# UNIVERSITY MATHEMATICS TEACHING – WHERE IS THE CHALLENGE?

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*Data from university mathematics tutorials were analysed on characteristics of teaching using a construct 'the teaching triad', derived from analyses of secondary mathematics teaching. Elements of 'management of learning', 'sensitivity to students' (from both affective and cognitive perspectives) and 'mathematical challenge' were sought and rationalised with earlier manifestations of these elements. Meanings were derived which made sense in the university context, taking into account the particular culture of university mathematics teaching and expectations of tutors and students. Whereas at first there seemed little challenge on which to remark, a reconceptualisation of challenge allowed an alternative perspective to be offered. The paper reports on this perspective and looks critically at processes and strategies in teaching mathematics at this level.*

Ways in which teachers offer and enable students to tackle appropriate challenges are important to students' engagement with mathematics and their development of mathematical concepts. Teachers' sensitivity to students' affective and cognitive needs is seen to be closely related to the effective nature of challenge[1]

The research reported here involves an analysis of tutorial teaching from a project – called the Undergraduate Mathematics Teaching Project (UMTP) - designed to characterise university mathematics teaching in first year tutorials[2]. Analysis has been done using *the teaching triad*, a construct deriving from earlier research into classroom mathematics teaching at secondary level (Jaworski, 1994) and used, subsequently, as a device to analyse teaching and as a developmental tool by teachers (Note 1). The triad has also been used to analyse the activity of mathematics teacher-educators in the professional development of mathematics teachers (Zaslavsky & Leikin, 1999)

## Theoretical Background

The UMTP was rooted in theory relating to mathematical learning at university level; in particular the difficulties experienced by students at this level (e.g. Tall, 1991; Nardi, 1996). It is distinct in its fine-grained focus on tutor-student interactions and tutors' expressed thinking relating to these interactions. It draws on previous research into secondary mathematics teaching, with a focus on teaching activity, its relationship with students' mathematical activity, and the associated thinking of the teachers in their planning of interactions with students (Jaworski, 1994).

So, for example, research into university learning suggests that students have difficulty in conceptualising cosets of groups, and relating them to notions of conjugation in groups (e.g; Dubinsky et al, 1994; Burn, 1998). We have shown how

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[1] See Jaworski and Potari (1998), Potari and Jaworski, forthcoming

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one tutor struggled with his students' lack of understanding of these concepts, and ways in which he personally addressed them and constructed his teaching to overcome them. As a result of such analyses, we were able to offer a number of indications for a theory of mathematics teaching at this level (Jaworski et al, forthcoming; Nardi et al, in preparation)

Current analysis of tutorial approaches is using the *teaching triad* with three inter-related elements: *management of learning*, *sensitivity to students* (from both affective and cognitive perspectives) and *mathematical challenge*. Briefly, *ML* describes the teacher's role in the constitution of the classroom learning environment. This includes classroom groupings; planning of tasks and activity; setting of norms and so on. *SS* describes the teacher's knowledge of students and attention to their needs; the ways in which the teacher interacts with individuals and guides group interactions. *MC* describes the challenges offered to students, to engender mathematical thinking and activity, in tasks set, questions posed and metacognitive encouragement. These domains are closely interlinked and interdependent. Analysis, reflexively, categorises tutorial data in relation to these elements and reassesses the meaning of the elements with respect to the data.

**Methodology**

The methodology of the UMTP was reported in detail in Jaworski et al (1999). Briefly, from interview data, factual summaries, or *protocols*, were constructed that were then analysed, using tested coding systems, for their teaching characteristics. Commonly occurring codes were inspected against key episodes from the data, selected for their significance by the researcher gathering the data. Sections of data highlighted by the codes were analysed in fine-grained detail to seek insights into teaching issues including teachers' decisions in conducting of tutorials and their relation to the particular mathematics that was the focus of observed tutorials (largely analysis and abstract algebra, with some topology and probability).

Subsequent analysis of tutorial data followed that of interview data, by a production of tutorial protocols: each a summary of the activity of the tutorial. Coding of these protocols, led to identification of (a) typical patterns of interactions; and (b) instances where activity or dialogue might be characterised by elements of the teaching triad. Particular episodes were identified to typify examples of (a) and (b) and these were studied in greater depth from the full tutorial text.

