started learning Trigonometry.

The interview was semi-structured. To set-up a cooperative and orientated atmosphere, the interviewer started all interviews by introducing herself and explaining "softly" the goal of the interview. The main tasks were pre-planned, but their specific order and the exact wording varied. Here is a sample of the pre-planned tasks:

(1) Give me a few examples of periodic phenomena or periodic functions that you are familiar with;

(2) Among your examples, is there any that you can extend as much as you desire? Extend it;

(3) Look at Figure 1. Which parts of Figure 1 represent a periodic series? Is there any part of Figure 1 that you can extend as much as you desire? Extend it;

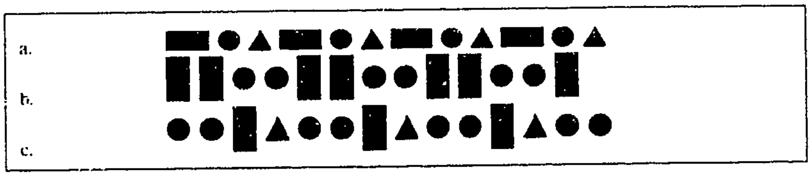


Figure 1: Periodic series presented by the interviewer.

(4) Can you find a periodic example which cannot be extended?

(5) Count the number of periods presented in each of the examples we looked at;

(6) Give me an example that you fill might be difficult for other students to determine whether it is periodic r not;

(7) Here is an example that a former student gave. Determine whether it is periodic or not.

Each interview lasted between one and two hours, according to the needs that evolved. Following the interviews transcribed protocols were segmented into units by task performance. Units were then classified into categories by type of task. Then various conceptions about periodicity were identified. Finally, further classification of the same units, was carried out, this time by type of conception about periodicity that the unit may point at.

Preliminary Results

Two of the conceptions resulted by the process described above. have special relevance to the problem addressed by this paper. These are:

1. Periodical phenomenon has a whole number of periods;
2. Periodical phenomenon extends without limit by a whole number of periods.

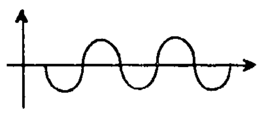
In the rest of this section we show evidences for these two conceptions.

<u>1. Periodical phenomenon has a whole number of periods.</u>

The total number of examples for periodic phenomena given by students was 86. Counting of periods was impossible in examples like: "The months of the year repeat themselves". In 54 of the examples counting the number of periods was possible. The students presented 42 (78%) of those examples by a whole number of periods. For example, two periods appeared in: "One, two, three, one, two, three."

In 12 (22%) examples the number of periods was not a whole number. Five of those were answers to task 6, namely examples that might cause difficulty for other students if asked to determine whether they are periodical or not. Two of these examples appear in Figure 2. Their originators stated that they actually were not periodical since the number of periods was not a whole number (The quotation appears below each example in Figure 2).

a. Anna's example:



Anna's argument:

"There is a periodicity here, but truly there is not. The period at the end is incomplete. I don't know if it is periodic."

b. Jane's example:

Jane's argument:

"In my opinion it is not a periodic phenomena because a triangle and a dot are missing "

Figure 2: Wrong identification of periodic phenomena as non periodic, since the number of presented periods isn't whole.

The students did not hesitate, nor gave any wrong answer in counting the number of periods in each example wherever the number of periods presented was actually a whole number. However, in cases where the number of periods presented was not an integer, only three students (Anna, Rick and Tammy) used fractions. This behavior was not consistent nor exhibited in a confident way. For example, in Figure 1b Tammy counted, "Three and a quarter", and in Figure 1c, "Three and a half."

Then she said: "it sounds very foolish, these numbers, a half, a quarter."

Interviewer: "Why?"

[Pause of several seconds].

Interviewer: "Why does it sounds foolish?"

Tammy: "I don't know. It just doesn't fit."

In other examples Tammy, like all other students except Anna, truncated the number of periods to the nearest lower integer, as shown in Figure 3.

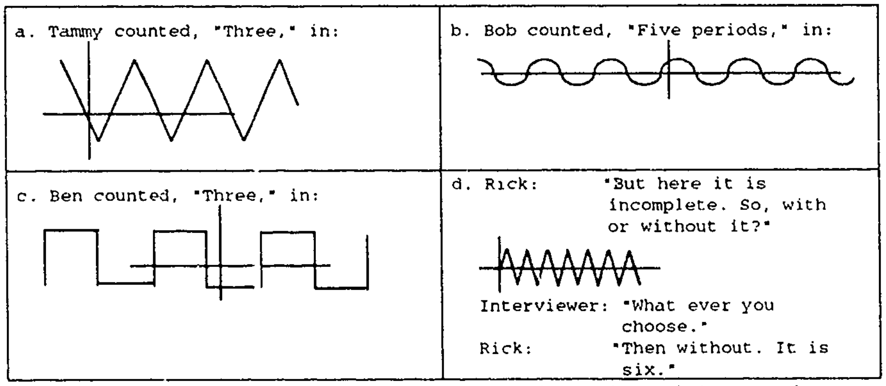


Figure 3: Students truncated the number of periods to the nearest lower integer.

Ben and Rick went even further and changed the examples so as to have a whole number of periods in them, as shown in Figure 4.

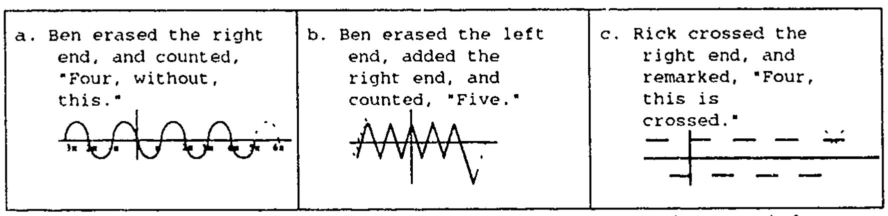


Figure 4: Students changed the examples so as to have a whole number of periods in them.

2. Periodical phenomenon extends without limit by a whole number of periods.

When asked whether an example was extendible (tasks 2, 3), all students approved that every given periodic example was extendible and they were able to actually extend it. However, many students mentioned the possibility of constrains to the extendibility. E.g., "Till death"; "Till the end of the world"; "And so on, till I get tired". Those were the only reasons for non extendibility of examples given by students (as a reply to task 4). An interesting constrain (held by Shira, Anna and Jane) had to do with completing the presented example so as to have a whole number of periods, before extending it further. Figure 5 presents this constrain.

Interviewer: "Is this example extendible?"

Anna: "That single one..." [She indicates the last element in the series] "Is it here on purpose?"
Interviewer: "What do you think?"
Anna: "If there were two more circles here" [pointing at the right end] "I could continue it, but as it is, I'm not sure."

Figure 5: Anna completed the presented example to have a whole number of periods, prior to the extension.

In the actual continuation (Tasks 2, 3) all students showed a demand for a whole number of periods, by stopping their constructions at the end of a complete period (see Figure 6).

a.
b.

Figure 6: Tammy, Anna and Ben completed the presented examples to have a whole number of periods.

One student even added a whole period to the examples on Figure 1 b and c, so that the extended series was not periodic (see Figure 7).

Figure 7: Rick extended the example by a whole number of periods.

Summary and conclusions

The observations obtained through the interviews of seven 11-th graders in this study were very consistent. They indicate that a whole number of periods is typical for one's mental representation of periodic phenomena. This may be either well acknowledged or go unnoticed by the beholder. It may be expressed explicitly (*stating that if the number of periods is not an integer, the phenomena cannot be periodic*) or it may be exhibited implicitly (*counting the number of periods using integers only*).

The results of the study also show that all interviewees maintain that infinity and whole numbers go together. It may also be either expressed explicitly (*stating that if the number of periods is not an integer the phenomena cannot be infinite*) or exhibited implicitly (*extending examples to or by a whole number of periods*).

These results suggest that many perceive infinity as a whole number. A possible way to explain it is by the gap between the physical, actual presentation of periodic phenomena, which often uses (finite) integers for the number of periods, and the abstract notion of periodicity, which is infinite. The mental representation one develops, draws from both: It combines the integer from the former, and the infinity from the latter. Thus an intuition that "infinity is an integer" evolves. Such a process of the development of an intuition is in line with Fischbein's theory (Fischbein 1987). What can be done about it through teaching, is remained to be investigated.

Further study.

While writing this paper we are analyzing similar interviews with students of three other age groups: third, sixth, and ninth grades. Among the younger students the misconception that periodic phenomena have a whole number of periods seems to be even stronger. We are also working on a further study, using a pen and paper test to establish the generality of this misconception.

References

Berry, J., Norcliffe, A., & Humble, S. (1989). *Introductory Mathematics Through Science applications. Chap. 3.* Cambridge: Cambridge University Press. 66-100.

Dreyfus, T., & Eisenberg, T. (1980). On Teaching Periodicity. *International Journal of Mathematics Education in Science and Technology, 11* (4). 507-509.

Dreyfus, T., & Eisenberg, T. (1982). Graphing Trigonometric Functions. *The Mathematics Teacher, 75.* 638.

Dreyfus, T., & Eisenberg, T. (1983). On the Solvability of x2=b(mod m). *International Journal of Mathematics Education in Science and Technology, 14* (4). 415-421.

Eylon, B., & Rosenfeld, S. (1990). *Cultivating Visual Cognition in Young Children. A Research Report of the Agam Project.* The Agam Project, Department of Science Teaching, Weizmann Institute of Science, Rehovot, Israel.

Fischbein, E. (1987). *Intuition in Science and Mathematics, an Educational Approach.* Holland:Reidel Pub. Company.

Fraser, D. G. (1968). A Method for Finding the Period and Phase Shift of Trigonometric Functions. *The Mathematics Teacher, 61* (8). 762-763.

Glazer, B., & Strauss, A. L. (1967). *The Discovery of Grounded Theory: Strategies for Qualitative Research.* Chicago, ill.: Aldine.

Hersberg, J., & Talsma, G. (1990). Graphing Composites of Trig and Arctrig Functions. *Mathematics Teacher, 83* (3). 194-195.

Hirsch, G. R. (Ed.), & Nowak, R. S. (1987). Activities: Periodic Pictures. *Mathematics Teacher, 80.* 126-137.

Hornsby, E. J. (1990). A Method of Graphing f(x)=Asin(Bx+C)+D. *Mathematics Teacher, 83* (1). 51-53.

Huges, B. B., & Freier, L. W. (1991). The Sierra Curve- an Introduction to Periodic Concepts. *Mathematics Teacher, 84* (6). 434-441.

Jacobs, N. (1975). More on Repeating Decimals. *Mathematics Teacher, 68* (3). 249-252

Lichtenberg, D. R. (1978). Minicalculators and Repeating Decimals. *Mathematics Teacher, 71* (6). 524-530.

Lovell, R. D. (1982). Sharing Teaching Ideas. Slide and Strech your Functions. *Mathematics Teacher, 75.* 469-470.

McGinty, R. L., & Mutch, W. (1982). Repeating Decimals, Geometric Patterns, and Open-Ended Questions. *Mathematics Teacher, 75.* 600-602.

Olson, M., & Olson, J. (1988). Predicting the Tens Digit of the Square of a Number. *School Science and Mathematics, 88* (7). 569-572.

Romberg, T. A. (1992). Perspectives on Scholarship and Reasearch Method. D. A. Grouws (Ed.). *Handbook of Research on Mathematics Teaching and Learning.* New-York: macmillan. Chap 3. 49-64.

Sconyers, J. M. (1986). Sharing Teaching Ideas. A Simple Periodic Function. *Mathematics Teacher, 79* (8). 619-621.

Seybel, K. (1979). Reader Reflections. Cyclic Fractions. *Mathematics Teacher, 72* (8). 567-568.

Sherard, W. S. (1982). Graphing the Functions y=asin(bt-c) and y=acos(bt-c). *Mathematics Teacher, 75.* 309-311.

Tirosh, D. (1991). The Role of Students' Intuitions of Infinity in Teaching the Cantorian Theory. In D. Tall (Ed.), *Advanced Mathematical Thinking.* Dordrecht: Kluwer Academic Pub. *Chap. 12.* 199-214.

V. N. R.: Van Nostrand (1977). *V.N.R. Concise Encyclopedia of Mathematics.* New-York: Van Nostrand.

Wagner, W. J. (1982). Circles, Polar Graphs and a Computer - Some Unexpected Results. *Mathematics Teacher, 75.* 323-328.

Stimulating Student Elaboration of Mathematical Ideas Through Writing

Mal Shield and Kevan Swinson, Centre for Mathematics and Science Education, Queensland University of Technology.

In this study, students wrote about the mathematics they had learnt recently in class. The protocols were analysed to identify the language and elaborations used. Classroom activities were introduced in an attempt to stimulate elaboration in later writing episodes. These aimed at relating the mathematics to everyday contexts. Students continued to write mostly about the procedures of mathematics with little elaboration.

Introduction

Understanding in mathematics learning involves knowing the concepts and principles related to the procedures being used and making meaningful connections between prior knowledge and the knowledge units being learned (Baroody & Ginsburg, 1986). Swing and Peterson (1988) described mathematical knowledge as "characterised by logical connections among knowledge units" (p.54) and stated that "making these connections is an integral part of learning and understanding mathematical knowledge" (p.54).

The term elaboration has been used, particularly in the study of reading processes, to describe the linking and integration of information being learned. For example, Hamilton (1989) described elaboration as follows.

> Elaboration can be defined as any enhancement of information that clarifies the relationship between information to be learned and related information, e.g., a learner's prior knowledge and experience, contiguously presented information. (p.205)

Investigations into the effects of learner generated elaborations (Reder, 1980; Anderson, 1990) have demonstrated improved comprehension and retention in a variety of contexts. Elaborative processing by learners appears to make new material more meaningful and allows for more effective integration of the ideas with existing knowledge structures. Of interest to mathematics educators is the finding in a number of studies (e.g., Mayer, 1980; Hamilton, 1989, 1990) that elaborative processing enhances the ability of learners to apply the new knowledge to novel problem situations. Hamilton (1990) suggested that improved problem solving performance may be "due to the nature of the generative activity and its similarity to problem solving activity" (p.8).

Writing in mathematics

One method of stimulating elaboration in learning mathematics is the use of writing. Abel and Abel (1988) indicated that writing activities involve a way of thinking that requires students to organise their thoughts. When suitable writing activities are used in mathematics lessons, Burns (1988) found that cognitive development was enhanced as these activities helped students to reflect on their reasoning. Furthermore, she found that writing in mathematics enabled students to explore, clarify and extend their thinking. While student writing tasks have been used to identify misconceptions (e.g., Keith, 1988; Miller, 1990) and to ascertain attitudes (e.g., Borasi & Rose, 1989), there appears to have been little attempt to describe the qualities of student writing in a systematic way.

Describing the writing

The description of the qualities of student writing used in this study is in two parts. Firstly, the various parts of the student "texts" are described using textual analysis categories developed by van Dormolen (1985). In his work, van Dormolen was describing the presentations in school mathematics textbooks. The following categories have been adapted for this work.

kernels generalisations, rules and definitions which have to be learned
aspects of mathematics ways in which mathematical ideas and procedures may be presented - theoretical, algorithmic, logical, methodological, communicative
levels of language styles of language - exemplary (demonstrative), relative (generalised), and within each, procedural, descriptive

Secondly, the nature of the elaborations present are described by the following categories developed for this study by surveying the representations and embellishments used in mathematics textbooks to support the *kernels*. Some of the terminology matches that used by Kaput (1989) in his discussion of algebraic representations.

verbal words of explanation used to add meaning
graphical any type of mathematical graph (line, bar, etc.)
pictorial picture (drawing or photograph) of a real object
diagrammatic line figure showing features of a concept
worked example demonstration of a procedure or algorithm
exercise practice of a procedure for students to attempt
data set set of data, a table, related to the concept

Each of these elaborations are further characterised as being:

everyday set in a recognisable real life context, or
theoretical symbolic mathematics not related to a context

Any given text unit receives a number of codes. For example, a verbal elaboration may express an algorithmic aspect of mathematics and be in exemplary language.

The experiment

The aim of this experiment was to use writing activities to help students to elaborate and integrate their mathematical knowledge. This project concerned firstly seeing if there was any evidence of this type of thinking in student writing in mathematics and secondly, if there were ways of stimulating such thinking through writing.

A year 8 class of 25 girls at a private girls school participated in the experiment. The teacher was already making regular use of writing activities with the class. The four activities described below took place at approximately monthly intervals when the two researchers attended the class with the regular teacher.

Individual writing without pre-writing activity

The students were asked to write a letter to a classmate who was absent from school. The letter was to provide sufficient information for the absentee to enable her to understand and apply the concepts of highest common factor and lowest common multiple. These had recently been covered in class time and familiarity with the concepts was expected. The students were given about 20 minutes to complete the writing task. No prompting or discussion took place. The letters were collected at the end of the writing session.

This set of protocols showed a high degree of similarity across the class, consisting of a numerical worked example for one or both of the ideas with varying amounts of verbal elaboration. This verbal elaboration, when present, consisted of exemplary procedural language describing the steps needed to find the HCF or LCM. The mathematics being expressed was algorithmic in nature with no justification of why any of the steps were necessary. The following is a typical example of the style of writing.

> *4 - 1, 2, 4*
> *16 - 1, 2, 4, 8, 16*
> *4 would be the highest common factor. You have to start by putting down 4 and you write its factors then you put down 16 and you write its factors. You have to match up the two highest factors and that's how you get the highest common factor.*

Only four of the students attempted to write a kernel in the form of a definition of the terms HCF and LCM. Five students made links with prior knowledge by discussing the meanings of terms such as factor and multiple.

The elaborations present consisted of one or more worked examples with varying amounts of verbal support. These were theoretical in all cases. Kernels were inferred rather than stated in most cases. In summary, the writing in these protocols was categorised as exemplary procedural discussion of an algorithmic aspect of mathematics.

Group elaborative discussion and writing activity

A second set of protocols was generated using the same letter format, the topic this time being the addition and subtraction of directed numbers. This occurred shortly after class work on the topic and immediately followed a group discussion activity. Students, in groups of three, discussed the following questions with the assistance of the teacher.

1. You have probably heard of the range of temperatures that James Scott had to endure while lost in the Himalayas. What temperatures was he subjected to? Write down three examples where special numbers which are like negative numbers are used in everyday life.
2. Make up two examples from everyday life in which subtraction of a quantity leads to an answer like a negative number.
3. Do you think that negative numbers are important in everyday life? Why?

Each group wrote answers to the questions and generated examples involving temperature, money transactions and lifts. The aim of the exercise was to provide students with experience of elaborations linked to everyday contexts. The teacher had used money examples in her expositions and had also made use of a model involving walking on a number line.

This set of protocols was again consistent across the class and very similar in style to the previous set. All but one of the students used a diagrammatic elaboration of a number line as used by the teacher and the textbook, as well as at least one numerical worked example for addition and, where they were able, one for subtraction. The descriptions again expressed algorithmic aspects of mathematics in exemplary descriptive language, for example:

> *Say if you were at +7 and the sum said +7 - -3 = ? you would hop back three spaces which brings us to +4. You don't have to put the positive sign in front of a positive number.*

Many of the students were incorrect in their working of the subtraction operation, as in the case above. Notice that the second sentence expresses a communicative aspect of mathematics in relative language. Fifteen of the students produced statements in relative language expressing aspects of mathematics as in the following examples.

theoretical	*A negative number is below zero.*
methodological	*When doing directed number, you must always start at zero on the number line.*

Twelve of the students attempted to link with prior knowledge by providing a definition or explanation of negative numbers in relative language as in the following examples.

> *A negative number is one that is less than zero. eg, -1, -2, -3 etc.*
> *The negative numbers (-?) go to the left and the positive numbers go to the right (+?).*

The pre-writing discussion had a limited impact on the writing of these students. Three made mention of money in their explanations. One student mentioned a time-line which had not been discussed by the teacher and :. not mentioned in the textbook. Another discussed the operations on a calculator. However, no one made any mention of temperature which figured strongly in the pre-writing discussion. This was surprising : ecause of the proximity of the writing exercise to the discussion. Six of the students made use of words such as "walk", "steps" and "turn around" in reference to the model used by the teacher in developing the ideas. Of these, only two actually invited the reader to "think of a person on a number line", the others simply using the analogy without introduction.

With this particular topic, elaborations other than those used in the first protocols were possible. Numerical examples could be related to the everyday contexts of money and temperature. However, few of the students chose to include such elaborations in their explanations. Seven students provided from one to six practice exercises for the reader. In general, the letters were similar to the previous set with their emphasis on algorithmic mathematics.

Comparison of two written presentations of a mathematical idea

For this activity, students were provided with two written presentations of an introduction to the topic matrices and the addition of matrices. This topic was chosen as it was something unfamiliar. The "Method 1" presentation was based around an example using the deliveries of three types of milk to four houses. The terminology and operation of addition were developed using this example. The "Method 2" presentation had no everyday context, merely a series of definitions of terms and a statement of the method of addition using letter symbols.

Each student was provided with a copy of both methods and asked to read them. They then had to select which one they preferred and to state their likes and dislikes of each. Following this, the students compared and discussed their responses in groups. Twenty-two stated that they preferred Method 1. The most prevalent comments about what they liked mentioned the real-life example, fuller explanations, easier, the diagram, and easier wording. The reasons given for not liking Method 2 included complicated, boring, confusing letters, and hard to follow. Three students indicated that they preferred Method 2. The reasons given were that it went straight to the point, it was more precise and factual, it showed with letters what you have to do and it was quicker to write down. Reasons given for not liking Method 1 included confusing the issue, providing unnecessary information, and not explaining the correct terms.

Small group and large group discussion of written mathematics followed by a writing activity

In this session, students were formed into groups of 3 or 4 and presented with the question:

> When you are presented with something that's written in mathematics, what do you like to see to make it more meaningful to you?

Each group discussed the question for several minutes before reporting their conclusions to the whole class. A discussion of the suggestions then took place with all students and the researchers involved. In this discussion, the two sample pages used in the previous activity were again presented to the students. At the conclusion of the discussion, students were asked to individually write down their answer to the question about what they like in written mathematics presentations.

Student responses, without exception, stated that a worked example with each step clearly explained was the feature that made the written mathematics meaningful. More than half also suggested that the use of real life examples was helpful. A few suggested the use of a diagram or "change the words to numbers and symbols" would help in making it meaningful. Here are two typical answers.

> *I like to see an example set out so that I can understand the work.*
> *When I look at a maths question I really like to see examples which use familiar things such as eggs, lollies, clothes, books, etc. By something that you know you can look at the question and make a picture in your head.*

The students were then asked to write in the same letter format explaining the multiplication of common fractions, making the letter meaningful. The letters and the written answers to the previous question were collected at the end of the session. The letters were without exception presented in the same format, namely, a numerical worked example. In this, the letters were similar to those written in the previous activities. However, these letters did have more verbal elaboration in exemplary procedural language accompanying each step of the worked example. Half the students wrote a general rule or kernel in relative language, mostly very similar to the example that follows.

> *It is very simple you just multiply the numerator by the numerator and the denominator by the denominator.*

No real life examples or other elaborations such as diagrams were used.

Discussion

In this experiment, attempts were made to stimulate students to elaborate and integrate their mathematical knowledge through writing activities. The first writing showed that in this medium the students expressed their mathematical knowledge as algorithms illustrated with theoretical examples. In spite of a pre-writing discussion session with an emphasis on elaborating the ideas just learned, the second writing produced similar responses. In the third session, students were asked to compare two presentations. There was general agreement that the presentation elaborated with a real-life example was more meaningful although a few students preferred the more "to the point", unelaborated version. These materials were again viewed and discussed in the fourth session as a pre-writing activity. For the final writing, students were asked to attempt to make their presentations as meaningful as possible for the reader, after discussion of ideas about what makes writing meaningful. The resulting protocols, while displaying a wider use of language in some cases, were limited in the range of elaborations presented.

It should be noted that the students involved in this project used writing in mathematics classes with their teacher on a regular basis. The teacher's aim was to improve the students' reflection on their mathematics knowledge and consequently their understanding of it. Thus, writing in mathematics was not unfamiliar to the students.

It is difficult to know from these writing activities whether students are using elaborative thought processes while learning their mathematics. Even if such processing took place during learning and in the pre-writing activities, when asked to provide a written explanation for another learner, it is clear that these thoughts are not expressed. Other published examples of expository writing by students in mathematics (e.g. Miller, 1990; McIntosh, 1991) appear to be mainly procedural as well. Either students find it difficult to elaborate their expression of mathematical ideas or they don't see any value in such processes. They appear to value the essential "how to do it" information, in line with the findings of Quilter and Harper (1988).

Conclusion

The case for writing as a method of developing students' understanding of mathematics has been advanced strongly in recent years.

> Creating an original piece of writing requires students to analyse and synthesize information, focus their thoughts, and discover new relationships between bits of information. (Pearce & Davison, 1988, p. 6)

However, the examples of writing generated in this and other studies do not appear to demonstrate deep processing of information by students. While expository writing may provide a vehicle for some reflection by the students about what they know and a window for the teacher on errant procedures used by students, it appears that this type of writing is not providing the intended result. Even with the deliberate pre-writing activities used in this study, the writing remained largely procedural. It could be concluded that it is the students' view of mathematics and what it means to learn and know mathematics that is having an over-riding effect. This is the result of seven years' exposure to school mathematics instruction through teaching and textbooks. Further research needs to be carried out to examine the relationship between teaching style and students' mathematical writing.

References

Abel, J. P. & Abel, F. J. (1988). Writing in the mathematics classroom. *The Clearing House, 62*, 155-158.

Anderson, J. R. (1990). *Cognitive science and its implications*. New York: W. H. Freeman.

Baroody, A.,& Ginsburg, H. (1986). The relationship between initial meaningful and mechanical knowledge of arithmetic. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 75-112). Hillsdale, NJ: Erlbaum.

Borasi, R. & Rose, B. (1989). Journal writing and mathematics instruction. *Educational Studies in Mathematics, 20*, 347-365.

Burns, M. (1988). Helping your students make sense out of math. *Learning 88, 16*(5), 31-36.

Hamilton, R. (1989). The effects of learner generated elaborations on concept learning from prose. *The Journal of Experimental Education, 57*(3), 205-217.

Hamilton, R. (1990). The effect of elaboration on the acquisition of conceptual problem-solving skills from prose. *The Journal of Experimental Education, 59*(1), 5-17.

Kaput, J. J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieren (Eds.), *Research issues in the learning and teaching of algebra* (pp. 167-194). Reston, VA: NCTM.

Keith, S. Z. (1988). Explorative writing and learning mathematics. *Mathematics Teacher, 81*(9), 714-719.

Mayer, R. E. (1980). Elaboration techniques that increase the meaningfulness of technical text: An experimental test of the learning strategy hypothesis. *Journal of Educational Psychology, 72*(6), 770-784.

McIntosh, M. E. (1991). No time for writing in your class? *The Mathematics Teacher, 84*, 423-433.

Miller, L. D. (1990). When students write in algebra class. *The Australian Mathematics Teacher, 46*(2), 4-7.

Pearce, D. L. & Davison, D. M. (1988). Teacher use of writing in junior high mathematics classrooms. *School Science and Mathematics, 88*(1), 6-15.

Quilter, D. & Harper, E. (1988). 'Why we didn't like mathematics and why we can't do it'. *Educational Research, 30*(2), 121-134.

Reder, L. M. (1980). The role of elaboration in the comprehension and retention of prose: A critical review. *Review of Educational Research, 50*, 5-53.

Swing, S. & Peterson, P. (1988). Elaborative and integrative thought processes in mathematics learning. *Journal of Educational Psychology, 80*(1), 54-66.

van Dormolen, J. (1985). Textual analysis. In B. Christiansen, A. G. Howson & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 141-171). Dordrecht: D. Reidel Publishing Company.

THE CONSTRUCTS OF A NON-STANDARD TRAINEE TEACHER OF WHAT IT IS TO BE A SECONDARY MATHEMATICS TEACHER.

Pinder Singh
University of North London

This paper focusses on the early stages of a study of non-standard-entry trainee teachers. The research is concerned with the shifts in their construction of a what it is to be a teacher in an inner city secondary school. In future the study also intends to explore the factors that cause these shifts.

INTRODUCTION:

Research in the past has identified a definite correlation between teachers' perceptions and belief systems and in the learning and teaching oof mathematics (Silver(1985), Thompson(1982)). An attempt has been made by several researchers to explain the contributory factors of teachers' existing attitudes and belief systems. The perspective of the teacher's whole life is recommended by Woods, P.(1984) since 'self" in his opinion, is derived from several elements in personal history. The influence of wider society has been emphasized by Stigler and Perry(1987). Moreira(1991) claims that the belief profiles of teachers from different countries would be different. She attributes these differences to their different educational and social situations.

Hoyles(1992) reported on the extensive research highlighting inconsistencies to be found in the belief systems of an individual. Woods(1984) believes that apparent inconsistencies are acts of survival whereby teachers' behaviour changes constantly. In his opinion the constant change is like a pendulum moving between bi-modal features such as tolerance and intolerance; it is a result of interaction with different situations.

Hoyles(1992) expressed a concern that due to an inconsistency in methodologies and theories related to research in teachers' belief systems, it was often difficult to make comparisons. Qualitative research methods had been shown to to be used extensively. Chapman(1992) uses the method of autobiography which enables teachers to explore and understand their belief systems. She has found this method effective as it allows the teachers to make relevant changes themselves.

BACKGROUND:
The trainee teachers on a two year B.Ed course in Secondary Mathematics(11-16 years) come from a wide variety of backgrounds - social, cultural and educational. They are non-standard mature students, many of whom have followed other careers in the past and have not studied for some time. The entry requirement to the course specifies equivalent to one year undergraduate study in mathematics resulting in a wide range of mathematical experience in the group. The trainee teachers on the course learn mathematics through an investigative approach which is unfamiliar to most of them. They are being trained to teach in inner city schools which have a culture quite different from the one they are familiar with. They are required to be effective mathematics classroom practitioners by the end of the course, they need to be able to survive afterwards, and to keep developing.

Before they start the course, each trainee teacher is operating with a(not necessarily explicit or articulated) construct of what it is to be 'maths teacher'. It is important to recognise that these constructs will be contigent, and not necessarily stable over time. Neither consistency nor coherence need be assumed. Over the two years of an intensive period of training - both at Higher Education Institution and in schools - these constructions may undergo significant shifts. This study illuminates these constructs and any changes they undergo over this period.

The theoretical framing for this study draws upon the work of psychologists such as Potter and Wetherall(1987) and Billig(1988) who have suggested that the traditional notion of an 'attitude' may not do justice to either the complexity of the belief systems of individuals or the social nature of the constuctions themselves.

METHODOLOGY:
The concern of this work is with teachers' constructions of the "teacher" and "mathematics teaching" as I have already stated. These constructions are not nessarily stable over time, nor are they always well-articulated, explicit or even conscious. They may also be joint constructs, held for a time by more than one trainee teacher. Therefore I have utilized several strategies including: semi-structured interviews, trainee teachers' autobiographies, structured observations, documentation e.g. their block practice files, intervention instruments e.g. game and questionnaire and narratives written by trainee teachers during each of the two block practices.

The changes in the teachers' constructions are monitored and linked to factors such as school/college experiences, personal crises, political events, professional demands etc.

COLLECTION OF DATA:

A) Each trainee teacher was interviewed before the course started. Two questions were asked at semi-structured interviews which were audio-taped and later transcribed. The two questions asked are as follows:

(i) You have recently visited two secondary schools and had the chance to observe mathematics teachers. How do these differ from your teachers when you were at school?

(ii) What do you currently believe the role of a mathematics teacher to be? What skills/characteristics are important?

Most of the trainee teachers had not been able to visit a secondary school due to unavoidable reasons. Hence they discussed about their own teachers.

B) During the second week of the course each trainee teacher was asked to write a mathematical autobiography.

Data from the semi-structured interviews and from their mathematical autobiographies was sorted into three main categories. These are as follows:

(i) Reasons why they enjoyed mathematics.
(ii) Reasons why they wanted to do well at mathematics.
(iii) Reasons why they felt negative feelings about learning mathematics.

A profile of the whole group emerged from this data.

C) An Intervention Instrument was devised in the form of a Questionnaire consisting of twenty statements. A sample of the responses from each of the three above-mentioned broad categories constituted statements relating to what a good teacher does. Each trainee teacher had to tick only one of the four options numbered 1-4.

1 represented 'not needed'
4 represented 'essential'

Data from this questionnaire was collected and a total score for each possible answer was tabulated. This was then represented graphically.

D) The trainee teachers wrote a narrative about their first experience in an inner city school during a serial day placement visit. This took place seven weeks into their course.

RESULTS AND DISCUSSION:

At this early stage in the three year research study, any comparisons and findings have inevitably a tentative flavour. However, I am able to offer a snapshot of the teachers' initial constructions. I shall start by providing the responses made by two trainee teachers.

Tolay is a male student from a different cultural and educational background to that of an inner city English School. Kay is a female student from UK but not familiar with inner city schools. (Names changed).

Snap shot one: Tolay

Mathematical biography- summary:

He enjoyed the subject-

Mathematics - challenging, interesting, right answers.

Teacher - gave different examples, involved us, made things clear.

Teacher's enthusiasm and interest in the subject.

Negative feelings:

Although Tolay suggested rote learning, reciting tables and the fact there was no visual representation, he did not view this as negative. He was not excited about it either.

He wanted to do well:

praised - 'a clever boy'

rewarded - right answers.

The construction of a good mathematics teacher:

Essential that a good teacher:

- encourages and rewards those that get right answers
- makes maths visual
- makes the subject interesting.

Relevant that a good teacher:

- explains a concept in more than one way
- creates a disciplined environment.

Sometimes a good teacher may

- instruct the children in particular procedures.

A good teacher does not need

- to encourage a competetive atmosphere.

His account of the first day in a school:

Initial shock

Some difficulty in relating to pupils

Discipline - authority of teacher, non-existent

Difficult to understand the attitude of pupils towards learning

Shock and confusion - in relating to things

to find his orientation.

All the above data is given using the trainee teacher's own words.

Snapshot two: Kay

Mathematical biography- summary:

She enjoyed the subject:

Teacher - enthusiastic, stimulated the pupils

Atmosphere - relaxed

Mathematics - right answers.

She had negative feelings:

Teacher - sarcastic

Mathematics - traditional approach

did not understand.

She wanted to do well:

Competetiveness - challenge.

Her construction of a good mathematics teacher:

Essential that a good teacher

- creates a relaxing environment
- creates a disciplined environment
- makes the subject interesting.

Relevant that a good teacher

- recognises the emotive nature of the subject
- encourages play in class.

Sometimes a good teacher may

- need to encourage competetive atmosphere
- encourage and reward those that get the right answers.

A good teacher never considers
- taking children outside the classroom for doing maths.

Her account of the first day in school:

Experience likened to 'Alien'
No blackboard in some of the rooms
Teacher did not teach the class as a whole
His(teacher) presence should be stronger
Not used to movement in the class.(trainee teacher)
Multicultural nature of the class - alien.
e.g. clusters of young moslem girls with their heads covered
Noisy - paid little attention to teacher
Pupils not as timid as in my time
SMILE - Individualised scheme, fragmented with no continuity
felt frustrated and confused.

CONCLUSION:

Both of the trainee teachers have been shocked by their initial visit to schools. Two specific points can be made.
(1) For both students the exposure of their first day in school contrasts sharply with their expectations and with their beliefs about what good maths teaching involves.
(2) Kay has an idea of "presence in the classroom". This appears to be absent from Tolay's construction. This idea in Kay's construction of the teacher may help to formulate what strategies she will need to employ. Also, it provides a point of contact between her own expectations and those of the tutor's(Singh, 1993).

General points:

Overall the research supports the tutor's initial experience that non-standard entry trainee teachers find great difficulty in matching their own experience of mathematical teaching and their expectations of what a teacher does and how s/he behaves with the "reality" of the 1990's classroom in an inner city school. As one of the college tutors put it , "After their first visit to school, they suffer from shell-shock!" A majority view at this point in time would be that very little learning takes place and the children have very low ability. It appears to these trainee teachers that not much mathematical work is going on at all. A fair summary of this data confirms the impression that the conflict between these teachers' own educational experiences and their classroom visits results in a low opinion of current mathematical teaching styles and pupils' behaviour.

After varying periods of time, the trainee teachers start to re-evaluate these classrooms which they visit. Kay commented that she had sat down to help a pupil and that she began to see all the small things happening which, " I had failed to see the first time I visited the classroom." The trainee teachers often remark on the teacher's casual approach and their acceptance of "stroppy behaviour from the pupils". They strongly suggest that the teacher does not convey the right impression to the pupils, that this attitude - less authoritarian and clear cut than that which they expect - is not conducive to learning. However, as they progress they begin to pin-point the skills and strategies which underlie this supposedly "laidback" approach . "I find it very difficult to relate to the pupils - but the teachers have found a way of relating to the pupils that works."

Another factor which has arisen in majority of interviews concerns the "low ability" of the pupils in these inner city schools as contrasted by the trainee teachers with "how things used to be for them". "The level of maths is very poor. I am shocked if I compare it to what I knew at this age." The implication here is strongly that this is the fault of the teacher. However, as the trainee teachers continue to work in the classroom, they speak of the possibility of other factors affecting pupils' apparent low achievement. The materials and resources, in particular SMILE, came in for more sustained criticism and the teacher's approach is likely to be more appreciated. Sometimes the fact that the classes contain pupils working at such a variety of levels within the maths curriculum is commented upon. "I liked the mixed-ablity classes but I find it easier to help the pupils in a streamed class where all the pupils are at a similar level".

SUMMARY

It is as if, over a period of time, these trainee teachers come to "see" the classroom in which they are located through different spectacles. The distance between their own personal educational experiences as articulated in their biographies, and their views of what a good mathematics classroom should look like, gets appreciably greater as time progresses. Other factors have become much more important such as the presence of investigations is recognised and the extent to which teachers are "on the children's wave length". The question which remains to be answered is whether the narrowing of this gap between their initial impressions of the "messiness" or even "ineffectiveness" of the 1993 mathematics classroom and their beliefs about what good teachers should look like and achieve, is actually a reflection of a clearer perception of what is going on, or if it is rather a measure of the extent to which, as one of the tutors put it, "we(the college) have corrupted them to see things like us".

Bibliography:

BILLIG, M. (1988) Ideological Dilemmas: A Social Psychology of Everyday Thinking , London: Sage

CHAPMAN, O. (1993) Facilitating In-Service Mathematics Teachers Self-Development, Proceedings of PME17, Tsukuba, Japan, Vol.I, pp.228-235.

HOYLES, C. (1992) Illuminations and Reflections- Teachers, Methodologies and Mathematics in Proceedings of PME16, New Hampshire, USA. Vol.III

MOREIRA, C. (1991) Teacher Attitudes Towards Mathematics and Mathematics Learning: Perspectives Across Two Countries, Proceedings of PME15, Assissi, Italy, Vol.II, pp.17-24

POTTER, J. AND WETHERALL, M. (1987) Discourse and Social Psychology: Beyond Attitudes and Behavoiur, London: Sage.

SILVER, E.A. (Ed) (1985) Teaching and Learning Mathematical Problem Solving: Multiple Research Perspective, London:Lawerence Erlbaum Associates.

SINGH, P. (1993) Having Presence in a Mathematics Classroom, Proceedings of a one-day conference of BSRLM, Manchester, Nov.1993

STIGLER, J. AND PERRY, M. (1987) Cross-cultural Studies of Mathematics Teaching and Learning: Recent Findings and New Directions, Proceedings of PME11, Montreal, Vol.III,pp.421-422

THOMPSON, A.G. (1982) Teachers, Beliefs and Conceptions: A Synthesis of the Research, In GROUWS, D.A. (Ed.) Handbook of Research on Mathematics Teaching and Learning, New York:Macmillan.

WOODS, P. (1984) Teacher, Self and Curriculum, in GOODSON,I.F. and BALL,S. (Ed.) Defining the Curriculum, Lewes,Sussex: Falmer Press.

ALGEBRAIC SUMS AND PRODUCTS: STUDENTS' CONCEPTS AND SYMBOLISM

Kaye Stacey & Mollie MacGregor[1]
University of Melbourne, Australia

This paper presents evidence on the difficulties that students have in symbolising sums and products algebraically. A cluster of obstacles related to the conjoining error have been identified in the literature and are seen by many to be the principal causes of students' notational difficulties in elementary algebra. This paper reports on the incidence of the conjoining error in a variety of new situations. The incidence of the error was found to be very variable. It was low in solving linear equations and did not arise at all in formulating an equation. Many of the incorrect responses are explained not by the published obstacles, but by examining the non-mathematical parallels which students draw on to write algebraic expressions. At all levels there were students who did not make clear distinctions between addition, repeated addition, multiplication and exponentiation.

It is well known that the algebraic symbolism for algebraic sums and products is puzzling for beginners, who must learn that the sum of x and y can only be written as $x+y$ or $y+x$ whereas the product can be written as something other than $x \times y$. A textbook popular in the 1950's, for example, warns students that the notation for algebraic sums is their "first difficulty". This paper explores how students represent and interpret algebraic sums and products in a variety of situations and examines the adequacy of current explanations for students' errors.

Writing conjoined terms for sums

Many research studies have documented the tendency for students to use the notation for algebraic product for the algebraic sum, i.e., to use the conjoined expression ab where $a+b$ is appropriate. Kuchemann (1981), working with the CSMS project, found that the error was extremely prevalent, and Booth (1984), after investigating the underlying causes of the errors that the project had exposed, concluded that conjoining in algebraic addition was a serious problem. The prevalence of the error in British students was confirmed by the monitoring of the Assessment of Performance Unit (1985). Examples of the items used in these British studies, showing the very high rates of conjoining errors on some items, are given in Table 1. These authors see the conjoining error as a significant error which is hard to remediate and with an important cause, as discussed below.

As with other common errors and interpretations in mathematics, there are several different reasons why students may believe that $a+b$ should be written as ab. Kuchemann (1981) noted that students who made errors derived from this misunderstanding (e.g., writing $8ab$ as a simplification of $2a+5b+a$) were generally at the lowest level of understanding on the CSMS test. He saw this error as an important indicator of "acceptance of lack of closure", a characteristic which Collis had identified as developing through the stages of cognitive growth. For Collis, an individual requires different levels of closure during the progression from concrete to formal thinking. In the early stages,

[1]This research was funded by a grant from the Australian Research Council to Kaye Stacey for the project "The Cognitive and Linguistic Aspects of Learning to Use Algebra". We wish to thank the teachers and students who participated in the testing for their valuable assistance.

children can only work on tasks where each expression can be evaluated. In Collis' later work (Collis & Biggs, 1979) acceptance of lack of closure is used in a wider sense, noting a mature student's tendency not to restrict an answer, but to generalise from the given, making links with new data. In the psychology literature, "closure" refers to the completion of a Gestalt when parts of the stimulus are missing, in order to make sense of it.

Kuchemann (1981) and Booth (1986) also observed that many students may write $7n$ as the answer to the question "Add 4 onto $3n$" because they believe the expression $4n+3$ is not acceptable as an answer. In the terms of Sfard (1991), students see the expression as representing the process of adding, not the object that results from the adding. Tall and Thomas (1991) have summarised these observations by identifying a cluster of obstacles involved with the conjoining error: those related to parsing, process/product confusion, lack of closure and the nature of the expected answer.

