DO INTUITIVE RULES HAVE PREDICTIVE POWER?

A REPLICATION AND ELABORATION STUDY ON THE IMPACT OF 'MORE *A*–MORE *B*' AND 'SAME *A*–SAME *B*'

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Abstract. *In recent years, the intuitive rules theory has received growing attention in the mathematics and science education research community because of its capacity to explain and predict various kinds of responses of students to a wide variety of tasks from scientifically different content domains. Two major intuitive rules show up in comparison tasks: 'More A–more B' and 'Same A–same B'. Although these two rules have been extensively studied by Israeli researchers, the relation between these rules and their gradual impact on students from countries with different traditions in mathematics and science education require further research. In this paper, we describe two consecutive studies aimed at re-examining the predictive power of 'More A–more B' and 'Same A–same B' in situations wherein both rules lead to different erroneous judgements.*

Theoretical and empirical background

Building on the pioneering work of Fischbein (1987) about the role of intuitive thinking in mathematics and science, Israeli researchers recently established the intuitive rules theory (Stavy & Tirosh, 2000; Tirosh & Stavy, 1999a; 1999b). According to this theory, students' responses to various conceptually unrelated tasks in mathematics and science can be explained and predicted by several common, intuitive rules that are activated by some salient, external features of a task. Two such rules show up in comparison tasks. The first is called 'More *A*–more *B*': students comparing two objects differing with respect to a certain salient quantity A ($A_1 > A_2$), intuitively argue with respect to another quantity B that $B_1 > B_2$. In many everyday life and scientific situations, such an intuitive reaction leads to conclusions that are accurate (e.g. squares with a larger perimeter have a larger area), but students of different ages strongly tend to use this rule also beyond its range of applicability (e.g. plane figures with a larger perimeter always have a larger area). The second rule is called 'Same *A*–same *B*': students comparing two objects being equal with respect to a certain salient quantity A ($A_1 = A_2$), intuitively argue with respect to another quantity B that $B_1 = B_2$. Clearly, this intuitive rule can also lead to both correct (e.g. squares with the same perimeter have the same area) and incorrect responses (e.g. plane figures with the same perimeter always have the same area). Stavy and Tirosh provided a variety of examples of students' erroneous and correct responses that are in line with these intuitive rules. Furthermore, they collected an impressive amount of research data demonstrating how strongly students of different age groups and study streams are influenced by these basic rules.

Although the intuitive rules theory is widely known in the international mathematics education community nowadays, replications of the results obtained by Stavy and Tirosh in other countries with different cultural and educational traditions, are quite rare. Furthermore, we need more convincing evidence that the reasoning *process* of students who respond 'in line with' a given intuitive rule, is actually affected by that rule. Especially for older students, who can generate a particular answer by simply applying an intuitive rule, but also by relying on a more advanced mathematical notion or strategy learned at school, this point needs to be clarified. Finally, connections between these intuitive rules and other misconceptions and systematic errors known from the mathematics education research literature (see, e.g., Davis, 1989; De Bock, Verschaffel & Janssens, 1998), has to be examined more thoroughly.

Study 1

We set up a first study to arrive at a better understanding of the nature and predictive power of the intuitive rules 'More *A*–more *B*' and 'Same *A*–same *B*'. More specifically, we aimed at providing answers to the following research questions: (1) do Flemish students react in a similar way as Israeli students to comparison tasks eliciting the intuitive rules, (2) do students' explanations contain explicit references to 'More *A*–more *B*' or 'Same *A*–same *B*', and (3) do their answers and/or explanations yield evidence for other erroneous strategies or misconceptions known form the literature. Because both 'More *A*–more *B*' and 'Same *A*–same *B*' were targeted, we looked for tasks wherein both rules lead to a different but inaccurate answer.

Seventy students of grade 10 (15–16-year olds) of a Flemish secondary school participated in Study 1. All participants were divided in three equivalent subgroups and each subgroup was confronted with two problems. All problems were variants of a geometrical problem about the ratio between the surface area and volume of differently-sized cubes, – a task introduced by Livne (1996), and mentioned in several publications of Tirsoh and Stavy (see, e.g., Stavy & Tirosh, 2000; Tirosh & Stavy, 1999a) (see Figure 1). The first group (Group I) received Livne's original problem about cubes and its two-dimensional variant about the ratio perimeter/area of differently-sized squares. The second group (Group II) was confronted with conceptually the same two problems, but formulated with spheres and circles. Also in the third group (Group III), students received two mathematically equivalent problems, but formulated with irregular two- and three-dimensional figures (maps and bottles, respectively). The problems were formulated in a multiple-choice format with three alternatives. One alternative proposed an incorrect answer in line with the rule ‘More *A*–more *B*’ (the *larger* figure has the *larger* ratio). A second (also incorrect) alternative was in line with ‘Same *A*–same *B*’ (‘*same* shape, *same* ratio’). The third alternative, which did not involve one of these two intuitive rules, was the correct one. Students were not only instructed to select one of the three alternatives (larger than, equal to, or smaller than), but also to explain their choice. As in Livne's study, drawings of the geometrical figures were given.