**Typical patterns of interaction**

The lecture-tutorial culture in mathematics in this university requires students to tackle problems set in a lecture and give written solutions to their tutor for comments. Tutors' marking of the solutions forms the basis of tutorial activity

In analysis of activity in tutorials, the codes, *tutor explanation*, *tutor as expert*, *tutor questioning* were abundant. Tutor *explanation* [TE] was usually a straightforward exposition of some aspect of mathematics. Tutor *as expert* [TEx] included the offering of key methods of solution, or proof, or mathematical tricks or routines that are perceived to be part of what one tutor expressed as the "mathematical armoury" that students need. Tutor questions were mainly of three types: enquiring about students' thinking or difficulties [QS]; specific mathematical questions relating to the mathematics being addressed [QM]; and prompting or leading questions directed at filling gaps in an argument [QA]. The following episode represents these patterns. The codes are illustrated in the transcript, and used in the microanalysis of the episode in relation to the teaching triad. The mathematical focus is a question concerning the *orders* of elements of the group $A_4$. Students (two here) have shown difficulty in their solution to the question, and the tutor is highlighting aspects of the concepts that he perceives as important. Voices of the students are not distinct.

**Episode 1: Words from tutorial transcript**

1 T The first thing is what's the order of $A_4$? [QM]

2 S It's 12

3 T So if H is a subgroup of $A_4$; - we quite often write $\leq$ to mean subgroup of [TEx] - if H is a subgroup of $A_4$, then what can we say about the order of H? [QM]

[Tutor writes as he talks. Students observe, listen and respond.]

4 S It must be a factor of 12

5 T Right, so what are they [QA]

6 S 1,2,3,4,6,12

7 T And what result did you use to deduce that? [QS] Pause (10 secs) [Inaudible words omitted]

8 S Lagrange's Theorem

9 T That's the one. So we now know what sizes to look for: 1 and 12 are dead easy. Right? Order 1, H just has to be the identity. Order 12, it has to be all of $A_4$. Those are easy.

[Students make 'understanding' noises: ah, mm]

Order 2. A little while ago, you worked out that there was only one Cayley table for groups of order two. Another way to look at this is what orders of elements can you have? Well, in fact we've already proved this, haven't we here? [Reference to an earlier question and solution] 2 is prime; a group of prime order must be cyclic, so any of these would have to be generated by an element of order 2, and any element of order 2 will generate one of these. So the subgroups of order 2 are precisely *e* and an element of order two. [TE]

Three. The same argument works. [A few inaudible words omitted here]

Four Now what orders could elements have inside a group of order 4. [QM/A]

10 S 2,1,4

11 T Is there any element of $A_4$ of order 4? [QS] Pause (5 secs)

12 T No, there isn't. They come in one of three types

[Tutor continues with an explanation of subgroups of order 4, and moves on to subgroups of order 6. Student participation is of a similar degree to that recorded above.]

13 T So, there's no subgroup of order 6.

The two reasons that this is interesting are that first of all you *can* have several subgroups of the same size; the other thing is there may be a size which by Lagrange's theorem is *allowed* for a size of subgroup, but for which there is no subgroup. [TE] Neither of you wrote this but some other people did, they wrote, by Lagrange's theorem there *must* be a subgroup of size n for every n that divides the order of the group. And that's *not* true. Here was an example. Right. [TEx][3]

A considerable part of tutorial activity can be characterised in the form exemplified above. Of course details change. Different bits of mathematics cause different problems for different students, and different tutors interact with their students in idiosyncratic ways. The pauses in the dialogue above are short compared to those of another tutor, who usually waits a much greater length of time (20-30 secs in some cases) for his students to say something; and much shorter than some tutors who wait hardly any time at all. "*Answering own questions*" is one coding category of tutor activity, in which a question is asked by the tutor, and almost immediately the tutor supplies the answer. It is a rhetorical form in which the question is an almost seamless part of tutor exposition of the concept. Tutors clearly want to offer their students good explanations of the mathematics they consider important.

