THE CONCEPT OF PARAMETER IN A COMPUTER ALGEBRA ENVIRONMENT

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Abstract

The study that is presented here concerns the learning of algebra in a computer algebra environment and, more specific, the concept of parameter. Students of 14 – 15 years old used a TI-89 symbolic calculator during a five week period. They studied the parameter in different roles such as placeholder, changing quantity and generalizer. The results indicate that using parameters in the computer algebra environment requires a clear view on the roles of the different letters. Also, the reification of a formula seems to be important for an appropriate instrumentation.

The research project

The study presented here is part of an ongoing research project called ‘The learning of algebra in a computer algebra environment’. The general research question of this project is:

> How can the use of computer algebra promote the insight in algebraic operations and concepts?

This question is specified in two sub-questions:

1. How can the use of a computer algebra system contribute to a higher level understanding of parameters as they appear in algebraic expressions and functions?
2. How does instrumentation of computer algebra take place and what is the role of the relation between machine technique and mathematical conception?

Why parameters?

A reconsideration of algebra education is currently taking place in the Netherlands as well as in many other countries. The transition from the informal, reality-bound algebra at the lower secondary level (students of 12 – 15 years old) to the more formal and abstract algebraic skills that are required at the upper secondary level is difficult for many students in the Netherlands.

The conception of the parameter can be a suitable topic to try to bridge this gap. Variables and parameters are in the heart of algebra. The parameter is an ‘extra’ variable in an algebraic expression or function that generalizes over a class of expressions, over a family of functions, over a sheaf of graphs. The parameter can be considered as a meta-variable: the a in $y = a.x + b$ can play the same roles as an 'ordinary' variable, such as placeholder, unknown or changing quantity, but it acts on a higher level than is the case for a variable. For example, a change of the parameter value does not affect one single point locally, but the complete graph globally. The different roles of the variable are resurfaced, but now at a higher level, and the generic function becomes the object of study. The concept of parameter, therefore, is adequate for enhancing the abstraction of concrete situations, so that the more formal and general algebraic representation can become a natural part of the students' mathematical world.

Why computer algebra?

Taking into account the affordances of technology in general, and the algebraic capacities of computer algebra in particular, it seems obvious to use a computer algebra system (CAS) for the purpose mentioned above. It can serve as a powerful and open algebra environment that allows students to concentrate on the concepts and the problem solving strategy. We conjecture that performing procedures in the computer algebra environment contributes to the development of insight in algebraic operations and concepts such as substitution and the distinction of the different roles of letters. Using the machine, the students don't have to worry about the calculations and this may enhance a more global conception of the problem solving procedures.

On the other hand, however, computer algebra can be demanding in its use. Guin and Trouche (1999, p. 205) pointed out that an adequate use of computer algebra tools requires making explicit the different roles of the letters to a further extent than is the case for paper and pencil work. This kind of explicitness that computer algebra demands is a burden but in the mean time it can stimulate the student to handle the operations more consciously.

Furthermore, from the perspective of Realistic Mathematics Education the integration of computer algebra is not a trivial matter. In Drijvers (2000) the issue is raised whether the development of informal strategies and the process of vertical mathematization, that are so important in this educational theory (e. g. see Gravemeijer, 1994), are stimulated in a computer algebra environment. As far as the informal strategies are concerned, it seems that the computer algebra environment does not support them. For vertical mathematization the CAS seems more appropriate.

Previous research and theoretical framework

Much research has been done into the concept of variable. Usiskin (1988) classified the different roles of variables as unknown, indefinite, generalized number, dynamical variable and parameter.

Not so many studies have been published on the learning of the parameter. Bloedy-Vinner (1994) stresses the hierarchy of substitution and the implicit quantifier structure that is often involved while using parameters. These ‘hidden quantifiers’ also are described by Furinghetti and Paola (1994), who state that parameters are conceptually more difficult than variables.

Several studies have been devoted to the use of computers for the learning of the concept of variable. Graham and Thomas (2000) successfully used the graphing calculator to stress the placeholder-role of the variable. Boers-Van Oosterum (1990) also claimed to improve the conception of the variable by using different software packages. Brown (1998) used a computer algebra environment for generalization of patterns, for solving equations step by step and for solving number problems. None of these studies used information and communications technology tools (ICT-tools) for the learning of the concept of the parameter, as is done in the project described here.

