REPRESENTING CHANGE

Robert Speiser and Charles N. Walter (Brigham Young and Rutgers Universities); Carolyn A. Maher (Rutgers University)[1]

We report on data from a summer institute for high-school students, part of a 13-year longitudinal study of the development of mathematical ideas. The students were invited to discuss the motion of a cat, given 24 time-lapse photographs taken in less than a second. As students explain, justify*, and convince others, a re-examination of previous explorations, including prior reasoning, is often triggered. The student data are analyzed through detailed examination of how students work with a variety of representations in order to build arguments, through extended social interaction.*

The work presented here is part of an extended longitudinal study of the development of mathematical ideas of a focus group of students, beginning in first grade.[2] The members of the focus group are now first-year university students. The objectives of the study, now in its thirteenth year, are: to provide in-depth case studies of the development of proof making in high-school students; and to investigate the relationship of students' earlier ideas and insights to later justifications and proof building; to trace the origin, development, and use of representations of student ideas, explorations, and insights relating to explanation, justification, and proof building. Within the context of the learning community, additional goals are to investigate the nature of teacher intervention in the growth of student mathematical ideas, and to study individual cognition in the context of the movement of ideas through the community of learners.

Significance. It is generally accepted that traditional approaches to "teach" students how to reason mathematically (for example, proof making) have not been successful. Most secondary teachers know this; few elementary teachers have been able even to address the issue, given the curricula with which they currently work. Robert B. Davis (1994) distinguished between what mathematics students should learn—that is, what "content" or "topics"—and what kind of knowledge students need to develop, domain by domain.

> If one takes seriously the various new suggestions about the teaching and learning of mathematics–-if, for example, one takes seriously the NCTM Standards (1989)—then one is faced with asking teachers to play a quite new role… It will not be easy for teachers to shift to the new role—working alongside students, trying to be aware of the student's thinking, working to help the student modify that thinking in an appropriate way... (Davis, 1994, p. 17)

In our view, specific, detailed knowledge about how students build mathematical ideas is central to successful teaching. To build instruction based on knowledge of

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student thinking requires detailed research into how students reason and communicate under particular conditions.

Theoretical Framework. As students engage in mathematical investigations, they frequently reconsider prior constructs[3] while attempting to make sense of new experience (Davis, 1984; Davis & Maher, 1990; Davis & Maher, 1997). As they explain, justify, and convince others, a re-examination of previous explorations, including prior reasoning, is often triggered (Maher & Speiser, 1997). In this way, learners come to emphasize certain features, for example, of the problem situations under study. Challenges to students (by each other and perhaps the teacher) to explain their ideas may lead to modification and/or rejection of some prior knowledge, or to generalizations which can be supported by convincing arguments. Further, emerging theories can be subsequently modified, extended or refined, often through long-term critical discussions by the learners.

The theoretical framework that guides this study comes from research on what students do when they do mathematics (Davis, 1984; Davis & Maher, 1990; Davis & Maher, 1997), recent work that traces the intricate and complex pathways for individual students' understanding within a larger community (Maher & Speiser, 1997), Kieren and Pirie's Dynamical Theory for the Growth of Mathematical Understanding (Kieren, Pirie & Reid, 1994; Pirie & Kieren, 1994a, 1994b), and some recent work by Dörfler (2000) on the construction of meaning in the course of social interactions.

To investigate the history, development and use of learners' arguments, we identify *events*, i.e., connected sequences of utterances and actions which invite explanation by us, by the learners, or by both. An event is called *critical* when it demonstrates a significant advance from previous understanding (also recorded as events), in the context of an emergent narrative. We may refer to critical events (moments of insight) as conceptual leaps (see, for example, Maher & Martino, 1996b). They are obvious and striking moments, in which learners demonstrate compelling intellectual power, to each other and to us, by having wonderful (Duckworth, 1996) ideas and putting them across in forceful (Maher & Martino, 1996a) ways. In large measure, we see education as creating situations which elicit critical events and then support reflection on their consequences.

Following the same learners for several years, under conditions where instruction has built carefully on what learners have already built, helps us to understand more clearly what students might do cognitively, when proposing justifications that reflect their growth of understanding. It is clear that we need to learn much about the cognitive processes involved in making proofs, especially as students, over time, advance toward higher mathematics.