After observing and recording this tutorial, the researcher interviewed the tutor, starting with a question about his tutorial agenda. The protocol reads as follows:

> **Researcher:** You mark their work. How does this form your tutorial agenda?
>
> **Tutor** It tends to be that all of them have made the same mistakes, so it helps to form a scheme of help, e.g. noticing that they all believed the converse of Lagrange's theorem to be true. Then, once this general scheme is formed, you have to tailor it individually, e.g. noticing that a particular student is writing left cosets for right and that gives her odd results. [TEACH; REC STU PRO;KNOW STU]

The bracketed words, e.g., [TEACH; REC STU PRO;KNOW STU] are part of the coding system used to analyse interview protocols. They characterise the associated dialogue as being about *planning for teaching*, *recognising students' problems*, and *knowledge of students*. These were some of the most commonly occurring codes in the analyses of *interview* protocols. They indicated common elements of tutors' thinking about planning and teaching of tutorials. In this case, the teaching focus, on Lagrange's theorem, is designed around students' misconceptions of the truth of the converse of the theorem, and particular errors in tackling associated questions. Tutor's knowledge of students' particular needs comes from marking their work, and seeing them in tutorials. An important part of the coding of both interview protocols and tutorial protocols, is a rationalisation of the two, providing insights into how a tutor interprets his thinking about teaching into practice in the tutorial.

### Characterising elements of activity or dialogue using the teaching triad.

The teaching triad arose from teaching that engaged students overtly in *questioning* and *inquiry* in mathematics (Jaworski, 1994). A teacher's questions that challenged *sensitively* the particular thinking of a student were seen to be fruitful in enabling conceptual development. In micro-analysis, *harmony* between sensitivity and challenge was seen to be a characteristic of a successful teaching interaction. Micro-

3 We might argue about distinctions between QM, QS, and QA; and between TE and TEx. It is hard to find clear distinguishing examples in one piece of dialogue. The insertion of the codes here is indicative only of their use and meaning.

analyses were then scrutinised against a macro-analysis taking into account wider sociocultural issues of the learning environment (Potari and Jaworski, forthcoming).

In the episode above, at the micro level, management of learning, *ML,* is evident in the tutor's recognition of difficulties and planning of the focus of the tutorial to address these difficulties [*Interview protocol*]. Sensitivity to students, *SS*, (influencing *ML*) is seen both at an affective level, basing the focus in aspects of mathematics that are clearly relevant to students difficulties, and at a cognitive level, focusing on the example that will highlight clearly the concept that the tutor wants students to consider (converse of Lagrange's theorem). The tutor's questions invite student participation, but in some cases this is of a minimal degree [*statements 2, 6, 10*] , and in others, students seem unable to respond. The tutor often waits for a response [*7, 11*] and sometimes supplies the answer himself [*12*]. The dialogue seems to encompass little mathematical challenge, *MC*. Largely, questions seek students' knowledge of particular mathematical results and teacher exposition provides the substance of the interaction. Any challenge is left to the students themselves in making sense of concepts in their own personal study.

At a macro level, mathematical enculturation might be seen as a tacitly agreed basis of interaction. Students have met Lagrange's Theorem – one of them is able to name it in response to a prompt from the tutor. However, it is important mathematically that they perceive the difference between theorem and converse, both in terms of Lagrange, but as a consideration in theorems more generally. The tutor's choice of example might be seen as a clever management strategy, enabling students to perceive truth relations between a theorem and its converse. In this we might discern a (tacit) element of challenge. Students are being confronted with a challenge to their perceptions, and it is up to them to go away and make sense of it.

### An alternative pattern of interaction

The episode that follows is chosen to show an approach that seems to incorporate *challenge* in a different form. Unusually the tutor here was also the lecturer of the Abstract Algebra course, so he had, himself, set the problems on which the students worked, including a question about quotient groups. In the lecture, a theorem had been proved and the lecturer had asked the student to prove for themselves its converse. This in itself is potentially a mathematical challenge for the students who tackle it. The tutor is working with two students, one of whom is ill (S1) and does not say much. The tutorial protocol and coding for the episode read:

> **Tutorial Protocol:** Tutor asks S2 a question. S2 explains at the board. Tutor offers advice and asks questions when S2 gets stuck. S1 offers suggestions. When S2 has finished his proof, tutor explains a quicker method he [the tutor]would have used.
>
> **Coding:** tutor questioning [QS/M/A]; student-led explanation [SE]; tutor-students interaction [TSI]; tutor as expert [TEx]; rapport between tutor and students [R].

**Episode 2: Words from Tutorial Transcript**
**Conventions**: () inaudible words omitted; … repetitive or irrelevant words omitted.

