COGNITIVE LOOPS AND FAILURE IN UNIVERSITY MATHEMATICS

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This paper explores one case study, typical of a large number of students, in the transition from school to university. This category of student has developed a view of mathematics as fundamentally procedural. This means that they are able to reason about specific objects, but cannot (and do not attempt to) acquire a meaningful conceptual understanding of university mathematics. We show that this procedural approach, adapted to the university level, leads to a loop in which the student's focus keeps him removed from the central ideas of university mathematics.

Background

Tom is good at doing mathematics, just not at university mathematics!

Tom was one of eighteen first year students taking part in a study of the effects of two different styles of teaching first term Analysis at a UK university (Alcock & Simpson, 2001). One was a standard lecture course to over 100 people, supported by assignments. The other involved students in small groups working through carefully structured questions which built up the analytic theory in the course (Burn, 1992). The study gathered data in numerous ways, though the data presented here comes from biweekly semi-structured interviews which the student attended in pairs. These covered their feelings about the course, general discussions about the mathematics they were encountering and a task-based section in which they were presented with a problem related to recent material. The data were examined using a grounded theory approach (Strauss & Corbin, 1990) and classes of common features emerged under categories such as the student's informal facility with the material, their approach to reasoning about general objects and their view of their own role as a learner.

Patterns in the data across these categories led to the classification of "types" of learner, with the largest factor in a student's mathematical development being their view of their own role as a learner and their resulting approach to the course. One clear category in this sense is those whose view we might call procedural (or instrumental in the sense of Skemp, 1976) – these students see mathematics as set of fixed procedures to be learned. This paper will examine the consequences of this procedural approach using Tom as an exemplar.

School and university mathematics

The procedural approach is associated with beliefs that are well documented in students at school level (Schoenfeld, 1992). While this may not be desirable, in school it is viable to believe that mathematics consists of procedures to be learned, that problems have only one "best" solution and that teachers are the expositors and arbiters of correctness (all of the eighteen students in the study attained an A in A-level mathematics and half of them exhibited some or all of these beliefs). In contrast university mathematicians see mathematics as being about concepts to be understood, problems with many forms of solution which demonstrate different insights, and correctness warranted by formal deductions (Tall, 1995, Moore, 1994).

Tom is typical of this procedural category. We will use his case (and illustrations from similar individuals) to demonstrate that those who bring these beliefs from school may attain some facility at the level of handling specific objects, but beyond this they tend to have a focus on detail, to be inflexible in their use of procedures and to have little meaning associated with them.. We then consider how these characteristics lead such students to ignore and avoid changes in requirements at university level.

Tom's background

In the first interview Tom credits his eventual success in A-level to the fact that:

> "…we got to practise exam-style questions, and got used to it and just went through lots of examples really. I think that's what you needed, lots of examples to practise."

and says he thinks the best way to learn a new piece of mathematics is:

> "Definitely step by step examples going through it. Like the teacher gives you step by step examples, so you can go away and look at them when you do questions and say 'oh this is how you do it'. And get used to the method."

This background was quite typical of this category of learner. Zoe, another procedural learner, expresses similar ideas:

> "I just like the way we were taught at school…Where we, we have our notes, and then we practise so many times that it's just sort of drummed into us. You just don't forget it that way, it's like, I don't know…"

From this viewpoint there is an implicit reliance on the fact that teachers will provide procedures, and little onus the student to assess their suitability in novel situations. There is considerable security for the student in this approach; the responsibility definitely lies with an outside authority and a student who diligently learns what they are told can expect to do well (cf. Perry, 1970, Copes, 1982). However this ceases to be the case at university level, where these beliefs actively interfere with any learning on a more conceptual level, as we now see.

Effects of procedural beliefs

Focus on detail, not concepts

One result of procedural learning is that such students tend to focus on remembering the detail of what they have done rather than thinking about what this means in terms of the concepts studied. For example in week 3 Tom is asked what they have been working on in Analysis recently and says:

> "Erm, we're learning about, different proofs with erm, modulus. All of those, like *a* minus *b*, all in bars equals - doesn't, is less than, *a* minus *b* outside and things like that."

