

# THE DEVELOPMENT OF CHILDREN'S UNDERSTANDING OF THE QUOTIENT: A TEACHING EXPERIMENT

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**Abstract:** The conceptual development of the Quotient in four children in the US was studied through a series of parallel individual teaching experiments. At the beginning of the study, children viewed whole number division and fair sharing as two different domains: Division and fractions. By requiring children to confront fractional and division situations involving similar contexts, quantities and symbolization, children moved from a Whole Number Quotient scheme to a Fraction-as-Division scheme. At the end of the study, children were able to think about fractions in terms of division and were able to solve problems involving division of fractions.

## Introduction

Research on understanding children's rational number concepts has been of keen interest to mathematics educators in the last two decades. Children's solutions to problems involving fractions and ratios have shown that children use additive schemes to solve such problems (Lamon, 1993; Resnick and Singer, 1993), and that children have difficulty in conceptualizing the rational number as *a* number (Mack, 1990, 1995; Behr et al., 1985). Children's difficulties with rational numbers can be attributed to the fact that rational numbers take different meanings across different contexts (Kieren, 1976; Behr, et al. 1983). Rational numbers are used to represent and control part-whole relationships; they are fundamental to measuring continuous quantities; in particular, they are involved where quantities are divided; and they are used in multiplicative comparisons of two quantities. Therefore, when we consider the variety of interpretations of rational numbers, it is not surprising that children have difficulties with rational number concepts.

One of these personalities that rational numbers takes on is the quotient interpretation (e.g., Kieran, 1993). Building phenomenologically from the contexts of fair sharing, under the quotient interpretation, rational numbers become quotients of whole numbers. More specifically, a rational number can

be defined as the yield of a division situation. Considering this meaning, the relationship between division as an operation and fractions as quotients becomes evident. It is reasonable to assume, therefore, that interpreting rational numbers as quotients of division situations requires one to have a sound understanding of the division concept. Moreover understanding of the division concept beyond whole number partitioning should address the relationship between quotients as numbers and the operation of division as embodying a multiplicative relationship.

Although the quotient is considered the phenomenological source of rational numbers (Freudenthal, 1983; Streefland, 1991; Kieren, 1993), the question of how children conceptualize the quotient subconstruct still remains unanswered. Within the quotient subconstruct, any fraction can be seen as the result of a division situation that requires the distribution of the quantity represented by the numerator equally into the number of groups represented by the denominator. For example,  $6/4$  can be thought as the quotient of 6 things distributed into 4 equal groups.

Children first learn the concept of division partitioning whole numbers—e.g., fair sharing—and they express quotients as whole number partitions. Division situations with remainders are found to be difficult because instruction fails to engage students in the interpretation of resulting fractional quotients appropriately within the given problem context, instead putting off fractional quotients for later study. However, in research on division, there is a lack of study addressing the relationship between rational numbers and division of whole numbers and whether this traditional separation of whole number quotients-with-remainder and fractional quotients is pedagogically sound practice. Moreover, developmentally, there is little research that documents plausible trajectories by which children come to see the two as isomorphic conceptually (see Behr, Harel, Post, & Lesh, 1992, p. 308 which illustrates the limits of our current understanding in rational number research).

At any rate, current curricular offerings in the United States present whole number division and fractions as separate and distinct topics. Division is treated as an operation one performs on Whole numbers, and fractions are taught almost exclusively as Part-Whole concepts. Only in the late middle grades, when an understanding of quotient becomes necessary for dealing with algebraic entities such as rational expressions, are children expected to connect these two heretofore distinct topics. At no point in the preceding 7 or 8 years is explicit attention focused on teaching fractions as division, nor on teaching division as expressing a rational number: A quotient.

The purpose of this research was to study four fifth graders' conceptualizations of the quotient subconstruct under instructional conditions that expressly required them to confront and connect the isomorphisms inherent in thinking about fractions and in thinking about division. Specifically, it was intended to understand the transitional understandings children constructed as

they moved from division of whole numbers to the description of a fraction as the quotient in division situations. Therefore, a special focus was put on how children conceptualized the remainder as the fractional part of the quotient in whole number division and how they moved to the conceptualization of a fraction as a quotient itself. In short, the research was designed to understand how children come to understand fractions as division, and concomitantly division as a number.

