MECHANICAL LINKAGES AND THE NEED FOR PROOF IN SECONDARY SCHOOL GEOMETRY

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Mechanical linkages, which abound in everyday objects as well as in historic "mathematical machines", provide a rich source of geometry appropriate for secondary mathematics. Explaining why these linkages work the way they do offers a rationale for mathematical proof. The results of a small pilot study suggest that, with the emphasis on the underlying geometry, linkages provide a visually-rich, motivating environment in which to encourage students to explore, conjecture and construct geometric proofs.

Recent decades have seen a general decline in Euclidean geometry in Australian school mathematics curricula, with the geometry that remains becoming largely empirical and students having little idea of the significance of proof. One outcome of this has been renewed debate amongst mathematics educators concerning the inclusion of geometric proof in school mathematics, a debate partly driven by the development and introduction into schools of dynamic geometry software, such as Cabri Geometry II™ and The Geometer's Sketchpad®. While concern has been expressed that dynamic geometry software is contributing to the empirical approach to school geometry (for example, Healy and Hoyles, 1999), there is also the strong feeling that the dynamic imagery associated with use of the software has the potential to play a significant part in geometric reasoning.

Although mathematical proof has several functions, one of its roles is verification, that is, conviction or justification of the correctness of mathematical statements. However, while some students may have a cognitive need for proof as conviction, many see little point in proving something which they already 'know' to be true. De Villiers (1998) has criticised the emphasis on the verification aspect of proof in school mathematics, arguing that a more meaningful activity for secondary school students is to focus on proof as explanation. Bell (1976, p. 24) claims that the emphasis on proof as verification/conviction/justification often results in a neglect of other important aspects of proof, noting that "a good proof is expected to convey an insight into *why* the proposition is true" and that "conviction is normally reached by quite other means than that of following a logical proof".

Healy and Hoyles (1999, p. 1), reporting on the project, *Justifying and Proving in School Mathematics*, note that the National Curriculum in the UK

> prescribes an approach to proving … in which the introduction of formal proofs is reserved for 'exceptional performance', and thus delayed until after students have progressed through early stages of reasoning empirically and

> explaining their conjectures. Most of the requirements to explain and justify take place within investigations driven by numerical data.

Garuti, Boero and Lemut (1998) note that students who are actively involved in the conjecturing process in geometry problems are more successful in constructing proofs than students who have been asked to construct a proof without prior involvement in producing a conjecture. They use the term "cognitive unity" to signify the continuity which they assert must exist between the production of a conjecture and the possible construction of its proof. Hoyles (1998, p. 171), cautions against the assumption that the computer and dynamic geometry software will automatically "help to build bridges between the empirical and the deductive", noting that "there remains the question of how to develop a 'need for proof'". She suggests that

> we need to design new learning contexts which require the use of clearly formulated statements and definitions and agreed procedures of deduction but which also allow opportunities for their connection with empirical justification and the conviction this engenders. (p. 169)

The research focus

The proposed research to be conducted in 2001 aims to explore the potential of mechanical linkages, or systems of hinged rods, to engage students in conjecturing and deductive reasoning and to foster a greater understanding of the need for geometric proof. Mechanical linkages occur in many everyday items, for example, umbrellas, folding tables and car jacks, as well as historical "mathematical machines", such as Pascal's angle trisector. Many linkages are based on simple geometry, such as isosceles triangles, similar triangles, parallelograms and kites (see Vincent and McCrae, 2000). The linkages used in the research have been carefully selected because their particular geometry is within the understanding of above-average students in Year 8 (approximately 13 years of age). Dynamic geometry software models of the linkages, which permit accurate measurements and tracing of loci, form a bridge between the concrete and the theoretical, assisting students to visualise the linkage as a geometric figure.

