Learning environment for mathematics in school: towards a research agenda in psychology of mathematics education

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1. *A little historical introduction...*

In February 16th 1630, a Dutch naval fleet arrived at the coast of Pernambuco, by this time a province of Brazil, owned by Portugal. The Dutch army took possession of Pernambuco and three other neighbor-provinces; in Pernambuco, the historical village of Olinda (by this time the capital of the province) was easily dominated, but Dutch military responsible soon realized that this village would be very hard to keep military, because of its topological irregular relief with many hills. Because of this, Dutch governor Waerdenburch decided to burn Olinda, and to move to a small village of fishermen at about 10 kilometers at the south.

Three aspects pleased the Dutch and made this specific village their final choice for establishing a fortified military nucleus: it was topologically regular, had a very good natural sea port and was situated at the estuary of two rivers, Capibaribe and Beberibe. Besides, the medium height of the village considering the sea level was negative in some regions, what made it very familiar to people coming from Holland, “the country stolen from the North Sea” (Gonsalves de Mello, 1978).

Count Maurício de Nassau is an important character in this historical scenario: he arrives at Pernambuco in 1637, as a general governor sent by Dutch Western Indians Company. Once arrived, he decides to build a real town, *his* town, in the village occupied by the Dutch after the burning of Olinda. This town took the name of “Mauritsstadt” or “Stadt Mauritia”, the first Dutch name of the village of Recife, today capital of the state of Pernambuco. Among many important initiatives in economy and political administration, Count Maurício de Nassau was also interested in public education, since he believed that the Dutch domain over Brazilian lands could not be completely established without an educational effort addressed to Brazilian natives (and Portuguese descendents born in Brazil). Because of this point of view, Nassau founded the very first public school of Recife (by this time Mauritsstad), open to Dutch children (descendents of free Dutch citizens – “*vrijeluijden*”), Brazilian-Portuguese children, Brazilian natives (Brazilian indians – specially from the tribes Tapuia and Tupi) and even Brazilian-African black children, by this time slaves. A public day school was created, where Brazilian and Dutch boys could start learning from 5 years of age; in these schools they could learn Dutch language, religion, and handicraft works (“*hantwerken*”). During this first period of education, mathematics and science were not offered: Dutch authorities believed that a long period of preparation would be necessary before offering such school subjects; besides, they believed that only few students would be able to go ahead, facing mathematical and scientific learning. There are some historical evidences showing that mathematical and scientific knowledge were considered crucial to the Dutch military and political project for Mauritsstadt and the whole province. This approach allows important Brazilian historians to consider that Maurício de Nassau founded "(...) the most modern and original cultural center not only in Brazil but also in the hole XVIIth century America (Gonsalves de Mello, op. cit.). The Dutch, nevertheless, had no time to test their political educational and strategically propositions in Pernambuco: in 1654, a Brazilian-Portuguese army has successfully taken Pernambuco back to Portuguese domain.

2. *Learning environment for mathematics in school: from past to future*

Dutch conquerors seemed to be the very first group in the history of Pernambuco to demonstrate a clear preoccupation towards education as civil right to be largely offered to all citizens (including native indians and black slaves - and not a *privilege* to be restricted to very few aristocrats and their descendents). They highlighted some other important aspects: 1. Efforts in order to offer an adequate curriculum at school; 2. Dialogue between school/formal knowledge and outside-school knowledge; 3. Place of mathematics and science in school curriculum (*what* mathematics to offer, and *when*). Today, 347 years later, these aspects are far from being implemented, particularly when we think about schooling in the context of a poor state (Pernambuco) in a so-called "emergent country" (Brazil). Important psychological aspects are connected to these questions, being good issues to a research agenda in psychology of mathematics education. These aspects are mentioned and discussed in two main sections below. Discussing them in this Dutch PME, nowadays, is an amazing task for a psychologist from former Mauritsstadt, today Recife (Pernambuco), with far family roots in the Dutch community that tried the *Mauritsstad utopia* in these tropical and luxuriously beautiful lands.

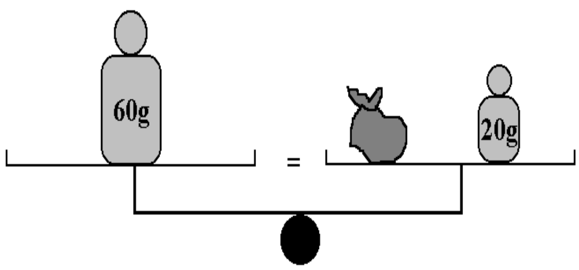
2.1. *Conceptualization in mathematics*

2.1.1. Mathematical concepts as *models*, instead of taxonomic categories.

- There are not "mathematical essences" (radical anti-Platonic perspective).
- There are not exemplary-concrete cases for mathematical concepts: mathematical concepts are metaphorical models.

