The Teacher's and Students' Important Roles in Sustaining and Enabling Classroom Mathematical Practices: A Case for Realistic Mathematics Education

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Abstract

This paper is a preliminary report of a second-grade classroom teacher experiment. The primary aim of this paper is to address the important role the classroom teacher and the students play in sustaining the classroom mathematical practices. To advance our ideas, we use excerpts from one discussion to illustrate the teacher's keen ability to consider aspects of the children's self-generated models. By capitalizing on such instances, the teacher not only provides possible learning opportunities for individual students but also makes it possible for the researchers to anticipate the emergence of subsequent more sophisticated practices.

In this paper, we provide a preliminary report of a second-grade classroom teaching experiment that was conducted during the 2000-01 academic year. The overall aim of this project was to support the children's flexible, mental reasoning with two-digit quantities. In particular, we explored the possibility of accounting for the children's learning as they moved from using informal to more formal ways of interpreting problems and recording their thinking. As a secondary goal, we hoped to refine an instructional sequence, Aunt Mary's Candy, to support the children's arithmetical reasoning. Our reasons for doing so were quite pragmatic. Because the classroom teacher was also using a newly adopted curriculum (that was compatible with the instructional sequence), the teacher needed to determine how she could infuse the sequence with the regular curriculum without threatening the integrity of either.

In our discussion here, we elaborate the significant role the classroom teacher and students can play in sustaining and the classroom mathematical practices. The teacher, for her part, must recognize important aspects of the students' interpretations and associated representations as they work individually, with partners or when they engage in whole class discussions. As she does so, she must employ strategies to make these interpretations explicit for the students. More generally, she must develop ways to infuse the children's idiosyncratic methods with the collective ways of interpreting and communicating mathematical ideas. At the same time, she must be flexible enough to advance ideas that support the emergence of new mathematical practices. As such, the teacher is faced with enormous challenges as she supports her students' mathematical learning. The students, for their part, must make sense of their activity and develop ways to notate and communicated their ideas to others. Before we address these issues, however, first we couch our discussion in theoretical assumptions and methodological issues that framed our efforts. We then provide examples from one classroom discussion to illustrate how the teacher and students can sustain and enable the classroom mathematical practices.

Theoretical Considerations

Instructional Design Theory of Realistic Mathematics Education. With regard to Realistic Mathematics Education [RME], Gravemeijer (1999) makes a strong case for the role curriculum might play in supporting children's formal mathematical reasoning. In particular, he elaborates several heuristics for developing and implementing instructional activities. One such heuristic is the role students' models can play in supporting their mathematical learning. These models are thought to emerge from the students' own understanding of a problem situation that is couched in a rich context. As students engage in problem solving within this context, they develop informal ways of representing their interpretations and solution methods. After some time, they develop more sophisticated ways to model their interpretations. Eventually their models take on a life of their own and can be used to reason mathematically without referring to the original problem context, that is, these models are transformed into more formal ways of reasoning.

In our view, mathematical meanings are socially accomplished by reflective individuals (Lave & Wenger, 1991; Cobb & Bowers, 2000). These meanings are negotiable as individuals reflect on their own and others' actions. That is, these meanings become social objects as individuals mutually orient themselves to one another's activity. One the other hand, these meanings have their roots in individual students' sense-making as they participate in mathematical activity that is valued by the teacher, the school district and the community at large. As such, we assume that the individuals' sense-making activity and the social situations in which they engage are inseparable and situated in various communities of practice. Although we acknowledge that social situations contribute to what constitutes knowing and doing mathematics in various communities, because our interest is that of mathematics learning in classrooms, our primary focus here is to elaborate the socially-situated nature of mathematics learning within this particular classroom microculture. (e.g., Cobb & Bowers; Lave & Wenger).

By coordinating RME with this situative view, the models that the students manipulate, adapt, or transform, although idiosyncratic per se, contribute to sustaining and enabling the taken-as-shared mathematical practices that emerge during whole class discussions.

