CONFLICT BETWEEN PERCEPTION, COGNITION AND VALIDATION AS YEAR 12 AND UNIVERSITY STUDENTS ANALYSE THE PROBABILITY OF AN EVENT

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Eighteen Year 12 students and 2 cohorts of final-year BEd students (74 students) were shown a "fair" (equiprobable outcomes) spinner with three noncontiguous colours and asked whether each of the three colours had the same chance of "being spun". Half of the Year 12 students either gave unequivocal incorrect responses derived from inappropriate considerations of sector size or number of sectors per colour, or vacillated between correct and incorrect responses and were unable to make a decision (equivocal). These findings were echoed with the university students although their incorrect responses tended to be more unequivocal than equivocal. Validation through trialing (with the university students) did not help as the results did not show exactly $^1/_3$ for each colour and, in fact, were interpreted as supporting an incorrect response.

Hawkins and Kapadia (1984) identified four types of probability, namely: (1) *theoretical* – derived from making assumptions of equal likelihood; (2) *frequentist* – calculated from observed frequencies; (3) *intuitive* – generated from personal belief and perceptions; and (4) *formal* – calculated precisely from the mathematical laws of probability. Of interest to this study are the latter two which relate to intuitive and analytic cognitions (Fischbein & Schnarch, 1997). They defined the intuition cognition as "self evident, directly acceptable, holistic, coercive and extrapolative" (p. 96) which was distinguished from the analytic cognition by "the feeling of obviousness, of intrinsic certainty" (p. 96).

Probability ranges from 0 (impossible event) to 1 (certain event) so possible events are represented numerically by fractions (part of a whole). It is well-documented (Behr, Harel, Post, & Lesh, 1992; Nik Pa, 1989; Payne, Towsley, & Huinker, 1990) that continuous area models are more conducive to facilitating construction of the part/whole notion than discrete set models. Therefore, when developing the part/whole notion of probability, it seems reasonable to begin with spinners (continuous area model in which all possible outcomes are visible) than with coins, dice, marbles, tickets, or playing cards (discrete set models where all possible outcomes often need to be held in memory).

The literature is replete with misconceptions in students' probabilistic thinking (e.g., Fischbein, 1975; Fischbein & Schnarch, 1997; Hawkins & Kapadia, 1984; Jones, Langrall, Thornton, & Mogill, 1999; Kahneman & Tversky, 1972; Piaget & Inhelder, 1975). Fischbein and Schnarch's (1997) study set out to determine whether 7 main known misconceptions (e.g., representativeness, negative and positive recency effects, compound and simple events) diminished, increased or remained stable across the years (Grades 5, 7, 9 & 11). They found that the only stable (and frequent) misconception across the ages was related to compound and simple events. However, the example given refers to the comparison of events in two similar sample spaces. Jones et al. (1999) referred to this type of comparison of events as Level 3 whilst the probability of an event in one sample space was classified as Level 2. They found that even after persistent instruction, misconceptions at Level 2 remained stable for some Year 3 students.

Apart from the subjective and experiential beliefs that students invoke when analysing probability tasks, one of the major problems related to the teaching/learning of probabilistic notions is the difficulty of validating responses because of the large number of trials required. This can be extremely time-consuming and the results are not always persuasive for students who have a deterministic view of mathematics. (See *Discussion* for elaboration of this point.)

This study explores a very elementary probabilistic notion, namely, the probability of an event in a single sample space (Level 2 – Jones et al., 1999) using a spinner (continuous part/whole area model) with Year 12 students who were individually interviewed on the task to determine the extent of their conceptions (including misconceptions). The study was replicated, to some extent, with university students but was extended to include validation of their conceptions.

Study 1 – Year 12 students

***Background**.* Sixteen Year 12 students comprising 8 students from an algebra-based university entrance mathematics (designated as UM) and 8 students from a "social" mathematics course that involved no algebra (designated as SM) were involved in this study. Within each of the mathematics categories, there were 4 males and 4 females with 2 high- and 2 low-achievers in each gender group.

This paper reports on one (see Figure 1) of several elementary probability tasks that were undertaken with the students in semistructured individual interviews conducted out of school in the student's home. The tasks incorporated both continuous area models (spinners) and discrete set (marbles) models.

