ALGORITHMIC AND MEANINGFUL WAYS OF JOINING TOGETHER REPRESENTATIVES WITHIN THE SAME MATHEMATICAL ACTIVITY: AN EXPERIENCE WITH GRAPHING CALCULATORS

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In designing mathematics learning with the mediation of computerized tools, one of the crucial questions to be considered is how much and in what way we would like the tool "to do the work" for the students. In problem situations where the solution is achieved mostly via graphical representatives, and algebraic models are mostly used as "keys" for obtaining graphical representatives on the screen, the algebraic representatives and their form seem minimized in importance, and students may tend to generate them from tables by mechanistic-algorithmic procedures. The above questions will be mainly demonstrated within the description and analysis of a case study involving a group of three 10th graders (about 16 years of age) working together to investigate and solve a problem situation on the topic of functions, having graphing calculators (TI-83 Plus) at their disposal. The role of contextual factors is highlighted by means of contrasts with the work of students on the same problem from another country.

Introduction

In designing as well as in studying a classroom learning activity in a computerized mathematics learning environment, one should consider contextual factors of various origins, like: (a) the mathematical content to be learned and its epistemological structure; (b) the learners, their mathematical knowledge, culture, and the history with which they started the researched activity; (c) the classroom culture and norms, the role of the teacher, the learning organization--in small groups or individually--etc.; and (d) the potential "contribution" of the computerized tool.

This study discusses questions related to the above factors and how they lead students to use a computerized tool and benefit from it mathematically or the opposite. The questions will be mainly demonstrated within the description and analysis of a case study involving a group of three 10th graders working together to investigate and solve a problem situation on the topic of functions, having graphing calculators (TI-83 Plus) at their disposal. The protocol analysis of the group work will reveal a dialectical problem-solving process that develops between two ways of making use of the computerized tool: a mechanistic-algorithmic one and another one that is led by students' search for meaning.

Contrasting the work of the group against the work of groups in a classroom from a different country will shed some light on the roles of the above contextual factors in students' making beneficial use of the computerized tool.

Some Theoretical Comments

During the past 20 years or so, the potential of computerized tools to develop and support *mathematization* by students working on problem situations has disclosed many positive aspects of their integration into school curricula. This can take the form of amplification and reorganization (Pea, 1985; Dörfler, 1993) and of experiencing new "mathematical realism" (Balacheff & Kaput, 1996). A common claim is that the computerized tools can play an advantageous role in assisting students to make connections between and within various representations of the same mathematical entity (e,g., Kaput, 1992). Teasley and Roschelle (1995) have documented instances of computer environments serving to disambiguate student thinking in the exploration of novel mathematical phenomena, by easy flexible transformations between representations. The power of a grapher to smoothly transform a function from its algebraic to its graphical representation, and the availability of the corresponding numerical data directly from the graph (by "walking on it"), make it possible to deal with problem situations involving complicated functions at an early stage of learning.

However, students have also been found to use technological tools in non-thinking and non-productive ways. Goldenberg (1988) has, for instance, warned of the ways in which students can misinterpret computer-based graphical representations of functions because they have not chosen an appropriate window; while Guin and Trouche (1998/1999) have argued that a surprising result produced by a graphing calculator does not necessarily induce a question on the part of students.

Looking closer at the interaction that learners have with a computerized tool in the classroom, one should take into account the epistemological power of the technology (Balacheff & Kaput, 1996), for example, the ways in which this power emerges from the multi-representational nature of the tool, the availability of different representations and different *representatives* of the same mathematical object (i.e., displays within the representations, Schwarz & Dreyfus, 1995), and the smooth transformation between them. Schwarz and Hershkowitz (in press) investigated how representatives can mediate the construction of meaning in mathematics learning. They claimed that from an epistemological point of view the relations between the mathematical entity and its representatives are inherently ambiguous. Representatives either may represent mainly the most prototypical examples of the mathematical entity or, because they are partial, may often be ambiguous in the sense that only some of the critical properties of the entity are displayed in the representative.