Using both physical models and pre-prepared dynamic geometry (Cabri Geometry II) models, students will be encouraged to conjecture about the behaviour of each linkage, propose a conjecture which seems fundamental to the operation of the linkage, and so prove *why* the linkage operates in the observed way. Such proofs embody the multiple roles of *verification*, for example, that apparent invariants are in fact invariant, *understanding* of geometric relationships, and *explanation*, that is, giving an insight into *why* the linkage works the way it does.

The main research questions are:

1. Does the modelling of mechanical linkages provide a motivating environment in which to introduce secondary school students to conjecturing and proof?
2. Does the static and dynamic imagery provided by dynamic geometry software models of mechanical linkages assist students to produce relevant conjectures?

3. Does active involvement in the conjecturing process facilitate secondary school students' transition to construction of proofs?

Methodology

The participants in the multi-case study research to be conducted in 2001 will be approximately 20 students from an extension Year 8 Mathematics class at a private girls' school in Melbourne. The students are selected for this extension class on the basis of their non-verbal reasoning (Ravens' Progressive Matrices) scores, their performance on mathematics tests throughout Year 7, and teacher recommendations. They have no previous formal exposure to deductive reasoning and could be expected to benefit in some way from the research tasks. Prior to commencement of the research the following geometry will be taught/revised: properties of angles in parallel lines, triangles and quadrilaterals; Pythagoras' theorem; similar and congruent triangles. Pre-testing will include a Proof Questionnaire (Healy and Hoyles, 1999) and a van Hiele level test (Levels 1-4) (see Vincent, 1998).

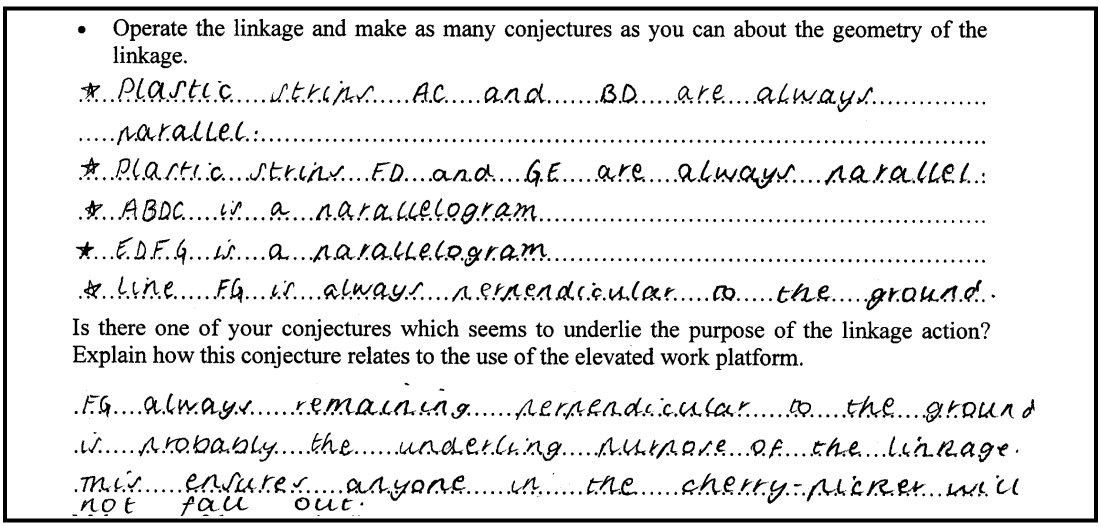
The students will work in matched ability pairs according to high/low Ravens' Progressive Matrices scores and van Hiele levels, and will be videotaped and interviewed during their work on some of the tasks. For each linkage, worksheets will direct the students to operate the linkage, record their conjectures, select the conjecture which seems to underlie the design purpose of the linkage and then construct a proof of that conjecture. Students will also complete a questionnaire for each linkage, requiring them to explain how they obtained the data on which they based their conjectures (for example, physical/Cabri model, measurement, loci of points) and which aspects of the models assisted them in constructing their proofs. Unless directed otherwise, students will be free to work with their constructed physical geostrip models and the Cabri models provided, perhaps, for example, checking the action of the physical linkage again after exploring the computer model. Post-testing will include conjecturing and proof tasks as well as re-testing with the van Hiele test and the Proof Questionnaire (Healy and Hoyles, 1999).