This appears to be a description of the triangle inequality, although it is hard to tell because he does not state the final result. Again Tom's response is typical of this category of learner. In this excerpt Wendy is describing a homework question that she found difficult:

> Wendy: You had, you had to erm, find an increasing and a decreasing sequence that erm, converged to the same limit,
> Xavier: Subsequence, wasn't it?
> Wendy: Yes, subsequence and prove that the, sequence would converge to the limit. And it was like, we had to find one that, a decreasing sequence that converged to the same limit as an increasing sequence. And all this other stuff going round that, you had to use and…oh no!

Everyone tends to become less coherent when trying to describe something that they didn't understand well. However Wendy is very focused on the instructions they "had to" follow; at no point does she state the result that this was supposed to lead to.

This lack of focus on conceptual relationships also shows up when Tom does attempt to use concept terms in sentences:

> Tom: Erm, he's just been giving us sequences, and we're supposed to be proving them.
> *Laughter*.
> Interviewer: Proving what?
> Tom: Just, proving that they're true. The sequences tends to a limit, and…

His meaning is relatively clear at the second attempt, but his first does not make sense and he clearly is not fluent in describing the work in these terms.

Informal use of the ideas from the course

This is not to say that procedural learners don't learn anything. Again Tom is typical, reasoning informally using the ideas from the course when asked to think about specific objects.

For example in week 5, considering the sequence given by $a_n = \frac{1+\cos n}{nx}$, he says:

> "Well, because it's cos n, the greatest cos n can be, is, 1, so the greatest that this can be, is, on the top is 2, so, whereas the bottom, is always going to get larger and larger."

And later:

> "Then the bottom's just going to, increase and increase and increase and even if, x is negative it's going to, be increasing by so much, that the top is going to become irrelevant, and it's just going to tend to zero."

He does not notice the problem with $x=0$, but does reach a correct conclusion about the eventual behaviour of the sequence for the remaining cases using strategies of considering the extreme values of the cosine function and making informal comparisons between variable quantities.

Procedures as immodifiable wholes

Difficulties arise for Tom when he tries to prove his assertions algebraically. Typically of the procedural learners in this study, he has procedures available, but if they are not applicable in very standard form he does not try to adapt them to fit the new situation. This is frustrating to listen to since he can identify problems with some precision. In this excerpt he is explaining why he doesn't know how to write down an answer to the above question:

> "Because we can't really, bring that nx up to the top, on the other side, because we don't know, whether x is positive or negative. And we can't really use the squeeze rule, because, what happens - we don't know what x is. It could be like 0.0001. And in that case, with it being on the bottom, it would increase the number."

His first approach is, in his own words, to "make n the subject", showing some awareness of the routine for showing that the definition of convergence is satisfied (though whether he understands that this is what he is doing is open to question). Further thought would make either this or his other suggestion viable: he could treat positive and negative values of x as separate cases, or use the fact that if (a_n) tends to zero then so does (ca_n) (for any real c) in conjunction with the "squeeze rule". What seems to happen is that when a particular algebraic manipulation routine does not give him a straightforward path to the desired result, Tom simply stops. We suggest that he does not think his role involves being creative in mathematics, so that once a procedure is invoked he simply follows this and will not attempt to adapt it.

Lack of meaning associated with a procedure

In the week 9 interview the students are asked to establish for which values of x the series

$$\sum \frac{(-x)^n}{n}$$

converges.

By this time Tom's earlier level of meaningful understanding has collapsed under his search for procedures. Whereas before we saw him instantiate a suitable procedures but lack the inclination to adapt them, here he attempts to use an inappropriate and incorrectly recalled "procedure" on the basis of surface similarities with the given question. He says:

> "Oh right, I know, I remember, that the sum of $1/n$, tends to log n plus, lambda n. Or, it's the sum of, brackets minus 1 close brackets, to the power of n. Divided by n, tends to, log n, plus lambda, n. And I think it's the first one."

He appears to be explaining a confused memory of the process of finding the partial sums of the series $\sum \frac{1}{n}$, which form a sequence $\log n + \lambda_n$ and so

demonstrate that the series is unbounded. What he says would not be correct even if it were appropriate, since it would make no sense for anything to tend to the variable quantity $\log n + \lambda_n$, but Tom apparently remains unaware of this. Also he says he "thinks" that the first expression is the one he wants, but although he is uncertain of this there is no attempt to do any reasoning to check; he seems to rely entirely on recall. He goes on:

> "Okay this is just the formulas, not what they tend to. Formula 1 is the sum of 1 over n, and formula 2 is the sum of brackets minus 1 close brackets n, over n. To the power of n or whatever. So, I think, formula 2, is very similar to the question, except for, x has been given a number, 1, and we've just got to use, formula 1, to create formula 2, to see what it tends to."

