METHODOLOGICAL PROBLEMS IN ANALYZING DATA FROM A SMALL SCALE STUDY ON THEORETICAL THINKING IN HIGH ACHIEVING LINEAR ALGEBRA STUDENTS

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Abstract

The paper gives an outline of a research on theoretical thinking in a group of 14 high achieving linear algebra students. The research was instigated by our hypothesis that one of the reasons why, even in the most 'friendly' environment, students' understanding departs in many ways from the theory is that students try to grasp the theory with a practical rather than theoretical mind. We were interested in knowing if the highly successful students in linear algebra, i.e., those most likely to have a 'good understanding' of the basic linear algebra concepts, indeed think in ways that can be characterized as strongly theoretical. The paper proposes a characterization of theoretical thinking and discusses certain quantitative methods of analyzing the students' mathematical behavior in an interview based on mathematical and epistemological questions.

Introduction: the research question

Our purpose in this paper is to briefly outline our recent research focused on theoretical thinking in a group of high achieving linear algebra students and point to some questions related to the methodology of this research.

Our research was instigated by the need we felt to refine our hypothesis that one of the main reasons why certain students' understanding of linear algebra substantially differs from the theory is that these students try to grasp the theory with a practical rather than theoretical mind (Sierpinska 2000). Taking the contrapositive form of our hypothesis, and *assuming that highly successful students in linear algebra are likely to have a 'good understanding' of the basic linear algebra concepts, we asked ourselves if the thinking of these students could be characterized as 'strongly theoretical'.*

The research of this question called for (a) a definition of theoretical as opposed to practical thinking on which our judgment of students' mathematical behavior could be based; (b) a research instrument capable of externalizing students' thinking in some observable form; (c) analytical means for the clarification of what we meant by 'strong' or 'weak' theoretical thinking tendency in a student or a group of students.

Research procedures

We interviewed 14 students who obtained A grades in the first university linear algebra course (vector spaces and linear transformations). Twelve of these students took the second linear algebra course (Jordan canonical forms and inner product spaces), and 6 of those obtaind A grades also in the second course.

Our interview was aimed at revealing what we considered, at that time, the features of theoretical thinking in the students. However, our analysis of the students' responses led us to refine our definition of theoretical thinking, and eventually we found ourselves analyzing the students' behavior against a set of criteria that was slightly different from the one that guided us in the design of the interview.

The interviews were audio-recorded and transcribed. For each student, a 'story' was then written of his or her mathematical behavior, highlighting behaviors that would attest to the theoretical or practical features of the student's thinking. These stories were revised several times, as our definition of theoretical thinking was refined. A third step was the question-by-question description of the students' behavior, first individually and then as a group. The fourth step was to dress a 'profile' of the theoretical thinking in the group of students, from the features that were the most strongly represented in the group to those that were the weakest.

Theoretical framework: a definition of theoretical thinking

Mathematics education has always been concerned with theoretical thinking in its various aspects and forms. However, there has been a renewed interest in this kind of thinking in the past years (e.g., Steinbring 1991; Boero, Pedemonte, Robotti 1997). Boero et al.'s definition of theoretical knowledge was inspired by Vygotski's distinction between scientific and everyday concepts.

So was, at the beginning, also our definition. However, the concept could be derived from such classics as Aristotle, and his epistemological categories of empirical knowledge, art, craft, productive science, and theoretical science. 'Theoretical science' aims at the development of wisdom or 'knowledge of certain principles and causes' (*Metaphysics*, Book I (A), 982a). Aristotle proposed that the motor of theoretical science is the need to know for the sake of knowing rather than knowing for the purpose of carrying out some other action (ibid., 983a). In recent mathematics education research, Steinbring's work (ibid.) refined this feature, postulating the 'self-referential' character of theoretical knowledge.