*The episode transcribed here is 10 minutes of the tutorial.*

1 T: You did the part of question 3 I did in lectures, but the part I left as an exercise, namely the converse, er, you forgot about (students laugh). [R]
One wants to show that for a congruence … - well its an equivalence relation, - that satisfies $g_1 \sim k_1$ and $g_2 \sim k_2 \Rightarrow g_1 g_2 \sim k_1 k_2$ ; and $g \sim k \Rightarrow g^{-1} \sim k^{-1}$
So its an equivalence relation that respects the group operations inversion and congruence … we're told that H is a normal subgroup of G and we're given this equivalence relation by $g_1 \sim g_2$ if and only if they represent the same coset. So lets try proving these two things.
Let's say we want $g_1 g_2 = k_1 k_2$ and this means saying that $g_1 H = k_1 H$ and $g_2 H = k_2 H$ then this means saying that $g_1 g_2 H = k_1 k_2 H$ [On board is also written $g_1 g_2 \sim k_1 k_2$ ]
I said in lectures, although you may not remember, that the work had been done already because the quotient group is well defined, you can actually multiply cosets together in a well-defined fashion. … Why do you think those cosets at the end are equal? [TEx]

2 S2: Well for the things we're getting at the top –
from $g_1 H = k_1 H$ [T: yes] you can say that, for any h in H, $g_1 h =$ [SE]

3 T: Go on, go for it [tutor invites him to the board. S2 writes, talking as he writes] [R]

*[The next portion of the tape is hard to follow as symbols are being written (evidenced by noise of chalk on board) but not necessarily articulated. The researcher's fieldnotes (FN) help with what was written. Reconstruction is as faithful as possible under these circumstances.]*

4 S2: If you've got h and l in H, then this one tells us that $g_1 h = k_1 l$ (pause) for some l in H (S2 and tutor say this together) and same sort of thing for the twos, (he writes $g_2 h = k_2 l$) (pause) [SE]

5 T: call it h' and l', well, call it l', you might want the same h as () (pause) [TE]

6 S2: erm (pause)

7 T: well what's a general element of the left hand side downstairs look like? [QS]

8 S2 erm that's gonna be $g_1 g_2$ and that's the same h [chalk on board and inaudible speech: FN suggest he writes $g_1 g_2 h$ and wants to continue with $k_1 k_2$ ] [SE]

9 T: well, just expand - don't write $k_1 k_2$ now - keep expanding …what you had there – so we can rewrite that can't we … [TE]

10 S2: well $g_1$ and $g_2$ can be combined to another g [SE]

11 S1: Can't you write as something else, can't you say $g_2 h = k_2 l'$ [SE]

12 T: Yes, go for that I'd say [R]

13 S2: Say that $k_2 l'$ equals (pause, hesitancy) can you say from that one, that is equal to er (pause) $k_1$ (pause) before, that was only true cause that was in H [SE]

14 T: well you know now that

15 S2: [inaudible, but FN give us the following symbols: $g_1 g_2 h = g_1 k_2 l'$]

16 T: you haven't really written much up there yet that requires, uses, normality. The fact that er H is normal means that $k_2 l'$ can be written in a different way. (Pause) [TE]