The ambiguity and the power of the computerized tool encourage the production of various representatives upon need, and also stimulate students' requirement for and ability with interweaving representatives together. In so doing, students may extract the invariant properties of the mathematical entity and thus overcome the ambiguity within the single representative. This process is part of the students' search for mathematical meaning. On the other hand, technological environments can induce students to reach "false representatives" (those that do not represent the critical properties of the mathematical entity at all) or to interweave representatives together in a non-meaningful, algorithmic fashion, as for example in the automatic extending of the numerical values of a spreadsheet column (Ponte & Carreira, 1992).

In this paper, we discuss two ways in which students interweave together the tool-based representatives: a mechanistic-algorithmic way (where students combine representatives in non-thinking, rote ways), and a meaningful way. Moreover, we tell the story of the use of two kinds of representatives in the process of problem solving: representatives that do not represent the properties of the mathematical objects involved at all, and representatives that do.

The Study

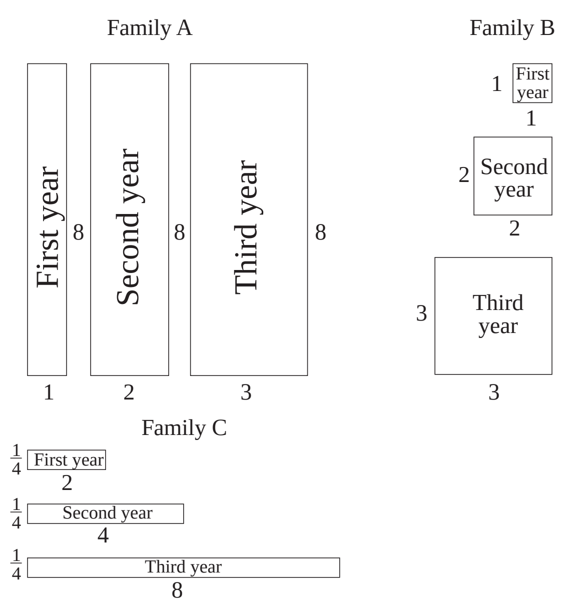
The case study involves three 10th graders from a Montreal high school working together to investigate and solve a mathematical problem situation, having graphing calculators (TI-83 Plus) at their disposal. The analysis of their activity is described in four rounds. On the whole, the investigative activity done by this group and their solution strategies are monitored by their need for a search for meaning. But, in Rounds 1 and 2, the process of "joining together tool-based representatives," which the students engage in, is mostly of the first kind, the mechanistic-algorithmic, which leads students towards some “false representatives.” Rounds 3 and 4 provide evidence of a different way of joining together representatives, one that is clearly characterized by a search for meaning. The progress of the group from Rounds 1 and 2 into Rounds 3 and 4 is led by critical thinking and is supported by the students' control of the "joining-together-representatives" functions, which are part of the strength of the graphing calculator.

The problem situation and some comments about its history

The following are the parts of the problem situation that are relevant to this paper.

Growing Rectangles

Each of the following three families of rectangles has its growth pattern:



*In family **A** the width grows each year by one unit; the length remains constant at 8 units. In **B** the width and the length of the rectangle grow each year by one unit. In **C** the length doubles each year, and the width remains equal to 1/4.*

Please investigate the problem in groups. At first try to generate various hypotheses concerning the following questions:

1. *Please compare the areas of the 3 families of rectangles over the years. What are their initial situations? Which family (or families) takes over the other families (or family) and when?*
2. *In which years will the area of each family exceed 1000 square units?*

Now check your hypotheses with mathematical tools (the help of the graphing calculator is recommended). Try to be as accurate as possible.
Please write a report as a common product of your group.
Try to describe your conjectures and what they were based on. What kinds of debates did your group have?

