COMPOSING ANALOGICAL WORD PROBLEMS TO PROMOTE STRUCTURE ANALYSIS IN SOLVING ALGEBRA WORD PROBLEMS

Edna Hallel & Irit Peled

University of Haifa, Israel

A short instructional unit was constructed to promote awareness of problem structure. The instructional unit included problem mapping, schema abstraction and analogical problem composition. The unit was tried with students and teachers. In this paper we report the results for 75 students, in 8^{th} grade and 11^{th} grade. It was hypothesized that following instruction students would improve in constructing analogical problems and would be less affected by context in sorting problems. The change in the sorting task was mostly in the expected direction but was not significant. Both age groups improved in problem construction but the change was significant only for the 8^{th} graders.

Word problem solving was considered an Achilles' heel in mathematics education. In the last decade, however, it is perceived as an opportunity to acquire understanding of mathematical structures.

Traditionally, algebra problems were organized in topics such as transportation problems, or work problems (Mayer, 1981). These methods resulted in the categorization of problems by context in students' minds. Nowadays math educators promote the development of meaningful knowledge through the construction of connections between mathematical ideas (Hiebert and Carpenter, 1992). These links can be achieved, according to English (1997), through a process of analogical reasoning.

Analogical reasoning in problem solving occurs when previous problem solving experience (source) is transferred to a new problem solving situation (target). Such transfer requires, first of all, good understanding of the structure of the source problem (Gentner, 1983). As it turns out, this is not sufficient and more conditions have to be met. The transfer process includes three crucial and related actions that have to be performed: recognition of a potential problem to transfer from, abstraction of source problem structure, and mapping of corresponding elements and relations in the source and target problems.

Quite a lot of researchers detail conditions that promote transfer. Some of these works suggest, for example, that the abstraction of a general schema is facilitated by comparison and mapping of several examples in different contexts, and by the use of a combination of simple and complex examples (Gick and Holyoak, 1983; Reed and Bolstad, 1991; English and Halford, 1995).

In this work we try to promote the abstraction of problem schema by asking children to compare elements of pairs of problems, and requiring a generalization of problem components. We then ask them to compose analogical problems of their own. The task of composing a problem subject to given constraints is usually used for detecting students' conceptions (Rowell & Norwood, 1999). We use it here as a part of the instruction unit and view it as an important tool in encouraging focus on structure.

It should be noted that *structure*, much like *schema*, has different definitions. Reed (1987) defines similar-structure algebra problems as problems that have the same equation. Weaver and Kintsch (1992) suggest that equations play a lesser role and that the dominant factor is some deeper conceptual structure.

In this work we used problems that have the same equation and the same conceptual structure as examples of similar-structure problems. We use the terms *analogical* and *isomorphic* interchangeably to denote similar-structure problems. The construction of problem space and similarity tasks that test whether context or structure affect children's conceptions was based on ideas from Reed's (1987) work on a structure-mapping model.

METHOD

The study was conducted in the course of the school year. It took 6-7 class sessions and consisted of three parts: a pretest, an instructional unit, and a posttest. Both pretest and posttest included a problem sorting task and an analogical problem composition task.

The purpose of the sorting task was to identify the criterion children use in categorizing problems. Students were given 12 problems created by the product matrix of 4 context categories: transportation, work, pipes, and mixtures, and 3 different problem structures.
Using Reed's (1987) terms, three types of relations exist between these problems:
Isomorphic (or analogical) problems – same structure, different context
Similar problems – different structure, same context
Unrelated problems – different structure, different context

According to the research hypothesis children were expected to sort by context in the pretest. A shift towards sorting by structure was expected in the posttest as an outcome of applying the instructional unit.

A problem composition task, given several times along the study, checked children's ability to write an isomorphic problem to a given (or composed) source problem. The composition task had a significant role in the instruction. It was expected that children would perform better in this task in the posttest, and that this would be part of the explanation for the change in the sorting task.

