CAN YOUNG STUDENTS OPERATE ON UNKNOWNS?[1]

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Algebra instruction has traditionally been delayed until adolescence because of mistaken assumptions about the nature of arithmetic and about young students' capabilities. Arithmetic is algebraic to the extent that it provides opportunities for making and expressing generalizations. We provide examples of nine-year-old children using algebraic notation to represent a problem of additive relations. They not only operate on unknowns; they can understand the unknown to stand for all of the possible values that an entity can take on. When they do so, they are reasoning about variables.

Mathematics educators have long believed that arithmetic should precede algebra because it provides the foundations for algebra. Arithmetic presumably deals with operations involving particular numbers; algebra would deal with generalized numbers, variables and functions. Hence instructors of young learners focus upon number facts, number sense, and word problems involving particular values. Algebra teachers pick up at the point where letters are used to stand for unknowns and sets of numbers. Although there are good reasons for this natural order it lends itself to discontinuities and tensions between arithmetic and algebra.

The difficulties adolescents show in learning algebra (Booth, 1984; Filloy & Rojano, 1989; Kieran, 1985, 1989; Sfard & Linchevsky, 1994; Steinberg, Sleeman & Ktorza, 1990; Vergnaud, 1985) has led to an even starker separation of arithmetic from algebra. Many have believed that algebraic reasoning is closely tied to and constrained by students' levels of cognitive development. For them, algebraic concepts and reasoning require a degree of abstraction and cognitive maturity that most primary school students, and even many adolescents, do not yet possess. Some have suggested that it would be developmentally inappropriate to expect algebraic reasoning of children who have not reached, for example, the period of formal operations (e.g. Collis, 1975). Others (Filloy & Rojano, 1989; Sfard, 1995; Sfard & Linchevsky, 1994) have drawn upon historical analyses such as Harper's (1987) to support the idea that algebraic thinking develops through ordered and qualitatively distinct stages. Filloy and Rojano (1989) note that western culture took many centuries to finally develop, around the time of Viète, a means for representing and operating on unknowns; they propose that something analogous occurs at the level of individual thought and that there is a "cut-point" separating one kind of thought from the other, "a break in the development of operations on the unknown (op. cit., p. 19)". Herscovics and Linchevski (1994) proposed that student's difficulties are associated with a *cognitive gap between arithmetic and algebra*, "the students' inability to operate spontaneously with or on the unknown" (p. 59). Function concepts and their associated algebraic notation are postponed until adolescence for similar reasons.

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We do not wish to deny that there are developmental prerequisites for learning algebra. And we agree that there is a large gap between arithmetic and algebra in mathematics education from Kindergarten to grade 12. The question is: does it have to be this way? Is the gap set by developmental levels largely out of the sphere of influence of educators? Or is it to a great extent a matter of learning? Or of teaching itself? Does the fact that there presently *is* a large gap signify that there m*ust always be* such a gap?

It is crucial for students to learn to represent and manipulate unknowns. However, we believe it is a mistake to attribute the late emergence of this ability to developmental constraints. We believe it emerges late because algebra enters the mathematics curriculum too late and at odds with students' knowledge and intuitions about arithmetic.

Arithmetic As Inherently Algebraic: Functions and Unknowns

Arithmetic derives much of its meaning from algebra. The expression, "+ 3", can represent both an operation for acting on a particular number and a relationship among a *set* of input values and a *set* of output values. This is borne out by the fact that we can use functional, mapping notation, "$n \rightarrow n +3$", to capture the relationship between two interdependent variables, n and n plus three (Schliemann, Carraher, & Brizuela, 2000; Carraher, Schliemann, & Brizuela, 2000). So the objects of arithmetic can be thought of as both particular (if $n = 5$ then $n+3 = 5+3 = 8$) and general ($n \rightarrow n +3$, for *all* values of n); arithmetic encompasses number facts but also the general patterns that underlie the facts. Word stories need not be merely about working with particular values but working with *sets of possible values* and hence about variables and their relations.