Try to describe the ways in which you solved the problem and in what ways you were using the graphing calculator

This activity was first tried in an introductory one-year-long course on functions with the mediation of graphical calculators (TI-81) in Grade 9 in Israel (for details see Hershkowitz & Schwarz, 1999). It took place during the sixth week of the course. At that stage, students had practiced the actions and passages between and within representations and representatives. They were aware of the fact that obtaining a graph representing a given phenomenon required an algebraic model.

The students in the Israeli class were first invited to suggest hypotheses without using the computerized tool, then to use it to check them. Students tried to figure out which family would eventually take the lead by using intuition and/or by computing "by hand" the areas of the three families of rectangles for a few values. Then students (in most of the groups) translated the situation into algebraic representations (with some difficulties in generating the function-area for Family C, $y=1/4*2^x$), and then obtained the graphs with their graphical calculators (see Figure 1 below for a stylized representation of the graphing window).

Figure 1

After the groups had finished, the teacher discussed with the class the mathematical findings and strategies. Students reported that in the eighth year ($x=8$), the three rectangles had the same area, and that from that year on, Family C took the lead from Family A. Family B remained in between. The evidence provided by the different representations was accepted even if, for some students, it was unexpected; no student declared the computer wrong. Nevertheless, they tried to reinterpret the situation, and even to overcome wrong intuitions, by matching together representatives from different representations; the algebraic, the numerical, the graphic, and the phenomenon itself. The sociomathematical norm of what constitutes evidence in problem situations was formed here as a consequence of students' interactions with the tool.

What happened in the case-study group

Before setting off to work in groups of threes on the problem situation, the research class from the Montreal school was asked also to read the problem and to vote on whether they thought that Family A, B, or C would have the largest area over the long term. A couple of students voted for Family A; most voted for Family B; and a few abstained. No one predicted that Family C would eventually have the largest area. Teams then started to work on the problem situation. They also had to produce a written report on the problem-solving process, as a common product of the team, and then to present it orally in front of the whole class during the concluding discussion. This activity was two periods long (an hour and a half in all).

The team to be featured will tell the story of the dialectical process between the mechanistic and the meaningful, with the search for meaning as the guiding thread of the dialectical process.

Round 1: Making the technological tool generate the algebraic expression for the situation.

After reading the problem questions, the group of Kay, Ema, and Sam (two girls and a boy) began immediately to create a table of values on paper. They entered the numbers 1 to 10 in the left-most column, headed "year." They labeled the next three columns "Family A," "Family B," and "Family C." To fill the Family A column, they calculated 8 x 1, 8 x 2, 8 x 3, and continued by increasing each entry by 8. Kay remarked: "*It is going by 8 each year*." To fill the Family B column, Sam suggested "*the year where it is, just square that number and you will get the area."* When the Family B column was completed, by computing the area of the growing rectangles in their heads, Sam began to compare the 1st and 10th entries for Families A and B. Kay remarked that "*so far, B has more over the long term*." She then stated: *"We can do equations for each one and compare on the graph of the calculator to see where they all intersect."* Sam nevertheless wanted them to first complete their table.

To fill the Family C column, they multiplied the given length of 2 for the first year by 1/4 (with the help of the calculator); they then took the length of the previous year, doubled it, and multiplied by 1/4. Thus, for the initial values of each of the columns, the procedure used for filling them reflected the operations suggested by the text of the problem situation (e.g., "*in C the length doubles each year and the width remains equal to 1/4*"). But the filling-in of the table was not done with the aim of detecting the relationship between x and y values so as to yield an expression for the function.

It was done simply to have 10 values for each Family so as to be able to calculate the differences between the 1st and 10th values, and check their initial hypothesis as to which Family had the largest area over the long term. Their attention was on the global differences. It is worth noting that while the calculations for Families A and C were done in a recursive fashion, the calculations for family B were based on an explicit generalization for x (the year number) – Sam said: "*the year where it is, just square that number and you will get the area,*" and yet he did not reach or use a closed-form algebraic expression.

