# CHAPTER 5: ANALYSES OF MATHEMATICAL INTERACTION IN TEACHING PROCESSES

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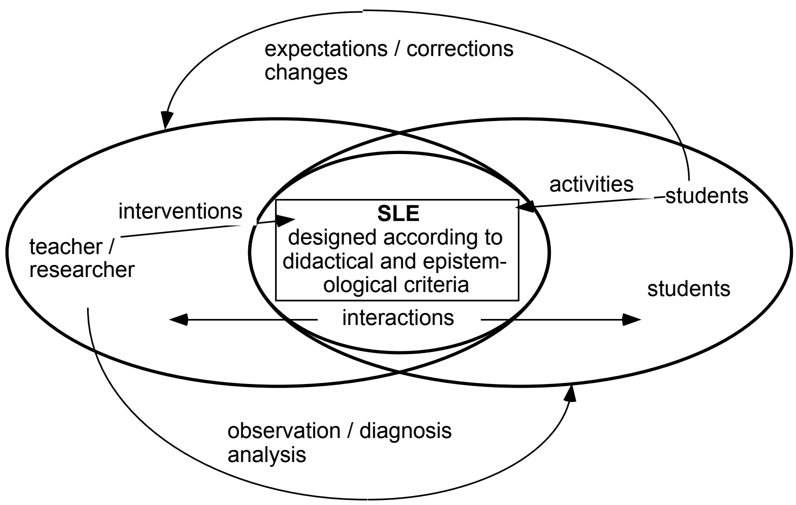
## About the Epistemological Nature of Substantial Learning Environments

Mathematical concepts are no empirical things, but represent relations. "... there is an important gap between mathematical knowledge and knowledge in other sciences such as astronomy, physics, biology, or botany. We do not have any perceptive or instrumental access to mathematical objects, even the most elementary ... The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations." (Duval, 2000, p.61). With regard to this epistemological position, mathematical knowledge is not simply a finished product. The (open) concept-relations make up mathematical knowledge, and these relations are constructed actively by the student in social processes of teaching and learning.

Mathematical learning environments concern the particular character of mathematical knowledge in the following way: On the one hand, it is a matter of concrete mathematical problems which are given in a situative - not formalized - context as immediate learning offers (cf. Wittmann, 2000 & chap. 1). On the other hand, it also becomes obvious that it is not only a matter of concrete mathematical activity, but with the exemplary treatment, something should be learned at the same time, something which is not directly visible, e.g. a mathematical relation or a generalized structure. Essentially, mathematical learning environments develop only in the active construction by the learner. Also, one is not dealing with concrete objects which one could touch, but with *embodiments of mathematical structures*.

## The Role of Learning Environments in the Analysis of Classroom Interaction

Mathematical learning environments essentially represent an open, structural system which was constructed according to didactic and epistemological design criteria, and which can then become an environment filled with life only in classroom interaction through the students' activities and the teacher's interventions. The wired structure of learning environments is illustrated in the following diagram:




When interactively treating a learning environment, a similar problem arises in communication: The fact that the invisible mathematical relations in learning environments cannot be directly communicated by naming visible attributes means that the students have to construct relevant mathematical relations in the present exemplary learning environment with their own mathematical conceptions. This construction by the students is not directly readable in their descriptive statements; one has to analyse the intentions meant in the children's descriptions, and to discover which general mathematical relations the children aim at with their exemplary words.

In the interaction, the children have to deal with the not directly palpable mathematical knowledge and with the hidden relations by means of exemplary, partly direct interpretations - and not by means of abstract descriptions, notations, and definitions. By means of epistemological analysis (cf. Steinbring, 2000a; 2000b) it is to be found out whether the exemplary description used in the documented statement aims at a generalizing knowledge construction or whether it is a statement in the frame of the old, familiar knowledge facts.

