A MODELING PERSPECTIVE ON STUDENTS' LEARNING THROUGH DATA ANALYSIS

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A modeling approach to problem solving shifts the focus of the learning activity from finding an answer to a particular question to creating a system of relationships that is generalizable and re-usable. In this research paper, we discuss the nature of tasks that can be used to elicit the development of such systems. We present the findings from one classroom-based case study of Australian children and a summary of findings from all U.S. and Australian classes in our studies. Student reasoning about the relationships between and among quantities and their application in related situations is discussed. The case study suggests that students were able to create generalizable and re-usable systems (or models) for selecting and ranking data.

Introduction

Data analysis is increasingly recognized as an important topic within school mathematics and has gained an increasingly visible role in the K-12 curriculum (English, Charles, & Cudmore, 2000; Greer, 2000). The justification of such significance is generally made by an appeal to the usefulness of such skills in everyday life as well as in a wide range of work-related settings. The analysis of data as part of school mathematics is sometimes justified on the basis of contexts which are motivating and of interest to students. In this research, the context is seen as providing a site that is rich for the sense-making activities of learners and that holds the potential for the development of more generalized understandings across a range of contexts. In this paper, we discuss the results of an analysis of a sequence of modeling activities in which middle-school students investigated non-routine problem situations where the core mathematical ideas focused on the creation of ranked quantities, operations and transformations on those ranks, and, finally, the generation of relationships between and among quantities to define explanatory and predictive relationships.

Theoretical Framework

A modeling approach to the teaching and learning of mathematics focuses on the mathematization of realistic situations that are meaningful to the learner. The emphasis on modeling involves three important shifts in the approach to teaching and learning mathematics: (1) the nature of the quantities and operations that are useful, (2) using contexts that will elicit the creation of useful systems (or models), and (3) developing and refining such models in ways that are generalizable. We briefly discuss each of these aspects in turn.

The quantities and operations that are needed to mathematize realistic situations often go beyond what is usually taught in school mathematics (namely quantities such as counts, measurements, ratios, rates and proportions, shapes, and the four basic operations of arithmetic). Often in realistic situations, the kinds of quantities that are needed include accumulations, probabilities, frequencies, ranks, and vectors. The operations needed include sorting, weighting, organizing, selecting, and transforming entire data sets rather than single, isolated data points. In solving typical school "word problems," students generally engage in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. In most cases, the problem information has already been carefully mathematized for the student. The student's goal is to unmask the mathematics by mapping the problem information in such a way that an answer can be produced using familiar quantities and basic operations. In modeling tasks, the student's goal is to make sense of the situation so that s/he can mathematize it in ways that are meaningful to her/him. This will involve a cyclic process of selecting relevant quantities, creating meaningful representations, and defining operations that may lead to new quantities (Lesh & Doerr, in press).

A modeling approach to problem situations explicitly uses meaningful contexts that elicit the creation of useful systems (or models). Modeling begins with the *elicitation* stage, which confronts students with the need to develop a model to describe, explain and predict the behavior of an experienced system. Principles for designing such model-eliciting problems have been put forward by Lesh and colleagues and are described elsewhere (e.g., Lesh, Hoover & Kelly, 1992). Models are systems of elements, operations, relationships and rules that can be used to describe, explain or predict the behavior of some other experienced system. We are particularly interested in those models in which the underlying structure is of mathematical interest. The sequence of data analysis problems (described in more detail below) provide us with a setting in which we can examine the development of students' interpretations of the problem situation, their reasoning about relevant elements of the system, their selection of quantities, operations, and representations, and their multiple cycles of interpretation.

Engaging in this kind of model building is not seen as finding a solution to a given problem but rather as developing generalizations that a learner can use and re-use to find solutions (Bransford, Zech, Schwartz & The Cognition and Technology Group at Vanderbilt University, 1996; Doerr, 1997). To this end, we argue that the students need multiple experiences that will provide them with opportunities to explore the mathematical constructs, to apply their system in new settings, and to extend their model in new ways. Each of these stages of the model development process includes multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and re-constructed by the learner, ordinarily interacting with other learners. This view of student's conceptual development through modeling is shaped by earlier research that posits a non-linear, cyclic approach to model building (Doerr, 1997). Generalizing and re-using models are central activities in a modeling approach to learning mathematics. Thus, a modeling perspective leads to the design of an instructional sequence of activities that begins by engaging students with non-routine problem situations that elicit the development of significant mathematical constructs and then extending, exploring and refining those constructs in other problem situations leading to a generalizable system (or model) that can be used in a range of contexts.