Aunt Mary's Instructional Sequence. As one of the goals of the project, in collaboration with the classroom teacher, Ms. Wilson, we developed a series of activities to support the students' flexible manipulation of two-digit quantities for addition and subtraction situations. We adapted tasks from previous classroom teaching experiments conducted by the Purdue Problem Centered Mathematics Curriculum Project (Cobb, Yackel, Wheatley, Wood, NcNeal, Preston, & Merkel, 1992) and the Mathematizing, Modeling and Communicating in Reform Classrooms Project (Cobb, Yackel & Gravemeijer, 1995). Using some of these previously developed materials, we collaborated with Ms. Wilson to design an ongoing context about candy that her Aunt Mary made and distributed at various community functions, or gave to family members and friends. As the students worked in this context, they first created physical collections (with multilink cubes) to represent packages (collections of ten) and pieces of candy (collections of ones) and later drew pictures to reason about addition and subtraction situations involving Aunt Mary's candy. Making physical and pictorial collections allowed the students to engage in informal problem solving situations. These experiences, in turn, eventually allowed them to develop ways of notating and symbolizing that fit with more formal ways of notating and interpreting addition and subtraction situations (cf. Gravemeijer, 1999).

The activities Ms. Wilson introduced were designed so that the students could develop their own notational methods to explain their thinking about collections of tens and ones. Yet, certain ways of speaking about and notating how they broke apart and recombined collections of tens and ones became commonly used by many of the students. Typically, students drew rectangles to represent packages of ten candies and small circles to represent individual, loose candy. When solving subtraction situations, they usually crossed off loose pieces and subtracted additional pieces to take away the amount they needed to subtract. Whereas there was some variation in how they marked the packages, their notational methods and accompanying interpretations appeared to be understandable to the other students.

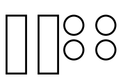
Methodology

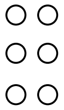
Data Collection and Analysis. Data collection techniques included videotaping the daily lessons, making field notes, and collecting samples of the students' independent and small-group work. We gathered additional information as we talked to children each day and observed their mathematical activity. In addition to these data collection techniques, we conducted clinical interviews with each child at the beginning and end of the project to document their progress.

To analyze the data, we first perused our field notes to identify key lessons. After watching videotape recordings of several daily lessons, the videotape recording of one representative lesson was transcribed so that we could conduct further analysis. We also analyzed the children's written work samples to characterize their notational methods. Analysis of the children's written work was then triangulated with the whole-class discussion to reconstruct to the events of the lesson.

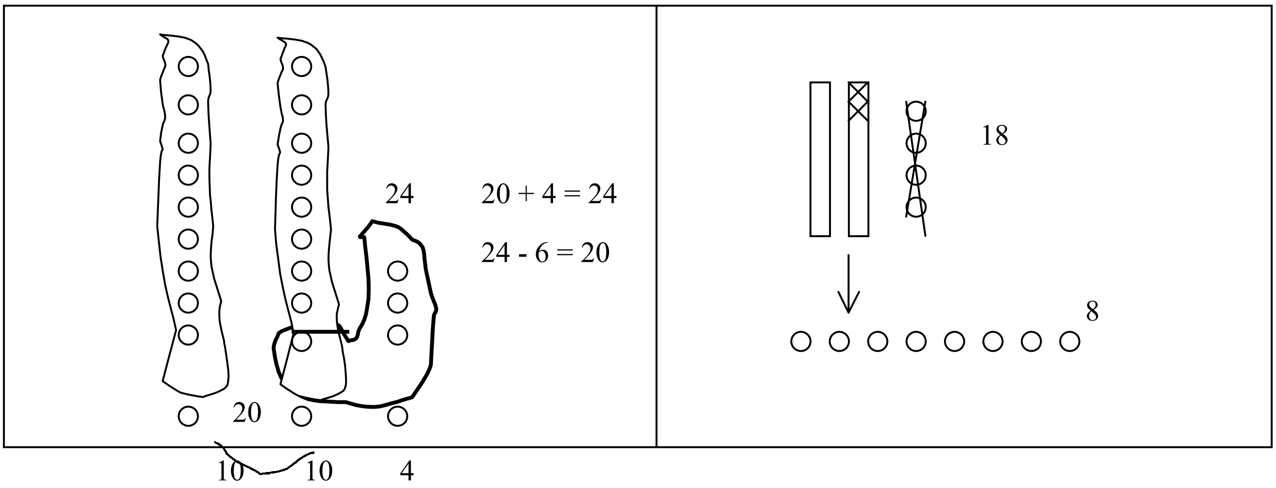
Classroom Episodes

The example we use here was taken from a lesson that occurred several weeks into the instructional sequence. The example is part of a whole class discussion that occurred after the children worked on several addition and subtraction tasks. After the children had completed several of the problems, Ms. Wilson asked them to join her in the front of the room to share their ideas. To begin the discussion, Ms. Wilson asked two children to come to the blackboard at the same time and show how they solved the following problem:

Aunt Mary has  pieces of candy on the counter. Uncle Johnny eats

 pieces of candy. Show how much candy she has now.