From a constructivist perspective, it is noted that students will draw on previous and concurrent learning from other areas to work with algebraic symbols. They will make parallels with other notation systems, such as in writing fractions where conjoining represents addition (e.g., $3\frac{1}{2} = 3+\frac{1}{2}$), in chemistry where CO_2 is produced by adding oxygen to carbon, and in music where ♩♪ lasts for one and a half beats. Chalouh and Herscovics (1988) observed Grade 6 and 7 students interpreting algebraic expressions in terms of other frames of reference (e.g., place value $53 = 50+3$).

Booth (1984) and others use the words "closed" and " unclosed" to distinguish answers such as $7n$ and $3n+4$. However, in this paper we refer instead to "conjoined terms" (for $7n$, etc.) because "closed" does not distinguish between observed responses and proposed causes.

The review above shows that many researchers believe that errors based on conjoining are particularly significant indicators of cognitive growth. There is, however, some disagreement. Sutherland (1991) has found that children working with Logo and spreadsheets accept "unclosed" expressions such as $x+7$ without difficulty and she questions the claim that the need for closure is a major obstacle in learning algebra. Tall and Thomas (1991), also working in a computer environment, noted that there needs to be "a reassessment of fondly held beliefs of what is hard and what is easy" (p. 145).

In this paper we present data which extends information on the prevalence of conjoining errors. Previous studies used only items requiring simplifying or formulating algebraic expressions. We present further data on similar items but we also look for evidence of the conjoining error in students' attempts to formulate and solve equations. We examine the data to see to what extent students use conjoining only for addition and we examine the adequacy of the published cognitive obstacles as causes of students' responses.

Method

This paper brings together data from a number of separate studies. Tests which included the items used in this paper were constructed and marked by the authors and administered by volunteer teachers in 24 Australian secondary schools in 1991-93. Although the participating schools were not randomly selected, the sample includes schools from two states, from all socio-economic areas and a mix of government and private schools. Almost all of the participating schools take students of all abili :es from the local area and teach them in mixed-ability mathematics classes. In order to expose important difficulties that are widespread, the results have been analysed by class and by school as well as by individual as is reported in this paper and were found to be reasonably consistent across classes. In this way we hope to reduce the limitations of this form of sampling to be confident that the results are not confined to any particular school, category of school or textbook. The tests contained items testing ordinary reading competence, the results of which established that literacy in English was a limiting factor for no more than 3% of the students involved.

Table 1: The prevalence of the conjoining error in published studies

Source	Item	Percentage correct	Percentage who conjoined for addition and used exponents for multiplying
Kuchemann 1981 Item 13(iv)	Simplify $2a+5b+a$	60% (14 yr olds)	20% ($8ab$)
Kuchemann 1981 Item 4(ii)	Add 4 onto $3n$	36% (14 yr olds)	31% ($7n$)
APU 1985 Item A4	Peter's age is represented by x. Alan is 2 years older than Peter. How can we represent Alan's age?	49% (15 yr olds)	10% ($2x$)
APU 1985 Item A5b	How do we represent the number which is three less than n?	45% (15 yr olds)	3% ($-3n$)
APU 1985 Item A5c	How do we represent the number which is twice n?	44% (15 yr olds)	23% (n^2)
APU 1985 Item A6	I have x pence and you have y pence. How many pence do we have altogether?	48% (15 yr olds)	34% (xy)
APU 1985 Item A7	I have x pence and you have 3 pence. How many pence do we have altogether?	46% (15 yr olds)	36% ($3x$)
APU 1985 Item A8	A bar of chocolate costs x pence and a packet of crisps costs y pence. What is the cost of 2 bars of chocolate and 3 packets of crisps?	59% (15 yr olds)	9% ($5xy$, $2x3y$)

Results and Discussion

The items used are shown in Figure 1. The number of students fo each item and the number of schools from which they came is given in Table 2 along with the pe.centage correct and making conjoining errors. The final column also records the percentage of students who used exponential notation to represent multiplication. Because of space limitations, the table gives results for students in Years 7 and 10 only. The students in this sample have performed similarly to students in the APU samples on some items, but the overall rates of conjoining are markedly lower.

Item MS1. Write in mathematical symbols:
(i) Add twelve to x. ....
(ii) Multiply x by three. ....

Item MS2. Write the following in mathematical symbols:
"Add 5 to an unknown number n, then multipl · the result by 3".

Item MS3.
(i) David is 10 cm taller than Con. Con is h cm tall. What can you write for David's height?
(ii) Sue weighs 1 kg less than Chris. Chris weighs y kg. What can you write for Sue's weight?
(iii) Tina has twice as much money as Dino. Dino has $\$n$. What can you write for the amount of money Tina has?

Item MS4. Which of the following expressions can be written as $x+x+x+x$?
(Circle one or more of the answers below)

$x+4$ $\quad$ $x\times 4$ $\quad$ $4x$ $\quad$ x^4 $\quad$ 4^x

Item MS5. [Presented following table in vertical format, asked students to calculate three values, describe the rule in words and then asked them to "Use algebra to write a rule connecting x and y."

x	1	2	3	4	5	6	7	8	.	.	.	.
y	5	6	7	8	9	10	11	12	.	.	.	.

Item MS6. Solve the following equations [sample only]:
$4x - 7 = 3$, $\quad$ $16.4x - 7.9 = 0.3$, $\quad$ $34 - 8x = 10$, $\quad$ $2 + 0.6x = 2x$

Item MS7. Given $H=3f^2g$, find f in terms of the other letters. [Then similarly find g]

Figure 1: Items used in testing

1. Conjoining in translating into algebraic expressions

Item MS1 in Figure 1 requires students to translate statements in words directly into algebraic expressions. Table 2 shows that these items were correctly answered by a high percentage of students. A popular response for part(i), which was counted as correct in the table, was "$x + 12 =$ ". Explicit use of the multiplication sign ($3 \times x$ or $3 \times x =$) was the most common correct response for part (ii). The use of the "equals" sign suggests that students regard the expressions as incomplete and needing to be evaluated.

The very low incidence of conjoining in this item is surprising given the APU and CSMS results for Items A6 and 4(ii) shown in Table 1. Possibly Item MS1 asked more directly than did the British items that the procedure "add twelve", rather than the answer, be symbolised, .

The conjoined term was written for addition by only 24 individuals (3.5% of the total sample), but it was not confined only to younger students and it occurred in 17 of the 21 classes. Three of these students used conjoining again (now correctly) for the multiplication in part(ii) and four used division. However, fourteen used the multiplication sign explicitly and three used exponents. Thus about 3% (i.e., 14+3) of the total sample showed that they used conjoining for addition but not for multiplication.

Table 2: Success rates on items and incidence of conjoining

Item	Sample	% correct Yr 7 - Yr 10	% who conjoined for + or - Yr 7 - Yr 10	% who used exponents for x Yr 7 - Yr 10
Item MS1(i) Add 12 to x	678 students (9 schools)	68% - 93%	2% - 2% ($12x$)	
Item MS1(ii) Multiply x by 3	678 (9 schools)	73% - 91%		2% - 2% (x^3)
Item MS2 $(n+5) \times 3$	1806 (18 schools)	14% - 47%	14% - 12% ($15x$, $5x \times 3$)	about 1%
Item MS3(i) David $h+10$	1463 (16 schools)	39% - 74%	14% - 5% ($10h$)	
Item MS3(ii) Sue y -1	1463 (16 schools)	36% - 64%	8% - 3% ($1y$) 1% - 1% ($-1y$)	
Item MS3(iii) Tina $2n$	1463 (16 schools)	38% - 65%	2% - 1% (nn)	17% - 14% (n^2)
Item MS4 $x+x+x+x$	1034 (12 schools)	47% - 60%		40% - 34% (x^4 or 4^x)
Item MS5 $y = x+4$	951 students (11 schools)	29% - 58%	0% - 0%	
Item MS6 Solving one variable equations	97 (3 schools, Yr 10 only)		3% ($2+0.6x = 2x \Rightarrow 0.6x = x$)	
Item MS7 Transposing equation $H = 3f^2g$	48 (3 schools, Yr 10 only)		16% ($H-g = 3f^2$)	

2. *Conjoining in a multiple step expression*

Item MS2 had the same stem as Item MS1 but required students to use two operations. The most common error was the omission of brackets (11% at Year 10, 22% at Year 8). Conjoining was very prevalent, although still not as prevalent as in several of the APU or Kuchemann items in Table 1. Common conjoined answers were $15n$ (about 8%) and $5n \times 3$ (about 4%), and another 4% were responses such as $3(5n)$, $5n^3$, $15n^3$, $8n$, $18n$. In another test this item had been replaced by an item identical to MS2 except that 5 was replaced by 14. Although there was a similar amount of conjoining overall, the relative frequency of responses was very different. Only 3 of the 517 responses were $42n$ (down from 8%) but $14n \times 3$ was more common. Many features of an item determine students' responses, one being the size of the numbers and the ease of automatic response.

The incidence of conjoining for "add 5 to n" here was markedly higher than occurred in Item MS1(i), which was identical except for the second stage of multiplication by 3. Presumably many students felt the need to work out an answer before proceeding to multiply by 3. The higher incidence of conjoining in this item is in marked contrast to the conclusion reported in the APU survey. On the basis of comparison of the incidence of conjoining in Items A6, A7 and A8 (see Table 1), the APU report concluded that "when two operations are involved (multiplication and addition), fewer pupils represent addition by juxtaposition and more get such items right." (p 331). We did not find this.

3. *Conjoining in formulating expressions*

In Item MS3, students had to deduce which operation was required and write the answers algebraically. In the three parts of this item, results for our sample (especially the 15 year old Year 10 students) were similar to the results for the parallel APU items A4, A5b and A5c (including the number of omissions), although a little better. The incidences of conjoining for addition and subtraction were similar, and in both cases many students used exponents for multiplication.

However, less than 3% overall used conjoining only for addition. We consider that if students had clear concepts for the three operations involved and believed that conjoining is the convention for addition, then they would write $10h$ or $h10$ or 10C for DAVID, $1\text{-}y$, $y\text{-}1$, $-1y$ or similar for SUE and $2 \times n$ or n^2 for TINA. There were 28 students (2%) who did this. An additional 8 students gave the answers $10h$, $1y$ and n^2 indicating that they wished to symbolise multiplication differently to addition. This percentage is similar to the estimate of consistent students from MS1.

Forty students (3%) wrote down the numbers and the letters without concern for the operations linking them. They all wrote $10h$, $1y$ and $2n$. Although their response to DAVID suggests that they have used conjoining for addition, the other two responses indicate that they have used it for subtraction and multiplication as well.

The variety of incorrect responses to this item indicates that students draw on different ideas about letters and abbreviations to construct their algebraic expressions. Year 10 students in the sample used in their algebraic expressions letters as abbreviations for words ($Dh = h+10$, where Dh stands for David's height, $h = D\text{-}10$, or $C+10 = D$); letters standing for procedures ($h = +10$ or $h = -10$); letters as objects and objects as letters (h = David-10, h = Con-10). As these examples from Year 10 show, many of the wrong answers were expressed as equations rather than expressions. The students who wrote $x\text{-}10 = h$, $h\text{-}10 = x$, and $h = h+10$ show the need students felt to have a second variable.

Item MS4 is included to investigate further the confusion between addition, repeated addition and multiplication and exponential notation, which was evident in MS3(iii)-TINA. Adding like terms is one of the first skills taught in school algebra, so the low success rate (see Table 2) is disappointing. However, the choice by 40% of Year 7 students and 32-34% of Year 8, 9 and 10 students of an exponential form supports the result of MS3(iii) and the APU Item A5c. The improvement from Year 7 to Year 10 was minimal.

4. *Conjoining in formulating equations*

Item MS5(iv) required students to write the equation $y = x+4$ (or equivalent) to describe a relationship evident in the table of two variables (see Figure 1). Although the success rate (see Table 2) was very low, and there was great variety in the answers, there was no conjoining. No student wrote $y = 4x$, $y=x4$, or simply $4x$. One wrote $4x+$ and another wrote $x=4y$. The absence of the conjoining error is surprising given the variety of answers that were produced. Examples included $1x=5y$, $x=y$, xy, $x+y$, $x+4y$, $x=y+4$, $xy+4$, $x=1y$, $x+5$, $(x=1,y=4)$, $x=1+4y$. In some of these examples, students are simply "connecting" x and y as instructed in the Item (e.g. xy, $x+y$). Some tell a story in abbreviated form (e.g., x=1+4y may say that "you start with x equal to 1 and add 4 to get y", and x+4y may say that "you take x and add 4 to get y"). Many of these responses constructed by students themselves are far from the syntactically well-formed expressions and equations that could demonstrate a conjoining error.

5. *Conjoining in solving equations*

A sample of 97 Year 10 students were given a total of 251 equation solving questions (Item MS6 in Figure 1) as part of another study and its pilot (Bell, MacGregor & Stacey, 1993). Their solutions were examined for evidence of interpreting conjoined terms as sums. Only seven such errors were made (e.g., deducing $0.6x = x$ from $2+0.6x = 2x$, or $x = 160$ from $13x = 173$), each by only one student and each in only one equation. Some of these errors are what Carry, Lewis & Bernard (1980) classify as the very common *deletion errors*. We conclude that conjoining errors are not common in familiar numerical equation solving (by Year 10students), a "microworld" where addition signs are common and routine procedures are often practised. However, in transposing the literal equation $H = 3f^2g$ (Item MS7), the conjoining error rate was high (see Table 2). This item was less familiar (indicated by a relatively high omission rate) and more abstract. Presumably these characteristics prompted the conjoining error.

Conclusions

Where parallel items were used, the results of this testing are similar to but a little better than the APU testing reported in Table 1. However, none of the items provoked conjoining to the same extent as the items reported there. There was substantial variability in the incidence of conjoining from item to item. The greatest incidence of conjoining came from the two step item (MS2) and the difficult transposing of an equation (Item MS7). In formulating an equation, there were no instances of conjoining errors at all. The students who were not correct very often wrote ill-formed equations (discussed further below).

For items MS1 and MS3, it was estimated that about 3% of the sample used conjoining to represent addition and only addition. About half of the students who used conjoining for addition used it for subtraction and multiplication as well. These students are simply putting together the numbers and symbols in the question to make an answer that looks like algebra. They are not applying consistent laws of logic to well distinguished concepts, but are probably achieving closure only in the sense of completion of a Gestalt.

Items MS3(iii) and MS4 revealed a substantial percentage of students using exponential notation for ordinary multiplication or repeated addition, and little improvement from Year 7 to 10. We suspect that this is due to the concepts of repeated addition, multiplication and repeated multiplication (exponentiation) not being sufficiently distinguished by students. This is a very important aspect of arithmetic that needs considerable emphasis in pre-algebra courses.

The literature reviewed above on students' use of algebraic symbolism has listed several obstacles related to the conjoining error. Students' responses to this testing have provided examples of all of these. However, many of the responses which students have constructed (e.g., to Items MS3 and MS5) are not explained by these obstacles. Instead, they seem to arise from students' drawing parallels with notation systems other than algebra. The published obstacles seek to explain students' errors from within mathematical systems and ways of thinking and symbolising and from an assumption that operations are clearly conceived, but students seem instead to often use a personal shorthand that only looks on the surface like algebra. For example, the conjoining error results in a syntactically well formed answer, but students' answers are often ill-formed. The personal shorthands draw on principles from a variety of other (informal) systems, such as the practice of abbreviating words and leaving out unimportant words when writing a note. Personal shorthands are not used consistently from question to question, nor is one symbol (or lack of symbol as in conjoining) kept for one meaning. Students often achieve closure in the sense of completion of a Gestalt, rather than applying a formal rule. Because their personal shorthands use the same surface features as does algebra, it is difficult for teachers to explain that the meaning is quite different.

References

Assessment of Performance Unit (1985). *A review of monitoring in mathematics 1978 to 1982.* London: HMSO.

Bell, A., MacGregor, M., & Stacey, K. (1993). Algebraic manipulation: actions, rules & rationales. In B. Atweh, C. Kanes, M. Carss & G. Booker (Eds.), *Proceedings of 16th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 101-109). Brisbane: MERGA.

Booth, L. R. (1984). *Algebra: children's strategies and errors. A report of the Strategies and Errors in Secondary Mathematics Project.* Windsor, Berks.: NFER-Nelson.

Booth, L. R. (1986). Difficulties in algebra. *The Australian Mathematics Teacher, 42*(3), 2-4.

Carry, L. R., Lewis, C., & Bernard, J. (1980). *Psychology of equation solving: an information processing study* (Final Technical Report). Austin: University of Texas at Austin, Department of Curriculum and Instruction.

Chalouh, L. &. Herscovics, N. (1988). Teaching algebraic expressions in a meaningful way. In A. F. Coxford (Ed.), *The ideas of algebra, K-12* (1988 Yearbook, pp. 33-42). Reston: NCTM.

Collis, K. F., & Biggs, J. B. (1979). *Classroom examples of cognitive development phenomena: the SOLO taxonomy.* Report prepared at the conclusion of an Educational Research and Development Committee funded project. Hobart: University of Tasmania.

Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics:11-16,* (pp. 102-119). London: Murray.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1-36.

Sutherland, R. (1991). Some unanswered research questions on the teaching and learning of algebra. *For the Learning of Mathematics, 11,* 40-46.

Tall, D. & Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics,* 22, 125-147.

THE INTUITIVE RULE
"THE MORE OF A - THE MORE OF B"

Ruth Stavy and Dina Tirosh
School of Education, Tel-Aviv University

Abstract

In the last 20 years, researchers have studied students' mathematical and scientific conceptions and reasoning. Most of this research is content-specific. It was found that students hold ideas that are often not in line with accepted scientific notions. In our joint work in mathematics and science education, it became apparent that many of these alternative conceptions hail from the same intuitive rules. We have so far identified two such rules: "The more of A - the more of B", and, "Everything can be divided by two". The first rule is reflected in students' responses to many tasks, including all classical, Piagetian conservation tasks, in all tasks related to intensive quantities, and in tasks related to infinite quantities. The second rule is observed in responses related to successive division of material and geometrical objects and in successive dilution tasks. In this paper we describe and discuss the first rule and its relevance to science and mathematics education.

Throughout the past 20 years, researchers have studied students' conceptions and reasoning in the context of science and mathematics. Many have pointed out the prevalence of persistent alternative conceptions which are often not in line with the accepted scientific notions. Such conceptions covered a wide range of subject areas in physics (astronomy, mechanics, electricity, heat, light, the particulate nature of matter), in chemistry (mole concept, concept of equilibrium), in biology (growth, health, photosynthesis, heredity) and in mathematics (function, infinity, area). (For surveys of students' conceptions, see Driver, Guesne, & Tiberghien, 1985; Eylon & Lynn, 1988; Fischbein, 1987; Hart, 1981; Osborne & Freyberg, 1985; Perkins & Simmons, 1988). Most of this research has been content-specific, and aimed for detailed descriptions of particular alte[illegible] concepts (Thijs & Berg, 1993).

In our work in mathematics and science education, it has become apparent that many alternative conceptions are based on common, intuitive rules. We have so far identified two such rules: "The more of A, the more of B", and "Everything can be divided by two". In this paper we describe and discuss instances of the first rule.

Conservation Tasks

Conservation refers to the understanding that quantitative relationships between two objects remain invariant (are conserved) in the face of irrelevant changes in one of the objects. In the standard conservation task, the subject is first shown two objects that, in addition to being perceptually identical, are known to be equivalent with respect to a certain quantitative property. The experimenter then proceeds to deform one of the objects in such a way that its perceptual identity is lost while maintaining the quantitative relationship. Finally, the subject is asked whether the objects are still quantitatively equivalent.

We shall now consider some student responses to scientific and mathematical conservation tasks.

A. Scientific Tasks

1. Conservation of Quantity of Matter - Piaget and his colleagues' studies on the development of the principle of conservation of quantity of matter are concerned with children's dependence on shape or form when asked to judge the relative quantity of matter. When comparing the (equal) amounts of water in two differently shaped cups, children up to about age five or six, paid attention only to the relative heights of the water in the two cups, arguing that "there is more water in the taller cup" (Piaget, 1965/1952). In a similar experiment involving the conservation of clay, responses were split. Some children argued that the thicker slice had more clay than the thinner one, while others responded that "the longer piece of clay has more clay than the shorter one" (Piaget & Inhelder, 1974/1941).

2. Conservation of Weight

(a) Changes of state. Stavy and Stachel (1985) studied children's understanding of the invariance of weight during the process of solid to liquid state transformation. They presented children aged 5 to 15 with two identical candles, one of which was then melted. The child was asked about the equality of weight (whether the solid candle and the melted one weighed the same). Many children between ages six to ten argued that "the solid candle weighs more because it is harder or stronger than the liquid candle."

(b) Expansion. Megged (1978) studied children's understanding of the invariance of weight during the process of heating water. In this case, the volume of the heated water is actually larger than that of the unheated water, but its weight remains constant. Many children aged six to ten, however, argued that "the heated water is heavier because its volume is larger."

3. Conservation of Volume - Lovell & Ogilvie (1961) asked children to compare the amount of water spilled when two cubes of exactly the same shape and size, one of which was made of wood and the other of lead (and therefore much heavier), were lowered very carefully into a full pint pan. They found that many of the children between age six and ten thought that the heavier cube displaced more water than the lighter cube of the same size and shape.

B. Mathematical Tasks

1. Number conservation - Piaget's (1965/1952) studies on the development of the number concept are concerned with children's reliance on length and density information when they are asked to compare the number of objects in two parallel rows, containing the same number of objects. Young children (until the age of about five or six) only paid attention to the relative length of the two rows. Rows of the same length were said to have the same number of objects; otherwise, the longer row was said to be more numerous than the shorter row. Some older ages, based their judgments on the relative density of objects on the two rows, stating that the denser row was more numerous.

2. Conservation of area - Piaget, Inhelder, and Szeminska (1960) presented children with two identical rectangular arrangements of six blocks. One arrangement was altered by removing two blocks from one end and placing them on the other. The children were then asked if the rectangles still were the same size and "have the same amount of room". Many children aged five to six tended to argue that one configuration was larger because it looked longer.

3. <u>Conservation of angle</u> - Noss (1987) and Foxman & Ruddock (1984) presented children with two identical angles, one of which had longer arms than the other. They found that many children between the ages of ten and fifteen argued that "the angle with the longer arms is bigger".

In all these, as well as in other cases of conservation (e.g., length [Piaget, Inhelder & Seminska, 1956]), children's responses seem to derive from a general rule according to which "more of A implies more of B". A similar phenomenon was observed with tasks related to intensive quantities.

Intensivity Tasks

Intensive quantities are formally defined as follows "If a,b,c,... are parts of a system, and y is a property such that y(a) = y(b) = y(c)..., then y is said to be an intensive quantity. The value of y for the entire system may be defined by y(system) = y(any part). Clearly y(system) is independent of the size (or extent) of the system" (Canagaratna, 1992, p. 957).

Intensivity tasks involve presenting the subject with two systems which are identical with respect to a certain intensive quantity but different in size. The subject is asked to judge or to compare the values of the intensive quantity in the two systems.

A. <u>Scientific Tasks</u>

1. <u>Concentration</u> - Stavy, Strauss, Orpaz & Carmi (1982) presented children with three cups of sugar water of the same concentration. The contents of two of these cups were poured into one empty cup, and the children were asked to estimate the sweetness (concentration) of the combined cup compared to that of the original third cup. It was found that the majority of children aged six to ten claimed that the combined cup was sweeter. Two types of justifications were presented: "The cup with <u>more sugar</u> is <u>sweeter</u>" and "The cup with <u>more water</u> is <u>sweeter</u>". Both of these justifications share the structure of "the more of A, the more of B". In this case the perceptual change in the quantity of water was salient and directly elicited "the more water, the sweeter" response. Most likely, this salient difference in the amount of water also indirectly encouraged "the more sugar, the sweeter" response. The reasoning behind this response was probably that "the more water, the more sugar," therefore "the more sugar, the sweeter.".

2. <u>Temperature</u> - Strauss, Stavy and Orpaz (1977) presented children with three cups containing equal amounts of hot water at the same temperature. The water from two of the cups was poured into an empty cup, and children were asked to compare the temperature of the water in the combined cup with that of the third cup's contents. The majority of children aged six to eight claimed that the water in the combined cup was warmer. These incorrect judgments were justified with reference to the amount of water -- namely, "the <u>more water, the warmer</u>". A similar response was observed with older students, who were asked to refer to the same problem, and were presented with numerical values of temperature (e.g., 40°C in each cup). Most children between ages seven and eleven argued that the temperature of the combination was higher than that of the third single cup. These responses were accompanied by an arithmetic calculation (e.g., 40+40=80).

B. Mathematical Tasks

1. Comparing infinite sets - Students aged 13 to 25 were asked to compare the number of points in two line segments. According to Cantorian set theory, any two line segments contain the same number of points. Yet about half of the students across the entire age group claimed that "the longer line segment contains more points" (Tirosh, 1991). Similarly, students of the same age group were asked to compare the numerosity of two infinite, enumerable sets: The set of natural numbers and the set of odd numbers. Many of them argued that "Set A contains more elements than set B and therefore its numerosity is bigger".

Reactions to these as well as to other intensivity/density tasks seem to evolve from the application of the same general rule: "The more of A, the more of B". This response was observed in both children and adults.

Other Tasks

The intuitive rule "The more of A, the more of B" seems moreover to operate in many instances other than those involving conservation or intensivity tasks. We shall provide some examples.

A. Scientific Tasks

1. Time - Levin (1982) studied the nature and the development of the time concept in young children. She asked nursery school children and kindergartners to judge whether two lights were lit for the same amount of time, or which light was on for more time. The lights were lit for the same duration but differed in size and/or brightness. Her findings indicated that children tended to associate the longer duration significantly more often with the bigger than with the smaller light, with the brighter than with the dimmer light, and with the bigger and brighter than with the smaller and dimmer light. She concluded that this phenomenon may be based on a mediational mechanism of "any more is more time".

2. Free fall - Gunstone & White (1981) presented first-year physics students with a problem: An iron sphere and a plastic sphere of the same diameter were held next to each other, two meters above a bench. Subjects were asked to compare the time it would take for the metal sphere to fall to the bench with the time it would take the plastic sphere. Some claimed that the metal sphere would fall faster because "a bigger weight will cause bigger acceleration". Champagne, Klopfer and Anderson (1979) presented first year physics' students with a similar task related to free-fall of two rectangular prisms of equal volume but different masses. Seventy-two percent of these students argued that "the heavier an object, the faster it falls, and that since lucite is denser than aluminium, it falls faster".

B. Mathematical Tasks

1. Arithmetic Operations - Tirosh, Wilson, Graeber, & Fischbein (1993) presented college students with several tasks concerning arithmetic operations. One such task is related to the relationship between factors and products in multiplication expressions. Many of these students argued that when one of the factors increases, the product always increases. While this statement holds for natural numbers, it does not for negative numbers [e.g. $2 \times (-4) \geq 8 \times (-4)$]. It seems that

the students who made this overgeneralization were operating according to the rule "the bigger the factor, the bigger the products".

2. Percentage tasks - Rachmani (personal communication) presented students at different age levels with several problems in an attempt to assess their understanding of percentages. One of these problems was the following: "Joe saved 25% of his salary. Maya saved 50% of her salary. Can you determine which of them saved more money?" Obviously, the answer to this problem is "no", because Joe and Maya's salaries are not known. Yet many students responded that "Maya saved more money, because she saved a higher percentage."

Discussion

We have presented students' responses to a variety of tasks in different subject matters -- some deal with physical objects while others refer to mathematical ones, some focus on extensive quantities while others involve intensive ones. All these tasks, though, have some common features. In each of them, two objects (or two systems) which differ in a certain, salient quantity are described. The student is then asked to compare the two objects or systems with respect to another quantity. In all these cases, a substantial number of students responded according to the rule "The more of A (the salient quantity), the more of B (the quantity in question)".

In all the cases described above, "The more of A, the more of B" led to inappropriate answers. In many tasks embedded in everyday as in scientific situations, however, a response based on this rule leads to correct accurate conclusions. For instance, "the more money you have, the more candy bars you can buy". People of different ages tend to use this rule in many situations, sometimes appropriations and sometimes not. The question which naturally arises, then, is why the use of this rule so far outspans its applicability.

One possible explanation is that the rule is used as a first approximation in solving a certain problem. This may serve as the starting step in a process of exploration. Often, approximation is made in the absence of a solid understanding of the situation. An example of such behavior was the reaction of a science educator from our department when asked to judge whether two given angles were equivalent. She explained that she was unfamiliar with this problem, but that it seemed to her that the most sensible answer should be "the angle with the longer arc is larger".

Another possible explanation is that using this rule gives the solver a grip over the situation, as it creates a causal/logical relationship between the various components of the system. Such a relationship gives the sense (though often erroneously) of an ability to predict.

There are several possibilities as to the nature of this causal relationship: (a) a qualitative relationship -- as A increases, B also increases, (b) a quantitative relationship of the form B=kA. It could also be that in some cases the relationship between A and B is perceived as qualitative and in others as quantitative. This issue should be further investigated.

It was also observed that the solver often views "The more of A, the more of B" as self-evident, and that its use is frequently accompanied with a sense of confidence. Self-evidence and confidence are two major characteristics of intuitive reasoning (Fischbein, 1987). In fact, this rule has some additional characteristics of intuitive reasoning: immediacy, globality and coerciveness.

Though it seems intuitive, there is still a need to explain why this rule is activated in the situations described above. Let us recall that in each of these tasks, the two objects or systems differ in one particular quantity. Our claim is that the intuitive rule is directly triggered by immediate perceptual differences or by salient differences between symbols associated with perceptual images, such as numbers. These are often visual, as in the conservation and intensivity tasks. A source of support for this claim is provided by Bruner's (1966) experiment related to conservation of quantity of liquid. Here, children aged four to seven were verbally asked about the conservation of liquid. The task was carried out behind a screen. It was found that half of the four-year-olds and all the others correctly replied that the amount of water was the same when the screen was used. When no screen was present, though, most four- and five-year olds said that "the higher the water, the more water to drink".

We have suggested so far that "The more of A, the more of B" is an intuitive rule which is activated by specific, perceptual input. What could be the origin of this intuitive rule?

At this stage we can only suggest several speculative possibilities.

(a) It is an innate, intuitive rule. The following, surprising excerpt, taken from Tinbergen's (1951) book on the study of instinct is in line with this possibility:

> Oystercatchers preferred a clutch of 5 eggs to the normal clutch of three. Still more astonishing is the oystercatchers' preference for abnormally large eggs. If presented with an egg with normal oystercatcher size, one of herring gull's size, and one double the (linear) size of the herring gull's egg, the majority of choices fall upon the largest egg (p.45).

It seems that the oystercatchers' decisions are determined by the implicit probably innate laws of "the more, the better" and "the larger, the better".

(b) Overgeneralization from successful experiences. As mentioned above, often in everyday life and in math and science problems, the rule "The more of A, the more B" is applicable. Children, starting from early infancy, encounter many situations in which perceptual differences between two objects go hand in hand with quantitative parallel differences in another property of these objects. It is reasonable to assume that children generalize these experiences into a universal maxim: "The more of A, the more of B".

At this stage in our inquiries, it is impossible to determine which of the above possibilities is the source of this intuitive rule. However that may be, repeated experiences reinforce the rule, enhancing its use in other seemingly similar situations.

We have mentioned before that this rule is often used in situations in which it is not applicable. Yet, with regard to each of the tasks described above (and to other tasks as well), children at different ages and/or with different levels of instruction at some point start using it selectively.

Most four-year-old children, for instance, argue "the higher the water level, the more water" in the conservation of the quantity of matter task, while practically all ten-year-olds know that the amount of water is preserved in this task. How can such changes in response be explained? Why do people use this rule in certain situations and discard it in others?

With age and/or instruction, schemes, rules, and bodies of knowledge related to a specific task and groups of tasks are developed or reinforced. As this occurs, people realize the inappropriateness of the rule "The more of A, the more B" in some of these specific tasks. Consequently, in respect to these, the rule loses its power in favor of other, competing knowledge. For instance, in the case of conservation of quantity of matter, "The higher, the more" is replaced by identity or compensation considerations. In addition, with age and/or instruction children become aware of the need to examine their initial responses, to consider other factors which might be relevant to the task, and to avoid conflicting arguments. Thus, they gradually learn the boundaries within which "The more of A, the more B" is applicable.

Although over time children cease to use the rule in certain instances, it does not altogether disappear and continues to dominate in various other situations. In fact, in many of the previously described instances (e.g., comparing segments, comparing angles, free fall) older children and adults kept using the rule. Thus it seems that like many other intuitive rules, this rule persists and retains its dominance.

The status of the rule however, is unclear with regard to those cases in which subjects seem to overcome its dominance and give judgments not in line with it. For instance, when a child correctly judges conservation tasks, is this because the rule "The more of A, the more B" has ceased to exist for these tasks, or does it exist but fail to compete with other bits of knowledge? Support for the possibility that the intuitive rule continues to exist comes from the following example. A very distinguished physics professor, an expert in astronomy, when answering a question on a television science program, was heard to explain that: "the force that the larger star exerts on its moon is larger than the force that the smaller moon exerts on the star". Of course this professor was familiar with Newton's third law (the law of action and reaction), according to which any two bodies exert the same force on each other. The next day, he explained that, tense because appearing on a television program, he had momentarily lost concentration and given an immediate uncontrolled response. Such slips are very common.

What is the relevance of this type of intuitive rule to science and mathematics education? First, it is crucial that teachers, policy makers, and curriculum developers be knowledgeable about the common roots of many seemingly unconnected, content-specific alternative conceptions. Such knowledge has predictive power: it enables teachers and researchers to foresee students' possibly inappropriate reactions.

It is equally important to improve our understanding of how children overcome the coerciveness of such rules. This knowledge can help teachers and curriculum developers plan sequences of instruction which take the role of intuitive rules into consideration and look for ways to overcome them. Finally, this paper points strongly to the importance of raising students' awareness of the need to be consistent, and to boost their critical thinking. All of these are necessary to the formation of boundaries with regard to the use of intuitive rules.

References

Bruner, J.S. (1966). On the conservation of liquids. In J.S. Bruner, R.R. Oliver, P.M. Greenfield (Eds.), *Studies in cognitive growth* (pp. 183-207). New York: J. Wiley & Sons.

Canagaratna, S.G. (1992). Intensive and extensive - Underused concepts. *Journal of Chemical Education, 69*, 957-963.

Champagne, A.B., Klopfer, L.E., & Anderson, J.H. (1979). *Factors influencing the learning of classical mechanics.* Research report: LRDC, University of Pittsburgh, USA.

Driver, R., Guesne, E., & Tiberghien, A. (1985). *Children's ideas in science.* Milton Keynes, England: Open University Press.

Eylon, B., & Lynn, M. (1988). Learning and instruction: An examination of four research perspectives in science education. *Review of Educational Research, 58*, 251-301.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach.* Dordrecht, the Netherlands: Reidel.

Foxman, D., & Ruddock, G. (1984). Concepts and skills: Line symmetry and angle. *Mathematics in school, 13*, 9-13.

Gunstone, R.F., & White, R.T. (1981). Understanding of gravity. *Science Education, 65*, 291-299.

Hart, K. (Ed.). (1981). *Children's understanding of mathematics; 11-16.* London: John Murray.

Levin, I. (1982). The nature and development of time concepts in children: The effects of interfering cues. In W.J. Friedman (Ed.), *The developmental psychology of time* (pp. 47-85). Academic Press.

Lovell, K., & Ogilvie, E. (1961). The growth of the concept of volume in junior school children. *Journal of Child Psychology and Psychiatry, 2*, 191-202.

Megged, H. (1978). *The development of the concept of density among children ages 6-16.* Unpublished Master's thesis. Tel-Aviv University, School of Education. (Hebrew)

Noss, R. (1987). Children's learning of geometrical concepts through LOGO. *Journal for Research in Mathematics Education, 18*, 343-362.

Osborne, R., & Freyberg, P. (1985). *Learning in science: The implications of children's science.* Auckland, NZ: Heinemann.

Perkins, D.N., & Simmons, R. (1988). Patterns of misunderstanding: An integrative model for science, math, and programming. *Review of Education Research, 58*, 303-326.

Piaget, J. (1965/1952). *The child's conception of number.* NY: Norton.

Piaget, J., & Inhelder, B. (1974/1941). *The child's construction of quantity.* London: Routledge & Kegan Paul.

Piaget, J., Inhelder, B., & Szeminska, A. (1956). *The child's conception of geometry.* London: Routledge & Kegan Paul.

Rachmani, L. (personal communication).

Stavy, R., & Stachel, D. (1985). Children's conception of changes in the state of matter: From solid to liquid. *Archives de Psychologie, 3*, 331-344.

Stavy, R., Strauss, S., Orpaz, N., & Carmi, G. (1982). U-shaped behavioral growth in ratio comparisons, or: That's funny I would not have thought you were U-ish. In S. Strauss & R. Stavy (Eds.), *U-shaped behavioral growth* (pp. 11-36). New York: Academic Press.

Strauss, S., Stavy, R., & Orpaz, N. (1977). *The development of the concept of temperature.* Unpublished manuscript, Tel-Aviv: Tel-Aviv University.

Thijs, G.D., & Berg, E. van den (1993). Cultural factors in the origin and remediation of alternative conceptions. Paper to be presented at the *Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics,* Cornell University, Ithaca, New York, USA.

Tinbergen, N. (1969). *The study of instinct.* Oxford: Clarendon.

Tirosh, D. (1991). The role of students' intuitions of infinity in teaching the Cantorian theory. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 199-214). Dordrecht, the Netherlands: Kluwer Academic.

Tirosh, D., Wilson, J., Graeber, A., & Fischbein, E. (1993). *Conceptual adjustment in progressing from whole to rational numbers: The case of preservice teachers* (final report). Tel Aviv: Tel-Aviv University. (Hebrew)

CHILDREN'S INVENTED STRATEGIES AND ALGORITHMS IN DIVISION

Ruti Steinberg
State Teacher Training College,
Seminar Hakibbutzim, Tel Aviv, Israel

Thomas C. Carpenter, Elizabeth Fennema,
University of Wisconsin, U.S.A

This paper describes procedures and algorithms for solving division problems with large numbers that were invented by fourth grade students in a Cognitively Guided Instruction (CGI) class.

Division with large whole numbers is one of the most difficult arithmetic operations taught in elementary school. The standard "long division" algorithm is not transparent and it is hard to understand why it works. Many students learn the algorithm in a rote manner, without meaningful understanding and without relating the steps of the procedure to their conceptual knowledge of division. Simon (1993) found that prospective elementary school teachers could not give meaningful explanations why the long division algorithm works or what the different steps mean.

Lampert (1992) analyzed the complexities of the long division algorithm and suggested ways to teach it meaningfully. One question that Lampert posed was: "Is it possible to teach division (with large numbers) in ways that support students' creative engagement with mathematical ideas?" (p.222) This paper suggests one way of answering this question in which students construct their own knowledge.

We report here strategies that fourth grade students developed to solve division problems with large numbers. We were interested in students' invented procedures and in their move towards the use of more symbolic representations. The use of symbols might help the children record their actions and serve as tools for developing and inventing new procedures and algorithms (Carpenter, Fennema, & Franke, 1992). When children invent their own procedures and algorithms, they usually have a better conceptual understanding of them and avoid errors that are common when children learn procedures in a rote fashaion (Crpenter, Fennema, Franke, 1993a; Greeno, Riley, & Gelman, 1984; Putnam, Lampert, & Peterson, 1989).

__Development of Division Strategies__. Children can construct meaningful ways to solve division and multiplication problems from an early age (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993). At first children model the semantic structure of the problems closely, with manipulatives. Thus they model partitive and measurement problems differently (Carpenter, Fennema, & Franke, 1993b; Kouba, 1989). Later, children use more flexible counting strategies that are abstractions of their modeling strategies. To solve measurement problems, children can skip count or add or subtract until they reach the given number and then count the number of counts. In partitive division they use similar strategies based on guessing and adjusting the number needed in each group. Children then use derived facts strategies which usually exploit or involve some known multiplication facts and counting on or adding. Most of the strategies beyond modeling the problem with manipulatives for 1- or 2-digit numbers are conducted orally, sometimes with the help of tallies to keep track, or writing the sequence of numbers in the patterns.

Murray, Olivier, & Human (1992) described the development of third grade students' strategies for division problems with 2- and 3-digit numbers. They found that groups of children in the class progressed differently and that with time, the children depended less on the structure of the division problem for their solutions and solved all of them similarly.

In this paper we describe a few strategies invented by fourth grade children and one third grade child for solving division problems with large numbers. We especially concentrate on strategies which use symbolic representations.

__Methodology and Classroom Setting__. This paper is part of a case study of the relationship between instruction and the learning process in a fourth grade class The teacher taught mathematics using Cognitively Guided Instruction (CGI) (Carpenter, & Fennema, 1992). In this approach the children have opportunities to solve a variety of problems, construct their own solution strategies, discuss them and present them.

__Sample__. One teacher and her 21 students were followed intensively for 5 months (December, 1992 to April, 1993) in a mid-west city in the U.S. The class was diverse racially and socio-economically. A strategy of one third grade child in Israel will also be discussed here.

Methodology. Thirty lessons were observed and the teacher was interviewed 13 times. Each student was interviewed twice on solution strategies for a variety of problems, students' journals were examined, and their strategies were recorded during classroom observations. The standard division and multiplication algorithms were not explicitly taught and no textbook was used.

Results and Discussion

For many reasons, many of the children in the class used quite basic strategies at the beginning of the study. During the study most students shifted their strategies for division problems from directly modeling the situations with single counters units or base-10 blocks to using counting and symbolic procedures. For partitive problems, for example, children who solved problems by dealing by 10's and 1's with base-10 blocks at first, started to just write 10's in each "box" they drew. For example, here is a solution to a partitive problem (253:23) described by a child who began with rather basic strategies (begining of January):

> He drew 23 "boxes," he wrote 10 in each and counted by 10's: "10, 20...230," he then drew another tally in each box and got the answer 11 in each box.

In the second interview, in April, only one child dealt by 1's for the problem 42:3 (par), but about a third of the class still dealt 10's with blocks for this problem. Interestingly, when the children solved a problem with larger numbers (226:8, par) at that time, they tended to just write the 10's in each "box" and didn't use the blocks. Also more children tried in this situation to start by guessing a number and checking it (realizing it will take a lot of labor to construct the 226 in the problem by dealing). The checking and adjusting was done either by patterns of counting, or by adding and multiplying.

For example, Jill solved the problem by guessing 50. She counted by 50's and realized it is too much. She tried 30:

$\underbrace{30\ 30\ 30}_{90}\ \underbrace{30\ 30\ 30}_{90}\ \overset{30}{\underset{20}{\cancel{30}}}\ \overset{30}{\underset{40}{\cancel{30}}}$ -->240 240-16=224->answer 28(R2)

Bob solved this problem by guesing and checking by mental multiplication: 8x20->160 8x30->240 too much.
8x25->160+40->200 3x8->24 so it is 28 and 2 left over.

Guessing and testing was also used to solve measurement problems. On Febuary 2 David solved the problem: "One box of pancake mix makes 24 pancakes. If I made 175 pancakes, how many boxes would I have?" He solved it by guessing how many times he should add 24. He tried 7 24's and added them vertically, adding pairs of 4's and 20's (breaking numbers to make it easier):

140
28
168

80 (40 (24, 24) 8 ; 40 (24, 24) 8) 16
20 ~~40~~ (24, 24) ~~8~~ 10
40 ~~24~~ 2
140 28

He tried another 24 -> 192 (too many), needed another box for the extra 7->8 boxes

Another symbolic strategy that involved guessing used the child's knowledge of multiplication. To solve 253:23 (par.), the child assumed that there are no remainders in this problem, and looked for what number to multiply 23 to get 253. He immediately saw it will be more than 10: 10x23=230. He looked for the ones digit of the number that, when multiplied by 23, makes the ones digit end in 3.

