CHAPTER 1: DRAWING ON THE RICHNESS OF ELEMENTARY MATHEMATICS IN DESIGNING SUBSTANTIAL LEARNING ENVIRONMENTS

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By means of an example (arithmogons) it is shown how the design of substantial learning environments for both students and student teachers is based upon the progressive elaboration of the epistemological structure of elementary mathematics.

Progress in group theory depends primarily on an intimate knowledge of a large number of special groups.
Graham Higman (1957)

In the first volume of the ”mathe 2000” textbook the learning environment “Rechendreiecke” is introduced in the following way:

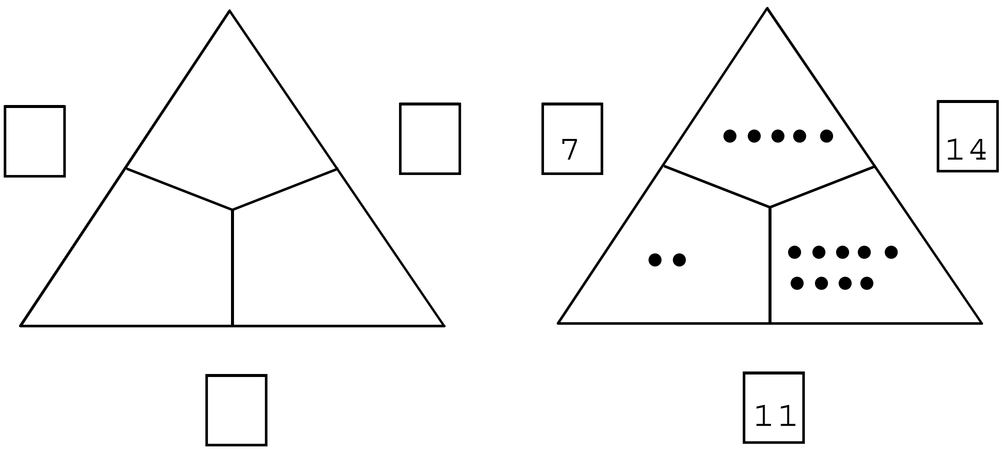

Fig. 1

A triangle is divided in three fields. We put counters or write numbers in these fields. The simple rule is as follows: Add the numbers in two adjacent fields and write the sum in the box of the corresponding side (see Fig. 1).

Various problems arise from this context: When starting from the numbers inside, the numbers outside can be obtained by addition. When one or two numbers inside and respectively two or one number outside are given, the missing numbers can be calculated by addition or subtraction. When the three numbers outside are given, we have a problem that does not allow for direct calculation but requires some thinking. Firstgraders can find the solution by (more or less) systematically varying the number of counters in the inner fields. There are, however; also systematic solutions. By studying the behaviour of a ”Rechendreieck” one understands why there is exactly one solution. Interestingly, the solutions are not always natural numbers. If the sum of any two of the outer numbers is smaller than the third number at least one of the inner numbers is negative. If one outer number or all three are odd the inner numbers are fractions (denominator 2). So ”Rechendreiecke” is a nice context to cross borders even in grade 1.

It is obvious that ”Rechendreiecke” forms a substantial learning environment in the following sense (Wittmann 2001):

(1) It represents central objectives, contents and principles of teaching mathematics at a certain level.

(2) It is related to significant mathematical contents, processes and procedures ***beyond*** *this level, and is a rich source of mathematical activities.*

(3) It is flexible and can be adapted to the special conditions of a classroom.

(4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.

In order to illustrate the driving forces behind the design process it will be explained in the following how in ”mathe 2000” learning environments come to life and find their way into the curriculum: elementary mathematics provides the raw material and the above properties of a substantial learning environment serve as a checklist.

The following problem of elementary geometry is well-known (Fig. 2): For a given triangle with sides a, b, c construct three tangent circles around the vertices.

