REVEALING AND PROMOTING THE STUDENTS' POTENTIAL: A CASE STUDY CONCERNING INEQUALITIES

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ABSTRACT: This paper presents a study concerning novices' cognitive apprenticeship in the field of inequalities. A specific educational context was designed with the purpose of revealing and enhancing the students' potential in dealing with inequalities according to a functional approach. A preliminary analysis of students' solutions is provided.

1.Introduction

In many mathematics domains, mathematics education research must face widespread, strong difficulties in teaching and learning specific subjects. Difficulties met by teachers and students frequently bring to postpone those subjects and/or reduce their teaching to procedural aspects. In some cases epistemological, didactical and cognitive analyses can help planning teaching experiments which allow researchers to better understand the reasons for these difficulties and, possibly, reveal students' potential in dealing with those subjects. Innovative educational choices are an expected reasonable outcome (cf Arzarello and Bartolini, 1998). In particular, didactical analyses may point out conditions under which difficulties take place and peculiar didactical choices related to them; while epistemological analyses may enlinghten the nature of involved mathematical concepts. Cognitive analyses seem to be necessary in order to detect crucial mental processes underlying specific mathematical performances, with the aim of enhancing them through classroom work.

In this report we present the guidelines and some preliminary results of a research program conceived according to the above perspective and concerning the approach to inequalities in 8^{th} - grade. Our working hypothesis is that a functional approach to inequalities (i. e. an approach based on the comparison of functions), when suitably managed by the teacher, can reveal (from the research point of view) and allow to exploit (from the curriculum design point of view) a students' potential which goes far beyond the mathematics content involved (inequalities). Our preliminary results support this hypothesis and enlighten some conditions concerning the educational setting which students' success seem to depend on.

2. Inequalities: a challenge for teaching and research

In most countries, inequalities are taught in secondary school as a subordinate subject (in relationship with equations), dealt with in a purely algorithmic manner that avoids, in particular, the difficulties inherent in the concept of function. For instance, in Italy and some other countries students are taught to deal with second order inequalitities depending on a parameter (e.g. $x^2+Kx+1>0$) in a very rigid, prescriptive way: they must solve the equation $x^2+Kx+1=0$ (distinguishing between the three cases: no real solution, two coincident real solutions, two distinct real solutions); then they must build up a schema where "concordances" and "variations" of signs, in relationship with the values of the parameter K, provide the student with the solutions for the given inequality. We observe that this approach implies a "trivialisation" of the subject, resulting in a sequence of routine procedures, which are not easy to understand, interpret and control.

As a consequence of this approach, students are unable to manage inequalities which do not fit the learned schemas. For instance, according to different independent studies (cf Boero, 2000; Malara, 2000), at the entrance of the university mathematics courses in Italy most students fail in solving easy inequalities like $x^2-1/x>0$. In this task, proposed to a sample of 58 students entering the Faculties of Science of Genoa and Pisa Universities, less than 60% engaged in solving the inequality (the others answered *"I am not able", "I did not study it")*; most of them performed the following transformations: from $x^2-1/x>0$ to $x^2>1/x$ to $x^3>1$. Few students took care of the case $x=0$; less than 10% of the whole sample made a distinction between the case $x>0$ and the case $x<0$. In general, graphic heuristics were not exploited and algebraic transformations were performed without taking care of the constraints deriving from the fact that the > sign does not behave like the = sign (see Tsamir et al., 1998, for a deep analysis of such behaviour). Similar phenomena were described in some studies concerning the French situation (see Assude, 2000; Sackur and Maurel, 2000).

This brief presentation brings to the following conclusion: the prevailing manner of teaching inequalities in school neither is efficient (as concerns the results, in terms of capacity of dealing with a large set of rather simple inequalities) nor results in acquisition and/or reflection about the mathematics concepts involved. We may ask ourselves what are the reasons of this situation.