Arithmetic also involves representing and performing operations on unknowns. This is easy to forget since arithmetic problems are typically worded so that students need invest a minimum of effort to using written notation to describe known relations. The relations tend to be expressed by students in *final form*, where the unknown corresponds to empty space to the right of an equals sign. Were arithmetic problems sufficiently complex that students could not straightaway represent the relations in final form, it would become easier to appreciate how central algebraic notation is to solving arithmetic problems.

We are suggesting that arithmetic can and should be infused with algebraic meaning from the very beginning of mathematics education. The algebraic meaning of arithmetical operations is not an optional "icing on the cake" but rather an essential ingredient of the cake itself. In this sense, we believe that algebraic concepts and notation are part of arithmetic and should be part of arithmetic curricula for young learners.

During the last three years we have been working with children between 8 and 10 years of age to explore how to bring out the algebraic character of arithmetic (see Brizuela, Carraher, & Schliemann, 2000; Carraher, Brizuela, & Schliemann, 2000; Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, 1999). Our work focuses on how 8 to 10 year-old students think about and represent functions and unknowns, using both their own representations and those from conventional mathematics. This work is guided by the ideas that: (1) children's understanding of additive structures provides a fruitful point of departure for "algebraic arithmetic"; (2) additive structures require that children develop an early awareness of negative numbers and quantities and to their representation in number lines (3) multiple problems and representations for handling unknowns and variables—including algebraic notation—should become part of children's repertoires as early as possible; and (4) meaning and children's spontaneous notations should provide a footing for syntactical structures during initial learning even though syntactical reasoning based on the structure of mathematical expressions should become relatively autonomous over time.

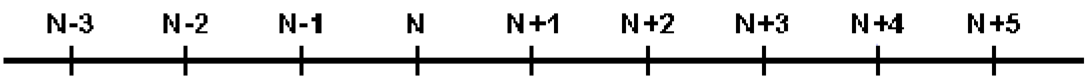
Here we will look at evidence that young children can represent and operate on unknowns. Our examples are taken from our longitudinal investigation with the students of three third-grade classrooms in a public elementary school from a multi-cultural working-class community in Greater Boston. When the children were 8 and 9 years of age, we held eight 90-minute weekly meetings in each of three classes, working with additive structures. Descriptions of our class materials are available at www.earlyalgebra.terc.edu. Our examples come from the seventh lesson we held in one of the three classrooms. There were 16 students in the class that day. Our team of researchers included a teacher, Bárbara, and two camerapersons who occasionally interviewed children as they worked through problems. The students' regular classroom teacher was also present. The following problem served as the basis for discussion and individual work:

Mary and John each have a piggy bank.
On *Sunday* they both had the same amount in their piggy banks.
On *Monday*, their grandmother comes to visit them and gives 3 dollars to each of them.
On *Tuesday*, they go together to the bookstore. Mary spends $3 on Harry Potter's new book. John spends $5 on a 2001 calendar with dog's pictures on it.
On *Wednesday*, John washes his neighbor's car and makes $4. Mary also made $4 babysitting. They run to put their money in their piggy banks.
On *Thursday* Mary opens her piggy bank and finds that she has $9.

We initially displayed the problem in its entirety, so that the students could understand that it consisted of a number of parts. But then we covered up all days excepting Sunday.

Representing An Unknown Amount

The student's first problem sheet contained information only about Sunday. It also contained the following variable number line (or *N*-number line):



After reading what happened each day, students worked alone or in pairs, trying to represent on paper what was described in the problem. During this time, members of the research team walked up to children, asking them to explain what they were doing and questioning them in ways that helped them to refine their representations.

Sunday: After Kimberley reads the Sunday part for the whole class, Bárbara asks whether they know how much money each of the characters in the story has. The children state a unison "No" and do not appear to be bothered by that. A few utter: "N" and Talik states: "N, it's for anything". Other children shout "any number" and "anything".

When Bárbara asks the children what they are going to show on their worksheets for this first step in the problem, Filipe says "You could make some money in them [the piggy banks], but it has to be the same amount". Bárbara reminds him that we don't know what the amount is. He suggests that he could write N and Jeffrey says that this is what he will do. The children start writing and Bárbara reminds them that they can use the *N*-number line on their worksheets if they so wished. She also draws a copy of the line on the board.