The study sample consisted of 100 participants, studying or teaching mathematics, including: 75 students (61 students in 8^{th} grade and 14 students in 11^{th} grade); 12 student-teachers in mathematics; 13 mathematics teachers in middle school and high school. In this paper we report the results for the 75 middle school and high school students. We focus on the analogical problem composition task, and discuss the changes in students' performance.

The instructional unit was especially designed for the study and consisted of different types of activities:

1. Mapping between two given problems
2. Abstraction of problem components
3. Practice in writing analogical problems
4. Matching a general schema to a given problem

Examples of some of these tasks are detailed in the following section, as a part of the description of individual children's performance.

RESULTS

We will present quantitative results and proceed with examples that demonstrate the change in problem composition.

In the pretest sorting task students identified (put in the same category) more similar problems than isomorphic problems as being related. This trend was significant in the 8^{th} grade group. Following instruction the 11^{th} graders identified fewer connections between similar problems than they did before instruction, and both groups identified more connections between isomorphic problems than they did before instruction. These changes are in line with our hypothesis, but they are not significant.

Table 1 presents percentages of students who successfully composed an isomorphic problem in the first composition task (a simple problem given in the pretest), and in the last task (a complex problem given in the posttest).

Table 1: Isomorphic problem composition before and after instruction.

	Pretest	Posttest
8^{th} grade	37%	71% *
11^{th} grade	67%	80%

* significant difference

A comparison of problem composition in the pretest and posttest shows a significant change for 8^{th} grader, where the difference in performance was 34%. In the 11^{th} grade the difference in performance was 13% and non significant. However, as will be discussed later, the problem composition tasks included problems that differed in structure. As the student progressed, the structure for which he was asked to build an analogical problem became more complex.

It is interesting to note that the change in 8^{th} grade occurred mainly for the girls. In the pretest 17% of the girls and 50% of the boys could construct an isomorphic problem. In the posttest 65% of the girls and 75% of the boys could compose an isomorphic problem.

The shift in ability to construct analogical problems can be seen in the work of an 8^{th} grader and an 11^{th} grader. It should be noted that with the students, for instructional purposes, we used (and defined by examples) the term *similar problems* instead of *isomorphic problems.*

Hilla's case (8^{th} grade):

Table 2: Hilla's first problem composition task.

task	Source (given)	Composed by Hilla
Pretest: compose a similar problem using machines that produce parts.	*Two persons started walking one towards the other on foot and at the same time from 2 different cities, the distance between which is 96 km. One person was walking at the rate of 5 km/h, and the second at the rate of 3 km/h. How much time will it take before they meet each other?*	*Two machines in a factory that produces parts were operated at the same time. One machine works at the rate of 25 km/h and the other works faster than the first by 5 km/h. I have no idea how to continue this.*

During the instructional stage Hilla improved gradually and managed to compose a part of the analogical problems correctly. In the last task, which presented a complex source problem, she exhibited good structure analysis (as seen in Table 3).

Table 3: Hilla's last problem composition task.

	Source	Composed by Hilla
Posttest Task: write a similar problem with different context (not specified)	*When a student reads a favorite book at the rate of 50 words/min it takes him a certain amount of time. Once he read for 8 minutes at his usual pace and then increased it to 60 words/min and finished 2 minutes earlier than usual. How much time does it usually take him to read the book?*	*A cyclist usually rides at the rate of 10 km/h. After 12 minutes of riding at his usual pace he increased it to 14 km/h. How much time did he ride if we know that he arrived 4 minutes earlier?*

Hilla is also able to specify the similarity between the problems: *They both deal with the same thing that starts at a certain rate and changes to a new rate.*

Stav's case (11th grade):

Table 4: Stav's first problem composition task.

task	Source (given)	Composed by Stav
Pretest: compose a similar problem using machines that produce parts.	*Two persons started walking one towards the other on foot and at the same time from 2 different cities, the distance between which is 96 km. One person was walking at the rate of 5 km/h, and the second at the rate of 3 km/h. How much time will it take before they meet each other?*	*Two machines in a computer factory produce CD-s. One produces 4 records/hour and fills up a CD case in 16 hours It works 24 hours and then rests. The other machine produces 6 CD/h and it too works for 24 hours. How much time will it take the machines to fill up the same amount of cases?*

Stav's composed problem, presented in Table 4, is different in structure from the given problem. Stav does not really use the simultaneous work of the machines. There is no focus on a certain amount that is produced by the two machines working together.