One reason could be the fact that equations (and inequalities) are considered (in most of European countries, including Italy) as a typical content of school Algebra; this subject matter is distinguished from Analytic Geometry and does not include functions. This might explain why inequalities (and equations) are not dealt with in those countries from a functional point of view. But even in countries where functions (and Analytic Geometry) belong to school Algebra (cfr. NCTM Standards, 1989) the procedural, algebraic approach prevails in many curricula and even in innovative proposals (cf Mc Laurin, 1985; Dobbs and Petersen, 1991).

Another possible hypothesis is that inequalities are a very complex and demanding subject; dealing with few and well codified cases in an algorithmic way appears as a consequence of these intrinsic difficulties. In order to support this interpretation we may observe that (from an epistemological point of view) the functional aspect plays a crucial role, both for equations and inequalities. Indeed, let us analyse mathematicians' work when they solve equations with approximation methods, deal with the concept of limit or treat applied mathematical problems involving asymptotic stability: the functional aspect of inequalities plays a crucial role. This fact is often neglected in traditional teaching: as suggested at the beginning of this Section, we may recognize that the traditional teaching of inequalities avoids the "function" concept and reduces the difficulties ineherent in the "variable" concept and the complexity of the solution process by treating inequalities as a "special" case of equations.

Under the same perspective we can make the hypothesis that an alternative approach to inequalities based on the concept of function could provide an opportunity to promote the learning process of the difficult concepts involved and the development of the inherent skills. It could also ensure an high level of control of the solution processes of equations and inequalities (Sackur and Maurel, 2000).

Finally, we must remark that in spite of the importance of inequalities in mathematics and of the difficulties met by teachers and students in dealing with them, few studies in mathematics education concern the school approach to inequalities (see References). It is like if the subalternity of inequalities to equations in the teaching of mathematics, was reflected in a scarce relevance for inequalities in mathematics education research; in particular, as concerns the functional approach to inequalities.

3.Method

Keeping the previous analysis into account we have planned a teaching experiment in two VIII-grade classes with two rather limited aims: investigating the feasibility of an early functional approach to inequalities; and revealing students' potential and difficulties in dealing with this subject as a special case of comparison of functions. We choose to guide VIII-grade students in a cooperative, gradual enrichment of tools and skills inherent in the functional treatment of inequalities. Then we have analysed how (in relatively complex tasks) they were able to use their knowledge and increase their experience in an autonomous way.

3.1. The educational context

36 VIII-grade students (divided into two classes) were involved; as usual in Italy, they had started to work with the same mathematics and science teacher in grade VI. The didactic contract established in grades VI, VII and at the beginning of grade VIII was coherent with the methodological choice of a cooperative, participated, guided enrichment of tools and skills in the planned activity. A rather common routine of classroom work consisted in individual production of written solutions for a given task (if necessary, supported by the teacher with 1-1 interventions), then the teacher guided classroom comparison and discussion of students' products; possibly, the adoption of other students' solutions in similar tasks followed. Another aspect of the didactic contract included the exhaustive written wording of doubts, discoveries, heuristics, etc.

3.2. Specific content and educational choices

As concerns the **content,** the concepts of function and variable have been approached through activities involving tables, graphs and formulas. At the beginning the geometrical context (area, perimeter, etc.) was prevailing, then it has been progressively left aside. At the beginning the function was presented as a machine trasforming x-values into y-values (*machine view* in Slavit, 1997), then classroom activities focused on the variation of y as depending on the variation of x (*covariance view*). By this way a dynamic idea of function gradually prevailed on the static consideration of a set of corresponding pairs (*correspondence view*). As a consequence, a peculiar aspect of the concept of variable was put into evidence (a variable as a "running variable", i.e. a movement on a set of numbers represented on a straight line) (cf Ursini, 1997). Finally, the approach to inequalities was realised by comparing functions.

The didactical contract demanded to compare functions as global, dynamic entities. Students knew that they had to compare functions by making hypotheses based on the analysis of their formulas. The point-by-point construction of graphs was discouraged. As a consequence, the ordinary table of x, y values was sometimes exploited as a tool to analyse how y changed when x changed (column-vertical analysis) and not as a tool to read the line-horizontal point-by-point correspondence between x-values and y-values. Finally, we can remark that the algebraic and the graphical settings were strictly related (formulas were read in terms of shapes in the (x,y) plane, while graphs evoked formulas).