After completing column C, Sam announced that they had been wrong in their prediction that Family C would be ahead. He then said that *"by having all this, you can now just make up equations for each one*." It is hard to know what Sam meant by this; either he was ready now to generate from the situation and the tables the algebraic expressions for the growth of the three families, or something else. In any case, the objective of the three of them seemed to be to look for intersection points on a graph. Kay, at that point, seemed to have a very clear strategy in mind: *"Let's do a linear regression*." Sam added: "*So, if they want to know when they take over, you have to do an equation and find out where they meet."* Kay continued his sentence: *"And you have to find the point of intersection."*

The group had finished answering the front-end question that we had posed regarding their initial hypothesis and could now get on with the main task at hand. As in the Israeli class, the objective of all of them seemed to be to draw the graphs in order to be able to compare the three families by having intersection points. They were also aware that, in order to obtain a graph representing a given phenomenon, it is necessary to have an algebraic model. However, the way in which the group members went on to find the equation for each Family was quite unexpected. With the experience we had from the first class, we had thought that they would either analyze the given problem situation with its patterns of rectangles, or look at the numerical values they had generated for their tables in order to find an expression for the relation between the x values (the years) and the corresponding y values (the areas). The way in which they chose to join the paper-and-pencil and technology-based representatives in order to obtain the graphs and the intersection point/s is a crucial part of our story and is described in the following paragraphs.

After rapidly entering the first five values of each column from the paper table of values into STAT-Edit lists of her graphing calculator, and then selecting LinReg (ax + b) of the STAT-Calc menu, Kay obtained the following three LINEAR functional expressions for Families A, B, and C: $8x$; $6x$ - 7; $1.8x$ - 2.3. She suggested to her team-mates that they enter these "equations" into the *Y= editor* of the calculator and then ask for the graphs, which they did. She, who was the fastest at using the calculator, soon announced, "*OK, guys, these are the graphs,*" while turning her calculator toward them so that they could see her screen (see Figure 2).

It is noted that this team was not alone in its use of the calculator's linear regression tool as a means of generating the area expressions for the three families of rectangles.

The teacher, when questioned, disclosed that the class had used regression for two or three weeks, about half a year earlier when they were dealing with problems involving real-world data. Students had learned to model (i.e., obtain equations for) these "real-world" situations with the help of the linear and quadratic regression tool of their calculator.

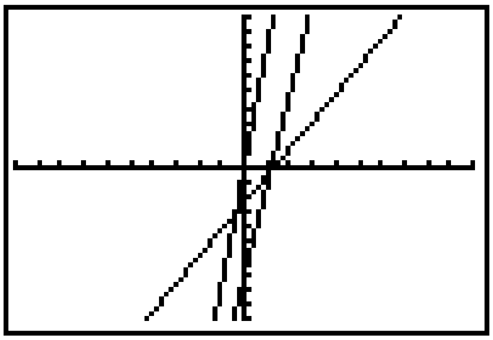


Figure 2

Figure 3

We can conclude that the above regression procedure, which is often the only way to get an algebraic rule for the so-called regularities of real-life phenomena, was adopted by these students as the most efficient way to get the algebraic model of a problem situation, even if this particular problem situation involved idealized mathematical data. The students appeared to trust the algorithmic routine involved in using the regression tool of the calculator and did not feel any immediate need to verify empirically the correctness of the expressions that had been generated by the calculator with the data that they already had in their paper-and-pencil table.

Nevertheless, one is led to ask why the group mechanistically used the calculator to generate and to join together the representatives (the numerical and algebraic representatives) instead of looking more carefully at their table where the answer to the question "WHEN?" was there all along (in Year 8, all the families had an area of 64). Did they consider the graphical representation more reliable? Or is a functional algebraic rule, and its joined-together graph, a stronger argument than mere numerical evidence? Or was it important for them that what they had been able to conclude by means of the numerical table (that Family C had the larger area over the long term) be supported by confirmatory and matching evidence from the graphical representation as well, that is, that they be able to join together the paper-based numerical representatives with the graphical tool-based representatives? Or were they just carried away by the above algorithm that begs to be executed once one has generated tables of data for a given phenomenon?

