A STUDENT‘S UNDERSTANDING OF MATHEMATICAL INFORMATION

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The possible causes of elementary and secondary students' difficulties with grasping arithmetic information embedded in a word problem are studied using the "decompose and reconstruct" experimental setting (modified from one type of message games) and theory of meaning making. Concepts of elementary datum and elementary image are defined for research purposes. Seven obstacles inhibiting a student's grasping of the word problem or making its grasping impossible have been identified and analysed.

1. Introduction and framework

In everyday life, numbers play many different roles (Verschaffel, De Corte, 1996, Freudenthal, 1983): to quantify (cardinal aspect), to identify an object's location in a sequence or in a group (ordinal aspect), to measure, to reckon, or to name things. The better the student's discernment of these roles, the more effectively he/she can use his/her arithmetic knowledge in everyday life and the better is his/her insight into the world of arithmetic.

The diversity of number notions in real world is most richly reflected in word problems. Many studies have dealt with the classification of addition, subtraction, multiplication and division situations (e.g. Vergnaud, 1983, Nesher, 1988, Verschaffel, De Corte, 1996, Verschaffel, Greer, De Corte, 2000). The effect of the format of a word problem (verbal or pictorial) on the solution to the problem has been studied e.g. by Minato, Honma, Takahashi (1993). A student's ability to pose (word) problems was studied by English (1997) before and after a carefully prepared teaching programme which brought to student's attention all aspects of word problems (their structure, context, semantic relations, their "critical information units", etc.). The readability factors in the ordinary language of mathematics texts for (second language) learner are investigated in Adetula (1990) and Prins (1997). Prins (1997) identified a variety of phenomena influencing readability and consequently a student's solution to a problem stated in words. The following are particularly related to our research: difficult vocabulary, text structure, obscure information (i.e. confusing information, culturally biased contexts, contradictory and senseless information).

We hypothesise that the core of the difficulties that (not only) Czech and Slovak students have with word problems lies in their inability to grasp them, i.e. to transform (to model) a situation described in words into the language of equations. A teacher trying to "teach" students to solve word problems often presents them with instructions how to transform a particular type of problem (e.g. word problems on common work, on age, on filling up a swimming pool, on movement, etc.) into equation(s). These instructions, however, replace, rather than promote real understanding. They may help a student solve a standard task but they do not enable him/her to create a clear and rich image of a situation.

We have been observing and analysing processes of students' grasping of situations described in words for many years. In this contribution, we shall present one type of such analysis, based on the atomisation of numerical information and the corresponding number image.

2. Methodology

2.1 Instructional Setting

In our research, the centre of attention is a set of students' semantic images of a number(s). Its starting point is our pedagogical experience – a database which we have acquired through classroom observation and/or experiments. The experimental setting used in this research has been designed as a modification of *an instructional setting* used in our experimental teaching for the development of a student's ability to grasp a mathematical text.

The setting, tentatively labelled "decompose and reconstruct", belongs among so called message games and is based on the following sequence of students' activities: student A gets a certain mathematical text, for instance a word problem, and he/she will decompose it into a set of elementary information units (by an elementary information unit, we mean a sentence or a phrase including one number); student B then tries to reconstruct the original text from them. For instance, this instructional context was used in a game based on the familiar game Chinese Whispers. Groups of four take part, each group has two "decomposers" and two "recontructors". They work on their own. A teacher prepares text *T* and everyone is given the number of words of *T* and the number of words a decomposer can use. The teacher hands over to the student A a text which will undergo a series of transformations.

$$T \underset{A}{\overset{d}{\longrightarrow}} T_1 \underset{B}{\overset{r}{\longrightarrow}} T_2 \underset{C}{\overset{d}{\longrightarrow}} T_3 \underset{D}{\overset{r}{\longrightarrow}} T_4$$

Where T_i are texts, d = decompose, r = reconstruct and letters A to D stand for students.

Each team is evaluated according to the information value of their texts T_4, i.e. to what extent the information of text T_4 corresponds to that of the original text *T.*

A similar setting was investigated in the plenary panel in PME20 (Puig, Gutierrez, (eds.), 1996, pp 53-84). One pair of students was first asked to solve a presented word problem and then to write a message explaining their solution to their friends that have to solve a similar problem, *without using any number* in the message (later specified as "to write it in mathematics using mathsymbols"). The second pair of students was then asked to solve their problem according to the method described in the message.

2.2 Experimental Setting

The instructional setting "decompose and reconstruct" has been modified for research purposes and used in research in two ways. First, pairs of students took part in experiments during which one of them was given a text and asked to decompose it into elementary information units so that there was exactly one number in each unit. Then the second student was asked to reconstruct the original text. Experiments were recorded and observed by experimenters.

Second, experimenters chose fourteen concrete word problems from Czech, Slovak and Polish textbooks (some of them will be given below as examples) and decomposed them into a list of elementary information units. In some cases, the decomposition was done by other people as well, e.g. by students and teachers. During this process, the need arose for the specification of basic concepts, e.g. the concept of decomposition itself or the concept of elementary information unit. For instance, during an interview one student decomposed the following information "I have 25 crowns in my piggy-bank", which we considered to be elementary, into two elementary information units "I have a twenty-crown note in my piggy-bank" and "I have a five-crown coin in my piggy-bank". In other words, the quantity of 25 crowns, which was considered as one number by most students, was regarded to be two numbers.

