INVESTIGATING THE MATHEMATICS SUBJECT MATTER KNOWLEDGE OF PRE-SERVICE ELEMENTARY SCHOOL TEACHERS

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This paper reports recent findings from a project which investigates the mathematics subject knowledge of prospective elementary school teachers, and how this relates to classroom teaching performance. The project was initiated in 1997 in the context of UK government policy to introduce subject content knowledge as an explicit dimension of the 'standards' for the award of Qualified Teacher Status in England. We present some findings about topics that trainees find difficult and show that the extent and security of their subject matter knowledge is related to their teaching competence.

BACKGROUND

Recent changes in the curriculum for Initial Teacher Training (ITT) in England incorporate a stronger focus on trainees'[1] subject matter knowledge (SMK). Notwithstanding the complex relationship between SMK and pedagogical content knowledge (PCK), there is evidence from the UK and beyond which would seem to support this shift of emphasis (Ball, 1990; Kennedy, 1991; Alexander, Rose and Woodhead, 1992; Ofsted, 1994; Simon and Brown, 1996). Government Circular 4/98 (DfEE, 1998) sets out what is considered to be the "knowledge and understanding of mathematics that trainees need in order to underpin effective teaching of mathematics at primary [elementary] level", and charges ITT providers with the audit and remediation of students' SMK.

> Providers must audit trainees' knowledge and understanding of the mathematics content in the National Curriculum Programmes of Study for mathematics at KS1 and KS2[2], and that specified in paragraph 13 of this document. Where gaps in trainees' subject knowledge are identified, providers of ITT must make arrangements to ensure that trainees gain that knowledge during the course … (DfEE, 1998, p. 48)

In this paper, we describe our approach to the audit of the mathematics SMK of 173 primary trainees in 1998-99. This was the first cohort of students following the one-year Postgraduate Certificate in Education (PGCE) course to whom the requirements of Circular 4/98 applied by statute. We had, however, piloted the audit and a draft version of the 'standards' on a voluntary basis the previous year. We present here some findings related to the trainees' knowledge and understanding of proof. Currently, there is evidence for concern in the UK about students' facility with mathematical proof, both at school and at university level (Coe and Ruthven, 1994; London Mathematical Society,

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[1] The official discourse in England refers to students undergoing pre-service preparation for school teaching as 'trainees'. In this paper, we speak of 'students' and 'trainees' synonymously.

[2] In England and Wales, Key Stage 1 (KS1) is the first phase of compulsory primary education, between the ages of five and seven. Similarly, KS2 covers ages seven to 11.

1995; Healy and Hoyles, 1998). One argument suggests that curriculum and assessment reforms in the 1970s and 1980s promoted investigational approaches to school mathematics at the same time as Euclidean point-line geometry went into decline, favouring inductive reason at the expense of deduction. One requirement of Circular 4/98 (detailed later) can be seen as an attempt to address a deficit in the current generation of prospective primary school teachers.

A number of PME papers have considered aspects of elementary teachers' SMK, such as divisibility (Zazkis, 1994), ratio (Klemer and Peled, 1998), place value (McClain and Bowers, 2000), with comment on the relevance of SMK to the professional role of their participants.

OVERVIEW OF GOALS AND METHODS

The project sets out to investigate:

1. those areas of SMK required by Circular 4/98 which prove to be problematic for significant numbers of trainees;
2. whether the expectations of Circular 4/98 are well-founded insofar as secure SMK (or otherwise) is reflected in classroom performance;
3. ways in which trainees' practice in school-based placements is informed by their SMK;
4. the process and effectiveness of SMK remediation through peer tutoring.

Some findings with respect to the first year and the first three goals of the project, incorporating trainees' ability to perceive and express generalisation, can be found in Rowland, Martyn, Barber and Heal (2000). Preliminary findings concerning the fourth goal were reported in Barber, Heal, Martyn and Rowland (1999).

The structure of the primary PGCE under consideration is such that by the middle of January, with fully six months of the course remaining, the main content areas – number concepts and operations, data handling, mathematical processes, shape and space, measures, algebra, probability – have been 'covered' in lectures and workshops, giving the trainees opportunity to recall those topics they have forgotten (for lack of use) since they did mathematics at school. A 90-minute written assessment consisting of 16 test items in mathematics is administered at this point of the course. Each trainee's response to each question includes a self-assessment of their ability to complete it successfully.