**Exemplary Analysis of a Classroom Interaction in a Learning Environment**

On the basis of the following short interaction scene, the joint and situative construction of an "invisible" mathematical relation in a learning environment about number walls (in a 4th grade class) shall be illustrated exemplarily. The children are working on four-stage number walls; the four base stones 35, 45, 55, and 65 have been interchanged several times, which led to number walls - written on the board - with different top stones. The teacher asks about particularities, whereupon the children give descriptions and first reasons for big or small top stones. The student Timo explains his reasons first:

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| 102 Timo | The reason is also that if the sixty-five is in the middle, it is counted twice. Once with the outmost and once with the one next to, that is with both next to it. If it is at the edge, it is only counted once. |

For the time being, Timo stays at his seat; he talks about the base number 65 "in the middle". This is not a mere description, but Timo develops linguistic-conceptual designations: "in the middle", "the outmost", "both next to it", "at the edge". These are not names for the objects ("stones" or "numbers"), but relations between these number-places which are important for the application of the rules for number walls.

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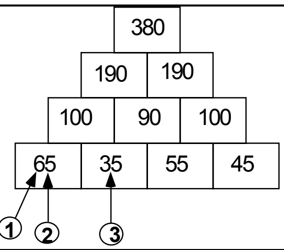
| 103 T | Timo, I would like you to come up to the board and to show that to the other children |

Timo goes to the board, talks, and shows:

| | |
|---|---|
| 104 Timo | Here, (*points at the first stone "65" of the lower number wall with the top stone "380"*) when the sixty-five is here, it is counted only once. (*points at the first and second stone, "65" and "35"*) Here. One moment, plus thirty-five. #   |

Now, Timo does not continue using his "concepts" constructed before; he points at the stones and talks about the numbers. Thereby, his description becomes more concrete and tied to the example. The importance of the attribute "count a base number once" can only become clear to other students through additional, active allusions by Timo, who points at the stone "65" and designates an edge stone at the same time. Timo constructs new "mathematical signs" and symbolic relations with the sequence of designating numbers and showing stones in a wall.

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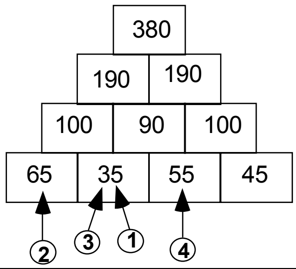
| 105 T | # Please show the children where you are counting to!# |

After the teacher‘s request, Timo points at the calculated "result", the number "100" in the 1st stone of the second stage of the wall.

| | |
|---|---|
| 106 Timo | # Yes, there (*points at the first stone "65"*) is the sixty-five, there (*points at the second stone "35"*) plus that are a hundred. (*points at the stone "100"*) But if the sixty-five stands here, (*points at the second stone "35"*) that one there and there the thirty-five, (*points at the first and second stone, "65" and "35"*) this one here plus thirty-five and there plus fifty-five (*points at the third stone "55"*) is counted once more. |




In the second step of his argumentation, Timo refers to the 2nd stone - a middle stone - of the same number wall. He uses the same number wall, but varies the position of the "65": he moves it mentally from the left edge stone to the middle stone next to it. This is what he intends by pointing at the corresponding stones. By means of the possibility that "65" and "35" could switch positions, Timo stresses that it is mainly a matter of the different positions (edge or middle stone); in this way the example contains aspects of a general interpretation. Furthermore, Timo emphasizes that this number now has to be added twice: "... counted once more." The teacher asks Timo to show this possibility for "65" as a number in the middle in an appropriate example.

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| 107 T | Great. But now, please, choose an example where the sixty-five really is in the middle, then we can imagine even more easily how you mean that. (*Timo points at the second stone "65" of the lower number wall with thetop stone "440"*) yes. Mhm. |

Timo points at the lower wall with top number "440":

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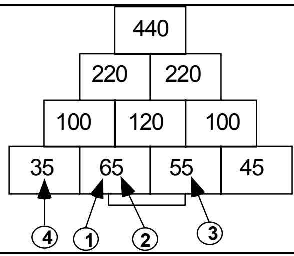
| 108 Timo | Plus fifty-five (*points at the second and third stone "65" and "55"*) and plus thirty-five. (*points at the first and second stone "35" and "65"*) |




Here, the rather "general" interpretation "if the 65 stands here" (on the 2nd stone) is concretized. Timo now only names the readable addition tasks. The possibility that, not only the concrete calculations and results are meant hereby, but that e.g. the middle number "65" is counted *twice*, cannot be inferred from the statement, but has to be constructed actively by the other participants.