Jennifer used *N* to represent the initial amount in each bank. She draws two piggy banks, labeling one for Mary, the other for John, and writes next to them a large *N* along with the statement "Don't know?" David points to "*N*" on her handout and asks:

David: Why did you write that down?
Jennifer: Because you don't know. You don't know how much amount they have.
David: […] What does that mean to you?
Jennifer: N means any number.
David: Do they each have N, or do they have N together?
Jennifer: (no response).
David: How much does Mary have?
Jennifer: N.
David: And how about John?
Jennifer: N.
David: Is that the same N or do they have different N s?
Jennifer: They're the same, because it said on Sunday that they had the same amount of money.
David: And so, if we say that John has N, is it that they have, like, ten dollars each?
Jennifer: No.
David: Why not?
Jennifer: Because we don't know how much they have.

The children themselves propose using *N* to represent an unknown quantity. We had introduced the convention before in other contexts but now it was making its way into their own repertoire of representational tools. Several children seem comfortable with the notation for an unknown and with the idea that they could work with quantities that might remain unknown. Some start by attributing a particular value to the unknown amounts in the piggy banks but, as they discuss what they are doing, most of them seem to accept that this was only a guess. Their written work shows that, by the end of the class, 13 of the 16 children adopt *N* to represent how much money Mary and John started out with. One of the children chose to represent the unknown quantities by question marks and only two children persist using an initial specific amount in their worksheets.

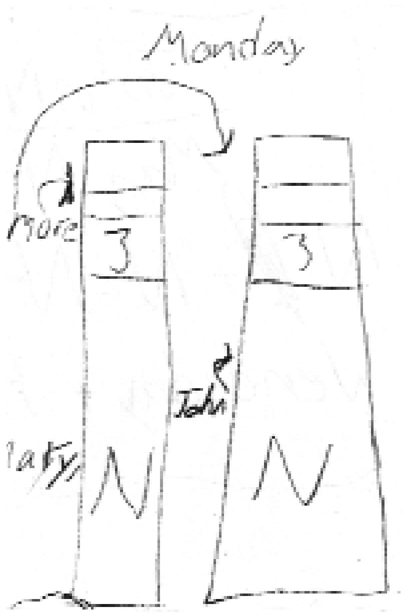
Talking About Changes in Unknown Amounts

<u>Monday</u>: When the children read that on Monday each child received \$3, they inferred that Mary and John would continue having the same amount of money as each other, and that they both had \$3 more than the day before. As Talik explains:

Talik: … before they had the same amount of money, plus three, [now] they both had three more, so it's the same amount.

Bárbara then asks the children to propose a way to show the amounts on Monday. Nathan proposes that on Monday they would each have *N* plus 3, explaining:

Nathan: …because we don't know how much money they had on Sunday, and they got plus, and they got three more dollars on Monday.



Talik proposes drawing a picture showing Grandma giving money to the children. Filipe represents the amounts on Monday as "? + 3". Jeffrey says that he wrote "three more" because their grandmother gave them three more dollars. But when David asks him how much they had on Sunday he incorrectly answers, "zero". Max, sitting next to him then says, "you don't know." The drawing in this page shows Jeffrey's spontaneous depiction of N +3. Note the 3 units drawn atop each quantity, N, of unspecified amount.

James proposes and writes on his paper that on Sunday each would have "N + 2" and therefore on Monday they would have N + 5. Carolina writes N +3. Jennifer writes N + 3 in a vertical arrangement with an explanation underneath: "3 more for each". Talik writes N +3 = N +3. Carolina, Arianna, and Andy write N +3 inside or next to each piggy bank under the heading Monday. Jimmy, who first represented the amounts on Sunday as question marks, now writes N +3 with connections to Mary and John's schematic representation of piggy banks and explains:

Jimmy: Because when the Grandmother came to visit them they had like, N. And then she gave Mary and John three dollars. That's why I say [pointing to N +3] N plus three.

