PROTOTYPICAL USES OF FUNCTION PRESENT IN SEVENTH- AND EIGHT-GRADE TEXTBOOKS FROM FIFTEEN COUNTRIES

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Conceptions of function at stake in problems and exercises in mathematics textbooks for early secondary students were investigated using Biehler's prototypical uses of function. Following a constant comparative analysis method that involved 2304 exercises and problems, five main categories were identified: rule, set-of-ordered-pairs, social, physical, and figural. Decontextualized uses—rule and set of ordered pairs—were by far the most frequent uses given to functions with contextualized uses receiving less attention. Such trend invites reflection about current practices associated with the introduction of functions in the early secondary school grades.

The evolution of the concept of function in mathematics as a discipline has followed an interesting path, changing how people understand mathematics (Buck, 1970, p. 237). The rapid changes that occurred in mathematics once the set-theoretical definition of the concept was introduced were echoed in school mathematics, generating difficult problems for the mathematics education community, problems that stimulated new lines of research (Eisenberg, 1991, p. 141). Research has provided descriptions of students', teachers', and prospective teachers' understanding of function, illustrating that their views are shaped by teaching practices, mathematical discoveries, and people's cognitive capabilities (Cooney & Wilson, 1993; Harel & Dubinsky, 1992; Sfard, 1991). The research has also suggested approaches in which less formal presentations are fostered, with technology playing an important role (NCTM, 1989, 2000; Tall, 1991). Textbook is an object that has received little attention from researchers on functions. Textbooks synthesize what is known about a concept from multiple perspectives: historical, pedagogical, and mathematical. As documents, they provide valuable information about the potential learning that could occur in a classroom. An investigation of textbook content is relevant not only to complement the set of views of function but also to help explain its relation to the difficulty of learning the concept. Obviously, what students learn from textbooks and the practicality of that learning are mediated by the school context, including teacher, peers, instruction, and assignments (Stodolsky, 1989). The textbook, as a source of potential learning, expresses what has been called the *intended curriculum* (the goals and objectives for mathematics intended for learning at a national or regional level; Travers & Westbury, 1989, p. 6), which implies that an analysis of textbook content becomes in some ways a hypothetical enterprise: What *would* happen *if*…? becomes the beginning of the inquiry. What would students learn if their mathematics classes were to cover all the textbook sections about functions in the order given? What would students learn if they had to solve all the exercises in the textbook? Would they learn what a function is? Would that learning work well in characterizing *function*? Such hypothetical questions leave open the space for different things to happen in reality: as said before, teachers mediate (and sometimes dismiss) textbook content. The answers to the questions, however, act as an *a priori* analysis, which help to determine the plausibility of different alternatives that could occur in classrooms.

Because practices within a country are usually quite similar, looking at more than one country offers the possibility of making contrasts to highlight aspects of the concept that are taken for granted within a culture. In this study I selected lower secondary school textbooks assuming that in grades 7 to 9 function would begin to appear explicitly in school mathematics. My interest in the findings of the Third International Mathematics and Science Study (TIMSS), and the availability of the textbooks used in TIMSS led me to choose those textbooks from participating countries written in a language that I could read. In the larger study I explored the *conceptions* (Balacheff, in press) of function suggested by the seventh- and eighth-grade mathematics textbooks of selected countries participating in TIMSS. In this paper I present the process by which the set of problems that put a conception at stake was operationalized and some associated results.

Theoretical Framework

Different situations generate different interactions between the subject (a person's cognitive dimension) and the milieu (those features of the environment that relate to mathematics), and in consequence lead to different meanings. The different interactions explain the coexistence of multiple knowings by a subject. Contradictory knowings can coexist, either at different times in a subject's history or because different situations enact different knowings. In both cases, what is isomorphic for the observer—probably the teacher—is not for the learner (Balacheff, in press). Balacheff has characterized conceptions as a quadruplet consisting of problems, representations, operations needed to solve the problems, and verification and validation activities needed to determine that an answer has been obtained and to establish its correctness. Different sets of problems require different representations, operations, and verification and validation activities that would correspond to different conceptions as described by an observer. The analysis of a conception as a 4-tuple of different but interconnected elements allows for a description of subtle differences in conceptions that otherwise could not be distinguished. However, the problems are at the core of the issue, once they are chosen, an observer can associate to them particular conceptions. I used Biehler's (in press) *prototypical domains of application*, to assist in characterizing the set of problems.

