ARITHMETIC AND ALGEBRA, CONTINUITY OR COGNITIVE BREAK? THE CASE OF FRANCESCA

Michele Cerulli & Maria Alessandra Mariotti

Dipartimento di Matematica - Universita di Pisa

***Abstract**. Taking the 'operational/structural' perspective, as introduced by Sfard, this paper analyses the passage from computing with numbers and computing with letters. The discussion is based on a case study, taken from long term research project, still in progress. A key aspect, characterizing the transition from the two types of computation, will be highlighted: the change of role of operation properties.*

Introduction

High school Algebra activities are often characterized by the use of symbolic manipulation. According to the Italian tradition, symbolic manipulation constitute a basic element of secondary school curriculum and in school practice. Usually, students are introduced to the manipulation of algebraic expressions only after experiencing large amounts of computation of numerical expressions: therefore the problem of transition from numerical to algebraic computation arises. Analysing the connection between Algebra and Arithmetic, Lee & Wheeler (1989) showed that the relationship between calculation with numbers and calculations with letters is not so direct and transparent. They argue that, in spite of the use of common operation signs, the activities of writing and manipulating expressions in algebra and in arithmetic are quite different.

In this paper we shall point out and discuss some aspects characterizing the difference between calculating in Algebra and in Arithmetic; the following analysis will be carried out in terms of Sfard's *operational-structural* theory (Sfard, 1994).

Both the necessity and the difficulty of achieving such a duality is clearly expressed by the author:

"The formula, with its operational aspect (it contains 'prompts' for actions in form of operators) must be also interpreted as the product of the process it represents."

"[…] our intuition rebels against the operation – structural duality of algebraic symbols, at least initially."

(Sfard, 1994, p.199)

The operational character of pupils' conceptions related to algebraic formula and expressions tends to persist; at the same time, although symbolic manipulations of algebraic expressions is largely present in school practice, the absence of "structural conceptions" appears evident (Kieran, 1992, p. 397).

This paper aims at analysing the relationship between the two levels of computation:

computation with numbers and computation with letters. We will argue that, contrary to what books and teachers usually state, the transition from computing with numbers and computing with letters is not so smooth and in fact, it may present a cognitive break: as suggested by Francesca: *"Our teacher says that with letters it [computing] is the same as with numbers, but to me it doesn't look the same, it looks very different [It: non mi sembra la stessa cosa]"*.

Methodology

The results we are going to discuss are part of a long term project (Cerulli & Mariotti, 2000) concerning the introduction of pupils (aged 14-15 years) to algebra and in particular to symbolic manipulation. A 9^{th} grade class was split into two separate groups, one following the project, the other following a traditional approach to algebra. Comparison between the groups is interesting because the pupils have the very similar school experiences: apart from the class of mathematics, and they share the same courses for all the other subjects.

The following discussion concerns only some results related to the exemplary case of Francesca, a medium-high level student who attended the traditional course.

1 Before the algebra course

At the beginning of the teaching experiment, before splitting the class, a test was submitted to the students. Some answers given by Francesca are analysed. The first item of the test concerned the correctness of some equalities between numerical expressions.

> **T1** Observe the following statements, for each of them explain why you think it is correct or why you think it is wrong.

Let us consider the following answers given by Francesca:

T1.2 $17 + (6 + 9) = (17 + 6) + 9$

This statement is correct because of the associative property.

T1 .5 $8 + 9 \cdot (3 + 2) - 17 = 8 + 27 + 18 - 17$

This is not right because one can't get rid of the brackets and compute 9•3 and 9•2 and then add [the terms] because the result changes.

T1. 8 $3 + 6 \bullet 73 + 6 \bullet 8 + 13 = 3 + 6 \bullet (73 + 8) + 13$

It is right because adding few numbers, which are multiplied by the same number, or multiplying them by the previously defined [common] number the equivalence remains unchanged.

Unlike other students, Francesca seems to tackle the problem within a structural, instead of a operational approach (Sfard, 1994). Both aspects actually seem to be present: on the one hand the operations properties are considered rules which determine whether or not it is possible to pass from one expression to another (*"...keeps the equivalence unchanged"*); on the other hand, they are considered equivalent relationships between computing procedures. Answers to items 5 and 8 are clearly in contrast but both justifications have the same nature: Francesca is so strongly convinced about the acceptability of the transformation rule, that she performs no computations to check the equality statements.