Pilot Study

A small pilot study was conducted with 29 students from the corresponding extension Year 8 class in 2000 with the aim of testing the feasibility of using mechanical linkages to stimulate conjecturing and deductive reasoning. Before undertaking any of the proof tasks the students were asked to give a brief written explanation of what "proof" in mathematics meant to them. While many students were unable to demonstrate any understanding of mathematical proof, 16 students were able to articulate at least one aspect of proof. The collective responses of these students illustrate the many facets of proof, for example, *verification* (10 responses)– "If you have proven something you have given evidence that undeniably shows that something is the truth. Proof cannot be denied"; *acceptance* (2 responses) – "Proof is required for people to believe and use a new mathematical theory"; *explanation* (4 responses) – "Proof is essential in order to explain a solution or idea. It should enable a viewer to comprehend the meaning". These responses may be compared with those of the high-achieving Year 10 students studied by Healy and Hoyles (1999), where approximately half of the sample of 2459 students recognised the role of proof in establishing the truth of a statement, just over one third believed that proof should be explanatory and over one quarter of the sample had little or no sense of proof.

While most of the 16 appropriate responses of the pilot study students focused on the role of proof, several referred to how proof is achieved. However, with the exception of one student who wrote: "Proof in mathematics is like proof in anything: using facts to support an argument. Any answer must be backed up using existing common knowledge and theorems", the responses referred to empirical, rather than deductive, methods. Typical responses included: "Proof in mathematics is when someone may have a theory and they have shown that it actually works by trying out every possibility (though that is very unlikely). They have tried or used something that shows that it works in every situation", "Proof of a theory needs to be tested over and over again to make sure it's true so you don't make a theory that doesn't work". Some responses indicated uncertainty about the generality of a proof, for example, "A proof has to prove that it works most of, if not all of, the time, not just in a handful of cases".

The familiar linkage of the elevated work platform, commonly known as a "cherry-picker", was used to introduce the class to conjecturing in a geometric context. Using plastic geostrips, card templates and paper fasteners, the students worked in pairs to construct the linkage and explore its behaviour. They enjoyed operating the linkage and their observations were accurate, as reflected in their individually-written conjectures (for example, see Figure 1). All students conjectured that there were two parallelograms in the linkage (see Figure 2) and that the safety cage remained vertical (or its base remained parallel to the ground). Most were able to represent the linkage as a geometric diagram, although there was variation in the degree of accuracy of portraying the lengths of equal links and in the positioning of pivot points (for example, see Figure 2), despite the parallelogram conjectures.


- Operate the linkage and make as many conjectures as you can about the geometry of the linkage.

★ Plastic strips AC and BD are always parallel.

★ Plastic strips FD and GE are always parallel.