For Biehler a concept may have different meanings in different disciplines, and those meanings are determined by the differences in practices in each discipline. Three elements are constitutive of the meaning of a mathematical concept: the domains of application of the concept (its use inside and outside mathematics), its relation to other concepts and its role within a conceptual structure (a theory), and the tools and representations available for working with the concept. Using as an example the concept of function, he identifies the "prototypical ways of interpreting functions (prototypical domains of application) which summarize essential aspects of the meaning(s) of functions." These are *natural law* (e.g., a parabola as a representation of the curve of a cannon ball), *constructed relations* (e.g., to express a price depending on a quantity), *descriptive* (e.g., functions involving time-dependent processes), and *data reduction* (e.g., functions in statistics). He notes that the concept of causal relation has been abandoned in mathematics in favor of a "'functional relation' between two quantities (Sierpinska, 1992)… due to philosophical reasons [and] to pragmatic ones:

If we have a 1-1 correspondence, we can invert the cause-effect functional relation to infer the ‘causes’ from the effects.” The decision to invert the relation is rooted in the academic practice of mathematics; in disciplines such as physics, it might not make sense. Biehler’s characterization of prototypical domains of application of function, that is, its uses, was instrumental for me in initiating a characterization of the problems in a textbook that eventually can be solved by the student. These different uses gave me a stepping stone to use in characterizing the problems needed to define the conceptions that could be elicited by textbook exercises. With these tools, I looked for answering the following question: What are the prototypical uses of function present in the seventh- and eighth-grade mathematics textbooks of selected countries participating in TIMSS?[1]

Method

The original sample for the study consisted of 35 textbooks from 18 countries chosen from the TIMSS data base according to the following criteria: the textbook was intended for 7th, 8th, or 9th grade; the textbook was written in English, Spanish, German, French, or Portuguese; and the textbook contained references to *functions, linear functions, graphing in two coordinates, graphing in the Cartesian plane, tables, patterns,* or *relations.* All the exercises, hereafter *tasks*, from such sections (for a total of 2304 tasks) constituted the corpus of data for the study. Each task received a 4-tuple code. The first code, P, identified the prototypical use of function present in the task. The second, O, contained all the operations that were needed to solve the task. The third, R, contained all the representations that were needed to solve the task. Finally, Σ contained all the activities available for the student to verify that a solution was obtained and that it was correct.

The development of the categories for coding each element of the quadruplet was accomplished in four steps. First, I selected one task from the first section of each textbook to analyze in depth (35 tasks). I worked each one, following as much as possible the textbook presentation that preceded the exercise section and developing categories for each element of the quadruplet. Second, I used the resulting categories to code the remaining tasks in all the first sections of each textbook, looking for new categories and refining the properties of each. I used the *constant comparative* method (Glaser & Strauss, 1967) in which I described the salient features of the categories for an element and at the same time looked for possible breaks or mismatches that could lead to the creation of a new category. This second step involved 518 tasks and resulted in 133 categories. Because there were so many categories, the third step consisted in merging categories within common groups, thus yielding a smaller, more manageable number of categories for each element. The final step was to test the coding system by having other raters use it to code tasks, which helped to further refine and validate the categories of the coding system. The final system consisted of 10 codes for uses of function, 36 codes for operations, 9 for representations, and 9 for controls. I report the results associated with the first element of the conception, the prototypical uses of function present in the tasks.

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[1] The larger study from which this paper is derived tackled the issue of *conceptions* suggested by textbooks (see Mesa, 2000).

Results

When working with the first set of 35 tasks, I found that Biehler's initial classification (natural law, constructed relations, descriptive, and data reduction*)* did not include tasks lacking a real context: namely, when the function was treated as a set of ordered pairs (e.g., "Represent in the Cartesian plane the relationship whose solution is given by the set $R = (x, y) \mid x, y > 0 \wedge x, y \in \mathbf{R}\}$"), when it was treated as a rule, when a pattern with numbers or figures was sought, or when there was a proportion involved. Biehler's categorization was not accounting for phenomena that are particular to mathematics (Puig, 1997). In addition, in some tasks that suggested relations that could be classified as constructed using Biehler's characterization, the content used geometrical definitions or principles (e.g., similarity), that suggested an additional category. The following task is an illustration of such cases:

> The slide projector puts a picture on the screen. The size of the picture changes as you move the projector. The picture gets bigger and bigger as you move the projector further away. When the projector is 300 cm from the screen, the picture is 120 cm high. Here are figures for other distances [a table with six values for distance and height is given].
>
> Draw two axes on graph paper. Mark the across axis from 0 to 500 and the up axis from 0 to 200. Label the across axis 'Distance from screen in cm'. Label the other axis correctly. Use the figures in the table [given] to plot points.
>
> (a)What do you notice about the points you have plotted? (b) Use your ruler to draw the graph through the points. (c) Use the graph to find the height of the picture when the projector is 350cm from the screen. (d) How far is the projector from the screen when the picture is 50cm high?

