Relationships between embodied objects and symbolic procepts: an explanatory theory of success and failure in mathematics

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In this paper we propose a theory of cognitive construction in mathematics that gives a unified explanation of the power and difficulty of cognitive development in a wide range of contexts. It is based on an analysis of how operations on embodied objects may be seen in two distinct ways: as embodied configurations given by the operations, and as refined symbolism that dually represents processes to do mathematics and concepts to think about it. An example is the embodied configuration of five fingers, the process of counting five and the concept of the number five. Another is the embodied notion of a locally straight curve, the process of differentiation and the concept of derivative. Our approach relates ideas in the embodied theory of Lakoff, van Hiele's theory of developing sophistication in geometry, and the process-object theories of Dubinsky and Sfard. It not only offers the benefit of comparing strengths and weaknesses of a variety of differing theoretical positions, it also reveals subtle similarities between widely occurring difficulties in mathematical growth.

Introduction

The theory presented here builds on work that has developed steadily over the last two decades (Gray, 1991; Tall & Thomas, 1991; Gray & Tall, 1994; Tall, 1985, 1995). But it is not a simple restatement of earlier theories. A simple switch of viewpoint is seen to reveal powerful insight into very different ways in which individuals construct mathematical concepts. To gain insight into this viewpoint, we consider the situation in which *embodied objects* are perceived by and acted upon by individuals. (The precise nature of embodied objects will be discussed in more detail shortly, but essentially they begin with human perceptions using the fundamental senses, and become more mentally based through reflection and discussion over time.) Our viewpoint then compares the developing embodied meanings of the objects and their configurations with other mental constructions relating processes and concepts through the use of symbolism. Our purpose is to compare the meaning embodied in the objects and their configurations on the one hand with process-object abstraction on the other. We seek a theory with the power of both explanation and prediction of the varied nature of cognitive development throughout mathematics. We require a viewpoint that is theoretically sound yet has a simple and practical meaning relevant to the spectrum of practitioners from teachers of young children to university mathematicians.

One of our hypotheses is that the theorised encapsulation (or reification) of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon (which we henceforth refer to as *base objects*). We observe that the embodied configurations are more primitively meaningful than the encapsulated mental objects and yet lack the flexibility and power of the distilled essence of the symbolism that links dually to both mathematical concept and mathematical process. The consequence is that the embodied approach can give fundamental meaning to mathematical ideas but that such embodied representations prove to be complex to handle when they are applied to increasingly sophisticated problems. Progress to more subtle levels of mathematical thinking requires eventual access to the powerful and compact use of symbolism. We observe that a practical, real world understanding of simple mathematics can very well benefit from a focus on the operations on the base objects, and such a perspective is satisfactory, even insightful in everyday situations. However, an *exclusive* focus at this level can act as an epistemological obstacle barring the way to the more sophisticated theory that is required for subtle technical and conceptual thinking. As was observed in Gray, Pitta, Pinto & Tall (1999), those students who consider only their perceptions of embodied objects remain at a more primitive level, whilst those who succeed move on to more sophisticated levels, with an easy movement forward to focus on the symbolism or back to consider the configurations of the base objects. Some of those with a developing hierarchy maintain the full range (being 'harmonic' in the sense of Krutetskii (1976)), others become successful by focussing on the higher levels (increasingly 'analytic' according to Krutetskii), losing contact with the real world and becoming 'formal' thinkers (Pinto & Tall, 1999).