Description of the Study

The model development sequence (described briefly below and elaborated at http://soeweb.syr.edu/mathed/HMDproject/Main.html) was designed by the first author and has been investigated in several middle-school classrooms (students aged 12- 13) in the United States. The sequence was later revised by the second author for use in Australian classrooms with students aged 11-12 years. The Australian children addressed here were from a grade 6 class who participated in the sequence of model development activities described below. The activities were implemented in the children's classroom each fortnight over a period of 11 weeks. The children worked in small groups of 3-4 in sessions of approx. 90 minutes duration.

Description of the Modeling Tasks

The overall sequence of model development activities consists of four problem situations centered on the core mathematical ideas of ranking, weighting ranks, and selecting ranked quantities. Since the problem situation is focused on ranking, of necessity the students analyze and transform entire data sets or meaningful portions thereof, rather than single data points. The students had no specific formal exposure or instruction on these ideas prior to the unit. Rather, the unit was designed so that the students could readily engage in meaningful ways with the problem situation and could create, use and modify quantities (e.g., ranks) in ways that would be meaningful to them and in ways that could be shared, generalized, and re-used in new situations. The sequence of problem situations was designed to be completed by a small group of students, thus providing a social setting for the negotiation of conjectures and justifications and for the clarification of explanations. In addition to the sharing which would take place within the small groups, discussion of specific systems created by students provided a forum that could potentially lead to the sharing of either common or multiple systems within the whole class. Each of these tasks is described briefly.

The sequence begins with the Sneakers Problem, which was designed to elicit the notion of ranking in the first place and the multiplicity of factors that would lead to the need for selecting based on ranks. This is followed by two tasks, namely, the Weather Problem and the Summer Camp Problem, which were designed to explore and extend the constructs elicited in the Sneakers Problem. The final task in the sequence, the Crime Problem, provides the opportunity for students to apply the constructs developed in the earlier tasks, to symbolize the systems they have developed, and to refine their notions about weighting factors.

In the Sneakers Problem, the students encounter the notion of multiple factors that could be used in developing a rating system for purchasing sneakers and the notion that not all factors are equally important to all people. Students were asked "What factors are important to you in buying a pair of sneakers?" This generated a list of factors where not all factors were equally important to the students; the students then worked in small groups to determine how to use these factors in deciding which pair of sneakers to purchase. This resulted in different group rankings of the factors. The teacher then posed the problem of how to create a single set of factors that represents the view of the whole class; in other words, the group ranks needed to be aggregated into a single class ranking.

The second problem, the Weather Problem, was designed to extend, explore and refine the idea of ranking, sorting, selecting, and using quantitative and non-quantitative data. The context of the problem is a travel agency that provides a re-location service for clients who specify certain climatic factors that govern their choices of possible destinations. Students are given a table of data on the climatic conditions of various cities, together with letters from two clients (extracts shown below). The students' task is to develop a generalised rating system for comparing the climates in different places, and to write a letter to the travel agency recommending the first and second best, and worst cities for each client. The students are to explain to the agency how their rating system works and why it is a good one.

Climatic Information

City	Clear Days	Days below 15°	Days above 30°	Average yearly rainfall (mm)
Sydney	85	12	15	1220.4
Alice Springs	195	40	169	274.5
Hobart	36	184	6	516.3

Letter from client: *Dear Global Travel, My wife and I are retiring in several months and would like to relocate in a warm and sunny area. We don't care if there is a lot of rain and we definitely don't want to be too cold. What are some cities we should consider living in? Sincerely, Mr & Mrs Johnson.*

The Summer Camp Problem (designed by Lesh and colleagues) is a structural analog to the Weather Problem and asks the students to create a generalized system for selecting teams for track and field events in a summer recreational program. In the Crime Problem, students are presented with crime rates on both violent and property crimes for various cities. They are asked to devise a system that can be used to determine if a particular city (in the present instance, Ipswich) on the list of cities is "safe enough" or warrants increased expenditures for the police budget. Intended in the mathematization of this data set is the notion that although violent crimes against persons have lower overall rates, they would be considered more significant than property crimes.