According to G. Lakoff and R. Núñez, there is a strong "mythology" concerning conceptualization in mathematics, a kind of "romance of mathematics" (sic), according to which mathematics would be "abstract and disembodied" (yet it *is* certainly real), having "objective existence", providing "structure to the universe" and being "independent of and transcending the existence of human beings or any beings at all" (Lakoff & Núñez, 2000, pp. xv). We refuse this belief about mathematics, proposing, in accordance with these authors, that mathematics comes from *us* (human beings), instead of from heaven or the outer universe. Mathematics is definitely a human construction, not the way towards platonic-transcendent truths of the universe. By human construction we mean, quoting once more the authors above, not (...) a *mere historically contingent social construction"*, but "(...) a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, *and our long social and cultural history"* (Lakoff & Núñez, op. cit., pp. 09; italics added). Two important conclusions emerge from this epistemological assumption: first, mathematical concepts are rooted in human bodily realities, as well as in socio-cultural activities; second, the learning of these concepts can not be limited to the experience of receiving axioms and developing (by direct imitation, perhaps) the "right" way of thinking (mathematical proof).

We agree with philosopher E. Cassirer propositions about conceptual development, specially when he stresses the role of *modeling* in concept formation and development (Cassirer, 1977). In fact, concepts in general are much more than descriptive, taxonomic categories based on common aspects, as proposed by many psychologists in the past (see, for example, the classic research work started by E. Heidbreder (1946), followed by Bruner, Goodnow & Austin (1956), and many other psychologists up to present days). This is especially true in the case of mathematical concepts; from a mathematical standpoint, mathematical concepts express *functions*; from a psychological perspective, they express *relations* issued from perceptual input and gathering. These relations are psychologically represented through symbolic tools like natural or formal language, in many degrees of combination. These combinations open the way to the didactically important psychological activity of *metaphorization*, as illustrated below:



$\mathbf{60g = x + 20g}$ then
$\mathbf{60g - 20g = x + 20g - 20g}$ (because of the metaphorical connection between the balance scale and the principle of equivalence in the treatment of algebraic equations).

$\mathbf{40g = x}$ (then, the unknown weight is 40g, which is the final step of the algebraic algorithm).

Figure 1: The metaphorical connection between balance scales and algebra (Reproduced from Da Rocha Falcão, 1995).

Symbolic representation is a key psychological aspect in the development of algebra and many other *conceptual fields* (Vergnaud, 1990) in mathematics because of two points: first, it is not a result or superstructure of operational structures, as proposed in the context of Piagetian theory (Piaget, 1970) but rather a constituent of concepts, with operational invariants and situational links that gives socially shared meaning to knowledge (Vygotsky, 1985); second, it opens to a particular individual a wide range of symbolic cultural tools that, as *cultural amplifiers* (Bruner, 1972), enables one to access new instances of conceptual construction. So, representations provide ***metaphors*** that can be useful as pedagogical tools; these metaphors help in amplifying pre-existing schemes, since they provide semantic links between structured knowledge and new pieces of information. In this process of enrichment of meaning, a quite important psychological sub-process is represented by the explicitation of *theorems-in-action* (Vergnaud, op. cit.), upon which are established many practical competencies exercised in daily life. The proposal of the two-pan balance scale represents an effort of offering a metaphor of algebraic equivalence between equations, based in the conservation of a pre-established functional equality between each side of an equation. The construction of meaning for the equivalence of equations (essential aspect for the comprehension of algebraic algorithms) is initially connected to the familiar idea of *equilibrium*, in the context of a culturally familiar artifact, the balance-scale. This idea of equilibrium is frequently poorly explicited, although people can make a competent use of a two-pan balance scale in order to sell or buy fish in Brazilian popular markets; nevertheless, equilibrium as theorem-in-action is based upon *explicitable* principles. As a metaphor, the balance-scale offers a context of cultural functioning where complex mathematical concepts (algebraic equivalence and algorithmic manipulation) can be initially rooted in competencies and theorems-in-action, enriching pre-existing schemes. This last aspect, concerning the mobilization of competencies-in-action for conceptualization in mathematics, led us to the next point of this brief theoretical navigation.

2.1.2. Concepts as schemes are not only limited to the language domain, but also cover *competencies-in-action*.

- Mathematical competence (as many other complex social competencies) cannot be restricted to the domain of language: language constitutes cognition, but cognition is larger than language.

The theoretical position defended here tries to establish the relations between thought and language in the context of what we propose to call the *psychology of human gesture* (Vergnaud, 1998). The human gesture, in its functional richness, was once part of H. Wallon's theoretical contributions, who distinguished three levels of evolution development: the *automated* or *habitual* gesture, the *adaptative-motoric gesture* in interaction with the external ambient, and finally the *symbolic gesture*, connected to an objet not empirically present (Wallon & Lurçat, 1962). The gesture, as a theoretical subject in cognitive psychology, is directly connected to the notion of *scheme*, as defined by Gérard Vergnaud: it is a prototype of human cognitive activity, present in all periods of human development, from childhood to adult life (Vergnaud, op. cit.). As an organizing and organized structure based on schemes, the human gesture is characterized by the following aspects: a) a *goal*, eventually divided in secondary or sub-goals: a pre-linguistic 18 months old child can look for a small ball which has disappeared under a table, making bodily gestures with his/her head towards the direction he/she imagines the small ball is going to re-appear (the opposite side of the table under which the child saw the ball disappearing); b) *an ordered set of actions:* the same child in the example above will be able to organize his/her actions in order to achieve his/her main goal; c) *identification of material tools and empirical information*: in his/her efforts to recuperate the small ball, the same child will be able to take into account situational variations and constancy, like those concerning the speed of the ball, length of the table under which the ball moves, and so on; d) *calculations concerning the actions to be performed, information to be coordinated, controls to carry out:* a gesture action, when performed many times with the same goal, can be reformulated in the course of the action itself, in order to take into account a certain degree of situational variation. In other words, the child is able to modulate his/her gesture actions in the context of a certain (and necessarily limited) spectrum of variations.