11
23
23
230
253

1x23=23 will give 3 in the ones place

The multiplication algorithm the child used here is very meaningful to him and he is able to play with it and solve the "puzzle" by manipulating the numbers in the algorithm flexibly.

Pam's Algorithms Involving Remainders We will describe here the development of Pam's algorithms over the 5-month period. Pam is a very able child in mathematics. When asked a problem with 2-digit numbers in December (42:3, partitive) she solved it very quickly by guessing a number and checking it with multiplication: 12 x 3 = 36 (remembers), 13 x 3 = 39 and 14 x 3 = 42, so it is 14. Only when she had to work with larger numbers did she try her new algorithm. In the beginning of January the children were given the problem 253:23 (partitive). Pam wrote a division number sentence. She separated 253 into 200 and 53 and checked how many times 23 goes into each one of them. She did that by adding 23's (and doubling the results). She found that 23 goes into 53 2 times with remainder 7 and 8 times into 200 with remainder 16. She added the two results and got 10 with remainder 23, which gave her anoter "whole" 23, so the answer is 11(R0) or 11. Similarly she solved the problem 108:6 :

253:23
200 53
: 23 :23
8(R16) 2(R7)
another 23
11(R0)

108:6
16(R4) 1(R2)
18(R0)

6x4=24
6x10=60
60+24=84
6x2=12
96

Pam shared her invention with the class and found that she needed to explain it and represent it in clearer ways so her classmates and the teacher will understand. The teacher took the time to listen to her strategy individually and to understand it. Pam improved her explanations and the way she represented the writing and it developed to the examples above. The teacher asked her questions in class to help her clarify her algorithm for the children. For example, for 108:6, Pam said: "Take the 8 and divide it in 6," the teacher asked: "What does that mean?" Pam: "How many 6's go into 8? 1 and remainder 2" (January 21). The children were used to regularly presenting their strategies to the class. The social norms and expectations of this class enabled a very diverse group of students to each develop solution strategies suitable to their understanding and development.

Pam extended the algorithm to a 4-digit problem (1542:3 partitive) on February 2nd. She separated 1542 into 1000+500+42 and checked separately how many 3's go into each and what were the remainders. She then added the remainders and adjusted the result if the remainder was over 3:

****

1542:3 -> 3's in 42
15x3=45
less 3 is 42
so it is **14(R0)**

3's in 500
3x33=99
3x66=198 (double)
3x132=396 (double)
(add 33->3x165=495 (396+99=495)
(another 3) 3x166=498 -> **166(R2)**

1000:3 -> double the 500: 3x332=996 3 more-> 999 (3x333)
so 1000:3=**333(R1)**

together: 14(R0)
166(R2)
333(R1)
513(R3)-->**514(R0)**

****

Pam used this algorithm every time she solved a division problem with large numbers (both for partitive and measurement problems), until she realized she can use easy multiples of the first result to find the other parts without investing so much labor. On March 1, she wrote a word problem for herself (a regular activity in the class): 9786:14. She stared by finding how many 14's go into 86, as usual, and found that it is 6(R2). (She added 28+28+28=84 (vertically) and 2 left to get to 86.) Then, instead of finding the number of 14's in 700 and 9000, which she realized would be tedious, she took 9800 instead. She found out that there are 7 14's in 98 (84+14=98) and concluded that there are 700 14's in 9800. 98:14=7 -> 9800:14=700

She still needed to subtract thenumber of 14's in 100, since she needed 9700 and not 9800. She already knew this result from the 98 found before:
100 : 14 = 7(R2) -> 700 - 7(R2) = 692R(12) (see how easily she manipulated the remainders - 7 from 700 is 693, taking away the remainder 2 gives 692 and remainder 12.) -> So 9700:14=692R(12). Adding the two parts she found:

$$\begin{array}{r} 692(R12) \\ \underline{6(R2)} \\ 699(R0) \end{array}$$

gives the result 699 (another 14 from the remainders).

While Pam's division procedures might not be the most efficient and quick ones, she had the opportunity to invent them, to understand their origins and to be engaged in a worthwhile mathematical investigation. While doing so she showed flexibility in manipulating the numbers and the remainders in a meaningful way. Pam also had the opportunity to have her invention recognized and appreciated by the other children and the teacher who supported her in her explorations. Talking, explaining, answering questions and writing about her strategy in words and in symbols, helped her develop symbolic representations and meta-cognitive processes to explain her thinking.

Divide and Keep Dividing the Remainders. Dan, a third grade Israeli boy, descibed his invention for the problem 777:8 (He posed the problem to himself symbolically.) -> 7 hundred divided by 2 is 3. The 1 hundred that is left from the 7 you also divide into 2, giving 5 tens. 7 tens divided by 2 is 3 tens, plus the 5 tens from before is 8. The 1 ten that is left from the 7 you also divide into 2, for 5 ones. 8 divided by 2 is 4 plus the 5 from before is 9. The answer is 389. We can write this strategy schematically, writing the 5's that the child remembered on top of the dividend:

```
  55           778:2
 778:2    or   334
 389            55
               389
```

Although Dan developed his strategy for specific numbers that are divisible by 2, we can generalize it by continuing to divide the remainders and writing the results above the appropriate places. For example, to divide 778 by 3, we can write:

```
    (2)
   3(1)
  33(1)
 778:3
 222
 258(R4)->259(R1)
```

7 hundreds divided by 3 is 2 hundreds. 1 hundred is left. We divide 100 by 3, which is 3 tens (write the 3 above the 7 tens to remember), 10 is left. Divide the 10 by 3, which is 3 and remainder 1. You write 3 above the ones to remember (or you can think about 100:3 as 33(R1) and write it). 8 divided by 3 is 2 (you write it), and remainder 2. (you can write it). You add the numbers for the hundreds, tens, ones and remainders, adjust the remainders for another 3, and get the answer. This algorithm is relatively transparent.

Conclusions

Enabling children to develop their own solution procedures for division problems with large numbers can help them make sense of the problems, develop meaningful understanding of division and promote progress toward the development of symbolic representations. In this paper we have described some of the more interesting strategies that we have seen children invent. They are a sample of the many invented strategies that we have observed. Classroom discussion of the children's invented strategies further helps them to understand the properties of division procedures and to compare different methods of solution.

References

Carpenter, T.P., Ansell, E., Franke, M.L., Fennema, E., & Weisbeck, L. (1993). Journal for Research in Mathematics Education, vol. 24, 5, 428-441.

Carpenter, T.P. & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. In W. Secada (Ed.), Researching educational reform: The case of school mathematics in the United States (pp. 457-470). Special Issue of International Journal of Educational Research.

Carpenter, T.P. & Fennema, E., & Franke, M.L. (1992, April). Cognitively Guided Instruction: Building the Primary Mathematics Curriculum on Children's Informal Mathematics Knowledge. Presented at the annual meeting of the American Education Research Association. San Fransisco.

Carpenter, T.P. & Fennema, E., & Franke, M.L. (1993a). Cogitively Guided Instruction: Base ten number concepts and multidigit operations. Wisconsin Center for Educational Research, University of Wisconsin-Madison.

Carpenter, T.P. & Fennema, E., & Franke, M.L. (1993b). Cogitively Guided Instruction: Multiplication and Division, Wisconsin Center for Educational Research, University of Wisconsin-Madison.

Greeno, J.G., Riley, M., & Gelman, R. (1984). Conceptual competence and children's counting. Cognitive Psychology, 16, 94-143.

Kouba, V.(1989). Children's solution strategies for equivalent set multiplication and division word problems. Journal for Research in Mathematics Education, 20, 147-158.

Lampert, M (1992). Teaching and learning long division for understanding in school. In Leinhardt, G. Putnam, R., & Hattrup, R. A. (Eds.). Analysis of arithmetic for mathematics teaching (pp. 221-278). Hillsdale, NJ: Lawrence Erlbaum Associates.,

Murray, H., Olivier, A., & Human, P. (1992, August). The development of young students' division strategies. In Geeslin, W., & K. Graham (Eds.). Proceedings of the sixteenth conference of the international group for the psychology of mathematics education, Durham, NH.

Putnum, R., Lampert, M., & Peterson, P.L. (1989). Alternative perspectives on knowing mathematics in elementary schools. In Cazden (Ed.), Review of research in education (vol. 16, pp. 57-152). Washington, DC: American Educational Research Association.

Simon, M. (1993). Prospective elementary teachers' knowledge of division, Journal for Research in Mathematics Education, Vol. 24, 3, 233-254.

STUDENTS' ABILITY TO COPE WITH ELEMENTARY LOGIC TASKS — THE NECESSARY AND SUFFICIENT CONDITIONS

Jonathan Stupp and Shlomo Vinner

Hebrew University of Jerusalem, Israel

ABSTRACT

This paper describes an experiment conducted with 10th grade students. The aim of the experiment was to examine whether students in a self-instruction approach can acquire some concepts of logic presented to them by formal definitions and a few examples. The students were asked to construct their own examples, to answer questions about relations between the concepts and to prove certain statements. Another aim of the experiment was to examine the students' intuitions about certain principles of logic and their ability to learn by means of symbols in a general way. The results show that the students can succeed with mathematical reading comprehension tasks but they have difficulties with logical manipulations.

1. **INTRODUCTION**: Logic and formal symbols constitute a part of the many tools we use in mathematics. In the high-school level the use of these tools is rather scarce, as it seems that it adds more problems to the students in addition to the many problems which they already encounter. The constructivist approach, which is used nowadays by many educators, evades as much as possible the use of formal definitions and formal logical deductions. Yet, we think it is important to examine how difficult it really is for high-school students to deal with logic and formal definitions — at least for students with above average abilities — since in their future studies they may encounter these aspects of mathematics.

H. Freudenthal (1973, Chapter 6) speaks also in favor of teaching the elements of logic because "thinking situations in every day life usually are too simple." But he cautions against teaching it in a way which might deter the students from using logical argumentation.

We chose the method of self-instruction. We wanted to find out if students can learn the formal definition by themselves with only the help of guiding questions. In this way there is a good chance to expose many cognitive aspects which naturally appear when a student has to cope by himself with learning new concepts. For example: Can students deal with general statements devoid of specific content, which are represented by letters? Or to what extent do the students regard logical truths as truths which are accepted intuitively and therefore do not need any formal proof? Do the students possess any logical schemata in the sense of Skemp (1971)? When the students use intuition in logical reasoning, is it distorted and misleading? These aspects were already investigated by E. Fischbein (1981) in the case of infinity.

As we cannot prevent students from the natural urge of using intuition, we have to check whether their logical intuition is not distorted. With wrong intuitions mathematical argumentation will be a very difficult task.

Also, Skemp (1971) speaks in favor of introducing the use of symbols because only by using them one can better communicate his ideas and can express his mathematical arguments in a clear way.

As the subject of our experiment we chose the concepts of necessary and sufficient conditions. These two concepts occur frequently during the Mathematics lessons and students tend to confuse between these concepts. There is a tendency, and not only in Mathematics, to consider every condition as both necessary and sufficient. "If A then B" is often automatically construed also as "If B then A." In non-mathematical every-day language we usually phrase statements as a sufficient condition, although we mean both sufficiency and necessity. For example: When a father tells his son, "If you wash the car you'll get $10," he means of course that the son will get the money if and only if he washes the car. But in dealing with mathematical statements the distinction between these two concepts is crucial, and, therefore, an additional aim of this study was to clarify this distinction to the students. It is easier for the student if a mathematical condition is both necessary and sufficient, it then saves thinking too much. How easy it would have been for the student if the vanishing of the derivative at a point were a necessary and sufficient condition for an extremum point. Therefore, students often fail to discern a stationary point which is not an extremum point.

O'Brien (1973, 1975) investigated the problematics which arise in understanding conditional statements and found difficulties both by adolescents and college students even after they have taken a basic course in logic.

2. **METHODOLOGY**: Our experiment took place at the High School for Sciences and Arts in Jerusalem. This is a boarding school for students who excel either in the field of science music or the plastic arts. Each year the School accepts about 70 students out of approximately 300 applicants. They have to pass several tests before they are accepted. The working sheets were given to 44 10th grade students. These students were not the students who excel in Mathematics (They are grouped in a different class), although most of them intend to matriculate in the higher level of matriculation test.

The atmosphere in this School encourages non-routine ways of studying, and hence produced a positive attitude towards the experiment. We mention this in order to emphasize that the difficulties students had with the working sheets during the experiment, were a result of real cognitive difficulties and not a result of lack of motivation.

The working sheets were written so that the formal definitions appeared at the beginning and the students were immediately asked to give appropriate examples. In this way we could check whether the definitions were understood at least at an elementary level. Then we continued with more complicated questions which examined the deeper understanding of the definitions, examining especially whether the students perceived the relation between the definition of a necessary condition and the definition of a sufficient condition.

The first 5 exercises do not deal with Mathematics but with the general meaning of the concepts. Only the 6th exercise deals with a purely mathematical instance of necessary and sufficient conditions.

The students were given 90 minutes for the task. In order to ease the atmosphere and to avoid unnecessary pressure they were told that this was not an examination. We allowed them to consult the teacher if they confronted any difficulty.

The Working Sheet

A,B will represent attributes, situations or events. We'll call them by the mathematical term: 'conditions'.

First Definition: Condition A is necessary for condition B, if B cannot exist without the existence of A.

An Example: A – the sky is cloudy; B – it is raining.

Remark: A can exist without B. Perhaps the sky is cloudy, but it doesn't rain. So even if A is necessary for B, one cannot derive the existence of B from the existence of A.

Exercise 1: Give two examples of pairs A and B, so that A is necessary for B.

Exercise 2:

a) A is necessary for B. A does not exist. Is it possible that B exists? Justify your answer.

b) A is necessary for B. B does not exist. Is it possible that A exists?

Second Definition: Condition A is sufficient for condition B, if from the existence of A, one can conclude the existence of B. Compare with the first definition.

An Example: A – Ron is a mathematics teacher. B – Ron knows the Pythagorean theorem.

Remark: If A is sufficient for B, B can exist even without the existence of A. Maybe a person knows the Pythagorean theorem, without being a mathematics teacher. This is precisely the difference between a necessary and a sufficient condition.

Exercise 3: In the first example (A – cloudy sky, B – it is raining), which condition is sufficient to which?

Exercise 4: Give two examples of pairs A and B, so that A is sufficient for B.

Exercise 5:

a) Is A necessary and sufficient for itself?
b) Give two examples of pairs of conditions, A and B, so that A is both necessary and sufficient for B. Try to give a non-mathematical example. For a mathematical example consider the congruence theorems.

Exercise 6: This time we will deal with a mathematical example. Let A stand for: 'N is an even number,' and let B stand for: 'N is a multiple of 4.'

a) Is A necessary for B?
b) Is A sufficient for B?
c) Is B sufficient for A? Give a mathematical reason for your answers.

Exercise 7: Which of the following statements is true for any conditions A & B?

a) If A is sufficient for B, and B is sufficient for A, then A is both necessary and sufficient for B.
b) If A is both necessary and sufficient for B, then so is B for A.
c) In order to show that A is both necessary and sufficient for B it suffices to show that A is sufficient for B, and B is sufficient for A.

Exercise 8: (Taken from O'Brien (O'Brien 1975/76).

Four cards are presented to you. Each card has a number written on one side and a letter on the other side. A sufficient condition for an even number to be written on one side, is that the letter on the other side will be taken from the letters A-I. Which of the cards must be turned over in order to either affirm or contradict the above statement?

G	18	7	T

3) **ANALYSIS OF THE STUDENTS PERFORMANCE**

First, we would like to note that concerning the non-mathematical examples we were not particular about facts but about logical correctness. For example, if a student wrote that the shining of the sun is a sufficient condition for the fact that it is still daytime, we considered it as correct, disregarding the rare phenomenon of the northern midnight sun.

We divided the performances into three basic categories.

Category I: Here we included answers which indicated that the student understood the logical aspect and also indicated a satisfactory ability to express his reasoning in a formally correct language.

Category II: Here we included answers which were logically correct, but lacked formal reasoning or if there were any reasoning, it was poorly formulated. Answers in this category indicated that the student relied on his natural logical intuition. He did not justify his answer with a formal proof either because it seemed absolutely obvious to him or because he could not do it. Fischbein (1987) claims that students use a kind of "Primary Intuition" as a basis to rely on. Their intuition serves them like an axiom system and therefore there is no need to justify it. As long as the student's intuition leads him in the right direction there is no harm in using intuition. Answers in the 2nd category indicated that the students possess the correct primary intuition. However, they lack an analytical ability which can establish this intuition.

Category III: Here we included wrong answers, or answers which were officially correct (from the "yes" or "no" aspect), but were followed by a totally wrong argumentation.

Answers in this category indicate that the student lacks the logical elements needed for the context of the task involved.

Since the various exercises were not on the same level of difficulty, we will treat in our analysis only those exercises the analysis of which is interesting from the cognitive point of view.

Exercise 2: a) Almost all students (40 out of 44) concluded correctly that if A is necessary for B and A does not exist, then B cannot exist. They argued correctly that it is a direct conclusion from the definition. We included answers of this type in Category I. The 4 remaining students were students whose mother tongue was not Hebrew. Since in formal definitions mastering the language in which the definitions are given is crucial, this might explain the wrong answers. b) Here there is no definite conclusion, since if A is necessary for B and A exists, B may or may not exist. Thirty students gave the right answer, but didn't give examples for each case. It was absolutely obvious for them, and they probably did not realize that if a statement may be true or not according to the specific instance, a fully correct answer is an example for each case. This might explain the difference in the distribution among the categories between part a and part b.

We deliberately changed the role of the letters in the definition of a sufficient condition. This time A stood for the sufficient condition. We wanted to check whether the students assigned a particular meaning to the particular letter, or they understood that the choice of the letter is arbitrary.

Exercise 3: Thirty-six students answered correctly and the choice of the letter did not confuse them. They suggested correctly that in the first example the statement B ("It is raining") is sufficient for the statement A ("The sky is cloudy"). It is interesting to note that 2 students interchanged the letters representing the statements in order to avoid confusion. Eight students focused on the letters and not on the content of the statements and gave wrong answers.

Exercise 5: a) Thirty-four students claimed correctly that A is both necessary and sufficient for itself, but 21 of them gave no justification. These were classified as category II students. Only 13 showed that if we set A=B in the definitions they are trivially fulfilled. These answers were included in Category I. Among the remaining 10 answers there were answers like: "A stands for different conditions." This clearly shows no ability to cope with formal symbols. The student is rejecting the question because he believes that A cannot be both sufficient and necessary.

Exercise 6: Here we wanted to check if the students understood necessity and sufficiency concerning pure mathematical concepts. In this case A stands for: N is an even number, B stands for: N is a multiple of 4. Thirty-one students answered correctly that A is necessary but not sufficient for B. Only 4 of them gave a counter-example of an even number which is not a multiple of 4. Once again, the answer was probably intuitively obvious for them so that they did not bother to give a fully justified answer. We included only the 4 fully justified answers in category I, including the remaining 27 under category II. Among the 13 wrong answers was one of a particular interest. A student wrote that if A is necessary for B, it also must be sufficient for B (this despite all the previous examples). Here we probably see the influence of the every day semantics of the words. "Necessary" has a much stronger connotation than "sufficient". Therefore, if something is necessary it is also sufficient.

Exercise 7: In this exercise the students were tested for their ability to work with general logical principles about general conditions, denoted by letters and devoid of any specific content. Till now letters denoted numbers, geometrical points, functions, etc. Now it is a totally new experience for them to regard letters as denoting statements. The students' performance in this exercise was very poor. They had to conclude by themselves that if A is sufficient for B, then B is necessary for A, and that the relation of being both necessary and sufficient is a symmetric relation. Only 6 students answered correctly all the parts of the exercise and were included in category I. There were 18 correct answers to part a, 11 to part b, and 14 to part c, but they were given in terms of "true" or "false" without any formal argumentation. The students showed they possess the correct intuition but lacked the ability to reason in a formal symbolic language. About half of the students were unable even to give a correct "true" or "false" answer. This confirmed Freudenthal's thesis that at this stage, in general, students cannot work with formal logic.

Exercise 8: This is a modification of O'Brien's 4-cards problem (O'Brien 1975).
Twenty-two students, exactly half, answered correctly, namely that only the cards $\boxed{7}$ and $\boxed{G}$ must be turned over. But only 2 of them argued why cards $\boxed{T}$ and $\boxed{18}$ are irrelevant. Since the argumentation why we have to turn over cards $\boxed{7}$ and $\boxed{G}$ — was correct, we included all the 22 answers in category I.

Five students claimed that cards $\boxed{G}$ and $\boxed{18}$ have to be turned over and argued that if on the other side of $\boxed{18}$ there is a letter not among A-I, the statement is refuted. They confused here sufficiency with necessity. Five students claimed that only $\boxed{G}$ has to be turned over. Among them were students who gave the interesting reason why cards $\boxed{7}$ and $\boxed{T}$ were irrelevant. "Seven is not an even number and T is not among A-J." For them the relevance of the card stemmed from the actual words appearing in the statement. Only even numbers and letters from A to I were mentioned, therefore 7 and T are irrelevant. This is a problem where one has to deduce logically, because there is no concrete experience to rely on. So the students use the words appearing in the text to rely on in their reasoning.

Seven students wrote that cards 18 and T must be turned over. This would have been the answer if we had written "necessary" instead of "sufficient."

Yet, relative to the poor results of this exercise reported by different authors (O'Brien 1973, Adi, 1980), we think that the results in our case were pretty good, and we relate this partly to our usage of the way of self instruction. In this way students internalize the new concepts, and can use them later in a correct way.

4. CONCLUSION: In this research we wanted to investigate several issues. We wanted to check the ability of high-school students to learn by themselves the concepts defined in a rigid formal way (à la Bourbaki). As can be seen from the table (given at the end), most answers belong to categories I and II, showing that the students understood the concepts and were able to work with them. Except in exercise 7, more than 2/3 of the answers belong to these categories (on the average, as not all the exercises have the same level of difficulty).

In Exercises 2,5,6,7 and 8, where formal reasoning was required, most students knew the right answer but had difficulties in using formal arguments and symbols. We conclude that the majority of the students who participated in this experiment possess the correct logical intuition but it is pre-mature for them to deal with symbolic logic.

The students possess certain logical schemata which help them to deduce correctly but without being able to formalize their arguments. This was especially prominent in Exercise 7, which dealt with abstract conditions presented by letters devoid of specific content. Therefore, the students had nothing to rely on except their intuition (see Fischbein (1987)), and this is in contrast to Exercise 6, where the students could use their knowledge about numbers.

We believe that introducing logical tasks in high-school, at least in small doses, will enhance the developing of logical schemata in their thinking, and it will be easier for them to reach, as Piaget put it, the level of propositional or formal operation, "where thought no longer proceeds from the actual to the theoretical, but starts from theory so as to establish or verify actual relationships between things. Instead of just coordinating facts about the actual world, hypothetico-deductive reasoning draws out the implications of possible statements and thus gives rise to a unique synthesis of the possible and necessary." (Piaget, 1953, p.19).

Table 1
Distribution of Students to the Cognitive Categories I-III (N=44)

	Cat I	Cat II	Cat III		Cat I	Cat II	Cat III
2a)	40	0	4	7a)	11	18	15
2b)	10	30	4	7b)	7	11	26
3	36	0	8	7c)	8	14	22
5a)	13	21	10	8	22	10	12
6	4	27	13				

Table 2
Distribution of Students Answers to the Reading Comprehension Tasks (N=44)

	Correct	Wrong
1)	44	0
4)	37	7
5b)	31	13

References

Adi M. Karplus, R. and Lawson A.E. (1980). "Conditional logic abilities on the four-card problem: assessment of behavioral and reasoning performances," *Educational Studies in Mathematics* 11(4).

Fischbein, E. (1987). *Intuition in Science and Mathematics*, D. Reidel Publishing Co.

Fischbein, E, Tirosh, D., Melamed, U. (1981). "Is it possible to measure the intuitive acceptance of mathematical statements?" *Educational Studies in Mathematics*, 12, 491-512.

Freudenthal, H. (1973). *Mathematics as an Educational Task*, D. Reidel Dodrecht.

O'Brien, T.C. (1973). "Logical thinking in college students," *Educational Studies in Mathematics* 6, 23-29.

O'Brien, T.C. (1975/76). "Definition and the Four-Card Problem." *Educational Studies in Mathematics* 6, 23-29.

Piaget, J. (1953). "Logic and Psychology," Manchester University Press.

Skemp, R.R. (1971). *The Psychology of Learning Mathematics*, Harmondsworth U.K., Penguin.

Skemp, R.R. (1977). "Relational understanding and instrumental understanding." *Mathematics Teaching*, 77, 20-26.

PRACTISE WHAT YOU PREACH: INFLUENCING PRESERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS.

Kevan Swinson and Mal Shield. Centre for Mathematics and Science Education. Queensland University of Technology.

School experience forms the basis of preservice teachers' beliefs about teaching and mathematics. Many of these beliefs are quite negative. The results of an intervention program aimed at using practical experiences to modify beliefs are reported. The study is based on interview, written survey and student writing. Results suggest some success was achieved in modifying preservice teachers' beliefs.

Preservice teachers' beliefs about mathematics teaching and learning develop during their own schooling and are influenced by the experiences they have as mathematics students (Ball,1988). Beliefs such as you are either good at mathematics or you are not, answers in mathematics are either right or wrong, and memorising facts and procedures is the way to learn mathematics have been found to be prevalent among preservice mathematics teachers (Frank, 1990). As the beliefs teachers and beginning teachers hold influence their teaching and student learning (Kagan, 1992; Crawford, 1992) it is essential that graduates from preservice courses hold appropriate beliefs about teaching and how students learn mathematics, that are congruent with those given in the NCTM Professional Standards.

These include challenging and extending student ideas, encouraging students to work independently or collaboratively, using concrete materials as models, and using a variety of tools to reason, make connections, solve problems and communicate.

Reviewing research on mathematics teachers' beliefs Thompson (1992) concluded that beliefs are resistant to change as they have been reinforced over a long period. There is evidence (Crawford, 1992) to suggest "that the traditional forms of teacher education have been largely ineffective in changing ....student teachers' beliefs about teaching and learning" (p.162). Programs specifically aimed at modifying pre service teachers' beliefs have been undertaken (Grover & Kenney,1993) and their results offer some hope preservice teachers' beliefs may be beneficially modified. Moreover, the results suggest that if we are to have prospective teachers rethink and modify their beliefs, we must create situations where these beliefs are faced and reconsidered (Wilcox, Schram, Lappan & Lanier, 1991). The obvious place to attempt such an intervention program is in a mathematics methods course.

The study described here is an attempt to alter pre service teachers' beliefs by having them:

1. reflect on and elaborate about their conceptual understanding of the mathematics they would teach;
2. in a small group situation, use materials and aids to actively construct a conceptual understanding of the mathematics they will teach and through discussion and reflection gain an insight as to how children learn mathematics; and
3. throughout the program reflect on their experiences to gain an awareness of their understanding of the difficulties experienced by children learning mathematics.

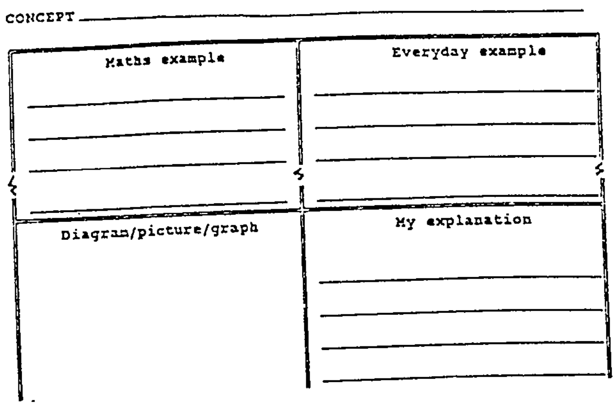
The Study

Methodology

The study involved nine students enrolled in a one semester preservice methods unit for secondary mathematics teaching. The students had selected secondary mathematics as their second (minor) teaching subject. All had been awarded degrees in fields such as music, drama or biology with a minor strand in mathematics. At their first meeting the preservice teachers completed a questionnaire and two problem solving tasks. The questionnaire, developed from those of Schoenfeld (1989) and Pehkonen (1992), contained 62 closed and 3 open questions concerning beliefs about mathematics and its teaching and learning. The problems were (1) finding the maximum area enclosed by a rope and (2) determining the fairness of a payout in a game of chance. One week later they were interviewed, audio taped, and questioned about, their school experiences, metacognitive processes, and their attempts at solving the two problems. Based on the information gathered, three of the preservice teachers who exhibited a range of beliefs about mathematics and its teaching and learning were selected for closer study.

Supplemental data was obtained from journals, written work, elaboration worksheets, observation and discussion of their practice teaching, and observation during the workshops. The elaboration worksheets were developed to aid the students' conceptual understanding of the mathematics discussed and to provide researchers with data concerning the preservice teachers' thinking.

Figure 1 Elaboration worksheet



Description of the workshop program

In real world situations, users of mathematics communicate and share ideas for their common good. This new program, first taught in the second semester of 1993, provided a similar setting where student-student interaction and communication was an essential ingredient, with students working collaboratively toward common goals as they cooperated in small group problem solving activities. During every class the preservice teachers participated in small group problem solving activities. In this way it was envisaged that the course would not only prepare students to teach in a mathematics classroom, but would also be a forum where their personal beliefs about mathematics and its teaching would be challenged and developed.

Discussion, clarification, elaboration and reflection were used extensively throughout the program, students were encouraged to think quietly about a concept or problem, write down their ideas, move into a group mode for discussion, clarification and elaboration then finally undertake individual reflection. To assist them to examine their conceptual development of the mathematics, elaboration worksheets were used at the beginning and end of each topic to enable changes in the conceptual structure to be noted and analysed.

Experience with other groups of secondary methods students suggested that the present group would not be familiar with the use of materials for learning or teaching mathematics and would have negative or neutral beliefs concerning their use. This was supported in the questionnaires and interviews, for instance, Ida a music\mathematics student commented:

> "in mathematics we never cut out anything or did anything with our hands except write."

Students brought prior beliefs about mathematics processes which were incomplete or erroneous. In order to challenge these procedures, materials were provided so that concrete irrefutable instances were seen to be in conflict with their prior beliefs. This form of cognitive conflict was seen as an essential element in bringing about student change. All activities in the course were selected or constructed to allow the use of concrete materials. The use of materials was preceded by a discussion of where the mathematics is used in the real world of children, and the relationship between the materials, the language used and the mathematical symbolism was then closely examined, discussed and reflected upon. Initially students reluctantly used these aids, however by the end of the program they considered them an integral part of every lesson and searched for alternative and different materials to use.

Tracing the change in student beliefs

The three students, Jan, Ann and Ida, chosen for closer investigation, indicated that the mathematics program they had experienced at school was "basically straight from the textbook" consisting mainly of "facts and procedures to be memorised" with the teacher demonstrating the steps to be followed. Group work and the use of materials were not a part of their high school mathematics. The influence of these experiences on their beliefs concerning mathematics and mathematics teaching is highlighted by their response that knowing you understand a piece of mathematics is based on "getting the right answer" and "knowing the steps to go through." Student beliefs about mathematics and its teaching and learning were determined at different times in the program by examining and discussing their attempts at problem solving, reviewing the elaboration worksheets, and by comparing the pre- and post- questionnaires.

Problem solving

Problem 1. In the corner of a rectangular room we are going to "fence off" a section of the floor. To do this we are going to use a piece of rope 16m long. What is the maximum floor space that can be fenced off with this piece of rope.

Problem 2. To play a game three disks are used, two red on both sides and one disk has a red side and a blue side. The game is played by tossing the disks: if they all turn up red Bill gives Tom $1, if the disks turn up two red and one blue Tom gives Bill $2. Is the game fair?

Student problem solving protocols and interviews identified a number of preconceived beliefs about mathematics such as the product and not the process was considered the important part of solving a problem. For example Jan, a biology\mathematics student, stated her first thoughts were "I've got to get the answer." While Ida said "the whole thing was going through my head. Oh my goodness what if I get this question wrong. What if it is incorrect."
Both Ann and Ida did try alternative strategies after obtaining solutions but only because they believed the answers may be wrong.

The three students believed diagrams aided visualisation and assisted with finding solutions. Ida stated "I try to visualise what I am doing" and Jan, when discussing her diagram said, "a bit of visualising." The effect of school experiences on beliefs is illustrated clearly by Ann and Ida. Ann, when discussing her solution to question one, indicated she realised her strategy was inappropriate but continued with it. When questioned about this she replied "well I differentiated but really didn't know why I was doing that, its because we were supposed to differentiate" and "I realised it was wrong." Ann had only previously attempted optimisation problems in calculus and believed calculus must be used. Ida, discussing the probability problem which she had successfully solved using logic, explained "I'm a music student and I'm very...er...I don't think in mathematical terminology very much and so I thought maybe I should think mathematically. I wasn't sure er just seemed I'd rather do it in a logical way than to do it with all that confusing terminology and that sort of thing." For Ida mathematics and its terminology are not logical.

Elaboration worksheets

The major purpose of these worksheets was to assist students clarify their thoughts and to act as a starting point for discussion. From classroom observation, in this role they were very successful. A secondary use of the worksheet was to gather data on change in student beliefs and concepts. A comparison of the elaboration worksheets done before and after each topic suggested student concepts had remained poorly developed and only minimal change occurred.

Ida's initial elaborations on algebra were in terms of labels but these changed to discussing unknowns. Ann's description of algebra moved from "using letters to represent unknown values" to "I think algebra is the part of mathematics which is the most fun! It can be applied to a wide range of problems.....the unknown is represented by a letter." No student described algebra in terms of modelling or generalising.

For decimals Jan initially equated the idea with just the notation "it is a dot used in a number", later describing it as "a different way of expressing fractions." Ida's conception expanded to seeing "the decimal point divides the wholes from the parts."

Questionnaires

An examination of the responses on the pre- and post- questionnaires suggested that students' beliefs concerning good mathematics teaching underwent some modification during the program. Jan changed from agreeing to disagreeing good mathematics teaching involves "doing computations with pencil and paper" with a similar change for "doing word problems." This change is unexpected as the use of word problems and pencil and paper computations had been discussed in the program and their positive and negative attributes highlighted. To the question "when solving problems the teacher explains every step exactly" both Jan and Ann moved from agreeing to disagreeing. Ann altered her view from being undecided on "much will be learned by memorising rules" to full disagreement. Similarly she changed from agree to disagree "that everything will always be reasoned exactly." Her beliefs that "some people are good at mathematics and some are not" and "in mathematics something is either right or it's wrong" changed from sort of true to a definite not true at all.

Ida's initial beliefs about good mathematics teaching were fairly congruent with those of the NCTM Standards. Minor modification did occur, such as moving from agreeing to strongly agreeing to "there is usually more than one way to solve problems" and similarly for "the idea that games can be used to help pupils learn mathematics" and "pupils can sometimes make guesses and use trial and error." Moreover her belief that "everything will be reasoned exactly" changed from undecided to disagree.

Discussion

A one semester course is somewhat restrictive in terms of providing sufficient time for significant assimilation and accommodation which will alter beliefs. The three students in this study commenced the program with the preconception that mathematics is learned without the aid of materials or discussion and that demonstration is the acceptable mode for teaching. For Ann and Jan these beliefs altered to a position where group work with materials was seen as an integral part of learning mathematics. Ida's summary about how she would like to teach mathematics highlighted the students' new beliefs. She wrote "In groups with lots of games, problem solving, visual aids, colour and exciting ideas. Not chalk and talk but hands on experiencing and learning."

While there were some positive modifications of students' beliefs about teaching and learning mathematics, an examination of their elaboration worksheets suggested the mathematics concepts are very poorly developed.

This study has several implications for preservice teacher education courses. The students prior beliefs about mathematics and its teaching and learning should be determined. Programs can then be implemented to provide active contexts for students to undertake mathematics in ways contrary to past experience. However, the present study also highlighted the deficient conceptual development of the mathematics the students will teach. How this deficiency can be overcome in a methods program warrants further investigation.

References

Ball, D. (1988). Unlearning to teach mathematics. *For the learning of mathematics, 8*(1), 40-48.

Crawford, K. (1992). *Applying theory in teacher education: Changing practice in mathematics education.* Proceedings of the 16th annual conference of the International Group for the Psychology of Mathematcs Education pp. 161-168). University of New Hampshire, Durham.

Frank, M. (1990). What myths about mathematics are held and conveyed by teachers. *Arithmetic Teacher, 37*(5), 10-12.

Grover, B. W. & Kenney, P. A. (1993). *The effects of a course emphasising teaching techniques congruent with the NCTM Standards on preservice teachers' beliefs and assessment practices.* Proceedings of the 15th annual meeting of PME-NA (pp. 182-188). San Jose State University, San Jose.

Kagan, D. (1992). Professonal growth among preservice and beginning teachers. *Review of Educational Research, 62*, 129-169.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: NCTM.

Pehkonen, E. (1992). *Problem fields in mathematics teaching, Part 3. Views of Finnish seventh-grades about mathematics.* Research report 108, Department of Teacher Education, University of Helsinki.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal of Research in Mathematics Education, 20*(4), 338-355.

Thompson, A. (1992). Teahers' beliefs and conceptions: A synthesis of research. In D A. Gwouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning.* New York: Macmillan.

Wilcox, S., Schram, P., Lappan, G., 7 Lanier, P. (1991). The role of a learning community in changing preservice teachers' knowledge and beliefs about mathematics education. *For the Learning of Mathematics*, 11(3), 31-39.

A TRAINING PROCEDURE FOR PROBLEM SOLVING: AN APPLICATION OF GAGNE'S MODEL FOR DEVLEOPING PROCEDURAL KNOWLEDGE

Margaret Taplin
University of Tasmania at Launceston, Australia

This paper describes a small-scale training program which explored whether the management of problem solving strategies could be improved by teaching children to use a procedure commonly used by persevering, successful problem solvers. The training program followed Gagne's model for developing procedural knowledge and was implemented individually with a sample of six twelve-year-olds. Descriptive analysis suggested that it was possible to train these children to use the problem solving procedure and that its use enhanced their managementt of problem solving strategies and their likelihood of success.

Although a great deal has been written and debated about mathematics problem solving, it is still clear that there are difficulties associated with teaching people how to succeed at it. One of the lines which has been followed in the attempt to contribute to an understanding of the problem solving process is the comparison of the problem solving strategies of experts and novices in the hope that the heuristics used by the experts can be taught to the novices (Newell & Simon, 1978, Schoenfeld, 1985). However, teaching the use of heuristics alone does not appear to have enhanced problem solving performance (Schoenfeld, 1985) so there is also a need to explore managerial strategies used by successful problem solvers (Schoenfeld, 1985, Lester, 1985, McLeod, 1988).

One of the managerial strategies related to problem solving is concerned with perseverance, in particular with spending time on task effectively, rather than giving up too soon because of not knowing what to do next, or "overpersevering" when it might be more efficient to use another strategy such as help seeking. There has been little attention paid to the question of how time is most effectively managed once a student has exhibited willingness to persevere with a task. Mason, Burton and Stacey (1987) caution that perseverance can be a hindrance if the problem solver persistently pursues the first idea which comes to mind. They emphasize the importance of reflective, flexible thinking, particularly in planning strategies to be used and in seeking insight or fresh ideas.

Taplin (1992) addressed this notion of effective perseverance by exploring managerial strategies used by children in their attempts to solve mathematics problems. The focus was on those students who were termed "perseverers" because they reached a stage in their problem solution where they recognised that they had not reached a satisfactory answer and decided to take some action - start again, modify their strategies or change to different strategies rather than give up immediately. The study suggested that "perseverers" who were ultimately successful were more likely to be flexible in trying different approaches with the same data, or cues, provided by the problem, whereas those who eventually gave up were more likely to spend their time repeating the same approach. A model was developed (Figure 1) which described the sequence of strategies used most consistently by successful students.

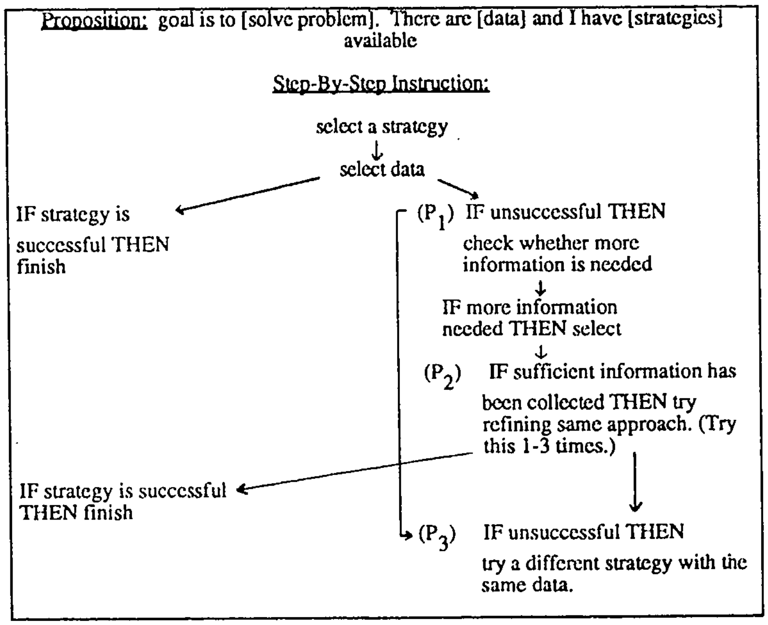
The small-scale project described in this paper investigates the effectiveness of applying this theoretical model to a teaching experiment with a small sample group (Uprichard and Engelhardt, 1986). Consistent with Uprichard's and Engelhardt's recommendation, evaluation of the instructional sequence is based primarily on observation, with comparisons being made between pre-treatment and post-treatment performances. Because the training program was aiming to train children to use a particular strategy pattern, or procedure, it was based on Gagne's (1985) model for developing procedural knowledge.

Theoretical Basis: A Model for Developing Procedural Knowledge

Gagne defines procedural knowledge as knowing "how to classify and [having] specialized rules for manipulating information" (Gagne, 1985, p.103). She suggests that the availability of procedural knowledge is one of the main distinguishing factors between experts and non-experts. She identifies two categories of procedural knowledge: pattern-recognition and action-sequence. Pattern-recognition refers to classifying information so that the learner is able to recognise specific examples of concepts by relating them to general patterns. Action-sequence requires "that an individual not only recognize the pattern specified by the conditions, but also carry out a sequence of actions - either covert ('mental') actions or both covert and overt ('physical') actions" consisting of "a series of steps in their correct sequence"(p.103). Gagne suggests that these two categories are closely linked together in the acquisition of procedural knowledge.

The first step in acquiring procedural knowledge is the creation of a propositional representation for the procedure, that is direct exposure to the procedure which enables the learner to generalise patterns and recognise action sequences. This involves the student being made aware of the steps involved in the procedure, either through reading about it, having it described, or observing others modelling the action or modelling a thinking aloud protocol. Gagne suggests (p.118) that an appropriate modelling aloud protocol could begin with, "My goal is to [solve the problem]. There are [data] and I have [strategies] available". The next step in developing procedural knowledge is to translate this information into "productions" in the form of IF/THEN statements. One production is created to represent each step in the action. Figure 1 shows the propositional representation developed for the training program. This was derived from the model used most frequently by successful problem solvers in Taplin's (1992) study.