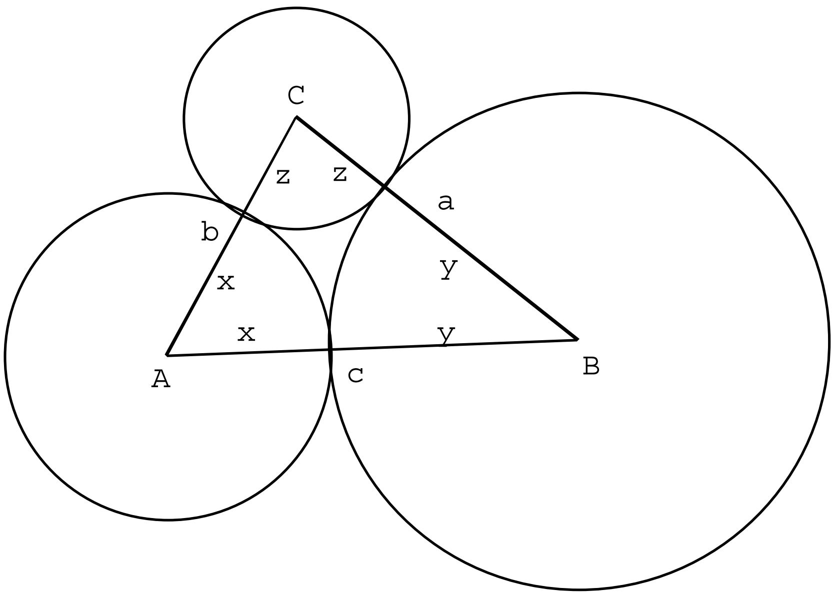

Fig. 2

Solution: The unknown radii x, y, z fulfil the relations $x + y = c$, $y + z = a$ and $x + z = b$. Therefore $c + a = x + z + 2y = b + 2y$ and $y = \boxed{}(c + a - b)$. Correspondingly $x = \boxed{}(b + c - a)$ and $z = \boxed{}(a + b - c)$ are derived.

The analogous problem for quadrilaterals has a somewhat different structure: It is solvable if and only if the quadrilateral has a circumcircle. In this case the set of solutions is infinite. A closer investigation shows that the case of triangles is typical for polygons with an odd number of vertices and the case of quadrilaterals typical for polygons with even vertices.

This geometric problem appeared in Sawyer 1963 (p. 150 ff.) in the disguise of the ”bears problem”: Someone wants to mount gates on each of three posts A, B, C such that gates x and y close side c, gates x and z close side b, and gates y and z close side a (in order to keep bears coming from various directions away).

The problem is also generalized by Sawyer to more than three posts.

McIntosh&Quadling (1975) chose another algebraic setting for this problem: An equal number of circles and squares are arranged like the vertices and sides of a polygon. Numbers have to be filled in such that each square carries the sum of the numbers in he adjacent circles (Fig. 3).

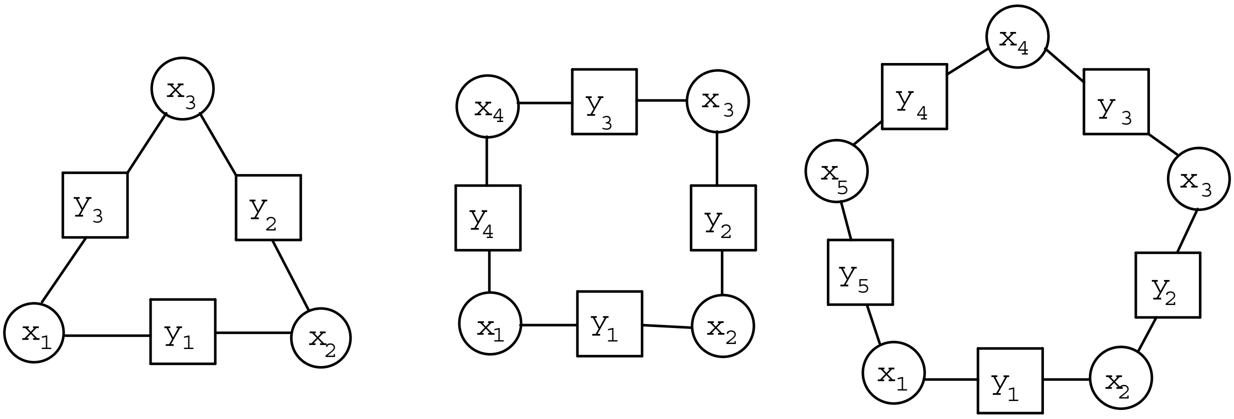

Fig. 3

The authors elaborated the mathematics underlying this structure: The mapping $(x_1,...x_n) \rightarrow (y_1,...y_n)$ is a linear mapping from R^n into R^n. The corresponding matrix is non-singular for odd n and singular for even n (rank $n-1$).