Background. We build on an existing and extensive data base, and on considerable experience in the conduct of studies of this type. Further, our work connects to and draws from other investigations by members of a broader community of researchers with whom we have substantial common ground. Long-

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[3] More precisely, learners can retrieve and critically examine previously built notations and constructions as new knowledge enters.

term studies of the type that we conduct (and with the depth of data collection we accumulate on the same learners) are almost unprecedented in mathematics education. While there are aspects of this research that are significantly different from other studies, important related research includes work by Susan Pirie, Tom Kieren, Leslie P. Steffe, Andrea diSessa, Alan Schoenfeld and his colleagues, and Jim Hiebert. Further, there are similarities between our presentation of tasks developed by David Page, and methods of Paul Cobb and Erna Yackel. Our interpretation of student performance, in some ways, resembles work by Susan Pirie and Tom Kieren, although, in other respects, our approach is different. Our research supports and then extends investigations of learning which have emphasized the central and essential role of social interaction in the growth of learners' understanding.

In the school year 1998-99, when the research subjects were third-year high-school students, after-school explorations were designed to elicit the construction of mathematical models, the making of connections, and the construction, not only of specific proofs, but of the idea of mathematical proof, at increasing levels of abstraction. In the context of this study, models and proofs were found to emerge from concrete explorations which include demands for careful and convincing explanation. In practice, the drive for explanation requires learners to support their ideas, which often triggers connections across different problem contexts.

In order to help prepare for calculus, the student subjects, during the July 1999 Summer Institute, were offered two extended explorations which drew deeply on precalculus mathematics.[4] We discuss the second exploration here.

Toward Calculus: the Catwalk task. Calculus, historically, arose to study change and motion. In the task we now describe, called Catwalk, one challenge is to build connections between local rates of change and total changes, based on real-world data.[5] The Cat task was designed originally to expose some of the complexity inherent in the use of mathematics to examine motion, and so to open opportunities for students to discuss the issues raised. Work on this task by college calculus students, and by a study group of university faculty, has been reported in three papers by Speiser and Walter (1994a, 1994b, 1996). The questions which arose seem very challenging.

At Kenilworth, the Catwalk was tried with high school students for the first time. In the context of the present study, the task was chosen to elicit rather than simply illustrate the construction of important underlying concepts, such as the distinction, and connections, between average and instantaneous rates of change.

> **The photographs** (Muybridge, 1957, Plate 124). These consist of 24 frames of a single cat, entitled Cat in Walk Changing to a Gallop.[6] The photos were made in 1880 by Eadweard Muybridge, using 24 cameras, activated successively at intervals of .031 second. They show the cat against a background grid, composed of lines

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[4] Only one of the 17 student subjects (Victor) had studied any calculus before the Cat activity.

[5] We emphasize that this example illustrates only one facet of how calculus is useful for such problems, but does so in depth.

[6] The photographs are reproduced in (Speiser & Walter 1994a, 1994b).

spaced 5 centimeters apart. Every tenth line is darker. The 24 photographs show the cat over a total time of action of .71 second.

The task (Speiser & Walter, 1994a, 1994b, 1996). Project the Muybridge photos from the overhead. Distribute copies of the photos to the students. Discuss briefly how speed is measured. Then pose the following two questions: How fast is the cat moving in Frame 10? How fast is the cat moving in Frame 20?

The teaching team at Kenilworth was led by Walter.

Data. The data we present are from July 15, 1999, from the fifth hour of work on Catwalk. Just before the first segment below, the students had built a linear representation of the cat's position, in the form of a horizontal line of masking tape, 65 meters long, in the hallway where the students had their lockers. Along the line, two students marked locations for the cat's position (scaled by a factor of 50) in each frame. Beating a makeshift drum,[7] several students[8] ran the course, striving to reach each mark just as the drumbeat sounded.

In the first data segment, attention focuses on the whiteboard, where Romina's scatter plot of velocity vs. time has been projected. Earlier, she had plotted this graph in her calculator, using data found with members of her group by measuring the Muybridge photos. Matt has just walked to the whiteboard.