### **Method**

A set of four parallel individual teaching experiments were conducted in studying the development of four fifth graders in an inner city school in the southwest United States. The primary goal of this teaching experiment was not aimed at uncovering how children thought about a given set of tasks, but at determining how children went about making sense of, and structuring their experience towards learning novel content, and in particular, the sense that they made in confronting the subdivision of the remainder of whole number division. The teaching experiment lasted approximately 8 weeks.

One of the students was Anglo, the rest were Hispanic and bilingual. Subjects consisted of two boys and two girls. A clinical interview at the beginning was conducted to determine baseline data and another clinical interview at the end was conducted to determine changes in children's schemes over the entire teaching experiment. Between the clinical interviews, children participated in teaching episodes. The researcher constantly modified the tasks to draw finer-grained details of schemes children were constructing at a particular moment of time (e.g., Steffe & Thompson, 2000).

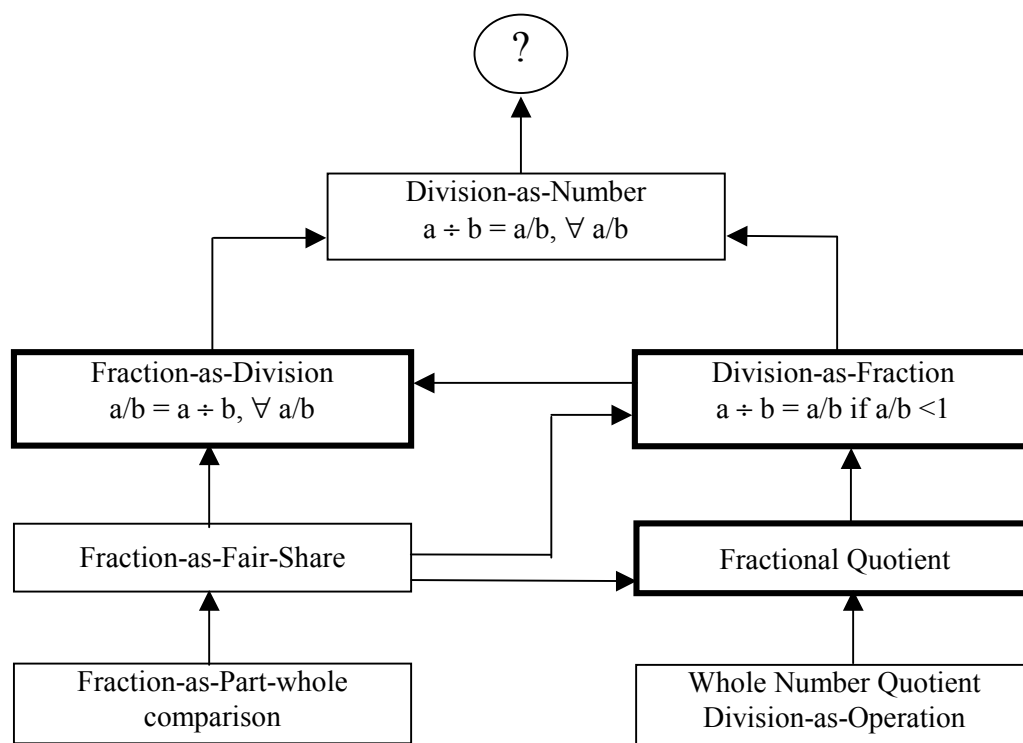
Each teaching episode lasted approximately 60 minutes. Tasks were presented either in the form of a word problem or a division number sentence. The general strategy we employed was to present word problems that involved fair sharing contexts, which we felt would have the highest probability of eliciting quotient thinking in children, wording the problems to guide the students to both fractional and whole number division solutions. By pairing mathematically isomorphic problems that afforded these different interpretations, we forced students to confront their common qualities, and to draw connections between symbols in their respective notational forms. With symbolic problems, we paired problems that involved the same quotient relationship, but that utilized different notation (e.g.,  $13/4$  vs.  $13 \div 4$ ). In particular, we presented problems that required students to deal with the partitioning of the remainder in whole number division, or reconceptualizing the unit in the case of improper fractions.

