Algebraic understanding and the importance of operation sense

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This paper examines young children's ability to generalise from their early experiences in arithmetic. A semi-structured interview was conducted with 87 children who had just completed their first three years of formal schooling. The purpose of this interview was to ascertain their understanding of 'turn arounds'. The results of the interviews indicated that many children are experiencing difficulties in reaching correct generalisations from their classroom experiences. These difficulties seem to be related to incorrect 'sense making', misleading teaching materials, and interference of new learning.

Introduction

With the recognition that students continue to experience many difficulties with algebraic concepts (Third International Mathematics and Science Study, 1998), the focus has moved to introducing algebraic ideas in the elementary grades. Kaput (1999) claims that we need to begin algebra early with an emphasis on sense making and understanding. Algebraic understanding evolves from viewing algebra as a study of structures abstracted from computation and relations, and as a study of functions (Kaput, 1999). Both these themes are believed to be appropriate for young children (NCTM Standards, 2000). As beginning algebra students progress from arithmetic thinking to algebraic thinking, they need to consider the numerical relations of a situation, discuss them explicitly in simple everyday language, and eventually learn to represent them with letters (Herscovics & Linchevski, 1994). This transition involves a move from knowledge required to solve arithmetic equations (operating on or with numbers) to knowledge required to solve algebraic equations (operating on or with the unknown or variable). It is believed that young children should be involved in making generalizations, using symbols to represent mathematical ideas, and in representing and solving problems (Carpenter & Levi 1999). This paper investigates young children's ability to generalise from their early experiences in arithmetic.

Early algebraic understanding

Two aspects are considered to be crucial in the transition from arithmetic to algebra. These are first, the use of letters to represent numbers and second, explicit awareness of the mathematical method that is being symbolised by the use of both numbers and letters (Kieran, 1992). This involves a shift from purely numerical solutions to a consideration of method and process. Yet many students experience difficulties in achieving this transition (Boulton-Lewis, Cooper, Atweh, Pillay & Wilss, 1998). Kieran and Chalouh (1992) suggest a reason for this is that most students are not given the opportunity to make explicit connections between arithmetic and algebra (Kieran, 1992). It seems that knowledge of mathematical structure is essential for successful transition (Boulton-Lewis, Cooper, Atweh, Pillay, & Wills, 2000).

An understanding of algebraic structure is typically derived from knowledge of the structure of arithmetic. In this instance, knowledge of mathematical structure is knowledge about the sets of mathematical objects, relationship between the objects and properties of these objects (Morris, 1999). It is about relationships between quantities (e.g, equivalence and inequality), properties of quantitative relationships (e.g., transitivity and equality), properties of operations (e.g., associativity and commutativity), and relationships between the operations (e.g., distributivity). In a beginning algebra course it is implicitly assumed that students are familiar with these concepts from their work with arithmetic. From repeated classroom experiences in arithmetic it is assumed that by inductive generalisation students arrive at an understanding of the structure of arithmetic. Thus, knowledge of structure is considered to be at a meta-level, derived from experiences in arithmetic. How do classroom experiences impact on students' ability to derive structure?

Previous research has documented ways in which students' arithmetic experiences constitute obstacles for the learning of algebra. Most of this research has focussed on the differences between the two systems, for example, differing syntaxs (Lodholz, 1993), closure (Kieran, 1992), use of letters as shorthand (Booth, 1989), manipulations (Booth, 1989), and equality (Wagner & Parker, 1993). Recent research has begun to focus on the development of young children's algebraic thinking (Falkner, Levi, & Carpenter, 1999), with a focus on children's understanding of equality. This paper adds to this research by considering young children's understanding of the properties of the operations.

The specific aim of this paper was to investigate young children's ability to recognise the generality of the commutative property and to discuss this property in everyday language.

Method

Sample

The sample comprised of 87 children from four elementary schools in low to medium socio-economic areas. The children are all participants in a three year longitudinal study investigating early literacy and numeracy development. The average age of the sample was 8 years and 6 months and all had completed the first three years of formal education.

