INTEGRATION, COMPENSATION AND MEMORY IN MENTAL ADDITION AND SUBTRACTION

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This paper reports on a study of Year 3 children's addition and subtraction mental computation abilities, and the complexity of interaction of cognitive and affective factors that support and diminish their ability to compute proficiently (accurately and flexibly). In particular, the study investigated the part played by number sense knowledge (e.g., numeration, number facts, estimation and effects of operations on number), metacognition, memory and affects (e.g., beliefs, attitudes). It found that proficient mental addition and subtraction was a consequence of the integration of all factors, but that accurate mental addition and subtraction could occur when some factors were impoverished if there was compensation.

Skilled mental computers are disposed to making sense of mathematics; they use a variety of strategies in different situations (depending on numbers and context) (Sowder, 1994). Thus, proficiency in mental computation involves both accuracy (achieving the correct answer) and flexibility (using a variety of strategies as efficiency requires). This paper looks at why some children are more proficient (more accurate and flexible) at mental addition and subtraction than others. In particular, a fundamental aim was to identify factors, and relationships among factors, that influence this proficiency.

Research on mental computation has proposed specific connections among mental computation and aspects of number sense, in particular, number facts and estimation (e.g., Heirdsfield, 1996). Other research relating to computation (in particular, children's natural strategies) has reported connections with numeration (e.g., place value) and effects of operation on number (e.g., Kamii, Lewis, & Jones, 1991).

Relationships have been posited between mental computation and affects (e.g., Van der Heijden, 1994), where affects cover beliefs (with respect to mathematics, self, teaching, and social context), attitudes (including self efficacy and attribution) and emotions (McLeod, 1992). Beliefs about the nature of mathematics can be manifested in a student's disposition – mastery orientation or performance orientation (Prawat, 1989). In relation to computation, mastery oriented students would aim for understanding and flexibility. Here, monitoring, checking, and planning might be evident. Whereas, performance oriented students would tend to aim to complete a task as quickly as possible, and not attend to understanding and reflection.

Proficient mental computers are flexible in their choice of strategies. Such effortful, reflective and self-regulatory behaviour should involve metacognition (e.g., Sowder, 1994). Metacognition can be considered to have three components: *metacognitive knowledge* (knowledge of own thinking), *metacognitive strategies* (planning, monitoring, regulating and evaluating), and *metacognitive beliefs* (perception of own abilities and perception of a particular domain) (Paris & Winograd, 1990). It is believed that metacognition, particularly metacognitive knowledge, is domain specific or even task specific (Lawson, 1984).

With regard to memory, Hope (1985) argued that superior short term memory was not necessary for proficient mental computation; rather, interest, practice, and knowledge were more important factors. Heirdsfield (1999) found that a superior short term memory was unnecessary for a student who was accurate and flexible with mental addition and subtraction. A well-connected and accessible knowledge base and efficient mental strategies were sufficient for the student. However, Heirdsfield also found that the *mental image of pen and paper algorithm* strategy always used by accurate/inflexible students tended to place heavy demand on short term memory.

Two aspects of memory seemed to be significant for mental computation: retrieval of facts and strategies, and concurrent calculation. Hunter (1978) suggested that the first aspect, demand for retrieval of facts and strategies, is met by long term memory. In his study of expert mental calculators, Hunter posited that these experts not only build up vast resources of numerical equivalents (e.g., number facts and other more complicated numerical equivalents), but also a vast store of ingenious strategies. In this way, complex calculations can be handled more easily by accessing long term memory for facts and strategies, thus eliminating the need for massive calculations and demands on temporary storage. However, his model did not account for the second aspect, concurrent processing of calculations. To encompass this, a model for working memory (Baddeley, 1992; Logie, 1995) consisting of a central executive, a phonological loop and a visuospatial scratchpad has been proposed. The *central executive* provides a processing and co-ordinating function, including information organisation, reasoning, retrieval from long term memory (access), and allocation of attention. The *phonological loop* (PL) is responsible for storage and manipulation of phonemic information, for instance, rehearsal of interim calculations. The *visuospatial scratchpad* (VSSP) deals with holding and manipulating visuospatial information. This may involve representation of numbers in the head, or positional information of algorithms.

In summary, research on mental computation and number has proposed connections among mental addition and subtraction, number sense (e.g., number facts, estimation, numeration, effects of operations on number), affective factors (including beliefs, attributions, self efficacy, and social context in classroom and home); and metacognitive processes. Further, it appeared that memory might have an effect on mental computation.

The study

The research consisted of two studies, a pilot study and a main study. Both studies were based on interviews developed to investigate mental computation (strategies and accuracy) and other aspects that were identified from the literature. The findings of the pilot study informed the main study. For the purposes of this paper, findings of both the pilot and main studies will be combined.

Subjects. The subjects were Year 3 students from two Brisbane independent schools that served high and middle socioeconomic areas. The students (13 in all) were selected (from a population of 3 classes, 60 students in all) after participating in a structured mental computation selection interview. As proficiency in mental computation was defined in terms of both flexibility and accuracy, both these factors were considered when selecting the students. As a result of their performance on the selection items, students were identified as accurate and flexible (4 students), accurate and inflexible (2 students), inaccurate and flexible (3 students), and inaccurate and inflexible (4 students).