Such uses of function were labeled, s*et of ordered-pairs, rule, pattern, proportion*, and *geometrical* respectively. Biehler's "descriptive relation" was renamed *cause and effect* and was used to characterize the cases in which the task dealt with physical phenomena not dependent on time. After the final coding step, a new category appeared, *graph*, which was used to characterize those tasks in which the relation was given by a graph in a Cartesian plane that did not have any marks (e.g., a graph of a function $f(x)$ is given; the student has to identify the graph that corresponds to $f^{1}(x)$). I kept a record of all the different instances of uses within each category. These examples of uses were crucial in fully characterizing the categories for uses of function (see Appendix).

To simplify the presentation, the uses that referred to physical phenomena, cause-and-effect relations, and time relations were grouped into a new category called *physical* to capture the character of these relations. Because they relate to human activity, data-reduction relations and constructed relations were grouped into a new category called *social*. Geometrical relations, graph-defined relations, and pattern relations were grouped into a new category called *figural,* to highlight the crucial role of images and patterns for defining functions in these relations. Rule and direct proportion/proportion relation were grouped together into the category *rule*. *Set of ordered pairs* was left as a separate category.[2] Table 1 presents the frequencies and

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[2] The reorganization is not only a practical one; I wanted to highlight particular characteristics of the use of function, which make several categories look as equivalent. Different criteria would produce a different reorganization.

percentage of occurrence of these categories. The results refer only to grades 7 and 8, yielding a sample of 1319 tasks, in 24 textbooks from fifteen countries.

Table 1: Frequency and Percentage of Prototypical Uses of Function

Uses	Frequency	%
Rule	556	42
Set of Ordered Pairs	319	24
Social	227	17
Physical	136	10
Figural	81	6

As the table shows, the most frequent uses were rule and set of ordered pairs. Only one third of the uses corresponded to those involving concrete contexts: namely, social, physical, and figural. Social uses were almost twice as frequent as physical uses, which suggests that at these grade levels physical phenomena in which functions can be defined do not play a very important role. Almost 25% of the tasks had a set-of-ordered-pairs use of function, which implies that such definition still plays an important role for introducing the notion. In contrast, the figural use of function accounted for only 6% of the tasks which implies that at these grade levels it is not a common practice to present functions based solely on mathematical phenomena. Six textbooks showed important differences from this pattern: in three textbooks (from two countries, about 5% of the tasks) there were no tasks with a set-of-ordered-pair or rule use (one textbook had only social uses and the other two had mainly physical uses). The other three textbooks (from three countries, about 8% of the tasks) had more than 40% of social and physical uses.

Discussion and Conclusion

One possible reason for the high frequency of rule uses in textbooks might be didactical: Because the idea of correspondence is so fundamental to the (modern) notion of function, and because the seventh and eighth grades mark the transition period from arithmetic to algebra, transformations of numbers by means of basic operations seem to fit the double purpose of defining valid functions—with a notion of correspondence as *transformation* or *constrained variation*—while at the same time linking known operations with the new idea of correspondence. In this way the burden of considering unrealistic situations in which the correspondence can be *arbitrary* (as is the case with the Set-of ordered-pairs use) is overcome. In other words, such uses of function are actually serving the didactical purposes of smoothing the transition from arithmetic to algebra and of introducing the idea of correspondence. The somewhat large proportion of tasks that with a set-of-ordered-pairs use of function may be due to authors' interest in keeping the textbooks updated mathematically; also, because textbooks change slowly (Farrell & Heyneman, 1994), the influence of the new math movement was still operating in this set of textbooks (with one exception, textbooks with copyrights in the 70s and early 80s contained more of such uses). The modest contribution of tasks with social or physical uses seem to be a consequence of reform movements that suggest the use of applications in presenting mathematical concepts. The difference between social and physical uses might be related to the complexities that the latter bring to the problems (Monk, 1992). One possible reason for the few instances of Figural use might be linked to the separation between geometry, arithmetic, and algebra in school mathematics curricula. The textbooks tended to contain separate chapters for geometry, and it seems likely that within those chapters, functions did not get much attention. The low frequency of geometrical uses could be also a consequence of the new math movement, which almost eliminated geometry from school mathematics in several countries (Ruiz & Barrantes, 1993).