Data Sources and Analysis

Data sources from both the Australian and American classes included audio- and videotapes of the students' responses to each of the tasks, together with their work sheets and final reports detailing their models and how they developed the models. Transcripts were analyzed for evidence of students' mathematical development over the course of the tasks. Field notes and audio-tapes were analyzed for the varying strategies that were developed across classroom settings.

The researchers shared their data analyses across the two sites and the analysis is on-going.

Results and Discussion

First, we consider the development of one group of 3 children from the Australian classes (Ben, Erin, and Rose). The group began the Sneaker Problem by listing factors using non-mathematical arguments and decisions that reflected their own experiences in buying sneakers ("*I think style then size*;"commented Rose, to which Erin replied, "*But what if it doesn't fit them*?"). Ben soon realized that they needed to be more objective ("*It doesn't matter*!"). This informal negotiating strategy was gradually replaced by a more mathematical one when the group faced the problem of aggregating several group lists into one list. They used a frequency-based strategy (described later) in combining the lists until they faced a difficulty with ranking two of the factors (brand and grip). The girls tried to negotiate here on a subjective basis but Ben rejected this approach and began the shift from subjective negotiating to mathematizing the elements in the lists as mathematical quantities in their own right. He totalled the ranks of each of the two problematic factors and then re-ranked their totals. The group reverted to their earlier frequency-based strategy, however, to complete their aggregation of the lists. Nevertheless, it was interesting to observe Ben's comments in the subsequent whole-class sharing session. After one group explained, "*We put grip* [before brand] *because there's two grips there and there's only one brand at* 6" [the student was referring to the sixth place on her list]), Ben asked, "*What if it was the opposite? What if grip was where brand was, and brand was where grip was. Would you choose grip or brand?"* Some children responded in a subjective way ("*Use your own opinion*"), while one child suggested adding the ranks across the lists and finding the average (although his group had not done so). Ben explained how he solved this dilemma: "*I had a bit of trouble so I added up all the brands and got 27 and when I added up grip I got 25, and 25 means it's, like .... grip before brand.").*

On the Weather Problem, which the students completed two weeks after the Sneaker Problem, Ben's group displayed two major mathematical developments. At the beginning of the Weather Problem, the group appeared focused on finding a generalized system. Rose and Erin recommended using an averaging system, which they had developed independently on a small analogous task at the end of the Sneaker Problem session (this task involved working with data from a McDonalds' survey of people's top 5 reasons for coming to McDonalds. Students had to develop a master list that ordered the reasons from most important to least important.). Ben wanted to create his own system, however, his initial ideas were rather unsophisticated ("*Why don't we take each city and say what we think of it, like we could say it's pretty cold, pretty hot…*"). The girls kept reminding Ben that they had to find a generalized system ("*Method, method, because it has to work for all of them*."). After some negotiation, the group started to develop their system by ranking cities according to the number of clear days. Rose stated that they should "*rank from lowest to highest and then you average it*." The group then proceeded to add the data in one of the columns, namely, the number of clear days, and then divided by 9 (there were 9 cities listed in the table). Following Ben's comment, "*If we wanted a perfect city, we should go about getting all the averages for a start*," the group found the average of each of the remaining climatic factors in the table of data. After deciding that, if the clients "*want a hot one and it's above the average that should mean it's hot*" (Ben) and "*yeah, if it's below it's cold*" (Erin), the group proceeded to draw a table to record those cities that were above or below the average for each climatic factor. In trying to refine the group's explanation of their system, Ben stressed, "*If people want a hot day, you find a city that, hang on, hang on, just listen, you find a city that has over 118 days over 30 degrees,"* and "*you wouldn't look for cities that are the closest to the average*." In returning to the clients' letters, Ben directed the group to list the cities that were above/below the average according to what the client requested. In scanning their list for cities that met the criteria, the group became somewhat bogged down, until Ben decided that the use of an elimination strategy might help: "*Write down all nine cities and then cross out the ones we don't think it is and go right down to the last three*." As the group was doing so, Rose was working at a "meta-level," re-focusing the group on the goal of their efforts: "*I'm going to re-read the question again…we don't need to look at every single one..we don't really have to worry about rain…..they definitely don't like cold days so we definitely don't want to be cold*."

