MOVING BEYOND PHYSICAL MODELS IN LEARNING MULTIPLICATIVE REASONING

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A key stage in learning multiplication and division is a capacity to move beyond reliance on physical models of problem situations and to form mental images to seek solutions. Some longitudinal data are presented to suggest that young children (5 to 8 years) progress through identifiable key stages in learning multiplicative concepts. One of these key stages is represented by movement away from a need to refer to physical models, and some children progress to this stage within the first three or four years of schooling. A clear finding is that teacher interventions facilitate this progression.

This paper reports results from the Early Numeracy Research Project[1] (ENRP) that examined the effect on student learning of a whole school approach to improvement of teaching and learning (Hill & Crevola, 1998). As a measure of student learning, the project collected data across nine domains of mathematics, one of which was multiplication and division. The data suggest that a key stage in the learning of multiplicative concepts, termed here *abstracting*, presents a significant barrier to many students, but that this barrier can be overcome with teacher support. This key stage, *abstracting*, is characterised by students moving beyond a need to create physical models as a prerequisite to solving multiplicative problems. It is conjectured that the necessary steps include developing a conceptualision of multiplication and division that allows students to deal with different situational contexts (e.g., partition and quotition) and generalising the concepts in a way that prepares them for future learning (Sullivan & Beesey, 2000).

It seems, for example, that there are many students in the later primary years (ages 9 to 12) who can cope with multiplication and division concepts with natural numbers, but who experience difficulty not only with multiplication and division of decimals but also with the very nature of fractions and decimals (e.g., Baturo, 1997). It is possible that the semantic complexity of the question forms and associated physical models used to assist the learning of multiplicative concepts in the early years themselves contribute to these difficulties (e.g., Mulligan & Mitchelmore, 1997; Verngaud, 1988). This paper suggests that students will develop more robust

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[1] The project, titled *Early Numeracy Research Project* (ENRP) was established in 1999 by the (then) Victorian Department of Education as a collaborative venture between Australian Catholic University, Monash University, the Victorian Department of Employment, Education and Training, the Catholic Education Office (Melbourne), and the Association of Independent Schools Victoria. The ENRP Project is directed by Doug Clarke, and the team includes Barbara Clarke, Jill Cheeseman, Ann Gervasoni, Donna Gronn, Marj Horne, Andrea McDonough, Pam Montgomery, Anne Roche, Glenn Rowley and Peter Sullivan.

conceptualisations of multiplication and division if teachers pose problems that gradually but explicitly remove physical prompts or supports, and encourage students to form mental images, in multiplicative situations.

Early Learning of Multiplication and Division

Over the past decade, there has been considerable attention to research on multiplication and division concepts in early mathematical learning (e.g., Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, l989; Mulligan & Mitchelmore, 1997; Steffe, 1994; Wright, Mulligan, & Gould, 2000). Studies focused largely on the analysis of counting, calculation and modelling strategies from children’s solutions to problem solving tasks.

There has also been some emphasis on the importance of developing conceptual structures for multiplication and division (Greer, 1992). In longitudinal analyses of young children's intuitive models for multiplication and division problems, Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular problem did not necessarily reflect any specific problem feature but rather the mathematical structure that the student was able to impose on it. Students acquired increasingly sophisticated strategies based on an equal groups structure, and calculation strategies that reflected this.

The acquisition of an equal-grouping (composite) structure is at the heart of multiplicative reasoning. For example, a composite is a collection or group of individual items that must be viewed as one thing. To understand multiplication and division the child needs eventually to co-ordinate a number of equal sized groups and recognise the overall pattern of composites of composites, such as "three sixes". Steffe (1994) described the demand on students as follows:

> For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over elements of the other composite unit. (p. 19)

The key issue is that co-ordinating these two composite units is complex, and physical models can help initially. Clearly students must move beyond physical models partly because such models do not easily represent all multiplicative situations (e.g., Greer, 1992), and partly because these models become less feasible with large numbers and inappropriate with rational numbers. We suspect that some teachers avoid these difficulties by using limited situational contexts of multiplication and division and continuing to rely on physical models, generally restricted to repeated addition. Since the form of the models is likely to be representative of the problem structure, we argue that it is preferable to encourage students to create models or mental images of a variety of multiplicative situational contexts and to use these models or images in solving the problems. It seems desirable to pose tasks that specifically remove elements of physical models, even within the first three or four years of schooling, and to emphasise movement towards use of mental images.