★ ABDC is a parallelogram

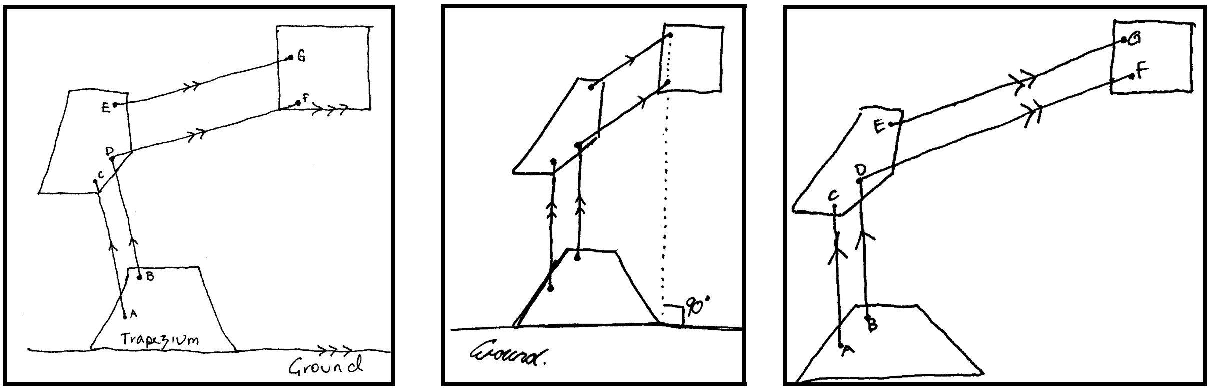
★ EDFG is a parallelogram

★ Line FG is always perpendicular to the ground.

Is there one of your conjectures which seems to underlie the purpose of the linkage action? Explain how this conjecture relates to the use of the elevated work platform.

FG always remaining perpendicular to the ground is probably the underlying purpose of the linkage. This ensures anyone in the cherry-picker will not fall out.

Figure 1. Student A's conjectures for the "cherry-picker" linkage.



Student A Student B Student C

Figure 2. Students A, B and C: drawings of the “cherry-picker” linkage.

Tchebycheff’s linkage (see Figure 3) was then introduced to the students as a means of demonstrating that visual evidence could sometimes be misleading. The linkage consists of a crossed quadrilateral where A and B are fixed and AB = 4 units, CD = 2 units and AC=BD = 5 units. The students constructed the linkage from geostrips and were directed to trace the loci of various points on the strips. After conjecturing that P, the midpoint of CD, moved in a straight line parallel to AB, they were provided with a Cabri construction of the linkage. By tracing the locus of P in the Cabri model and accurately measuring the distance PX (Figure 3) the students realised that the motion of P was only approximately linear. In this case empirical evidence was accepted, as the mathematics involved in determining the path of P was beyond these Year 8 students.

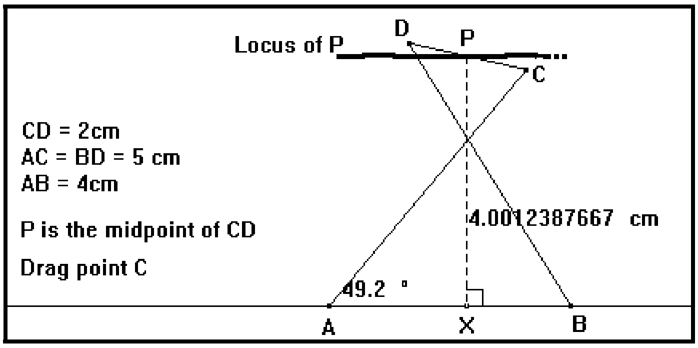


Figure 3. Cabri model of Tchebycheff’s linkage.

The students were then introduced to the car jack linkage, shown diagrammatically in Figure 4, where AB=PB=BC, A is fixed and P moves horizontally.

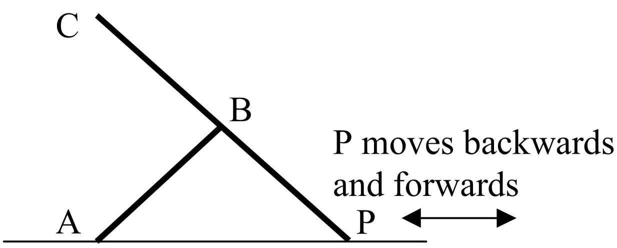


Figure 4. Diagram of the triangular car jack linkage.

Students D and E, who worked with the actual jack (see Figure 5) while the other students constructed geostrip models, explained their observations of the jack:

> We found that when you draw a line from this point [attachment point for car] to this point [point directly below attachment point] this line is always perpendicular to the ground. So even when it's moving up and down, the .. er … sort of line down here is always perpendicular even though it's attached and this here [base of jack] holds it in place and so the part where the screw is is always the same. This part [pivot at centre of long arm of jack] rotates.