Matt: This is time and this is velocity. This is …[points with a marker to the first few scatter points] Each of these…little dots here, these are all his velocities at a certain time… So … [begins to draw a line connecting all the scatter points] If you see how the line goes, that's …

Victor: This is …

Matt: … his climb in velocity.

Victor: … acceleration graph

Matt: Yeah. This is, like… This is, like, his acceleration.

Victor: Alright. That's, uh, vel… its velocity, uh, over time. So then its, like, uh, you know, dt over t. You know, its, like…

Matt: Acceleration.

Victor: … dt squared… Right?

Matt: It shows acceleration.

CW:[9] So, you said you saw that as an acceleration graph, Matt. Could you … could you tell me a little more about that?

Matt: It just …that … It just basically shows you, like, how far … how fast. Like, you can tell, like, [indicating the segment connecting the first seven points] from here to here he goes, his acceleration goes down, and [indicates the steep segment through the next four points] from here to here, it starts to skyrocket, up like that. Then [indicates the next six points] he evens out for a while, then it goes down a little bit and [refers to the next five points] then it skyrockets up again.

CW: Well, where is the acceleration?

Victor: Right there where he lands, he's slowing down, right?

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[7] Roughly 70 beats per minute.

[8] And two of the investigators. At the given tempo, keeping pace was strenuous. Most participating students were athletes. They chose the scale themselves, after rejecting a shorter model.

[9] Charles Walter.

Matt: These lines. [Draws arrows pointing to the two sharply increasing segments of his graph.] This line here and this line here. Where it starts to slant up.

Magda: … he's going at a constant speed.

Matt: At these … At these ones here [indicates the two relatively flat segments] he's going at a constant speed.

It might be helpful to consider first the surprising range of representations now in play.[10] Reference to videotaped evidence,[11] as well as to observers' field notes[12] leads to the following list of nine representations, just for the segments we discuss. For each representation, we define a code in boldface, to facilitate analysis

1. The Muybridge photographs. **Ph**
2. Graph of position as a function of time. **P-T**
3. Romina's graph of velocity, as a function of position, projected on the overhead. [The horizontal axis, in effect, is a scaled replica of the marked line on the floor.][13] **V-P**
4. Romina's graph of velocity, as a function of time, projected on the overhead. [The scatter points for the first few frames, for example, are now further apart than in the previous graph.] **V-T**
5. References to standard mathematical or scientific notations or formulas. **F**
6. Gestures. [For example, Victor's hand, tracing the graph of speed against time, moving from left to right as seen by the viewer.] **Ge**
7. Onomatopoeia. [Sounds which mimic actions.] **O**
8. Mime, enacting motion. **M**
9. Drawing and writing, at the whiteboard, on projected images. **DI**

In the first data segment, above, we see V-T, F, Ge and DI. Romina's graph has anchored Matt's discussion, while Victor's attempt to introduce a calculus notation has been set aside. Matt focuses on changes in the cat's velocity, by drawing on the whiteboard onto which Romina's graph has been projected. Almost immediately after running the linear model in the hallway, momentary speed changes seem especially evident. In our second segment, hardly four minutes later, several students respond to an observer's question about how many steps the cat has taken in the photographs.

Brian: No way he can take four steps in a …three fourths of a second. [referring to the total time of action for the cat photos, .0713 sec.]

Romina: Tch, I don't know. Maybe he did only take, like, a step.

Brian: Three fourths of a second is fast. That's [pauses] starting … now. Now! [Timing three fourths of a second.]

Benny: [At the overhead projector, adjusting the display of the cat photos.] There we go. Now, if you look at the, at the first step. [points to the cat in the first frame.] He begins it, he begins his step right here, and if you look, [points successively to frames 2 through 6] he is lifting his leg up gradually, gradually,

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[10] By representation, we mean *presentation*, but in the very particular sense that the students, by inventing a notation, for example, simultaneously create a structure which the notation represents *for them* by presenting it to them. Our usage is similar to terminology of Dörfler (2000).

[11] The session was filmed by four cameras: three following the students group discussions, and a fourth, hand-held, operated by a documentary film-maker, to record some of the flow of conversation, especially between the student groups.