Figure 1: Propositional representation of problem solving model



The first two steps of the model involve the student deciding what is to be done and which of the data given or implied by the problem are needed in order to do it. If the chosen strategy leads to immediate success, then the solution would obviously be completed. If unsuccessful, however, the next step is for the problem solver to refer back to the problem to check whether any information has been overlooked (P_1). Once this has been checked and necessary adjustments made, the problem solver has between one and three attempts at refining the initial strategy, perhaps by checking for arithmetical errors or trying different numbers (P_2). If this is still unsuccessful, the problem solver switches to a different strategy (P_3). The steps P_1 to P_3 are repeated as many times as is necessary until a solution is obtained.

The model shown in Figure 1 was modified into a format and language which proved to be easier than the full version for children to understand. This adaptation is shown in Figure 2.

Figure 2: Simplified problem
solving model

1.	Try an approach.
2.	Try it 2-3 times in case using different numbers or correcting errors might work.
3.	Try something different. (You might decide to come back to your old way later.)

Procedure

A randomly selected sample of six twelve-year-old children, three boys and three girls, participated in the training program. All were regarded by their teachers as being of average problem solving ability. Each student worked individually with the experimenter and was asked to complete four non-routine number problems. The first of these problems was given to the student as a pre-test to be worked through without any guidance. The purpose of this was two-fold: to determine whether the student was prepared to engage with the task for more than one attempt, and to investigate whether the student was already intuitively using the proposed model. In some instances, when it was obvious that the student was not going to solve the pre-test problem, the experimenter intervened and suggested that it might be a good time to look at the model to see if it would make the solution easier. On other occasions the student made several attempts at the problem then decided to give up, which was when the experimenter suggested looking at the model.

The first stage of Gagne's model is to show subjects the procedural representation. The experimenter explained the model, then worked through a sample procedure, verbalising each step as it occurred. After the explanation and demonstration of the model, students were asked to repeat the first problem while the experimenter guided them to use the model, prompting them to change to a different approach after a maximum of three repetitions of the previous approach. To facilitate this the experimenter intervened when it was time to implement a change of approach.

At the next stage of Gagne's model the learner is the one who attempts to use the procedure. However, Gagne suggests that there is a need to have verbal guidance and constant reminders of the sequence. She suggests that many attempts need to be made before the learner is able to perform the task smoothly and rapidly. After successful completion of question 1 the students were given questions 2 and 3. The experimenter monitored the strategy pattern and reminded students, when necessary, to follow the model. Finally the students were given the fourth problem and asked to attempt a solution, following the model, without any prompting from the experimenter.

Gagne cautions that at this stage of acquisition, there are two major obstacles. One is the limitation on working memory and the other is the learner's lack of pre-requisite skills. She suggests that it is more useful to provide lists of steps in the sequence or other prompts rather than expecting the learner to memorise the steps. This is why, in this study, the experimenter gave verbal prompts as well as displaying the model. The other obstacle described by Gagne is the learner's lack of pre-requisite skills. In the case of the present study, this would refer to a lack of a sufficient range of strategies to enable the child to be flexible. It was possible that a student might recognise the need to change to another strategy, but not to have any others available. Consequently students were told that, if they wanted to change approaches but did not have any ideas about what to do next, the experimenter would offer some ideas. When this request was made, the experimenter would offer a number of suggestions and students would then choose from these alternatives.

Summary of Outcomes

There were three distinct stages in the procedural acquisition process described above. In the first the students needed to be told when and why it was a good idea to change to a different strategy at a particular time. In the second they were able to recognise the need for change on some occasions, but were inconsistent and still needed prompting at times. The third stage was one of independence, where the student was able to follow the procedure and make appropriate changes of strategy without any prompting. The rate at which each student progressed through these stages in shown in Table 1. Emphasis was placed on observing how effectively the students were able to exhibit use of the problem solving model rather than just on the correctness or otherwise of the answer (Silver, 1985).

Table 1: Summary of students' acquisition of problem solving model

	Needed prompting to change strategy	Inconsistent in need for prompting	Able to change strategy at appropriate time without prompting
Michelle	1(i)*,2,3		4
Angela		1(i),2	3,4
Jodi	1(i)	4	3
William	2,3	1(i)	4
Greg	1(i),3		4
Craig	1(i)	2,4	3

*1(i) refers to second attempts at Question 1, after discussion of the training model

Problem 2 is not included in the table for either Jodi or Greg. Both solved this successfully at their first attempts, so did not need to use the model. An example of a student's progress through the stages is Michelle's. She needed prompting for each of the first three problems, but suddenly became able to use the model independently for the fourth. Overall it can be seen that four of the six students needed frequent prompting to change strategies on the first problem, with the other two needing it intermittently. However all students were able to use the model independently or almost independently by the third or fourth problem. These patterns in the development of students' use of the model suggest that it is feasible to "train" children to use it, at least on an individual basis.

As well as considering the patterns of students' acquisition of the training model, it was important to consider the extent to which they were able to select different approaches when they either chose to do so themselves or were prompted to do so. Table 2 indicates, for each question, the extent of the student's independence in selecting new strategies.

Table 2: Summary of students' ability to select new strategies independently

	Needed guidance to select new strategy	Inconsistent in ability to select new strategy	Able to select new strategy without assistance
Michelle	1(i)*,2	3	4
Angela	2	1(i)	3,4
Jodi	1(i),4	3	
William	2	1(i),4	3
Greg	1(i)		3,4
Craig	1(i)	2,4	3

*1(i) refers to second attempts at Question 1, after discussion of the training model

From Table 2 it can be seen that five of the six students required assistance with selecting new strategies at appropriate times when first introduced to the model. However it is apparent that all six became more independent in their abilities to select their own "new strategies" without assistance as they became more experienced with using the model. This also suggests that flexibility in problem solving is a skill which can be developed through training.

Limitations of the Training Program and Implications for Further Development

The training program took place over a short period of time, which did not allow for investigation of consistency in strategy patterns or long term retention of the model. Also it did not provide any insight into how effectively the training program could be integrated into the classroom.

When introducing the strategy sequence model to students for the first time, it was necessary to make some minor modifications in order to reduce the loads on their working memories (Gagne, 1985). Such a discrepancy may indicate that details identified by research are not always straightforward to implement because the children do not always have the capacity to incorporate all details at first. Refinements of the model would therefore need to be introduced as competency in its use developed. Nevertheless, in spite of the simplifications which had to be made to the model to make it "usable" by the target group, there seemed to be evidence to suggest that the students benefitted as the result of the experience.

Two main questions have arisen from the training program.

(i) Would students automatically refine their strategy sequences over a period of time, e.g. from grade six to grade ten? If so, what would be the implications for the training program? A long term exploration could shed some light upon the development of children's capacities to deal with the more complex models suggested by the research.

(ii) What long-term effects would the training program have? Would students use it automatically and easily in a variety of problem solving situations and retain its use over a period of time? Could it be successfully implemented with children who are initially non-perseverers?

To date this study has developed a model to enable children who are willing to spend time on a problem solving task to improve the effectiveness of the way they spend this time. In the small-scale training program which explored the extent to which use of the model could be "taught", it was found that the original model was too complex for the working capacities of children. It had to be simplified to account for the amount of information which they appeared to be able to use. However, this does not detract from the fact that the simplified version of the model seemed to make some contribution to the students' problem solving skills. This development is consistent with the aims of action research.

The ultimate aim of any training program such as the one developed in this study is for it to be effective when implemented in a natural classroom setting. Thus the training model described and modified in this study, and particularly the questions outlined above, would need to be implemented in a group experimental situation (Uprichard and Engelhardt, 1986). Similarly the questions listed above would need to be investigated in classroom as well as one-to-one situations.

References

Gagne, Ellen D. (1985). The Cognitive Psychology of School Learning. Boston: Little, Brown.

Lester, Frank K. (1985). Methodological considerations in research on mathematical problem-solving instruction. In E.A.Silver (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives (pp. 41-69). Hillsdale, N.J.: Lawrence Erlbaum.

Mason, John, Burton, Leone, & Stacey, Kaye (1987). Thinking Mathematically. Great Britain: Addison-Wesley.

McLeod, Douglas B. (1988). Affective issues in mathematical problem solving: some theoretical considerations. Journal for Research in Mathematics Education. 19, 134-141.

Newell, A., & Simon, H.A. (1978). Human Problem Solving. USA: Prentice-Hall.

Schoenfeld, Alan H. (1985). Mathematical Problem Solving. London: Academic Press.

Silver, E.A. (1985). Research on teaching mathematical problem solving: some underrepresented themes and needed directions; in Silver, E.A. (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives (pp.247-266). Hillsdale, N.J.: Lawrence Erlbaum, Hillsdale.

Taplin, Margaret (1992). An Investigation of Perseverance in Number Problem Solving. Unpublished Ph.D. Thesis, University of Tasmania.

Uprichard, A. Edward and Engelhardt, Jon (1986). A research context for diagnostic and prescriptive mathematics; Focus on Learning Problems in Mathematics, 8,1,19-38.

EXAMINATION OF A RELATIONSHIP BETWEEN CHILDREN'S ESTIMATION OF PROBABILITIES AND THEIR UNDERSTANDING OF PROPORTION

John Truran

University of Adelaide

There is some debate as to whether asking children to compare two urns containing different proportions of balls of two colours is a test of probabilistic understanding, of perceptual responses, or of proportional reasoning. This paper reports the results of clinical interviews with 32 children aged from 8 to 15. It argues that children's language does indicate that they are fully aware of the probabilistic nature of such situations and that in the case of certain and impossible events, where proportional reasoning is not possible, some children can be seen to be moving spontaneously towards using formal mathematical language. Strategies used in more general cases are shown to be far more idiosyncratic than have been reported in other research. It is suggested that there is room for further research to see whether probability scales can provide a useful unifying approach for learning consistent ways for comparing proportions.

One technique used in the assessment of children's understanding of probability has been to present to a subject two urns containing different proportions of balls of two colours and to ask which urn would be the better to choose if one wanted to draw out at random a ball of a specified colour. This technique has been used by Piaget & Inhelder (1951), Siegler (1981), Green (1982) and Singer & Resnick (1992). Others, such as Hoemann & Ross (1972), have done similar work involving the comparison of sectors of spinners. It has been argued by some (e.g., Fischbein, 1975, pp. 82 - 89) that in such experiments children may be making perceptual rather than probabilistic judgements.

This paper presents results of an investigation into this matter using clinical interviews. Full details are contained in Truran (1992). Clinical interviews in general, and Piaget's work in particular, have been criticised for being biased towards articulate subjects. However, the clinical interview provides a valuable exploratory mechanism for finding out something of what some children are thinking in order to develop more precise approaches to a wider selection of children. It is particularly valuable when beginning research into children's understanding of a specific topic.

In this paper examples will be provided to show that children's language does indicate that they are fully aware of the probabilistic nature of comparing urns and that in the case of certain and impossible events, where proportional reasoning is not possible, some children can be seen to be moving spontaneously towards using formal mathematical language and towards the development of a probability scale. In more complex situations, however, they use a very wide variety of proportional strategies, usually inconsistently. Research evidence is quoted which suggests that the use of a probability scale to assist in proportional comparisons can reveal a more complex pattern of proportional reasoning than is usually reported. It is suggested that there is room for further research to see whether probability scales can provide a useful unifying approach for learning consistent ways of comparing proportions.

METHODOLOGY

Two primary schools and two state high schools were selected from suburban Adelaide, South Australia. One primary school was a Catholic school in a low socio-economic area, the other was a state school in a middle to low socio-economic area. Both high schools were in middle to low socio-economic areas; one had a strong academic tradition, the other had both a strong academic stream and a significant number of non-academic students.

Eight students from each of Years 4, 6, 8, and 10 were interviewed. For each level teachers at each school were asked to make available two boys and two girls of "average" academic ability who would be reasonably articulate. It is my impression that the students provided were often of above average ability, particularly in the high schools. All children were adequately articulate; only one or two showed marked signs of nervousness.

Each student was individually seated at a large table and interviewed for about 30 minutes. The section summarised here occupied about 10 minutes near the end of the interviewing time. The interviews were tape recorded, and in almost all cases were conducted in privacy with few interruptions. No rewards were offered at any stage. There was no evidence that subjects were unwilling to be co-operative or to give of their best, though some did show signs of low self-esteem. But some clearly regarded escape from their timetabled activity as a definite bonus.

The younger subjects had usually not met probabilistic ideas in any formal way as part of their school studies. However, the Year 10 students had all had some formal teaching in the topic by the time of the interviews.

Subjects were presented with beads of four colours (red, green, yellow and blue) and asked to nominate which colours he or she liked most and least. These two colours were discarded and all subsequent work was done with the two colours about which the subject had less strong feelings. Subjects were presented with two urns containing balls of two colours and asked which urn would be the better to choose if one wanted to draw out at random a ball of a specified colour. I used a table of random numbers before each situation was presented to the subject to randomise variables like the desired colour, the box in which the correct answer was placed and the side of the table on which the box containing the correct answer was placed.

For convenience, in this paper all questions and answers have been standardised. Green (G) is always the desired colour, and blue (B) the other colour. The contents of Box X are presented first, and then the contents of Box Y. The question asked is standardised to "If you want to draw out a green ball which would be the best [sic] box to choose, Box X or Box Y, or are they both the same?" In cases where the contents of the two boxes do not have equal proportions, then the box where G has the higher probability is given first.

Before administering the interviews a list of possible pairs of boxes was drawn up. The list was a substantial expansion of the list used by Piaget & Inhelder (1951). During the interviews I attempted to establish whether the subject was using any particular rule for comparing proportions, and to ask questions which would test whether that rule was being used consistently. So there was no constant administration of a fixed set of questions to each subject. Some of the questions were quite difficult to answer, but subjects were not pressed if they seemed to be becoming uncomfortable. The clinical interview was seen as a flexible instrument allowing me to select which areas of the child's thinking needed investigation and to use the limited available time as efficiently and as gently as possible. It had the further advantage that it was able to obtain both verbal and non-verbal responses from the subjects, and to be able, in some cases, to compare these for consistency of thinking.

Siegler (1981) has investigated children's comparison of proportion in several concrete situations, one of which involved comparing probabilistic urn models. He found some differences between children's strategies when comparing urns and when comparing the other situations. This suggests that probabilistic situations were perceived as different in some way. Similarly, while the children in this sample answered many of these questions by comparing proportions, their language and the way they approached certain and impossible events show that they were aware that they were dealing with probabilistic situations. So, if it can be shown that children do have some intuitive ideas onto which a deeper understanding of proportion can be built, this may prove to be a valuable way of planning instruction in two different but related topics.

CHILDREN'S PROBABILISTIC LANGUAGE

Some children show in their comments an awareness that they want to maximise their luck. Several examples will be quoted here.

Many students, and almost all the older ones, use language like "you've got a better chance".

PL (Year 4, M, 8:8) 3G 3B v 2G 1B[1]

> Box Y. You've got a better chance of picking up more Gs because its got 2 Gs and 1 B. If you pick from this box they're the same so you could pick up a B. So if you picked from this box, you'd get a G more easily.

CG specifically related his response to his experiences in a "drunken walk" game at the beginning of the interview.

CG (6) (Year 6, M, 10:5) 3G 5B v 2G 1B

> Because there's 2 Gs and only 1 B, but in box X there's 3 Gs and 5 Bs and if you reached in and pulled out a B one and put it back in, you could reach in and pull out

1 This abbreviations means that subject PL, who is in Year 4, male, aged 8 years and 8 months was presented with box X containing 3 green balls and 3 blue balls and box Y containing 2 green balls and 1 blue ball and asked which box would be the best one to choose if he wanted a green ball, or whether they were both the same.

another B one. In box Y you could reach in and pull out a G and give it another shake, reach in and pull out another G and shake but then you could pull out a B one which only puts you back one and you could put it back and take a G one and keep on going vice versa.

HD frequently used words like "easier" and harder":

HD (Year 6, F, 11:4) 2G 1B v 5G 2B

> X, because it's only got one B and 2 Gs so it wouldn't be hard to get the Bs, and Box Y has got 2 Bs and 5 Gs so that would be easier than box X because it's got 2 and B's got only 1.

AM used language involving "sometimes":

AM (4) (Year 4, F, not recorded) 0G 3B v 1G 2B

> Y, that's got one G. That's [X] got no G. Sometimes you might end up with one G.

DR explicitly stated that she was making a judgement based on what would happen after the box was shaken:

DR (Year 4, F, 8:9) 1G 2B v 3G 2B

> I think I'd go for Y. ... Because in Box X there are 2 Bs and you want the colour G and there's only one G in there, and unfortunately you might shake it and get the colour B out, so you wouldn't be very happy then. If you had Box Y you'd just shake it and get G out because with the B in it you've only got 2 Bs and in that box you've got 2 Bs and a G.

In some cases, as will be shown below, the difficulties of doing the calculation placed the idea of comparing probabilities into the background. JM used phrases like "you've got more of a chance" for several of the easier questions. However, in more difficult cases she seemed to be dealing just with a rule which she has devised:

JM (Year 8, F, 13:7) 3G 5B v 4G 7B

JM I would say they are equal.
I Why?
JM Because in Y they all have 2 Bs to every one G except one which is only one B to one G. And in this one, they have all got two Bs to one G except one, 1 is to 1 so both the same.

So a surprisingly large number of subjects do make explicit statements which suggest that they are aware of the probabilistic nature of what they are doing. However, even the older subjects who had been taught probability in class did not use technical terms, and when more difficult numbers were involved they tend to forget about the probabilistic aspects of the situation.

APPROACHES TO CERTAIN AND IMPOSSIBLE EVENTS

Children's responses to situations involving probabilities of 0 and 1 have the potential to illustrate ideas strictly about probabilities as opposed to ideas about proportions. Fischbein *et al.* (1991) have suggested that the concept of "possible" may develop before the concept of "certainty". The results from these interviews suggest that some children develop mathematically mature language for certain and impossible events before they can use it for possible events.

TI (Year 8, M, 12:7) 2G 0B v 0G 3B

Box X.
I Can you say why?
TI Because you've only got a chance of getting a G, you can't get any other.

Some children may even be able to quantify such absoluteness.

LH (Year 8, M, 13:1) 2G 0B v 3G 0B

They are both the same.
I Why?
LH Because you've got G in X and Y and you ' ve 100% chance of getting a G .

Not all children, however, see things as clearly as this. They may opt for the lesser or the greater number and they may not be totally consistent.

NP (Year 8, F, 13:1) 4G 0B v 2G 0B

X.
I Why?
NP Because it's got more than in Y.

NP 2G 0B v 3G 1B

Box X.
I Why is that the best one?
NP Because whichever one you pick you are going to pick G.

The opportunity for concrete practice helped to clarify doubt.

ML (Year 4, M, 8:10) 5G 0B v 2G 1B

Both the same.
I Can you tell me why?
ML That one's got two and if I do this [demonstrating removing beads from Box X] I get Gs and Gs and Gs, 1,2,3 and you've got it. And this one's easier; you just stick your hand in and pull it out.
I So which one would be the best one?
ML Box X.
I Why have you changed to Box X.
ML Box X, you haven't got no Bs in it.

Examination of all 32 interviews shows that while there were no wrong answers to situations like aG 0B v 0G dB ($a, d \neq 0$), some children did not appreciate that situations like aG 0B v cG 0B ($a \neq c$) were equi-probable. In such simple cases some children spontaneously quantify probabilities of 1. There are no examples of children's spontaneously quantifying probabilities of 0. Some children may not be able (at least without reflection) to order simple cases like aG 0B v cG dB (c, d $\neq$ 0).

A small set of clinical interviews does not enable generalisations to be made. But it can show that some children do possess certain intuitive ideas which may mean that these ideas are fairly widely held by children. I would argue that these results do suggest that ideas about probabilistic reasoning and language used may be quite sophisticated by Year 8. In the next section I will show that the use of proportional reasoning in probabilistic situations is very inconsistent, and then suggest that the intuitive ideas which children do possess might be used as the basis for alternative instructional strategies when teaching proportional reasoning.

GENERAL STRATEGIES FOR COMPARING PROBABILITIES

An examination of the 32 interviews has revealed at least 19 different strategies of proportional reasoning which children used to compare probabilities. These strategies are used inconsistently, according to the perceived complexity of the situation. Examples of each kind are provided in Truran (1992, pp. 211 - 216); there is only space to list them here.

(1) No obvious reason.

(2) Mere description of content.

(3) "Intuition".

(4) Use of different strategies for each box.

(5) Bias towards the lesser number of a pair.

(6) Strategies involving 'more', etc, but without quantification.

(7) Comparing a and c if b and d are equal.

(8) Comparing a and c if b and d are unequal.

(9) Comparing b and d if a and c are equal.

(10) Comparing b and d if a and c are unequal.

(11) Decomposition of c into a, and c - a, so that b and d - b may be more easily compared.

(12) Approximating (e.g. "2G 3B is about even").

(13) Comparing odds with 1:1.

(14) Comparison of $|a - b|$ and $|c - d|$.

(15) Comparison of $|c - a|$ and $|d - b|$.

(16) Comparing with even proportions.

(17) Comparison of $\frac{a}{b}$, $\frac{c}{d}$ (the "within" strategy described by Freudenthal, 1978).

(18) Comparison of $\frac{a}{c}$, $\frac{b}{d}$ (the "between" strategy described by Freudenthal, 1978).

(19) Comparison of probabilities (i.e. $\frac{a}{a + b}$, $\frac{c}{c + d}$).

This list is significantly longer than, and in some cases different from, lists produced by workers who have sought to understand children's understanding of proportions, such as Karplus & Peterson (1970), Karplus & Karplus (1972), Karplus *et al.* (1974), Siegler (1981) Hart (1984) and Koch (1987) and, for probability, such as Falk *et al.* (1980), Green (1982) and Fisher (1988).

It is argued here that the very wide differences in results which have been obtained by such workers suggests that an instructional strategy is likely to meet with many difficulties. Hart's (1984) strategy specifically attempted to show students the fallacy of some of the more common inappropriate methods, but if there is such a diverse number of methods used so inconsistently, then such an approach may be simply too hard.

APPLICATION TO TEACHING

The relationship between research into children's naïve understanding and appropriate classroom practice has not always been well developed. Hart's (1984) work was effective in showing one way to eliminate the use of additive strategies, but less effective in replacing them with alternative strategies. On the other hand, Lamon's (1993) study, like many others mentioned above, concentrated on how children co-ordinate two aspects of the ratio, be they parts or wholes.

However, it has been argued by Acredolo *et al.* (1989) that a methodology which asks a subject to make a specific choice between two ratios discourages the subject from integrating the relative importance of the numerator and denominator of each ratio and is more likely to lead to an overemphasis on one part, usually the numerator. They developed a methodology which in effect asked subjects to locate ratios on a number line and they found that such an approach showed that children are capable of greater and more accurate integration of numerator and denominator than is revealed when merely asked to choose between two ratios.

It has been shown in this paper that children do have some probabilistic understanding when comparing urns and that at least some children develop sound mathematical language to describe impossible and certain events. This suggests that the teaching of probability measures with a number line may be effective and that it may also be effective in providing helpful strategies for comparing proportions in general.

The way is open to see whether probability scales can also provide a unifying structure for the teaching of ratio and proportion.

REFERENCES

Acredolo, Curt, O'Connor Jacqueline, Banks, Lesley, & Horobin, Karen (1989) "Children's Ability to Make Probability Estimates: Skills Revealed through Application of Anderson's Fundamental Measurement Methodology" *Child Development* 60: 933 - 945

Falk, Ruma, Falk, Raphel, & Levin, Iris (1980) "A Potential for Learning Probability in Young Children" *Educational Studies in Mathematics* 11: 181 - 204

Fischbein, Efraim (1975) *The Intuitive Sources of Probabilistic Thinking in Children* Dordrecht, Holland: D. Reidel

Fischbein, Efraim, Nello, Maria Sainati & Marino, Maria Sciolis (1991) "Factors Affecting Probabilistic Judgements in Children and Adolescents" *Educational Studies in Mathematics* 22: 523 - 549

Fisher, Linda C. (1988) "Strategies Used by Secondary Mathematics Teachers to Solve Proportion Problems" *Journal for Research in Mathematics Education* 19 (2): 157 - 168

Freudenthal, Hans (1978) *Weeding and Sowing: Preface to a Science of Mathematical Education*

Green, David R. (1982) "A Survey of Probability Concepts in 3000 Pupils aged 11 - 16 Years" in Grey, D.R., Holmes, P., Barnett, V. & Constable, G.M. (eds) (1982) *Proceedings of the First International Conference on Teaching Statistics* (pp. 766 - 783) Sheffield, UK: Organising Committee of First International Conference on Teaching Statistics

Hart, Kathleen M. (1984) *Ratio: Children's Strategies and Errors—A Report of the Strategies and Errors in Secondary Mathematics Project* Windsor, Berkshire: National Foundation for Educational Research-Nelson

Hoemann, H.W. & Ross, B. (1972) "Children's Understanding of Probability Concepts" Child Development 35: 265 - 273

Karplus, Elizabeth F., Karplus, Robert and Wollman, Warren (1974) "Intellectual Development Beyond Elementary School IV: Ratio, the Influence of Cognitive Style" *School Science and Mathematics* 74: 476 - 482

Karplus, Robert & Karplus, Elizabeth F. (1972) "Intellectual Development Beyond Elementary School III: Ratio — a Longitudinal Study" *School Science and Mathematics* 72: 735 - 742

Karplus, Robert & Peterson, Rita W. (1970) "Intellectual Development beyond Elementary School II: Ratio — A Survey" *School Science and Mathematics* 70 (9): 813 - 820

Koch, Laura Coffin (1987) "Strategies used by College Students Enrolled in Developmental Mathematics to Solve Proportional Reasoning Problems" in *PME 11 - Proceedings of the International Conference of International Study Group for the Psychology of Mathematics Education* 19/7 - 25/7 1987, pp. 296 - 302

Lamon, Susan J. (1993) "Ratio and Proportion: Connecting Content and Children's Thinking" *Journal for Research in Mathematics Education* 20 (1): 41 - 61

Piaget, Jean & Inhelder, Bärbel (1951) *La genèse de l'idée de hasard chez l'enfant* Translated by Leake, Lowell jr, Burrell, Paul D. and Fischbein Harold D. (1975) *The Origin of the Idea of Chance in Children* London: Routledge & Kegan Paul

Siegler, Robert S. (1981) "Developmental Sequences within and between Concepts" *Monographs of the Society for Research in Child Development* 46 (2): 1 - 74

Singer, Janice Ann, & Resnick, Lauren B. (1992) "Representations of Proportional Relationships: Are Children Part-Part or Part-Whole Reasoners?" *Educational Studies in Mathematics* 23: 231 - 246

Truran, John M. (1992) *The Development of Children's Understanding of Probability and the Application of Research to Classroom Practice* M.Ed. Thesis: University of Adelaide

COMPARING INFINITE SETS: INTUITIONS AND REPRESENTATIONS

Pessia Tsamir & Dina Tirosh

Tel-Aviv University & Kibbutzim Teaching College

Abstract

Initial investigations suggest that students' intuitive decisions concerning the equivalency of two given infinite sets are largely determined by the way these sets are represented. So far the effects of two types of representations were investigated: the numerical-horizontal and the numerical-vertical representations. Our study was mainly aimed at determining which representations (numerical-horizontal, numerical-vertical, numerical-explicit or geometric) yielded higher percentages of 1-1 correspondence reactions. For these purposes, 189 middle class 10th to 12th graders were asked to react to 12 problems dealing with comparing infinite sets. The problems presented different representations of the same infinite sets. It was found that 1-1 correspondence justifications were mainly elicited by numerical-explicit and by geometric representations. The discussion suggests ways of adjusting these findings to two different approaches to teaching: Analogy and conflict.

Introduction

Though the concept of infinity is at the heart of mathematics (Goetz, 1966), and though it is generally known to be extremely difficult to grasp (both from the phylogentic and ontogentic aspects), only several studies have attempted to take into account the accumulating knowledge available on cognitive obstacles related to this concept. One such commonly reported obstacle is related to comparing infinite sets. Unlike with finite sets, here the use of criteria for comparing infinite sets, such as "Each element in one set can be paired with one element of the other set" (the 1-1 correspondence criterion), and "the whole is greater than its parts" (the part-whole criterion), often leads to contradictory solutions. For instance, in the case of comparing the number of elements in the set of natural numbers with that of the set of multiples of four, the use of the 1-1 correspondence criterion would lead us to conclude that the sets have the same number of elements, whereas the "part-whole" idea suggests the conclusion that the set of natural numbers has more elements than the other set.

Hence, using both of these criteria to determine if two infinite sets have the same number of elements, leads to inconsistencies. In order to avoid this, one criterion should be selected, while other criteria may be used if and only if they lead to the same conclusion. The 1-1 correspondence is the criterion which was elected, within the Cantorian set theory, to determine if two given infinite sets contain the same number of elements.

Research consistently shows that when asked to compare infinite quantities, naive students often base their responses on their acquaintance with finite sets. These students often use a variety of criteria to respond to various tasks related to comparing infinite quantities. In fact, it is widely reported that students tend to use incompatible criteria for comparing infinite quantities, including the part-whole criterion, the 1-1 correspondence criterion, and other notions such as "All infinite sets have the same number of elements, since there is only one infinity", and "Infinite quantities are incomparable" (Duval, 1983; Sierpinska, 1987; Tall, 1980; Tsamir & Tirosh, 1992).

Some initial investigations (Duval, 1983; Tsamir & Tirosh, 1992) suggest that students' intuitive decisions concerning the equivalency of two given infinte sets are largely determined by the way these sets are represented. So far the effects of two types of representations were investigated. The first one is the numerical- horizontal representation, e.g.

$\{1, 2, 3, 4, ...\}$ $\{4, 8, 12, 16, ...\}$

This representation often yielded part-whole reactions.

The second type of representation is the numerical- vertical representation of the two infinite sets, for example,

$\{1, 2, 3, 4,...\}$
$\{4, 8, 12, 16,...\}$

This representation elicited the use of the 1-1 criterion for comparing the sets more often than did the numerical-horizontal representation.

The effects of other representations such as the numerical- explicit representation, e.g.,

$\{1, 2, 3, 4, ...\}$
$\{1^2, 2^2, 3^2, 4^2, ...\}$

have not yet been studied.

From an instructional perspective, it is important to determine which types of representations of a given problem elicit the use of the 1-1 correspondence criterion. This study aimed to investigate this issue.

Method

Sample

The sample consisted of 189 children from three different grade levels: Sixty-eight tenth graders, 60 eleventh graders, and 61 twelfth graders. Different achievement levels were represented in this sample: About a third were high achieving students, a third were middle level students, and another third consisted of low achievers.

Instruments and Procedure

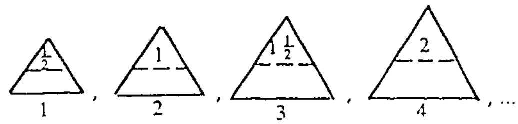
Twelve problems were developed for this study, in each of which two infinite sets were described. Subjects were asked to determine whether the number of elements in the first set was equal to, greater, or smaller than the number of elements in the second set, and to explain their answers. All sets presented in the problems were enumerable ones.

Arithmetic	Geometric

Pair I

A= {1, 2, 3, 4, 5, 6, ...}

B= $\{\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, ...\}$

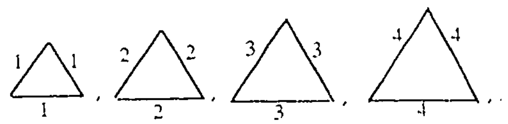


A: Sides of equilateral triangles
B: Midsegments of equilateral triangles

Pair II

C= {1, 2, 3, 4, 5, 6, ...}

D= {3, 6, 9, 12, 15, 18, ...}

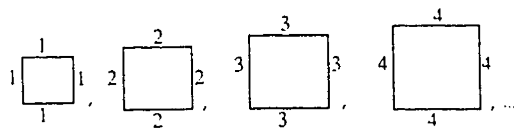


C: Sides of equilateral triangle
D: Perimeters of equilateral triangle

Pair III

E= {1, 4, 9, 16, 25, ...}

F= {4, 8, 12, 16, 20,...}



E: Areas of squares
F: Perimeters of squares

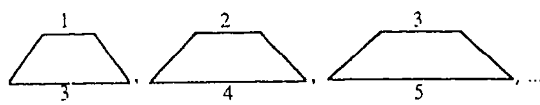
G= {1, 2, 3, 4, 5, ...} H= {3, 4, 5, 6, 7, ...}

•••

Triple I

G= {1, 2, 3, 4, 5, ...}

H= {3, 4, 5, 6, 7, ...}



G: Upper bases of trapezoids
H: Bottom bases of trapezoids

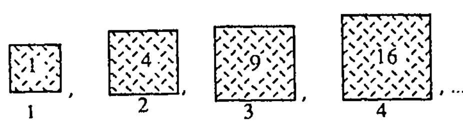
I= {1, 2, 3, 4, 5, ...}

J= $\{1^2, 2^2, 3^2, 4^2, 5^2, ...\}$

Triple II •••

I= {1, 2, 3, 4, 5, ...}

J= {1, 4, 9, 16, 25, ...}

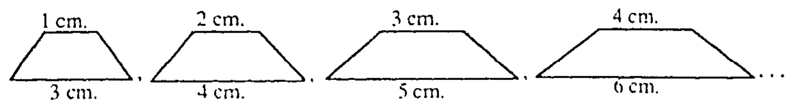


I: Sides of squares
J: Areas of squares

Figure 1: Schematic Representation of the Problems

The 12 problems consisted of three pairs and two triples. In each pair, two identical sets were presented, one in its numerical-vertical representation and the other in its geometric representation. Each of the two triples consisted of one geometrical and two numerical representations of the same problem (see Figure 1). The geometric representation consisted of a schematic drawing describing the two sets, a verbal explanation, and a vertical representation of the sets. The problems appeared in a fixed order: First the seven numerical problems and following them, the geometric ones.

Let's refer to an infinite set of trapezoids:



(1) In each trapezoid the upper basis is 2 cm. longer than the bottom base

(2) Each base is 1cm. longer than that of the trapzoid on its left

Set G is the lengths of the upper bases (in cm.), namely G= {1, 2, 3, 4, 5, 6, ...}
Set H is the lengths of the bottom bases (in cm.), namely H= {3, 4, 5, 6, 7, 8, ...}

(a) **Circle your answer:**

The number of elements in set G is greater than / equal to / smaller than / the number of elements in set H.

(b) **Explain your answer.**

Figure 2: Example of a Geometric Representation of a Problem

Results

Table 1 shows that students' decisions as to whether two given infinite sets have the same number of elements depend largely on the way these problems are represented. The highest percentages of "The same number of elements" reactions were given to the numerical-explicit representation of the problem described in Triple II (86%), and to the geometric representations of all problems (75%-86%). Four of the five numerical-vertical representations yielded about 50% "The same number of elements" responses. The only exceptional problem that yielded more such responses in its numerical-vertical representation (61%) was the last in the series of the numerical representation, succeeded by a problem that referred to the same set, but was presented geometrically. Thus it seems that higher percentage of "same number of elements" responses has to do with the location of this particular problem in the questionnaire. Support for this explanation is provided by the fact that during administration of the questionnaire the experimenter observed that some students erased their initial, non-equal reactions in favor of the same number of elements reactions after responding to the problem in its geometric representation.

Table 1
Frequencies (in %) of "Same Number of Elements in Both Sets" and of "1-1 Correspondence" Reasoning

	The Sets Compared	The Representation	Same Number of Elements*	1-1 Correspondence
	A= {1, 2, 3, ...}	numerical - vertical	47	6
Pair I	B= $\{\frac{1}{2}, 1, 1\frac{1}{2}, 2, ...\}$	geometric	80	45
	C= {1, 2, 3, ...}	numerical - vertical	52	9
Pair II	D= {3, 6, 9, ...}	geometric	76	49
	E= {1, 4, 9, ...}	numerical - vertical	52	5
Pair III	F= {4, 8, 12, ...}	geometric	82	37
	G= {1, 2, 3, ...}	numerical-horizontal	25	
Triple I	H= {3, 4, 5, ...}	numerical - vertical	49	
		geometric	75	33
	I= {1, 2, 3, ...}	numerical-explicit	86	52
Triple II	J= $\{1^2, 2^2, 3^2, ...\}$	numerical-vertical	61	13
		geometric	85	33

* The other most frequent "equal" answer was: "All infinite sets have the same number of elements".

The lowest percentage of "The same number of elements" responses was given to the only problem which was represented in a numerical-horizontal manner (problem 2- 25% correct). This finding is consistent with the findings of other studies which reported that numerical-horizontal representations of two sets, one of which is a proper subset of the other set, elicit "The sets do not have the same number of elements" responses.

Not less striking are students' justifications to these tasks. Basically, two main kinds of explanations were used to justify "The same number of elements" judgment: (a) There is only one kind of infinity, and (b) there is a 1-1 correspondence between the two sets. The judgment:

"The number of elements in the two sets is not equal" was mainly justified by the argument "One set is a subset of the other one". In some of the problems, when one set was not a proper subset of the other, (for instance, the sets {4, 8, 12, 16,...}, {1, 4, 9, 16, ...}), students used the "differences" strategy, namely, they examined the differences between pairs of consecutive elements in each set, and compared the matching differences. For instance, in the case of the sets described previously, they noted that the differences in the first set are always four, whereas the differences in the second set constitute the series {3, 5, 7, ...}. They concluded that the first set contained more elements because the gaps between its elements was constant whereas in the second set the differences increased, so that the set was less condensed.

As our major interest is in representations that evoke the use of the 1-1 correspondence criterion, we shall look mainly at the extent to which students referred to it. Table 1 shows that the percentages of "1-1 correspondence" justifications matched those of "The same number of elements" reactions, in the sense that the highest percentages of these justifications were given to the numerical- explicit representation and to the geometric representations. The lowest percentages of these justifications, on the other hand appeared on the numerical-horizontal representation. It is notable that while the numerical (horizontal and vertical) representations yielded only up to 13% of 1-1 correspondence justifications, the matching, geometrically- based representations drew significantly higher percents of these justifications (33 to 49%).

Another issue of interest was students' consistency in using a certain justification across tasks. Our data revealed that about 25% of the students consistently came up with "All infinite sets have the same number of elements, since there is only one infinity". A substantial number of students (about 50%) used the part-whole criterion and the the 1-1 correspondence criterion alternatively, depending on the representation of the problem: 1-1 correspondence justifications for numerical-explicit and geometric representations, and part-whole justifications when responding to the same problems, in their numerical-vertical and horizontal-representations.

Discussion

A main aim of this study was to determine which representations of a given problem that dealt with comparing infinite quantities yielded higher percentages of 1-1 correspondence reactions. Much like other studies, our data suggest that students' responses are largely affected by problem representation (Even, 1993; Silver, 1986; Arcavi, Tirosh, & Nachmias, 1989). It was found that geometric and numerical- vertical representations elicited "equal number of elements" reactions. One-to one correspondence justifications were mainly elicited by two kinds of representations: numerical-explicit and geometric representations.

The question that naturally arises is why do certain representations of the same problem elicit certain responses, while other evoke other responses? Our conjecture is that the visual presentation of the problem had a great impact on students' responses. In both the numerical-explicit and the geometric representations, the two infinite sets were arranged so that pairing elements stood one below the other, and consisted either of the same numbers (e.g., 1, 1 square, 2, 2 square), or were associated by an illustrated geometric relation, (i.e., the upper and bottom bases of trapezoids). Such presentations invited the student to grasp pairing elements of the sets, which lead to 1-1 correspondence reactions (much like in the case of the dancing hall, when all people are dancing in male-female pairs, and no one stands alone, in which case you immediately know that the number of men and women in this hall is the same). However, other representations, especially the numerical- horizontal ones, lead students to examine each of the sets as a separate entity, and then to the attempt to compare the total number of elements in one set to those in the other. Such an attempt naturally leads to the examination of the inclusion relationship between the sets (or, in the absence of such a relationship, to other criteria based on the notion of inclusion, such as comparison of the consecutive differences, or, again to the conclusion that both sets have the same number of elements because they are infinite).

What could be the contribution of this study to mathematics education? Information concerning the interrelations between various modes of presentations of a given problem and the typical reactions elicited by these is of crucial importance for instruction planning. Such information is invaluable for instruction related to complex, counter-intuitive topics such as the comparison between infinite sets. These topics neccessitate not only analytical instruction of the theorems, definitions, etc. but also draw special attention to intuitive schemes that may well compete with the formally learnt materials.

Two approaches to instruction related to such topics are teaching by analogy and the conflict teaching method. The information gathered in this paper is valuable for designing instruction on comparing infinite quantities according to both these approaches. When teaching by analogy, both the explicit numerical and the geometric representations could be used as anchoring tasks since they intuitively trigger the desired response. All that is left is to convince the learners that these anchoring tasks are identical to the target tasks (i.e., the horizontal representations of the same problems that commonly elicit part-whole responses). Within the conflict teaching approach, on the other hand, a conflict may be derived from presenting the same problem in two different modes: an horizontal representation which elicits "part-whole" responses, and a geometrical representation triggering 1-1 correspondence reactions. Our initial investigations suggest that in such conflicts, a substantial number of students tend to trust 1-1 correspondence justifications. They view the geometric representation as more convincing and, in a way, as a proof that the two sets contain the same number of elements.

Returning to our initial reasons for taking up this study, it is notable that so far most research in mathematics education that deals with students' conceptions has tried to pinpoint certain cognitive obstacles in learning specific subject matters. Such research is necessary for planning instruction. Yet, the time seems ripe now to proceed to the next stage: that of using this knowledge for actual instruction. This paper clarifies that in the case of comparing infinite quantities, both the numerical-explicit and the geometric representations are preferable (as they intuitively yield 1-1 correspondence reactions) to the numerical-horizontal representation (which is often used when teaching this theory). Further, this study suggests how to adjust the finding related to the interrelationships between intuitions and representations to two different approaches to teaching: Analogy and conflict. It is hoped that such information may help the sensitive teacher to plan his or her instruction of the complex, non-intuitive world of infinite sets.

References

Arcavi, A., Tirosh, D., & Nachmias, R. (1989). The effects of exploring a new representation on prospective teachers' conception of functions. In S. Vinner (Ed.) *Science and mathematics teaching: Interaction between research and practice*. Jerusalem, Israel: Hebrew University.

Duval, R. (1983). L'obstacle du dedoublement des object mathematiques. *Educational Studies in Mathematics, 14,* 385-414.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge; Prospective secondary teachers and the function concept. *Journal of Research in Mathematics, 24.* 94-116.

Goetz, B.E. (1966). The usefulness of the impossible. In W.L. Schaaf (Ed.), *Our mathematical heritage* (pp. 211-235). NY: Collier Books (reprinted from *American Association of University Professors Bulletin, 42,* 1956, 275-287).

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18,* 371-387.

Silver, E. (1986). Using conceptual and procedural knowledge: A focus on relationships. In J. Hilbert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp 181-198). Hillsdale, NJ: Lawrence Erlbaum.

Tall, D.O. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics, 11,* 271-284.

Tsamir, P., & Tirosh, D. (1992). Students' awareness of inconsistent ideas about actual infinity. *Proceedings of the Sixteenth PME Conference, Vol. III* (pp. 90-97). Durham, NH, USA: University of New Hampshire.