Figure 1. Spinner used to determine the robustness of student's probability notions.
("Fair" is used to denote equiprobable outcomes.)

The spinner used in this study was designed to be provocative, that is, to provoke conflict between intuitive and analytic cognitions. To help students invoke the part-whole fraction notion of probability, a simple area model (familiar to Queensland students) was used but was made more difficult because students were required to *reunitise* (Behr et al., 1992; Baturo & Cooper, 1997, 1998, 2000) either: (1) the red and green sectors as two parts, each of which was equal to the yellow part, thus realising the spinner was actually partitioned into sixths; or (2) reunitise the two yellow sectors as one sector, thus realising the spinner was actually partitioned into thirds. Therefore, although the spinner had only three colours to consider, the *noncontiguous* nature of the colours increased the difficulty level of the task (Jones, 1974). Furthermore, to provoke conflict between intuitive and analytic reasoning, the first question was designed to promote analytic reasoning (a consideration of "fairness"– equal chances) whilst the second question was designed to promote the intuitive reasoning that is often invoked by games and winning.

Thus, the task had three main purposes: (1) To determine the robustness of the students' analytic reasoning in determining the probability of an event; (2) To ascertain whether the students displayed any conflict between visual perception/intuitive cognition (the amounts of colour do not look equal) and analytic cognition (knowing that if the two yellow parts were adjacent, they would cover the same amount of area as each of the other two colours); and (3) To determine whether the student's dominant form of processing was intuitive or analytic.

Each interview was videotaped then transcribed into protocols for analysis in terms of intuitive or analytic cognitions.

Results. The students either responded with a firm conviction regarding the correctness of their response or they vacillated with their answers. To indicate the conviction or the vacillation, the responses for this task were categorised as *unequivocal* (immediately stated and incontrovertible) or *equivocal* (ambivalent, indeterminate) with correct and incorrect subgroups within each category. (See Table 1 for the results.)

Table 1

Task Results in terms of Equivocation and Correctness of Responses

	Form of response	
	Unequivocal	**Equivocal**
Correct	Andrea & Michelle (UM/High)	Matthew (UM/High)
	Ben & Camille (UM/Low)	John (SM/High)
	Brendan (SM/Low)	
	Sarah (SM/High)	
	Cognitive processing dominant	*Visual perception strong but not dominant*
Incorrect	Karoline (UM/Low)	Eddy (UM/High)
	Malcolm (UM/Low)	Jane (SM/High)
	Nicholas (SM/High)	Kerri (SM/Low)
	Marney & Joe (SM/Low)	
	Visual perception dominant	*Conflict between types of processing*

Only half of the students gave the correct response (including those 2 in the equivocal correct category). Of the two students in the *equivocal correct* category, Matthew's (UM/H) initial response was negative, but his change to a positive response was almost instantaneous, perhaps indicating that, to him, intuitive reasoning is still a powerful factor in processing information but not so powerful that it dominates his analytic cognitive processing. The other student, John (SM/H), took about 10 seconds before responding but his explanation was rather interesting.

. . . because there's an even amount of each colour; like these two [red and green] *have the same odds, right, but . . . these two* [2 yellow] *have got to vary between each of their coordinates* [indicating the width of each yellow section with the fingers of each hand]*; add these* [the 2 yellow] *both add up and I think they would equal the green and the red.*

Of these 8 students who gave correct responses, there was an equal number of the higher-level UM and lower-level SM students, an equal number of high and low-performing students, as well as an equal number of females and males. Therefore, this study did not find that course, achievement, or gender impacted on the ability to process the probabilistic notion of equally likely outcomes analytically.

The incorrect responses were based on strategies related either to number of like colour sectors or to sector area. For example, of the five students in the *unequivocal incorrect* category, Karoline (UM/L), Marney and Joe (both SM/L) maintained that yellow had more chance of being spun than either red or green because there were two yellow portions and only one red and one green portion. Nicholas (SM/H) and Malcolm (UM/L), however, had no doubts that red or green would have a greater chance of being spun than yellow because each of them had a larger area for the needle to land on than yellow.

