ARGUMENTATIVE PROCESSES IN PROBLEM SOLVING SITUATIONS: THE MEDIATION OF TOOLS

Catia Mogetta
Graduate School of Education
University of Bristol

Abstract

Argumentative processes enacted in problem solving situations involve both concrete and discursive operations. An analysis of the processes of solution needs to take into account three main components: the problem, the agent and the context of solution (including all the tools available, other individuals and the situation for the devolution of the problem). When the context incorporates a dynamic geometry software, additional aspects related to the availability of dynamic tools need to be considered. The paper illustrates and analyses the role of heterogeneous tools, such as concrete tools and conceptual and theoretical tools in the process of elaborating a conjecture and proving it. Possible conflicts due to the static nature of the theory and the dynamic nature of the tools for exploration are discussed.

Introduction

Solving a mathematical problem requiring the elaboration and proof of a conjecture is a complex activity, made up of a number of phases and involving discursive, concrete and mental operations. The literature accounts for a number of studies focusing on the proving process and especially on the possible relationship between the production of the conjecture and the construction of a proof for it (Duval, 1992-93, Mariotti et al. 1997). Despite the undoubted differences between the argumentative and the proving discourse, a cognitive continuity between them seems possible, under specific constraints on the problem situation. An Italian research group (Boero et al. 1996, Mariotti et al.1997, Garuti et al.1998) has elaborated the theoretical construct of cognitive unity, that attempts to describe and analyse such continuity:

> **CU**: during the production of a conjecture, the student progressively works out his/her statement through an intensive argumentative activity, functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organising some of the previously produced arguments according to a logical chain. (Garuti et al, 1998)

For the purposes of the present paper, the cognitive unity (CU) construct will be considered as an analytical tool to interpret and explain some of the processes students engage in when striving to organise the informal arguments produced during the solution process, into a logical chain that corresponds to accepted mathematical rules. Concrete and discursive operations involved in the production and enchaining of arguments have a cognitive counterpart in mental operations for which they are the external signs. In order to trace the evolution of arguments toward a deductive discourse within the solution of a problem, I will set the issue in the mediated activity framework. From this perspective, in any problem solving situation three main components may be identified: the problem, the agent, i.e. the individual-acting-with-mediational-means (Wertsch, 1991) and the context of solution (including all the tools available, other individuals, be they peers or teachers, and the situation for the devolution of the problem). The study reported in this paper focuses on geometrical, open problems tackled by 11th and 12th grade students in a context including the dynamic geometry software Cabri-Géomètre (Baulac et al.,1988).

The mediation of tools: the notion of toolkit.

Within the context outlined in the previous section, the word 'tool' incorporates many different meanings and refers to both concrete and psychological tools. Drawing on the seminal work of Vygotsky, tools and signs (i.e. psychological tools) may be distinguished according to their function:

> The tool's function is to serve as the conductor of human influence on the object of activity; it is *externally* oriented; it must lead to changes in objects. […] The sign, on the other hand, changes nothing in the object of a psychological operation. […] the sign is internally oriented. (Vygotsky, 1978, p. 55)

In some cases the distinction cannot be neatly drawn: within a certain activity, since some of the externally oriented tools may be internalised and function as psychological tools. The internalisation process, as well as the relationships among the tools used within a certain context, are complex and manifold. In order to describe them I have introduced the notion of **toolkit** (Mogetta, to appear) as an organised set of (both externally and internally oriented) tools that each individual develops and uses in a particular context. The idea of toolkit is meant to account for:

- *the diverse and manifold nature of its components*. Verbal or written signs, symbolic systems of notation, drawings, constructions and changes of configuration, dynamic manipulation through dragging, measurement and gestures may all be subsumed under the category of externally oriented tools, since the actions they mediate aim at changing the external, phenomenological world. Language, strategies of solution, theorems and definitions and all sorts of conceptual tools belong to the category of internally oriented tools, which shape and influence the mental processes enacted along the solution;
- *the relationships among the components, that may change and evolve continuously*. Mutual relationships among tools that are heterogeneous in nature may be developed, along the solution of a problem, either through a process of internalisation of externally oriented tools or through a joint use of different tools, that are necessarily re-interpreted (and, in some cases, re-conceptualised[1]) in a new context, as they may acquire a new functionality.

[1] For instance, a student might use a theorem as an exploration tool in an initial phase of the solution, seeking possible properties of the configuration at hand, and later reuse the same theorem in order to justify the conjecture. In this case the same conceptual tool 'theorem' is functionally related to the phase of solution and assumes the value of argument when a proof is constructed.