[12] There were six observers, two per student group.

[13] Square brackets denote researchers' comments.

gradually, gradually and he begins to put his foot down here and then his foot touches base right here [frame 6]. That's one step! Now, [points to frame 7] he begins another stride right here. Gradually lifts his back leg up. You can see. This, this is the foot we're going to judge it by, the back leg. Then [pointing sequentially to frames 8 through 12] he lifts his back leg up again, then he lifts it up higher, higher, higher, then he bends it, bring it down, touches base again [frame 12]. That's his second step! Then as his speed picks up, it starts to take him less time to take a step. So, [points to frame 13] in this, in this third one, his, his other leg, his front leg, picks up, picks up, picks up. We see he touches base here 'cause he's picking up speed, so he touches base earlier, at a quicker time. So, that's his third step! So then he's pickin' up his back leg, back leg, back leg, back leg, back leg. Back leg touches base here and then he begins another stride, right here.

Matt: Basically he takes, like, four steps.

Benny: Four steps.

Victor: All right.

Benny: [Starts back to his seat but stops.] You ask? [retrieves the cat transparency and walks slowly to his seat, his voice trailing off as he moves] Its gone. Its gone, gone, gone, gone! [Laughter.]

Matt: [At the overhead, which projects the scatter plot onto the whiteboard.] Like, this is what, this is his first two steps here, where he's walking at a constant speed. Then all of a sudden, on that third step he gains his speed and this [indicates the flat segment in the middle of the graph] is when he lands. And then that's his speed again.

Benny: True.

Matt: When he pushes off.

Benny: And did you remember what we did with the other graph? Its coming back to me. Yes!

Matt: Oh yeah!

Benny: You remember what we did with the other graph? [Refers to the position-time plot derived earlier from Romina's data.] When we weren't moving, the line was flat. [His hand traces a line in the air, resembling the first segment of the position-time plot.]

Matt: Uh huh.

Benny: So when his foot comes down, he's not movin' at that time. So, that's why the line is flat. That's why you see that gradual line.

Matt: And then when he pushes off again, he accelerates.

Benny: Acceleration then. [Leans to his left. Laughs.]

Matt: Oh, man.

Building from the first segment's discussion, the incremental changes in velocity, as graphed, have been connected to steps taken by the cat, as seen in the photographs. Again, the anchor seems to be V-T, together with Matt's added line (DI), which now extends beyond the given interval of time. Benny works directly from the photos (Ph), summarizing with an onomatopoeic "Gone, gone, gone, gone, gone," as if to sound the rhythm of the cat's steps: first slow, then faster (O). Matt shadows Benny at the scatter plot (V-T) for velocity, checking every step against the graph. Then, suddenly, Benny connects the cat's low velocity before frame 10 (walking) to a prior discussion of position vs. time (P-T). In each of these two data segments, at least four different representations were in play. In the second segment, however, ideas built in the discussion of velocity connect both to the cat photos and the earlier discussion of position. The narrative the students build seems to demand not one but several different visual, numerical and graphic anchors.

Discussion and Conclusion. The next morning, student discussion would begin about frame 10, where the largest change in interval velocity (the spring into the gallop) had already been identified. In this way, grounded in a *global* picture of the motion of the cat, triggered in part by the experience of running, the two questions given in the task were broadened, deepened and then reshaped, to yield a powerful and detailed argument that the velocity at Frame 10 cannot be known, based on what might be gathered from the photographs.[14]

These students build their understanding through collective effort, in which students continually interpret and reframe each others' claims and contributions. This process of interpretation and reframing leads to clearer and more widely shared conclusions. Our data show how students jointly build from prior knowledge, with little teacher intervention, either before or during the discussions we present. Our analysis suggests that mathematical proficiency requires shared experience in social situations that promote both careful reasoning and reflective reconstruction of past knowledge. The students' work presented here is notable for the number and variety of presentations, as well as for its powerful response to the task's questions. We find these students' work astonishing.

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[14] For Frame 20, the situation is different. As several students emphasized, the cat is in the air, the change in interval velocity around Frame 20 is quite slight, and so an average should (unlike Frame 10) suffice.

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