Cube problem. Consider two differently-sized cubes. Is the ratio between the surface area and volume of Cube 1 smaller than/equal to/or larger than/the ratio between the surface area and volume of Cube 2? Explain your answer.

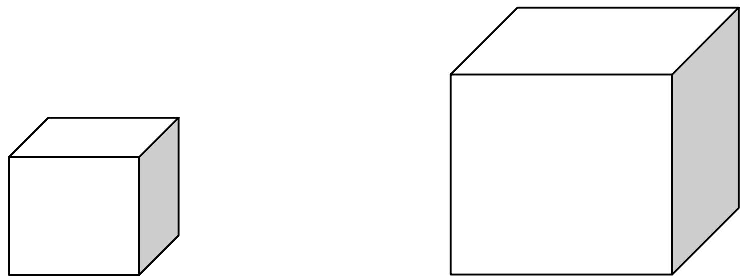

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Figure 1. Task by Livne (1996)

Livne (1996), who confronted biology majors in grade 10, 11, and 12 (15–18-year old students) with this problem, reported the following results: (1) Substantial percentages of students in grades 10, 11, and 12 (41%, 45%, and 55% respectively) incorrectly argued that the ratio surface area/volume is the same in both cubes. Typical explanations were: ‘Cube 1 and Cube 2 have the *same geometrical shape*, hence *the ratio* surface area to volume is *the same* regardless of their size’; ‘The surface area and the volume in Cube 1 are proportionally smaller than in Cube 2 and therefore the ratio is constant’. (2) Only 24%, 19%, and 24% of the students in grades 10, 11, and 12 respectively claimed that the ratio surface area/volume is larger in Cube 2 than in Cube 1.

Table I gives an overview of the results of our study.

Groups		I $N = 22$	II $N = 24$	III $N = 24$	Total $N = 70$
Plane figures	'Same *A*–same *B*'	36	41	50	43
	'More *A*–more *B*'	27	37	4	23
Solids	'Same *A*–same *B*'	36	17	38	30
	'More *A*–more *B*'	23	71	10	36

Table I. Distribution (in %) of the answers of the three groups of students in Study 1

These quantitative data do not reveal a clear trend: the student answers seem to be more or less uniformly distributed. For plane figures, most students tended to select the alternative in line with 'Same *A*–same *B*', but this tendency was not confirmed for the solids. With respect to the geometrical figure involved, most correct answers were given for the items about squares and cubes (Groups I) and for the items about irregular figures (Group III) (39% and 49%, respectively); least correct answers were given for the items about circles and spheres (Group II) (17%). Remarkably, 43% of the students selected a different answer for task 1 than for task 2. Probably, as a result of the 'experimental contract' (Greer, 1997), many students expected different answers for two tasks given in the same test.

A qualitative analysis revealed that 24% of the explanations contained an explicit reference to proportional reasoning. Although '*k* times *A*, *k* times *B*' (with $k > 1$) can be seen as a straightforward quantification of 'More *A*–more *B*', students' response sheets contained no empirical evidence that students' misuse of proportionality was the direct result of a prior 'More *A*–more *B*' judgement. Above all, this finding confirms students' tendency to apply proportional reasoning 'everywhere' (see, e.g., De Bock, Verschaffel & Janssens, 1998), – a misconception that seems to be affected by students' instructional histories rather than by their general intuitions. Only 3% of the students referred explicitly to shape or shape similarity and not a single student reportedly reasoned that 'the larger figure must have the larger surface/volume ratio'. Apparently, Flemish students do not *readily* verbalise the 'Same *A*–same *B*' or 'More *A*–more *B*' rule. If these rules affected students' reasoning, it seems they were not aware of it.