The Data Collection

In order to explore various aspects of numeracy learning, the ENRP project created a framework of key "growth points" that can be thought of as conceptual signposts on the road to children's development as mathematical thinkers. The focus of interest here is on what the data can tell us about the learning of multiplicative concepts in the early years of schooling.

The source of data was a one-to-one interview over a 30 to 40 minute period with every student in the first three year levels in 35 trial schools at the beginning and end of the school year. Note that the Australian school year is February to December.

Although the full text of the interview involves around 50 tasks (with several sub-tasks in many cases), no student moves through all of these. Given success with a task, the interviewer continues with the next task in the given mathematical domain as far as the student can go with success (see Clarke, Sullivan, Cheeseman, & Clarke, 2000 for a fuller discussion). Many of the interview questions invited the children to solve problems using small plastic teddy bears.

Interviews were conducted by the classroom teacher, who was trained in all aspects of interviewing and recording. As well as moving carefully through the 18-page interview schedule, the teacher completed a four-page Student Record Sheet.

Some Growth Points and Items for Assessing Multiplication and Division

The ENRP project developed sets of growth points in nine domains of mathematics, one of which is multiplication and division. The multiplication and division domain includes seven growth points, only four of which are relevant here.

Growth Point 0 – Not apparent
Not yet able to create and count the total of several small groups.
Growth Point 1 – Counting group items as ones
To find the total in a multiple groups situation, refers to individual items only.
Growth Point 2 – Modelling multiplication and division (all objects perceived)
Can successfully determine totals and shares in multiplicative situations by modelling.
Growth Point 3 – Abstracting multiplication and division
Can solve multiplicative problems, where objects are not all modelled or perceived.

These are presented as a conjectured sequence of development. It is accepted that students can follow different pathways in their learning, but nevertheless the intention is to describe a learning trajectory (Cobb & McClain, 1999) of the majority of the students.

The four questions that addressed these aspects of multiplication and division are shown in Figure 1. It is noted that these questions address only two of the four multiplicative situational contexts proposed by Greer (1992). Multiplicative comparison (I have 3 times as many as you), and Cartesian product (2 cones, 3 flavours, how many possible combinations?) were not included.

23. Teddy cars
Put four matchboxes in a line.

24. Sharing teddies
Show the child the picture of four "teddy

Here are four teddy cars.
Please put two teddies in each car.
a) How many teddies is that altogether?
b) Tell me how you worked that out.
c) *If the child appear to be counting all, ask*: Could you do that another way, without counting them one by one?

25. Dots task

Here are some dots. *Show card (4 x 5 array of dots) for an instant*. I'm going to hide some. *Cover the bottom 3 x 3 section.*
a) How many dots are there altogether on the whole card?
b) How did you work that out?

mats". Put out 12 teddies.
a) Here are 12 teddies. Share the 12 teddies between the four mats so that there is the same number of teddies on each mat.
How many teddies go on each mat?
b) How did you work that out?

26. Teddies at the movies

Here comes another story.
a) 15 teddies are sitting in rows at the movies. The teddies are sitting in three equal rows. How many teddies are in each row?
b) How did you work that out?

Figure 1: Multiplication and division questions.

Materials were provided for the first three questions. The interviewers asked the first two questions for all students, but only proceeded to the latter two if the responses to both the first two were correct.

The instruction provided to the coders for this domain was to rate a student at the:
- *Counting group items as ones* growth point if they responded to the *Teddy cars* and *Sharing teddies* questions correctly;
- *Modelling* growth point if they used a non-count-all strategy in *Teddy cars* and answered *Sharing teddies* correctly; and
- *Abstracting* growth point if responses to *Dots task* and *Teddies at the movies* questions were correct, although count by ones strategies were not allowed.