TRADITIONAL MATHEMATICS CLASSROOMS - SOME SEEMINGLY UNAVOIDABLE FEATURES

Shlomo Vinner
The Hebrew University of Jerusalem, Israel

ABSTRACT

Some typical classroom behaviors in mathemataics lessons are described and analysed. These behaviors are not something that the educational system is proud of or wants to encourage.The behaviors might be considered as a result of the common teaching and testing styles. It is suggested, however, that both the teaching and the learning styles are implied by very deep social and psychological reasons . The classroom behaviors which the mathematical education community wants to eliminate reflect, perhaps, some basic cognitive and emotional needs and inclinations. These might be the need for cognitive security, the inclination to prefer procedures to concept analysis (because procedures are simpler) and the need to survive through mathematics classrooms and mathematics exams.

§1. The Traditional Classroom. Most of the mathematics education research does not deal with the traditional classroom as a whole. There might be several reasons for that: 1. The cognitive research focuses on student concepts and thought processes independently of the classroom context. (Here is an implicit assumption that the cognitive structure is an invariant of the context.) 2. The classroom reality is too complex to be studied in a systematic way. There are too many factors and too many aspects involved in each classroom event. 3. The traditional classroom has already been labelled as negative. It is the cause of the common mathematics education practice, namely, mathematics as a collection of unrelated prescriptions for solving common test items. Our aim should be the elimination of the traditional classroom. Therefore, we should use our time, money and energy to create alternatives to the traditional classroom and to examine them.

If you add to this that a moderated version of (3) is also the assumption of the majority of the research and development funding agencies then it becomes quite clear why there are relatively few mathematics classroom studies. Without rejecting the above (1) - (3) I would like to raise some doubts about them. it might be true that cognitive structures are situation invariants but the researcher must deal with them in a given context It is quite possible that in that given context subjects will react in such a way which will not reflect their cognitive structures. In addition to that, there are factors which are not cognitive that determine the subjects' behavior (emotional, social and so on) and these factors strongly depend on the context. It is true that the classroom reality is extremely complicated but it is also extremely important to understand. Therefore, it is preferable to use research methods which are not "true experimental" and to study the classroom reality than to ignore it. Finally, the traditional classroom and all its associated factors (supervision, evaluation, state examinations, matriculation, social relations and so on) may be the cause of the poor practice that we all denounce but it may well be also a result of very serious psychological, social and economical reasons. If it is so, our chance to change the practice is very little before we understand these reasons. My impression is that some essential elements of the human nature are the cause of the current mathematics education practice. Whether it is possibie to overcome these elements is not quite clear. Being less cautious I would even say that it is doubtful. Therefore, I suggested the "seemingly unavoidable features" in the title of the paper

If I have to charecterize the approach of this study it is the *in vivo* approach. A possible paradigm for this approach is the didactical contract by Broussau (for details see Balacheff, 1990) Another possible paradigm is the meaning negotiation paradigm by Bauersfeld and others (for details see Bauersfeld et al., 1988). Steinbring (1991) has a similar approach. I will not refer directly to these in my paper. I mention them here in order to give some coordinates to my work and also in order to keep them as background paradigms to which the reader can refer as an additional interpretation tool for some of my findings. The episodes which I will describe and analyze in this paper are either classroom episodes or classroom related situations. Namely, they are "natural enviroment" episodes. All of them occur in regular mathematics classrooms or at office hours, in homework assignments and tests. There was no external intervenience in the natural course of a lesson, an office hour meeting and so on. Hence, this sudy can be considered as a classroom study. In most of the cases I was also the teacher but not in all of them. The students that I taught were average science colllege students (not mathematics majors). Other students were regular high school students. I am not discussing remedial classes here. I am saying this in case some readers will raise a question about the students' mathematical abilities.

§2. How to please the teacher? - The pseudo-conceptual behavior. In any traditional classroom, the students, if they want a passing grade, have to please the teacher in one way or another. This will probably be the case also with non-traditional classrooms. As long as students depend on an external authorities for evaluation and grading they will have to do something in order to please these authorities. The attempts to please the teacher can be implied by different reasons. For some students it is the desire to be loved and appreciated. For other students it is the understanding that the teacher's evaluation is important for their future. Also, good grades might increase their parents' warm affections, might establish their social prestige and so on. Answering a question is the most common way to please a teacher. Of course, the teacher is interested in answers based on mathematical knowledge. The teacher looks for meaningful answers, conceptual understanding and analytical thinking. These notions are not necessarily familiar to the students. They might have a more *behavioristic* approach to the entire learning process. They look for the right answer (*the response*) to a given question (*the stimulus*). The internal processes are not necessarily important. The main concern is that the answer will make the desired impression. Therefore, all kinds of (intelligent and not intelligent) guessing are legitimate. The moment a student realizes what the desired answer is, he or she will repeat it, even without understandig , in similar situations. A behavior which might make the impression that it is based on conceptual thinking but in fact it is not, I would like to call a *pseudo-conceptual* behavior. This is in contrast to real conceptual behavior. Observe, for instance, the following class discussion in the beginning of a mathematics lesson in a 12-th grade class. The teacher tries to explain how to do the homework assignment on recursion, a topic with which the majority of the class has serious difficulties.

Teacher: *Give a recursion rule for the sequence:* $a_n=n^2$

Students: ...

Teacher: *What is a recursion rule?*

Student A: a_2-a_1

Student B: $2n-1$

Student C: $a_1+a_1=a_2$

Teacher: *How should you express* a_{n+1}*?*

Student D: 1

Student E: n

Teacher (writing on the blackboard): $(n+1)^2 = n^2+2n+1 = a_n +2n +1$

$$a_{n+1} = a_n + 2n + 1$$

The correct answer to the teacher's original question is: a_{n+1} *should be expressed by means of* a_n *and* n. Thus, the students, in a way, were "quite close" to the correct answer. However, there was very little thinking involved in the above dialogue. It was mainly blind guessing. On the other hand, if you do not know the particular topic (recursion, in this case) you might get the impression of a meaninful discussion. The charecteristic of the students' reactions is the following: The term expressed by the teacher evoked in their mind certain associations. Since they lack understanding of the topic they cannot examine these associations and tell whether they constitute a correct answer or not. Thus, there are two alternatives. The first one is to remain silent. The second one is to express what they have on their mind. At least some students do not think they risk anything by telling the teacher what they have on their mind. If you usually do not practice critical thinking or you lack the reflective abilities you cannot consider the uncontrolled reaction as a negative one. If you lack the mechanism of examining your associations and determining whether they make any sense in a given situation then remaining silent is not an option. In the above behavior the students miss only one stage: the control stage. All of us have associations when we hear or see a certain notion. We cannot control our associations. They are the spontaneous internal reaction to a given stimulus. We can control our behavior, which is the external reaction to a stimulus. This is what education (and in this case mathematics education) is all about. The students' behavior in the above event, in which a conceptual analysis did not occur although it should have occurred, is, according to the terminology I suggested earlier, a *pseudo-conceptual* behavior. I would like to emphasize that this is a characterization of a behavior and not of a person. Every person, including an outstanding mathematician, can demonstrate a pseudo-conceptual behavior. The above event, which started with the teacher's attempt to clarify the meaning of a recursion rule, ended with an action that does not relate to meaning. The teacher presented to the students a procedure for the right answer. In the next set of problems to be given to the students, this procedure will serve as a generic model for the desired answers to which the students will arrive by mere imitation. Thus, the teacher - perhaps unwillingly, perhaps because of the need to move on (and you cannot move on in a mathematics lesson without writing some symbols or numbers on the blackboard) - did not do anything to encourage a real conceptual behavior. Implicitly, the pseudo-conceptual behavior is encouraged in this event and in many similar events.

§3. The how to solve it syndrom. Many of the difficulties students have with mathematics are a result of communication failure. Sometimes the cause of the failure is *local*. By this we mean an altertnative understanding of a term or terms involved in a given context. Sometimes the cause of the failure is *global*. By this I mean an alternative understaning of an entire framework or even the entire mathematics. It is well known that for many students mathematics is a framework where you get formulae into which numbers are plugged. In order to solve a given problem one should carry out the calculations with the numbers which were plugged into the relevant formulae. Thus, the leading principle in mathematics is the following: *plug the numbers in the relevant formula and calculate!* Bearing this in mind, a student might have serious difficulties when some parts of mathematics which do not have the above features are presented to him or her. Here is typical case. After discussing in class the notion of an increasing function, I gave, in addition to the examples and the visual explanations, also the formal definition. That is: *A function f is increasing at a certain domain if for every x_1 and x_2 in this domain such that $x_1<x_2$ also $f(x_1)<f(x_2)$.* A week later a student came to my office with a desperate expression on her face. She opened her notebook at the above definition and said with an apologetic voice: *I have thought about it through the entire week and I still don't understand how you solve it.*

A declaration that there is nothing to be solved in this case would have very little effect on the desperate student. Such a deeply rooted conception cannot be changed by one sentence. Here is another example:

Teacer: *Can you explain what is* $\lim_{n\to\infty}\frac{n-1}{n}$?

Student: *That's 1.*

Teacher: *I didn't mean a numerical answer. I meant an explanation.*

Student: *I don't understand.*

Teacher: *I mean what is this number that you are supposed to find. We talked about it many time. in class.*

Student: *I have no idea.*

Teacher: *It's the number to which* $\frac{n-1}{n}$ *tends (gets closer and closer) when n increases without any restriction (namely, you substitude n by numbers which get larger and larger).*

Student: *Will you ask such questions on the final?*

Teacher: *Yes, of course.*

Student: *In my entire life I haven't written a math exam that had words in it.*

Here, again , the belief that mathematics involves only calculations implied by substitutions in given formulae blocked the student's way to other, much more important. aspects of mathematics. The student does not realize that a mathematical expression has a meaning and one can ask about its meaning without asking about the numerical result of this expression. The student is not mentally prepared to relate to any question which is not directly related to the question how to solve it. The last question means, almost in all cases, how to obtain a numerical answer.

§4. **The search for rules as an expression of the need for security .** Many mathematics teachers know that very often, while explaining concepts or while being involved in mathematical reasoning, the concentration level of their students goes down significantly. On the other hand, when they present operational rules (mathematical prescriptions) the students become quite alert. This is quite plausible behavior if the assumption is that mathematics is calculations of numerical expressions obtained from given formulae by substitution. The bottom line of any mathematical discussion should be a "how to do it " or a " how to solve it" rule, as indicated in the last section. The students hardly accept, and even reject the idea that a mathematical situation will be given to them and they will have to use their own considerations in order to determine what the answer should be. They always expect a universal rule which will guarantee an answer. Moreover, the answer is supposed to be obtained mechanically. Although all this can be considered as a result of the common teaching style and the common testing style it might be also a result of some basic cognitive and emotional factors. The cognitive factor is that procedures, in case there are not extremely complicated, are easier to handle than concepts. It is part of the human nature, with some exceptions, of course, to prefer simple tasks to complicated tasks. The emotional factor is probably related to the need for security. The fact that somebody has tools (procedures) to deal with a situation contributes to his or her security. They have something in hand to start with. It feels insecure to face a problem without knowing how it should be solved. This requires to create a solution procedure, not only to carry it out. This might be considered as a threat by many students. Here is an episode that may illustrate it:

Student : *What is* $\lim \frac{\sqrt{n^2+1}}{n}$ when $n \to \infty$?

Teacher: *How do you deal with it ?*

Student: *It's 1, isn't it ?*

Teacher: *Let's see. You can write* $\frac{\sqrt{n^2+1}}{n} = \frac{\sqrt{n^2(1+1/n^2)}}{n} = \frac{n\sqrt{1+1/n^2}}{n} = \sqrt{1+1/n^2}$

After some more verbal explanations the teacher concludes that the above limit is 1.

Student: *But this is exactly what I said.*

Teacher: *Yes, you did, but I wanted to show you the logic behind it.*

Student: *I don't need your logic. I have my own one. I just wanted to know whether my answer is correct*

Teacher: *And what is your logic ?*

Student: *It's ∞/∞ ; that's 1.*

In the above situation the student made up an *ad hoc* rule to deal with the task. Her problem was that this rule was not suggested to her by the teacher. Therefore, her personal security was not complete. She came to the teacher to get an approval to the rule she invented. The approval was supposed to be given by the acknowledgement of her numerical result. When she failed to get this approval she became quite frustrated. She expressed her frustration by two implicit statements: 1.Your explanations were totally unnecessary since they led to a numerical answer that I already knew. 2. I do not need your logic because I already have logic which is better than yours. It is shorter and it led me to the same answer which was obtained by your obscure and compliocated logic that I do not want to adopt.

§5. **Concept Substitutes.** I have spoken earlier about the commom tendency in many students to avoid concepts. A typical class situation in which students have to face concepts is when the teacher asks them "what is ...?" For instance: What is a rectangle? What is a function? What is a vector? and so on. Many students, in many cases, develope what I call *concept substitutes* which they use, perhaps, in their inner speech (see Vygotsky, 1962) and offer them to the teacher when being asked. Here, I would like to mention two typical concept substitutes. Because of space problems I will give only one example for each one of them.

1. *The criterion as a concept substitute.* A common answer to the question "what is an increasing function?" is : *it is a function whose derivative is positive.* I even know some teachers who accept this answer as a correct one. To justify their point of view they would claim that in all the basic courses we deal only with differentiable functions and there are at most finite number of points in which an increasing differentiable function is not positive (equal to 0, in this case). Therefore, there is no need to insist on the original definition and to mess with it. No harm will be caused if the student uses the criterion for the concept instead the concept itself. Even if you accept the pedagogical principle that it is alright sometimes to sacrifice the mathematical accuracy for the sake of clarity and simplicity, in this case you just encourage the tendency to avoid conepts. (There are cases in which a criterion for a concept is a sufficient and necessary condition. In these cases, it is possible to claim that since the concept definition and the concept criterion have the same extension there is no

point to insist on fine distinctions between same extensions but different intentions. I am even ready to argue about this point of view although I would not declare a war against it). The fact that we compromise so easily about situations like that is an additional indication to the general claim that both teachers and students will prefer simple paths and shortcuts to facing difficult problems which require high level of thought processes.

2. *The example as a concept substitute.* What is a derivative? I asked a student who came to me to ask for an exemption from my college calculus course. Her request was justified by the claim that she had a calculus course in high school. Her answer to my question was: *if* $y=ax^2$ *then* $y'=2x$. *Can you give another example?* I kept on asking in order to see how rich was her pool of examples. *Yes*, she replied, *if* $y=ax^2+bx+c$ *then* $y'=ax+b$. *But what does it mean to differtiate?* I refused to stop my inquiry at this point. After a short pause she said: *To differentiate means to divide by x.* I: *And what is the derivative of* $y=c$? She: $y'=0$*...here* $(c/x)=0$*...the result is as if you get 0 ...to differntiate is as if you divide by x.* (The emphasis was in her intonation).

The student tried to construct the meaning of the derivative out of her (mistaken) examples. She came up with something which was more or less coherent with her examples. When I embarrassed her with my question about y=c she "invented" a system in which the result of dividing a constant by x is 0. As a matter of fact, she knows that this cannot be the case. Therefore she says with a special emphasis: *as if you get 0.* By this she raised an additional wall between the world of numbers with which she is familiar and the world of mathematical symbolism which she, unfortunately, does not understand.

References

Balacheff, N. (1990). Towards a problematique for research on mathematics teaching. *Journal for Research in Mathematics Education*, 22 (4), 258-272.

Bauersfeld, H. , Krummheuer, G. and Voigt, J. (1988) .Interactional theory of learning and teaching mathematics. Related micro-ethnological studies. In Steiner, H. G. and Vermondel, A. (Eds.) . Foundations and Methodology of the Discipline Mathematics Education. Proceeding of the 2-nd TME Conference. Antwerp, 174-188.

Steinbring, H. (1991). Mathematics in teaching processes. The dispairity between teacher and student knowledge. *Recherches en Didactique des Mathematiques*, 11 (1), 65-107.

Vygotsky, L.S., Thought and Language. The M.I.T. Press, Cambridge, Massachusetts, 1962

CHILDREN'S NOTIONS OF UNITS AND MATHEMATICAL KNOWLEDGE

Tad Watanabe
Towson State University
Dept. of Mathematics
Towson, Maryland
USA

Young children's unit-related notions have been shown to play significant roles in their construction of mathematical knowledge. However, the theoretical framework with a specific focus on units is still under development. The goal of this paper is to initiate a more systematic discussion and research on children's unit-related notions and their roles in children's construction of mathematical knowledge.

Introduction

Steffe and his colleagues (Steffe et al. 1983) investigated young children's counting and its relationship to the types of units the children constructed. The research was based on the belief that the construction of countable units is a prerequisite for counting. In fact, they claim that "(a)t the root of all quantification and of all numerical thinking and operating lies the construction of discrete, repeatable units and their conjunction." (p.4) They were able to identify a variety of units children counted. Furthermore, they argued that these unit types were closely related to children's mathematical development. Their subsequent works showed that children's notions of units play significant roles in their understanding of other elementary school mathematics (Cobb and Wheatley, 1988; Steffe and Cobb, 1988).

More recently, a number of researchers have reported that children's notions of units, and other unit-related ideas, play significant roles in their construction of a variety of mathematical concepts, such as fractions, ratios and proportions, and geometric problem solving. The fact that researchers have identified children's unit-related notions as an important factor in such a wide variety of mathematical concepts suggests that children's unit-related notions are "recurrent, recursive, and increasing complexity across mathematical domains." (Lamon, 1993, p.133)

Although some researchers have suggested that there may be a link between children's unit-related notions and a variety of mathematical concepts, much of the existing research focuses on specific mathematical concepts. Furthermore, in some cases, it is not even clear that the researchers' notions of units are compatible. However, if children's notions of units are an important part of their growth of mathematical power, then it is crucial that researchers come to a consensus on these issues.

In this paper, I will review, briefly, the major existing research that focuses on children's unit-related notions and try to point out some of the differences among these researchers' viewpoints. I will further suggest problems associated with children's unit-related notions as they relate to mathematical knowledge construction. However, the purpose of this paper is not to discount the usefulness of these ideas; rather, it is hoped that this paper will start a more cohesive effort among researchers to study and analyze children's unit-related notions as a fundamental constructive mechanism of mathematical knowledge.

Counting

Steffe and his colleagues (Steffe et al., 1983) identified several types of units: perceptual, figural, motor, verbal, numerical composite, and abstract composite. Their subsequent investigations (Cobb & Wheatley, 1988; Steffe & Cobb, 1988) identified different types of units of ten children may construct such as an iterable unit and an abstract singleton. It was clear that these different types of units were central to understanding (whole) numbers because "(a) natural number, in our view, is no more and no less than the result of operations that constitute the abstract entity we call "unit", carried out so many times, and now considered as one composite whole." (Steffe et al., 1983, p.11)

A crucial aspect of the unit construction process in their theoretical framework is what these researchers termed the integration operation. It is the integration operation that will allow children to form a unit composed of existing units. They argue that "the operations that constitute experiential items (which then manifest that aspect of 'oneness' or 'discreetness') are analogous to what we called the mental act of 'taking together' - except that what is being taken together to create experiential items is a plurality of sensory signals." (p.3) Therefore, the unit construction process is a building-up process. A child starts with a unit, then a group of the unit is integrated to form a "larger" unit.

This perspective helps us to understand, and appreciate, many difficulties young children encounter as they try to make sense of our numeration system. From this perspective, the construction of a large unit, say a hundred, requires multiple applications of the integration operation. In other words, a hundred is a unit of units (tens) of units (ones). Furthermore, in order to deal with large numbers, a child must be able to deal with a number of these units simultaneously. That this is not a simple task is often evidenced by a young child producing a counting sequence like "... fifty, sixty, seventy, eighty, ninety, a hundred, two hundreds, three hundreds,..." Such an "error" is caused by the fact that children must coordinate at least three different units.

Rational Number and Proportional Reasoning

Watanabe (1991) found that second graders' strategies to coordinate different units were related to their understanding of simple fractions like one-half. In the study, he identified four different types of unit coordination schemes: one-as-one, one-as-many, many-as-one, and many-as-many. Children who had developed quantitative understanding of one-half also had developed the one-as-many and many-as-one schemes, but only when the many-as-many scheme was available was the abstract understanding of one-half evident. Although his study only shows correlation between children's understanding of one-half and their unit coordination strategies, it nevertheless suggests the potential explanatory power of a theoretical framework that focuses on children's unit-related notions.

Behr et al. (1992) pay very close attention to the notion of composite units as they analyzed children's rational number knowledge. They not only provided an extensive semantic analysis of fractions using the notion of composite units but also invented a notation system that was consistent with the mathematics of quantity that made different unit types explicit. Using the notation system, they were able to suggest a variety of constructive paths for simple fractions like 3/4. Their analysis again showed how complex an understanding of fractions really is.

Lamon (1993) and Lo and Watanabe (1993) suggest that children's understanding of ratios and proportions require them to construct "a ratio as a unit." They argued that the ability to take a ratio such as "9 to 2" or "4 quarters to 6 candies" as a unit and act upon it played a significant role in young children's proportional reasoning. For example, Lo and Watanabe documented (1993) how a fifth grader used her ability to freely move among different (yet equivalent) ratios she had constructed to solve a variety of proportion tasks. However, this "ratio as a unit" notion raises a fundamental question: *What is a unit?* It appears that the "unit" in this sense is not quite identical to the unit in counting, or even rational number, contexts. This issue will be addressed later in this paper.

Exponential Function and Splitting

In her study of students' understanding of exponential functions, Confrey (1990) proposed the notion of splitting. She defines splitting as "an action of creating simultaneously multiple versions of an original, an action which is often represented by a tree diagram." She argues that the geometric character of splitting, its relationship to similarity, distinguishes it from counting. Confrey believes that splitting can be the basis of an alternative construction of positive real numbers. She further suggests that counting-based approaches to rational numbers too often reduce multiplication as repeated addition and a fraction as a part of a whole, both rather limiting perspectives, while splitting can facilitate children's construction of more sophisticated multiplicative perspectives.

In her report, Confrey defines a unit as "the invariant relationship between a successor and its predecessor; it is the repeated action." Thus, in the counting world, the unit is the action of adding one while in the splitting world the action involved is multiplication by N. Thus, a two-split starting with one results in the sequence, 1, 2, 4, 8, 16, 32,... The constant relationship between any two consecutive terms is that the successor is always the totality of two versions of the predecessor. Furthermore, in the splitting world, establishing the relationship 8/2 = 32/8 is accomplished by noting that both 8 and 32 are two "growth units" away from their respective predecessor, 2 and 8. Resnick and Singer (1993) pointed out the importance of the factorial structure of whole numbers in children's understanding of ratio reasoning, and the splitting naturally leads to such a structure. Thus, the notion of splitting seems to suggest not only an alternative constructive path for whole numbers but also fractions, ratios, and proportions.

However, like the notion of "ratio as a unit," this notion of unit as a "repeated action" does not appear to be consistent with the notion of units proposed by other researchers. So, are these researchers using the same word to describe different constructs? Or is there a way to reconcile this apparent differences? In the following section, I will present a short discussion on some of the fundamental issues with the notion of units.

Fundamental Issues

The first issue I would like to discuss is the question: *What is a unit?* As it has been discussed above, there does not appear to be a consensus as to what a unit is. There are two words that are used by a variety of researchers to denote this notion of unit: unit and unity.

In the realm of counting (and whole numbers), these two words appear to be almost interchangeable. According to McLellan and Dewey (1909), "(i)n the simple recognition, for example, of three things as three the following intellectual operations are involved: *The recognition of the three objects as forming one connected whole or group* - that is, there must be a recognition of the *three* things as individuals, and of the *one*, the unity, the whole, made up of the three things." (p.24, emphasis original) Thus constructed three, in turn, can become a unit that composes another unity, say nine. If there is any distinction between a unit and a unity, it seems to lie in the person's action. If the person does not act upon the whole, it is a unity, but when a person acts (or intends to act) upon the whole, it is a unit. However, many researchers do not appear to make such a distinction. For example, Lamon (1993) describes how 3/4 of 16 objects is a unit of units of units: i) 16 one-units, ii) 4 four-units, and iii) 1 three-(four-unit) The first two units are constituent parts of larger wholes; however, the last unit does not share that characteristic. If we distinguish a unit and a unity, the first two are units while the third is a unity. A natural question to ask is if such a distinction is important. I will leave that question for the reader to consider further.

Although the notion of counting units is already complex, the ideas like "a ratio as a unit" or a unit as a "invariant relationship between the successor and its predecessor" further complicate the issue. On the surface, these "units" are not units in the same way one is a unit. In fact, these "units" are really a second order unit in that the relationship exists (in the mind of the individual) between objects which must be recognized themselves as unities. So, in what sense are these relationships units?

It is clear that there are a variety of ways for a person to construct a unit. For example, the integration operation Steffe et al. (1983) suggested is a way for a person to "build up" larger units from what have been constructed. However, the integration operation does not explain the construction of the unit of one. The nature of the constructive process for the unit of one seems to be qualitatively different from that of such units as ten and hundred. Here is where an apparent paradox in the "unitizing" operation lies. McLellan and Dewey (1909) argued that, in order for a person to recognize three colored cubes as *three*, the person must view the cubes, simultaneously, similar and different. They must be similar so that they can be counted but they must be different or there is no three-ness.

Such a paradox was also discussed by Steffe et al. (1983).

> Number is a conceptual creation. A particular number is produced as a double act of abstraction - for, as Euclid observed, a number is a unit (a multitude) that itself is composed of units. There is a first act of abstraction that produces units from sensory-motor material ... and there is a second act of abstraction that takes these units as the material for the construction of a unit that comprises them ... (p.1)

Thus, in order to construct a unit, there must be a multitude. Yet, what is a multitude if it is not a collection of units? Furthermore, the direction of such unit construction process is opposite of the integration operation. That is, a new unit is constructed from an old one when its components are recognized as a unit. Thus, the unitizing operation is more of a decomposition than an integration.

This perspective is consistent with the idea suggested by Maturana and Varela (1980). They argued tnat

> (t)he basic cognitive operation that we perform as observers is the operation of distinction. By means of this operation we specify a *unity* as an entity distinct from a background, characterize both unity and background with the properties with which this operation endows them, and specify their separability... If we recursively apply the operation of distinction to a unity, so that we distinguish components in it, we respecify it as a composite unity that exists in the space that its components define because it is through the specified properties of its components that we observers distinguish it. (p. xix)

Thus, the unitizing process is really a further breaking up of the existing unit. The decomposition as a unitizing process is especially useful as we try to understand children's rational number concepts. Furthermore, a unit constructed as a result of decomposing a different unit is a unit, not simply a unity (if we make this distinction).

A Hypothesis

So, what does this apparent paradox about the construction of a unit tell us about the notion of a relationship as a unit? It is clear that no unit, even (or especially) the unit of one, exists in a vacuum. A unit is constructed as it relates to a whole which is composed of the unit. Thus, it is not unreasonable to say that all units are relationships.

Maturana and Varela's idea of distinction may provide a framework to describe the relationship between Confrey's notion of splitting and Steffe et al.'s integration. Although the integration and splitting operations are not inverse of each other, they are complementary in the following sense. The splitting operation may be needed to construct units from a multitude and the unit thus constructed can be integrated. On the other hand, in order to apply the splitting operation repeatedly, the result of a split must be unitized, and this process may be through the integration operation. Unfortunately, we know very little about such mental operations and further research is needed.

Suggestions for Further Studies

If children's unit-related notions are truly "recurrent, recursive, and increasing complexity across mathematical domains" another important question is: *Do all unit-related notions provide a useful framework for all mathematical knowledge? If not, which unit-related notions are important across a variety of mathematical topics and which are specific to a certain topic?* In the existing literature, there are two ideas that may be of importance for a variety of mathematical ideas: unit coordination and unit conversion.

Steffe and Cobb (1988) argued that as young children construct more sophisticated number worlds, relationships between units go from unrelated, co-existing, to coordinated. Watanabe (1991) presented evidences that children's coordination schemes may be related to their understanding of simple fractions like one-half. Lo and Watanabe (1993 a & b) documented how children's ability and inability to coordinate different units impacted their proportional reasoning. Wheatley and Reynolds (1993) also indicated that individuals' ability to coordinate units influenced the quality of their geometric problem solving. It thus appears that unit coordination is relevant across a variety of mathematical topics.

One of the most critical ideas in Behr et al.'s (1992) semantic analysis is the notion of unit conversion. The ability to convert 4(1-unit) to 1(4-unit) plays a significant role in their analysis. They point out that further research on the unit conversion principle is essential. It is not clear if this unit conversion principle is related to the notion of unit coordination presented by Watanabe (1991). However, there seems to be an agreement that additional research is needed to further our understanding of these ideas.

Other important ideas suggested in a variety of research reports include the notions of decomposition of units (Behr et al., 1992; Piaget et al., 1960), re-unitizing (Behr et al., 1992), and re-initialization of units (Confrey, 1990). Understanding of these notions is essential if we are construct a rigorous theoretical framework on units and unit-related notions.

References

Behr, M.J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D.A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning* (pp.296-333). New York: Macmillan

Cobb, P. & Wheatley, G.H. (1988). Children's initial understanding of ten. *Focus on Learning Problems in Mathematics, 10(3),* 1 - 28.

Confrey, J. (1990). *Splitting, similarity and rate of change: A new approach to multiplication and exponential functions.* Paper presented at the annual meeting of the American Educational Research Association, Boston, Mass., April 16-20, 1990.

Lamon, S.J. (1993). Ratio and proportion: Children's cognitive and metacognitive processes. In Carpenter, Fennema, & Romberg (Eds.) *Rational numbers: An integration of Research* (pp.131-156). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Lo, J. & Watanabe, T. (1993). *Young children's construction and coordination of units in proportional tasks.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta, Georgia, April 12-16, 1993.

Maturana, H.R. & Varela, F.J. (1980). *Autopoiesis and cognition: The realization of the living.* Dordrecht, Holland: D. Reidel.

McLellan, J.A. & Dewey, J. (1909). *The psychology of number.* New York: Appleton. (Originally published in 1895).

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry.* New York: Basic Books.

Resnick, L.B. & Singer, J.A. (1993). Protoquantitative origins of ratio reasoning. In Carpenter, Fennema, & Romberg (Eds.) *Rational numbers: An integration of Research* (pp. 107-130). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Steffe, L.P. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory and application.* New York: Praeger.

Watanabe, T. (1991). *Coordination of units: An investigation of second grade children's pre-rational number concepts.* Unpublished doctoral dissertation, Florida State University.

Wheatley, G.H. & Reynolds, A. (1993). *The construction and coordination of mathematics units.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta, Georgia, April 12-16, 1993.

MULTIMODAL FUNCTIONING IN UNDERSTANDING CHANCE AND DATA CONCEPTS

Jane M. Watson and Kevin F. Collis

This paper presents some of the results from the pilot stages of a large project studying Australian children's understanding of concepts related to chance and data. The model used for the analysis of responses is that developed and extended by Biggs and Collis (1982, 1991). The appearance of multimodal functioning in some responses and its relationship to the level of functioning within the concrete symbolic mode are of particular interest. The specific topics covered in this report include comparison of two groups from graphical presentations, interpretation of a bar chart and decision-making about the fairness of dice.

An important theoretical goal of this study is to gain an appreciation of the types and levels of cognitive functioning which occur when students solve problems involving chance and data. For some time now researchers (e.g., Davis, 1984; Johnson, 1987) have been suggesting intuitive and visual influences on the problem solving process which cannot be identified directly with logical mathematical functioning. The work of Fischbein (e.g., 1975) and Tversky and Kahneman (1971) over the years suggests that the study of probability may be an area where intuitions play a particularly significant role. Also recent work of Berenson, Friel and Bright (1993) in statistics shows a strong visual component in many decisions related to graphical interpretation, and research of Campbell, Collis and Watson (1993) has shown that the purely imaginative and story-creating aspect of mental functioning may be at least a motivating factor in some problem solving activities. While it is acknowledged that basic levels of logical functioning are required to solve certain problems, it is of interest to explore the interaction with other forms of functioning exhibited by problem solvers.

A very useful model for considering this type of functioning is that of Biggs and Collis (1991). Evolving from the neo-Piagetian SOLO Taxonomy (Biggs & Collis, 1982) which stressed the sensori-motor, ikonic, concrete symbolic, and formal modes of operation within which learning cycles operate, the model incorporates an acknowledgment of the existence and importance of intermodal and multimodal functioning in many types of learning experiences. Hence it is to be expected from this model that earlier-developed modes continue to develop in parallel with later modes and provide opportunities for interaction which may facilitate intellectual functioning in general. Of particular interest here are the ikonic and concrete symbolic modes, these being associated with intuitive functioning and the symbolic learning which takes place in school and which is usually rooted in concrete materials. Results demonstrating multimodal functioning in the acquisition of understanding of fractions (Watson, Campbell & Collis, 1993) and the work of others in the field of stochastics (Berenson, *et al.*, 1993; Fischbein & Gazit, 1984; Watson & Collis, 1993) lead to the belief that such functioning will also occur for concepts in statistics and probability.

The study reported here was designed as a pilot to examine the hypothesis of multimodal functioning in chance and data. Three different types of chance and data stimulus materials were selected because it was speculated that the different contexts would draw on different aspects of both the ikonic mode and the concrete symbolic mode in each case. This should give at least a partial view of the range of thinking strategies used by children and, at the same time, make it possible to note any underlying similarities between the modes of thinking utilised.

Procedure

The data for this report were extracted from the larger study (see Watson (1992) and Watson, Collis & Moritz (1993) for a fuller description of the study and data collection). Interviews from 18 students, six each in Grades 3, 6 and 9, were analysed to explore visual, imaginative and intuitive aspects of ikonic functioning and their relationship to functioning in the concrete symbolic mode. The students were chosen from larger groups on the basis of responses to a paper-and-pencil questionnaire; a range of apparent abilities was selected. Three interview protocols were used: a four-part problem on the comparison of two groups which invited visual ikonic functioning in conjunction with concrete symbolic reasoning, a series of questions on the interpretation and manipulation of a large bar chart with moveable parts, and a question about the fairness of dice which allowed for intuitive beliefs to be expressed. These will be discussed in turn.

Comparing Groups

The interview protocol used is shown in Figure 1. In succeeding parts the questions were the same for each graph, asking the students to determine which class had done better or whether they had done the same. The idea for the item was adapted from Gal, Rothschild and Wagner (1990). Assuming that problems of this kind could be solved at a high level (i.e. detecting a universal principal involved) by intuitive (in this case basically data visualisation) means or by taught mathematical symbolic techniques (concrete symbolisation), or by means of an interaction between the two, it is possible to speculate that by taking some well thought-out problems and analysing a range of students at different age levels we should be able to find the following methods of solving the problem at the relational level: (i) by visualisation only, (ii) by a mathematical technique only, or (iii) by a combination of (i) and (ii).

There were three approaches to solving these problems. The Visual method involved looking at the graphs and n...king some type of visual comparison of the two classes. Nearly everyone used this approach for Part 1. Only two students, however, used a visual comparison successfully for Part 4: a Grade 6 student said the Black class had less people and were doing pretty well, and a Grade 9 student asked if the class sizes were the same and then said it looked as if the Black class had more "right". Half of the students used visual comparison only to answer the questions.

A second method, termed Total, referred to those who added scores to obtain total scores for each class and then decided the better class on the basis of the highest total. Six students used this approach for at least one part; for one student it was the only strategy. All students who used it for Part 4 incorrectly chose the Pink class because of unequal class sizes.

The third method employed the Average to compare the two groups. Three students used this for one or more parts and all were successful at all parts of the problem. The results on the four parts of the question for the 16 students who completed it are shown in Figure 2. They have been ordered by the number of parts successfully completed. The ordering displays a perfect Guttman type hierarchy except in two instances, one of which resulted from a counting error and the other a visual error. That is, generally students completed each part correctly up to an error but none correctly after that.

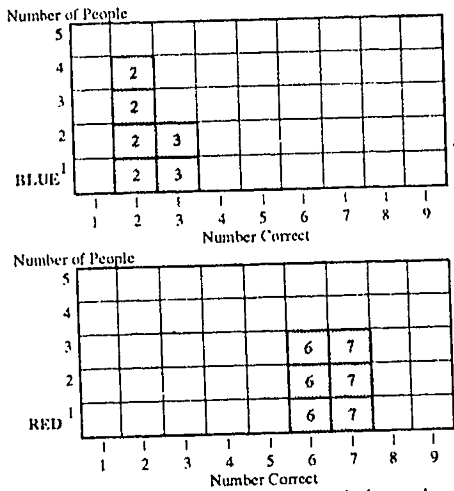
Two schools are comparing some classes to see which is better at quick recall of 9 maths facts. In each part of this question you will be asked to compare different classes.

1. First consider two classes, the BLUE class and the RED class. The scores for the two classes are shown on the two charts below.

Each box is one person's test, and the number inside is their score

In the BLUE class, 4 people scored 2 correct and 2 people scored 3.

In the RED class, 3 people scored 6 correct and 3 people scored 7.



Now look at the scores of all students in each class, and then decide, did the two classes score equally well, or did one of the classes score better? Explain how you decided.

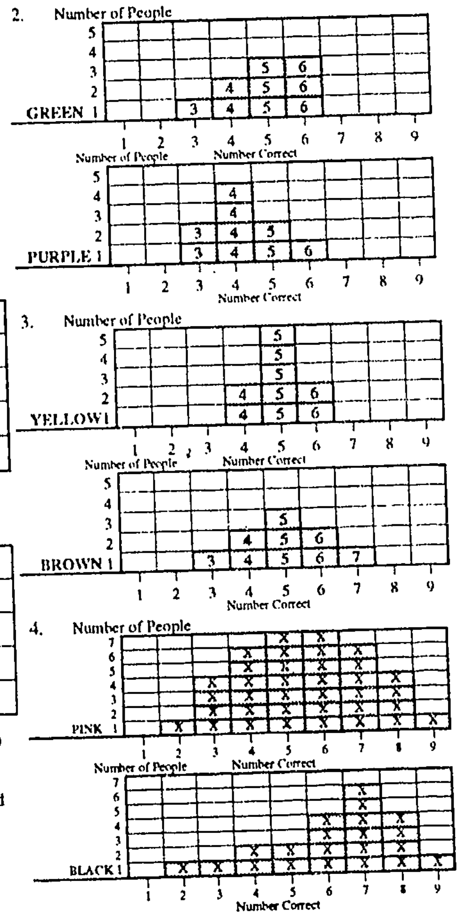


Figure 1. Comparison of Groups

The responses to this item are of interest for two reasons. First is the aspect of multimodal functioning demonstrated by the use of both ikonic (Visual) and concrete symbolic (Total and Average) processing. Second is the level of concrete symbolic processing associated with an increasing number of correct responses to the sub-parts of the protocol. As all responses appreciated the numerical aspect (even if only single, bigger, fewer, etc.) of the problem, there are no totally ikonic responses. Such responses would be of the form "the Blue class because Blue is my favourite colour" or "Green because it is fatter than Purple". A couple of younger students did make comments about classes being their favourite colours, but none made a decision based on this criterion. In terms of the unistructural (U), multistructural (M), and relational (R) levels of the concrete symbolic mode, it would appear (as was true for fractions) that ikonic support was available and increased in sophistication with the level of concrete symbolic response. A summary of the hypothesized types of responses and their ikonic and concrete symbolic cha... is given in Figure 3 in the format of Watson, *et al.* (1993). It shows the increasing sophistication of multimodal functioning (labelled A and B for ikonic) used in responding to this protocol. It is interesting that even when the average is a commonly used tool, the presentation of the problem invites multimodal functioning.

Int. No.	Approach	Q1 (Red)	Q2 (Green)	Q3 (Equal)	Q4 (Black)	Comments
6.3	Visual (1,2,3,4)	✔	a	Brown	Pink	(3) more bigger mix, (4) same for 7,8,9, less for 6,5,4,3,2,1
3.4	Visual (1,2,3,4)	✔	✔	Brown	Pink	(3) they've got a 7
3.3	Visual (1,2,3,4)	✔	✔	Brown	Pink	(3) they've got a 7, (4) it's got the most things
3.6	Visual (1*,2*,3,4)	✔	✔	Brown	Pink	(1*,2*) adding up could decide ; (3) more high numbers, despite favourite colour
6.5	Visual (1,2,3,4)	✔	✔	Yellow	Pink	(3) more 5s & 6s, (4) more people and more scores
9.6	Visual (1,2,3,4)	✔	✔	Brown, Yellow	Pink	(1) still 6 people, but all 6 got a better mark, (4) probably, but more people
3.1	Total (1,2,3)	✔	✔	Yellow	·	(3) counting in fingers, error in adding
3.5	Visual (1*), Total (1*,2,3,4)	✔	✔	Brown	Pink	(1) decided, THEN added; (3) counting error (48/47), also counts group sizes, (4) counted group sizes
9.1	Visual (1), Total (2,3,4*)	✔	Purple	✔	Pink	(2) adding error, 34 instead of 44, (4) Pink 36, Black 21 no of students
6.2	Visual (1*), Total (1*,2,3,4)	✔	✔	✔	Pink	(1) decided, THEN added, (4) 198 better than 130
3.2	Visual (1,4*), Total (2,3,4*)	✔	✔	✔	Pink	(4) Begins adding. Then decides Pink higher, more people
6.4	Visual (1,2,3,4)	✔	✔	Brown	✔	(4) less people and doing pretty well
9.3	Visual (1,2,3,4)	✔	✔	✔	✔	(4) asks if class size the same, looked like more right
6.1	Total (1,2,3), Average (4)	✔	✔	✔	✔	(1) notes group sizes the same; (2) total, but notes visual clues to this, (3) symmetry, (4) Pink better, but... average.
9.5	Visual (1), Average (2,3,4)	✔	✔	✔	✔	-
9.4	Visual (1,2), Average (3,4)	✔	✔	✔	✔	-

Figure 2. Comparison of Groups

Type of response	Level and Ikonic Support	
	Ikonic	Concrete Symbolic
Visual comparison only	I_A	U
Totals only	·	M
Visual and totals	I_A	M
Visual comparison in complex case	I_B	R
Visual, totals and average	I_B	R

Figure 3. Comparison of Groups - Multimodal Functioning

Interpretation of a bar chart

The protocol for this interview was based on a large bar chart with coloured moveable columns showing the four ways children in a class travel to school (car, bicycle, bus or walk). It was devised after discussion with L. Pereira-Mendoza. Eighteen children were represented on the chart at the beginning of the interview with the highest frequency for travel by bus. Students were asked a series of questions and allowed to change the chart to show answers where appropriate. The questions were (i) 'how many children are in the class?', (ii) 'how many more walked than rode bikes?', (iii) 'what is the most likely way a new child would travel to school?', (iv) 'how might the chart look if the bus didn't come?', and (v) 'how might the chart look on a rainy day?'. With this problem it was speculated that the nature of the questions in the main would determine the strategy. The first two gave very little option but to use numbers and operations (concrete symbolic) with minimal visualisation or intuition required; the last two again required the use of numbers and operations but this time with considerable ikonic (visualisation) support. The third question gave the children the option to use either mode, depending upon their interpretation of the question.

This protocol was used as a foundation question of particular interest in relation to Grade 3 children. It was asked of all Grade 3 and 6 children. Several Grade 3 children claimed to be unfamiliar with the chart shown but only two needed the details of the frequency on the left side explained. Figure 4 shows the results for all 12 children and summarizes in the first two columns the manner in which children coped with the mathematical aspects related to the basic operations covered and in the third column the results of compensating when changing the chart for a response. These again show a Guttmann type hierarchy. The remaining columns indicate the method of response when the child was asked how a new child came to school, with a brief note of imaginative elaborations in the final column. In relation to question (iii) one child would make no guess, two based a decision on the fact that the bus column had the most children (frequency approach) and the rest made up stories to justify a choice. Nine students used ikonic support in the form of imaginative stories from outside the mathematics class experience.