Embodied objects

We take the notion of 'embodied object' to begin with the mental conception of a physical object in the world as perceived through the senses. Examples include a Jersey cow, a hamburger, a paper bag, geometric objects such as triangles, arrays of objects such as the dots on a domino, the drawing of a graph of a function, a Venn diagram, and so on. In addition to our direct perceptions through our physical senses, we also think about what we perceive, compare our sense of one embodied object with another and share these ideas with others. In this way our perceptions take on an increasingly subtle meaning. On the one hand our mental conception may be in the form of a "skeleton" or a prototype, having general properties that provide a basis for communication until the addition of specific properties lead to the particular. (For instance, the word 'dog' may bring to mind a domestic animal with fur that barks; but as we consider new information, such as a dog whose fur is cropped into artistic shapes, we might then home in on a poodle, or to a specific poodle belonging to a friend.) On the other—and this is highly relevant to the development of mathematics—our perceptions may become abstractions which no longer refer consciously to the specific objects in the real world. An example of the latter is the idea of a 'straight line' which is initially seen as a line drawn with a tool such as a ruler that makes it 'look straight'. By talking about the idea we move on to consider a mental concept that is ‘perfectly straight’, ‘having no width’, ‘arbitrarily extensible in either direction’. None of these properties is true of an actual line in the real world, but it is *based* on real-world perception and can only be constructed mentally by building on the human acts of perception and reflection. In this way we see an increasing sophistication in the notion of ‘embodied object’ that begins with sensory perception and is refined in mental thought through the use of language to give increasingly refined precision and hierarchies of meaning. This gives an increasingly sophisticated conception of embodied objects in a general manner which has been specifically described by van Hiele (1985) with reference to geometric objects.

We use the term ‘embodied object’ in a manner similar to the theory of Lakoff and his colleagues who speak of ‘embodied cognitive science’ (Lakoff & Johnson, 1999; Lakoff & Nunez, 2000). However, we note that Lakoff’s theory does not explicitly use the notion of ‘embodied object’—the term does not appear in the index of either Lakoff & Johnson (1999) or Lakoff & Nunez (2000).

Our approach makes a closer analysis of the nature of mathematical concepts and sees a significant distinction between embodied objects (such as a triangle or a graph) on the one hand and the symbols of arithmetic and algebra on the other. The latter symbols act as pivots between processes and concepts in the notion of procept (Gray & Tall, 1994), providing a link between the conscious focus on imagery (including symbols) for thinking and the unconscious interiorized operations for carrying out mathematical processes. In particular, we empathise with Dörfler (1993) who claims that, although he can imagine five *objects*, nowhere in his mind can he imagine a mental object for the number ‘‘five’. From the perspective we are adopting in this paper, we agree that the imagery for the number ‘five’ is not an embodied object, although a mental image of ‘five fingers’ clearly is. This emphasises that thinking involving embodied objects is likely to differ significantly from the kind of thinking involved in the successful development of arithmetic and algebra. However, it does not mean that we cannot call a number an ‘object’ to manipulate, simply that it is not an *embodied* object. In fact, our linguistic use of number as a noun—‘five is a number’—gives it a semblance of being an entity, even though this is no more an embodied object than the gerund ‘running’ in ‘running is good for you’. We refer to a number as ‘it’, we operate on ‘it’ and with ‘it’ in arithmetic and—far more important—the symbol for the number allows us to switch flexibly between mental concept and mental process.

Encapsulation of procedure - process - procept

Gray & Tall (1994) adopted the distinction between procedure and process of Davis (1983, p. 257) whereby the term *procedure* is a step-by-step algorithm in which the individual needs to complete each step before taking the next. A *process* occurs when one or more procedures (having the same overall effect) are seen as a whole, without needing to refer to the individual steps, or even the different procedures. For example, “count-all”, “count-on”, “count-on-from-largest”, “known fact”, are all different procedures for the process of adding two numbers. When the symbols act freely as cues to switch between mental concepts to think about and processes to carry out operations, they are called *procepts*, These can be composed and decomposed at will to derive new facts. For instance, 8+6 may be calculated by decomposing 6 into 2+4, composing 8 and 2 as 10, and 10 and 4 as 14, or as decomposing 8 into 4+4, then recomposing 4+6 as 10, and then the other 4 plus the 10 makes 14. More particularly, it is now relatively easy to see the implications of the distinctions we make between the process of addition and the concept of sum. The former suggests *doing* the arithmetic whereas the latter emphasises a proceptual structure that consists of a theory of related procepts, including the base objects on which the processes act, the symbols as process and object, and the concept image related to the use and meaning of the procepts. Thus, in the example above, the procepts are symbolised whole numbers with the related process of addition; the base objects are initially physical objects, but then become figural objects and later become redundant as they are subsumed in a counting process which is itself compressed into the concept of sum.