To get a more complex picture, some of the analysed word problems were given to students and their solutions were recorded.

2.3 Definitions of Concepts and Our Assumptions of Understanding Texts

Karl Popper (Lorenz, Popper, 1994), in line with Bertrand Bolzano, speaks about three worlds: the World of Objects (mass and energy), the World of Culture (everything created by the humankind) and the World of Mind (everything which is present in the minds of individual people). In concordance with this view we will consistently distinguish between *information* which belongs to the World of Objects and *image,* evoked by the information, which belongs to the World of Mind. These two concepts are interconnected through a pair of projections:

grasping	information $\rightarrow$ image
articulation	image $\rightarrow$ information

The wide space of information will be narrowed down to numerical information embedded in the World of Objects. Such information will be called *a datum*. Let us specify the basic terms of our considerations even further.

By *an elementary datum*, we mean a statement which

(a) includes at least one known or unknown number (i.e. the question in the word problem will be considered an elementary datum, too),

(b) has an unambiguous and if possible also briefly described connection to the world of objects, and

(c) cannot be further decomposed.

For instance, the statement "I bought things for 18 crowns" is an elementary datum, while neither "16 is an even number" nor "Peter had one more crown" nor "in the classroom there are 5 girls and 11 boys" are elementary data; the first one has no connection to the real world, the connection of the second one is not clear (we do not know with whom Peter is being compared) and the last one can be further decomposed.

The concept of elementary datum has two components: (1) a number or numbers, (2) its (their) embedding in the world of objects. By grasping an elementary datum, *an elementary image* originates. The concept of elementary image has three components. Besides (1) and (2), it also includes its location in the mind of a concrete person.

By the *decomposition of text T* we mean a sequence of data $D_1, D_2, ..., D_k$ such that each number (known or unknown) of text T is contained in exactly one of these data and the semantic meaning of a number in the text T and corresponding data D_i is the same. The inverse process to decomposition is the *reconstruction of the text*. The decomposition is considered to be good if it is brief and if someone else is able to reconstruct the original text or at least its mathematical layer from it.

We have seen that decomposition is a subjective activity – two persons can make two different decompositions of the same text. Moreover, the above characterisation can rarely help us to decide which of several decompositions made by teachers or students is the best. Yet, the given characterisation helps us both in analysing and describing the investigated cases.

The process of decomposition has brought to light several interesting phenomena which play an important role in a student's grasping of a mathematical text and may sometimes function as obstacles in this process.

3. Some Phenomena Elucidating a Grasping Process

In this section, we will describe seven phenomena identified as important elements in some grasping processes.

3.1 Non-verbal Information

By non-verbal information, we mean information whose main carrier is a picture, table, graph, scheme, etc. The importance of this kind of information is emphasised by the fact that pictorial information often appears at a pre-school and early school age and thus plays a key role in the process of developing a student's attitude towards mathematics. The example of non-verbal information is the picture in example 1 below.

Non-verbal information can be characterised with five criteria.

1. Adequacy: information falls into a student's experience.
2. Comprehensibility: information is presented in such a way that a student is able to create its image for him/herself.

3. Non-ambiguity: information does not allow for several different interpretations (see the next section).
4. Memorisibility: information as a whole and its individual parts contribute to its holding in a student's memory.
5. Motivation: information increases/decreases a student's personal interest.

What we call non-verbal information falls into the category of (external) representations which has been given a considerable attention (e.g. Verschaffel, De Corte, 1996) and it can be a source of serious learning difficulty as "such devices do not speak for themselves" and "their meaning must be constructed by the learner" (Becker, Selter, 1996) and as will be seen in the section below.

3.2 Vague (Ambiguous) Information

Example 1 (Kovácik et al, 1995):

I purchased three things for 18 crowns.
Draw them.

When asked by their teacher, both Eve and Mike said that a roll costs 2 crowns and cheese 8 crowns. It appeared that contrary to other students who gave various different interpretations of the picture Eve and Mike had the same image of the picture. This assumption proved to be wrong. Later when solving the problem, they disagreed. Eve saw the picture as a shop window offering six different kinds of goods. In view of this image, she included the case 5+5+8 (two bananas and cheese) in her solution. Mike, who understood the picture as a set of six different things, rejected this solution and gave only solutions in which each object was used once (like 2+5+11 = a roll, a banana, yoghurt). This illustrates the fact that a person's image is a multilayered concept. Two images which seem to be the same in one layer may differ in another. Thus we can say that the information in the picture is *vague (ambiguous)*.