The remaining students, those in the *equivocal incorrect* category (Eddy, Jane, and Kerri), fluctuated between intuitive and analytic processing. Ultimately, though, the intuitive cognition was more dominant than the analytic cognition. Their protocols reveal the conflict invoked by the task.

Eddy: *No, maybe because these two colours* [yellow] *are right opposite so these two* [yellow] *would have more chances. But if these two* [yellow] *joined together are the same as these two colours* [red and green] *then it would be a fair spinner but these two* [red and green] *would have more chances.*

Jane: *I think if you put those two* [yellow] *together, they'd probably be the same as the others but I think that the red and the green are probably more dominant. Like your object* [indicating the needle on the spinner] *is more likely to land on the red or the green.*

At this stage, Jane's (SM/H) understanding of fair and unfair as they applied to spinners was investigated. She was first shown a spinner which was half blue and half orange and asked if this was a fair spinner to use in a game. Jane said that it was because you had the same chance of landing on either colour. She was then shown another spinner which had all equal parts (4 blue, 3 red, 1 green) and all colours were contiguous. Jane said that this was a fair spinner, too. When asked if she were playing a game and could only win if she spun green, she said: *Oh, no, it wouldn't be fair then. Oh, do you mean to look at the colours? No, not fair* [referring to the original spinner in Figure 1] *because there's two of them* [yellow].

Kerri's responses were indeterminate on all the tasks and were therefore difficult to probe as the following protocol reveals.

Kerri: Um . . . that is not a 50% fair spinner. It's probably 2 thirds.

I: *What colour would you prefer to have in a game?*
Kerri: *Prefer or most likely?*
I: *Well, if you say that's not a fair spinner, then one colour must have more or less chance of occurring.*
Kerri: *Well, it'd be the two yellows because they're smaller and there'd be either the red or the green.*
I: *Are you saying that red or green would have more chance than yellow or – ?*
Kerri: *Definitely* (interrupting). *Yes.*

Study 2 – BEd students

Background. The task in Study 1 was tendered for discussion in a tutorial/workshop with 2 cohorts of final-year BEd students (39 and 35 students). As for the Year 12 students, responses were either correct, unequivocally incorrect or equivocally incorrect. (The conflict provoked vociferous and robust arguments as each group of students tried to convince the others that their thinking was appropriate.) With respect to the incorrect responses, the university students had the same misconceptions as the Year 12 students. No new misconceptions were proffered.

Validation results. The spinner (with colours) was shown on an overhead transparency and then a transparency copy of the spinner partitioned into sixths but without colour was placed on top (see Figure 2). The students were then asked what the probability was of getting red, green or yellow. For these students, the fact that they could see that each colour had 2 sixths (or 1 third) of the area did not offset their initial intuitive cognitions regarding the fact that one of the colours was split and therefore red or green had a better chance (because of sector size) or yellow had a better chance because there were two parts, albeit smaller parts.

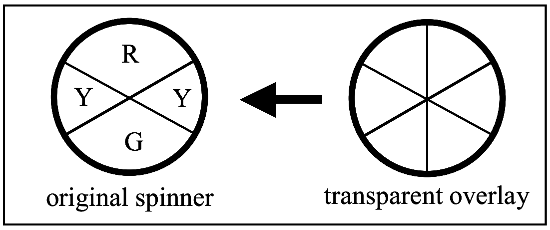

Figure 2. Attempt 1 to "prove" that all outcomes are equiprobable on the given spinner.

The students were then shown the spinners in Figure 3 and asked if any of them were "fair". All agreed that Spinners A and B were fair (A because the colours were contiguous; B because the noncontiguous allocation of the colours was "even") but continued to maintain (or be indecisive) that Spinner C was not.