It is clear from this discussion that the spectrum of procedure-process-procept is not a classification into disjoint classes; we explicitly mentioned the *ambiguity* of the symbolism as process or concept in the title of our paper (Gray & Tall, 1994). It is a categorization into a spectrum of improving sophistication in which the categories blend one into another, even regressing on occasion to a more primitive case. One of us remembers adding up marks in mathematics examinations and getting to 'know' most of the required facts, yet regressing to add 89 and 2 with a quick count-on as '89, 90, 91'. What matters with the increasing sophistication is that a 'process' usually (but not always) may be performed by a specific finite procedure (a counter-example lies in the general process of convergence to a limit). A 'procept' relates to a thinkable concept and a process carried out by its corresponding procedures.

What is clear, however, is the steady development of entities operated on, from physical objects including fingers, to imagined fingers or configurations of counters, to mentally operations with the number symbols themselves. The increasingly sophisticated arithmetical knowledge developed by children (see Steffe *et al*., 1983) is exemplified by an increasing detachment from immediate experience, the development of different aspects of counting and a change in the form of unit counted. Within four of the counting types, the perceptual, figural, motor and verbal we may see the gradual shift in the nature of the base object from a perceptual unit to a mental embodied object. Cobb (1987), has suggested that it is "only at the level of abstract counting that number words or numeral signify conceptual entities that appear to exist independently of the child's actual or represented sensory motor activity" (p.168). We suggest that it is at this stage that the transition from process to concept can occur that forms the basis for understanding the numeration system. Though the system is straightforward for those who understand it, it remains a source of difficulty for many, particularly when shifted beyond whole numbers and extended to decimals. We suggest that it is the formation and reliance upon embodied configurations in the whole number context that is the basis for this difficulty. The recognition of proceptual structures provides the flexibility.

Sophistication and a spectrum of performance

Figure 1 (expanded from figure 1 of Gray, Pitta, Pinto & Tall, 1999) shows the possible outcomes of different levels of sophistication from pre-procedure through to procept. It shows that a problem requiring only a routine procedural solution will distinguish between failure and success only in terms of the change from pre-procedure to procedure. The availability of different routes at the process level introduces the possibility of alternative methods allowing checking for possible errors in execution, even to an underlying unconscious feeling that something is wrong when an error is made (Crowley & Tall, 1999). The procept level moves to a higher plane where the symbols act dually as process and concept, allowing the individual to think about relationships between the symbols in a manner which transcends process alone.

For example, we may recognise that the procedure 'add 3 to a number and double the result' and the procedure 'double a number and add 6' both give the same process. Symbols can be effective for these two procedures in terms of the expressions $(3+n)\times 2$ and $2\times n+6$, or the more standard notations, $2(3+n)$ and $2n+6$. These all represent the same input-output *process* operating on the (value of) the number n. A student still at the procedure level might find these various expressions and their procedural meanings a considerable barrier to understanding expressions as processes.