Indicators of Student Growth

To examine the way the growth points portray the nature of the increasing sophistication of the students' strategies, the following presents a profile of students' achievement over the grade levels. Table 1 shows the percentage of students at their highest achieved growth point both by grade level and overall in the March 1999 interview. "Prep" refers to students about 5 years old, in their first year of school; Grades 1 and 2 are the next years.

Table 1: Students (%) Coded at the *Multiplication and Division* Growth Points (March '99)

	Prep (n=1237)	**Grade 1** (n=1233)	**Grade 2** (n=1168)	**Total** (n=3638)
Not apparent	71	37	12	41
Counting group items as ones	24	26	14	21
Modelling	5	36	66	35
Abstracting	0	1	7	3
Basic strategies +	0	0	1	0

There are significant numbers of students at each of the first three growth points overall and by grade level. It can be inferred that the first four points, at least, are necessary to describe the growth of such students.

To consider whether the growth points represent a sequence, it is appropriate to consider the way the students develop. Given that the questions were asked in such a way that if students made an error in an early item, they were not asked the latter ones, it is not possible to draw inferences on the sequence merely from the percentages of students answering the questions correctly. To allow consideration of the growth, Table 2 presents the ratings of all students in November 1999.

Table 2: Students (%) Coded at the *Multiplication and Division* Growth Points (November '99)

	Prep (*n*=1257)	**Grade 1** (*n*=1225)	**Grade 2** (*n*=1170)	**Total** (*n*=3652)
Not apparent	26	8	2	12
Counting group items as ones	27	12	6	16
Modelling	44	73	61	59
Abstracting	1	6	18	8
Basic strategies +	0	2	14	5

An indicator of the sequential nature of the growth points is the extent to which students progress from one point to subsequent points. Note that a better sense of the growth of the students can be gained by comparing Tables 1 and 2 than merely by comparing across grade levels within either table because the comparisons are between the same groups of students.

At each of the three levels, students progressed through the growth points. Few students in either Prep or Grade 1 progressed to *Abstracting*, and only one third of the Grade 2 students reached that point by the end of the year.

The *Abstracting* Barrier

The project team examined whether the conjectured growth points are sufficient to describe growth within each domain or whether more growth points are needed in between. One possible indicator of the need for an additional growth point could be that students take too long to move from one point to the next. It can be noted that the points as conjectured represent quite major growth stages since it takes the group, on average, just over 12 months to progress one growth point.

Of a total of 3410 students, 841 were rated at *Modelling* in both March and November. This represents 24% of all students, and 70% of the students rated at *Modelling* in March were still rated at that level in November. This implies that these students were able to represent the *Teddy cars* question and skip count or use other multiplicative strategies for calculating the total, and to represent and solve the *Sharing teddies* task, but were not able to answer both the *Dots task* or *Teddies at the movies* in a non count-by-ones manner.

Not only is the next growth point, *Abstracting*, an important goal for most students, it seems also to present a significant barrier. There are a number of components of this barrier. These could include the problem structure, the semantic subtlety of words like each and between, the calculation demand, and the need for the students to form some sort of mental image of the problem statement.

To explore further the nature of the development needed between these two growth points, the following tables present some characteristics of the 841 students who were rated at *Modelling* in both March and November.

One of the possible contributors to the barrier is the counting demand of the tasks. To examine this, Table 3 presents the responses of the students on the *Counting* growth points, from the same interview, for the 841 students rated as *Modelling* on the *Multiplication and Division* domain in both March and November.