Int. No.	Able to take in Information	Perform basic operations	Compensate for totals	New student: No guess	New student: Story	New student: Frequency	Ikonic Support
3.5	✓ (help)				✓		live close
3.4	✓ (help)	✓ (all count)				✓	
3.6	✓	✓			✓		live close
6.2	✓	✓			✓		might have bike
3.1	✓	✓	✓ (count error)		✓		doesn't know way
3.3	✓	✓	✓	✓			
3.2	✓	✓	✓		✓		more shoes than cars
6.3	✓	✓	✓		✓		poor, used to walking
6.5	✓	✓	✓		✓		house further, rain gear
6.6	✓	✓	✓		✓		not safe on bus
6.4	✓	✓	✓		✓		many elaborations
6.1	✓	✓	✓			✓	

Figure 4. Interpretation of Bar Chart

It is likely that the first listed response is 'prestructural' with respect to this task, that those who could perform the basic operations but not compensate for totals are unistructural, and that those who could compensate are multistructural. It is of interest that at both levels some responses included ikonic support and others did not. This is summarized in Figure 5 in a similar fashion to Figure 3.

Type of response	Ikonic Support	Level of Concrete Symbolic Response
Difficulty interpreting information	I_A	P
Basic Operations only	I_A	U }
	-	U }
Ability to compensate for totals	I_A	M }
	-	M }

Figure 5. Bar Chart problem - Multimodal Functioning

Fairness of Dice

The protocol for this interview began by asking if the student believed that one number comes up more often than others when dice are thrown. It then went on to explain about fair and unfair dice and presented the student with two (Grade 3 and 6) or three (Grade 9) dice with the task of determining which were fair and which not (again devised after discussion with L. Pereira-Mendoza). One die was fair (red) and one had been loaded on the side with a five on it (blue). The third die (for Grade 9) had two of each of the numbers one, two and three on it. Because of the nature of the problem expectations would be that ikonic strategies would involve the use of physical characteristics, all of which in this context would be inconclusive, and that these would form a set of strategies preliminary to trials of some sort. The highest (relational) concrete level would be represented by a few controlled trials. The dependence on *perception* (of physical attributes in this case) is one of the features of ikonic mode functioning, while the use of a limited set of concrete experiments upon which to generalise is one of the marks of concrete symbolic functioning.

Of the students asked, over half believed that some numbers came up more often than others, although a couple of Grade 9 students were unsure. In determining the fairness of the dice, there appeared to be four ..ages. One Grade 3 had an anthropomorphic view of the dice including comments like "3 lets other numbers come up", "[It] changed its way", and "It's trying to confuse me". Ten students used the physical characteristics of the dice as a means of deciding their fairness. This included noting the heaviness of the loaded die and using the fact that each die had numbers 1 to 6 to declare them fair. Of these ten students, three mentioned tossing the dice but could not instigate any systematic method of doing so. One other student tried tossing the dice but tossed them at the same time and could not distinguish later which numbers appeared on which dice. These four responses are considered in transition to a full understanding of using experimental trials to determine fairness. Finally six students (two in Grade 6 and four in Grade 9) successfully carried out trials as a basis of determining which dice were fair. These responses are summarized in Figure 6 where it can be seen that it is more likely that students who use trials to determine the fairness of dice will believe in general that no numbers are more likely to come up than others. As before, Guttman type progression is evident.

Int.	All numbers same chance	Dice decide	Physical Characteristics	Idea of trials only	Carry out trials
3.6		✔ (several rolls)			
3.2	(depends on method)		✔ (red, guess)		
3.1			✔ (appearance)		
3.3			✔ (heavier)		
3.5			✔ (same numbers - fair)		
6.3			✔ (heavier)		
9.3	(unsure)		✔ (heavier)		
3.4			✔ (bigger)	✔ (3 tosses, change to fair)	
6.2	✔ (changed from 6)		✔ (larger, rounder)	✔ (confused)	
6.1	✔		✔ (heavy)	✔ (few only)	
9.4	✔		✔ (nos., 1-6)		
6.5				✔ (together)	
6.4					✔ (10)
6.6	✔				✔ (6)
9.2	✔ (wishing)				✔ (10)
9.6	✔				✔ (4, plus)
9.1	✔				✔ (10,16)
9.5	✔				✔ (20)

Figure 6. Fairness of Dice

It is hypothesized that those who use physical characteristics to determine the fairness of dice are mainly relying on ikonic features rather than logical concrete symbolic reasoning. The hypothesized relationship between the ikonic support and the level of concrete symbolic functioning is shown in Figure 7.

In this particular problem, it would appear that ikonic support is for lower level concrete symbolic responses. This is unlike the previous two cases where ikonic support of different types occurred along with the highest level responses.

Type of response	Ikonic Support	Level of Concrete Symbolic Response
The dice decide	I_A	P
Physical Characteristics only to decide	I_B	U
Conflict of physical characteristics & unsuccessful trials	I_B	M
Unsuccessful trials	-	M
Successful use of trials	-	R

Figure 7. Dice problem - Multimodal Functioning

Discussion

In relation to the study of problem solving in the area of chance and data it would appear that not only will attention need to be given to multimodal functioning but also it will have to focus on the type of ikonic support which is used. The three problems presented here indicate that visual, imaginative, and intuitive ikonic support are all possible.

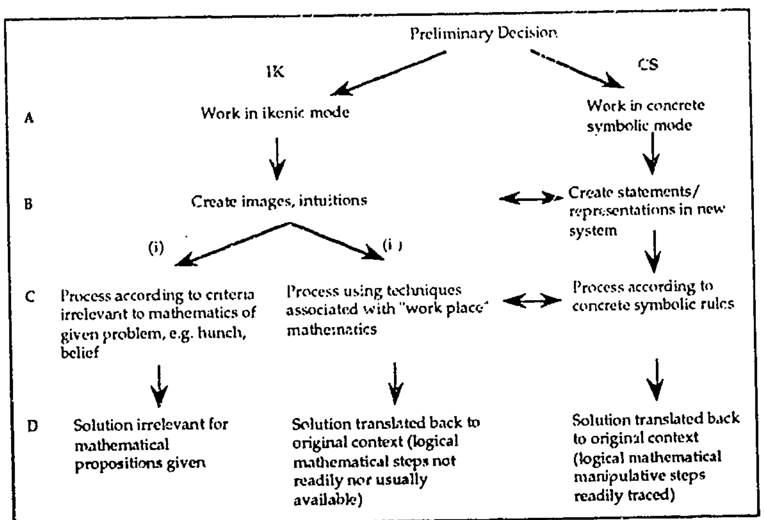


Figure 8. The Problem Solving Path (adapted from Collis & Romberg, 1991)

These aspects of ikonic functioning and its interaction with concrete symbolic functioning are illustrative of the problem solving path suggested by Collis and Romberg (1991) and shown in Figure 8. In the figure a respondent begins by following either an ikonic (IK) or concrete symbolic (CS) course of action. Some people continue down one path or the other. Consider several examples. Student 3.5 followed path IK (i) reaching an irrelevant solution for the third protocol: that the dice decided how to fall. In solving the problems set in the first protocol, two students (6.4 and 9.3) followed path IK (ii) using visual clues only to compare groups successfully. In all three protocols there were students (3.1 on the first, 3.4 and 6.1 on the second, and 6.6, 9.1, 9.5, and 9.6 on the third) who chose a straight CS course of action for response. On the other hand, some people used both modes, corresponding to the horizontal arrows in Figure 8. Nine students worked at level B creating images/stories in response to parts of the bar chart protocol, while both the first and third protocols elicited responses which can be associated with the interaction at level C in Figure 8. It would appear that the responses in this study would illustrate that all of the kinds of processing suggested by Collis and Romberg (1991) occur in the area of chance and data. That they do not all appear in the same problem is not surprising since context has a part to play in the selection of a solution strategy.

Conclusion

Two important findings are suggested by this study. First is the confirmation of the underlying structure suggested by Collis and Romberg (1991) which accounts for all responses obtained. In addition the responses appear to cover all possible paths in their model. Second is the appearance of different specific strategies according to the context of the protocol set. Not all protocols elicited all possible paths to solutions. Both of these results are of major significance if confirmed in more extensive studies. For mathematics educators, particularly those embarking on studies of the teaching and learning of chance and data, we must know *both* the underlying structure *and* the specific courses of action which are likely to be chosen, if we are to make intelligent suggestions for the curriculum and for methods of instruction. Further research is called for to confirm these findings and expand the contexts in which they may occur. In the area of probability and statistics, the ground is ready for tilling and planting.

Acknowledgments

This research was funded by the Australian Research Council, Grant Number A79231392. The authors wish to acknowledge the assistance of Jonathan B. Moritz in collecting and analyzing the data.

References

Berenson, S. B., Friel, S. & Bright, G. (1993, April). *Elementary teachers' fixations on graphical features to interpret statistical data.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA.

Biggs, J.B. & Collis, K.F. (1982). *Evaluation the quality of learning: the SOLO Taxonomy.* New York: Academic Press.

Biggs, J.B. & Collis, K.F. (1991). Multimodal learning and the quality of intelligent behaviour. In H.A.H. Rowe (Ed.), *Intelligence: Reconceptualization and measurement* (pp 57-75). Hillsdale, NJ: Erlbaum.

Campbell, K.J., Collis, K.F., & Watson, J.M. (1993). Multimodal functioning during mathematical problem solving. In B. Atweh, C. Kanes, M. Carss, & G. Booker (Eds.), *Contexts in mathematics education* (pp.147-151). Brisbane: Mathematics Education Research Group of Australasia.

Collis, K. F. & Romberg, T.A. (1991). Assessment of mathematical performance: An analysis of open-ended test items. In M.C. Wittrock (Ed.), *Cognition and instruction* (pp 82-130). Hillsdale, NJ: Erlbaum.

Davis, R.B. (1984). *Learning Mathematics: The cognitive science approach to mathematics education.* London: Routledge.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children.* Dordrecht: D. Reidel.

Fischbein, E. & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? An exploratory research study. *Educational Studies in Mathematics, 15*, 1-24.

Gal, I., Rothschild, K. & Wagner, D.A. '1990, April). *Statistical concepts and statistical reasoning in school children: Convergence or divergence?* Paper presented at the meeting of the American Educational Research Association, Boston.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination and reason.* Chicago: University Press.

Tversky, A., & Kahneman, D. (1971). Belief in the law of small numbers. *Psychological Bulletin, 76*(2), 105-110.

Watson, J.M. & Collis, K.F. (1993). Initial considerations concerning the understanding of probabilistic and statistical concepts in Australian students. In B. Atweh, C. Kanes, M. Carss & G. Booker (Eds.) *Contexts in Mathematics Education* (pp. 575-580). Brisbane: Mathematics Education Research Group of Australasia.

Watson, J.M. (1992). What research is needed in probability and statistics education in Australia in the 1990s? In B. Southwell, B. Perry, & K. Owens (Eds.), *Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia* (pp 556-567). Kingswood, N.S.W.: MERGA.

Watson, J.M., Campbell, K.J. & Collis, K.F. (1993). Multimodal functioning in understanding fractions. *Journal of Mathematical Behaviour, 12*, 45-62.

Watson, J.M., Collis, K.F., & Moritz, J.B. (1993, Sept.) *Assessment of statistical understanding in Australian schools.* Paper presented at the Statistics '93 conference, Wollongong, NSW.

WORKING WITH TEACHERS TO ADVANCE THE ARITHMETICAL KNOWLEDGE OF LOW-ATTAINING 6- AND 7-YEAR-OLDS: FIRST YEAR RESULTS

Robert J. Wright
Southern Cross University
Lismore NSW Australia

In each of six schools, four low attaining first graders (6- or 7-year-olds) were individually taught in an early arithmetic program consisting of four 30 minute lessons per week, during an eight-week teaching cycle. The progress of the 24 participants and that of 13 counterparts of similar ability levels is documented via pre- and post-assessments. In each school, the four participants were taught by a teacher undertaking an on-going, prototype, professional development program aimed at preparing specialist teachers in intervention in young children's arithmetical learning. The work of Steffe and colleagues in early arit!.metical learning provided the theoretical basis for the intervention program. Children's progress is documented using four models, each of a different aspect of early arithmetical ' nowledge. The progress of the participants notably exceeded that of their counterparts providing positive indication of the viability of the intervention program.

This paper focuses on results from the first year of a four-year (1992-5) research and development project which, in broad terms aims to: (a) establish an accredited, year-long, professional development course for teachers; (b) which has as its focus a program of early intervention in arithmetical learning; (c) through which selected first-grade children (6- or 7-year-olds) are placed in a long-term, individualised teaching program; (d) the purpose of which is to advance their arithmetical knowledge to a level at which they are likely to learn successfully in a regular class. In the first year of the project a teacher in each of six schools taught four children individually for 30 minutes daily, four days per week, for an eight-week teaching cycle. The specific focus of this paper is the progress of the 24 participants and its comparison with that of 13 counterparts as documented by pre- and post-assessments.

The professional development program. Both prior to and during the eight-week teaching cycle the six teachers undertook a 20-week, prototype, professional development program. In this program the teachers were released from their regular work for the morning half of each day for the duration of the 20-week period. The program consisted of: a six-week orientation; four weeks involving assessing children and planning teaching programs for four selected children; the eight-week teaching cycle which involved teaching the selected children individually four mornings per week and a professional development meeting on the fifth morning, and a final two weeks of post-assessment. The program commenced in mid-August 1992, and occurred during the last two terms

The research reported here has been funded by Grant No. AM9180064 from the Australian Research Council and by contributions in kind from government and Catholic school systems of the North Coast Region of NSW. Additional support has been provided by research grants from Southern Cross University. I acknowledge the work of Margaret Cowper, Rosalie Dyson and Garry Stanger who made significant contributions to the research project described in this paper. Additionally, I am particularly grateful to the children, teachers and schools involved.

of the 1992 school year. (The school year runs from early February until mid-December and has four terms in all.) Further details of this program are available elsewhere (e.g. Wright, 1993a).

Theoretical framework. Four models, each of a particular aspect of early arithmetical development, underlie the current study: (a) a five-stage model (Table 1) of early arithmetical meanings and strategies adapted from work by Steffe and colleagues (Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988; Steffe, 1992; Wright, 1989; 1991; 1992; in press); (b) a five-level model of facility with forward number word sequences (FNWSs) in the range one to one hundred (Table 2); (c) a five-level model of facility with backward number word sequences (BNWSs) in the range one to one hundred (Table 3); and (d) a four-level model of ability to identify numerals (Table 4). The models of number word sequence development draw on work by Fuson and colleagues (e.g. Fuson, Richards & Briars, 1982). The terms 'stage' and 'level' are used in the sense explicated by von Glasersfeld and Kelley (1982). These four models are applied in the instructional program as well as in the assessment (e.g. Wright, 1993b) of children's early arithmetical knowledge.

Methodology. The methodology used in this report has its basis in the theories of early arithmetical development of Steffe and colleagues (e.g. 1988) and has been used extensively by the current author to document development in young children's arithmetical knowledge (e.g. Wright, 1989; 1991; 1992; 1993b; 1993c; in press). The method allows for: documenting in detail, development over time of an individual's arithmetical knowledge; qualitative comparisons among individuals' developing arithmetical knowledge; and comparisons both within an individual and among individuals of developments of different aspects of arithmetical knowledge such as meanings and strategies, facility with number word sequences and ability to identify numerals.

The Study

The children. Across the six schools 94 first-graders (minimum 14 and maximum 17) were selected by the first grade teacher(s) for initial assessment (i.e. the pre-interview). In each school the first grade teacher(s) was/were asked to include if possible, at least 10 children they regarded as being amongst the least advanced of the grade cohort, in terms of their arithmetical knowledge. In each school four of the least advanced children were chosen on the basis of the initial assessment to participate in the eight-week teaching cycle. Ninety-two of the 94 children who were initially assessed were reassessed in the week immediately following the eight-week teaching cycle. This report focuses on the progress of the 24 participants and 13 counterparts during the teaching cycle. The group of 13 counterparts comprised all children who were initially assessed, did not participate in the teaching cycle, and did not attain at least Stage 3 of arithmetical meanings and strategies (Table 1) in their initial assessment, that is they were prenumerical (Steffe et al., 1983, p. 73). All but two of the participants were prenumerical on their initial assessment and thus the 13 counterparts are regarded as being of ability similar to that of the participants at the time of the initial interview.

Table 1. — Stages of Early Arithmetical Meanings and Strategies

Stage 0: Preperceptual. When attempting to count is unable to coordinate number words with items.

Stage 1: Perceptual. Can count visible collections.

Stage 2: Figurative. Can solve additive tasks involving screened collections but counts from one when doing so

Stage 3: Initial Number Sequence — Sequential Integrations. Counts-on to solve additive and missing addend tasks involving screened collections.

Stage 4: Implicitly-Nested Number Sequence — Progressive Integrations. Uses counting-down-to to solve subtractive tasks and can choose the more appropriate of counting-down-to and counting-down-from.

Stage 5: Explicitly-Nested Number Sequence — Part /whole Operations. Uses a range of strategies which include procedures other than counting-by-ones such as compensation, using addition to work out subtraction, and using known facts such as doubles and sums which equal ten.

Table 2.— Construction of FNWSs in the Range One to One Hundred

Level 0: Absence of FNWSs. Cannot produce the FNWS from "one" to "twenty", allowing errors such as omissions.

Level 1: FNWS from "one" as an unbreakable string. If errors such as omissions are ignored, the child can produce a number word sequence from "one" to around "twenty". The child cannot produce the number word just after a given number word. Dropping back to "one" does not appear at this level.

Level 2: FNWS from "one" to "twenty" as a breakable string. The child can produce the number word just after a given number word but drops back to "one" when doing so.

Level 3: FNWSs as chains in the range "one" to "ten". Produces the number word just after a given number word in the range one to ten without dropping back, but will typically drop back for number words after "ten".

Level 4: FNWSs as chains in the range "one" to "thirty". Produces the number word immediately following given number words in the range one to thirty without dropping back.

Level 5: FNWSs as chains in the range "one" to "one hundred". Produces the number word immediately following given number words in the range one to one hundred without dropping back.

Table 3.— Construction of BNWSs in the Range One to One Hundred

Level 0: Absence of BNWSs. Cannot produce the BNWS from "ten" to "one".
Level 1: BNWS from "ten" to "one" as an unbreakable string. Can produce the BNWS from "ten" to "one" but cannot produce BNWSs from number words less than "ten". The child cannot produce the number word immediately before a given number word and the "dropping back to 'one' " strategy is not available to the child.
Level 2: BNWS from "ten" to "one" as a breakable string. The child can produce the number word immediately before a given number word up to "ten" but will typically drop back to "one" when so doing.
Level 3: BNWSs as chains in the range "one" to "ten". Produces the number word immediately before a given number word in the range "one" to "ten", without dropping back, but will typically drop back to "one" for number words after "ten".
Level 4: BNWSs as chains in the range "one" to "thirty". Produces the number word immediately before given number words in the range "one" to "thirty" without dropping back.
Level 5: BNWSs as chains in the range "one" to "one hundred". Produces the number word immediately before given number words in the range one to one hundred without dropping back.

Table 4. — Numeral Identification

Level 0	Cannot identify some or all of the numerals in the range 1-10.
Level 1	Can identify numerals in the range 1-10 only.
Level 2	Can identify numerals in the range 1-20 only.
Level 3	Can identify one and two digit numerals only.
Level 4	Can identify one, two and three digit numerals.

Pre- and post-assessments. The arithmetical knowledge of each of 24 participants and 13 counterparts was assessed via a pre-interview held in September 1992 and a post-interview held in December 1992. In each of the six schools the assessment interviews were conducted by the teacher who was undertaking the professional development program and was responsible for teaching the children in the individualised teaching program. The interview tasks included simple additive and subtractive tasks involving screened and unscreened collections, saying FNWSs and BNWSs, stating the number word before or after a given number word, and identifying one-, two-, and three-digit numerals. Each interview was videotaped for subsequent analysis. This analysis included determining a level or stage according to each of the models of early arithmetical development referred to earlier. Detailed descriptions of this assessment procedure are provided elsewhere (e.g. Wright; 1993b; in press).

The teaching sessions. The prototype professional development program undertaken by the teachers included: developing a bank of instructional activities; understanding the purposes of the activities; and learning ways to present the activities in individualised teaching sessions. Each teacher used the bank of instructional activities to develop for each of the four children she taught, an individualised teaching framework, that is, a weekly plan containing around six sets of instructional activities which is reviewed and updated daily. The initial development of teaching frameworks occurred under the guidance of members of the research team as did weekly redevelopments of the teaching frameworks. The latter was informed by the teachers' on-going reviews and evaluations of their teaching sessions. All of the teaching sessions were videotaped, and these videotaped records together with written notes made during and after teaching sessions were used by the teachers for the on-going redevelopment of the teaching frameworks.

The instructional activities. The instructional activities were adapted from those used in constructivist teaching experiments (e.g. Steffe et al., 1983; 1988; Cobb, Yackel & Wood, 1992; Wright, 1989; 1991; 1993c) and included: (a) additive and subtractive tasks involving collections or rows of counters some or all of which are hidden; (b) activities involving copying or counting sequences of sounds or movements; (c) activities using numeral cards arranged either singly or in numeral tracks or grids; (d) ascribing number to spatial patterns which are briefly displayed; (e) activities using collections of tens and ones; (f) number word sequence activities such as reciting sequences and stating number words before or after a given number word.

Results and discussion. Table 5 shows for each of the six schools, the stages or levels on each of the four models for pre- and post-assessments of the four participants and the counterparts (zero to three depending on the school). The following observations concerning the progress in terms of the model of stages of early arithmetical meanings and strategies (see Table 1) of the 24 participants and their prenumerical counterparts can be made: (a) Twenty-three of the 24 participants advanced at least one stage, 10 advanced at least two stages and two advanced three stages. (b) Nine of the 24 participants advanced to at least Stage 4. (c) Eight of the 13 counterparts advanced at least one stage but none advanced two stages or more. (d) In each school the advancements of the participants notably exceeded those of the counterparts (excepting School C which had no prenumerical counterparts). (e) Five of the thirteen counterparts and one of the 24 participants did not advance to a numerical stage (i.e. beyond Stage 2). Also evident in the results is that the participants made notable progress in terms of the models of FNWS, BNWS and numeral identification and as before, their progress was significantly in advance of that of the 13 prenumerical counterparts. When one considers that the teachers in this study were undergoing a prototypical professional development program it seems reasonable to interpret the results described in this report as a positive indication of the viability of the intervention program. In period from August to December 1993 a revised version of the program was implemented in 10 new schools. This will continue in the new school year which commences in February 1994.

Table 5

Levels in Pre- and Post- Assessments for Participants and Counterparts in Six Schools

Gender M - Male F- Female	Arithmetic Strategies 0-5		Forward NWS 0-5		Backward NWS 0-5		Numeral Identification 0-4	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
School A — Participants								
M	2	3	3	3	0	1	0	0
F	2	3	3	4	0	3	0	*
F	2	3	3	5	1	4-5	2	*
F	2	4-5	3	4	3	4	1	3
School A — Counterparts								
M	2	2-3	4	4	3	3	2	2-3
F	2	3-4	4	4-5	4	3-4	3	3
F	2	3	4-5	5	2	4-5	3	3-4
School B — Participants								
M	1	3-4	3-4	5	3	5	1	3
F	2	4	3	5	2-3	4-5	3	4
M	2-3	3-4	3	4	3	5	2	3
F	2-3	3-4	4	5	3-4	5	3	4
School B — Counterpart								
M	2	3-4	4	4	3	3	3	3
School C — Participants								
M	2	4	3	5	3	5	2-3	3
F	2	3-4	4	4-5	3	4-5	3	3
M	2	3-4	4	4	3	4	3-4	3
F	2	3	4	5	3	3-4	3	3-4
School D — Participants								
F	2	4 5	3	4-5	2	4	0-1	2-3
M	2	5	3	4	3	5	3	4
F	2	5	3	5	3	5	3	3
M	2	3-4	3	5	3	5	3	4

Table 5 (cont.)

Levels in Pre- and Post- Assessments for Participants and Counterparts in Six Schools

Gender M - Male F- Female	Arithmetic Strategies 0-5		Forward NWS 0-5		Backward NWS 0-5		Numeral Identification 0-4	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
School D — Counterparts								
M	2	3	3	4-5	2	3	3-4	1-2
M	2	3-4	4	5	3	4		3
F	2	3	5	5	5	5	3	3
School E — Participants								
F	2	3	3	4	0	3-4	0	3
M	2	2-3	4	4-5	0	4	3	3
F	2	4	4-5	*	5	*	3	3
M	2	3	5	4	3	5	3	3
School E — Counterparts								
M	2	2	5	4	3	3	3-4	1
M	2	3	5	5	5	5	3	3
M	2	3-4	5	5	5	5	4	4
School F — Participants								
F	2	4-5	3	5	2	5	2	3
F	2	3	3	5	3	5	3	3
F	2	4-5	4	4-5	3	5	1	3
F	2	3	4	4	3	3	2	3
School F — Counterparts								
M	2	2	4	4-5	2	3	2	2
F	2	2	4	4	4	3	3	2-3
M	2	2-3	4-5	5	5	5	3	3

Note: Table entries in the form of a range, e.g. 2-3, rather than a single level, indicate that the precise level could not be determined.

* This datum was not available.

References

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics, 23(1), 99-122.*

Fuson, K., Richards, J., & Briars, D. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Progress in cognitive development: Vol. 1. Children's logical and mathematical cognition, (pp. 33-92).* New York: Springer-Verlag.

Steffe, L. P. (1992). Learning stages in the construction of the number sequence In J. Bideaud, C. Meljac & J. Fischer (Eds.), *Pathways to number: Children's developing numerical abilities.* Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P., & Cobb, P. (with E. von Glasersfeld). (1988). *Construction of arithmetic meanings and strategies.* New York: Springer-Verlag.

Steffe, L. P., von Glasersfeld, E., Richards, J. & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger.

von Glasersfeld, E. & Kelley, M. (1982). On the concepts of period, phase, stage, and level. *Human Development, 25, 152-160.*

Wright, R. J. (1989). *Numerical development in the kindergarten year: A teaching experiment.* Doctoral Dissertation, University of Georgia.

Wright, R. J. (1991). An application of the epistemology of radical constructivism to the study of learning. *The Australian Educational Researcher, 18(1), 75-95.*

Wright, R. J. (1992). Number topics in early childhood mathematics curricula: historical background, dilemmas, and possible solutions. *The Australian Journal of Education, 36(2), 125-142.*

Wright, R. J. (1993a). *An overview of a project organisationally similar to Reading Recovery focusing on intervention in the mathematics learning of at-risk six-year-olds.* Paper presented at the annual conference of the New Zealand Association for Research in Education, Hamilton, 2-5 Dec.

Wright, R. J. (1993b). *Interview-based assessment of young children's arithmetical knowledge.* Paper presented at the Second Conference on Assessment in the Mathematical Sciences organised by the Australian Council of Educational Research, Gold Coast, 15-18 April [To appear as as monograph chapter]

Wright, R. J. (1993c). *Problem-centred mathematics in the first year of school: Results from a longitudinal, classroom-based project.* Paper presented at the annual conference of the New Zealand Association for Research in Education, Hamilton, 2-5 Dec.

Wright, R. J. (in press). A study of numerical development in the first year of school. *Educational Studies in Mathematics.*

SCHOOL CULTURES AND MATHEMATICS EDUCATION REFORM

Erna Yackel

Purdue University Calumet

This paper investigates the differences in the processes by which elementary school teachers at two different research sites revised their instructional practices and established an inquiry approach to mathematics in their classrooms. These differences are accounted for by relating the teachers' beliefs and values to both the microculture of the classroom and the more encompassing culture of the local school community. The paper concludes with a discussion of what appear to be critical issues for reform in mathematics education, based on the experiences at the two research sites.

For the past eight years my colleagues and I have been involved in a research and development project at the elementary school level that has pragmatic goals that are highly compatible with the current reform movement in mathematics education. These include that students be actively involved in sense-making and that they develop increasingly powerful mathematical concepts as they engage in mathematical explanation, justification and argumentation. For many students and teachers engaging in mathematics teaching and learning that has these goals requires a reconceptualization of their views about what constitutes mathematical activity, about what it means to learn (and teach) mathematics, and about the nature of mathematics itself (Shifter & Fosnot, 1993; Wood, Cobb, & Yackel, 1991).

Initially, we developed an approach to teacher development by reflecting on our experiences of interacting with teachers and their students at one research site. This approach appeared to be generally successful in helping the teachers at that site realize what we have called an inquiry form of practice in their classrooms (Cobb, Wood, Yackel, & McNeal, 1992). However, the difficulties we initially encountered when working with teachers at the second research site indicated that some of our suppositions and assumptions about teacher development needed to be revised. In particular, it became apparent that several implicit, taken-for-granted assumptions that we and the teachers at the first site made about learning, teaching, schooling, and mathematics in fact reflected culturally-situated beliefs and values. This in turn indicates that some of the seemingly self-evident assumptions that underpin current reform

---

The research reported in this paper was supported by the National Science Foundation under grant numbers DMS-9057141and MDR 855-0560 and by the James S. McDonnell Foundation. All opinions expressed are solely those of the author.

Several notions central to this paper were elaborated in the course of discussions with Paul Cobb at Vanderbilt University.

recommendations might be in direct conflict with the beliefs and values of some pedagogical communities. Clearly, to the extent that this is the case, it poses a major challenge to all interested in reform in mathematics education.

Establishing a Basis for Communication with Teachers

Elsewhere, we have argued that a first priority when working with teachers is to help them become aware of and make problematic aspects of their current practice (Cobb, Wood, & Yackel, 1990). Only then would they have reason to attempt to reform their instructional practices while working with us. The approach that we developed initially was influenced by our experiences of interacting with the classroom teacher prior to and during the first teaching experiment and uses children's understanding of place value and their use of the standard two-digit addition algorithm as an initial setting for discussion. Previously, we have documented the success of this approach when working with second grade teachers at the first site:

> ... [the teachers] began to differentiate between correct adherence to accepted procedures and mathematical activity that expressed conceptual understanding.
>
> ... We did not have to convince them [the teachers] that children should learn with understanding. Rather, they had assumed that this kind of learning was occurring in their classrooms. *A shared desire to facilitate meaningful learning and a general concern for children's intellectual and social welfare constituted the foundation upon which we and the teachers began to mutually construct a consensual domain* [emphasis added]. (Cobb, Wood, & Yackel, 1990)

In retrospect, we can say that not only was "a shared desire to facilitate meaningful learning and general concern for children's intellectual and social welfare" the basis upon which we could construct a consensual domain, but that we and the teachers had a taken-as-shared understanding of what constitutes children's intellectual and social welfare. This taken-as-shared understanding was central to the basis for communication that we established with the teachers. We now realize that our notion of what constitutes children's intellectual and social welfare reflects assumptions about *what it means to be a child in school.* Both we and the teachers assumed that our taken-as-shared view was self-evident. The very language used in the above passage indicates that we had not relativized the notion for ourselves. In Walkerdine's (1988) terms, it was part of our own and the teachers' regime of truth.

In her critique of the child-centered British primary schools, Walkerdine argues that central concepts such as *what a child is* and how children become *normal* are social constructions. This perspective is extremely useful in attempting to explicate some of our implicitly held assumptions.

> 'The child' as a sign within the child-centered pedagogy is not simply a description of a pre-existing child. The practices themselves, in their regulation, produce what it *means* to be a child: what behaviours, words, etc. are used and those are regulated by means of an apparatus of classification, and a grading of responses. 'The child' becomes a creation ... (Walkerdine, 1988, P. 203,204)

Following this argument, children come to be seen in light of the practices that define them. Thus, "Mathematics becomes cognitive development. Cognitive development becomes a description of the child. This exists as a regime of truth, a system of classification in which what counts as a properly developing child may be recognized, and in which certain behaviours are required and produced" (p. 205).

In this view, *what it means to be a child in school* is not a pre-existing notion, independent of history and culture. Rather, it is continually regenerated by the members of a pedagogical community as they participate in the practice of schooling. This notion became important for us when we attempted to make sense of our interactions with teachers at the second site.

The Second Site as a Contrasting Case

The research at the second site began three years later. The school system at the second research site placed a great deal of emphasis on computational proficiency but this was not viewed as an insurmountable difficulty given that we had worked with teachers who employed a drill-based approach at the first site. During the induction sessions at the first site, the teachers realized the incompatibility of an emphasis on drill and practice with their desire to foster their notion of children's intellectual and social welfare and understood the need to develop alternative instructional approaches. Unlike the first classroom teacher at the first site who viewed herself as weak mathematically and who was dissatisfied with her textbook-based instruction, the teacher at the second site, Mr. K., viewed himself as strong mathematically. He was aware that his principal and his colleagues thought of him as an exceptional mathematics teacher. In prior years he deviated from his textbook to include much more drill and practice than the textbook called for and had been recognized by the local newspaper the previous year because of this drill-based approach to mathematics instruction. Prior to the school year I met with him for three days to induct him into the project. It soon became apparent that attempts to help him view his practices as problematic were ineffective. In the following paragraphs I attempt to account for this ineffectiveness.

Like the teachers at the first research site, Mr. K. was interested in students' intellectual and social well being and wanted them to express themselves. However, for him there was no apparent conflict between his mathematics teaching practices and these other goals. This lack of conflict can be explained by returning to Walkerdine's contention that concepts such as *the child* or *what it means to be a child in school* are themselves constituted by the local pedagogical community.

In this school, strictly enforced discipline was highly valued. As the principal said, "This is what the parents want." He did not clarify what he meant by this. However, as the year progressed it became apparent that it meant that the children should follow, without question, rules and regulations set out by adult authorities, including the principal, the teachers, and other school staff. Two features of this conception of discipline are noteworthy. First, from my perspective, a number of the rules appeared to be arbitrary. Second, there was no discussion of the rationale for any of the rules. (There were discussions of whether or not a rule was violated in a given instance but not of whether or not the rule was appropriate.) This emphasis on following rules *sans* rationale both expressed and contributed to the taken-as-shared view of the chid in school as one who follows prescribed instructions without question. Since, as Walkerdine points out, children are evaluated in terms of the taken-as-shared conception of the normal child, children who followed rules and instructions were considered to be acting appropriately. Thus, in this school setting, adults showed their concern for children's intellectual and social welfare by helping them learn and follow rules. Further, to know or make sense of things meant to be able to state the appropriate rule.

Given this view of the child in school, it was natural for Mr. K. to want children to express themselves by saying the right things in the right way. And for him, showing concern for the children's intellectual and social well being meant helping them produce correct answers by following rules and procedures. There was, for him, no conflict between the consequences of traditional mathematics instruction discussed during his induction into the project and his view of the child in school.

Renegotiation of Classroom Social Norms

By considering the ways in which the teachers at the two research sites negotiated classroom social norms it is possible to further clarify the beliefs and values of the two pedagogical communities. It became apparent during the first teaching experiment that both the teacher and the children reorganized their beliefs about what constitutes mathematical activity, their roles in the classroom, and about the nature of mathematics. This reorganization was made possible by and itself contributed to the constitution of the classroom social norms that characterize inquiry instruction. A retrospective analysis of the video-recordings made during this experiment indicate that the teacher used several types of interventions to initiate and guide the negotiation of social norms (Cobb, Yackel, & Wood, 1989; Yackel, Cobb, & Wood, 1991). The critical issue for this discussion is that, in all cases, the teacher discussed her rationale for particular expectations. There is every indication that she took it as self-evident that the negotiation of classroom social norms necessarily involves a discussion of the underlying rationale. For example on the first day of class a student gave an incorrect answer as the entire class discussed a problem. The teacher explicitly told the students that it was acceptable to make a mistake in her class and went on to give a rationale for why it was acceptable, "we learn from our mistakes, a lot."

This emphasis on a rationale was quite typical. As a consequence, the social norms established in this classroom were not mere directives that the children were expected to follow. Instead, they were integral aspects of an encompassing perspective on what it meant to know and do mathematics in school.

In the analysis of the process by which the teacher initiated and guided the mutual construction of the classroom social norms for inquiry mathematics, we were aware that the teacher had taken the initiative and had made interventions that reflected pedagogical beliefs that she had constructed prior to her participation in the project. Thus, we had influenced the particular social norms the teacher attempted to establish in her classroom, but not the process by which she did so.

The claims made earlier concerning the pedagogical practices institutionalized at the second research site are further substantiated by the manner in which Mr. K. typically initiated the negotiation of social norms. For example, he, like other teachers at both research sites, had rules to govern behavior in the classroom, which were posted so the children could see them. Several of these rules conflicted with expectations that characterize inquiry mathematics. In our meetings and interactions with Mr. K. we emphasized the importance of establishing the expectations that characterize inquiry mathematics. He agreed with us completely. However, it soon became apparent that he assumed he could do so within the framework of his usual practice. The absence of any conflicts between these rules and attempts to develop an inquiry form of practice is quite reasonable given the taken-as-shared beliefs and values of his pedagogical community. His role as a teacher involved specifying rules and ensuring that children followed them. From his point of view, the proposed changes in his instructional practices involved adding rules *sans* rationale to his list. And, in the absence of a rationale, the possibility that the rules might be in conflict does not arise.

In light of these arguments, it would be legitimate to question whether or not it is possible for teachers whose beliefs and values are similar to Mr. K.'s to develop an inquiry form of practice. My experience of working with him indicate that the answer is a resounding, "Yes." By the end of the year, he had radically revised his instructional approach (Yackel, Cobb, & Wood, in press). His students actively engaged in mathematical argumentation and became particularly adept at questioning and challenging each other's explanations (Yackel, 1992; Yackel & Cobb, 1993).

Summary

I have argued that "the desire to facilitate meaningful learning and a general concern for children's intellectual and social welfare" were critical in making it possible for the teachers at the first research site to make their traditional instructional practices problematic. We have also seen that what it meant to facilitate learning and to be concerned about children's intellectual and social welfare differed significantly between the two sites. It was for this reason that Mr. K, as a representative of his pedagogical community, did not come to see his drill-based instructional approach as problematic during the induction into the project. Further, the changes that Mr. K. and other teachers at the second research site had to make to develop an inquiry instructional approach were far more radical than those required of the teachers at the first site. The latter group of teachers were able to develop forms of practice compatible with current reform recommendations by guiding the establishment of alternative social norms in their classrooms. In contrast, Mr. K. and his colleagues had to transcend their pedagogical community's taken-as-shared view of the child in school and change the very process by which they initiated the renegotiation of social norms. Even with hindsight, I find Mr. K.'s efforts to reform his practice quite remarkable and, in large measure, attribute this to his relatively deep understanding of mathematics and to his mathematical values.

We have seen that, at the first research site, the compatibility between our own and the teachers' conceptions of children's intellectual and social welfare enabled us to develop a basis for communication. In contrast, it was the compatibility between our own and Mr. K.'s mathematical values that eventually made it possible for us to communicate effectively. In addition, it was his desire to further encourage mathematical activity that involved the creation and conceptual manipulation of mathematical objects that led him to see his traditional instructional practices as problematic. It was as he and the students renegotiated what counted as a mathematically insightful solution and an appropriate mathematical explanation that the social norms characteristic of an inquiry approach began to emerge. Thus, the combination of Mr. K.'s mathematical expertise and values, and the students' spontaneous development of solutions that he judged as insightful drove their interactive constitution of an inquiry mathematics tradition.

Implications for Reform in Mathematics Education

In the course of the analysis I have argued that teacher's and researchers' activities both reflect underlying understandings and conceptions about schooling in general and the child in school in particular. It is with this in mind that I join Apple (1992) in calling for mathematics educators to consider the ideological and social groundings and effects of their reform recommendations. The issue I raise is: what are the implicit assumptions underlying the reform, as represented by documents such as the Standards (National Council of Teachers of Mathematics, 1989). In responding to Apple, Romberg (1992) states that the NCTM documents were based on the belief that, "The notion that mathematics is a set of rules and formalisms invented by experts, which everyone else is to memorize and use to obtain unique, correct answers, must be changed" (p. 433). The analysis in this paper shows that developing a practice compatible with this belief requires greater change for members of some pedagogical communities than for others. In some cases, there is a general compatibility between this belief and a pedagogical community's taken-as-shared views of what it means to be a child in school. That compatibility can be used as the basis for initiating change. In other cases, there is a basic and deep-rooted incompatibility.

Given that the first research site was a rural/suburban white middle class community and the second site was an inner city minority community, this raises the very real possibility that reform efforts in which researchers assume that their culturally-situated communities are ontological certitudes might well result in even greater disparities in the type of mathematics education children experience than is now the case. Such a consequence would be tragic given the explicit acknowledgement of the issue of diversity in documents such as the Standards. In my view, Apple is right when he calls for us to clarify the ideological, social, and political dimensions of our efforts to initiate reform in mathematics education. Only then can we guard against the possibility that we will unknowingly foster even greater inequities.

References

Apple, M. W. (1992). Do the Standards go far enough? Power, policy, and practice in mathematics education. Journal for Research in Mathematics Education, 23(5), 412-431.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist views on the teaching and learning of mathematics (pp. 125-146). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. American Educational Research Journal, 29, 573-602.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), Affect and mathematical problem solving: A new perspective (pp. 117-148). New York: Springer-Verlag.

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: NCTM.

Romberg, T. A. (1992). Further thoughts on the Standards: A reaction to Apple. Journal for Research in Mathematics Education, 23(5), 432-437.

Shifter, D. & Fosnot, C.T. (1993). Reconstructing mathematics education: Stories of teachers meeting the challenge of reform. New York: Teachers College Press.

Walkerdine, V. (1988). The mastery of reason: Cognitive development and the production of rationality. London: Routledge.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. American Educational Research Journal, 28 (3), 587-616.

Yackel, E. (1992, August). The evolution of second grade children's understanding of what constitutes an explanation in a mathematics class. Paper presented at the Seventh International Congress of Mathematical Education, Quebec City, Canada.

Yackel, E., & Cobb, P. (1993, April). Sociomathematical norms, argumentation and autonomy in mathematics. Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, GA.

Yackel, E., Cobb, P., & Wood, T. (1991). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education, 22(5), 390-408.

Yackel, E., Cobb, P., & Wood, T. (in press). The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. Journal of Mathematical Behavior.

SYMBOLIC AWARENESS OF ALGEBRA BEGINNERS

Michal Yerushalmy — University of Haifa
Beba Shterenberg — Center of Educational Technology

Algebraic Patterns is one component of a new algebra curriculum built to support technological intensive guided inquiry. Together with activities designed for it, Algebraic Patterns form a microworld to motivate the use of algebraic statements and argumentation to describe and explain numerical phenomena. It presents the user with numerical lattices, 'local' and 'global' algebraic tools and feedback about algebraic statements made by the user. In this paper, we describe the first stages of students' development of algebraic language, of symbolic fluency, students' preferences for single or multi variable description, and the first perceptions of symbolic equivalence.

Patterns have been assumed to be an appropriate springboard to the use of generalizations and algebra. The Standards (1989) and others (Kieran 1993) suggest that the inductive study of patterns should underpin the beginning of the algebra learning. However, a few recent studies suggest that the link between patterns and algebra is not trivial. Lee and Wheeler (1987) report that students can formulate excellent verbal generalizations without needing to use the algebraic language. MacGregor & Stacey (1993) report on difficulties of 14-15 years old students to formulate functional relationships between variables algebraically while being able to easily describe a computation procedure. Some evidence suggests that technology can motivate the involvement in describing patterns. Software that presents examples of partial sequences (such as King's Rule O'Brein 1985) motivates the creation of other examples which comply with the hidden rule. Spreadsheet can be used to create a large collection of numerical data by talking symbolically to the software. The first type of software emphasizes the creation of more examples, but does not require nor provide tools for, algebraic conversation. The second type focuses on 'pattern making' rather than 'pattern describing'.