Each child was interviewed individually. Children were asked to explain their thinking and show their work as they solved the problems. Interviews were audio-recorded. The researcher also took notes during the interviews. Therefore, there were three main sources of the data: Audio-recording of verbal interaction, student's written work and field notes of the researcher. All recordings were

transcribed fully and analyzed qualitatively. Each child formed an individual case. After multiple reading of the data, children's schemes of quotient subconstruct were constructed and generalities across cases were established.

## Results

The results of this study indicated that children constructed four schemes that embodied their progress in conceptualizing the quotient. These schemes appeared to follow a developmental order of 1) Whole Number Quotient, 2) Fractional Quotient, 3) Division-as-Number, and 4) Fraction-as-Division when children progressed toward an understanding of fractions as quotients. Based on the regularities in conceptual development evidenced by the children, and supported through the sequence of teaching episodes, a schematic illustrating how children coordinated these schemes to come to a more full understanding of the quotient subconstruct was generated (see Figure 1). More specifically, it describes children's progress from a disconnected understanding of fractions as representing part-whole comparisons and division representing a whole number operation, toward a more connected understanding of fractions as quotients of division situations.



**Figure 1. Schematic Illustrating Children's Developing Connections Among Fractions and Division Schemes**

In the rest of this section, each scheme will be described in detail.

*Fractions-as-Part-Whole Comparisons.* At the beginning of the teaching experiment, all four children conceived of fractions as representing part-whole relationships exclusively. When asked to describe the meaning of a number presented in fractional form (e.g.,  $\frac{1}{2}$ ), they attempted to describe the given fraction in terms of part-whole relationships by partitioning a whole into the

denominator and coloring the number of parts specified by the numerator. Beside part-whole comparisons, they were unable to provide a different explanation for the given fraction. Because in part-whole comparisons, the unit whole is limited to one, three of the children had difficulty in identifying the whole to explain a given improper fraction. One of these students, Juan, referred to these fractions as “bad fractions.”

*Whole Number Quotient Scheme.* The Whole Number Quotient scheme appeared when children tended to provide whole number quotients for all division situations. It was also evidenced by their inclination to write quotients with remainders even when they converted the remainder to a fraction. In cases when the problem situation allowed the equal distribution of the remainder among the groups, they avoided writing fractional quotients as answers to division number sentences. It appeared that when they conceived of a situation as division, they provided the quotient with remainder even when they realized the possibility that the remainder could be further subdivided.

*Fraction as a Fair Share.* The Fraction-as-Fair-share scheme was illustrated by children’s ability to solve given fair sharing problems in which the answer was less than one. In such situations, children were able to find the answer by partitioning the given quantity. However, when they were asked to write a number sentence representing their solution, they were unable to do so. Transcripts indicate that children did not initially conceive of fair sharing situations as cases of division even though they easily found the fractional quotients through partitioning. It was also evident that they held a conception of division in which the dividend is always bigger than the divisor. In such conception, the quotient of a division situation is never less than one. This was also supported by children’s tendency to reverse the dividend/divisor order in solving symbolically represented problems in which the dividend was less than the divisor.

*Fractional Quotient Scheme.* The Fractional Quotient scheme (which appears to be a key transitional phase) was characterized when children started to symbolize fractional quotient situations as answers to “division problems.” Children were comfortable with fractional answers when solving partitive word problems. However, for symbolically represented problems, they provided fractional quotients only after the problem was put in a partitive context. They started to conceptualize fair sharing situations as division because they recognized that fair sharing situations involved the creation of equal “groups” (i.e., partitions) similar to whole number division.