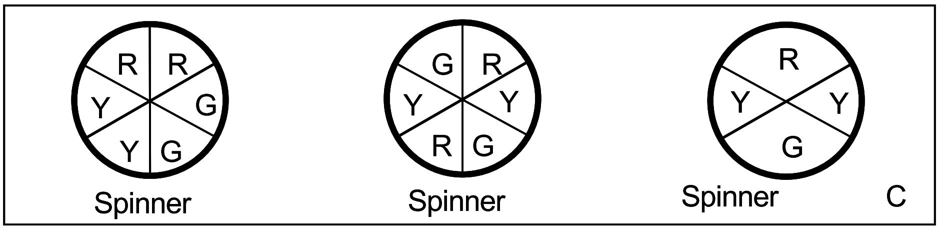

Figure 3. Attempt 2 to "prove" that all outcomes on the original spinner are equiprobable.

Each class decided that, if this occurred in their teaching career, they would ask their students to undertake an experiment with Spinner C. The BEd students were allocated to 10 groups and each group was provided with a model of the spinner. Each student in the group was asked to spin the needle 10 times and to record his or her result for each spin.. Table 2 shows the outcomes of this trial as well as those undertaken by a second cohort.

Table 2
Results of An Experiment Undertaken by Two Cohorts of BEd Students to Validate Predicted Outcomes for Spinner C

	Outcome		
Results	Red	Yellow	Green
1st cohort (n = 390)	136	117	137
2nd cohort (n = 350)	116	119	115

Note. In this table, n refers to the number of trials in the experiment.

With respect to the first cohort, the students who had thought that the spinner was not fair because red or green would be more likely to be spun than yellow felt vindicated by this result. They believed that the results supported their prediction that the spinner was not fair. With respect to the second cohort, the students who thought that the spinner was not fair because yellow, with its two parts, had more chance than either red or green also felt that the results vindicated their reasoning. The first cohort was unimpressed by the suggestion that there should have been more trials whilst the second cohort was unconvinced that the results were very close to a third of all trials (indicating a deterministic view of mathematics).

Discussion and conclusions

Cognition. As both studies showed, misconceptions with respect to the spinner in Figure 1, which had only three possible outcomes to consider, were evident in a large percentage of students. Half of the Year 12 students and many of the BEd students gave either an incorrect response or vacillated between correct and incorrect responses when shown the spinner. The incorrect responses revealed that students had two main misconceptions which were an artefact of the task, namely: (1) the larger sectors (red and green) had more chance because they were "dominant" (Jane); and (2) the two smaller yellow sectors had more chance because they gave 2 chances whereas the red and green sectors gave 1 chance only. The students who gave unequivocal incorrect responses appeared to be operating from intuitive cognitions based on comparing either the size of the parts or the number of like colours. That is, they were estimating chances using a part-part ratio schema rather than measuring probability with a part-whole fraction schema (Fischbein, 1975).

This result supported Fischbein and Schnarch's (1997) study in which they found that the probability of an event produced stable and frequent misconceptions across age levels. It also extended their findings by showing that these misconceptions continue into adulthood. Furthermore, the task used in this paper was more simple than the one used by

Fischbein and Schnarch because only one sample space had to be considered, indicating that the problem is deep-seated.

Validation. Validation of reasoning was not successful as the results of the 2nd study showed. Neither encouraging reunitising by overlaying a transparent replica of the spinner showing sixths (see Figure 2) nor a consideration of structurally isomorphic spinners (see Figure 3) was sufficient to persuade students to focus on the fraction schema embodied in the task (i.e., analytic reasoning). The university students all stated unequivocally that Spinner A and Spinner B (see Figure 3) had equally likely outcomes (red green, yellow) but maintained that Spinner C (task spinner) did not. Spinner A had all contiguous parts whilst Spinner B had all noncontiguous parts. However, Spinner C had some contiguous and some noncontiguous parts thereby producing the conflict between equality (2 red, 2 green, 2 yellow), inequality through sector size (red and green both larger than either yellow) and inequality through number of sectors (1 red, 1 green, 2 yellow). These inequalities appear to be linked to the part-part notion of ratio rather than to the part-whole notion of fraction.

Validation through experiment was equally unsuccessful for two reasons: (1) the insufficient number of trials produced skewed results (see Table 2), thus inadvertently supporting a misconception; and (2) the students' deterministic view of mathematics was so entrenched that they were dissatisfied with any result that did not exactly show 1 third of the trials for each colour (as for the 2nd cohort of university students).