Bárbara comments on Filipe's use of question marks. He and other children acknowledge that N is another way to show the question marks. She tells the class that some of the children proposed specific values for the amounts on Sunday. Filipe says nobody knows and James says that they're wrong. Jennifer says that it *could* be one of those numbers.

Only three children do not write N +3 as a representation for the amounts on Monday.

Operating on Unknowns with Multiple Representations

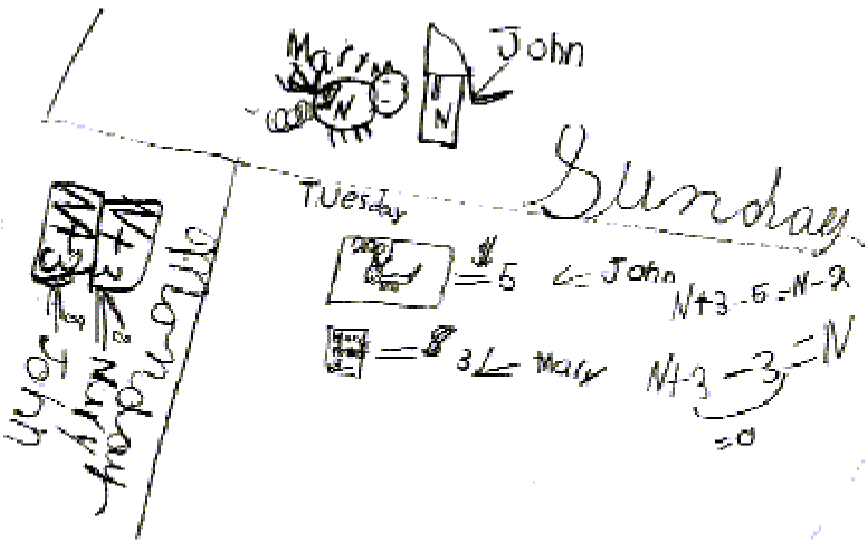
<u>Tuesday</u>: When they consider what happened on Tuesday, some of the students begin to feel uncomfortable because the characters have begun to spend money and the students feel the need to assure themselves that the characters have enough in their piggy banks. A child says that they probably have ten dollars. Most of the children assume that there must be at least \$5 in their piggy banks by the end of Monday; otherwise John could not have bought a \$5 calendar (they seemed uncomfortable with him spending money he didn't have).

Bárbara recalls for the class what happened on Sunday and Monday. The children agree that on Monday they had the same amounts. When she asks Arianna about their amounts on Tuesday, she and other children agree that they will have different amounts of money because John spent more money, leaving Mary with more money.

Jennifer then describes what happened from Sunday to Tuesday, concluding that on Tuesday Mary ends up with the same amount of money that she had on Sunday, "because she spends her three dollars." At this point Bárbara encourages the children to use the N-number line on the board. She draws green arrows going from N to N +3 and then back to N

again to show the changes in Mary's amounts. She shows the same thing with the notation, narrating the changes from Sunday to Tuesday, step by step, and getting the children's input while she writes *N* +3-3. She then writes a bracket under +3-3 and a zero below it, comments that +3-3 is the same as zero, and extends the notation to *N* +3-3=N+0=N. Jennifer then explains how the 3 dollars spent negates the 3 dollars given by the grandmother: "Because you added three, right? And then she took, she spent those three and she has the number she started with."

Using the *N*-number line Bárbara then leads the students through John's transactions, drawing arrows from *N* to *N* +3, then *N* -2, for each step of her drawing. While sketching each arrow, she repeatedly draws upon the student's comments to arrive at the notation *N* +3-5. Some children suggest that this is equal to "N minus 2". Bárbara continues, writing *N* +3-5=N-2. She asks Jennifer to point to, on the number line on the board, the difference between John and Mary's amounts on Tuesday. Jennifer first points ambiguously to a position between *N* -2 and *N* -1. When Bárbara asks her to show exactly where the difference starts and ends, Jennifer correctly points to *N* -2 and to *N* as the endpoints. David asks Jennifer how much John would have to receive to have the amount he had on Sunday. She answers that we would have to give two dollars to John and explains, showing on the number line, that, if he is at *N* -2 and we add 2, we get back to *N*. Bárbara represents what Jennifer has said as: *N* -2+2=N. Jennifer grabs the marker from Bárbara's hand, brackets the sub-expression, "–2+2", and writes a zero under it. Bárbara asks why it equals zero and, together with Jennifer, goes through the steps corresponding to *N* -2+2 on the number line showing how *N* -2+2 ends up at *N*. Talik shows how this works if *N* were 150. Bárbara uses his example of *N* =150 and shows how one returns to the point of departure on the line.