Each child then explained his or her drawings on the board. We enter the discussion as Alice, the first child, explained how she solved the problem. She had drawn two packages showing each of the pieces of candies and four loose pieces (See Figure 1a). She then labeled each package with the numeral ten, the four loose pieces, and



(a) (b)

Figure 1. (a) Alice's and (b) Luke's written records for 24 - 6 = ?

circled the six candies that she needed to take away. Interestingly, she also wrote two number sentences (20 + 4 = 24, 24 - 4 = 20). As her sentences indicated, she did not include the two loose pieces that she took away as she recorded her answer of 20. We enter the discussion as Ms. Wilson asked Alice to explain her drawing:

Teacher: Alice would you tell us about your drawing?

Alice: Yeah. Well first of all, I knew that if I had a package of ten (points to one of her packages) well um, what I did, I writed out and separated them and circled the ten and put a ten at the bottom. And then I made another package of ten…over here and then I circled that and put a ten and I added this ten with this ten and that made it twenty. And then I knew that I had to add the four [pieces] on (points to the four loose pieces) so I put the four over here and I got twenty-four.

We infer from Alice's explanation that telling about her drawing meant to describe what she had drawn. As the discussion ensued, Ms. Wilson asked Alice about her picture. In doing so, Ms. Wilson prompted Alice to provide a rationale for her drawing.

Teacher: And what you are showing is what Aunt Mary has on the counter right now. Right?

Alice: (Nods yes.)

Teacher: Can I ask you a question before you go on to explain?

Alice: (Nods yes.)

Teacher: You said that you laid them out as pieces that and you circled them so that we would know that they were packages. What made you lay them out in pieces like that?

Alice: Because I knew if I did it the other way, that it would be harder. [It would be harder] to draw it to make the package torn apart

Teacher: Okay, so you did it so that you could see the packages when you wanted to tare them apart. Okay. So you have your twenty-four sitting out there, so now what are you going to do?

Ms. Wilson's question to Alice about why she showed loose pieces in each package is very important. By asking this question, "What made you lay them out in pieces like that?," Ms. Wilson made it possible for Alice to explain why she needed to show all the individual pieces. In essence, Ms. Wilson implicitly communicated to Alice and the other students that her drawing was valued. By making all the pieces visible, Alice could easily manipulate the pictorial collections to reason sensibly. When we consider the fact that Alice predominantly used counting strategies during her interview session and had difficulty solving tasks involving numbers over 20, her rationale seems quite fitting. Further, by asking this question, Ms. Wilson made it possible for the other students to understand why Alice chose to show the packages this way. The students had an opportunity to consider the particular details of Alice’s drawing.

Following this exchange, Alice explained how her number sentences fit with her pictures but did not notice that her answer would be 18, not 20. As the discussion continued, Ms. Wilson prompted Alice to explain her answer:

Teacher Okay and then you circled some other pieces there. Um, what did you circle those pieces for?

Alice: Oh because, um, remember I said that Uncle Johnny ate six [pieces]

Teacher: Okay. And show where those six are that you [circled].

Alice: Well, I knew I had four, and then I knew that I had to use another package. So that I can get six because there is no other, um, there's no other way to get to six. So I took the four and put a line, and then I circled the four, I mean two, and then I counted them and figured out four, (points to the two pieces in the package and counts) and then five, and then six.

At this juncture, Alice moved beyond merely describing her drawing to giving a reason for why she circled the six loose pieces. We suspect that Ms. Wilson's question about the circled pieces was particularly significant in helping Alice explain her thinking. Initially, Alice may not have realized that she needed to use her picture to explain her thinking. However, at this point in the exchange, she explained in some detail her thinking using the drawing.

Following this exchange, Ms. Wilson asked Alice to determine how many pieces of candy Aunt Mary had left. After counting the remaining candy, Alice changed her answer to 18. She then changed her number sentence 24 - 6 = 18.