In conclusion, the strong 'Same *A*–same *B*'-tendency reported by Livne (1996), was not confirmed for the different geometrical shapes included in our study. With respect to the tasks involving solids, the global percentage of responses in line with 'Same *A*–same *B*' was even below chance level! This finding seems to question the predictive power of 'Same *A*–same *B*' for that kind of geometrical problems. But also students' explanations, especially those explicitly referring to shape or to shape-similarity, qualified as being 'typical' in the Livne-study, proved to be quite rare in our first study.

Study 2

Because (1) the quantitative data of Study 1, especially the 'total scores' did not show a convincing trend and (2) very few of students' explanations explicitly referred to the intuitive rules that, according to the theory of Stavy and Tirosh, would have affected their (erroneous) answers, we decided to set up a follow-up study on a larger scale.

In this study, fifty-eight, fifty-eight and fifty-six students of, respectively, grades 10, 11, and 12 were confronted with a written multiple-choice test consisting of five problems from different mathematical subdomains. As in Study 1, each problem was accompanied by three alternative responses. One alternative proposed an incorrect answer in line with the rule 'Same *A*–same *B*'. A second alternative (not necessarily in this order) was also incorrect, but was in line with 'More *A*–more *B*'. The third alternative, which did not involve one of these two intuitive rules, was the correct one. Students were instructed to explain their choice on their answer sheet. They were given 50 minutes to complete the test and were not allowed to use calculators.

Two of the five test items functioned as 'anchor items', allowing a straightforward comparison of our results with those obtained in Israel: the Cube problem that was already used in Study 1 (cf. Figure 1) and the 'Carmel problem' (see, e.g., Tirosh & Stavy, 1999b; Stavy & Tirosh, 2000) given in Figure 2. We completed these two items with three self-made tasks, which are also listed in Figure 2, together with an indication how the distinct alternatives are linked to the intuitive rules.

<u>Carmel problem</u>. The Carmel family has two children, and the Levin family has four children. Is the probability that the Carmels have one son and one daughter larger than/equal to/smaller than/ the probability that the Levins have two sons and two daughters?

> 'Equal to' is in line with the rule 'Same *A*–same *B*' ('*same* ratios ($\frac{2}{4} = \frac{1}{2} = \frac{1}{2}$), *same* probability').
> 'Smaller than' is in line with 'More *A*–more *B*': the larger family has a larger probability.

<u>Cake problem</u> (inspired by Lin, 1991). On Wednesday afternoon, Els and her father are baking together a cake. Both cakes have the same height. The diameter of Els' cake is 15 cm, the diameter of the cake of her father is 30 cm. Els adds 150 gram sugar to her batter, her father adds 300 gram sugar to his batter. Is the cake of her father less sweet than/as sweet as/or sweeter than/ the cake of Els?
(This problem was accompanied by drawings of the two cakes, having a cylinder form.)

> 'As sweet as' is in line with the rule 'Same *A*–same *B*' ('*same* ratio ($\frac{15}{150} = \frac{30}{300}$), *same* sweetness').
> 'Sweeter than' is in line with 'More *A*–more *B*': the cake with more sugar is sweeter.

Root problem. Is $\sqrt{10}$ larger than/equal to/or smaller than $\sqrt[3]{15}$?

'Equal to' is in line with 'Same *A*–same *B*' ('*same* ratio ($\frac{2}{10} = \frac{3}{15}$), *same* expression').

'Smaller than' is in line with 'More *A*–more *B*': 15 > 10 and/or 3 > 2.

Physics problem. The gravitational pull on a mass *m* at a distance *s* metres form the centre of the earth is given by a formula of the form: $F = c\ \frac{m}{s^2}$ (in which *c* is a constant).

Suppose a mass *A* is twice as heavy as a mass *B* and the distance of *A* to the centre of the earth is twice the distance of *B* to the centre of the earth. Is the gravitational pull on mass *B* smaller than/larger than/or equal to the gravitational pull on mass *A*?

'Equal to' is in line with 'Same *A*–same *B*' ('*same* ratio ($\frac{m_A}{s_A} = \frac{m_B}{s_B}$), *same* gravitational pull').

'Smaller than' is in line with 'More *A*–more *B*': $m_A > m_B$ and/or $s_A > s_B$.

Figure 2. The four new tasks from Study 2 (the fifth problem was given in Figure 1)

Table II gives an overview of the results.