The main change occurred on composing one of the problems in the instructional session task presented in Table 5.

Table 5: Stav's problem composition during instruction.

	Source (given)	Composed by Stav
Task: Compose a similar problem.	Back and Forth *A person leaves the city of Bally riding his bikes for 2.5 hours at the rate of 20 km/h he gets to the city of Gat. He does the way back on foot and walks for 10 hours from Gat back to Bally. What is his walking rate?*	 *Yesterday a student was preparing his homework at the rate of 16 problems per hour for 3.5 hours. At what rate did he work today to finish the same amount of problems, if it took him 6.5 hours?*

Stav solved his own problem by writing and calculating: (3.5x16)/6.5=8.6 and writing the answer in words: The student solved 8.6 problems per hour.

While Stav managed to perform well on the Back and Forth (see Table 5) problem, another 11th grade student, Jenny, exhibited some interesting difficulties. Jenny wrote: *A student prepares his homework in the afternoon and checks his work again in the evening. If he prepares his work at a rate of 1 problem per minute it takes him 10 minutes. But when he checks his work at the rate of 2 problems per minute it takes him 5 minutes. How many problems were there?*

Jenny wrote: 5x2xX=10x1xX She realized that the same person is doing two similar actions, each at a different rate. She also used (correctly) the same total production: in both cases the same amount of problems is used. However, she did not have a good mapping strategy for creating an analogical problem, as a result she had trouble in choosing the unknown.

In the above examples students composed an isomorphic problem to a given problem. Another type of task involved mapping problems. Students were given examples of pairs of isomorphic problems, and asked to map their elements. Following several examples they were asked to compose a pair of isomorphic problems and map their elements. In this task Sagi, an 8th grader, wrote the problem pair presented in Table 6.

Table 6: Sagi's composed pair of isomorphic problems.

Sagi's 1st problem:	Sagi's 2nd problem:
The Truck: A truck travels on a 100 km track. It travels in the rate of 50 km/h. How much time will it take it to finish the track?	The Fisherman: A fisherman has to catch 100 fish. He catches them at the rate of 50 fish per day. How many days will it take him to catch 100 fish?

	100 fish?

Sagi map of the corresponding problems' elements is detailed in Table 7.

Table 7: Sagi's problem mapping.

The Truck	The Fisherman
distance -	total number of fish
rate of driving -	rate of catching fish
number of hours -	number of days
In this problem we calculate: Distance/ rate = time -	In this problem we calculate: Amount/rate = time

Sagi's example demonstrates very good performance of a young child on a problem composition task which, as will be further discussed, is not an easy task for students (or, in fact, also for teachers).

DISCUSSION

Using a short instructional unit this research tried to change students' focus in problem solving from context-centered to structure-centered. The instructional unit included several ideas suggested by different researchers, such as mapping between problem elements and comparing given examples. In addition to these, we tried another task, different in nature, but also more difficult, the task of writing an isomorphic problem. As it turned out, when problem structure was simple, such as a rate x time = production, a third of the 8^{th} graders and two thirds of the 11^{th} graders could compose it with some instructional guide. When problems became more complex, it apparently became more difficult to perceive a complete picture of problem structure and compose a problem of the same structure. This is not surprising in light of what is mentioned in the introduction, that understanding of the source problem structure is a necessary condition for analogical reasoning.

Although the tasks were difficult, some change in the direction of more attention to problem structure and less to context similarity occurred. The students' awareness of structure was also expressed in class discussions, and brought up in a follow up questionnaire.

The instructional unit together with the pretest and posttest took very few class sessions, and can be considered as an exploration of the power of the unit's task. In view of the effect of this short experience, we suggest to use these tasks and construct a more comprehensive program.

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