The course includes two extended 'practicum' placements in schools in the latter parts of the second and third terms. Given these and other demands of the course, the major SMK remediation opportunity comes between the first and second placements.

Other aspects of the taught course based in the University	Maths SMK audit	Other aspects of the taught course based in the University	School placement 1	SMK peer-tutoring and remediation (with other aspects of the course)	School placement 2
Term 1 (autumn)	Term 2 (spring)			Term 3 (summer)	

Table 1: The chronology of the PGCE course

From this second project cohort, 34 students who were assessed as secure in nearly all of the topics audited became mathematics peer tutors. Following training for this task, they conducted peer tutoring sessions with those students (at most two per peer tutor) who experienced most difficulty with the audit. These peer tutors wrote a feedback sheet on the post-audit progress of each of their tutees. In addition, 12 students acted in a looser 'on demand' support capacity to a self-support group of about five students. Members of these groups self-reported their progress with mathematics SMK.

During school placements, each student works under the joint supervision of a school-based mentor and a university tutor. For the purposes of the project, the two supervisors agreed on assessments of the student's performance in teaching mathematics towards the end of (and in the context of) each placement, against the standards of Circular 4/98.

TRAINEES' MATHEMATICAL THINKING: ASPECTS OF PROOF

One dimension of our research has been to identify what mathematics (within the remit of Circular 4/98) primary trainees find difficult, and the nature of their errors and misconceptions in these areas. Facilities in the four 'easiest' and 'hardest' of the 16 items in the 1998-99 audit are shown in Table 2.

HIGHEST FACILITY		
% secure[1]	Mean score[2]	TOPIC
98[3]	1.96	Inverse operations
95	1.92	Ordering decimals
93	1.90	Divide 4-digit number by 2-digit
92	1.90	Problem solving in a money context

LOWEST FACILITY		
% secure	Mean score	TOPIC
52	1.17	Pythagoras, area
49	1.15	Generalisation
33	0.87	Reasoning and proof
30	0.73	Scale factors, percentage increase

[1] The written response gives a high level of assurance of the knowledge being audited.

[2] A secure response scores 2, which is therefore the maximum possible for the mean.

Table 2: Audit topics with the highest and lowest facility ratings

The table demonstrates some striking similarities with that for the previous 1997-98 cohort (Rowland *et al*, 2000) although facility with the more difficult items is lower. We discuss here aspects of the trainees' understanding of reasoning and proof as evidenced in the audit. At another UK university, Goulding and Suggate (in press) have found, as we did, that proof is a source of particular difficulty for trainees. They add that these difficulties are particularly resistant to remediation within the span of the PGCE course. Circular 4/98 requires that trainees demonstrate:

> That they know and understand […] methods of proof, including simple deductive proof, proof by exhaustion and disproof by counter-example (DfEE, 1998, p. 62)

The following item was designed to audit this 'standard'.

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[3] All cohort percentages have been rounded to the nearest integer.

A rectangle is made by fitting together 120 square tiles, each 1 cm^2. For example, it could be 10cm by 12 cm. State whether each of the following three statements is true or false for every such rectangle. Justify each of your claims in an appropriate way: (a) The perimeter (in cm) of the rectangle is an even number. (b) The perimeter (in cm) of the rectangle is a multiple of 4. (c) The rectangle is not a square.

More than one mode of justification is possible for each part, and a proof by exhaustion (listing the 8 possible rectangles) would meet the requirements of all three. We anticipated some deductive arguments for (a), counterexamples for (b) and perhaps contradiction ($\sqrt{120}$ is not an integer) for (c).

Table 2 shows that only one-third of the students made a secure response to the whole question. 30% gave insecure (or blank) answers to all three parts. The percentage of secure[4] responses to the three individual parts were 59, 44 and 52 respectively. In the self-assessment referred to earlier, students indicated a lower level of confidence in their ability to tackle this question than any of the others. Only two percent declared themselves confident to do it, whereas 35% reported that they didn't think they could do it, or didn't understand it, or were too terrified to think about it.

We proceed here with further consideration of part (b) of the question, which fewer than half of the students were able to manage to our satisfaction. The difficulty with counter-examples encountered by the majority of students is consistent with the findings of Zaslavsky and Ron (1998) with top-level 9th and 10th grade school students.

Each student's response to part (b) was categorised and assigned to one of eleven codes (the first column below, Table 3). The second column describes the type of response for each code; the third column gives the percentage of students making that response.