Task		Grade			
		10 $N = 58$	11 $N = 58$	12 $N = 56$	Total $N = 172$
Cube problem	'Same *A*–same *B*'	40	43	39	41
	'More *A*–more *B*'	16	20	16	17
Carmel problem	'Same *A*–same *B*'	33	24	36	31
	'More *A*–more *B*'	3	9	5	6
Cake problem	'Same *A*–same *B*'	48	41	43	44
	'More *A*–more *B*'	19	10	9	13
Root problem	'Same *A*–same *B*'	5	9	7	7
	'More *A*–more *B*'	24	17	14	19
Physics problem	'Same *A*–same *B*'	48	31	23	34
	'More *A*–more *B*'	5	19	13	12

Table II. Distribution (in %) of the answers by grade and by problem in Study 2

For five problems and the three grades together, 55% of the students selected the correct alternative. Responses in line with 'Same *A*–same *B*' and 'More *A*–more *B*' were below chance level (31% and 13%, respectively). For the five tasks together, the percentages of correct responses slightly increased with age: 52%, 55%, and 59% correct responses in grades 10, 11, and 12). The Root, the Carmel, and the Physics problem proved to be the easiest tasks (for the three grades together, respectively, 74%, 63%, and 54% correct responses). For these problems, the percentages of responses in line with each of the intuitive rules were not above chance level for all three grades. For the Cake and the Cube problem, the percentages of correct responses and those in line with 'Same *A*–

same *B*’ were more or less the same (between 41% and 44%), whereas the number of responses in line with ‘More *A*–more *B*’ were far below chance level.

Although it is problematic to compare test results form different countries on the basis of a rather small and not carefully matched sample of students, it seems that students in Flanders are less affected by the ‘Same *A*–same *B*’ intuitive rule than their Israeli peers. For the Cube problem, 41%, 45%, and 55% of Israeli students in, respectively, grade 10, 11, and 12 responded in line with this rule; in Flanders we found (slightly) lower percentages in all grades, respectively, 40%, 43%, and 39%. For the Carmel problem, the results in both countries diverge completely. In Israel, where about forty students in each grade were confronted with this task, an increasing percentage of students (57%, 50%, and 62% in grade 10, 11, and 12, respectively) responded in line with ‘Same *A*–same *B*’, incorrectly arguing ‘the ratio is the same, therefore the probability is the same’ (see, e.g., Tirosh & Stavy, 1999b); in Flanders, such an increasing trend could not be established and the corresponding percentages were much lower, respectively, 33%, 24%, and 36%. Correspondingly, the percentages of correct responses in Flanders (64%, 67%, and 59% for grades 10, 11, and 12) were significantly higher than in Israel (24%, 42%, and 30%). This last finding does, however, not necessarily mean that the Flemish students displayed a higher quality of probabilistic reasoning. A closer look at the response sheets of the Flemish students revealed that the majority selected the correct answer on the basis of another *erroneous* reasoning process! Some students misinterpreted the statements ‘one son, one daughter’ and ‘two sons, two daughters’ as, respectively, ‘first a son, then a daughter’ and ‘first two sons, then two daughters’, which leads to, respectively, the probabilities of 1/4 and 1/16. However, the majority of the participants that selected the correct response proved to suffer highly from the so-called *equiprobability bias* (see, e.g., Lecoutre, 1992), according to which all events were thought as equally likely. So, in the Carmel family, there are three possible events (‘two sons’, ‘one son, one daughter’, and ‘two daughters’), each having a probability of 1/3; in the Levin family, there are five possibilities (‘four sons’, ‘three sons, one daughter’, ‘two sons, two daughters’, ‘one son, three daughters’, and ‘four daughters’), each having a probability of 1/5.

Discussion

The results of two Flemish studies on the intuitive rules ‘Same *A*–same *B*’ and ‘More *A*–more *B*’ did not provide a strong confirmation of the predictive power of these intuitive rules. In both studies, percentages of responses in line with these rules never exceeded 50% and were mostly below chance level. In the explanations accompanying their answers, students seldom referred explicitly to these rules.

In our view, two points need further investigation. First, a more detailed and systematic picture of problem types and characteristics that elicit the intuitive rules is required. In this picture, there should be room for intercultural differences. Indeed, it seems that the intuitive rules and how they affect students' mathematical problem solving compete with alternative conceptions and strategies – both correct and incorrect ones – that are shaped by culturally bound instructional traditions. Second, there is a need of more fine-grained, process-oriented research that tries to unravel in which stage of the problem-solving process intuitive rules show up, and to detect under what internal and external conditions this intuitive thinking is replaced by more advanced mathematical reasoning. A possible research approach could be to administer individual interviews in which a student has to respond instantly to a comparison situation, and is then encouraged to reconsider his initial judgement.

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