Interview

Five tasks were developed for the semi-structured interview. The two tasks reported on in this paper probe students' understanding of the generality of the commutative property, which is commonly referred to in our elementary curriculum as "turn arounds". Given the age of the sample and the number of years they had been studying mathematics, it was decided to limit the questions to addition and subtraction situations. By this stage of their schooling all children had completed their formal introduction to the concepts of addition and subtraction and could add and subtract numbers involving tens and ones. Task 1 involves probing children's understanding of 'turn arounds' for addition and Task 2 focuses on gauging whether children saw subtraction 'turn arounds' as being different from addition 'turn arounds'. For each task students were presented with 2 cards (see Figures 1 and 2).

2 + 3 = 3 + 2	**31 + 16 = 16 + 31**

Figure 1 Cards used for Task 1

After the completion of task 1 children were given the two cards presented in Figure 2 and were asked the same sequence of questions (see Figure 3) as for Task 1.

2 - 3 = 3 – 2	**31 - 16 = 16 - 31**

Figure 2 Cards used for Task 2

The script for the interview was as follows:

Script for Task 1 and task 2

Ask

What can you tell me about the number sentences on these cards?
Are they true or not true?

(If the child says that they are true)	(If the child says that they are not true)
Explain why they are true.	**Explain why they are not true.**
Can you give me some more examples?	**Can you give me some examples that are true?**
Describe the pattern in your own word	

Figure 3 Script for the interview

The interviews were audio-taped and the scripts transcribed for analysis.

Results

An examination of the transcripts indicated that the responses to the two tasks fell into five broad categories. The next section describes each of the categories and includes a typical response for each.

Category 1 (Correct generalisation)

For task 1, the child stated that the two cards for addition were true, gave some more examples and clearly explained in their own words how to form such examples. For task 2, the child stated that the two cards for subtraction were false (not true) and gave a valid example of why they were false. A typical response was as follows:

Task 1

What can you tell me about these number sentences.

They are turnarounds

You can really only do turnarounds with addition.

Can you give me some more examples

5	4
+4	+5

Are these statements true

Yes

Can you explain to me what a turn around is?

A Turnaround is when the like say that 5 plus 9 then the turnaround is 9 plus 5 - you just turn the numbers around

Task 2

What about these two?

They are turnarounds.

Are they true?

Not really. *Why*

No can't take 3 from 2 or 31 from 16

Category 2 (False generalisation)

The child stated that all four cards for the two tasks were true, gave some more similar examples and clearly explained in their own words how to form such examples. A typical response was as follows:

Task 1

They mean that it is a turn around.

Are they true or false

True

Can you write some more turn arounds for me

16 + 13 13 + 16

Explain

It's a turn around because if you had 31 and add 16 I could turn it around so that it is 16 plus 31

Task 2

What about these two?

They are both turn arounds.

Are they true or false? True

Can you give me some other examples of turn arounds?

Wrote 18-6

6-18

How would you explain that to someone else?

Now it is take away and if I had 18 over there and 6 over there I would turn it around so that I would have my 6 over there and my 18 over there.

Could you give me a subtraction example?

Subtraction means you take away. Subtraction is like a give away - If I had 12 bricks over there 8 bricks over there I could have 8 over there and 12 here so it is a turn around

Category 3 (Interference of question's format)

For this category the child stated that all four cards for the two tasks were false. A very common reason for this stance was that there should only be one number after the equal sign. A typical response was as follows:

Task 1

Not true

Because the = is meant to go last and the plus first

Because = is in the middle and there is another plus after the equal and it doesn't equal 3 + 2 it is meant to equal one number. You can have 2+3=5

After the equals how many numbers do you think there should be? one

Should be 31+ 16 = should be 47 - you have to take away the plus sign

Task 2

Not true

Because 31-16= 15 and they have a take away sign there and the number 31 at the end of the sum so it should be 31-16=15. It meant to be 2 take away 3 is one but they have 2 take away 3 equals 3 take away 2. But if it was three take away two it would equal one.