Instruments. The students participated in a series of semi-structured clinical interviews that: (1) addressed mental computation strategies, number facts, computational estimation, numeration, and number and operations, (2) investigated metacognition and affect; and (3) administered memory tasks. The number sense, metacognition and affect tasks have been described elsewhere (Heirdsfield & Cooper, 1997). The memory tasks (Lezak, 1995) consisted of: (1) the Digit Span Test, a test of short term recall that requires verbal rehearsal and/or verbal recall; (2) a modified version of a short term retention test; and (3) a mazes test that addresses the central executive, for example, planning and decision-making.

The Digit Span Test specifically addressed the phonological loop (Gatherole & Pickering, 2000). Evidence from research investigating working memory in six- and seven-year old children found that existing visuospatial tests did not actually measure visuospatial memory (Gatherole & Pickering, 2000). Therefore, for the purposes of this study, no specific tests addressing visuospatial memory were administered. However, evidence of this component of memory was sought from observations of students' responses as they computed mentally. In a similar manner, evidence for the utilisation of the phonological loop and the central executive came from witnessing students' rehearsal of interim calculations, students' self-reports of "seeing things/numbers in the head", evidence of planning and choosing strategies, and other elicitations.

Interview procedures. The students were withdrawn from class to a quiet room in the school for the interviews. For most students, the series of interviews took four sessions of twenty minutes each. All interview sessions were videotaped.

Analysis. Students' responses on the interviews were analysed for: (1) accuracy and strategy choice for mental addition and subtraction (which, in turn, was used to determine flexibility, the use of a variety of strategies); (2) knowledge and strategies for numeration, number facts and computational estimation, and knowledge of the effects of operation on number; (3) metacognition and form and extent of affects; and (4) scores and strategies on memory tasks. For the purposes of identifying flexibility, mental computation strategies were classified using a scheme (based on Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995) that divided strategies into the following categories: (1) *separated* (e.g., 38+17: 30+10=40, 8+7 = 15 = 10+5, 40+10+5 = 55); (2) *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55); (3) *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55); and (4) *mental image of pen and paper algorithm* – following an image of the formal setting out of the written algorithm (taught to almost automaticity in the schools the students attended).

Each student's ratings for number sense, metacognitive, affective and memory factors were summarised. These summaries were combined for each of the computation types: accurate and flexible, accurate and inflexible, inaccurate and flexible, and inaccurate and inflexible to produce a composite figure to represent that type. Factors were identified as commonly present (representing), varying and not present for each type. Analysis then moved from within types to across types to identify factors which by their presence or absence would show a relationship to accuracy and flexibility. Analysis focused on what was not present during failure as well as what was present during success.

Results

In this study, accurate/flexible mental computers employed a variety of efficient mental strategies to alleviate demands on working memory, while accurate/inflexible students resorted to one automatic strategy (*mental image of pen and paper algorithm*). Only one other student reported using automatic strategies. She was an accurate/flexible student, but her "automatic" strategies included a variety of efficient mental strategies. Inaccurate/flexible students also employed a variety of strategies (but low level strategies), while inaccurate/inflexible showed little in terms of strategies. This last group of students possessed poor knowledge, metacognition and memory.

Comparing accurate/flexible and accurate/inflexible students' responses, accuracy in mental computation was found to relate predominantly to fast and accurate number facts. Those students who scored poorly in the number facts test (slow and/or inaccurate) were inaccurate in mental computation. This would make sense, as fast and accurate recall of number facts from long term memory would result in less load on working memory, when more complex calculations are involved. Thus, fast and accurate number facts were found to be essential knowledge for accuracy in mental addition and subtraction.

In contrast, comparing accurate/flexible and inaccurate/flexible students' responses, flexibility in mental computation was found to relate to a mixture of factors, to number facts strategies and numeration, and, in part, to understanding the effects of operations on number, and metacognition. Fast and accurate number facts were not found to be related to flexibility (inaccurate/flexible students often had poor number facts). Students who were flexible in mental computation employed a variety of efficient number facts strategies (derived facts strategies) in the number facts test. Some students (particularly those who were flexible/accurate) applied some number facts strategies to mental computation strategies. In the case of flexible/inaccurate students, using derived facts strategies in the test did not help them in mental computation, as derived facts strategies were not used in interim calculations – instead, *count* was often used.

Efficient mental strategies (e.g., *wholistic* and *aggregation*) were found to require good numeration understanding. Lower level alternative mental strategies (e.g., *separation*) also were found to require some numeration understanding (canonical and noncanonical). However, accurate/inflexible students, who tended to use the *mental image of pen and paper algorithm* strategy, did not require the same level of numeration understanding, although a threshold knowledge was essential for procedural understanding (e.g., canonical understanding). Inaccurate/inflexible students were found to have very poor numeration understanding.