As concerns the **educational choices** related to classroom management of the functional approach to inequalities, the following points were considered crucial:

- classroom discussions about "what do we loose and what do we earn" when a function is represented through formulas or graphs or tables or common language;
- different ways of describing given functions have been encouraged. For instance the discovery that the formula $y=2x^2$ corresponds (within the table of the x and *y* values, read vertically according columns) to an *"irregular"* increase of *y* which is different from the *"regular increase"* in the case of $y=2x$, and that the irregular increase results in a curved line when the graph is drawn (later on the teacher will call it *"parabola"*) allows to link different ways of representing functions (see Duval, 1984: coordination of different linguistic registers). They will become personal tools exploited and transformed (see Ex.1 in the next Section) to compare functions. Even the metaphors used by students to describe the role of different pieces of the same formula have been encouraged and discussed.

3.3. Individual task

We will consider the following task:

"Compare the following formulas from the algebraic and graphic points of view. Make hypotheses about their graphs and motivate them carefully, finally draw a sketch of their graphs. A) first function. B) second function"

Slightly different fuctions A and B were chosen according to the students' levels. In particular in this paper we will analyse the solutions of three students engaged in solving the problem with the following functions (of mean level of difficulty):

A) $y=x^2-4x+4$; *B)* $y=-x^2+4$

In every case the above task was significantly more difficult and complex than the previous ones: they concerned the comparison of functions like $y=x^2$, $y=-x^2+4$

We expected that (according to the didactical contract and previous experiences) most students could explore both functions in a dynamic way, trying to answer the following questions:

- *What does happen with y when* x>0 *or* x<0*? Does it increase?Does it decrease?*
- *Are there meeting points between the two graphs? Where? When?*
- *Does the first graph overcome the second one? Where?*

All individual solutions were collected (some protocols contain questions and comments written by the teacher during the 1-1 interactions). We collected also some recorded interviews performed after the end of the activity and concerning the strategies produced by students and their general ideas about inequalities.

4. Preliminary results

4.1. Some quantitative data

- 5 students out of 36 did not succeed in tackling the problem (they did not understand it, or were stuck)
- 7 students (out of 31 who were able to tackle the problem) arrived to incorrect conclusions (sometimes due to trivial mistakes in calculations).
- 3 students (out of 31) build up graphs point by point (against the didactical contract), while the others compare functions in a dynamic and global way with different strategies.

4.2. Some qualitative data.

From a qualitative point of view collected data are interesting for the following reasons:

a) 28 students (out of 31) show good skills in coordinating different linguistic registers (formula, graph, verbal language, etc.)

b) a plurality of strategies of comparison between the two functions, frequently consisting in a personal blend of strategies and moves compared and discussed in previous classroom activities. In particular we can find:

b1. Strategies based on the relationships between formula and shape of the graph (6 out of 28 who keep the didactical contract).

Formulas, or some of its parts, are associated with a peculiar shape of graph (Ex. 1, Davide).

In the case of Davide x^2 evokes the shape "parabola" and the other elements of the given formula are interpreted in terms of transformations of the prototypical graph: for instance the presence of a negative sign before x^2 suggests the "reversed U shape" (for a similar behaviour cf an example provided by R. Hershcowitz: the student detects the graph of $y=-x^3$ by referring it to the well known graph of $y=x^3$). The dynamical aspect of the solution consists in the interpretation of the different parts of the given formula as transformations of the prototypical formula.

Davide seems to ask himself questions like "What does it mean *+4* in the graph? And what about the sign - on the left of x^2 *? What does it mean –4x in the graph?*".The most interesting fact is that the graph seems to be "an object", a shape which can be moved as a whole.

We can ask ouselves how the relationship between the shape "parabola" and the formula " $y=x^2$ " was established by Davide. Considering the background of his class and his personal history we may make the hypothesis that such relationship was probably acquired through the activities of drawing graphs and that the dynamical dependence of y-values on x-values that was observed and discussed in the classroom (with the help of the table) could have "condensed" in a shape. This hypothesis implies that for Davide the shapes of the graphs are not mere shapes, because they bear the peculiar modality of increase of *y* when *x* increases. A careful analysis of his protocol seems to confirm our hypothesis.