Round 2: Failure at getting the technological tool to work for them: a turning point.

From the beginning of applying the regression option of the calculator tool and up to this moment, the group had appeared to be marching along a mechanistic path in a kind of automatic fashion, without doing very much in the way of reflection. When the graphs they generated did not look as they had expected them to (Sam: "*They don't all intersect at the same point* "--as seen in Figure 2), Kay changed the scales immediately. But, that did not help her to obtain the meeting point she wanted to see (see Figure 3). Repeated scale changes (even up to 1500 years) did not produce the elusive point of intersection that they had intuitively been expecting. Kay then suggested they try the CALC-Intersect option to have the calculator provide some information with respect to the point of intersection. The error message of "no sign change" suggested that there was no point of intersection in the given window and confirmed for them what had been evident from their reading of the graphs, graphs that had been produced by the expressions of the linear regression option. Yet, there was a sense of unease. They had failed in their attempt to get the technology to work for them. They had expected a single point of intersection somewhere in the first quadrant.

Round 3: A shift of attention.

This round starts with Kay beginning to think more about the situation: *"We try to find how much it increases, which one grows the most."* They returned to the text of the problem situation and reread it: "*Which family (or families) takes over the other families (or family) and when?"* Kay emphasized the *"AND WHEN."* Sam began to look at the paper-and-pencil table of values and said: *"We already figured out who is going bigger, but we can't answer when, because ..."* Then Sam and Kay noticed in the table that, at 8 years, all three families had an area of 64, and so Sam asked three times: *"So why aren't they meeting?"* They now paid attention to the hard numerical evidence that there was a single point of intersection. Their inability to match this evidence and the graphical representatives they had obtained became very clear and waited to be resolved and explained.

Kay substituted 8 into their algebraic expressions for Families B and C and obtained 41 and 12.1, instead of the desired 64, and she said: *"We did something wrong."* In an attempt to make sense of what was going on, the students restarted the process of joining-together-representatives, again with the regression option of the calculator tool. But, this time they were equipped with the critical thinking they had developed from the above comparison of the numerical and graphical representatives. They recalculated the linear regression for Family B and this time noticed that the value of the correlation coefficient was not 1, but .98. Sam remarked that the resulting expression was therefore not 100% sure. Kay insisted: *"Even so, we were way off."*

Sam began to systematically substitute values into the expression for Family B:*"6(1) - 7 is -1: that's wrong; it's not having a negative area*." With his substitution of 2 into the same expression, he became even surer that they were wrong. Kay wondered aloud: "*So do we make up our own equation; maybe it's not a linear regression."* Sam, who was still substituting, said, *"We're getting further apart; even C goes off."*

In this round, their mechanistic approach to joining-together-representatives was being put into question by a more meaningful one. It is not clear if they started to suspect the mechanistic routine itself or the way they had performed it. However, as will be seen in the next round, rather than abandoning the process carried out with the technology, they insisted that it be made to work for them.

Round 4: Insisting that the technology be made to work for them.

Kay decided to check the way that they had used the regression algorithm by trying other forms of regression available on the calculator tool. She went to cubic regression and realized that the correct equation was $y = x^2$ -- because of the neat parameters and the correlation coefficient of 1. She added: *1 squared is 1, 2 squared is 4, 3 squared is 9*. Soon Sam reacted: *"I don't know why I didn't figure it out from the beginning. I said it, why didn't I see it?"* Even Ema expressed that they had been carried away by the regression routine: *"It's because we were so absorbed by our calculators."*

Meanwhile, Kay ran through other regression choices for Family C until she hit upon the exponential regression, which yielded $y = 0.25 \times 2^x$ with a correlation coefficient of 1. Just to be sure, she entered an x-value of 1 to see if the calculator would produce the same y value as she had in her paper-and-pencil table. One value seemed to satisfy her that the equation was correct, but she stated that she did not really understand the equation.