Thus the rules known by Francesca are instructions which make it possible to move from one expression to another, from one computing procedure to another; the sign "=" is interpreted according to a fixed direction (from left to right) and this peculiarity may affect the acceptability of reversible transformations of algebraic expressions.

In conclusion, as concerns the operations properties, both the structural and procedural aspects can be found in Francesca's answers, but they do not seem to be stable and merged together.

2 After a traditional algebra course

After one year of activities within a traditional framework concerning algebra and symbolic manipulation, a number of interviews were designed, aimed at investigating the relationship between computation with numbers and computation with letters. In other words, we were interested in studying the evolution of the conception of "computing, taking into account the fact that in Italian a unique word "calcolo") includes both symbolic and numerical computations.

2.1 What is the meaning of "calcolo"?

When asked what she intends by the word "*calcolo*", Francesca refers to the primitive model of computation related to the four operations, natural numbers and eventually fractions; the meaning of "computing" ("calcolare") is *"finding something unknown [qualcosa che non si conosce]"*, she says. Changes occur when letters are introduced; consider the following excerpt of Francesca's interview.

30.A. Listen, and after these computations… are there any others?

31.F. With letters

32.A. With letters they are even more difficult, aren't they?

33.F. *(Laughs)* Yes.

34.A. Let's write a computation *(calcolo)* with letters, but a very difficult one …

35.F. (*She thinks and writes)* Mmm …something…how was it? *(while producing the expression she tries to remember some "prodotti notevoli"*[1]*, then she writes a cube)* Yes this one *(the cube)* that I could never work out, then the other one with fractions *(she writes the fraction line and denominator)*

$$\frac{((a+b)\cdot(a-b))^4\cdot(a-b+2)^3}{(a+b)\cdot(a-b)}$$

The construction of the example seems to be inspired by a strong model of computing with letters: the expression is obtained combining various "prodotti notevoli"; within

---

[1] A "prodotto notevole" is a standard equivalence statement used to speed up computations with letters, an example is **(a+b)□(a-b)=a^2-b^2**.

this model, success in computing depends on whether one knows/remembers the formula associated with computing chunks (35).

In the case of Francesca, this model contrasts with the original model of computing numerical expressions, based on the idea of "finding something", i.e. finding a result.

The common aspects shared by the two models are evident. Furthermore, the teacher states that they "are the same thing", but Francesca in convinced that computing with letters is quite different from computing with numbers, and she says:

64.F. Well, I know that…also the teacher always tells me that computing with letters is the same thing as computing with numbers, but to me it is not the same. Because if I am given 10+3 whilst…[if you give me] a•b+c+d I get stuck… *(laughs)* I can't work it out. With numbers we are back to something real, well … for me numbers are not real, but they are still more real than letters in mathematics

65.A. O.K. so the teacher said that it is the same.

66.F. Yes.

67.A. And you say "to me it is not the same", let's start from this point. Why isn't it not the same for you? That's what I am interested in.

68.F. Because!…because if you do 10+13 times, 25+3, you find it *[the results]*, but if you do a+b you can't do it, if you have to do it, for example *(she writes a numerical*

$$(10 + 13) \cdot (25 \cancel{+} - 3)$$
$$(a + b) \cdot (a - b)$$

expression and a literal expression)…here we put a minus *(she changes 25+3 into 25-3)*, if you do this, you get a number, but if you do that, you find "a square minus b square" *(she writes the "result" of the "prodotto notevole")*.

$$(10 + 13) \cdot (25 \cancel{+} - 3)$$
$$(a + b) \cdot (a - b) = a^2 - b^2$$

Francesca's words clearly show that the meaning of numerical computation is consolidated and based on concrete models, whereas *"to find something"* has a specific stable meaning, but this meaning cannot fit the case of letters. Francesca knows quite well how to cope with algebraic expressions, and knows how to transform them using standard formulas, but she still can't accept the supposed similarity between the two ways of manipulating expressions: the model of computing with letters is not derived from an evolution of the model related to numerical computations. As a consequence she feels a break between the two situations; in other terms we can recognise a break between the two meanings of "calcolo". the next part of the interview shows clearly where such a break occurs.