The group used the same "mean-split" system in working the <u>Crime Problem</u>. They found the average of the violent data of all cities listed in the given table and then repeated this for the property data. The group compared these two overall averages with Ipswich's rate for property and its rate for violence to make a decision on whether Ipswich is a safe place to live. They then decided to make a more fine-grained analysis by considering the individual components of property and violence for Ipswich, and explained, "*We added down every one of them* [the columns of data] *and murder is about average, and assault's better than average, and robbery's better than average, and burglary is better than average. All of them is better than average. But the overall trend* (for Ipswich) *is up* (the 5-yearly trends for each of the cities were listed in the table). Ben added, "*We found Ipswich is good against other cities but bad against itself*." With minimal intervention, Ben's group had developed a generalizable and re-useable system for selecting and ranking data; this sophisticated system was developed across three different problem contexts.

We now report the range of systems that we have found across all of the classes in our studies. In analyzing the aggregation of the ranked sneaker data, we have identified five distinct systems that students have developed. Similar to the development shown by the small group discussed above, these systems are further developed by the students as they progress through the modeling sequence. Here, we are summarizing just the variations in models that occur in the first task. In general, we have found that at least three of these systems will emerge in most classrooms.

The first system to determine an aggregated class rank is a frequency-based strategy. The factor with the greatest number of one's becomes the first ranked and so on for the second and third factors. The students were easily able to identify the highest and the lowest ranking factors by this system as well. What became problematic with this system was the selection and identification of a rank for the factors in the middle; this was the problem encountered by Ben's group, above. These middle factors were often assigned ranks using an estimation strategy that yielded a "close enough" rank from the perspective of the students. An alternative strategy, such as what Ben proposed, that is occasionally used is to use successive pairwise comparisons. So if two factors are competing for the same rank on the list, these two factors are compared in terms of the relative order in which they appear on all the original lists. The factor with the greater number of higher rankings is then ranked higher. We note that this notion of pairwise comparisons is mathematically powerful and useful. However, it does not become a strategy that persists across the sequence of tasks. Finally, we note that the difficulty in resolving the problematic issue of the middle ranks often precipitates a rejection of this model and a move to one of the other systems below. Hence, in most cases, this frequency-based model is an early way of thinking about the problem that is later abandoned as the students strive to accommodate all the data into their system.

A second system developed by students is to arithmetically average the ranks of each factor and then to explicitly re-rank those averages. Significantly, this system indicates a shift from seeing the ranks as labels or as positions on a list to seeing them as quantities that can be operated on by averaging. The factors now have explicit numerical values, rather than implicit places in a list. In turn, the newly constructed quantities of the average rank are assigned explicit numerical ranks. In one instance, we found that the occurrence of a tie in ranks led to a refinement of the system. The tied factors were assigned a common rank, and the next rank was skipped. At the time, we were surprised at this solution, since to the best of our knowledge students had not had any formal instruction in methods to resolve ties. In subsequent reflection, we surmised that the students may have had experience with this solution from sporting events in which this solution is a common strategy.

A third system developed by students is to simply total the ranks of each factor and then to explicitly re-rank those totals. This system reflects the same shift in thinking about the ranks as does the "averaging" system above. However, it would appear that the students are focusing on the meaning of the quantities in a slightly different way. Occasionally students will express the total as "the number of points" attributed to the factor.

A fourth system that is devised explicitly ranks the initial data so that a factor that is "highest" on the list (i.e. is ranked number one) receives the greatest number of "points." Similarly a factor that is "lowest" on the list is given the least number of "points," namely one. We see the students in this case explicitly shifting from a label or position on a list to a rank quantity ("points") that measures the relative merits of the factor. These ranks are then summed (similarly to the third system) and these totals are re-ranked in reverse order so as to keep the "best" factor in the first position, with the greatest number of points.

A fifth system used to determine an aggregated class rank simply ignores the group data and resorts to a “voting” strategy. This occurred in a classroom where the discussion of how to combine the lists was initiated not in the small groups, but in a whole class discussion. Hence, a salient feature of the ranks became the values given by each individual class member, rather than by the groups. Some students argued that this strategy was “fairer” than using the group data since it took into account the opinion of everyone in the class.

Concluding Points

The above five systems reflect the thinking of various groups of students about the task of aggregating data that are in ranked lists. These are the initial models that are elicited by the context of the task and importantly, are later developed by the students with minimal intervention from the teacher. As we saw in our case study, Ben’s group progressed from an initial frequency-based and at times, subjective strategy, to mathematizing elements in lists by totalling ranks, and then to a sophisticated “mean-split” system that was generalizable across tasks.

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