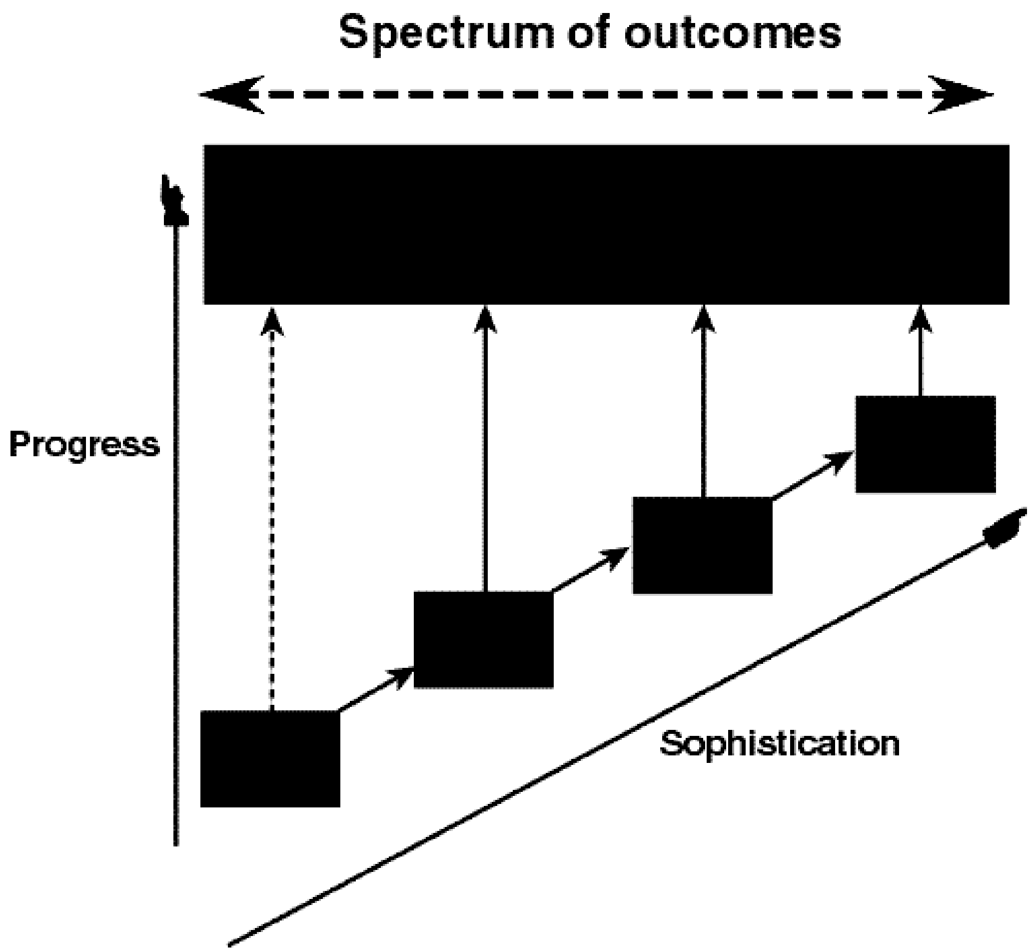

Figure 1: A spectrum of performance (taken from Gray, Pitta, Pinto & Tall, (1999, p.121)).

Refocusing – a possible explanation

How does this process of refocusing from operations on physical objects, to operations on mental entities, to operations *with* mental entities occur. A probable solution is given by Edelman & Tononi (2000, p.57) who report many studies that show that initial problem solving causes activity in a wide range of brain centres, measurable using brain scans, but, as the solution processes become more routine, the required brain areas become fewer as alternatives are no longer required. Edelman and Tononi hypothesise that conscious thought requires a combination of both high correlation between different areas of the brain (which they term 'integration') at the same time as there is a range of possible choices to make (termed 'differentiation'). Routine mathematics becomes unconscious because it requires little differentiation in parts of its activity, only coming to the surface when a particular decision must be consciously made. Thus it becomes possible—but not inevitable—that the focus on the basic objects being manipulated becomes less necessary, and the links, first to inner perceptions, then to increasingly unconscious processes, gives a natural sequence of development for the human brain.

The relationship between embodied objects and encapsulation of processes

It is at this stage that our theoretical positions begin to diverge from both embodied object theory and process-object theory. The former could be a viewpoint in which all mathematical concepts are embodied objects. Such a view fails because the concept of number is not an embodied object, although the concept of 'five things' is. In saying this we misinterpret Lakoff and his colleagues who say that all *thought* is embodied rather than *all we think about* is embodied. However, we consider it important to lay the ghost of the idea that all mathematical concepts are conceived as embodied objects. For several years now (for example, Tall, 1995; Gray *et al* 1999; Tall *et al* 2000), we have been homing in on three (or perhaps four) distinct types of concept in mathematics. One is the *embodied object,* as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. Another is the *symbolic procept* which acts seamlessly to switch from a 'mental concept to manipulate' to an often unconscious 'process to carry out' using an appropriate cognitive algorithm. The third is an *axiomatic concept* in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory. (Here the fourth type of concept might occur by distinguishing between those concepts evolving from embodied objects and those from encapsulated processes (Tall, *et al*, 2000).) Expanding the theory based on 'perception, action and reflection', we see the different kinds of mental entities arising as in figure 2 (overleaf).