As the discussion continued, Ms. Wilson redescribed Alice's picture to the class. As she did so, she communicated to the students that Alice had made an important contribution to the class. We share part of her comment to illustrate how Ms. Wilson capitalized on aspects of Alice's drawing:

Teacher: This is a really neat way of looking at it. [This is] something I hadn't really thought about, Alice. But it really makes a lot of sense to me. She knew that Uncle Johnny ate more loose pieces than she had loose pieces sitting on the counter. So what she decided to do is to go ahead and show the pieces loose, and then she could work with them real easily. But she wanted to make sure we knew that she still had two packages. So she just drew the candies as if Aunt Mary had not yet quite packaged them up yet. She [Aunt Mary] had them in rows of what she was going to package. So you can see the loose pieces (points to the two packages Alice has drawn)…

Ms. Wilson's comment here addressed to two important points. The first point relates to representing Aunt Mary's packages as loose pieces. Alice had a reason for drawing the loose pieces in each package. It was easier for her to work with the packages if she drew all the pieces. Also, by making the loose pieces, her pictures were understandable to the rest of the class. Second, we note that Ms. Wilson and Alice, together, made it possible for certain mathematical ideas to emerge during the discussion. For Ms. Wilson's part, she aligned Alice's reasoning with how the class thought about and represented Aunt Mary's candy. Alice also contributed to this process when she provided a rationale for making her packages as loose pieces and explained her solution process.

After redescribing Alice's drawing, the second child, Luke, came to the blackboard and explained his drawing (see Figure 1b):

Luke: I crossed off the two pieces to make six and so, um, so that that would then be eight little pieces. And then those x's are for showing that Uncle Johnny had already had eaten them. And then I did that [drew the 8 pieces] so that you could see that there were eight more pieces (points to the eight loose pieces he has drawn on the blackboard)…And, and that's how I got the number eighteen (points to the numeral 18).

Luke, interestingly, used his drawing to explain how he solved the problem. In response to Luke's explanation, Ms. Wilson again underscored the significance of his as well as Alice's drawings and associated interpretations:

Teacher: What Luke did was a really, really neat thing. Because what he did was he went ahead and drew what Aunt Mary had on the counter, okay? And kind of like Alice, he imagined this [package] (points to the package that Luke has used to cross off two pieces) being broken

> up into pieces I think. And he said (points to the fours loose pieces that are crossed out) “I have four pieces that I can take away for what Uncle Johnny eats plus I need two more from this package” (points to the two pieces crossed out from one of the packages). And then he shows us (makes a circle with her hand around the eight loose pieces left) the eight pieces what he has left when he takes them away.

As Ms. Wilson redescribed Luke’s picture, she specifically indicated how his picture was different from Alice’s. Luke did not show the individual pieces contained in each package. As she indicated, he “imagined” that he could break apart a package to subtract the two additional pieces. By making this distinction explicit, Ms. Wilson communicated that she particularly valued how he reasoned with his drawing. By redescribing his explanation, Ms. Wilson also implicitly communicated that Luke had offered what constituted an acceptable explanation. Liam, for his part, contributed to this process by explaining his thinking using the pictorial collections.

Final Remarks

As we revisit our classroom example, we note that this discussion constrained and enabled the classroom mathematical practices. On the surface of it, the children's contributions were not mathematically different–both children offered very similar methods for solving the task. However, as we consider the quality of the children's explanations, their drawings point to different ways they participated in these practices. Whereas both children subtracted the loose pieces by taking away two pieces from one of the packages, how they made their pictorial collections signified very different ways of reasoning with the packages. Alice needed to count individual pieces, whereas Luke appeared to act with the packages as abstract collections (Cobb & Wheatley, 1988). The challenge then for the classroom teacher was to align the students’ idiosyncratic interpretations with the emerging practices. As in our example, by juxtaposing these two children's explanations on the blackboard, Ms. Wilson could highlight the different interpretations the children might give as they decomposed and recomposed pictorial collections of tens and ones.

This example also points to the possible learning opportunities that arise during classroom discussions. Children like Alice might curtail how they made packages as pictorial collections. As a consequence of participating in this discussion, Alice had the opportunity to understand how other children reasoned with the packages. As such, she may develop interpretations that fit with how Luke acted with the pictorial collections. Luke could act with the collections without having to remake the individual pieces that composed each package. Further we clarify a collective conceptual shift that the class may make as they begin to act with their drawings in ways that are similar to how Luke solved the task. They may move from counting to making pictorial collections to reason with pictorial collections of abstract tens and ones.

These situations could also be learning opportunities for students like Luke. As they reflect on aspects of their drawings, they may curtail their methods by no longer needing to mark off individual pieces from a package. They may begin to reason mentally about the packages and use numerals to show how they partition collections of ten solve subtraction situations.

Finally, we note that the classroom mathematical practices are socially accomplished as the teacher and the students participate in these discussions. Whereas they contribute in different ways, it is clear that their contributions are equally important in advancing the mathematical practices. This is particularly the case in classrooms where children's self-generated models are valued and capitalized on. Such accounts as the one we have reported here remind us of the necessary role students' models play in learning mathematics with understanding.

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