Nathan's drawing (right) shows Sunday (top), Monday (bottom left), and Tuesday (bottom right). For Tuesday, he drew iconic representations of the calendar and the book next to the values \$5 and \$3, respectively, with the images and dollar values connected by an equals sign. During his discussion with Anne, a member of the research team, and using the number line as support for his decisions, he writes the two equations *N* +3-5=N-2 and *N* +3-3=N. Later, when he learned that *N* was equal to 5 (after looking at the information about Thursday) he wrote 8 next to *N* +3 on the Monday section of his worksheet.

<u>Wednesday</u>: Filipe reads the Wednesday step of the problem. Bárbara asks whether Mary and John will end up with the same amount as they had on Monday. James says "No." Arianna then explains that Mary will have *N* +4 and John will have *N* +2.

Bárbara draws an N -number line and asks Arianna to tell the story using the line. Arianna represents the changes for John and for Mary on the N -number line. Bárbara then writes out the notations, N +4=N+4, then N -2+4 = N +2. Talik volunteers to explain this. He says that if you take 2 from the 4, it will equal up to 2. To clarify where the 2 comes from, Bárbara represents the following operations on a regular number line: –2+4 = 2.

Bárbara asks if anyone can explain the equation referring to Mary's situation, namely, N+3-3+4=N+4. Talik volunteers to do so and crosses out the +3-3 saying that we don't need that anymore. This is a significant moment because no one has ever introduced the procedure of striking out the sum of a number and its additive inverse (although they had used brackets to simplify sums). It may well represent the meaningful emergence of a syntactical rule.

Bárbara brackets the numbers and shows that +3-3 yields zero. She proposes to write out the "long" equation for John, N +3-5+4 = N +2. The students help her to go through each step in the story and build the equation from scratch. But they do not get the result, N +2, immediately. When the variable number line comes into the picture they see that the result is N +2. Bárbara asks Jennifer to show how the equation can be simplified. Jennifer thinks for a while, Bárbara points out that this problem regarding John's amount is harder than the former regarding Mary. Bárbara asks her to start out with +3-5; Jennifer says –2. Then they bracket the second part at –2+4, and Jennifer, counting on her fingers, says it is +2.

Talik explains, "N is anything, plus 3, minus 5 is minus 2; N minus 2 plus 4, equals (counting on his fingers) N plus 2. He tries to group the numbers differently, adding 3 and 4 and then proposing to take away 5. Bárbara helps him and shows that +3+4 yields +7. When she subtracts 5, she ends up at +2, the same place suggested by Jennifer.

<u>Thursday</u>: Amir reads the Thursday part of the problem, stating that Mary ended with $9.00, to which several students respond that N has to be 5. Bárbara asks, "How much does John have in his piggy bank?" Some say (incorrectly) that he has two more; other children say that he has 7. Some of the students figure out from adding 5+2, others from the fact that John was known to have 2 less than Mary, since N +2 is two less than N +4.

Bárbara ends by filling out a data table that included the names of Mary and John and the different days of the week with the children's suggestions for how much money each one had on each of the different days. Some students suggest using expressions containing N and others suggest expressions containing the now known value, 5.

Some Reflections

Many students began by making iconic drawings and assigning particular values to unknowns. But over time, in this lesson, and in others like it, the students increasingly came to use algebraic and number line representations to describe the relations in stories.