Inspired by, among others, these ideas, we assumed that theoretical thinking is a voluntary mental activity which is 'self-serving' or thinking for the sake of thinking, goal-directed, and 'self-referential' or seeking meaning and validity within itself. To support its self-referential character, theoretical thinking is *analytical* and *systemic*, i.e., it distances itself from experience by filtering it through languages and conceptual systems, which it invents and uses as tools, as well as studies as objects in their own right. Aware of the existence of these 'filters', theoretical thinking has no aspiration to certainty. It considers its claims as *hypothetical* and tries to make their assumptions as explicit as possible. Rejecting any a priori assumptions about what is plausible or realistic, it aims at identifying and discussing all possible cases and implications within a system. It is concerned with the problems of *validation* of its claims. In particular, it *questions the scientific procedures* that may have become routine in solving certain type of problems and invents alternative methodologies of research and validation. The motor of such investigations is *wonderment, and doubt* about the existing explanations, irrespective of any practical use that the obtained answers may have.

The scope of this paper does not allow us to include a justification of the relevance of the above mentioned features of thinking for the learning of linear algebra, but this justification constituted an important part of our a priori analysis of the project.

For the purposes of our analysis of the students' behavior we codified the assumed features of theoretical thinking (TT), as follows:

1. TT is reflective, i.e., TT1.1 self-serving, TT1.2 self referential, TT1.3 voluntary
2. **TT is analytic,** i.e., TT2.1 mediated through language which is both an object of reflection and invention; TT2.2 aware of the conventional/symbolic character of language and mathematical notations and graphical representations in particular; TT2.3 aware of the possibility of inventing/designing an artificial language;
TT2.4 sensitive to syntax and mathematical syntax in particular, and especially to the quantification of variables; TT2.5 sensitive to the logical rules of drawing conclusions and negating statements.
3. **TT is systemic**, i.e., TT3.1 relational; TT3.2 has a definitional approach to meanings; TT3.3 has a systemic approach to validation; TT3.4 uses systemic categorization (see Bruner, Goodnow & Austin, 1960, p. 5-6).
4. **TT is hypothetical**, i.e. TT4.1 is aware of the conditional character of mathematical statements; TT4.2 is concerned not only with the plausible or the realistic but also with the hypothetically possible; TT4.3 believes in the relativity of truth.
5. **TT is concerned about validation**, i.e., TT5.1 is fuelled by doubt and uncertainty and hence considers validation as an important problem; TT5.2 considers proofs in mathematics as necessary for the establishment of knowledge
6. **TT has a critical attitude towards standard procedures**, i.e., it problematizes procedures and underlying concepts, does not take them for granted and does not accept them just because they have a stamp of authority.

The research instrument: the interview questions

There were seven questions in the interview. Question 1 asked the students to classify a set of 5 algebraic expressions into at least two groups according to their own criteria. Question 2 asked the students to comment on a flawed definition of linear independence of vectors. Question 3 cited a test question in which typographical mistakes were made: *Let **u**, **v**, and **w** be vectors in a vector* *space V over **R**. Show that the vectors **u - v**, **u - w**, and **v + w** are linearly dependent*. The students were asked to describe how they would approach the problem and then carry out their plan. They were expected to notice the flawed formulation and propose a correction. In Question 4 the students were shown two graphs in log-log base 2 scales, both looking like straight lines and asked if they think that these graphs represent linear functions. One of the graphs could represent a linear function and the other could not. In Question 5 the students were given 5 statements about a certain class of numbers called 'brillig numbers' (odd prime + 2) and asked to pick one they would consider as best suited for (a) a definition, (b) an explanation. They were also asked to tell if another statement about these numbers was true or false and to comment on whether they found these numbers interesting. In Question 6 a four-element set was given, T = {1, 2, 3, 4} in which an operation called 'vorpal' was defined by a Cayley table (the rows were:[[1,1,4,1], [2,2,2,4], [3,3,1,1], [4,4,4,3]]). The students were asked several questions about this operation; in particular, if the operation has a right (left) hand zero element. They were asked also if it is possible to define an operation in the set T with distinct left and right hand zero elements. Question 7 aimed at identifying the high achieving students' epistemological profile, i.e. their declared attitudes towards the basic epistemological questions related to the nature of scientific truth, the ways of arriving at scientific truth, and the ways of validating scientific statements.