Through our new development of a computer intensive secondary school curriculum in Israel which is based on guided inquiry learning (Yerushalmy Chazan & Gordon 1988) and on the function as its central concept, we developed a microworld to support the evolution of symbolic awareness by manipulating and describing patterns algebraically. While such awareness can be defined in many ways let us identify our major goals of the first stages of the development of symbolic language as:

-Identifying numerical phenomena and describing them with symbolic statement. (by statement we mean a function, a functional relationship, relation, or any other symbolic argumentation or description)

-Getting a sense of when a description by symbols is profitable.

-Being fluent and flexible in choosing alternative symbols.

-Revising statements according to changes in the phenomena

-Identifying equivalent meaning in non-equivalent symbolic description.

-Being aware of, and able to evaluate, the power (generality) of each symbolic statement.

A microworld that can support the creation of such symbols' awareness at a naive stage should include a large variety of numerical phenomena, tools to identify patterns and an easy way to express ideas in symbols. That was our motivation in designing Algebraic Patterns (Yerushalmy & Shterenberg, 1992).

DESCRIPTION OF THE MICROWORLD

The microworld, called Algebraic Patterns attempts to address the following needs:

-To provide a motivated arena for generalizations of numbers' phenomena

-To support generalization by providing tools of various levels.

-To allow dynamic changes of patterns to provoke higher levels of generality

-To support algebraic language as a language to conjecture about numbers

-To support algebraic language as a language of argumentations and proofs in algebra.

The microworld is built of three layers: The middle layer is the first layer the user sees. It consists of a dynamic numbers' lattice which is a finite collection of either a folded sequence or a matrix. The sequence can be folded in many ways, each affects the patterns found between the rows. The lattice can be arranged in a variety of geometric shapes and dimensions which affect the internal patterns in the lattice. The upper layer consists of stamps which can be put down on the lattice. The stamps act as 'lens' to help identify patterns. Local relations between the cells in the stamp (of size 2 to 9 cells in various geometrical shapes) can be described either by one variable, by many variables (according to the number of cells), by functional relationships between the variables or by any mix of the above methods. The bottom layer (usually invisible layer) is the index layer. It identifies a location, either in a sequence or in a matrix and can provide global language to describe the lattice: i.g. f(n)=n^2-5 or f(n)-f(n-1)=5 or f(i,j)=5-j.

A set of organizational tools is available to the user:

The organizational tools help students collect numerical examples, make algebraic statements, and digest the connection between numeric and symbolic phenomena. The tools are: a window of examples (randomly chosen or intentionally stamped by the user) that can be screened for generalizations, a table of values (in a single or two variables) of the stamped cells' which can be screened for global generalizations, an option to locate each example by its indices in the index layer and a window which assembles the list of statements written by the learner. In figure 1 we present the lattice, the stamps, symbols and generalization list.

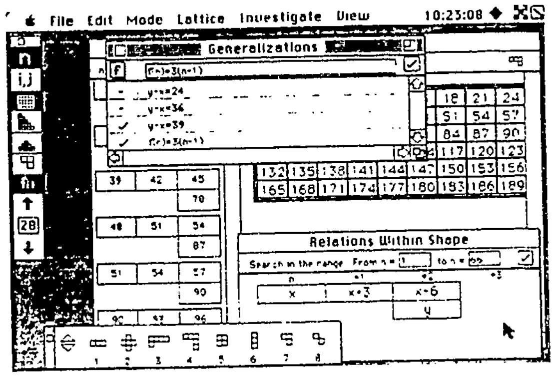


Figure 1: The major windows of Algebraic Patterns

A set of analysis tools serves the learner in this microworld:

Analysis of the correctness of algebraic conjecture: The statements' window presents judgmental feedback on each suggested algebraic conjecture. In addition, a non-judgmental, "mirror" format feedback allows the substitution of numerical examples into each statement.

Analysis of the 'generality power' of the conjecture: The software allows reconstruction of hidden lattices using the user's descriptions (algebraic statements). The 'power' of statement depends on the amount of variables used in a single rule. A function description may need only a single input to describe the whole collection of numbers while a rule connecting three variables will require two independent inputs. Another 'test of generality' can be carried out using geometric manipulations of the lattice. A statement that correctly describe more than a single variation of the lattice can be considered more general.

SOME OBSERVATIONS

During the year of 92-93 we studied groups of learners using the Algebraic Patterns. We report on an experiment conducted with 130 7th graders who had used the microworld through guided inquiry in pairs for about 8 weeks with the specific written activities and whole class discussions. This learning unit replaced the regular algebra introduction to variables and simple manipulations. The experiment included comparison groups (80 students at the same school) who (at the same time) learned the regular algebra I first units. The teachers were part of a team, which we met and worked with for a year, implementing our intensive computerized algebra curriculum. We observed the classes at various times and prepared the discussions with the teachers as well as their final tests. We present preliminary observations.

The need to use symbols to describe generalizations

A first striking impression raised in talking with and studying the students was that even when they were primitive, or even incorrect, symbolic statements turned out to be a popular way to express ideas. This impression was tested by asking students to describe the behavior of a few sequences without requiring a certain method or strategy to carry out this task. None of the groups (experimental and comparison) had directly dealt with such tasks (the lattice had never been presented as a sequence). Only three out of 80 comparison students chose to use symbolic description of the sequence, while 25% of the experimental group chose description by variables. These results fit the assumption that students do not have difficulties generalizing a collection of numbers, but that they will not naturally choose to do so in algebraic symbols. Some of the various uses of symbols are presented in figure 2 below.

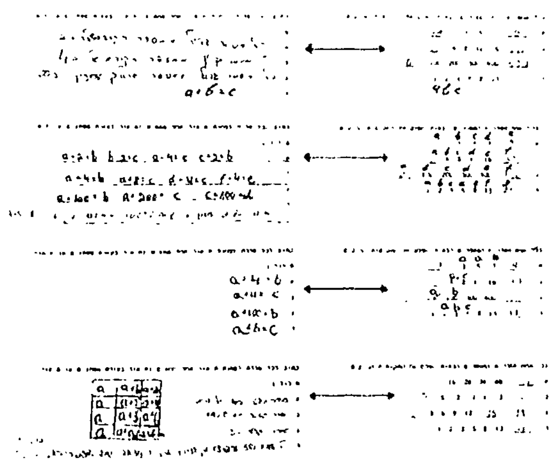


Figure 2: Four different uses of variables to describe given numerical sequences (experimental group's students)

The impact of the 'local tool' on generalizations:

The stamp which acts as lens zoomed into the lattice was introduced to motivate the search for 'interesting neighborhoods' of numbers, to ease the identification of examples and to support the exchange of ideas using variables.

Observations from students' tests suggest that the stamp had these effects.

In the 'sequences problem', mentioned above, the stamp was voluntarily used to identify connections among consecutive elements of the sequence; The most common used stamp was the horizontal stamp (of 2 or 3 cells) which yielded descriptions such as: a, a+2, a+2 (a mistake rooted in an attempt to describe infinite sequence with a local stamp while the functional description f(n+1)-f(n)=2 is more appropriate but not learned at that time yet) or x-10,x,x+10. While the stamp was used as an immediate lens into the pattern, there was no such use at the comparison groups.

In another item, the lattice problem, we asked to identify connections among numbers in a lattice and to describe them using variables. The answers of the groups were very different. While most experimental students formed stamps and described the relations algebraically, only very few of the comparison group used algebra to talk about the connections. One third of the comparison group described the connections verbally (interestingly not even one experimental student did that). The comparison group's students 'invented' stamps in order to ease the descriptions and the generalizations. In figure 3 we present a typical work from each of the groups.

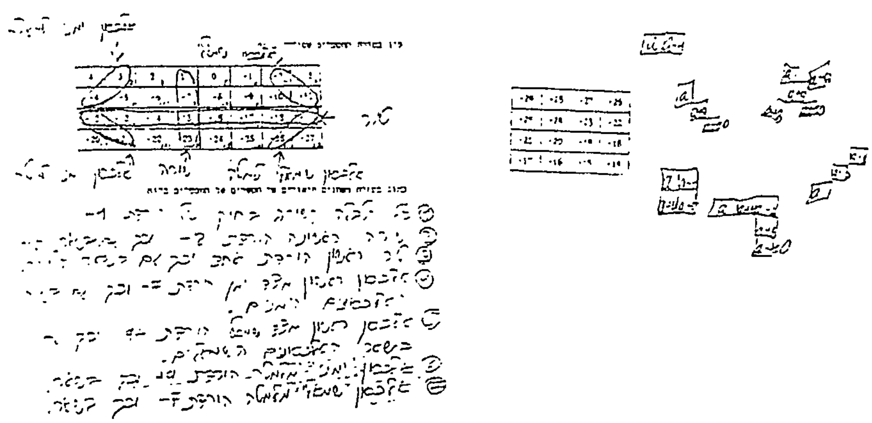


Figure 3: A lattice described by stamps and variables (right) by an experimental student and another lattice, described by words and 'invented' stamps.

The description of functional relationships in one or more variables

The literature suggests that in lower levels of understanding of variables students use letters as a replacement of a single number. It also mentioned however that it often seems to be an easy task to describe a connection between two adjacent terms in a sequence using a single variable (a,a+1) but difficult to describe the rule (or the function) that creates the sequence.

Since we watched students learning with the Algebraic Patterns in pilot studies and realized that they tend to use many variables in each stamp we planned the learning sequence start from their natural use (which we called the algebraic imitation of arithmetic) and gradually led them from the generalizations in many variables towards the reduction of the number of variables down to a single variable.

Teachers reported that the use of many variables was a benefit to students and especially to less able ones. It seems that for them writing conjectures after naming the cells in many variables was easier than expressing the cells directly with functional expressions. We consider this finding an important one and would like to investigate its possible dependency on the learning sequence we used.

Meanwhile we would like to suggest two explanations using data from the tests and the classroom: In the 'sequence problem' we found a noticeable difference between the expressions used to described the first three sequences (arithmetic sequences) and the Fibonacci sequence. In the fourth sequence, the use of variables increased. Students who described the first three sequences in words such as: "Jump 2 to get the next one.." could not find a description appropriate to the Fibonacci sequence and decided to use algebraic language. The use of many variables increased (the need to express that each number depends on the previous two and the lack of algebraic recursive language at that time forced them to use three variables c=b+a.)

In the 'lattice problem', we found a clear differentiation among different experimental classes. In one of them all students described the lattice using a single variable and the teacher reported that she had emphasized the advantages of using a single variable which other teachers did not do.

Teachers' priority are always an important factor of the students' actions however another factor seems to intervene here: the hidden message of the assessment. Quite often we defined the task as: "find as many statements possible..". Writing in a single variable makes it more difficult for beginners to write many statements; for example: when one fills the 'square' stamp with a,b,c,d one can write: d=c+1, d=b+6, b=a+1, (a+d)=(b+c) etc. However, filling it out with a,a+1, a+5, a+6 says a lot about the connections but makes it harder write other statements like: $a+(a+6)=(a+1)+(a+5)$ for those just learning to use symbols. Since at this preliminary stage students were not worrying yet about algebraic proofs, they wanted to create as many statements as possible and therefore did not use the single variable description.

The above examples suggest that at the first stage of developing symbolic awareness the use of multi variables description is useful. We assume that when the emphasize on algebraic argumentation replaces the emphasize on symbolic fluency the types of use of variables will change as well.

The perception of equivalence

Equivalence among statements, presented as a comparison of functions in a single variable, are formally introduced later in our curriculum, following the of symbolic rules. We did not plan to examine the equivalence among different statements in this introductory unit (more than that, the statements had never been named as functions). However, teachers kept telling us that students developed symbolic awareness of equivalence of statements. Here is a script taken from a group discussion in which the teacher suggested that the winner will be the one with maximum different statements to a given stamp and a given lattice:

I have $b = a+2$

I have more: I wrote $b = a+2$

$b+1 = a+2+1$

$b+1 = a+3$

$b+a = a+a+2$

Other students: "That is not fair, he has three same statements"

Why?

"Because all the statements describe the same thing. He is arguing the same thing 4 times!"

There had been no discussion about equations or legitimate operations or any other symbolic argument. This is an indication that the symbols our students used were meaningful to them.

CONCLUDING REMARKS

In the described experiment we concentrated only on the first stages of the shift from arithmetic to algebra. In our newly developed curriculum this shift is wider, deeper and longer. It is wider in the sense of other needs to establish the foundations of algebra, such as the shift from words to graphs, from numbers to graphs, from words to symbols, deeper in the sense of more serious work to do along the way with this specific microworld. (we would like especially to emphasize in later stages the need to carry out proofs and argumentations in symbols and to use global language (the language of functions)), longer in the sense of the impact of such a construction of algebra on symbolic awareness, as described above, at later stages.

References:

Kieran, C. (1993) Function, Graphing, and Technology: Integrating Research on Learning and Instruction. In Romberg, Fennema, Carpenter (Eds.) Integrating Reserach on the Graphical Representation of Functions Erlbaum Associates, NJ. pp. 189-238.

Lee, L. Wheeler, D. (1987) Algebraic Thinking in High-School Students: Their Conceptions of Generalization and Justification, Concordia University, Motreal.

MacGregor, M. Stacey, K. (1993) Seeing a Pattern and Writing a Rule. In the Proceedings of the 17th International Conference, Japan. pp. 181-188.

National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics, Reston, VA.

O'Brein, T. (1985) The King's Rule (software) Sunburst communications, NY.

Yerushalmy, M. Shterenberg, B. & Center for Educational Technology (1992) Algebraic Patterns (Software), Sunburst communications, NY.

Yerushalmy, M. Chazan, D. & Gordon, M. (1991) Guided Inquiry and Technology Education Development Center, Report 90-8.

Changing Attitudes to Mathematics through Problem Solving

<u>Yudariah Binte Mohammad Yusof</u> — Department of Mathematics, Universiti Teknologi Malaysia, Locked Bag No 791, 80990 Johor Bahru, Malaysia

David Tall — Mathematics Education Research Centre, University of Warwick, COVENTRY CV4 7AL, UK

University mathematics may be presented in a formal way that causes many students to cope by memorising what they perceive as a fixed body of knowledge rather than learning to think for themselves. This research studies the effects on students' attitudes of a course which encourages co-operative problem-solving coupled with reflection on the thinking activities involved. A pre- and post-test revealed that attitudes changed significantly during the course. Half the students stated beforehand that university mathematics did not make sense. A majority of these declared negative attitudes to mathematics as abstract facts and procedures to be memorised, reporting anxiety, fear of new problems and lack of confidence. After the course all measures investigated improved, confirming that appropriate problem-solving can alter students' perception of mathematics as an active thinking process

The teaching of university mathematics may be so formal that it teaches "the product of mathematical thought rather than the process of mathematical thinking" (Skemp, 1971). Over the last decade, a ten week, 30 hour problem-solving course has been taught at Warwick University based on the framework of Mason, Burton & Stacey (1982) to encourage students to improve their mathematical thinking. In written responses, students had often commented on their difficulties with formal mathematics:

> Maths education at university level, as it stands, is based like many subjects on the system of lectures. The huge quantities of work covered by each course, in such a short space of time, make it extremely difficult to take it in and understand. The pressure of time seems to take away the essence of mathematics and does not create any true understanding of the subject. From personal experience I know that most courses do not have any lasting impression and are usually forgotten directly after the examination. This is surely not an ideal situation, where a maths student can learn and pass and do well, but not have an understanding of his or her subject. *Third Year Warwick Mathematics Student, 1991*

The aim of the problem-solving course is to provide students with an alternative view of mathematics as a living activity. Each week students solve problems together in a two hour session then reflect on their actions in a one hour seminar. They are never given solutions, but are encouraged to share ideas and reflect on their effectiveness. Skemp's theory (1979) of goals and anti-goals proves valuable in allowing them to focus on the cognitive source of difficulties rather than the emotional symptoms.

The Universiti Teknologi Malaysia has similar problems. There is a wider range of ability (from the 50th to 90th percentile, with the top 10% going abroad for their education). The students are dutiful and eager to please their teachers by working hard and learning the procedures to pass the examinations. As one student commented – "To be good in maths requires good memory and lots of practice".

Based on a pilot study in the University of Warwick, it was hypothesised that the attitudes of UTM students could be fundamentally changed through a problem-solving course of the type outlined above. The first-named author translated the materials to Bahasa Malaysia (the language of instruction in the UTM) and taught a ten week, thirty hour course.

The 24 males and 20 females taking part in the research were a mixture of third, fourth and fifth year undergraduates aged 18 to 21 in SSI (Industrial Science, majoring in Mathematics) and SPK (Computer Education), covering the full honours degree range (see table 2 below). In the first two hour meeting each week, after an introduction by the instructor, the students spent the major part of the time working on the problems in their own in small groups of 3 or 4. After half an hour or so the instructor reviewed the situation, to see how well things are progressing, ensuring that everyone is solving the same problem and considering ideas generated by the students. She gave no clue nor made any attempt to lead the students towards a possible solution. They were encouraged to experience all aspects of mathematical thinking--formulating, modifying, refining, reviewing problems and their solutions, specialising to simple cases, generalising through systematic specialisation, seeking patterns, conjecturing, testing and justifying. During the one hour meeting, the students were encouraged to reflect on their mathematical experience and talk about their attempts to solve problems. The instructor commented on the effectiveness of the solutions, discussing where things may have gone wrong, where students may have failed to take advantage of certain things, and ended by summarising what the students had done.

The students' performance and attitudes were monitored by classroom observation, a questionnaire at the beginning and end of the course, and semi-structured interviews.

Classroom Observation

Initially, the students were very confused. They kept asking questions like "What shall I do now?" "Is this the right way of doing it?" when they became stuck after a frantic attack on the given problem. For a few weeks they showed enormous resistance. Little by little the resistance was worn away until, after four weeks, they were slowly able to make decisions and think for themselves. By this time they began to write a "rubric" commentary outlining their problem-solving activity.

Their knowledge of mathematics is sufficient to solve all the given problems. At first they were set simple problems which helped tremendously in giving them a sense of success to help build self-confidence, not only in those who were unwilling to tackle something new because they had failed in the past, but also those who were successful in regular mathematics courses. The instructor herself has not solved all the problems given in the course. On several occasions she solved a problem in front of the class, showing that even a mathematician does not produce a neat, straightforward textbook proof. This encouraged students to feel less reluctant to make conjectures which might prove to be wrong on the possible route to success. Their discussion became livelier as they moved into doing things that they could explain to their friends, rather than simply satisfying the course requirements or pleasing the instructor. Their problem-solving became " a more creative activity, which includes the formulation of a likely conjecture, a sequence of activities testing, modifying and refining." (Tall, 1991).

Questionnaire

An attitudinal questionnaire on mathematics and problem-solving, based on common responses in a Warwick pilot study, was given to the students in the first week of term and again following the course.

Section A: Attitudes to Mathematics

1. Mathematics is a collection of facts and procedures to be remembered.
2. Mathematics is about solving problems.
3. Mathematics is about inventing new ideas.
4. Mathematics at the University is very abstract.
5. I usually understand a new idea in mathematics quickly.
6. The mathematical topics we study at University make sense to me.
7. I have to work very hard to understand mathematics.
8. I learn my mathematics through memory.
9. I am able to relate mathematical ideas learned.
10. In a few sentences describe your feelings about mathematics.

Section B : Attitudes to Problem-Solving

1. I feel confident in my ability to solve mathematics problems.
2. Solving mathematics problem is a great pleasure for me.
3. I only solve mathematics problems to get through the course.
4. I feel anxious when I am asked to solve mathematics problems.
5. I often fear unexpected mathematics problems.
6. I feel the most important thing in mathematics is to get correct answers.
7. I am willing to try a different approach when my attempt fails.
8. I give up fairly easily when the problem is difficult.

Responses

Under each statement, students responded to a five-point scale: **Y, y, –, n, N** (i.e. definitely yes, yes, no opinion, no and definitely no). Their responses are given in Table 1. Rather than have separate columns for Y and y (which then have to be added to give the total "yes" response), the table has total "Yes" (Y+y) in **bold** print and a column for the subset "Y" who express the stronger definite opinion. The table reveals, for example, that 34 students (77%) regard mathematics as facts and procedures to be remembered, with a subset of 18 (41%) expressing this opinion strongly (a "definite yes").

Mathematics	Yes	Y	No	N	--
facts & procedures	34	18	8	2	2
solving problems	27	10	16	4	1
inventing new ideas	21	4	21	6	2
abstract	25	13	17	0	2
understand quickly	9	0	30	5	5
makes sense	22	4	22	5	0
work hard	37	15	5	1	2
learn by memory	30	1	12	2	2
able to relate ideas	24	8	18	2	2

Solving Problems	Yes	Y	No	N	–
confidence	26	7	17	2	1
pleasure	43	25	1	1	0
only to get through	16	4	27	8	1
anxious	17	1	24	4	3
fear unexpected	30	10	12	3	2
correct answers	21	4	21	3	2
try other approach	42	17	0	0	2
give up	19	3	24	9	1

Table 1 : Responses for 44 students on the Pre-test Questionnaire

The table in this form conceals valuable information about individuals. Responses to the statement that mathematics "makes sense" divides the students into equal groups, Group S consisting of the 22 students for whom it does and Group N, the 22 students for whom it does not. Table 2 shows the degree classification of the groups.

		Degree Class											
		Group N (22 students)						Group S (22 students)					
		I	II-1	II-2	III	P	F	I	II-1	II-2	III	P	F
Students	SPK year 5	2	5	2	1	0	0	0	3	3	0	0	0
	SPK year 4	1	5	3	0	0	0	2	6	4	1	0	0
	SSI year 3	0	3	0	0	0	0	1	2	0	0	0	0
	Total	3	13	5	1	0	0	3	11	7	1	0	0

Table 2 The distribution of students for whom mathematics makes sense (Group S) and does not (Group N)

The two groups have almost identical distributions, so there is no correlation between examination success and whether the students consider mathematics makes sense. Table 4 shows the data for the two groups on the pre- and post-test for Part A. Items underlined in column 1 show a significant (<5%) or highly significant (<1%) change in the total **Yes** response (compared with **No** and –) using the χ^2 test with Yates' correction. Underlining in the **Yes** columns refers to significant changes for each group.

Mathematics	Group N (22 students)										Group S (22 students)									
	Yes		Y		No		N		–		Yes		Y		No		N		–	
	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*
facts & procedures	20	*6*	16	*2*	1	*16*	1	*4*	1	*0*	14	*5*	2	*1*	7	*16*	1	*4*	1	*1*
solving problems	9	*21*	2	*10*	13	*0*	4	*0*	0	*1*	18	*21*	8	*11*	3	*0*	0	*0*	1	*1*
inventing new ideas	6	*18*	1	*8*	14	*2*	4	*0*	2	*2*	15	*19*	3	*7*	7	*3*	2	*0*	0	*0*
abstract	21	*11*	12	*7*	1	*8*	0	*0*	0	*2*	4	*4*	1	*1*	16	*18*	0	*3*	2	*0*
understand quickly	4	*8*	0	*0*	13	*12*	2	*1*	5	*2*	5	*12*	0	*3*	17	*9*	3	*1*	0	*1*
makes sense	0	*14*	0	*2*	22	*7*	5	*0*	0	*1*	22	*21*	4	*3*	0	*0*	0	*0*	0	*1*
work hard	19	*15*	6	*5*	1	*5*	0	*0*	2	*2*	18	*13*	9	*3*	4	*8*	1	*0*	0	*1*
learn by memory	17	*6*	1	*0*	3	*15*	1	*3*	2	*1*	13	*5*	0	*0*	9	*16*	1	*4*	0	*1*
able to relate ideas	6	*14*	1	*4*	14	*7*	2	*0*	2	*1*	18	*21*	7	*7*	0	*1*	0	*0*	0	*0*

Table 3 Responses to Part A of the Questionnaire

Figure 1 shows the percentage of total "Yes" responses (Y plus y) to the statement that "university mathematics makes sense". Group S remains at 100% positive before and after, but 64% of Group N change from "No" to "Yes".

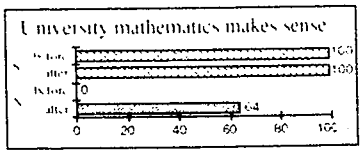


Figure 1 University mathematics makes sense

Figure 2 shows the same bar-chart layout for the other statements, re-arranged to place related statements side by side.

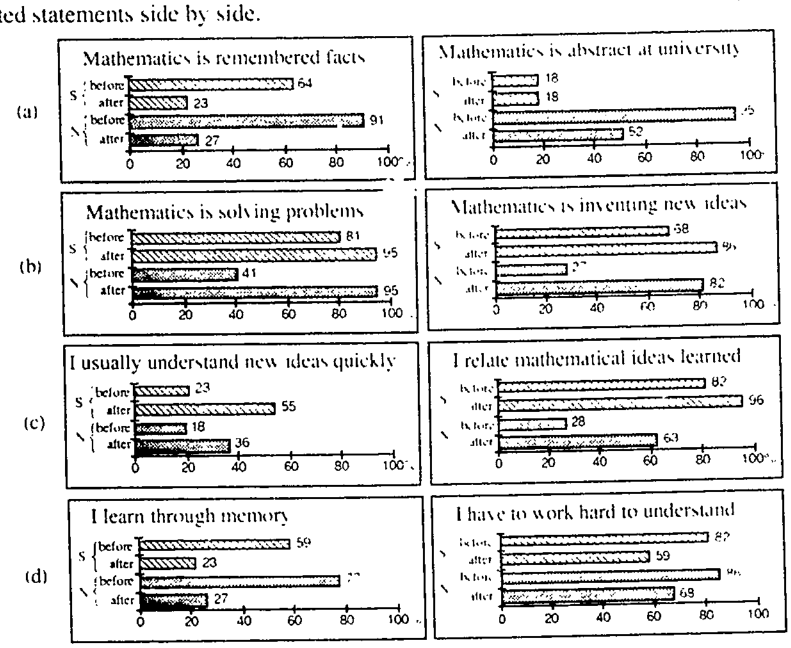


Figure 2 Attitudes to Mathematics

Each graph tells a consistent story, supported by significance tests from Table 1.

(a) There is a significant decrease in the perception of mathematics as facts and procedures to be remembered (from 91% to 27% in Group N) and the perception that university mathematics is "abstract" remains low in Group S, whilst diminishing significantly in Group N.

(b) A significant overall increase in perception of mathematics as solving problems and inventing new ideas, changing mainly in group N.

(c) Significantly more students overall now claim that they understand ideas quickly and can relate mathematical ideas together (mainly group N).

(d) Significantly fewer students overall claim they have to memorise ideas and fewer claim to work hard to understand (not statistically significant).

Results of Section B : Attitudes to Solving Problems

The data from section B is given in table 4.

Solving Problems		Group N (22 students)										Group S (22 students)									
		Yes		Y		No		N				Yes		Y		No		N			
		Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*	Pre	*Post*
confidence	ˆ	9	*16*	1	*3*	13	*5*	1	0	0	*1*	17	*20*	6	*9*	4	*1*	1	*0*	1	*1*
pleasure	ˆ	21	*21*	11	*9*	1	*0*	1	0	0	*1*	22	*21*	14	*12*	0	*0*	0	*0*	0	*1*
only to get through	ˇ	14	*4*	4	*0*	8	*16*	2	9	0	*2*	2	*0*	0	*0*	19	*21*	6	*8*	1	*1*
anxious	ˇ	12	*6*	1	*0*	8	*15*	1	1	2	*1*	5	*0*	0	*0*	16	*21*	3	*8*	1	*1*
fear unexpected	ˇ	17	*6*	7	*2*	3	*14*	1	2	2	*2*	13	*4*	3	*1*	9	*17*	2	*7*	0	*1*
correct answers	↓	13	*2*	3	*0*	8	*19*	2	5	1	*1*	8	*3*	1	*1*	13	*17*	1	*6*	1	*2*
try other approach	ˆ	21	*21*	8	*9*	0	*0*	0	0	1	*1*	21	*22*	9	*11*	0	*0*	0	*0*	1	*0*
give up	ˇ	13	*3*	3	*0*	8	*19*	3	8	1	*0*	6	*2*	0	*0*	16	*18*	6	*12*	0	*2*

Table 4 Responses to section B of the Questionnaire

A subset of these results is displayed pictorially in Figure 3, re-arranged as before.

(e) Confidence increases overall from pre-test to post-test and the pleasure in solving problems starts very high and stays there.

(f) Anxiety when faced with problems is reduced to a low level in both groups (zero in Group S), whilst fear of the unexpected falls significantly.

(g) Willingness to try a new approach remains very high in both groups; the tendency to give up with a difficulty reduces significantly to a low level.

(h) Extrinsic pressure—solving problems only to get through the course—reduces significantly overall, and (possibly due to a willingness to try out ideas) concentration on getting correct answers reduces significantly.

Before the course, Group N display many negative qualities, 64% are motivated to do problems "only to get through the course" and, though all but one derives pleasure from solving problems, 55% feel anxious, 77% experience fear of the unexpected, 59% give up easily and only 44% feel confident. In contrast, Group S has a majority of positive responses everywhere except for a fear of the unexpected —they are confident, take pleasure in getting solutions, have low anxiety, are willing to try a new approach without giving up too easily, and see mathematics as more than getting right answers.

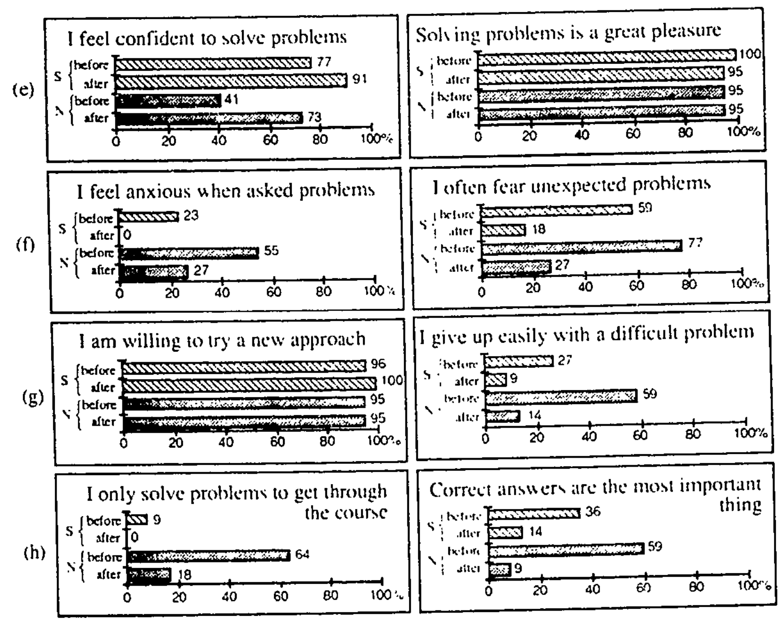


Figure 3 Attitudes to Problem-solving

On all items in both section A and B there is a positive change, except for some so extreme that little change is possible. In many cases the marked distinctions between Group N and Group S is considerably lessened. In particular, increase in confidence (graphs (e)) is associated with viewing the task as a positive goal to be achieved, and decrease in anxiety and fear (graphs (f)) is associated with the diminution of the negative feeling of wanting to avoid failure (an anti-goal in Skemp's theory, 1979).

Student comments

In the questionnaire, the students were asked to write a few sentences describing their feelings about mathematics. In the pre-test, the feelings of both groups seemed very much influenced by three factors: the nature of mathematics, personal feelings (such as motivation, interest, pressure etc.) and teaching methods. In Group N, of 7 responses related to the nature of mathematics; 5 were negative saying it is 'too abstract', 'seems pointless', and 'theory more difficult than practice'. Thirteen responses relating to personal factors included 10 negative, such as 'lack of motivation', 'put off by amount that needs to be done' and 'puzzled by what is going on'. The positive feelings were mostly about it being 'enjoyable and challenging', 'great sense of satisfaction when able to understand new concepts and to solve problems' and 'effort put in is worthwhile'.

Five responses relating to teaching were all negative—such as 'difficult to follow' and 'delivered in a dull atmosphere'. In Group S, there were 5 responses related to the nature of mathematics (3 positive), 14 responses about personal factors (12 positive) and 3 responses about teaching factors (all negative). Overall, in the pre-test, only 22% in Group N express positive feelings in comparison to 68% in Group S.

After the course the comments improve dramatically as is shown by Table 5. (The percentages do not add up to 100% due to responses including more than one factor.)

	Group N				Group S			
	Positive		Negative		Positive		Negative	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
nature of mathematics	9	23	23	5	14	32	9	5
personal	14	64	45	14	55	64	9	9
teaching	0	0	23	14	0	0	14	5

Table 5 Classification of written responses

Comments written after the course include:.

> Maths has always given me a lot of problems because I don't have the ability for memorisation. ... Now that I know about mathematical thinking, my interest and desire to learn maths have increased. (male, 4SPK)

> This is the first time that I have actually used maths to think. Before I just learnt maths to pass the exam (female, 4SPK)

> The way maths is taught here, it seems as though it is difficult and boring. ... There is no opportunity to display one's creativity. This makes it real dull and frustrating. (male, 5SPK)

> I am beginning to think instead of just doing the tutorial questions. ... I think I am learning more because I understand what is going on. (female, 3SSI)

> The course should have been introduced earlier. ... After following the course I am more confident to solve any maths problem that is given. (female, 5SPK)

These are consistent with the classroom observations and the changes intimated by the questionnaire, supporting the hypothesis that the course in problem-solving changes the attitudes of students from mathematics as a body of procedures to be learned to mathematics as a process of thinking.

References

Mason, J., Burton, L. & Stacey, K., (1982): *Thinking Mathematically*, Addison Wesley, London.

Tall, D. O., (1991): *Advanced Mathematical Thinking*, (editor), Kluwer, Dordrecht.

Skemp, R. R., (1971): *The Psychology of Learning Mathematics*, Pelican, London.

Skemp, R. R., (1979): *Intelligence, Learning and Action*, Wiley, London.

VYGOTSKIAN APPLICATIONS IN THE ELEMENTARY MATHEMATICS CLASSROOM: LOOKING TO ONE'S PEERS FOR HELPFUL EXPLANATIONS

Vicki Zack, St. George's School, Montreal, Canada

After discussions regarding non routine problems were concluded, the 24 students in the author's own Grade 5 classroom were asked to indicate in writing whether at any time during the discussions (which took place in pairs, then in groups of four, then in a small group of 12) they found an explanation helpful. If yes, they were to indicate who gave the explanation, and to try to state how it was helpful. The adept peers in the classroom garnered the majority of the votes. Several categories or types of helpfulness emerged from the children's responses. These related to help vis a vis (1) the parameters of the problem, (2) important but straightforward information, (3) help with problem-solving strategies/thinking, and mathematical concepts, and (4) alternate approaches to solutions.

How is knowledge acquired? and What role does social interaction play in that acquisition? As a teacher-researcher in my own elementary school classroom (10-11 year olds), I have been engaged in investigating the ways in which peers interacting together can contribute to the construction of their knowledge. The constructivist position is that "cognition is an invention, not a reproduction; we reassemble (construct) information taken from the outside world in ways that allow us to make personal sense of it" (Miles & Huberman, 1990). The most prevalent metaphor in the decade of the 80's was that of children as builders, actively filtering new information in light of what they already know in order to revise their personal, dynamic, ever-changing model of meaning (Pearson, 1993, p. 502). From Vygotsky, researchers have focused on the social nature of learning and the key role which peers and teachers play in facilitating the learning of individuals. Vygotsky's *zone of proximal development*-- the difference between the learning a child can construct on her own and what she can accomplish with the assistance of someone else (a teacher, a parent, or a knowledgeable peer)--might well be "the most popular learning construct of the 1980's" (Pearson, 1993, p. 503). Vygotsky's theory, and the investigation of the social construction of knowledge, is gaining a following among researchers in mathematics education (Yackel, Cobb & Wood, 1991; Sekiguchi, 1993; Olivier, Murray & Human, 1993; Forman, 1992). Moll has however pointed out that despite the fact that Vygotsky's ideas gained newfound visibility in the 1980's, there is a dearth of writings related directly to education (1990, ix).

Classroom set-up

St. George's is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds; the population is pre-dominantly English-speaking. The total class size this year is 24; however I always work with half-groups (12 children in each group) of heterogeneous ability. Problem solving is at the core of the mathematics curriculum in my classroom; non-routine problems are drawn from various sources (Charles & Lester, 1982; Meyer & Sallee, 1983, and others). Mathematics class periods are 45 minutes each day (and are at times extended to 90 minutes). Problem solving is the focus of the entire lesson three times a week. In class the children often work in Groups of Four teams: each teacher-selected heterogeneous team has 4 members, a problem is given, is either worked through by the members together from the start at their table, or begun by each child and then discussed together at their table (the members select the approach they favour). The children work first with a partner, and then when both pairs are ready, the two pairs discuss the solution to the problem together as a Group of Four. When the three teams are ready (they consider themselves ready when all team members have "understand" the solution--any member can be called upon to present, although they usually present on a rotation basis), they gather together. Presentations of the solutions to the whole group of 12 take place at the chalkboard. Each week the children also work on one challenging problem at home (Problem of the Week), and are expected to write in their log about all that they did as they worked the problem. The children present their Problem of the Week solutions to the class.

I videotape a pair and then their team (Groups of Four) on a rotating basis, and videotape all the presentations done at the board (Groups of Four team discussions, and Problem of the Week discussions), and observe and take notes during the sessions. Much of the class session is conducted by the children. Data sources: focused observations, videotape records, student artifacts (copybooks), teacher-composed questions eliciting opinions (written responses), class discussions regarding research topics.

A look at helpful explanations

I know from the results of previous inquiries I have made among the children I teach that they do find talk and peer-given explanations helpful when engaged in solving non-routine problems, especially when they are stuck, as for example, if they realize that they did not understand the problem, or that they got the wrong answer (Zack, 1993). The children value "clear" explanations, but to date we have arrived together at only a general definition of what constitutes a "clear" explanation (Zack, 1993). We will be continuing to finetune our definition. Most pertinent to the issue at hand here in regard to explanations, however, is that my adult perspective and the vantage point from which the children see the same peer-given explanation diverge at times. As classroom teacher I am constantly assessing the explanations and presentations given by the children as they work. In the course of doing so over the last four years, I have discovered that at times what I have judged to be a clear explanation is not necessarily considered to be so by the children. Yackel and Cobb (1993) have written recently about what they call *adequate* and *inadequate* explanations. A child-given explanation may sound eminently clear to the adult teacher, but if the child listening does not connect with it in a way that helps her make meaning of it, then it is *inadequate* for that listener.

Hence, I decided to inquire in order to see what the children would say. Which explanations would the children judge to be helpful to them? Which children are the ones seen to be giving the helpful explanations? What kinds of help do the children value? Would there be a pattern regarding when these helpful explanations are seen to be given, that is during the work done in pairs, in the foursome, or in the group of twelve?

And so, after the discussion of the problem was concluded, I asked the children to write a response to the following question:

> Was there an explanation given by a fellow student that helped you to understand the problem? _______ or helped you to understand the problem better?_________ If yes, tell me the student's name. If you can, tell me how or why the explanation helped.

I must clarify that when I first began, I posed the question as a recall question at the end of October 1993. I asked the children to think about the problems done in September and October (10 problems), and to answer the above question. In November and December, 1993, I asked the students to write a reply to the question above immediately after the conclusion to the discussion of each problem. There were 11 problems discussed during the November-December months. The child could nominate one, more than one, or no candidates, provided that they supported each one of the nominations with a reason. The total number of nominations (see Table 2) would have been higher had I used the November-December approach from the start in September. As it was, the number of November-December nominations outnumbered the September-October nominations by a ratio of 4 to 1.

The type of help the children felt that the explanations provided

As I considered the students' written responses, it was possible to see emerging a number of 'categories' of types of help the children felt the explanations were providing. The term 'categories' is not quite right in that they are not discrete; they overlap and at points are closely interconnected. All are important. The ones I was most interested in were the ones dealing with conceptual issues (3A, 3B, 3C, below), and alternate approaches to problem solutions (4, below). I know from previous findings that reference to alternate approaches and valuing of these approaches does not occur frequently (Zack, 1993), thus I was especially alert for evidence of any such occurrences.

TABLE 1

Brief description of types of help children identified

#1	**Parameters or conditions of the problem:** The child felt that the peer explanation had helped her to better understand what the problem was requiring her to do, and what the conditions were.
#2	**Factual information:** The child felt that the peer explanation had provided information needed for the solution; I coded it as #2 when I deemed it straightforward information. Eg. the distance between the bases in a baseball diamond is 90 ft. (This was the basic information needed in a problem which required that the children think about area, before they had learned the algorithm for area.)
#3	**How to solve the problem:**
#3a	**How to determine the problem-solving strategies to use, how to organize your information, your thinking.**
#3b	**How to represent your ideas with mathematical symbols,** eg. fractions or decimals, and what that means, eg. what does 1/3 represent (as part of a whole, as equivalent to 2/6, and as a way to solve the problem), or what is a right angle and is it still a right angle in different orientations.
#3c	**Focusing on the essence of the problem.** The children noted an element which I as teacher have termed the *essence* or *key idea* in a problem. It has much to do with the other conceptual elements and with the conditions of the problem noted above, but is central to the problem. I am not suggesting that the children when noting it are picking out that it is essential; in only one instance has a child presenter (highly adept) spoken specifically about *key ideas*.
#4	**Presenting an alternate solution to the problem,** a solution which is simpler (i. e. more economical) or aesthetically more pleasing. Two students last year, and two this year (all adept) stated that hearing an alternate solution helped them to better understand the problem itself or their own thinking. (See aesthetic reasons, Zack, 1993)
?	**Unable to classify.** Some responses were vague. For example, "She helped me understand it better" attested to the fact that the explanation was helpful but the child did not specify how, and when interviewed could not elaborate.

When interviewed, the children shed light on how the explanations helped them. What struck me was that some children were aware that the explanation helped them with only one component, and that they were still in the dark about the picture, the problem, as a whole. One child, Eric (less adept), said, for example, that his partner Janet had helped by telling him "what 1 sixth means", but he quickly added that he "didn't really understand the problem, just the two quarters"; "two quarters" functioned here as his shorthand for "fractions". He was able to show how what his group-of-four team-mate Janet did and said helped him recall having learned about the 'pie' chart (in sixths) back in his old school (i. e. three years ago).

In talking about the same problem as Eric was referring to above, another student, David, who was adept, credited a peer explanation (by an adept peer) with providing the missing puzzle piece that he needed to solve the problem. David impressed me with his grasp of the larger picture; he seemed confident about how it all fit together.

In another instance which caught my attention I saw that two adept students, Jeff and Abe, were sparked in their thinking by two less adept children, Sheree and Karen, whose explanations served as catalysts to Jeff and Abe (see the shaded boxes in Tables 2 and 3), although Sheree and Karen were not themselves able to solve the problem nor follow the thinking of those who did solve the problem correctly.

A high number of nominations for helpful explanations go to the "able" peer

As the data on Table 2 indicates, a substantial proportion of the votes went to the children who are adept. My first global assessment of the children's achievement was done in November, 1993, prior to my analysis of the children's responses regarding helpful explanations.