Timo begins a partially general interpretation of the *relations* between the positions of the base stones, which affect the number of calculations and therefore the size of the top stones, and which could lead to a mathematically full reasoning in the frame of the exemplary environment. The teacher requests Timo to concretize his considerations several times in the progress of the interaction. Timo follows this request, also where he in thoughts flexibly places the "65" instead of the "35", by now taking a number wall with the appropriate numbers. The span between the original exemplary conceptual generality in Timo‘s description and the teacher‘s request for concretization is narrowed down reciprocally until the general description has been substituted by concrete numbers. Timo‘s declaration seemed too abstract and therefore incomprehensible - although the teacher aimed at the just not directly visible structural relation in the environment "number walls", which cannot be described directly. In contrast to this, the designation of calculations with the given, concrete numbers and the written results, as they follow according to the construction rules of number walls, could, in tendency, only have led to a simply verifying confirmation of mathematical facts, without offering insights to the invisible relations. The "complementarity" of intended generality - in Timo‘s first description - and reducing concretization to the given numbers and examples - as requested by the teacher - seem to lead to an optimum of constructive interpretation and communicative understanding in this classroom interaction.

**The Specific Social Epistemology of Interactively Constituted Mathematical Learning Environments**

The particular epistemological character of mathematical knowledge consists in the concentration on *relations* which are neither openly visible nor directly palpable. (Duval, 2000). In order to develop these relations and to be able to operate with them, they have to be represented by signs, symbols, words, diagrams, and references to reference contexts (Steinbring, 2000c), learning environments, or experiment fields. Thereby, the scientific status of the mathematical knowledge does not depend on the choice or the abstractness of the means of representation; neither are there any universal means of illustration distinguished a priori which would automatically guarantee the epistemological quality of the mathematical knowledge (cf. Ruthven, 2000). The development of mathematical knowledge always occurs - be it in the academic discipline or in classroom learning processes - in social contexts which can, however, differ concerning their objectives and particular constraints (Steinbring, 1998).

Mathematical, substantial learning environments - the core element of the research project "mathe 2000" - represent such experiment fields which are suitable for inter-active, social learning and developing processes in different situations of learning and acquiring mathematical and didactic knowledge in the classroom or in the training and in-service training of teachers. Which epistemological conception of mathematical knowledge becomes relevant is not simply determined beforehand and objectively, but this is subject to the active construction in the communication and in the proceeding interactive processes. Exemplary, epistemological analyses show that conceptions of the knowledge of the following kind can be constituted at this point: Mathematical knowledge as a collection of single facts, or mathematical knowledge as isolated, formal structures (cf. Steinbring, 2000b).

In the exemplary episode, a third and essential epistemological attribute of mathe-matical knowledge appears: A situatively tied form of describing and constituting the relations of mathematical knowledge in the frame of the exemplary learning envi-ronment; using exemplary, independent descriptions and words, but with the inten-tions - identifiable in the analysis - of generalizing exemplary attributes of the situa-tion to the invisible general mathematical relations. In this regard, substantial learning environments represent a productive base for the interactive acquisition of knowledge, on which knowledge about mathematical knowledge can be acquired through the interaction at the same time, i.e., in the interaction, a specific, partly situation-bound, social epistemology of mathematical knowledge constitutes itself - which is not given by an independent authority from the outside.

This particular social epistemology constitutes itself in the proceeding of the according situation, for example during the treatment of a learning environment, and for this purpose, it needs situative, exemplary context conditions as well as words and relations already known and familiar for communication. In order to understand how relations in mathematical knowledge - which are not directly, empirically palpable - can actually be expressed and communicated in this way, a thorough epistemological analysis is required (Steinbring, 2000b). Such qualitative analyses of different situative epistemological interpretations of mathematical knowledge in interactive treatments of learning environments have different objectives and react upon the perspective of mathematical knowledge taken in the different chapters. So, feedback to the design and construction of learning environments, especially such modifications which make these environments become living systems, occur; furthermore, testing analyses of environments by teachers or students can increase the awareness (Selter, 1995) concerning the complex (professional) application conditions as well as the classroom interaction with mathematical learning environments.

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