The strategy of trying to manipulate a new situation so that it resembles a known one is good, and Tom's outline of a possible solution path is clearly explained. However, in addition to the apparent lack of meaningful understanding of what it is that he is suggesting, by this stage he has lost track of the question. The limit of the series is not required; only the values of x for which it converges. Failing to monitor progress toward a solution may be simply a weakness in this particular area; perhaps he finds the material hard and cannot keep sufficient in his working memory to be able to make this judgement as well. However the evidence in this study suggests that these factors are all related for a learner of the procedural type and, we will see, form a loop.

A cognitive loop

Tom uses the phrase "we've got to" twice in the above excerpts, and we have seen that his use of procedures is inflexible and sometimes inappropriate. This suggests a continued belief that there exist externally-provided rules prescribing what should be done when, rather than natural approaches which he himself could generate based on the intrinsic meaning of the mathematics. We suggest that his lack of progress is due to being caught in a loop involving two complementary aspects – lack of meaningful checking and attending to procedural detail rather than concepts.

Tom does not feel the need to check whether his suggestions "make sense" in a meaningful way. He assumes that the algebraic procedures he is shown are generally useful; having faith that these will work with similar-looking examples, he does not attend to the concepts that the algebra expresses. This actually renders him more or less incapable of performing such checks; he has no meaning for the manipulations so he cannot assess whether the concepts involved in a new situation are really comparable. However Tom does not notice this, because that isn't what he's trying to do, and so on.

The challenge for the teaching at this level is that this does not cause him any great distress. He finds enough in the new mathematics to maintain his beliefs; university mathematics is not all proofs, quite a lot of it involves applying techniques to particular examples and at least at the beginning of the course Tom can cope with this. Combined with the fact that he is not looking for links between concepts, this means that he does not feel he is failing to understand.

A challenge for teaching

A student's progress is limited by what it is that they are trying to do, and while procedural learners have individual strengths, on the whole they fare badly in this first course in Analysis. For example, Wendy can cope with the question about series, using generic examples to reason about different cases:

> Wendy: Erm, if, if you take *x* is between nought and 1, say *x* equal to a half, erm, and put it into the series, you'd get, minus a half, plus a half squared over 2, minus a half cubed over 3, and so on…
> Xavier: So that's going to be smaller.
> Wendy: Erm, the terms are decreasing in size, so…*x n*, is bigger than, *x n* plus 1 (*writing*)
> Xavier: And tending to zero.
> *Pause* (*writing*).
> Wendy: Converges?

This makes it appear that she is keeping up with the course, but in fact she shows serious weakness in forming more general arguments and a lack of understanding of important formal categories. For instance, it is very difficult to persuade her to consider sequences which are not monotonic (see Alcock & Simpson, 1999) and she ends the course believing that a convergent series have its sequence of partial sums eventually constant. Even Tom is not the weakest. Zoe, the other student mentioned earlier, has even less concern for meaning than Tom. Here she describes the approach she and Yvonne take to doing their assignments:

> "What we usually do is when we have our notes, like sprawled out everywhere (*laughs*)…copy bits from here and here and like, put it all together. And hope it turns out right."

Unsurprisingly, they find it difficult to get started even on the questions where Tom has some success, and it often appears that the interview is the first time they have given any meaningful thought to the work.

Of course there are students who have quite different goals. Jenny, for instance, expends a great deal of effort on her assignments and says that she “feels like a fraud” if she hands in something she does not fully understand. However the prevalence of procedural learners and their failure to attend to the aspects of mathematics their university lecturers see as central – the marriage of conceptual understanding and formal deduction (Yusof and Tall, 1995) – begs the question of the role teaching can play in helping students negotiate the transition from school to university.

The obvious direction in which we must now move is to address the question of breaking the loop generated by lack of meaningful checking and attendance to procedural detail. How do we help Tom to be as good at university mathematics as he is at *his* mathematics?!

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