Table 3: The *Modelling* students (%) at each *Counting* growth point (n=841)

	March	**November**
Not yet able to count to 20	1	0
Can say number sequence to 20	2	0
Can count a collection of 20 objects	50	13
Counts forwards and backwards by 1s	19	7
Counts from 0 by 2, 5, 10	26	54
Count from x by 2, 5, 10	2	25

These data suggest that over three quarters of these students are able to skip count and one quarter are able to skip count from variable starting points by November. While the *Sharing teddies* task prompts counting by 3, it seems that these *Modelling* students are able to calculate at a level sufficient for either the *Dots task* or *Teddies at the movies*. This suggests that the difficulty with those tasks may be related to the way the students interpreted the questions or their capacity to form the necessary mental images. To explore this further, Table 4 presents the growth points for *Addition and Subtraction* for these *Modelling* students.

Table 4: The *Modelling* students (%) at each *Addition and Subtraction* growth point (n=841)

	March	**November**
Not apparent	1	1
Count all	27	5
Count on	52	39
Count back	18	42
Basic strategies	2	12
Derived strategies +	0	2

To be rated at *Count on*, students find the total of nine teddies that are screened and four that are shown. This appears to require a similar imagining of the elements of the problem as the *Dots task*. Most of these *Modelling* students seemed able to do this.

To be rated at *Count back*, the students answer two questions about subtraction situations posed as stories but not modelled (8 – 3; 12 – 9). This seems to require similar interpreting and imagining of the representation as the *Teddies at the Movies* task. Of the *Modelling* students, over one quarter were able to do this.

It is possible that it is not so much the abstract dimension of the task, or the need to form some mental image, as it is the multiplicative conceptualisation that creates the barrier for these *Modelling* students. In other words, it might not be imagining generally that is required, but imagining of particular multiplicative situations.

A key issue, of course, is the extent to which the particular growth point *Abstracting* is defined and measured appropriately, and whether it has implications for teaching. On one hand, it is possible that there is an interim step between *Modelling* and *Abstracting*. On the other hand, it may be that the step is appropriate but the apparent barrier is as much an artefact of the curriculum and teaching approaches, or even that this is a single step but it takes time.

There are two further investigations that are appropriate to explore these possibilities. The first of these relates to the nature of any interim steps between modelling and abstracting. In examining the questions, it seems that the two division questions represent the growth being posed by the framework. However the growth between the two multiplication questions might have provided some unintended hurdles.

A second possible investigation relates to whether the issue is related to curriculum and teaching approaches. It appears that teachers do make a powerful difference. For each teacher, the number of students who moved from *Modelling* or below to *Abstracting* or above over the course of the year (March to November) was counted. Table 5 presents the number of students per grade, for just the straight Grade 2 classes and the Grade 1 and 2 composites, who progressed beyond this *Modelling* barrier over the year.

Table 5: Number of students per grade moving beyond *Modelling*

Number of students	Grade 1/2 classes	Grade 2 classes
Above 13	0	4
11 or 12	0	0
9 or 10	4	1
7 or 8	5	3
5 or 6	11	3
3 or 4	15	3
1 or 2	14	2
0	4	0

Clearly there is a broad spread. In some Grade 2 classes more than half of the students crossed the barrier, whereas in others it was only a few. It would be interesting to examine the approaches of the more successful of the teachers, in terms of the number of students progressing beyond *Modelling*, and whether this is a result of specific or intended actions on their part. Certainly these data suggest that the barrier is not impenetrable for students at these levels.

Summary and Implications

This paper reported one aspect of a project investigating the learning of mathematics in the early years of schooling. Data from individual interviews with over 3000 students confirmed that the conjectured growth points in multiplication and division represent key stages or goals for students. It seems that the *Abstracting* growth point represents a significant barrier, and that, to achieve this growth point, students need to move towards solving problems without using physical models. Further, students may need experiences both with forming mental images to solve problems, as well as with various multiplicative situational contexts. For students who can solve multiplicative problems by modelling, some specific activities prompting visualisation of multiplicative situations, broadly defined in groups, and arrays, multiplicative comparisons and Cartesian products seem desirable (see Sullivan et al. (2000) for examples of such tasks). Tasks that explicitly remove the materials seem desirable as a first step. It also seems that some teachers are much more successful than others in terms of the number of students who cross the *Abstracting* barrier.

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