The individual has a crucial role in the management of his/her personal toolkit: the evolution that occurs within the context of a problem situation is basically subjective and does not follow general rules. Wertsch (1991) refers to 'privileging' as a dynamic strategy to select one mediational means as more appropriate or efficacious in a particular socio-cultural setting. For example, in the context of the solution of a geometrical problem the choice of a theoretical tool, like a theorem, may be influenced by the exploration of the figure by means of the dragging tool (in Cabri): a particular configuration may be visualised and recall the mental image associated to a particular theorem. Nevertheless, in order for that theorem to function as a tool, the agent must have developed a modality of use for it. The basic idea is that any object or concept need to undergo a process of instrumental genesis (Rabardel, 1995, Verillon & Rabardel, 1995) in order to be fruitfully used for a certain purpose. Such process is described as turning an **artefact**, i.e. "the particular object with its intrinsic characteristics, designed and realised for purpose of accomplishing a particular task" into an "**instrument**, that is the artefact and the modalities of its use, as are elaborated by a particular user" (Mariotti, to appear). Schemes of use are individually developed and shape and organise the actions performed by an agent within any mediated activity.

Heterogeneous tools: is a harmonisation possible?

The re-organisation of tools within the specific context of the solution of a problem brings about a re-interpretation of previously acquired tools and the development of (possibly new) appropriate modalities of use for them. The process of re-interpretation is not necessarily successful, since there might be conflicts between the phenomenological and the theoretical worlds (Balacheff & Sutherland 1994), which coexist in the context where the problem is tackled. If the internal relationships among the elements constituting the toolkit mainly involve tools of the same type, and if the concrete and psychological tools are not related to an organised system of theoretical knowings[2], the world of theory and that of phenomenology can stay separated. Cabri is a microworld that incorporates the basics of Euclidean geometry as well as tools that allow a dynamic exploration and that give visual and conceptual feedback to the agent (Laborde, 1998). It may offer a good ground for the construction of a link between operations related to the phenomenological world and concepts and operations within the (Euclidean geometry) theory. Once they have been internalised, tools like dragging or even a generalised use of measurement, may control behaviour and shape the process of solution. As a consequence, two possibly conflicting situations might occur at the same time. On the one hand the mental processes based on an empirical approach and on visual evidence, that are spontaneously used by the agent, may be related to more rigorous and deductive processes and, consequently, the argumentative phase can end up in a proof. On the

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[2] The term knowing has been introduced by N. Balacheff to account for the distinction between the French *savoir* and *connaissance*, which is not possible in English. So knowledge is the English for savoir and knowing is the English for connaissance.

other hand, the co-existence of informal/empirical elements and formal/theoretical elements, can bring about cognitive conflicts that cannot be overcome spontaneously.

The changes and evolution of the internal organisation of each individual toolkit during the solution of a problem are important for the argumentative process. When the concrete (externally oriented) tools are used in harmony with the theoretical (internally oriented) tools, the elements and information gathered along the conjecturing phase may be linked systematically within a structured argumentation that fits with the rules of the theory (Mogetta, to appear), according to the CU hypothesis. Things are not so linear in the actual solution of a problem: the intermingled use of tools of different nature requires a continuous re-organisation of the toolkit and an internal re-negotiation of the meanings previously attached to some of the tools.

The case of Andrea: when tools of different nature are not harmonised

The data analysis carried out for the study[3] reported in the paper attempts to show how the personal toolkit of an individual agent may change along the solution of a problem and how the relationships among the tools can evolve. Such changes might head toward a harmonisation of empirical/perceptual and theoretical aspects, or rather cause (and show) a cognitive rupture between the argumentative process and the actual production of a proof for the conjecture. This section illustrates the case of a student who does not manage to harmonise tools of a different nature, thus staying at an empirical and perceptual level in the production and enchaining of arguments. Andrea (12th grade, Liceo Scientifico), had been asked to solve the following problem in the Cabri environment:

Two intersecting circles C1 and C2 have a chord AB in common. Let C be a variable point on circle C1. Extend segments CA and CB to intersect the circle C2 at E and F respectively.
What can you say about the chord EF as C varies on circle C1?
Which is the geometric locus of the midpoint of EF as C varies on the circle?
Justify the answers you provide.

and talk aloud along the whole process, the interview being video-taped. Andrea starts off with a dynamic exploration of the problem, dragging point C around the first circle in order to observe the behaviour of chord EF. The conjecture of the constant length of EF is formulated on the basis of the visual evidence from the Cabri screen and a first attempt of explanation involving the fixedness of A and B is provided:

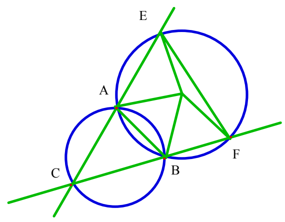
A: Yes, because ... EF always has to stay in the circle, the lines [CA and CB] *have a certain freedom of movement and not more than that and they have to go through those points necessarily. Hence when C is shifted in one direction* [he moves his hands showing the gap

[3] The study has been conducted within a PhD project, carried out at the Graduate School of Education, University of Bristol and funded by the ESRC.

between the two lines as a rotating angle that does not change in width] *I don't know how to explain ... hold o, I'll think about it*

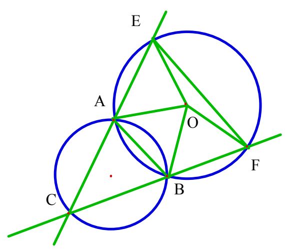
The rigid rotation of the lines through A and C and through B and C is perceived as the reason of the invariance of chord EF. The dragging of point C around the circle simply confirms that the intuition is correct and does not provide any other hint on the underlying geometrical reasons. The initial, purely exploratory, use of the dragging tool is partially modified when A. focuses on the quadrilateral in order to identify variable and invariant elements. It is the beginning of a dialectic between static and dynamic elements, respectively identified in constant angles and segments and in rotating triangles or lines, or a variable quadrilateral. Nevertheless, the basis of the actions is still empirical and the invariance is still sought at the perceptual level, with no reference to the possible geometrical reasons:

A: … I was thinking about this quadrilateral EFAB ... to see it as many triangles and then ... I don't know… I try that [he is constructing segments to form a quadrilateral split into triangles with vertex at the centre of circle] *and all the triangles then are constant ... well ... they are all isosceles triangles*



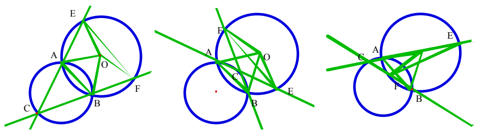
The fact that the conjecture is not refined in correspondence with the operations carried out on the figure, makes Andrea stick to the empirical approach. The lack of a phase of unpacking for the perceptually based conjecture in terms of geometrical relationships with other elements of the figure brings about a persistence in a random exploration by means of the dragging tool, with a slow and continuous movement of point C around the circle. The multiple changes in the strategies adopted to explore the conjecture involve different conceptual and concrete tools, spanning from dragging to additional constructions, to the search for invariants within sub-figures. The idea of looking at segments and angles that stay constant as C is dragged around the circle, seems to suggest the possibility of using Carnot's theorem (known in the English mathematical tradition as the cosine rule)

A: ... a cyclic quadrilateral, because as this angle here varies this other angle varies as well ... hence for Carnot's theorem ...[applied to triangle EOF] … *two sides and the angle in between ... a relationship for this one ... then a relationship for this other one with the angle ... this one is constant, this one is constant ... I can do it with Carnot's theorem!*
C: How?
A: To prove that EF is constant ... because it is enough to have two sides and the angle in between in order to find the length of EF … the two sides are both r *and the angle is always the same…*



Andrea's use of the theorem is nearly circular: in order to find the length of EF and show that it is constant Andrea wants to use the fact that angle EOF is constant, which has not been proved, and which is strictly linked to the invariance of EF.

Although the theorem seems to have been conceptualised correctly, its use in this specific context is not recalling other theoretical results or making a link between the perceptual and the conceptual aspects of the problem. Andrea seems to "let" the theorem take up the cognitive burden in the phase of justification: the elements involved in the theorem's hypothesis are assumed to have the properties required simply on the basis of the perceptual judgement coming from the dragging test. The dragging function of Cabri might potentially be the tool linking concrete and conceptual elements of the toolkit: but the use Andrea makes of it, even when he uses dragging in combination with theorems, is not successful in this respect. Dragging is mainly used at the perceptual level and invariance is sought and conceived of as a visual property of the figure, which has some obvious underlying reason in the rotation of the lines under the constraints of the fixed points A and B. Theorems are kind of "imposed" upon the dynamic figure, but they are not linked up with the preceding argumentative discourse based on the dynamic exploration carried out by means of externally oriented tools, such as dragging and measurement:



A: ... constant… it is 360° ... I put x this one does not move ... but with Cabri I can measure them…
C: You can measure ... do whatever ...
A: [he measures the angles with vertex at O] *and I see that when C is dragged the angle stays constant*

Tools of different types come to interact and to be used jointly, but the lack of theoretical control by Andrea does not lead to a fruitful interaction. The argumentation produced as a justification for the invariance of angle EOF, used as a given in the application of Carnot's theorem, is empirical and draws heavily on the exploration through dragging, as a tool ensuring the general validity of a property observed in the continuous motion:

A: I know for Carnot's theorem that instead of finding ... AB, I may find the angle and then in a triangle the angle… this one is a still triangle, it does not move and therefore angles cannot transform with no reason, since it stays fixed ... […] and there was the chord theorem… I do not remember how it goes ... the angle at the circumference ...