17 Together: T: $k_2 l'$ S2: some other thing - $k_1$ T: $k_2$ S2: Yes.

18 T: OK, so its $g_1 l''$ , let's say, $k_2$ – and what is

19 S2: so that $g_1$ T; Yes S2: l'' from this Together: is $k_1 h''$

| | | | | |
|---|---|---|---|---|
| 20 | T: | $k_2$, and then that is equal to | [TSI] | [R] |
| 21 | S: | erm, same sort of thing, because of normality we can say that this is something else | | |
| 22 | T: | oh, yes, you, why not go the other way | | |
| 23 | S2 | depends what we're aiming for | | |
| 24 | T: | well exactly, we're aiming for a $k_1k_2$, so why not instead swap them round | [TSI] | |
| 25 | Together: | $k_1k_2h'''$ OK [FN indicate that on the board now is the following: $k, l, l' \varepsilon H$ $g_1h = k_1l$ $g_2h = k_2l'$ : $g_1g_2h = g_1k_2l' = g_1l''k_2 = k_1h''k_2 = k_1k_2h'''$] | | |
| 26 | T: | You did the very hard work of essentially proving that multiplying cosets of a normal subgroup is well defined. I'd have been perfectly happy if you'd assumed that. [TEx] OK? So you've got that out, and that is absolutely fine, but, I'd have been happy with this: You've got two cosets – that coset's the same as that, that coset's the same as that; so $g_1Hg_2H = k_1Hk_2H$, and these cosets multiply to give that [$g_1g_2H$] and these cosets multiply to give that [$k_1k_2H$] [i.e. $g_1g_2H = k_1k_2H$] and that's all I was expecting. [TE] In the same way suppose g~k and gh=kh, so they have the same inverse in the quotient group and what is the inverse of this, we know it is represented by the inverse of the representatives [FN: writes $(gH)^{-1} = (kH)^{-1}$ $g^{-1}H = k^{-1}H$ $g^{-1} \sim k^{-1}$] so that would have done. So that was all really I wanted. What you proved was that multiplying cosets is fine when you have a normal subgroup; which is a good point to remember. [TE, TEx] | | |
| | | [Tutor now quickly explains the proof as he would have proved it] | [TE] | |
| 27 | T: | So it's a bit simpler … rather than getting lost in algebra. | [TEx] | |

The episode, while largely managed by the tutor, splits into 3 parts: (a) [*1*] in which tutor creates the problem-solving environment, stating the problem, and clarifying its context and parameters; (b) [*2-25*] in which we see tutor-students interaction: one student, mainly, constructs a solution with support from tutor instructions [*5,9*], questions [*7,22*], comments [*12, 14, 16, 18*], prompts [*20, 22, 24*], and expert input [*16*]. At points tutor and student seem to think together [*17, 19, 25*]; and (c) [*26-27*] in which we see the tutor in expert mode, explaining and demonstrating. Part (b) seems rather different from the style of interaction represented in Episode 1. There is evidence of student involvement in activity and thinking. Although we see in this episode, as above, considerable *TE* and *TEx*, there are new elements. Students join in the discussion in a more active way, encouraged by the tutor. Parts of the tutorial are student-led, albeit with tutor participation. Interaction is less responsive on the part of the students and more generative. There seems to be rapport between tutor and students which implies a degree of trust built up through experience of working together.

The teaching strategy of inviting a student to the board is a familiar practice here. In one interview this tutor acknowledged it might feel threatening to a student, but that students had come to realise that he would provide helpful support.

> I do promise to help; or will help ... they actually know I'll start them off. They won't just be stood at the board and me twiddling my thumbs. I might, after a few seconds, like 30 seconds, or something like that, or perhaps even less if they're looking panicky, I would suggest, er, "Well, OK, write down, what's the first line? What's it mean to say that?"

So, the intervention of the tutor shown above is a considered approach to encouraging students' participation in the mathematics of the tutorial. As such, it seems to constitute mathematical challenge, sensitively handled, albeit a challenge to think publicly, rather than to tackle a particular mathematical question. *ML* in Episode 2 seems more sophisticated than in Episode 1 in that it incorporates a teaching strategy that not only addresses the tutor's mathematical issues, but does so in a way that the students are seen, overtly, to be involved in the thinking. There is evident struggle to express ideas, and present them in accepted forms. Tutor interjections might be seen as tutor imposing his own explanations or expertise, but alternatively they can be construed as supportive input (*affective sensitivity*), according to his own words. As they are generally pertinent to the immediate thinking of the student they are also sensitive in a cognitive domain.

## Concluding Remarks

In our tentative theory of pedagogic development/awareness (Jaworski et al, forthcoming; Nardi et al, in preparation) Episode 2 is characterised at a higher level than Episode 1, indicating more sophistication of pedagogic awareness in addressing students' mathematical conceptualisation. However, a central part of the tutor role is seen widely as inducting students into appropriate ways of seeing and thinking mathematics. We see, above, examples of the various strategies that tutors use, which seem to form their pedagogical repertoire, and to be a practical manifestation of pedagogic thinking. Challenge to students is often implicit in the pedagogic approach, leaving its interpretation up to the students themselves. However, the second episode is more indicative of *harmony* in sensitivity and challenge. In previous research, harmony has been shown through teachers' questions being related to students' thinking, and through development of students' metacognitive activity [Note 1]. Here it is part of the teaching strategy (*ML*) that challenges students (sensitively) to address mathematical concepts. Further research might explore critically the relationship of harmony to students' conceptual learning.

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