I will note briefly my criteria for assessing a child as *adept*, *moderately adept*, or *less adept* vis-a-vis problem solving. The *adept* children (this year: 5 boys, 2 girls) often "get" the problem solution fairly early on or prior to the large group (group of 12) discussion, can see miscues and revise, can explain their thinking in talk and in writing and are often on the right track. The *moderately adept* children (this year, 4 girls) "get" a number of problems prior to the large group discussion, can sometimes see their miscues and revise, can often explain their thinking in talk and in writing and are sometimes on the right track. The *less adept* children (this year: 9 girls, 4 boys) cannot solve many of the problems, often cannot see miscues and revise, can sometimes explain their thinking in talk and in writing and are often not on the right track.

The analysis of the data indicates not only that a good proportion of the vote went to the adept peers, but also that a good proportion of these votes dealt with issues related to problem solving and to mathematical concepts. In regard to #3a, problem-solving strategies and ways of thinking about the problem, 29/40 or 72% went to the adept group; in terms of class members, this group represents 7/24 or 29% of the class.

TABLE 2

Number and types of nominations garnered by adept (A), moderately adept (MA), and less adept (LA) students

	Parameters of Problem	Information	Problem-Solving Strategies	Math Concepts	Key Idea	Alternate Approach	**Totals**	?
Adept (5 boys, 2 girls)	12	14	29	10	7	2	74	4
Moderately Adept (4 girls)	7	4	5	2			18	1
Less Adept (9 girls, 4 boys)	2	7	6		3		18	20
Totals	21	25	40	12	10	2	110	

TABLE 3

Who nominated whom

(A) nominating (A)	3	1	8	2	5	2		
(MA) nominating (A)	4	2	7	1				
(LA) nominating (A)	5	11	14	7	2			
(A) nominating (MA)	2	-	1					
(MA) nominating (MA)	-	1	1					
(LA) nominating (MA)	5	3	4	2				
(A) nominating (LA)	-	1	3		2			
(MA) nominating (LA)	-	-	-					
(LA) nominating (LA)	2	6	2		1			
Totals	21	25	40	12	10	2		

1492

Discussion of the findings

The findings seem to indicate that the children do nominate the explanations of the more adept more often than others. Sierpinska has suggested that "understanding is irrevocably linked with . . . explaining (which has traditionally been separated from understanding)" (1992, 78). Hence it may then follow that the ones who have a better grasp are the ones who will tend to give the better explanations. As one of my students stated last year: "If the person really understands the problem, they should have a good explanation." In the next phase of the study, I will be analyzing the videotapes of four of the children nominated as able explainers, in order to see whether I can identify some characteristics of good explainers and of good working partners, for as I have stressed elsehere, the cognitive, social and affective aspects are intertwined (Zack, 1993). I will also be taking note of the characteristics of the explanations of the less adept children who are paired with the afore-mentioned able explainers. We need to consider the attributes of the explanations of all the children.

It is possible that many votes went to the children whose explanations reflected the correct solution. The nominating was done after the discussion was concluded, and after the class had agreed upon what was the correct answer. Perhaps some of the discussions which had occurred along the way may have contributed to the children's construction of meaning but the children may not recall those as such. Explanations which are embedded in the flow of the talk, especially in the beginning phase, may be harder to distinguish than those offered towards the end. Two children last year made note of the fact that it is easier to follow the discussion during the large group (of twelve) session due to the fact that the possibilities have been narrowed down somewhat (Zack, 1993). When I asked the children in January to indicate whether the explanations they had cited as helpful occurred during the time they were working in a two-some, or when in four-some, or when they were part of the group of twelve presentation, the results were 23 occurrences when in pairs, 15 when in a four-some, and 58 when in a group of twelve.

It is also possible that the patterns of talk might be gender-based, with males tending more to 'holding the floor' presentations, and with females feeling more comfortables with collaborative, 'side-by-side' talk and building together (Edelsky, 1984, cited by Belenky et al., 1986, p. 145). The children may identify the notion of explanation with the 'holding the floor' stance, and as mentioned above, may not see the kernels of explanation when they are embedded in the flow of the talk. There must be more work done on how mathematical meaning is shared, with attention paid to the explanations given by children at all levels of achievement, so that we can better understand the children's understanding and the means they use to extend their ways of knowing mathematics.

References:

Belenky, Mary, Clinchy, Blythe M., Goldberger, Nancy R., & Tarule, Jill M. (1986). Women's ways of knowing: The development of self, voice and mind. Basic Books.

Charles, Randall, & Lester, Frank. (1982) Teaching problem solving: What, why & how. Palo Alto, California: Dale Seymour Publications.

Forman, E. A. (1992). Discourse, intersubjectivity, and the development of peer collaboration: A Vygotskian perspective. In L. T. Winegar & J. Valsiner (Eds.), Children's development within social context. Lawrence Erlbaum.

Meyer, Carol, & Sallee, Tom. (1983). Make it simpler: A practical guide to problem-solving in mathematics. Addison-Wesley.

Miles, M. B., & Huberman, A. M. (1990). Commentary on the papers by Barone and by Clark: Animadversions and reflections on the uses of qualitative inquiry. In Eisner, E., & Peshkin, A. (Eds.), Qualitative inquiry in education: The continuing debate. Teacher's College Press.

Moll, L. C. (Ed.) (1990) Vygotsky and education: Instructional implications and applications of sociohistorical psychology. Cambridge University Press.

Olivier, A., Murray, H., & Human, P. (1993). Voluntary interaction groups for problem-centered learning. Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education, University of Tsukuba, Japan, July 18-23, 1993, 73-80.

Pearson, D. (1993). Teaching and learning reading: A research perspective. Language Arts, 70, 6, 502-511.

Sekiguchi, Y. (1993). The complexities of refutation in the social interaction: An ethnographic inquiry in a mathematics classroom. Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education, University of Tsukuba, Japan, July 18-23, 1993, 270-277.

Sierpinska, A. (1992). Lecture notes on understanding in mathematics. Unpublished paper, Concordia University, Department of Mathematics, Montreal, Quebec.

Yackel, E., & Cobb, P. (1993). Sociomathematical norms, argumentation and autonomy in mathematics. Paper presented at the 1993 Annual Meeting of the American Educational Research Association, Atlanta, Georgia.

Yackel, E., Cobb, P., & Wood, T (1991). Small group interactions as a source of learning opportunities in second grade mathematics. Journal for Research in Mathematics Education, 22, 5, 390-408.

Zack, Vicki. (1989). Dyadic interaction in an elementary school computer lab classroom, and the learning of Logo geometry concepts. Unpublished doctoral dissertation, McGill University.

----------. (1993). Children's perceptions of the usefulness of peer explanations. Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education, University of Tsukuba, Japan, July 18-23, 1993, 286-292.

The author wishes to thank Carolyn Kieran for helpful discussions during the preparation of this paper

DIFFICULTIES WITH COMMUTATIVITY AND ASSOCIATIVITY ENCOUNTERED BY TEACHERS AND STUDENT-TEACHERS

Orit Zaslavsky
Technion - Israel Institue of Technology

Irit Peled
University of Haifa

This study explores difficulties that mathematics student-teachers and teachers have with the concepts of binary operation, associativity and commutativity and investigates their sources. These difficulties are encountered while they are engaged in a task of generating special examples of a binary operation. The findings indicate that the subjects have not formed an integrated concept of binary operation. Their difficulties can be traced back to earlier experiences, yet are not merely a result of exposure to over-emphasis of a limited example space of binary operations. Early experience which involves changing order of expressions seems to have caused confusion with regard to the associative and commutative properties.

Through their college education mathematics student-teachers and teachers have been exposed to many kinds of binary operations, and have been engaged in checking whether the associative or the commutative properties hold for the relevant mathematical sets they have been dealing with (Hadar and Hadass (1981) provide a broad collection of such binary operations). Nevertheless, when asked in a study by Tirosh, Hadass and Movshovitz-Hadar (1991) to generate examples of binary operations, which call for searching beyond the four basic operations, student-teachers failed to do so. Tirosh et al. (ibid) attribute their failure to the strong hold of early experience. This study investigates this issue further by looking at the types of problems the subjects encounter and by searching for a more detailed explanation which is not confined to dealing with their experience relating to the four initial operations.

The task used in Tirosh et al. (ibid) asks the subject to determine whether there exists a binary operation which is either commutative and not associative (C & NA) or associative and not commutative (A & NC). Such a case would differ from each of the four basic operations extensively dealt with in earlier experiences, for which either both properies hold or both do not hold.

Since this study does not focus on the issue of students' beliefs as much as it does on explaining the sources of difficulties, the task used by Tirosh et al. (ibid) has been changed to encourage subjects to make a stronger effort in search for special examples (as further detailed in the method section). The study discussed here is part of a larger study in which subjects were asked to generate counter-examples (Zaslavsky & Peled, 1994). The task of generating examples has been shown by Bratina (1986) to be powerful in revealing difficulties.

METHOD

Subjects

Two groups participated in the study. One group consisted of 36 in-service mathematics teachers, and the other group included 67 pre-service mathematics teachers, most of which were in the third year of their undergraduate studies. The study was carried out within the frameworks of an in-service professional development course for mathematics teachers and of two university level mathematics methods courses.

Research Instrument

The task analyzed in this paper with which the subjects were presented was part of a larger study on the generation of counter-examples. The subjects were given a false statement (supposedly suggested by a student), and explicitly were informed that the statement was false. Their task was to produce at least one example to convince this (supposed) student that his statement is false. They were also asked to specify in detail the process that led them to their response.

The false statement used in this study was:

Any commutative operation is also associative

RESULTS

Of the 67 student-teachers only 21 students (31%) suggested an example. The rest either did not give an answer or stated (11 cases) that the wrong claim is a correct one. However, only 3 of the 21 examples were correct. In the second group, the 36 teachers, 20 (55%) teachers gave at least one answer including 4 who gave more than one answer (yielding a total of 26 examples). The rest did not give an answer and 2 of them stated that the claim was correct. In this group out, of the 26 examples 16 were correct.

Altogether 47 examples were constructed, 28 (60%) of which were wrong. These 28 wrong examples were further analyzed in order to identify difficulties exhibited in the process of generating an example as required. Note, that two of the teachers constructed two wrong examples each. The rest of the subjects constructed at the most one wrong example each. Four main categories of difficulty were identified. These difficulties relate to:

1 *The Associative property.* This category includes three kinds of difficulties:

- Wrong translation of the formal definition of Associativity to the specific operation defined (e.g., for the operation $a*b=|a+b|$, refuting the associativity by showing that: $|a+b|+|c| \neq |a|+|b+c|$, instead of: $||a+b|+c| \neq |a+|b+c||$).

- "Over-generalization" of the associative property to hold for *two distinct* operations (e.g., defining the operation as $a*b=a\cdot b$ (multiplication), and trying to refute the associativity by showing that: $(a\cdot b)+c \neq a\cdot(b+c)$, instead of: $(a\cdot b)\cdot c = a\cdot(b\cdot c)$).

- Including a superfluous condition for associativity which involves changing the *order* in the expression (e.g., for the operation of intersection of sets, defining the associativity property by:

$(A\cap B)\cap C = (B\cap C)\cap A$, instead of: $(A\cap B)\cap C = A\cap(B\cap C)$).

2. *The Commutative property*. This category includes one main kind of difficulty:

- Confusing the operation sign with the number sign when changing the *order* of the elements (e.g., for the subtraction operation checking the commutativity by showing that: $-2-3 = -3-2$, instead of showing that:

$-2-3 \neq 3-(-2)$).

3. *The notion of binary operation*. This category includes one main type of difficulty:

- Producing an ill-defined or inappropriate operation (e.g., constructing an unary operation: $a*=a^2-1$, then checking its so-called commutative property by using an inappropriate expression as follows: $a*b=(a+b)^2-1$).

4. *Logical inferences*. This category includes one main kind of difficulty:

- Verifying that the defined operation is commutative by checking one special instance (e.g., for the non-commutative operation $a*b=a^b$, "verifying" that it is commutative by showing that: $2^4 = 4^2$).

Table 1 shows the distribution of difficulties exhibited in the responses to the task. Note, that the discrepancy between the number responses analyzed (8 teachers' and 18 student-teachers', a total of 26) and the total number of difficulties found (30) is due to the fact that in three responses more than one type of difficulty was identified.

Table 1: Types of difficulty

Types of Difficulty / Group	1 (associative)	2 (commutative)	3 (binary operation)	4 (logical)	other
Student-Teachers	6	3	3	2	6
Teachers	6	1	0	1	2
Total	12	4	3	3	8

Looking as a whole at the two distributions of types of difficulty in Table 1, no significant difference was found between the two sample-populations (on a Chi-square test, with value 3.37 (Pearson), $p>0.49$). Most of the difficulties were related to the associative property. This type of difficulty was particularly frequent among the teachers (6 out of 8 teachers). The eight responses categorized as "other" are cases in which an operation had been checked and found to be associative. Arriving at such an impasse the search was terminated without finding a counter example. These are cases in which it is not completely clear whether the subjects realized that no correct example had been generated.

DISCUSSION

The current study has shed some more light on the causes for the weak concept that student-teachers and teachers hold surrounding related properties of the binary operation. Tirosh et al. (1991) have suggested that extensive experience with special cases in the primary grades is the main cause for the creation of the belief that binary operations are either C&A or NC&NA. The findings of this study offer a more detailed explanation of this phenomenon which reaches beyond the mere learning and over-emphasizing of special cases of binary operations. Most of the difficulties identified as belonging to categories 1 or 2 (see Table 1) involve the issue of *order*. An examination of the mathematics curriculum suggests that the way in which the issue of *order* is interwoven could be a critical source of confusion. In the course of learning, there are many instances in which the two properties -- commutativity and associativity -- are applied simultaneously without making a clear distinction where exactly each one is used. This is not entirely farfetched, considering that the two properties deal with some sort of *change in order:* While commutativity involves the change in the order of elements, associativity involves the change in the order in which the operation is executed.

For example, when a child is asked to calculate the following: $6+7+4$, he is usually encouraged to do it more efficiently by using the following strategy: $(6+4)+7$. A careful analysis of this strategy shows that, in fact, the two properties are (implicitly) applied interchangeably. First, the associativity is applied, by: $(6+7)+4 = 6+(7+4)$. Then the commutative property is used in the following step: 6+(7+4)=6+(4+7). Finally, the associative property is applied again: 6+(4+7)=(6+4)+7. Since students are rarely expected to specify or justify each step, they often are offered a general explanation for this shortcut as a whole, by saying something like: *the order does not matter*. In higher grades similar shortcuts are common with algebraic expressions instead of numbers. These sort of experiences could easily contribute to the confusion between the two properties, by putting them under one roof. Stein (1986) discusses the use of shortcuts by elementary mathematics teachers, and suggests that some teachers introduce them without a clear idea of why they work. No wonder, therefore, that some students tend to think of commutativity and associativity as dependent properties, as indicated in Tirosh et al. (1991).

The issue of *order* can also account for difficulties related to the associative property in itself. The incorrect cases in which associativity was checked for two distinct operations could be traced back to discussions on the topic of conventions regarding *order* of operations. These discussions often involve checking the truth value of statements such as: (a+b)·c=a+(b·c). This expression looks much like the expression for associativity: (a*b)*c=a*(b*c), except that it deals with two distinct operations instead of the same one.

Difficulties connected to the commutative property (see category 2) can also be elucidated by past experiences. The treatment of two inverse operations as interchangeable (e.g., subtraction of a number is equivalent to the addition of its opposite), seems as a source of confusion regarding the commutative property. This principle could lead to the inappropriate substitution of one operation with its inverse (e.g., 5-2=5+(-2)=-2+5) in the course of checking whether one operation is commutative . A similar problem arises from the involvement in moving elements around in a given expression by attaching the operation sign to them (in the last example the subtraction sign could be seen as attached to the number 2). Consequently, these two approaches lead to the inference that a non-commutative operation (such as subtraction) is supposedly commutative.

CONCLUSIONS

In the current study it was established that both mathematics teachers and student-teachers struggle through the concept of *Binary Operation* , as well as the commutative and associative properties which relate to it. This struggle seems to have emerged from past experiences, in which much confusion had been created by the recurring theme of *Order* appearing in several different meanings and contexts. Considering that these salient difficulties were exhibited by grown-ups who have studied many advanced undergraduate courses in mathematics, it can reasonably be expected that children encounter similar, and perhaps even stronger, difficulties. Those undergraduate courses should have provided the students with an opportunity to resolve these difficulties. Clearly, these courses did not facilitate the integration of the scattered pieces into one full concept. The lesson to be learned is that this kind of integration should be done explicitly up front and not be left entirely to the individual.

REFERENCES

Bratina, T. A. (1986). Can your students give examples? *Mathematics Teacher, 79,* 524-526.

Hadar, N., & Hadass, R. (1981). Between associativity and commutativity. *International Journal of Mathematics Education in Science and Technology, 12,* 535-539.

Stein, M. K. (April, 1986). Shortcuts: Mindless procedures or opportunities for teaching? Paper presented at the annual meeting of the American Educational Research Association, New Orleans.

Tirosh, D., Hadass, R., & Movshovitz-Hadar, N. (1991). Overcoming over generalizations: The case of commutativity and associativity. In F. Furinghetti (Ed.), *Proceedings of the fifteenth conference of the International Group for the Psychology of Mathematics Education,* v.*3,* 310-315.

Zaslavsky, O., & Peled, I. (1994). Inhibiting factors in generating examples by mathematics teachers and student-teachers: The case of binary operation. Submitted for publication.

DIVISIBILITY AND DIVISION: PROCEDURAL ATTACHMENTS AND CONCEPTUAL UNDERSTANDING

Rina Zazkis and Stephen Campbell
Simon Fraser University

Abstract

This study contributes to a growing body of research on teacher's content knowledge in elementary mathematics. The general subject matter under investigation is elementary number theory. Specifically, we focus upon preservice teachers' understanding of concepts related to the multiplicative structure of whole numbers: divisibility, factorization and prime decomposition. Twenty clinical interviews were conducted with volunteers from a class of students enrolled in a professional development course, 'Foundations of mathematics for teachers'. We have adapted a constructivist-oriented theoretical framework for methodological guidance in data acquisition analysis and interpretation. Results from this study reveal in detail the participants' strong dependency upon procedures. Such procedural attachments appear to compromise and inhibit development of more refined and meaningful structures of conceptual understanding.

Introduction

This report is based upon results from the first few months of a three year project investigating preservice teachers' understanding of elementary mathematical concepts and operations. The specific concepts involved pertain to elementary number theory. Herein we present an abbreviated formulation of our theoretical and methodological approach and a preliminary sketch of part of the cognitive terrain we are attempting to survey. Our objectives for this study were:

(a) to explore preservice teacher's understanding of elementary concepts in the theory of numbers with emphasis given to concepts involving divisibility and the multiplicative structure of non-negative integers;

(b) to analyze and describe spontaneous cognitive strategies of solving unfamiliar problems involving and combining those concepts;

(c) to adapt and extend a constructivist-oriented theoretical framework for the analysis and interpretation of those strategies and the cognitive structures supporting them.

The results of this study provide a preliminary basis of a descriptive theory for the development of cognitive constructs involving elementary number theory concepts, their properties and relationships.

Theoretical Framework

Action-process-object framework

The particular interpretation of constructivism used in this study is based upon Dubinsky's action-process-object developmental framework (Dubinsky, 1991). Dubinsky developed this framework as an adaptation of ideas of Piaget to the studies of *advanced* mathematical thinking. Previously it has been used in studies of undergraduate mathematics topics such as calculus and abstract algebra (Ayeres, *et. al.*, 1988; Breidenbach, *et. al.*, 1992; Dubinsky, *et. al.*, in press). We suggest this theoretic... ·rspective is useful for investigating the development of mathematical understanding in general and is not limited to advanced applications. One of the goals of this study was to substantiate this claim.

The essence of Piaget's theory is that an individual, dis-equilibrated by a perceived problem situation in a particular context, will attempt to re-equilibrate by assimilating the situation to existing schemas or, if necessary, reconstruct particular schemas enabling the individual to accommodate the situation. Dubinsky holds that the constructions which may intervene are mainly of three kinds -- actions, processes, and objects.

An *action* is any repeatable physical or mental manipulation that transforms objects in some way. When the total action can take place entirely in the mind of an individual, or just be imagined as taking place, without necessarily running through all of the specific steps, the action has been *interiorized* to become a *process*. New processes can also be constructed by *inverting* or *coordinating* existing processes. When it becomes possible for a process to be transformed by some action, then we say that it has been *encapsulated* to become an *object*. We illustrate these notions below as we explore the extent to which this theoretical framework contributes to an understanding of number theoretical knowledge construction and development.

Method and Technique

Individual clinical interviews with preservice elementary teachers were conducted using an instrument that allowed for the flexibility to probe and clarify participant understanding of elementary number theoretical concepts. The instrument was designed to reveal the participants' ability to address problems by spontaneous construction of new connections within their existing content knowledge. The questions covered a spectrum ranging from elementary number concepts (e.g. What does it mean to you that a number is an even number?) to more subtle and sophisticated problems requiring deeper conceptual insight into elementary number theoretical properties and relationships (e.g. What is the smallest positive integer divisible by every integer, 1 through 10?). During the interviews, each of which lasted for about one hour, the participants were probed, when appropriate, for understanding that may not have been apparent from their initial response. In circumstances where participants experienced difficulties with a particular question they were encouraged to reflect upon and articulate, or otherwise express, the nature of those difficulties. In cases where such activity proved inadequate in leading the participant to a realization of a solution or, alternatively, a resolution of their difficulties, the interviewer would progressively allude to, or provide, additional information. This method proved conducive to uncovering and identifying the possible source and depth of conceptual and/or procedural difficulties. In addition, this method incorporates an important diagnostic dimension, described by Simon as a 'continuum of connectibility', demarcated by clear distinctions regarding the connectedness of an individual's knowledge as exhibited within the context of a clinical interview (Simon, 1993). The interviews were usually concluded with general questions regarding the participant's mathematical background, experience, attitudes, beliefs and mental processes.

Data Source.

Twenty preservice elementary school teachers participated in the study. They were volunteers from the group of students involved in a professional development course called "Foundations of mathematics for teachers". The concepts/topics of factorization, least common multiple, greatest common divisor, prime and composite numbers, prime decomposition and the fundamental theorem of arithmetic, divisibility and alternative 'divisibility rules' for the numbers 2, 3, 5, and 9 were part of their curriculum. The mathematical background and experience of the participants varied considerably, even though the interviews were conducted after the number theory related topics were 'covered' in their course.

Analysis

The interviews were transcribed and categorized in terms of different questions, their difficulty, and identifiable cognitive patterns of various degrees of sophistication exhibited by the participants. The action-process-object framework was used to guide the analysis of the interviews for the manners in which the participants appeared to think about the specific topics and problems presented to them. We illustrate an application of this model with an abbreviated phenomenological analysis or 'genetic decomposition' of the concept of divisibility:

Divisibility - an abbreviated illustration of genetic decomposition

A construction of the concept of divisibility as a conceptual object starts with specific examples of divisors. These divisors are usually small numbers such as 2, 3, 4 and 5. Initially, divisibility by 3, for example, is an *action*: a learner has to actively perform division and obtain a quotient of a whole number, with no remainder, in order to conclude *a posteriori* that a number is indeed divisible, or not divisible, by 3. Later, the activity of division may be *interiorized* as a conceptual *process*, such that the action is intended but not actually performed. In this case, the student has conceptualized the notion that it is the division procedure itself that determines whether or not a whole number satisfies the 'rule' or criterion for divisibility. In this way, the action/process distinction can be used to distinguish between *procedural activity* and *procedural* *understanding*. *Encapsulation* of divisibility as an *object* begins with an awareness of the concept of divisibility as an essential property of whole numbers independently of the procedural aspects of division. The concept of divisibility at this stage is conceived as a bivalent, yes or no, property of whole numbers. That is to say, where a and d are whole numbers, a is *a priori* either divisible by d or not divisible by d. More generally, 'divisibility by n', encapsulated as a conceptual object, may be indicated when used to determine 'compound' divisibility properties: For example, when divisibility by 2 and 3 is used to infer divisibility by 6; or when indivisibility by 2 is used to refute divisibility by any other even number. We defer considerations regarding coordinated and iterative structures, and the role of equilibration on encapsulation.

Interpretation of Results

We've identified several major issues and problematics from which we attempt to thematize a general schema for multiplicative number structure. Previous discussion of multiplicative structures (e.g. Vergnaud, 1988; Graeber, et. al., 1989; Greer, 1992) focused on contextual situations. In this research we are interested in the development of cognitive structures of number concepts and operations *per se*. In what follows we provide several examples which identify potential sources of difficulty relating to procedural attachments in attaining conceptual understandings.

Basic concepts/definitions: Mathematical vocabulary

It was found that a significant percentage of our participants experienced difficulties grasping conceptual aspects of mathematical definitions, seemingly invoking various forms of linguistic inference that provided a rational for preserving meaning in terms of procedural understanding. A frequent claim was, for instance, that 3 is a multiple of 18, since "you multiply 3 by 6 to get 18". In this case the meaning of the word 'multiple' was conflated with the concept of factor, apparently due to a linguistic association with the activity of multiplication: "A multiple is something you multiply with". Another common claim was that 5 is divisible by 2, since "you can divide any number by whatever you want" or "5 is divisible by 2, but the result isn't a whole number". In this case the concept of divisibility was conflated with the process of division. Given the use of similar terms in these cases and others, many participants appeared to be forcing the meanings of words like 'multiple', 'divisor' and 'divisible' into existent procedural schemas, rather than accommodating them with viable structural modifications.

Development of divisibility concepts in terms of action-process-object

The majority of participants were not able to discuss divisibility as a relation or property of numbers without performing division. This tendency is an indication that their construction of divisibility has not developed beyond action or process. This can be attributed to the absence of a conceptual basis that, when present, opens a path to deductive reasoning and proof strategies.

Furthermore, the absence of conceptual understanding justifies and supports an adherence to and a dependence upon specific examples that, in turn, reinforce a strictly empirical attitude towards mathematics. The following excerpts from interviews exemplify this.

I: Suppose I have an even number which is divisible by 7. Say I've now divided it by 7. Would I still end up with an even number?
S: You'd have to try. You'd have to try to see if it works.

The claim "you cannot be sure that the result is a whole number if you don't know what the result is" seemed to be typical in this group of teachers. Thinking of divisibility as an action is exemplified in the following excerpt:

Interviewer: Do you think there is a number between 12358 and 12368 that is divisible by 7?
Nicole: I'll have to try them all, to divide them all, to make sure. Can I use my calculator?
Interviewer: Yes, you may, but in a minute. Before you do the divisions, what is your guess, what is your bet?
Nicole: I really don't know. If it were 3 or 9 I could sum up the digits. But for 7 we didn't have anything like that. So I will have to divide them all. [Indeed, Nicole performed several divisions to find the number that gives a whole quotient when divided by 7 and only then answered the original question positively]
...
Nicole: Yes, there is one. 12362 divided by 7 is 1766 exactly. No decimal part. So this is the number.
Interviewer: Do you think there is another number in this interval that is divisible by 7?
Nicole: I'll just keep checking, 'cause I can't see a pattern happening. I don't know an easier way that you do it to find -- in a glance.

Thinking of divisibility as a process is exemplified in the following excerpt. This illustrates a procedural understanding, rather than just a procedural activity, of division.

Interviewer: Do you think there is a number between 12358 and 12368 that is divisible by 7?
Jennie: Let's see. [performs long division without calculator] 12359 divided by 7 gives remainder 4. So.. 60, 61, 62... 12362 will be divisible by 7.
Interviewer: It's interesting. How did you know? I haven't seen you doing division.
Jennie: If this one [12359] gave remainder 4, the next one will give remainder 5, and the next one--6, and the next one 7, which means zero or no remainder. So if you divide 12362 by 7 there will be no remainder, it will be divisible.

It is interesting to note Jennie's use of the future tense in the discussion of divisibility. Her statement "if you divide...it will be divisible" may be interpreted as a transitional dependence of the divisibility property of the number on the procedural activity of division. Indeed, the definition of divisibility is implicitly procedural: a is said to be divisible by b if and only if the quotient a/b is a whole number. But how can one be certain that the quotient is a whole number without knowing what this number is? Achieving this certainty is a step towards mathematical maturity as well as a step towards encapsulation of divisibility as a mathematical object.

Awareness of and ability to apply the 'divisibility rules' helps to make this step. Divisibility rules may help the learner to separate between performing division and considering divisibility as a property of a number. On the other hand there is a danger that divisibility as a property of a number may be reduced to seeking patterns of digits. For example, when one of our interviewees was asked "Do you think there is a number between 12358 and 12368 that is divisible by 7?" replied that 12358 was probably such a number since the sum of its digits is 14, which is divisible by 7. More common (repeated three times in a group of 20 students) answer was: "I would guess that 12363 is divisible by 7 since 63 is divisible by 7."

Roughly one half of the participants were thinking of divisibility as an action and one half of the participants have interiorized the process for divisibility. Only one student in this group demonstrated encapsulation of divisibility as an object. The following excerpt shows that Pam perceived divisibility by 7 not only as an *a priori* property but also could explain how often this property is found within a contiguous set of whole numbers.

Interviewer: Do you think there is a number between 12358 and 12368 that is divisible by 7?
Pam: I think there is.
Interviewer: Do you know which number it is?
Pam: Not yet, but I can find it if you want me to.
Interviewer: No, you don't have to find it. But if you don't know what it is, how do you know it is there?
Pam: Here we have 9 numbers. And I know that if I take any 7 numbers there will be one divisible by 7. And here I have 9, which is more than 7.
Interviewer: Are you saying that if I pick any 7 numbers I wish there will be one divisible by 7?
Pam: I didn't mean that, what I mean is if you take these numbers one after another there will be one of them divisible by 7.

The understanding of the modular distribution of numbers that share a certain divisibility property is further indication of 'objectivity' in Pam's divisibility concept. Pam's idea that "every seventh number is divisible by 7" was a singular exception in the repertoire of this group of preservice teachers. For example, Nicole, after finding the number divisible by 7 in the given interval, was asked whether she could have predicted the existence of such a number without calculating it. Her answer was negative, followed with the explanation: "the further you go, the more they grow apart". This description was accompanied with a hand-waving that indicated progressively increasing intervals. Has the process of repeated addition not been encapsulated?

<u>Divisibility: proof and refutation:</u>

In one of the interview questions participants were asked to consider the number $M=3^3 \cdot 5^2 \cdot 7$ and decide whether it was divisible by each of the numbers 7, 5, 3, 2, 15, 11, 9 and 63. Asking participants to consider the number given in its prime decomposition, we hoped to divert their attention from dividing when checking divisibility and motivate a focus towards the multiplicative structure of M. Still, there were several individuals who calculated the value of M and then performed division by 7. On the other hand, there were individuals who easily concluded divisibility by 7, 5, and 3, since "those were among the factors", but had to calculate the value of M in order to check its divisibility by 15 or by 63. Determining the decomposition of 63 as $3^2 \cdot 7$ and then contrasting the prime exponents of 63 to the prime exponents of M was a strategy used by only five of our participants.

Another point of attention was that the 'proof' of divisibility was in most cases more readily achieved than the refutation. It may seem obvious to a mathematically inclined person to claim that M is not divisible by either 2 or 11 since these are not factors in the prime decomposition of M. More than half of our participants inferred divisibility by 7 or 5 by considering the prime factors of M, but could not infer indivisibility by 2 or 11 using the same approach. For some participants, the question about 2 seemed easier than the question about 11, when they noted that "M is an odd number (as a product of odd numbers), so "2 can't go into it". For them, the mystery of divisibility by 11 remained unsolved unless the actual division was performed. We would like to suggest that these students may not 'believe', or at least not 'believe in practice', in the fundamental theorem of arithmetic that assures the uniqueness of prime decomposition. Most students could quote the fundamental theorem of arithmetic and exemplify how the factors in the prime decomposition of a given number will be the same (but for the change of order) and will not depend on which factor was found first. But when asked about divisibility of M by 11, it was apparent that many students did not over-rule the possibility of a "different decomposition". Another possible explanation for this phenomenon may be that divisibility has been encapsulated to an object whereas 'indivisibility' has not.

Complexity of structure:

Understanding 'divisibility by n' as a generalized object is likely to be proceeded by the encapsulation of separate processes of divisibility for specific numbers. These encapsulations need not occur simultaneously for all numbers. For example, in Nicole's interview we suggest that she may be thinking of 'evenness', or divisibility by 2, as an object; of divisibility by 3 as a process; and of divisibility by 7 as an action. At some point, one may speculate, a critical mass is obtained and a generalization of the process of encapsulating divisibility for specific numbers may serve as a catalyst for the encapsulation of 'divisibility by n'. These issues and others will be the subject of further refinements of our theoretical framework and empirical investigations.

New processes: inverting and coordinating.

One of the tenets of Dubinsky's theoretical perspective is that new processes may be obtained from existing processes by *coordinating* existing processes or by *inverting* existing processes. The tasks presented to participants in the interviews make it possible to observe their constructions and their struggles with coordination as well as with inverting. The process of divisibility by 15 is a coordination of divisibility by 5 and by 3. It was found that tasks involving coordination were problematic for many of our participants. Another major difficulty appeared when participants were asked to do 'reversed tasks'. The ability to check whether or not an object has a certain property appears to be easier than to construct an object that has such a property. In our study, participants who were able to check successfully whether a number was divisible by 15 using simple divisibility rules for 3 and 5, had significant difficulties providing examples of 6 digit numbers divisible by 15 that are quite readily constructible using those very same rules.

When analysed and interpreted in terms of this theoretical framework, response to questions and tasks such as these have worked particularly well in revealing the pervasiveness of procedural attachments even when some degree of conceptual understanding is in evidence.

Educational Importance

We believe with Steffe (1990) and many others that improvement of mathematics education starts with improvement of mathematical knowledge of teachers. Improvement of mathematical knowledge of teachers starts with a deeper understanding of their existing knowledge and its construction. This study provides details, or, using Shoenfeld's terminology (Schoenfeld, Smith, & Arcavi, 1992), serves to provide a finer granularity of knowledge in the domain of divisibility and factorization. Developing a conceptual understanding of divisibility and factorization is essential in the development of conceptual understanding of the multiplicative structure of numbers in general.

References

Ayeres, T., Davis, G., Dubinsky, E. & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education*, 19:3, 246-259.

Breidenbach, D., Dubinsky, E., Hawks J. & Nichols, D. (1992). Development of the process concept of function. *Educational Studies in Mathematics*, 23(3), 247-285.

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical thinking. In D. Tall (Ed.) *Advanced Mathematical thinking*. Kluwer Academic Publishers, Boston.

Dubinsky, E., Leron, U., Dautermann, J., & Zazkis, R. (in press). On learning fundamental concepts of group theory. *Educational Studies in Mathematics.*

Graeber, A., Tirosh, D., & Glover, R. (1989) Preservice teachers misconceptions in solving verbal problems in multiplication and division. *Journal for research in Mathematics education*, 20, 95-102.

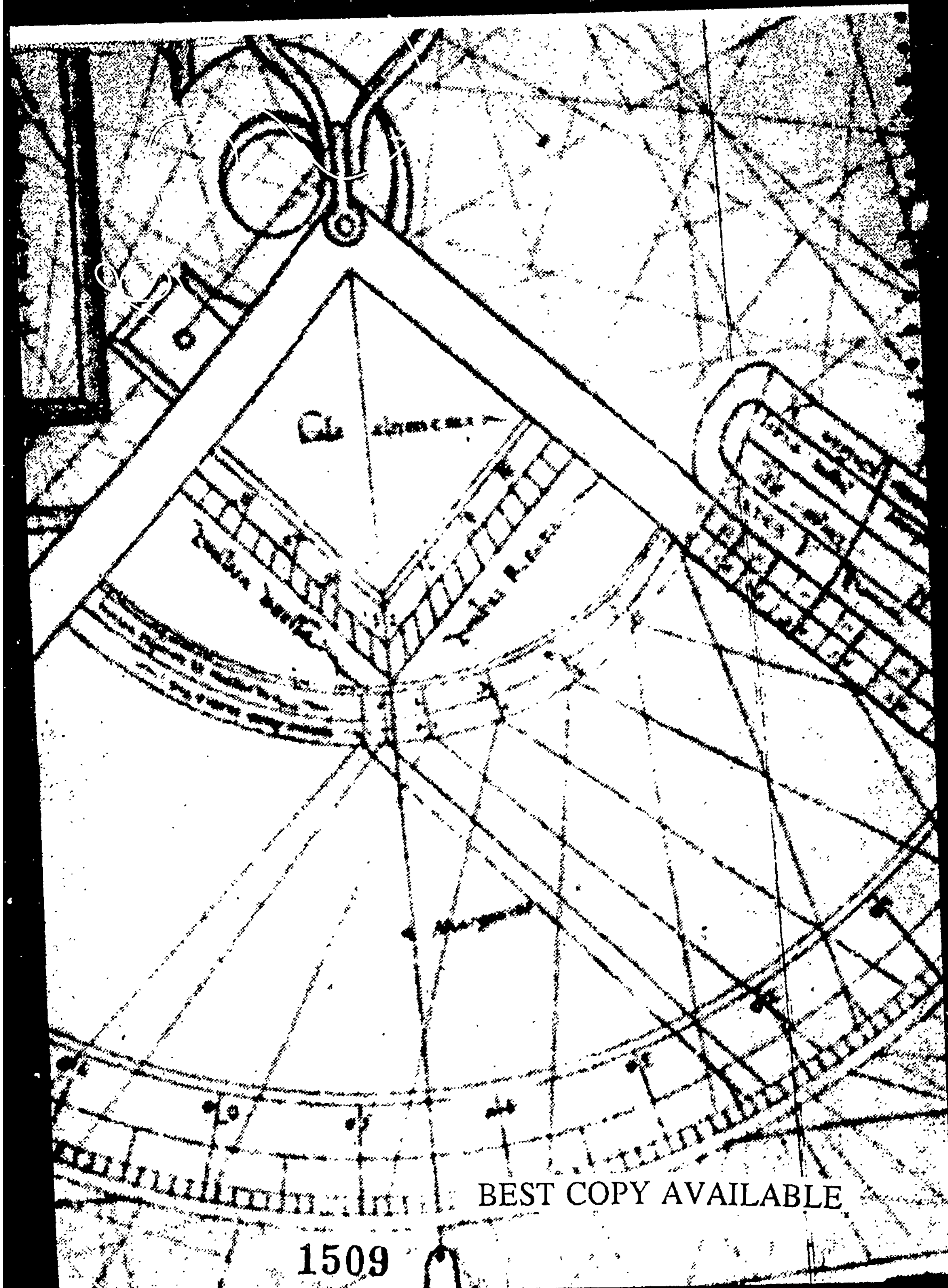
Greer, B. (1992). Multiplication and division as models of situations. In D.A Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning.* Macmillan Publishing Company, New York.

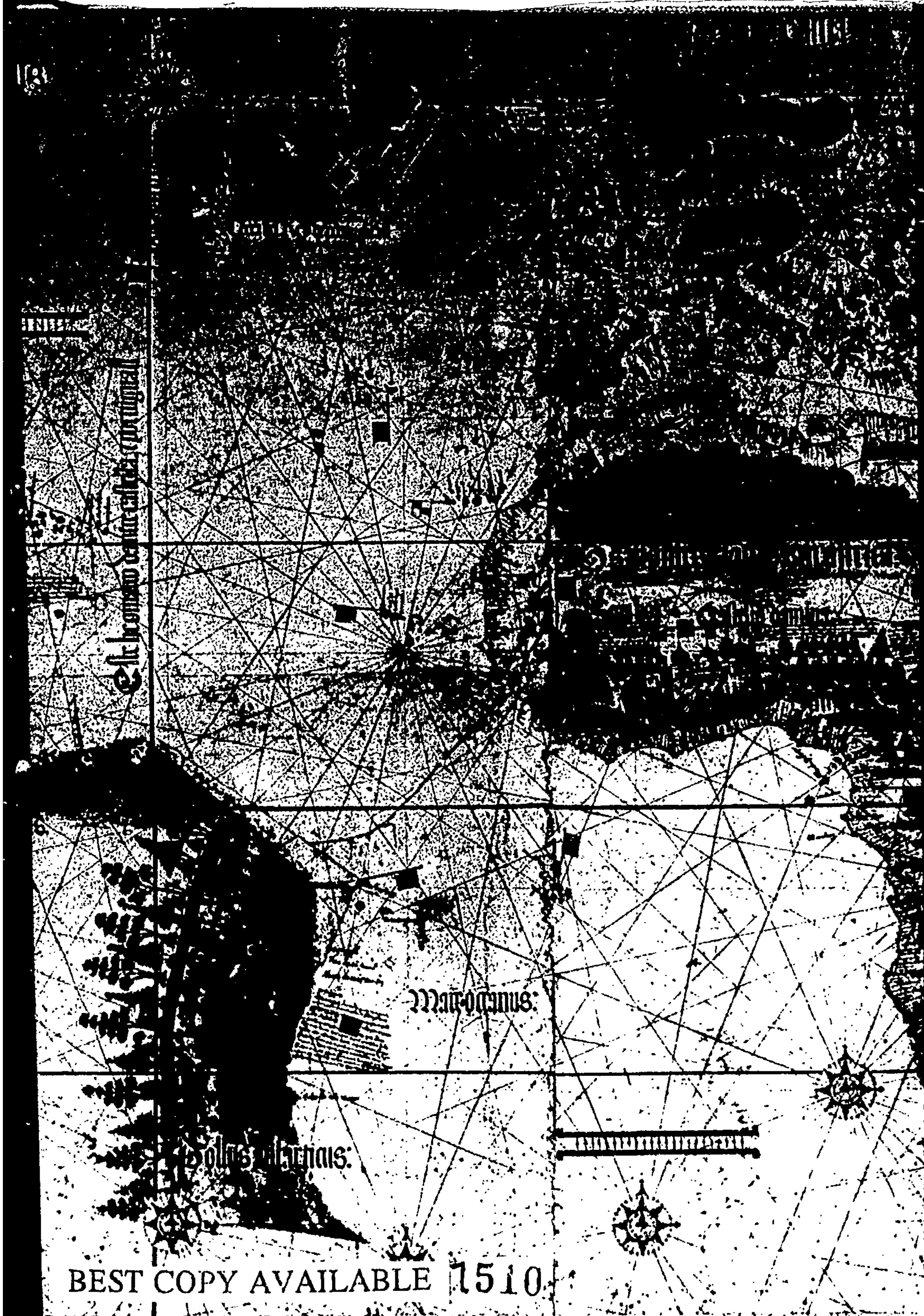
Schoenfeld, A. H., Smith, J. & Arcavi, A. (1992). Learning microgenetic analysis of one student's understanding of a complex subject matter domain. In R. Glaser (ed.), *Advances in Instructional Psychology*, Vol. 4, Hillsdale, NJ: Erlbaum.

Simon, M. (1993). Prospective Teachers' Knowledge of Division. *Journal for Research in Mathematics Education*, Vol. 24, No. 3, 233-254.

Steffe, L. (1990). On the knowledge of mathematics teachers. In Robert B. Davis, Carolyn A. Maher, Nel Noddings (Eds.). *Constructivist Views on the Teaching and Learning of Mathematics.*, JRME Monograph #4, (pp. 167- 184). Reston, Virginia: NCTM. .

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert and M. Behr (Eds.). *Number Concepts and Operations in the Middle grades.* (pp. 141-161).. Reston, Virginia: NCTM.




Mare oceanus
1510