An important part of the theoretical framework of this study is the theory of the *instrumentation* of ICT-tools. Following Guin and Trouche (1999) and Lagrange (1999), we consider the development of instrumentation schemes as a crucial and non-trivial step in the appropriation of an ICT-tool. In the acquisition of such schemes, technical skills and mathematical conceptions are interwoven. Examples of this complex relationship can be found in Drijvers & Van Herwaarden (in press).

A second part of the theoretical framework concerns the dual character of mathematical concepts, that have both a procedural and a structural aspect. Sfard (1991) uses the word *reification* for the gradual development of a process becoming an object. In the function concept, for example, the process of calculating function values may develop into the image of a function as an object that is represented by a formula or a graph. Sfard and Linchevski (1994) elaborated this for the case of algebra. Close to the idea of reification is the encapsulation that is described by Dubinsky (1991). Dubinsky states that encapsulation of processes into objects is an important step in reflective abstraction. He suggests that performing processes using a computer may stimulate its encapsulation. As a third means of representing the bilateral nature of mathematical entities we mention the procept that has been developed by Gray and Tall (1994). ‘Procept’ is a contamination of process and concept. The authors stress the flexibility that learners of mathematics need in order to be able to deal with the ambiguity of mathematical notations. In 3+5, the + may be an invitation to perform the process of addition, whereas in $a+b$ the + is only a symbol that defines the object ‘the sum of a and b’.

It is our conviction that the theories of reification, encapsulation and procept are very relevant to the learning of algebra and to the instrumentation of computer algebra tools. For the students, the mathematical entities in such an environment may tend to have a structural character, whereas the processes are more distant to the objects than is the case with work using the traditional paper-and-pencil.

Research design and methodology

As a research paradigm, the developmental research method was used (see Gravemeijer, 1994). According to this methodology, the researcher tries to develop (local) instruction theories by means of constructing and developing thought experiments and educational experiments in the classroom situation. This involves a cumulative process of consideration and testing.

The research method was mainly qualitative. The most important data consisted of audio recordings and field notes of classroom observations, audio recordings of mini-interviews with students, and written work of the students. The data were analysed by coding the classroom incidents and solution methods according to previously defined categories. While doing so the most dominant categories came up clearly.

Development of the didactical scenario

A conceptual analysis of the phenomenon parameter led to the identification of three essential steps in the learning trajectory: the parameter as a placeholder, as a changing quantity and as a generalizer. The table below summarizes these steps. Also, it indicates by means of what kind of activities the students are supposed to pass to the next phase and how computer algebra supports these activities.

parameter role	*a in y = ax+b*	*graphic model*
placeholder	*a* contains specific values, one by one	one graph, that can be replaced by another
changing quantity, 'sliding' parameter	*a* walks through a set dynamically	'comic' of the dynamic graph
generalizer, 'family' parameter	*a* represents a set, generalizes over situations	a sheaf of graphs

student activity	*CAS function*
systematic variation of parameter values	solve equations substitute animate graphs
generalization of situations and solutions	graph sheafs solve parametric equations

Classroom experiment: aim and situation

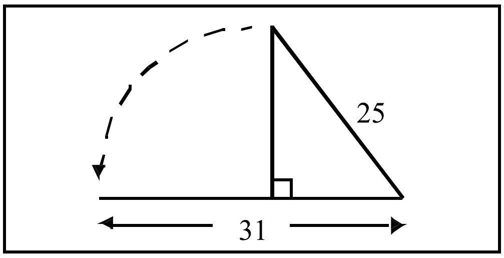
The aims of the classroom experiment were to investigate if the students' conception of parameter would develop according to the didactical scenario and if computer algebra serves as an aid for this. Also, we were interested in the process of instrumentation of the computer algebra tool with respect to the different roles of the letters involved.

The classroom experiment took place during a five week period in the spring of the year 2000. The subjects were 50 students of 14 – 15 years old, divided into two classes. The students were high achieving in general but not specifically in mathematics. As computer algebra tool the TI-89 symbolic calculator was used at school as well as at home. Each class had four mathematics lessons of 45 minutes each week. Because the students had no previous experience with technology such as graphing calculators, using this type of handheld technology was really new to them. Their knowledge of formal algebra was quite limited. For example, the general solution of a quadratic equation had not been taught so far, so the help of the symbolic calculator was needed in case such an equation was encountered.