Analysis of the interviews

In this report we shall focus on the third and fourth steps of our analysis and the encountered methodological problems.

We found it necessary to represent the TT features, as formulated in our definition, by features of the students' behavior in responding to the interview questions. For example, the feature TT2.5, i.e. sensitivity to the logical rules of drawing conclusions and negating statement) was assumed to underlie three kinds of behavior, which we called 'theoretical behavior' and labeled 'TB':

TB2.5a : Analytic-logical sensitivity to logical connectives - in the 'Linear dependence typo' and 'Vorpal' questions, the theoretically thinking student would correctly use and negate the connectives 'and', 'or' and 'if … then'.

TB2.5b : Analytic-logical sensitivity to circularity in reasoning - in deciding whether the graphs in the 'log-log scales' question represent linear functions, the theoretically thinking student would not assume that the functions are given by equations of the form y = mx + b and then try to prove that the graphs represent linear functions.

TB2.5c : Analytic-logical sensitivity to implications - in the 'Brillig numbers' question, the theoretically thinking student would distinguish between a statement with the necessary and sufficient conditions for a brillig number and a statement with only a necessary condition.

This analysis led us to a list of 33 TB features. We represented our analysis of the students' responses in two tables: the Question-by-question table (Q-b-Q) and the Feature-by-Feature table (F-b-F). In Q-b-Q, each question had a rubric, in which we collected information about how the students behaved on the features assumed to be revealed in this question. The behavior of a student with respect to a TB feature was coded as a vector [a, b], where a and b could be either 1 or 0. The vector tt = [1, 0] represented the fact that the student's behavior consistently had the TB feature; pt = [0, 1] - consistently the PB feature or the practical opposite of the TB feature, tt&pt = [1, 1] - a mixture of TB and PB features, and ø = [0, 0] was assigned whenever there was no evidence of either. The behavior of an individual student through the whole question was then represented by a coordinate-wise sum of the vectors obtained for each feature observed in the question: a vector [x, y]. In order to have a measure allowing to compare the individual students' behavior on a particular question, we used the number x/(x+y)*100% which we denoted by itt%. This number was not defined if both x and y were 0, i.e. if there was no evidence of any of the TB features in the behavior of the given student in the question. Had such case occurred, we would have discarded the data about the student, or about the question, from the study. Such case had not occurred, however. In Q-b-Q, for each TB feature in a question we calculated also a group index. The students' names were listed in a column. The individual student's itt% on a question was calculated by adding the vectors in the rows of the table. The group index for each TB feature in the question was computed, similarly as itt%, but applied to the vectors in the columns of the table: if the sum of the vectors in the column was [x, y], then gtt% = x/(x+y)*100%. The gtt% for the whole question was computed by adding the summative vectors for individual students. The indeterminate case of x=y=0 could not occur because we considered only those TB features for which there was some evidence in the group.

One could ask why not describe the group's behavior with respect to a TB feature just by counting the students in the tt, pt, tt&pt and ø categories. For example, for the feature TB2.4c (Sensitivity to the form of definitions), these numbers were, respectively, 9, 4, 1, 0. For the feature TB3.2c (Definitional approach to meanings in a graphical context) they were 7, 0, 7, 0. We could say that there were less students in the tt category for TB3.2c than for TB2.4c, but we would not be able to compare the strengths of the features on just this basis, especially that there were less students in the pt category in the latter feature than in the former. On the other hand, using the gtt% indices for the features did give us some ground for comparison. For both features, gtt% = 67%, so we could say that they were equally strongly represented in the group.