Teaching and learning. The major implication for teaching and learning is that probability schemata must be connected explicitly to fraction schemata through language, exemplars, and symbols. In this study, the students who vacillated between correct and incorrect responses clearly were unsure as to whether to trust their perceptual/intuitive processing (comparison/ratio) or their cognitive processing (fraction). Eddy, the top-performing mathematics student in his school was obviously perplexed by his inability to decide. He had a well-developed fraction schema but it seems as though he did not realise the validity of this cognition, possibly because his probability learning experiences did not focus on the connection between probability and fractions. As Study 2 showed, this situation is exacerbated by problems with validation. Neither logical argument nor experimentation may convince students of the errors in their answers.

Probability, possibly more than any other mathematical domain, is plagued by a plethora of informal and formal *language* to denote possible events. A diagram such as that in Figure 4 was found to be useful for the BEd students in this study because it provided an organisational framework for plotting the language "mathematically".

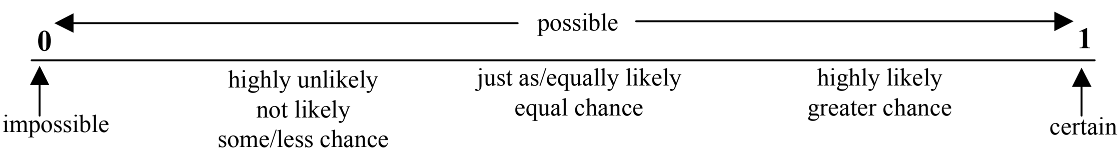

Figure 4. A continuum of formal and informal language ranging in meaning from impossible to certain

Teachers need to be aware of the strengths and weaknesses of common probability *exemplars* and should be guided in their use by sound pedagogical principles rather than by their real-world appeal. For example, the sequence of probability exemplars should follow the sequence of exemplars used to develop the part/whole notion of fractions, that is, continuous area models such as spinners before discrete set models such as marbles. However, spinners can be partitioned in different ways and the outcomes (colours, shapes or numbers) can be arranged either contiguously or noncontiguously. If contiguous parts only are used, students may inadvertently come to rely on the intuitive and inappropriate comparison/ratio schema. Therefore, provocative tasks such as the one in this paper should be incorporated to provoke conflict between intuitive and analytic cognitions to provide insights into the appropriateness of student's thinking.

In the early stages of learning, the common fraction recording facilitates connection to the fraction schemata required for processing probability tasks. Unlike decimals or percents, common fraction *symbols* indicate the total number of outcomes (denominator) and the number of outcomes under consideration (numerator).

References

Baturo, A., & Cooper, T. J. (1997). Reunitising hundredths: Prototypic and nonprototypic representations. In E. Pehkonen *(Ed.), Proceedings of the 21st conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 57-64). Lahti, Finland: PME.

Baturo, A. R., & Cooper, T. J. (1999). Fractions, reunitisation and the number-line representation. In O. Zaslavsky (Ed.), *Proceedings of the 23rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 61-67). Haifa, Israel: PME.

Baturo, A. R., & Cooper, T. J. (2000). Year 6 students' idiosyncratic notions of unitising, reunitising, and regrouping decimal number places. T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 57-64). Hiroshima, Japan: PME.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht, Holland: Reidel.

Fischbein, E, & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education, 28*(1), 96-105.

Hawkins, A.S., & Kapadia, R. (1984). Children's conceptions of probability: A psychological and pedagogical review. *Educational Studies in Mathematics, 15*, 349-378.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1999). Students' probabilistic instruction in thinking. *Journal for Research in Mathematics Education, 30*(5), 487-519.

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgement of representativeness. Cognitive Psychology, 3, 430-454.

Nik Pa, N. A. (1989). Research on children's conceptions of fractions. *Focus on Learning Problems in Mathematics, 11*, (3), 3-24.

Payne, J., Towsley, A., & Huinker, D. (1990). Fractions and decimals. In J. Payne & C.E. Clements (Eds.), *Mathematics for the young child*. Reston, VA: National Council of Teachers of Mathematics.

Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children*. London: Routledge & Kegan Paul.