Ex.1, Davide: compare the two functions $y= x^2- 4x+4$ and $y= - x^2+4$

In the graph x^2 and $- x^2$ are two parabolas: the former is over 0 and looks like U, the latter looks like a reversed U. In both functions there is +4, thus we can say that both parabolas stand over the x-axis of +4

(Fig. 1: he sketches the graphs of $y= x^2+4$ *and* $y= - x^2+4$*)*

<u>In function A there is an operation (-4x) which moves and transforms this parabola.</u>

A) when x=2, y=0, thus we can say that , when y=0, the parabola stands in x=2.

(Fig. 2: a parabola with the vertex in the point (2, 0))

This parabola does not go under zero when x is negative.

<u>This parabola starts slowly, because -4x decreases its speed</u> ,but successively goes up quickly.

(Fig. 3: superposition of parabolas A and B) [...].

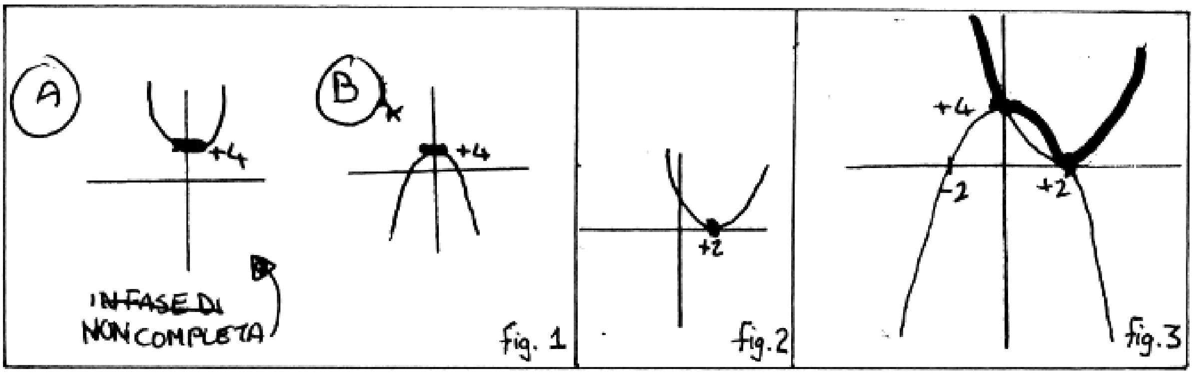

fig. 1 fig.2 fig.3

b2. Strategies based on the relationship between formula and increase or decrease of y when x increases (14 out of 28).

In this case students analyse the behaviour of the formula according to $x>0$ or $x<0$ (Ex. 2, Lorenzo). We can imagine that Lorenzo asks himself questions that are very different from those considered by Davide: *"What does it happen if x<0? And if x>0? Does it increase? Does it decrease?"*). In this case the only dynamical aspect concerns the analysis of how *y* changes in relationship with *x*. Also in this case the approach is global: it is the whole function that decreases or increases when *x* changes. For Lorenzo the graph represents a synthesis and the visual validation of his previous analysis.

Ex.2, Lorenzo: compare the two functions y= x^2- 4x+4 and y= - x^2+4

•In both formulas you have always found a +4. In the former x^2 you always have a positive result, while in the latter you never find it, because there is a minus before.

•If x<0: The former increases more because there is x^2 , always positive, and the sign changes also in -4x, which turns out +4x.

On the contrary, the latter dicreases only because I take out an x^2 and at a certain point the +4 cannot keep the - x^2 above the zero. There is no meeting point.

•If x>0: <u>The former decreases for a certain period (from 0 to +2)because the +4 and the x^2 don't keep the – 4x</u>, while after that *[the function]* increases.

The latter for a certain period stays above zero (from 0 to +2) because the +4 supports the - x^2 while after that it decrease below zero.