Sam too seemed a little uncomfortable with the expression for Family C. They had never before experienced exponential equations. Kay seemed prepared to go on to the next question. After all, they had earlier realized from their paper-and-pencil table of values that the point of intersection was (8, 64), and they now knew that they had equations that corresponded with the values of their paper table of values. However, Sam wanted more. He wanted to be sure that their newly-found equations did, in fact, yield graphs that intersected at the point (8, 64). He suggested that they enter the new equations in the *Y= editor* and graph them with a scale involving a y-maximum of 70: *"We should see that they are all meeting at 64."* He smiled visibly at the result. It is not clear, however, that Kay paid any attention to this last exercise in consistency.

The technology had now been made to work for them. The graphs, the equations, the paper-and-pencil table of values, and the situation all fit together. Doerr and Zangor (1999) have emphasized the importance of leading students to "develop a reasonable skepticism about calculator-generated results" and encouraging the establishment of classroom "norms that require results to be justified on mathematical grounds, not simply taken as calculator results" (pp. 271-272). It is noted that these students did not throw aside their calculator tool when the graphs it produced could not be justified on mathematical grounds; they continued with it until it could be made to deliver correct mathematical representations.

In short, we can conclude that the group completed its answer to Question 1 by means of actions involving "joining together representatives" which, at the starting point as well as at the end, were controlled by the need to have meaning. But the sequence as a whole was an intertwining one where both the mechanistic and the meaningful were dialectically connected. The regression routines were the mechanistic parts of the sequence. The to-ing and fro-ing between the mechanistic and meaningful joining-together-of-representatives was at times characterized by lengthy segments of mechanistic activity. Nevertheless, the search for meaning always prevailed.

Discussion

As mentioned before, when we speak of computerized tools in learning we usually speak of their "positive" potential in mediating learning. The above example showed that such mediation might raise dilemmas for learning. A crucial dilemma is how much and in what way we would like the tool "to do the work" for the students. And more specifically, do we value that students be able to express a problem situation with algebraic models, and that they produce themselves this algebraic model?

Both classes were driven by a search for meaning in comparing the growth of the three families and were looking for the three graphs and their intersection points, knowing that the algebraic models were the keys for obtaining the graphs. The first class, which had a more limited tool (TI-81), started immediately to construct the algebraic models of the three families from the problem situation itself. This was not so easy for Family C and a few groups in this class failed. Our group from the second class trusted from the beginning the mediation power of their more advanced tool (TI-83 Plus) to do the work of generating the algebraic models for them. But in the first round they failed to get the right algebraic models, even for Family B. During that stage, they had used the tool in a mechanistic way, without engaging critical thinking, and thus obtained "false representatives."

What is the source of the differences between the two classes? It is obvious that the tool itself may explain part of the difference because the TI-81 has only a quite limited regression option. But, the dissimilar history of mathematics learning in the two classes is likely a major contributing factor. As we mentioned above, our second class had the experience of dealing with real world problems with non-idealized data, which usually do not fit perfectly an algebraic model. This encourages the use of regression techniques as a means of obtaining an algebraic model. So the students imposed the same kind of modeling technique on the "Growing Rectangles" problem situation. For students who have difficulty in modeling the growth of an exponential function from the situation itself, such as was experienced with the Family C rectangle, this technique may serve as a temporary scaffolding.

Students may rely on a similar kind of scaffolding when using spreadsheets for modeling. We have observed younger students investigating similar problems of exponential growth with EXCEL (Hershkowitz, 1999). Some of these 7th graders tended to generalize phenomena like the growth of Family C into a recursive model by carrying out actions such as the following:

Enter 2 into cell A1, and then in cell A2 punch in "=A1+A1" and drag down to produce the lengths of the Family C rectangles; then in column B enter into the B1 cell "=1/4*A1" and drag down to obtain the area of the Family C rectangles. In this way, the spreadsheet tool allowed students to combine the use of recursive formulae and dragging, and thus to overcome the local property of recursion.