Vinner (1992), Norman (1992), and Even (1989) have documented that students, teachers, and prospective teachers view functions as defined by an algebraic expression that involves numerical variables only or as black-box machines (a rule use), with meanings that are contradictory with the formal Dirichlet-Bourbaki notion (a set-of-ordered-pair use). The results of this study suggest that these groups' understandings of function could have been expected, given the uses given to functions in textbook tasks. The characterization of problems and exercises of textbooks in this sample indicate that when functions are initially introduced to students, textbook authors tend to prefer situations in which the relation is defined trough transformation of an input to obtain an output. There is also a tendency to present a formal view of function as a set of ordered pairs in which the notion of arbitrary assignment is presented. Contextual uses of function do not seem to play a significant role in the majority of the countries. Such trend poses questions to us as researchers and as curriculum developers: is the same tendency observed in classrooms? Is this an appropriate strategy to follow for students' first encounters with the notion of function? To what extent is a more contextualized approach, as promoted by reform movements, more appropriate? How do we establish that appropriateness? As uses of function are just one of the constitutive elements of a conception, one can expect that these uses would not necessarily dictate what operations, representations, and control activities are enacted in a task. That is not the case, however (see Mesa, 2000). Tasks with rule uses tended to be associated with a very limited (in number and in requirements) sets of operations, representations, and controls. In contrast tasks with physical and figural uses offered the most variation, thus creating possibilities of enacting different meanings for functions but also illustrating that textbooks may promote the existence of several apparently contradictory views of function. Even though there are not clear answers to these questions, the actual situation should invite reflection on an intermediate strategy to follow if a variety of (uncontradictory!) meanings are to be promoted in the introduction of functions in early secondary school.

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Appendix: Characterization and Examples of Prototypical Uses of Function in Tasks

Cause/effect

Used to code content that refers to physical phenomena other than time related and in which the behavior of one variable is an effect of the behavior of the other (it is a directional relationship). *Atmospheric pressure vs. boiling point, Density of water/ice vs. temperature; Hooke's Law*

Constructed

Used to code content that refers to "real life" situations other than cause/effect, time, data reduction, and geometrical. In these relations it is somehow arbitrary which variable is called dependent and which one independent. An interchange of the roles of the variables produces equally valid (for the context) relationships. *Number of goods (gas, phone calls, book, etc.) vs. number of other goods or vs. cost, Conversions*

Direct proportion/proportion relation

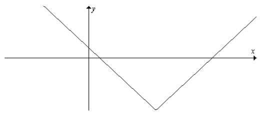
Used to code content where there is an explicit reference to a proportion or a direct proportion without context. *Fill a table in such a way that there is a direct proportion between the entries*

Data reduction relation

Used to code statistical situations; in situations involving two variables it may be possible to have more than one outcome for a given value of a variable. *Change of price of movie vs. year, Consumer price index vs. year, Diameter of sample of tree trunks*

Graph defined relation

Used to code content where the relation is presented in a graph whose two axes are neither labeled nor numbered.



Geometrical relation

Used to code content that refers to geometric figures and their characteristics. *Similarity; Height of a tower of cubes vs. number of cubes, visible or invisible faces, edges, and vertices.*

Pattern relation

Used to code content in which given a sequence the question is to find the general term (or an expression for the *n*th element) of the sequence. *Expression for triangular numbers, Number of sides of a polygon vs. number of diagonals*

Rule relation

Used to code content in which an input is transformed by certain procedure to obtain an output and in which a context is not provided.

All polynomial; rational, periodic; piece-wise; radical; step; trigonometric. Computer programming

Set-of-ordered-pairs relation

Used to code content where a list of ordered pairs is given or requested.

Any arbitrary pair assignment, Localization of points in a Cartesian plane, Relatives; numerical

Time relation

Used to code content that refers to physical phenomena where time is involved and the variable is treated continuously.

Speed) vs. distance; Speed vs. time; Distance vs. time.