The relationship of flexibility to understanding of number and operation was not so straightforward. Students who exhibited good understanding in number and operation were found to employ high-level strategies (e.g., *wholistic*). It appeared that both numeration and number and operation understanding was required for successful employment of the *wholistic* strategy. Research has found that an understanding of the effects of operation on number would be important for efficient mental computation (e.g., Reys, 1992). In particular, understanding how changing the addend and subtrahend affects the result of addition and subtraction examples is the basis of the ability to employ some *wholistic* strategies.

Similar to number and operations, the effect of metacognition on mental computation was mixed. In this study, metacognition was not directly related to either accuracy or flexibility, although accurate/flexible students showed evidence of metacognitive strategies, especially monitoring and checking. Research findings support a relationship with flexibility, that metacognition aids skilled mental computers (e.g., McIntosh, Reys, & Reys, 1992; Sowder, 1994). The reasons for this study not showing a clear relation between flexibility and metacognition might lie in the young age of the students and their lack of metacognitive knowledge (in particular, their unawareness of their metacognitive strategies). On the other hand, the students were able to verbalise their metacognitive beliefs (perceptions of their abilities).

Neither accuracy nor flexibility was found to be related to estimation. This finding was in contrast to the findings of Reys, Bestgen, Rybolt, and Wyatt (1982) and Heirdsfield

(1996). In the present study, even the most accurate/flexible mental computers did not exhibit proficiency in estimation. One reason could be the students were too young to have developed estimation strategies. Estimation is not part of Queensland's present Year 3 syllabus (Department of Education, Queensland, 1991). Heirdsfield (1996) found that, even in Year 4, most students with estimation strategies had developed them from out of classroom experiences.

Exceptional short term recall and retention were found not necessary for mental computation; however, threshold levels were necessary. These findings support those of Hunter (1978).

The expert (mental calculator) goes quite a way to meet these demands (of working memory), partly by the speed and quality of working, and partly by devising calculative methods which evade an excess of interrupted working. (p. 343)

Of course, poor working memory resources might contribute to a poor knowledge base in long term memory and poor connections between this knowledge, resulting in the diminished performance of inaccurate/inflexible students. Further, these students scored poorly on the working memory tasks. Thus, working memory might be a stronger influence of proficiency in mental addition and subtraction than the evidence of this study showed. To check this requires a look at the different tests used in the memory component of the interviews.

The results for Digit Span Test indicated that, for most students (other than inaccurate/inflexible), the phonological loop could support retrieval of number facts from long term memory, and holding and rehearsal of interim calculations. However, inaccurate/flexible students did not have number facts in long term memory, so the phonological loop could not retrieve these. The results from the other tests indicated that the visuospatial scratchpad only supported strategies such as the *mental image of pen and paper algorithm*, which meant there was little evidence of the use of the visuospatial scratchpad for accurate/flexible students, even though it was expected that some numbers would be represented in some visual form. However, because of their age, the students using strategies other than the *mental image of pen and paper algorithm*, might have been unaware of their use of mental imagery (or so preoccupied with their strategies, that they could not remember using any mental imagery).

Conclusions

The study showed that students proficient in mental computation (accurate and flexible) possessed integrated understandings of number facts (speed, accuracy, and efficient number facts), numeration, and number and operation. These proficient students also exhibited some metacognitive strategies and possessed reasonable short term memory and executive functioning.

Where there was less knowledge and fewer connections between knowledge, students compensated in different ways, depending on their beliefs and what knowledge they possessed. Accurate/inflexible students used the teacher taught strategy of *mental image of pen and paper algorithm* in which strong beliefs were held. Combined with fast and accurate number facts and some numeration understanding, their familiarity (almost automaticity) with this strategy enabled the students to complete the mental computation tasks with accuracy. Working memory was sufficient to use an inefficient mental strategy accurately. The visuospatial scratchpad was used as a visual memory aid. The inaccurate/flexible students compensated for their poor number facts and minimal and disconnected knowledge base by using a variety of mental strategies in an endeavour to find one that would enable them to reduce the difficulty of calculation. Although their limited numeration understanding and memory (including executive functioning) were sufficient to support the development of some alternative strategies, these were not high level strategies. In particular, access to *wholistic* strategies was only partially successful. Finally, the inaccurate/inflexible students who exhibited deficient and disconnected understanding tried to compensate by using teacher-taught procedures (similar to the accurate/inflexible students), but they were unsuccessful, as they possessed no procedural understanding and also had poor working memory.

The importance of connected knowledge for proficient mental computation demonstrates the need for teaching practices to focus on the development of an extensive and integrated knowledge base. Students can and do formulate their own strategies, but do not always use them accurately. Therefore, students should be encouraged to formulate their own strategies but in a supportive environment that assists them to use strategies appropriately. Because of memory load, students should be permitted to use external memory aids (e.g. pen and paper) to assist mental computation. This has a second payoff in that efficient mental strategies are, at times, also efficient written strategies. By having students formulate mental strategies, they have to call upon number sense knowledge, thus acquiring connected knowledge while they develop computational procedures. This is in contrast to students using teacher-taught procedures, which require little connected knowledge.

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