We should be careful not to interpret their behavior as totally spontaneous; in fact, children's behavior, even when indicative of their own personal thinking, expresses itself through culturally grounded systems, including mathematical representations of the various sorts we introduced.

Number line representations are a case in point. By the time our students had reached the class we analyzed above, they had already spent several hours working with number lines. They also learned to express such short cuts or simplifications notationally: "+7 –10" could be represented as "-3" since each expression had the same effect. We introduced the variable number line (*N*-number line) as a means of talking about operations on unknowns. "Minus four" could be treated as a displacement of four spaces leftward from *N,* regardless of what number *N* stood for. It could equally well represent a displacement from, say, $N+3$ to $N-1$. At the projector students interpreted values of *N* when the *N* -number line was set just above and slid over the regular number line. They also gradually realized they could infer the values of, say, $N+43$ (even though it was not visible on the projection screen) from seeing that $N+7$ sat above 4 on the regular number line. The connections to solving algebraic equations should be obvious to the reader.

We have found that children as young as eight and nine years of age can learn to comfortably use letters to represent unknown values and can operate on representations involving letters and numbers without having to assign values. To conclude that the sequence of operations "N+3-5+4" is equal to $N+2$, and to be able to explain, as many children were able to, in lesson 7, that *N* plus 2 must equal two more than what John started out with, *whatever that value might be*, is a significant feat—one that many people would think young children incapable of understanding. Yet we found such cases to be frequent and not confined to any particular kind of problem context. It would be a mistake to dismiss such advances as mere concrete solutions, unworthy of the term "algebraic". Children were able to operate on unknown values and draw inferences about these operations while fully realizing that they did not know the values of the unknowns.

In addition, we have elsewhere (Schliemann, Carraher, & Brizuela, 2000; Carraher, Schliemann, & Brizuela, 2000) found that children can treat the unknowns in additive situations as having multiple possible solutions. For example, in a simple comparison problem where we described one child as having three more candies than another, our students from grade three were able to propose that one child would have *N* candies and the other would have $N+3$ candies. Furthermore, they found it perfectly reasonable to view a host of ordered pairs, (3,6), (9,12), (5,8) as *all* being valid solutions for the case at hand even though they knew that in any given situation, only one solution could be true. They even were able to express the pattern in a table of such pairs through statements such as, "the number that comes out is always three larger than the number you start with". When children make statements of such a general nature they are essentially talking about relations among variables and not simply unknowns restricted to single values.

By arguing that children can learn algebraic concepts early we are not denying their developmental nature, much less asserting that any mathematical concept can be learned at any time. Algebraic understanding will evolve slowly over the course of many years; but we need not await adolescence to help its evolution.

Final Remarks

Over the last several decades several mathematics educators have begun to suggest that algebra should enter the early mathematics curriculum (e. g., Davis, 1985, 1989; Davydov, 1991; Kaput, 1995; Lins & Gimenez, 1997; Vergnaud, 1988; NCTM, 2000). Some have initiated systematic studies in the area and begun to put into practice ideas akin to those expressed here (Ainley, 1999; Bellisio & Maher, 1999; Blanton & Kaput, 2000; Brito Lima & da Rocha Falcão, 1997; Carpenter & Levy, 2000; da Rocha Falcão & al., 2000; Davis, 1971-72; Kaput & Blanton, 1999; Schifter, 1998; Slavitt, 1999; Smith, 2000).

Still, much remains to be done. "Early algebra education" is not yet a well-established field. Surprisingly little is known about children's ability to make mathematical generalizations and to use algebraic notation. As far as we can tell, at the present moment, not a single major textbook in the English language offers a coherent vision of algebraic arithmetic. It will take many years for the mathematics education community to develop practices and learning structures consistent with this vision.

We view algebraic arithmetic as an exciting proposition, but one for which the ramifications can only be known if a significant number of people undertake systematic teaching experiments and research. The ramifications will extend into many topics of mathematical learning, teacher development, and mathematical content itself. It will take a long time for teacher education departments come to realize that the times have changed and to adjust their teacher preparation programs accordingly. We hope that the mathematics education community and its sources of funding recognize the importance of this venture.

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