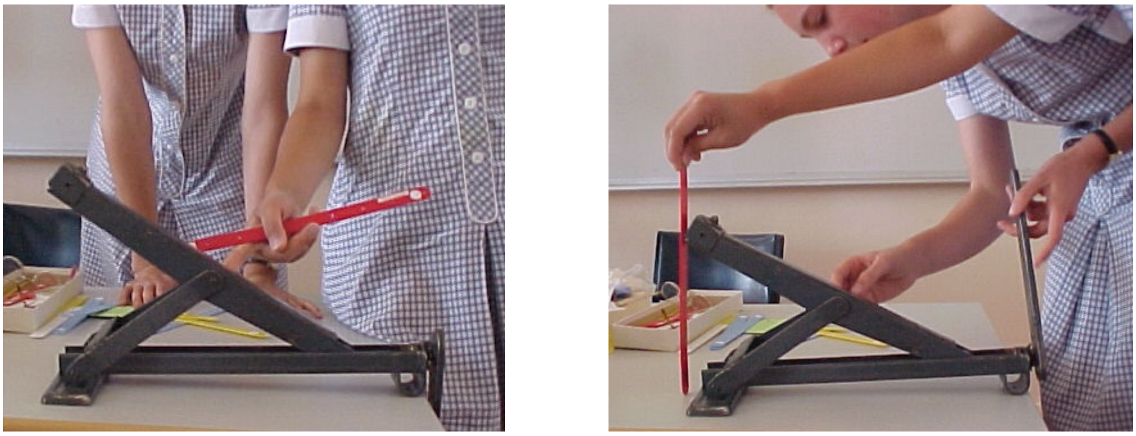


Figure 5. Students D and E: exploring the operation of the triangular linkage car jack.

Figure 6 shows the written conjectures of three students, A, F and G, each implying the same property of the linkage but expressed in different ways. Student A has focused on angle CAP (see Figure 4) without reference to a line AC, whereas students F and G observe that a line from C to A, drawn in the case of student F and imagined by student G, would be perpendicular to the ground.

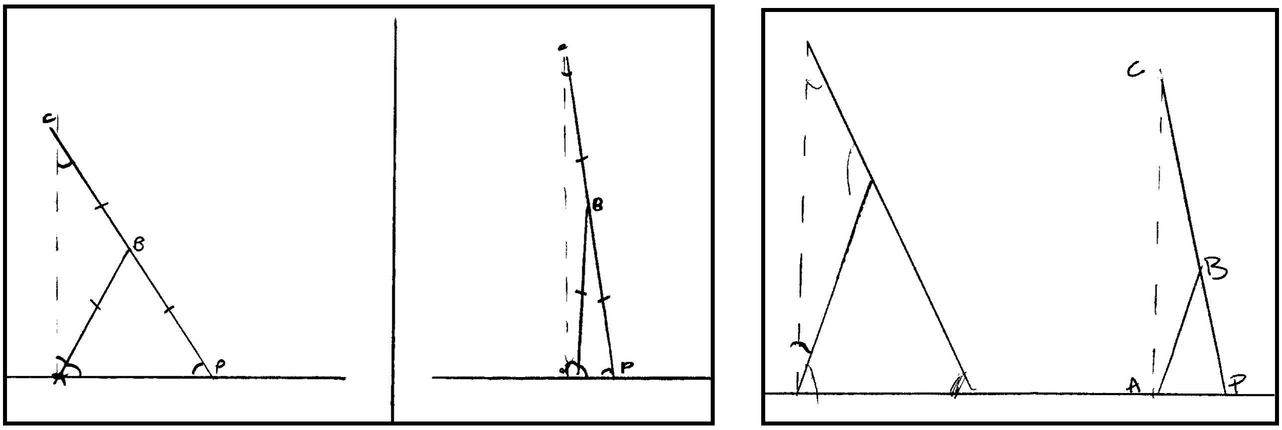
That angle $\angle CAP = 90°$

If you draw a line from A to C it would always be perpendicular from the base

Point C & point A is always in-line with each other and is an imaginary line perpendicular to the ground, no matter how you move it.