The previous considerations allow us to refine the definition of human gesture in terms of an invariant organization of behavior for a limited class of situations. It must be said that this same definition can be used to schemes in general and *concepts* in particular. Among human schemes, it is important to mention the *competencies-in-action,* like those shown by handicraft workers, but also some competencies of very high-level (in terms of their *educational capital*) workers (researchers, engineers, specialized technicians, and so on). These so-called competencies-in-action have two major characteristics: firstly, they are *effective,* in the sense that they are cultural tools for daily (and culturally situated) life; secondly, these competencies are very hard to *express* by any symbolic means (natural language, graphic representations, mathematical models, and so on). Taking these competencies into account in mathematical conceptualization implies in connecting school activity to other socio-cultural contexts, as discussed in the next section.

2.2. Learning mathematics as a cultural activity: interest of negotiations between teacher and students.

- ***Didactic contract is a very effective variable in the process of teaching and learning of mathematics***
- ***Argumentation (seen as the process of changing of opinion in an interlocutory context) is a key process in the learning of mathematics, but it is not the only effective (pedagogically speaking) process.***

Mathematical teaching and learning, as a specific cultural activity, is organized by certain consensual rules, most of them not explicitly expressed. These rules express previously established goals (both behavioral and attitudinal), giving rise to a *didactic contract* that guides social activity in the classroom (Schubauer-Leoni & Perret-Clermont, 1997). Examples of such rules are, for example, “a mathematical problem has always one (and only one) answer”, or “the teacher always knows the answers for mathematical problems”. There is a clear interest in the use of these rules as *didactic variables* (Brousseau, 1998), in a research context, since many psychological schemes can be addressed and amplified in the context of classroom activity. It seems to be possible, as shown by Da Rocha Falcão and colleagues, to propose certain activities by previous contract, even if these activities are completely discrepant concerning usual school habits and rules (Da Rocha Falcão et. al., 2000). These researchers were able to propose symbolic manipulation of literal symbols, in the context of algebraic sentences, to very young school children; the classroom contract that allowed this activity is illustrated below. The students were invited to compare two plastic boxes containing different and unknown quantities of small balls, these deposits being covered with paper in order not to allow visual inspection of the quantity of balls inside them. Two students were invited to hold the deposits in front of the group; each of these two students was "marked" by an icon of a happy face (☺) for the one owning more balls, and a sad face (☹) for the other owning fewer balls (the icons were drawn in sheets of paper and put on the floor, in front of each student: see illustration on the left). It was then proposed to the students that we would represent the unknown quantity of balls by a *letter* (they suggested to adopt the first letter of the name of the student holding the plastic box). All the group of students was able to model a relationship using letters to represent the quantity of balls in each deposit: $G > J$ as well as the passage to the equality considering the difference (if $G > J$, then $G = J + D$ or $J = G - D$, representing the unknown difference by the letter D). When the protocols of the activity above were closely examined in terms of argumentative exchanges, we could realize the occurence of three kinds of discoursive actions, according to the analytical model proposed by S. Leitão (see Leitão, 2000): 1. *Pragmatic actions*, proposing situational condictions in order to allow argumentation; 2. *Argumentative actions*, i.e., speech turns adressing the negotiation of divergent points of view; 3. *Epistemic actions*, consisting of offering specific information about the topic in discussion. These three actions were not equally distributed in the didactic sequence proposed (Araújo et. al., 2001): there was an preponderance of epistemic actions, in the context of which the teacher offered information and tried to justify what he was proposing (allowing the students to ask questions and to express their doubts). These data show that even when we make special didactic efforts to propose a certain topic, there is not any guarantee that argumentative speech-actions will occur. On the other hand, even though we strongly believe in the role of argumentative actions in knowledge building, these specific speech actions are one among other possibilities in the context of discursive (and effective) pedagogical exchanges in the context of classroom.

3. Conclusions and final remarks

Research about conceptualization is seen here as central in a research agenda proposed by psychologists, in the interdisciplinary field of mathematics education. The psychological nature of mathematical concepts, the context of development and learning of them and the place of symbolical representation in concept building are important theoretical domains for psychological inquiry. For Dutch conquerors, in the XVIIth century, education in general, and scientific / mathematical education in particular, were crucial to build a modern society, based on free opinion, trade and religion. For us, psychological researchers in mathematical education from a "melting-pot" country like Brazil, these issues are crucial for us to survive inside mathematics education research community.

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