2.2 Computation with numbers and computation with letters

Francesca tries to compare the two kinds of computations, and in order to express her uneasiness she starts to compute the expressions she has produced. In the numerical case, she first computes the sums in brackets, then multiplies the numbers obtained and finally obtains the result which is 506. In the case of the expression with letters, Francesca has already produced a result "a^2-b^2", thus she makes explicit the intermediate computation steps, previously skipped: she multiplies the two sums in brackets term by term, and finally sums the similar terms. At this point the interviewer asks:

74.Did you perform the same operations *(with numbers and with letters)*?

75.F. No I didn't

76.A. in this case and in that case *(she points to the two expressions)*?

77.F. No…well, yes, I multiplied, here *[expression with letters]* I multiplied everything, while here first I have to add, first I calculate the parts in brackets

78.A. And here, *(pointing to the numerical expression)* could you multiply everything or couldn't you? Is it forbidden?

79.F. You mean multiply 10 times 25, as I did here *[in the case of letters]*?

80.F. I never tried to!

81.A. Fine; it's ok that you never tried; do you think you would get 506? I mean do you think you would get the same result…or not?

82.F. I don't know…I don't think so…I don't know. *(she doesn't seem to be sure at all, she is curious, and starts to compute)*…250…13•25…

83.F. Aha! It is the same!

84.A. Is this a surprise for you?

*85.*F. Yes!(*laughs)*

$$250 - 30 + 325 - 39 = 545 - \frac{39}{506}$$

86.A. Well, you never tried: is it true?

87. (She nods)

88.A. And now are you convinced…that it would be always the same, if you had done …because this is a specific case, you might have been lucky…do you think it would always work, or not?

89.I don't know. How can I …? I didn't know the rule that if you do so you always get the same result, it is not as if it were an axiom that tells you that it is always the same, I thought it was just a case, and not a rule.

90.A. Bah!, you wouldn't trust it. So, here, this way to compute *(points to the literal expression)* why… are you sure that this is correct?

91.F. Yes I do

92.A. And why do you trust it here?

93.F. Because I was taught to do so *(laughs)*

94.A. Because you were taught, and does that make any sense?

95.F. The sense is that of not writing such a long computation, and to write only those two numbers *(points to a^2-b^2)*, but there is…

It is surprising to see how astonished Francesca is when she realizes that the result obtained by applying the rules of symbolic manipulation to a numerical expression is the same as that obtained previously by computing sums and multiplications. Furthermore it is surprising that, even after having verified such a phenomenon by executing computations, she is still not convinced of it's generality. It looks as if no link has been established between the two kinds of computations and the acceptability of the rules is strongly influenced by school practice: when pupils are required to transform an algebraic expression into another one, the validity of the transformation depends on the external control of the teacher ("I was taught to do so" 93). Nevertheless, Francesca is able to establish a link between numerical and algebraic expressions:

96.A. […] what is the relationship between the expression $(a+b)\cdot(a-b)$ and the expression $a^2 - b^2$? What is the relationship…

[…]

97.F. The res*[ult]*…well, if you put numbers instead of a and b, if you do this with 3 and 2, you get a number which is equal to…if you perform the other computations, there…

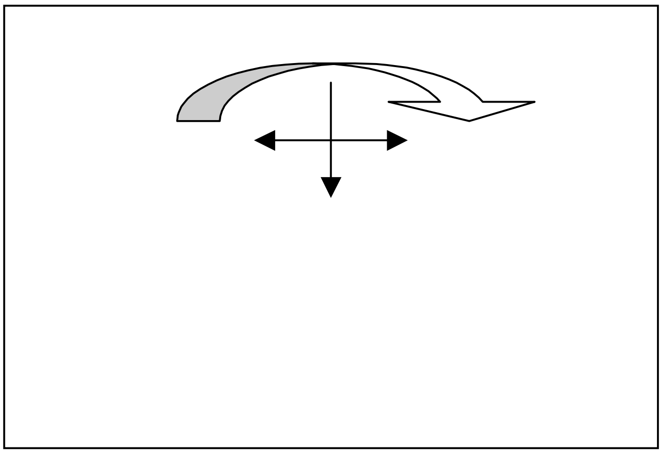
98.A. If I put numbers I get that result.