Vague (ambiguous) information can be seen from two points of view. (a) From a mathematical point of view it allows for different interpretations than that of the author. (b) From the didactic point of view it allows for two different but meaningful students' interpretations. We consider the latter vagueness to be desirable at school. We agree with Byers (1998) that "we tend to react to every presence of ambiguity by attempting to remove it rather than by working with it" which is often the case of both the authors of mathematical textbooks and teachers in the Czech Republic. The rich discussion which began over Eve's and Mike's interpretations can be very informative because:
1. it improves a student's sensitivity to possible ambiguity in the presented information, it teaches him/her not to be content with a protetic interpretation (e.g. to

use a key-word strategy solution (Verschaffel, De Corte, 1996)) but rather analyse it critically,
2. it shows students that the same situation can have several different interpretations and that the solution to any problem must begin with the specification of the situation (i.e. choosing one of the possible interpretations),
3. it helps overcome a widely held belief that a math problem has always one clear solution.

The prerequisite is that the teacher him/herself (1) finds the problem clear, (2) can see what different interpretations it can have, (3) is willing to monitor students' discussion.

3.3 The Word "About"

Example 2 (Repás et al, 1997): A bar of chocolate costs 12 crowns. A teacher bought 29 bars. He paid for the chocolate about .... crowns.

Example 3 (Cernek, Repás, 1998): My 10 steps are about .... metres.

In examples 2 and 3, the word "about" plays an important role. It indicates approximation. But while in example 3, it confirms the fact that there is no right solution to the problem, in example 2 there is an exact solution. In other words, in example 3 approximity is related to the result and in example 2 to the solving process. Students' solutions to the problems from examples 2 and 3 revealed four possible interpretations of the word "about":

1. I am to find an exact result and *round it off* (the answer was about 350).
2. I am to *guess* the result (i.e. $30 \cdot 10 = 300$).
3. I am to show how well I can *guess* the comparison of the lengths of "step" and "meter".
4. I am to *experiment* – I am to pace 10 steps and measure the distance. I can repeat the process and find the average.

The word "my" in example 3 shows that there is no given answer, that the result will be individualised. This phenomenon can be called **subjectivity of information.**

3.4 Comprehensibility

Example 4: The match ended with the result 3:2. In half-time, the score was 0:0. How many goals did the home team score?

As assumed above, the information is comprehensible for a student if he/she is able to create an image of it for him/herself. This phenomenon applies primarily to a student's image rather than to information written on paper. The same information can be comprehensible for one person and less comprehensible for another (e.g. Jane knows that one number means the number of goals of the home team and the second that of guests but she does not know which is which which became clear when she was asked to decompose the text) or totally incomprehensible.

3.5 Hesitation

By hesitation, we will mean a psychological state of an individual who is forced to decide a matter important for him/her but he/she lacks the necessary information.

Examples: (a) a student cannot remember whether 0 was assumed to be a natural number, or (b) if Pi=3.14 is an exact equality or not; (c) a student does not know if the formula for the area of a triangle $A=b{\cdot}h/2$ holds for the obtuse-angled triangle; (d) a student solves the problem in Example 3 and hesitates if 8 metres would not be considered by the teacher as wrong.

The examples (a) and (b) concern isolated facts which must be remembered and the reason for hesitation is memory failure. The third one is different. The student should be able to use his/her knowledge to resolve the problem, e.g. by relating the area of a triangle to the area of a quadrilateral. The last case is sad because the source of the student's problems is not his/her ignorance but fear that he/she would not guess the teacher's expectations right (see didactic contract – Brousseau, 1997). It is not a rare case in Czech schools. Consider the following example which highlights the difference between a mathematical and didactic conceptions of mathematics.

Example 5 (Kovácik et al, 1995): Ela took less than 17 steps. How many steps could she take?

The problem is in word "could". A mathematician can see no problem – the result can be any number between 0 and 16. However, the reaction of a student who believes that there must be a single correct answer to each problem can be different. He/she can be bewildered by this question.

3.6 Implicit information

Again, this phenomenon depends on the reader's experience. The space of implicit information can be classified according to two criteria:

1. *What* is hidden – a number, its meaning or relation.
Examples: (a) a number is hidden in the words "double-headed", "kilo" (meaning one kilo), "week" (one week or seven days), "goalless" (it hides either one number – no goal was scored in the match – or two numbers – guests scored 0 goal and the home team scored 0 goal);

(b) the meaning of a number is hidden in the label "the result was 3:2" (it is partly hidden for Jane above);

(c) the relation is hidden in the group of words "gross, nett, tare" as in the word problem "If gross weight is 58 dg and net weight is 35 dg, what is the tare?" (Demby, Semadeni, 1997).

2. The *carrier* of the implicit information – it can be a word, a group of words, a sign, a picture, a table, etc.

4. Conclusions

The "decompose and reconstruct" setting has been used in our experimental teaching for improving a student's ability to understand the text of word problems. It's research modification described above has been continuously elaborated. First, we try to standardise the techniques of analysis of material acquired through the "decompose and reconstruct" setting, second, we are looking for its other variants. The list of the seven given phenomena is being enriched by other phenomena and restructured.

A serious problem which has not been addressed in our research so far is how to apply this method in practice, how to convince teachers of its efficiency. In this respect, we direct our attention at university students – future teachers.

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