*Division-as-Fraction Scheme.* The Division-as-Fraction scheme was illustrated in children’s ability to anticipate the quotient of a given division situation without applying any analytic way of solving it. Children came to this understanding after they started to symbolize fair-sharing situations with an answer less than one. This notion can be summarized as: If any a quantity was divided into b equal groups then the resulting quotient is always  $a/b$ . Hence,

when a fair sharing problem was given, children first anticipated the answer using this rule. Once the answer was anticipated, they then were able to solve the problem conceptually. When children didn't get the same answer as the one they had anticipated, they had to verify whether their answer and the anticipated result were equivalent or not.

*Fraction-as-Division Scheme.* The Fraction-as-Division scheme was revealed when children reasoned about fractions in terms of division situations. For example, this scheme entails that  $\frac{3}{4}$  can be considered as 3 things divided into 4 equal groups, equivalently 6 things divided into 8 equal groups and so on. Children began to use this reasoning as a parallel explanation to Part Whole to explain what a given fraction meant. Under this scheme, children were able to reverse the Fraction-as-Division scheme. In other words, without probing, they were able to realize that if any division situation could be represented as a fraction, the reverse was also true.

*Division-as-Number Scheme.* This scheme was illustrated by the children's ability to generalize the two way relationship between a division situation and the fraction representing its quotient to all division cases. In other words,  $a \div b$  was equal to  $a/b$  for every  $a > b$ , where  $b \neq 0$ . However, only Kate constructed this scheme. The rest of the children did not generalize this relationship to all division cases. These children tended to write given any improper fraction as a mixed number so this hindered their ability to recognize this relationship.

## **Discussion**

Previous research has shown that young children deal with quotient situations- fair sharing- easily (Lamon, 1996; Kieren, 1988). In a fair sharing situation, a typical behavior is to partition quantities and write the resulting fractions as the quotient of a given situation. From their partitioning behavior, it is concluded that children conceptualize those fractions as quotients. The results of the present teaching experiment, however, suggest that ability to partition quantities into its parts didn't necessarily indicate that children actually conceptualized the resulting fractions as quotients. This was evidenced by the general reluctance in symbolizing quotients less than one as division number sentences.

Children were reluctant to symbolize the situations in this manner for a variety of reasons. First, for them, "division" always yielded a whole number quotient with or without a remainder. It was not that they didn't recognize the partitionability of the remainder, it was the fraction, the result of the partitioning that they didn't conceive of as a part of the quotient. Typically, in their minds, "fractions" resulted from fair sharing situations which involved cutting or splitting of whole quantities into parts less than one, whereas "division" resulted from partitioning quantities into groups with number greater than one. Second, students' resistance to consider a fraction as a quotient was due to the fact that children had such a strong understanding of the Part-Whole subconstruct

which, from prior experience, was the only way they were used to thinking about a fraction.

Moreover the existence of an understanding of fractions representing fair shares and part-whole relationships and division as a whole number operation as separate entities does not necessarily imply that children will connect these understandings into a coherent quotient scheme without intervention. Rather, the results of the teaching experiment showed that some explicit connections have to be made between these concepts. By making the commonalities in context and notation between whole number division and fraction situations problematic, children need to be encouraged to reflect on the equivalency of fair sharing situations and whole number division. Using a common symbolization; children's conceptions of division can be challenged. In this teaching experiment, division number sentences were used to show that the result of a division operation is a fraction represented by the common divisor/dividend notation).

The findings of this study suggest that a fragmentary approach to teaching quotients, which arranges instruction into two distinct and separate conceptual dimensions, i.e., division as an operation and fractions (as primarily part-whole quantities, but also applying to simple fair sharing), may lead children to develop a fragmentary understanding of the quotient. Instead, the present cases suggest that providing common problems in both fractional and division forms, and confronting children with the basic equivalencies of these forms may be fruitful in developing a more powerful understanding of the quotient subconstruct. In addition, the commonalities among cases in this study suggest that instruction on quotient should be cognizant of how the linkages between division as an operation and fractions as numbers are made. In particular, the order children followed as they moved towards the quotient subconstruct suggests that both whole number division and fraction instruction can proceed fruitfully in concert, intertwined but not parallel.

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