In fact, if we look at the process which our group used to fill the table for Family C in Round 1 -- doubling the previous length and then multiplying it by 1/4 -- we may conclude that the recursive approach is much easier than trying to generate a closed-form rule where the independent variable is the number of the year. Recursive approaches might even be more natural, at least in the case of exponential growth. In this sense, the tool--either the TI-83 or EXCEL--with its scaffolding affordances enables students to act meaningfully on quite high-level objects, such as exponential functions, even before they have learned formally about them.

The other side of this coin is that this scaffolding may stay longer then we, as mathematics educators, would like in the process of learning algebra. We had examples of students working on a problem situation in which exponential growth was investigated, where students were quite close to generating a closed-form exponential formula. But, when they discovered the EXCEL option of generating a recursive expression + dragging, as a quite efficient alternative for obtaining a whole set of data for the phenomenon, they curtailed their efforts to mathematize the problem situation in a higher level manner. While it might be argued that these students were engaging in a kind of algebraic thinking (Kieran, 1996), might the use of computerized tools in learning algebra reduce students' needs for high level algebraic activity?

In addition, we face an even more crucial dilemma; the algebraic representation and its form seem minimized in importance for students. Our case-study group, like groups in the other class, knew that the algebraic formula was the key, but perhaps because of their more advanced tool, and because of their learning history, the shape of the algebraic model seemed unimportant. In fact, students can now go from entering lists to a graphical representation without ever seeing or having to examine the algebraic representation of the situation. Does the use of these tools in algebra signal the beginning of the loss of the algebraic representation from our mathematical classes at the secondary level?

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References

Balacheff, N. & Kaput, J. (1996). Computer-based learning environments in mathematics. In A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde, (Eds.) *International handbook of mathematics education*, (pp. 469-501). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Doerr, H. M., & Zangor, R. (1999). Creating a tool: An analysis of the role of the graphing calculator in a pre-calculus classroom. In O. Zaslavsky (Ed.), *Proceedings of the 23rd International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 265-272). Haifa, Israel: PME Program Committee.

Dörfler, W. (1993). Computer use and views of the mind. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: mathematics education and technology* (pp. 159-186). Berlin: Springer-Verlag.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representations of functions. *Journal of Mathematical Behavior 7,* 135-173.

Guin, D., & Trouche, L. (1998/1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning 3,* 195-227.

Hershkowitz, R. (1999). Where in shared knowledge is the individual knowledge hidden? In O. Zaslavsky (Ed.), *Proceedings of the 23rd International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 9-24). Haifa, Israel: PME Program Committee.

Hershkowitz, R. & Schwarz, B. B. (1999) The emergent perspective in rich learning environments: Some roles of tools and activities in the construction of sociomathematical norms. *Educational Studies in Mathematics 39,* 149-166.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *8th International Congress on Mathematical Education: Selected lectures* (pp. 271-290). Sevilla, Spain: S.A.E.M. Thales.

Pea, R. D. (1985). Beyond amplification: Using the computer to reorganize mental functioning. *Educational Psychologist 20*(4), 167-182.

Ponte, J. P., & Carreira, S. (1992). Computer spreadsheet and investigative activities: A case study of an innovative experience. In J. P. Ponte, J. F. Matos, J. M. Matos, & D. Fernandes (Eds.), *Mathematical problem solving and new information technologies* (pp. 301-312). Berlin: Springer-Verlag.

Teasley, S. D. & Roschelle, J. (1995). Constructing a joint problem space: The computer as a tool for sharing knowledge. In S. P. Lajoie & S. J. Derry (Eds.), *Computers as cognitive tools* (pp. 229-258). Hillsdale, NJ: Lawrence Erlbaum.

Schwarz, B. B. & Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational Studies in Mathematics* 29, 259-291.

Schwarz, B. B., & Hershkowitz, R. (in press). Computer artifacts and construction of meaning in mathematics. *Mind, Culture and Activity.*