Along the whole solution process Andrea tries to use a number of theorems to justify his conjecture, without expressing it in relational terms: the invariance of chord EF is not related to the invariance of any other element of the figure. Although a theorem about chords is recalled, following the development of the previous reasoning around angles and segments, and, later similar triangles are sought, A. does not abandon his conviction based on the visual evidence. The conjecture, elaborated and formulated in terms of dynamic causality, remains linked to the phenomenological aspects of the problem; any attempt to justify it by means of theorems ends up in a list of results that do not correspond to the interplay of variable and invariant elements in the problems. The cognitive unity of conjecture and proof is thus broken, possibly due to the conflict between dynamic causality, pertaining to the phenomenological world, and static theory.

Discussion and concluding remarks

The analysis of the protocol presented in the previous section shows that the heterogeneity of tools within the context of solution of a problem may bring about difficulties in the elaboration of an argumentation and its possible evolution toward a proof. Andrea's personal toolkit includes externally oriented tools, such as dragging, measurement and additional constructions, as well as a number of theoretical knowings, exemplified by the use of Carnot's theorem, or the recalling of theorems about chords. The problem is that such heterogeneous tools are not harmonised: the joint use of dragging and theorems, for instance, is not fruitful because the dynamic, visual evidence of the (perceptual) invariance of elements under dragging is not interpreted in terms of the geometrical reason that would justify it within the theory.

Two main issues need consideration, also in view of future research: (i) the identification of possible reasons for the lack of harmonisation of heterogeneous tools and (ii) a deeper analysis of the features of dynamic geometry environments, with a particular focus on the issue of internalisation of the dragging tool.

As for the former issue, this paper suggests that a scarce theoretical control of the (concrete and mental) operations performed during the solution of a problem may end up in an unsystematic and nearly random use of theorems. The individual organisation of theoretical knowings may account for their status within the theory (hypothesis, premise, conclusion, derivation, postulate and so forth) as it has been conceptualised by the individual agent. Modalities of use of such tools necessarily reflect such organisation and require the explicit formulation of relationships among geometrical objects in terms of their status within the problem. When theoretical knowings are used in combination with tools of a different nature, they might be re-interpreted in the new context. On the other side, concrete tools might be used to refine the conjecture in relational terms and this might require the development of new modalities of use for them. In actual fact, when a conjecture is formulated in a compact form, on the basis of the dynamic, visual evidence of a property (e.g. the rigid movement of two lines seen as the cause of the invariance of a chord), a process of refinement is necessary in order to make explicit the (static) relationships among the involved objects. Often, the strength of the dynamic causality interrupts the refining process, thus provoking a cognitive rupture in the solution process. Results of the ongoing main study suggest a conjecture/hypothesis (Mogetta, to appear), with possibly strong educational implications. *The dynamic nature of the explorations carried out in the Cabri environment by means of the dragging tool, may conflict with the static nature of the theory (Euclidean geometry). Such conflict may bring about difficulties in linking the arguments elaborated during the conjecturing phase with those needed to construct a proof.*

Further studies are needed in order to test the hypothesis and to better characterise some of the features of dynamic geometry environments, in terms of the nature of the tools they make available to the individual-acting-with-mediational-means. One of the most important aspects to be analysed is the issue of the internalisation of the dragging tool. As suggested in this paper, the dragging tool has the potentiality to link up phenomenological and theoretical world. The point is to internalise it and use the dynamic variation of the figure in order to make a link between perceptual and geometrical invariance of objects.

Finally, more evidence is needed in order to establish whether and how an appropriate management of the individual toolkit each individual develops and uses in particular contexts, is linked to the idea of cognitive unity . My hypothesis, still to be tested is that *there is a tendency to establish, or re-establish CU once it is broken. In correspondence with possible different causes for the rupture to occur an appropriate management and organisation of the toolkit may help the agent overcome the rupture*. Further research is necessary to illuminate the issue and evaluate the actual impact of possible ruptures of the CU on the construction of a meaning of proving.

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