To illustrate the way we computed the itt% and gtt% indiced in Q-b-Q we include the results from the 'Classification' question (Table 1). As only three features were observed in this Question, the table fits within the frame of this short paper.

In our analysis, we formulated interpretations of the outcomes of our computations. Here is a sample of such commentary:

In Question 1, most students were not thinking of letters in algebraic expressions as names for concrete objects in concrete contexts: the group, as a whole, had a rather good analytic-representational approach to variables in linear algebra (71%). The students in the group were nearly as likely as not to be systemic in tasks of classification (65%). The group's sensitivity to the syntax of mathematical expressions was low.(38%).

In trying to grasp the students' theoretical behavior in the whole interview using Q-b-Q, we decided not to sum up the horizontal scores and then use an index similar to itt%, because such counting could privilege those TB features that were observable in more than one question over those that could be seen in one question only. A better statistic was, we thought, the average of the itt% obtained by a student in the seven questions. The list of average itt% for the students in the same order as in the table was [37, 36, 84, 50, 72, 67, 70, 56, 87, 62, 67, 70, 57, 74].

	Q. 1 '*Classification*'						Question 1 totals		
	TB2.2a		TB2.4a		TB3.4a				IttQ1%
O1	0	1	0	1	0	1	0	3	**0**
O2	0	1	1	0	0	1	1	2	**33**
O3	1	0	1	0	1	0	3	0	**100**
O4	1	0	0	1	1	0	2	1	**67**
V1	1	0	0	1	1	1	2	2	**50**
V2	1	0	1	0	0	1	2	1	**67**
V3	1	0	0	1	1	0	2	1	**67**
V4	1	0	1	0	1	0	3	0	**100**
S1	0	1	0	1	1	1	1	3	**25**
S2	1	0	0	1	1	0	2	1	**67**
S3	0	1	1	0	1	0	2	1	**67**
S4	1	0	0	0	1	0	2	0	**100**
N1	1	0	0	1	1	1	2	2	**50**
N2	1	0	0	1	1	0	2	1	**67**
	10	4	5	8	11	6	26	18	61
	gtt2.2a	71	gtt2.4a	38	gtt3.4a	65	gttQ1%	**59**	(aver)

Table 1: The Question 1 part of the Q-b-Q table.

The rubrics of F-b-F table organized along the TB features. If a feature was observable in several questions, it was counted only once. This analysis treated the interview as a whole, not as a set of questions. In this case, we thought it reasonable to compute the summative itt% index, adding the vectors horizontally. The list of summative itt% scores for the students was [45, 45, 84, 53, 76, 75, 62, 57, 86, 60, 74, 64, 56, 74]. Knowing the grades in the second course of the 12 students' who took both courses ([70, 50, 90, 63, 70, 85, 80, 60, -, -, 85, 77, 90, 85]), we could compute the correlation between these grades and the two lists of itt% scores, i.e., the average itt% from Q-b-Q and the summative itt% from F-b-F. The correlation was 0.71 in the former case and 0.67 in the latter. While both were quite high, it appeared that the average itt% from Q-b-Q was a better predictor of a student's success in the second linear algebra course.

A question for us was why it would be so. We thought that perhaps the reason was that the Q-b-Q results were structurally similar to the results of a test, where the students' performance is also evaluated on a question-by-question basis.

The analysis in F-b-F served different purposes in our study. This table appeared to give a better picture of the students as a group, independently of their individual successes in the courses. It allowed ranking the features from the strongest to the weakest and raising questions concerning the instruction in the courses. If a TT feature was weakly represented in the group it may mean that it is not a necessary condition for the success in the course. The question is, however, if this feature can be considered as negligible from the point of view of educational objectives and values.