Figure 6. Students A, F and G: conjectures about the operation of the jack.

When asked to draw two different positions of the linkage, some students, for example, student H (Figure 7), accurately demonstrated that point C moved vertically upwards as P moved closer to A and that the lengths PB, BC and AB were preserved. In contrast, student I shows neither the increase in height of C or preservation of lengths.



Student H Student I

Figure 7. Different positions of the jack linkage as drawn by two students.

Investigation of Tchebycheff's linkage had demonstrated that visual evidence, particularly if based on crude measurements, could not be trusted so the students were aware of the need for proof. Some students recognised that the linkage contained two isosceles triangles, but they were uncertain how to proceed from there, as illustrated by the recorded explanation by students B and J:

> The angle BAP [see Figure 4] equals the angle APB because that's an isosceles triangle. And the same thing happens with the angle BCA and BAC because that's an isosceles triangle. Therefore angle CAB and BAP should both [together] equal 90 degrees. But we need to work out exactly why and that's what we're still trying to figure out.

The failure of these students to construct a proof may be due merely to lack of familiarity with proofs, but may also be due to failure to recognise ΔACP. Figure 8 shows the proof constructed jointly by students A and H (written by student A). Having commenced their proof with angles ABC and ABP, they realised that they could instead use the angles of ΔACP. Their equation: $2x+2y=180$, thus led them to the proof that ∠CAP is a right angle. Student A (and H) had conjectured that ∠CAP was a right angle (see Figure 6) and they had now proved their conjecture. It may be significant that students F and G, whose conjectures referred simply to CA being perpendicular to the ground, were unable to construct proofs.

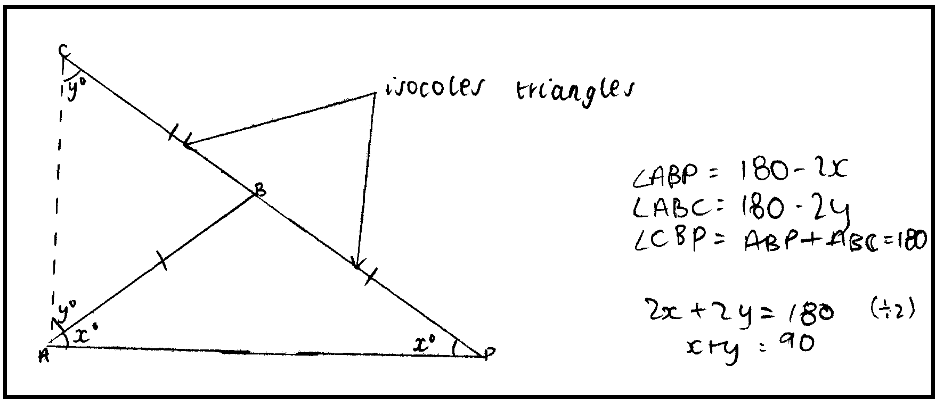


Figure 8. Students A and H: proof that ∠CAP is a right angle.

Conclusion

The results of this very brief pilot study are encouraging and suggest that Year 8 is an appropriate level for introducing able students to geometric proof. The students enjoyed working with the mechanical linkages, were able to produce conjectures relevant to the operation of the linkages and were motivated to try to find a geometric explanation for their observations. Not all students, however, were able to progress from conjecture to construction of proof when working with the physical (geostrip) models of the car jack linkage. This may be due to lack of familiarity with the concept of geometric proof, but may be related also to how a geometric figure is perceived. Further research may demonstrate whether the feedback from dynamic geometry software models can facilitate visualisation and analysis of geometric figures and assist the progression from conjecture to proof, particularly with more complex linkages. It is also important to investigate whether any positive effects of working with the linkages and with dynamic geometry software transfer to students' deductive reasoning and proof construction in pencil-and-paper proof tasks.

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