We begin by considering the classical situation where the individual performs *operations* on *embodied objects*. We have already considered the case of number concepts where the base objects are physical objects and the encapsulated concepts are number concepts represented by number symbols. Here we note an interesting fact. *Because the counting process operates on physical objects, the seemingly abstract* *concept of number, theoretically formed by a process of encapsulation, already has a primitive existence in the physical configurations of the base objects.* It is by elaborating this simple idea that we come to a distinct view of the role of base objects in the formation of the higher order encapsulated concept. Essentially we see this role of the base objects as a stepping stone to the higher order concept, whilst at the same time having specific meanings for some individuals that act as epistemological obstacles preventing the hierarchical development that is essential for progress to more sophisticated mathematics.

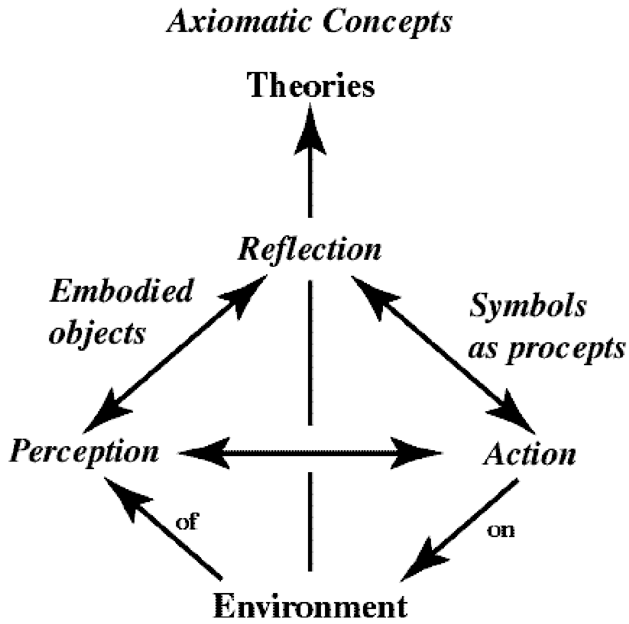

Figure 2: different kinds of mental entities arising through perception, action and reflection

Consider, as a second example, the idea of 'rate of change' and the subtle mathematical process of differentiation and its related concept of derivative. Here we see the picture of a graph as an embodied object that represents the function concept visually. It can be drawn and seen either with a pencil or with a wave of the hand in the air. This embodied action conveys the sense of the changing gradient of the graph as it changes slope. It proves to be natural for many students to develop an insight into the changing gradient by simply 'looking along the curve' and plotting the visual numerical value of the (signed) gradient as a graph. This can be done visually and enactively *without any numerical calculation or symbolic manipulation*. The more formal ideas can come (shortly) after the fundamental embodied activity has been constructed with support from the bodily movement of the individual.

This brings us to our major difference with theories of process-object encapsulation, particularly formulated in the sequence action-process-object-schema (Czarnocha et al, 1999; Sfard, 1991). Our observations of human activity reveal that the 'encapsulated object' is not simply produced by 'encapsulation' or 'reification' of process into object, but is greatly enhanced by using the configuration of the base objects involved as a pre-cursor of the sophisticated mental abstraction.

This is not to say that the higher sophistications of calculus and analysis always remain consciously linked to fundamental embodied objects. They don't. The connections may remain but become unconscious, so that the brain can move on to focus on essential details selected as the basis of axiomatic development. This starts with formal definitions (based on useful, generative, properties) and continues by a *process* of formal deduction of theorems. Each established theorem then becomes available as a *concept* for use in the proof of later theorems. Different students learn formal mathematics in different ways. Many do not develop beyond their existing embodied perception. Some build on their concept imagery, modifying and extending their conceptual hierarchy to grow naturally into the formal ideas. Others have grown in sophistication and no longer evoke conscious links to their embodied sense of the world, extracting meaning from the formal definitions by formal deduction (Pinto, 1998).

In this way we see abstractions rooted in embodied objects of the biological brain providing a basis not only for geometric thought in a developing van Hiele sense, but also a foundation for symbolic process-object encapsulation and on to axiomatic thought.

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