Taking the averages of the gtt% indices obtained for the categories of 'reflective', 'analytic', 'systemic', 'hypothetical', 'concern with validation' and 'critical' features of TT, we obtained the following ranking. The strongest feature in the whole group of students was systemic thinking (gtt% = 74%). This feature was even stronger in the group of the 6 students who achieved As in both courses (80%). Next best in the whole group was reflective thinking (72%), but the high achievers in both courses scored only 67% on reflectiveness. This implied, in particular, that these students were approaching problems more as students, trying to be effective in producing a solution that would satisfy the interviewers, than as theoretical thinkers whose aim would be to understand the problem and its implications for the sake of knowing alone. The next ranking feature in the high achievers in both courses was the concern for validation and a spontaneous need for proof (76%, compared to 65% for the whole group). Analytic thinking was also quite strongly represented in this subgroup (73%, compared to 66% for the whole group). Next came the critical approach to procedures (67%, compared to 59% for the whole group), and reflectiveness was the weakest, as mentioned above.

The question of the necessary conditions of high achievement led us to taking into account not only the itt% and gtt% indices but also the numbers of students in the tt, pt and tt&pt categories. If, for a given TB feature, one or more of the high achievers in both courses scored pt = [0,1], would this mean that the feature is unnecessary for high achievement? There were 11 such features. However, in many of these features, there was only one student in the pt category or (boolean 'or') the gtt% index was high. Perhaps only those features should be considered as unnecessary for the high achievement for which the number of students in the pt category was more than 3 *and* the gtt% was low? In this case there would be only two such unnecessary features: (a) approaching a problem not just as a student seeking to produce an acceptable solution, but with the goal of understanding the problem as part of a domain of knowledge; (b) being sensitive to mathematical terminology and trying to be articulate in formulating and communicating ideas. Ironically, these two features are perhaps those about which we care the most as mathematics educators.

Questions for further research

The quantitative approach we used in our research raises many questions. One of them is: would the ranking of the theoretical thinking features in the group of students be different if a different index of the strength of a feature was used? Indeed, if we look at the numbers of students in the tt, pt, tt&pt, and ø categories and apply a method of 'implicative analysis' suggested in the work of Gras (1992) then a slightly different ranking is obtained. Which of these rankings better reflects the theoretical tendencies in the group? Which one should we trust more? But, even more fundamentally, to what extent can we trust the outcome of any one of these methods in view of the fact that the assignment of the tt, pt, tt&pt, and ø vectors to individual students was based on our subjective interpretation of their behavior? These and similar questions led us to a certain skepticism about the relevance of statistical analyses in the domain of advanced mathematical thinking. We recognized, however, that, without the moderate quantitative approach that we used in our research, all we could do would be to tell 14 x 7 fascinating stories about the 14 students' behavior on the 7 questions, flavored perhaps with epistemological and historical reflection. But maybe these stories could have a bigger impact on our teaching of linear algebra than the dry numerical data? The question remains open: What is the relevance of each type of research?

References

Aristotle's Metaphysics, 1966: Translated with Commentaries and Glossary by Hippocrates G. Apostle. Bloomington and London: Indiana University Press

Boero, P., Pedemonte, B., Robotti, E. 1997: 'Approaching Theoretical Knowledge through Voices and Echoes: a Vygotskian Perspective'. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education, Lahti, Finland, 1997*, vol.2, 81-88.

Bruner, J.S., Goodnow, J.J., Austin, G.A. 1960: *A Study of Thinking*. New York: John Wiley & Sons, Inc.

Gras, R. 1992: 'Data Analysis: A Method for the Processing of Didactical Questions'. In R. Douady & A. Mercier (Eds.), *Research in Didactique of Mathematics: Selected Papers*. Grenoble: La Pensée Sauvage éditions, 93-106.

Sierpinska, A. 2000: 'On Some Aspects of Students' Thinking in Linear Algebra'. In J.-L. Dorier (ed.), *On the Teaching of Linear Algebra.* Dortrecht: Kluwer Academic Publishers., 209-246.

Steinbring, H. 1991: The Concept of Chance in Everyday Teaching: Aspects of a Social Epistemology of Mathematical Knowledge. *Educational Studies in Mathematics*, 22, 503-522.