99.F. Yes

100. A. So, why don't you trust the fact that if you had a numerical expression it wouldn't…

101. F. I don't know…because I was never told…thus *(she smiles embarrassed)* …well if I think it would have, well… if it was that, well… anyway everyone tends to shorten *[computations]* and do things as quickly as possible, but then I don't understand anything anymore…

Francesca knows that two expressions are equivalent when they have the same value if letters are substituted with numbers; but this equivalence relationship, in terms of "values of the expressions", is conceived considering the two expressions as autonomous entities. The equivalence relationship of this type does not concern the symbolic manipulation which transforms one expression into another. In Francesca's view, two algebraic expressions [$\mathbf{(a-b)(a+b)}$ and $\mathbf{a^2 – b^2}$] are two completely independent calculation procedures, which can be accomplished only by substituting letters with numbers. This shows that Francesca is conceiving the two expression according to a dual meaning (Sfard, 1994): both as calculation procedure and as two single entities that can be compared. Nevertheless she seems not to have related the two kinds of computation: "computing with numbers and "computing with letters"



This represents a rupture between the two meanings of computing ("calcolo"): the two procedures (for number on the one hand and for letters on the other) follow different rules which in Francesca's view have nothing in common. As a consequence, it is possible to accept the idea of equivalence in terms of values of expressions, but not to accept or believe that calculating with numbers "is the same as" calculating with letters.

3. Properties of the operations as instruments for symbolic manipulation

The case of Francesca is particularly interesting because it clearly shows the complexity of the relationship between the two meanings of 'computing' ("calcolo"). It shows that it is possible to access some key aspects of this relationship, such as a structural and operational conception of operation properties and algebraic expressions, equivalence between computing procedures, but still lack (CONTROLLARE) a comprehensive meaning of computing including both the case of numbers and that of letters.

As a matter of fact, grasping the link between computing with numbers and with letters, requires a radical change of perspective, of which the operation properties are the core.

According to Sfard's hypothesis, when computing with algebraic expressions a new operational level must be achieved, but this must be achieved without breaking the link with the previous one. The analysis of Francesca’s case shows that not only must the reification of an expression be accomplished (expressions can be acted upon as new objects), not only must the structural level be consolidated (equivalence between expressions must be stated in terms of their values), but also a relation between the two 'computing procedures' ("calcoli") must be constructed explicitly.

The key-point is that properties of the operations have to become *rules of transformation*, i.e. "instruments" of computation, and in order to do so, they must assume a dual meaning (structural and operational): properties state the basic equivalence relations and function as instruments for symbolic manipulation, i.e. instruments by means of which any symbolic transformation is derived.

Within the numerical context, operation properties do not play an operative role; they simply express the equivalence of computing procedures, but they are not necessary, and thus not usually employed for computation.

Within the algebraic context, operation properties must assume an operative role and must become the instruments for transforming expressions.

Such a change of role is not made explicit in school practice and focusing of attention on memorisation of particular shortcut procedures such as algebraic formulas ("prodotti notevoli") may definitely hide it.. In conclusion, it seems reasonable to take the hypothesis that this change of role becomes a first goal in introducing pupils to symbolic manipulation.

References

Cerulli M. & Mariotti, M.A.: 2000, *A symbolic manipulator to introduce pupils to algebra theory*. Proceedings of *Workshop W6 "Learning Algebra with the Computer a Transdisciplinary Workshop"*. ITS'2000, Montreal.

Cerulli M.: 1999, *Uso di manipolatori simbolici nell'introduzione all'algebra: realizzazione e sperimentazione di un prototipo*. Unpublished manuscript, Tesi di Laurea in Matematica, "Università degli Studi di Pisa", Pisa.

Kyeran, C.: 1981, *Concepts associated with the equality symbol*. "Educational Studies in Mathematics", 12. D. Reidel Publishing Co., Dordrecht,.

Kyeran, C.: 1992, *The learning and teaching of School Algebra*. In Handbook of Research on Mathematics Teaching and Learning. D. A. Grouws ed., N.C.T.M..

Lee L. & Wheeler D.: 1989, The arithmetic connection. *Educational Studies in Mathematics, 20,* 41-54.

Sfard A.: 1991, *On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin*. Educational Studies in Mathematics, 22 (pg. 1-36).

Sfard A. & Lincheski L.: 1994, The gains and pitfalls of reification – the case of Algebra. *Educational Studies in Mathematics, 25,* 191-227.