Episodes of student behaviour

Classroom observations indicate that it was important that students are aware of the different roles of the letters, especially if there are parameters in the equations. We saw that some students found it natural to use parameters to generalize a relation or procedure, whereas others seemed to be confused by several letters in one expression, each having a different role. The differences between the students were considerably, as is shown in the following episodes, that concern the phase when students use the computer algebra environment to solve (systems of) parametric equations.

John and Rob work at the following assignment:



The two right-angle edges of a rectangular triangle together have a length of 31 units. The hypotenuse is 25 units long.
a. *How long is each of the right-angle edges?*
b. *Solve this problem in case the total length of the two edges is 35 instead of 31.*
c. *Solve the problem in general, that is without the given values of 31 and 25.*

At question ***c*** John and Rob wrote down in their notebooks:

$$a^2 + o^2 = p^2$$

$$o + a = s$$

$$o = s - a$$

$$a = s - o$$

Then they entered into the TI-89:

$$\text{solve}(o^2 + a^2 = p^2 \mid o = s - a, o).$$

The wrong letter at the end. The response of the machine was:

$$0 = -s^2 + 2.a.s - 2.a^2 + p^2$$

The boys corrected this by solving this a second time, now with respect to a:

$$\text{solve}(0 = -s^2 + 2.a.s - 2.a^2 + p^2, a)$$

This time John explained the choice for the letter a at the end:

"You want to know the a".

This way they found the solution for a expressed in the parameters s and p.

John and Rob introduced the parameters themselves and generalized the problem solving strategy of the concrete cases of questions ***a*** and ***b*** without difficulties. Their solution schema did not seem to be confused by the presence of the parameters. However, there were some instrumentation problems at the start that may be related to a limited consciousness of the roles of the different letters.

For others, however, the presence of parameters was an extra complication, as illustrates the following observation of Sandra. The assignment was this time to calculate the dimensions of a rectangle with given perimeter and area. Using the viewscreen Sandra tried to demonstrate to the class how the corresponding system of equations could be solved:

$$b + h = s$$

$$b * h = p.$$

First Sandra entered a re-written version of the first equation in itself:

$$\text{solve}(b + h = s \mid b = s - h, b)$$

The machine replied: true.

Rob commented: "She did not use the p, she did not use the second equation."

Sandra changed the command into:

$$\text{solve}(b + h = 20 \mid b = s - h, h)$$

Before the generalization the value of the parameter s had been 20, and apparently she felt the need to return to the concrete case. The result, $s = 20$, is logical but Sandra does not notice that. Then she realises that Rob was right, but she entered $b + h = p$ instead of $b * h = p$:

$$\text{solve}(b + h = p \mid b = s - h, h)$$

The machine replied $s = p$, which is also reasonable. At the end Sandra entered

$$\text{solve}(b + h = s \mid b = p/h, h)$$

and that gave the right answer.

Sandra's behaviour gives the impression that the parameters were an extra, complicating factor in the problem solving process. In earlier situations she had shown that she was able to apply this solution scheme correctly in concrete cases without parameters. It is not clear whether she really perceived a formula such as $b = s - h$ as an object that can be substituted.

Conclusion

The conclusion of the above exemplary episodes is that the didactical scenario to use parameters for generalization was confirmed by some of the students such as John and Rob. Computer algebra was helpful to clarify the problem solving strategy. For others, the use of parameters complicates the situation to a greater extent. In order to be able to solve parametric equations, it seems to be important that students are able to perceive formulas as objects. If a formula invites to a calculation process in the eyes of the student, s/he will find a general solution containing parameters hardly satisfying. Some students were able to distinguish the roles of the different letters, whereas others seemed to be confused by the amount of variables.

The conjecture that performing procedures in the computer algebra environment would enhance the understanding of the global mathematical conceptions behind the procedures was confirmed only to a limited extent. Some students did not overcome the difficulties with the instrumentation scheme for solving systems of parametric equations. The equilibrium between paper-and-pencil work and machine work during the instrumentation process may be quite delicate, and we probably did not pay enough attention to the 'traditional' approach. Meanwhile, the teachers reported that they could benefit from the students' machine experience while treating the solution of quadratic equations after the experiment.

As a consequence for teaching, we would conjecture that the development of both the technical and the conceptual side of the instrumentation schemes deserve explicit attention. This can be done by means of student interaction, classroom discussions and demonstrations, so that the instrumentation process gets a more social character.

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