Category 4 (Interference of new learning)

The category represents responses where children seem to be experiencing some confusion due to over generalisations of new learning. For example, two children could now take 3 from 2 as they had recently been introduced to 'trading'. Another two posited responses as follows:

Task 1

True

Because you can add them up

What do you mean - can you write another one that is true

A plus or a take away

(wrote) 32+17=

The biggest number is at the top

What about this one here (16+32)

That's not true - you have to add more

Has the first number always have to be the biggest Yes

Is 2 plus 3 true No

Task 2

16 take away 31 and 2 take away 3 are not true because 2 can't take away 3

Because it is too small it is below 3

What about if I wrote down the plus one

16+4 =

2 + 28 =

Can I do the first one - YES

Can I do the second one - NO

Because 2 is too small.

Category 5 (unable to respond to the task)

Each response was coded according to the five categories. Table 1 summarises the frequency of responses in each category.

Table 1 Frequency of response for each category

Category	Category description	Frequency of response
1	Correct generalisation	23
2	False generalisation	24
3	Interference of the questions' format	31
4	Interference from 'new mathematics'	5
5	No response	4

The majority of children failed to reach the correct generalisations.

Discussion and conclusion

The results of this study present three differing themes. First young children certainly seem capable of reaching generalisations, even if these generalisations are mathematically incorrect. Most are engaged in sense making and understanding (Kaput, 1999). For example, Mitchell until recently could not make 'sense ' of 2-3. Children in his class have recently been introduced to the notion of trading. He stated *I can now do these. 2 take away 3 you can't do so you cross that one out (2) and put the 12 up there and then 12 take 3 is 9.* While this is not mathematically correct he felt very satisfied with his response. Young children are also capable of expressing their generalities in simple everyday language, a necessary step for progressing from arithmetic to algebraic thinking (Herscovics & Linchevski, 1994) and in most instances can offer further examples.

Second the types of classroom experiences that children are engaging in seem to interfere with them reaching valid generalisations. In this instance, teaching materials appear to be acting as cognitive obstacles to abstracting the underlying mathematical structure of arithmetic. A number of instances of these obstacles were presented in the data. The format of the cards themselves caused difficulties for some children. Samantha stated *I've never seen them written like this before. I write 31 take 16 = (and mimed the vertical algorithm). I don't know if they are true or false*, indicating that she had never seen expressions written in a horizontal format. The position of the equal sign also caused difficulties (see the response for category 3). Children's understanding of "=" also caused difficulties. Many of the responses in category 3 stated that *2+3 doesn't equal 3* and offered 2+3=5+2=7 as how the first card for Task 1 should be written. This confirms Anenz-Ludhow and Walgamuth's (1998) claim that many children in elementary grades generally think that the equal sign means that they should carry out the calculation that proceeds it and the number following the equal sign is the answer to the calculation. Again, this misunderstanding seems to be caused by teaching materials that predominantly involve getting answers to problems. In another section of the interview many children stated that *'equal means altogether, the answer'*.

Third children's misunderstandings also seem to be based on intuitive assumptions about applying new ideas to other familiar situations and interference from new learning in mathematics. Most of the children in the sample had had some classroom experiences with 'turn arounds' but, as indicated by the responses, these experiences seemed to be limited to addition situations. When presented with the subtraction task man simply applied their new understanding to this situation. Some tried to make sense of this intuitive assumption and changed their response during the interview. For example, Lucas stated *2-3=3-2 is true. 31-16=16-31 is true as well. I am thinking that they may not be true. Because 2 is not higher than 3 and 3 is higher than the 2.* But 36% were convinced that their intuitive assumption was correct. As indicated by the response to category 4, new learning seemed to also cause difficulties for some children. The introduction of subtraction interfered with their understanding of addition and the introduction of trading interfered with their understanding of subtraction.

From this research it seems from their classroom experiences with addition and subtraction young children have already developed misunderstandings with regard to the commutative property. When developing curriculum materials for the early years we must take into account that young children are engaging in sense making. We need to ensure that the ideas and materials we are presenting at this level help children abstract the structure of arithmetic rather than act as cognitive obstacles to future learning. From this study it seems that the misunderstandings these children are experiencing are based on pragmatic reasoning about new notions, the effects of misleading teaching materials and classroom experiences, and interference from new learning in mathematics.

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