Code 9	No response to part (b)	6%
	NOT SECURE, INCORRECT ANSWER ('true')	
Code 10	States without explanation/justification	6%
Code 11	Refers only to the 10x12 case	10%
Code 14	Because 4 divides 120 (including confuses perimeter with area)	6%
Code 15	Rectangle has 4 sides so 4 divides the perimeter	5%
Code 19	other	5%
	NOT SECURE, CORRECT ANSWER ('false') WITHOUT JUSTIFICATION	
Code 0	States without explanation/justification	3%
Code 2	Hints that a counterexample exists but doesn't give one	12%
Code 4	other	3%
	SECURE: CORRECT ANSWER ('false') WITH JUSTIFICATION	
Code 30	Gives one counterexample	34%
Code 32	Gives two or more counterexamples	10%

Table 3: Percentage responses to the proof item part (b), by code.

[4] We describe responses as 'secure' rather than 'correct', because some *answers* were correct (e.g. that statement (a) is TRUE) but not adequately justified (e.g. "because 120 is even"). A response that we are calling 'secure' would consist of a correct true/false judgement *and* a valid justification.

Each type of response other than code 9 corresponds to a student judgement concerning the validity of the statement that the perimeter of (every) rectangle is a multiple of 4, and – codes 0 and 10 apart – to a decision concerning an appropriate means of justifying that judgement. A variety of misconceptions seem to underpin invalid arguments, and we consider some of the more interesting ones.

The fact that one third of the students erroneously believe the statement to be true resonates strongly with the findings of Zaslavsky and Ron (1998). Ten per cent of the students seemed to base this conclusion on the fact that it holds true for the 10x12 rectangle given as an example. Perhaps some of these students genuinely believed that the question required them to consider only this particular rectangle, and interpreted "every such rectangle" to mean "every 10x12 rectangle", although this raises interesting questions about when they might consider two such rectangles to be different. Such an interpretation is supported by the response of students such as the one who wrote "The perimeter is 44. 44 is a multiple of 4. 44÷4=11 or 4x11=44." The peer tutor report on another such student read "(she) did not understand what was expected … (she) read the question as though it referred to only one possibility, now sees the need to investigate further". A rather different diagnosis is implied by those students who seemed to be drawing on a 'false conservation' misconception (Lunzer, 1968) i.e that once the area is fixed, so is the perimeter. One wrote "The perimeter has to stay the same otherwise the area will change … the perimeter is always in total 44."

Some students (code 15) argued that (b) must be true because a rectangle has four sides. Again, we can only speculate from their written responses, but these suggest an epistemological orientation which views mathematics as a non-empirical discipline, one in which truth can only be arrived at – or even guessed - by appeal to deductive argument, albeit argument of a spurious kind. There is little or no sense of mathematics as an experimental test-bed, in which they might confidently respond to an unexpected student question "I don't know, let's find out." Likewise, the suggestion that the perimeter is a multiple of 4 because 120 is a multiple of 4 (code 14) seems in some cases to privilege 'argument' over evidence. In others there is a clear case of confusing perimeter with area. Thus, one student wrote "the perimeter is always 120" and another "perimeter = $a \times b$". Confusion between perimeter and area is well-researched and documented e.g. Foxman, Joffe, Mason, Mitchell, Ruddock, and Sexton (1987).

The secure responses all gave counter-examples. Ten per cent chose to give more than one counter-example, even though one is sufficient. A quarter of these described the *general characteristics* of a counter-example, such as "As the addition of the two different side lengths does not have to be an even number (if one length is odd and the other is even, it won't be) the perimeter will not necessarily be divisible by 2(2)=4." Whilst such an analysis exceeds the requirements of the refutation, it seems to point to a desire for explanation – why it is that some perimeters are multiples of 4 and others are not. If a counter-example is deemed to be a kind of proof (that not $\forall x P(x)$), then a single example might typically fall short of one of the purposes of proof – to explain (de Villiers, 1990). Just under a fifth of the 44% who successfully refuted statement (b) with one or more counter-examples actually used the word ‘counter-example’ in their response, exposing some awareness of the ‘syntactic’ structure of the discipline (Grossman, Wilson and Shulman, 1989) i.e. the nature of enquiry in the domain of mathematics, and how new knowledge is introduced and warranted.