Another important step forward was made by Walther 1985 who transformed arithmogons into ”arithmetic chains” in order to get a richer variety of activities: When the sums (lower row) are given children can investigate how the target number (and the other numbers) depend on the start number (Fig. 4). In particular children can find out if it is possible to have the same number as start and target.

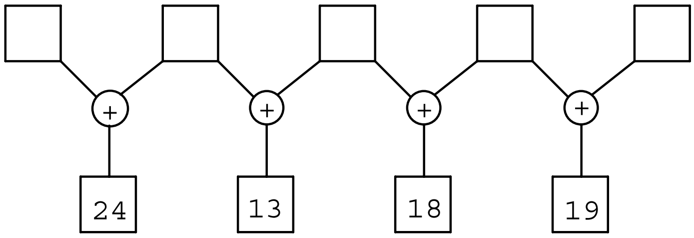

Fig. 4

Like McIntosh&Quadling Walther traced arithmetical chains through the grades up to the upper secondary level by unfolding richer and richer algebraic structures.

In subsequent papers the structure of ”arithmetical chains” was generalized to other chains and the analytical background (fixpoints of functions) was elaborated further.

There was no question that arithmogons because of their mathematical substance should become part of the mathe 2000 curriculum. However, the question was at which place and in which form they should be introduced. As in the project it is a general principle to introduce fundamental ideas as early as possible the above version of a ”Rechendreieck” was chosen as an appropriate format for grade 1: counters can easily be moved around and thus favour children’s exploratory activities and early reasoning.

The mathematical substance behind arithmogons (property 2) allows not only for revisiting this context over the grades but also for using them in teacher education. In a course on ”Elementary Algebra” for primary student teachers with mathematics as a major subject the structure of arithmogons has been generalized to polyhedra in an obvious way: Numbers are assigned to vertices and faces such that the numbers assigned to the faces are the sum of the numbers assigned to the vertices of this face.

In this way a rich universe of examples arises which can be explored by student teachers. All phenomena and all concepts relevant for the theory of systems of linear equations occur and can be studied and explained in this context: linear mappings, independence of a system of vectors, kernel of a linear mapping, image space, rank of a matrix, rank of the augmented matrix, basis and dimension of a subspace, Steinitz’ theorem, dimension theorem. The theory, however, does not hover in the air. It is developed ”just as far as is necessary to frame a certain class of problems” (Giovanni Prodi).

In this context even systems of linear equations over a finite field make sense: As shown above a ”Rechendreieck” does not allow for an integer solution if one outside number or all three are odd. As the properties ”even” and ”odd” can be represented by the numbers 0 and 1 of the field Z_2 we get a system of three linear equations over Z_2 (Fig. 5). The rank of the corresponding matrix (Fig. 6) is only 2, not 3 as before.

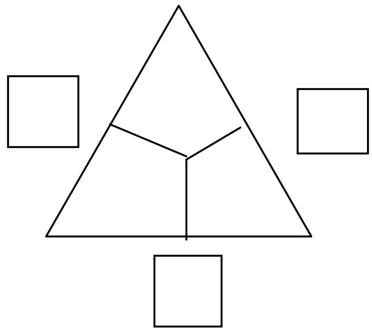

Fig. 5

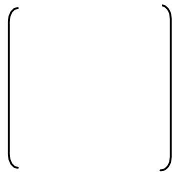

Fig. 6

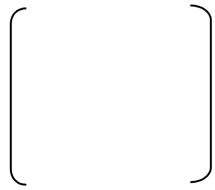

Fig. 7

When Gaussian elimination is applied to the augmented matrix (Fig. 7) we get $a + b + c = 0$ as a necessary and sufficient condition for the solvability of the system. In ordinary language this means that either none of the numbers or two of them are 1 (which means ”odd”). For student teachers it is intriguing to see how ”abstract” algebra applies to ”concrete” numbers.

Concluding remark

”Arithmogons” is an example of local design. However, the ”mathe 2000” research is not restricted to designing isolated learning environments. In fact the project’s central publication is the ”Handbuch” (Wittmann&Müller 1990/92) in which a comprehensive concept of teaching arithmetic in the primary grades is given in the form of coherent systems of substantial learning environments. The main emphasis in the ”Handbuch” is on the epistemological structure of the subject matter. This does not mean that we consider psychological and social studies into learning processes as less important, but that we consider a profound knowledge of the epistemological structure of the subject matter as a pre-requisite for teaching as well as for these studies. In addition we strongly believe that providing teachers with substantial learning environments and the necessary epistemological background strengthens their role as reflective practitioners.

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