Meeting points x=0, y=4 and x=2, y=0

[He draws the superposed graphs with a comment]

The latter is bigger than the former from 0 to +2

b3. Strategies based on the search for "remarkable points" (8 out of 28). (Ex.3, Laura)

These students through the discovery of remarkable points (*x=0 y=…, y=0 x=…*) perform an analytic-inductive analysis of the functions under comparison and get an idea of the shape of the graph. In particular Laura finds some points of the first function, then declares:"*It does not touch the origin and y is <u>never</u> negative!!*". This behaviour can be considered (in the case of the two functions) as intermediate between the preceding two. Indeed neither the use of the graph is prevailing.(like with Davide), nor the analysis of how *y* changes in relationship with *x* (like with Lorenzo), but these two aspects are intermingled.

Ex.3, Laura: compare the two functions y= x^2- 4x+4 and y= - x^2+4

Both formulas have +4 in the end, and thus we can neglect them (translate of +4 with respect to the y axis). They both are parabolas.The first one (A) is the greatest.

Now I proceed by means of "oriented" computation.

[......]

x=0 y=0; x=-4 y=+16 –(-16)=+16+16=32; x=+4 y=0; x=2 y=-4

between +4 and 0 the result is <0!! Now I add +4 !!

A) x=0 y=+4; x=+4 y=+4; x=+2 y=0

It does not touch the origin and y is never negative!!

B) x<0 y=negative, except for x= -2 , where y=0

x>0 y=negative, except for x=+2, where y=0

It does not touch the origin and it is positive (only) between x=-2 and x=+2.!

Curiosity: what is y like, when x is greater than –2 and smaller than +2?

x=-1 y=+3 x=+1 y=+3

The matching point of the two graphs is +4 in the y axis and the second point is in x=+2 y=0

(drawing: superposition of the parabolas A and B)

As I observed above, A is the greatest, but there is a little overcoming of B between x=0 and x=+2

c) Plurality of meanings of pieces of formulas

In the task of comparing the two functions $y= x^2- 4x+4$ and $y= - x^2+4$ a crucial difficulty concerns the role of *"-4x"* in the first fuction. We can observe how, according to the peculiar strategy of each student, *"-4x"* takes diferent meanings. For instance, in the case of Davide (Ex.1, underlined parts): *"-4x"* is responsible for *"moving and transforming the parabola"*; later on *"-4x"* takes another meaning: it *"diminishes the speed of increase"*. For Lorenzo (Ex. 2, underlined part) *"-4x"* is *"responsible for the decrease of the function between 0 and 2"*, in relationship with the other parts of the formula. Other students declare that *"if -4x did not exist the two parabolas would be equal: the former oriented upwards and the second oriented downwards"*. This diversity of meanings attributed to the same piece of the formula enriches the interpretation and representation of the whole formula. The same phenomenon occurs for other pieces of the same formula: for instance *"+4"*: for some students *"+4"* means that *"the parabola does not pass through the origin"* of cartesian axes, for other students it means that *"it moves the parabola upwards"*.

Conclusion

As a consequence of epistemological analyses (which reveal the importance of the functional aspects of inequalities in mathematicians' work), and didactical analyses (which show how the current approach to inequalities based on purely algebraic procedures produces very limited learning results), and keeping into account cognitive studies about functions, a teaching experiment aimed at exploring the feasibility of an early functional approach to inequalities with WIII-grade students was planned and analysed. Collected data seem to support our didactical and educational choices; in particular, inequalities appear an interesting issue to study the students' construction of different aspects of the concepts of function and variable, and a promising learning context for these concepts. This means that the functional approach to inequalities, originally intended to better understand and possibly solve some teaching and learning problems concerning inequalities, offers a possibility for studying and improving teaching and learning of functions.

A relevant aspect of our study concerns the educational choices related to the aim of a cooperative enrichment of tools and ways of reasoning useful to deal with inequalities. Here we may say that the approach to inequalities based on a global, dynamic approach to functions seem to fit very well with this aim: graphic representations, gestures, metaphors concerning functions were very easy to share in the classroom (once the teacher decided to encourage their use by students).

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