SUBJECT KNOWLEDGE AND CLASSROOM PERFORMANCE

We move on now to data which have enabled us to build on and update our earlier findings (Rowland *et al*., 2000) associated with the second of our project goals – investigating the relation between trainees’ SMK and their teaching competence. To summarise those findings: with the first project student cohort (N=154), the level of each student’s subject knowledge (based on the audit) was categorised as low, medium or high, corresponding to the need for significant remedial support, modest support (or self-remediation), or none. Towards the end of that course, specific assessments of the students’ teaching of number[5] were made on the second and final school placement (against the standards set out in Circular 4/98) on a three-point scale weak/capable/strong. These data did not support a null hypothesis that the spread of performance in the teaching of number was the same for the three categories identified in the subject knowledge audit. There was an association between mathematics subject knowledge (as assessed by the audit) and competence in teaching number. Further analysis (Goodman, 1964) pinpointed the source of rejection of the null hypothesis: students obtaining high (or even middle) scores on the audit were more likely to be assessed as strong numeracy teachers than those with low scores; students with low audit scores were more likely than other students to be assessed as weak numeracy teachers. In effect, there is a risk which is uniquely associated with trainees with low audit scores.

For the second cohort considered in this paper, more extensive data from school placements enabled comparison of mathematics subject knowledge with teaching performance (a) on both first and second placements (b) with respect to both ‘preactive’ (related to planning and self-evaluation) and ‘interactive’ (related to the management of the lesson in progress) aspects of mathematics teaching (Bennett and Turner-Bisset, 1993). For reasons of space, Tables 4 and 5 below show two of the four 3 by 3 contingency tables, those for Placement 2 (N=164: nine students had withdrawn from the course), together with expected frequencies (in parentheses) based on the null hypothesis that audit performance and teaching performance are independent.

Each table has df=4, and values of χ^2 less than 9.5 support the null hypothesis against the alternative that audit performance and teaching performance are in some way linked ($p<0.05$). The χ^2 values for the preactive and interactive data are 17.8 and 13.6 respectively. In fact, the association between audit score and teaching performance was significant for each of the four analyses. These results confirm our earlier finding and

[5] The restriction to ‘number’ rather than mathematics was a pragmatic decision determined by the fact that, in that year, the PGCE course was subjected to scrutiny by a government agency, the Office for Standards in Education. The inspectors’ brief was to focus on Reading and Number.

point to the positive effect of strong SMK in both the planning and the ‘delivery’ of elementary mathematics teaching.

	TEACHING PRACTICE PERFORMANCE		
SUBJECT KNOWLEDGE AUDIT	Strong	Capable	Weak
High	**12 (8.1)**	**18 (14.1)**	**4 (11.8)**
Middle	**20 (18.5)**	**33 (32.3)**	**25 (27.1)**
Low	**7 (12.4)**	**17 (21.6)**	**28 (18.1)**

Table 4: Placement 2, preactive

	TEACHING PRACTICE PERFORMANCE		
SUBJECT KNOWLEDGE AUDIT	1 (strong)	2 (capable)	3 (weak)
A (high)	**13 (8.5)**	**19 (18.2)**	**2 (7.3)**
B (middle)	**21 (19.5)**	**42 (41.9)**	**15 (16.6)**
C (low)	**7 (13.0)**	**27 (27.9)**	**18 (11.1)**

Table 5: Placement 2, interactive

CONCLUSION

We have chosen here to highlight the problematic nature of proof as a component of the mathematics SMK of pre-service elementary teachers, adding further weight to the doubts of Goulding and Suggate (in press) that much can be done to remedy trainees’ difficulties with proof within initial training, especially given the multiple demands on them in all areas of the curriculum in an intensely pressured course. We would expect that clarity of understanding of the nature of proof and refutation in mathematics would inform the trainees’ approach to questioning and enquiry with their students, and we are struck by the robustness under replication of our earlier finding (Rowland *et al*., 2000) that effective classroom teaching of elementary mathematics is associated with secure SMK at a level beyond the elementary curriculum. It may be that, even within the constraints of PGCE courses, greater priority could be given to syntactic dimensions of SMK, although inevitably this would be at the expense of substantive elements. In the light of Goulding and Suggate’s comment above, we observe that the second school placement occurred *after* the remediation sessions, yet the association between classroom performance and the audit some five months earlier was maintained. It seems clear that there is need for the development of teachers’ SMK as a component of longer-term continuing professional development. With this in mind, it might be more honest and realistic if the attainment of the full range of SMK standards (DfEE, 1998) were re-conceptualised as an ongoing professional process rather than a hurdle to be crossed in initial training.

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