FROM THE DECIMAL NUMBER AS A MEASURE
TO THE DECIMAL NUMBER AS A MENTAL OBJECT

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Abstract. *We have carried out some explorative studies about introduction of the concept of decimal number, in the upper elementary school. In order to achieve this objective, we have designed a classroom practice that engaged students in a sustained mathematical activity which requires an extensive use of the ruler to accomplish different functions (measuring, drawing segments, ordering and approximating decimal numbers). In this paper we presented an explorative study with fifth-grade children about the role of the ruler as concrete model as well as conceptual model in passing from the decimal number as a measure to the decimal number as a mental object. The emergence, in terms of prospective learning, of some of the properties of the decimal number, and in particular of the property of density on the real number line, is also presented.*

Framework

In recent years connections are advocated between mathematical content and the home cultures of learners, as well as between different branches of mathematics, various disciplines in which mathematics is used, historical roots of mathematical content, and connections with the real world and the world of work, see e.g. Civil, 1995. Some considerations are anyway important. Mathematics is a part of students' social and cultural lives, and the mathematics classroom has its own social and cultural life, see Boaler, 1993. Indeed, "academic" mathematics can also be viewed as a form of ethnomathematics, involving particular cultural practices, see Presmeg, 1998.

The task of connecting students' everyday contexts to the classroom is not easy. Many educators argue that transferring ideas from one context to another is hard because the two context differ in some significant ways. Just as mathematics practice in and out of school differs, so does mathematics learning. Schliemann, 1995, pointed out that "*school learning focuses on individual cognition, pure thought activities, symbol manipulation, and general principles, while out-of-school learning is characterized by shared cognition, tool manipulation, contextualized reasoning, and situation-specific competencies*". In particular in out-of-school mathematics practice, persons may generalize procedures within one given context but be unable to carry these procedures forth to another context since problems tend to be context-specific. Generalization, which is an important goal in school mathematics, is not usually a goal in out-of-school mathematics practice.

The role of cultural artifacts

The critical problem of how to manage at school the relationship between everyday knowledge and school mathematics has been the subject of our studies for some years now. How can we at the same time benefit from what children already know and avoid the limitations that are typical of the everyday mathematical experience? How can we design better opportunities for children to develop mathematical knowledge that is wider than what they would develop outside of school, but that preserves the focus on meaning found in everyday situations (see Schliemann, quot.)? How should we proceed to lead children to develop new understandings about underlying mathematical concepts and structures and their potential generalizability?

Although mathematics learning and practice in and out of school differ in significant ways, we deem that those conditions that often make extra-school learning more effective can and must be re-created, at least partially, in classroom activities. Indeed while some differences between the two contexts may be inherent, many can be narrowed if in the classroom we create and promote learning processes that are closer to the ones that occur in the out-of-school mathematics practices. That can be implemented in a classroom, for example, by encouraging the children to analyze some 'mathematical facts' that are embedded in opportune 'cultural artifacts' (Saxe, 1991). We are talking here of objects that have relevance for the children and that are meaningful because they are part of the children's real life experience and refer to concrete situations. This enables children to keep their reasoning processes meaningful, to monitor their inferences. As consequence, they can off-load their cognitive space and free cognitive resources to develop more knowledge (Arcavi, 1994). We can thus make use of the children's familiarity with the artifacts we have chosen as being objects that belong to the students' daily experience, and allow the children to express their intuitions and produce their own anticipations, in the sense of "*prospective learning*" as described by Freudenthal, 1991. These anticipations precede and can be functional to any systematic learning process. Further, if we use the artifacts in a certain way (cf. Basso & Bonotto, 2001, and Bonotto, 2001), we can develop classroom activities of "*realistic mathematical modeling, i.e., both real-world based and quantitatively constrained sense-making*" (Reusser & Stebler, 1997), and we can overcome the clear limitations of classical word-problems (cf. Verschaffel & De Corte, 1997).

Like in the Realistic Mathematics Education perspective of the Dutch school of taught (cf. Gravemeijer, 1994), we deem that progressive mathematization has to lead a student to algorithms, concepts, and notions that are rooted in the individual's learning history, a history that starts with informal, experientially real knowledge.

The formal approach to the introduction of decimal numbers in elementary-school classrooms

In previous studies (Bonotto, 1993; 1996), we analyzed the conceptual obstacles 10- and 11-year-old Italian children encounter in ordering decimal numbers. Our findings are consistent with those of classical research studies (Nesher, & Peled, 1986; Resnick, et al., 1989). It was hypothesized that such findings may depend not only on the inherent difficulties of the subject matter but also on the teachers' conceptions and educational strategies. Many teachers introduce decimal numbers by extending the place-value convention. They tend to spend little time to let the children understand the meaning of the decimal number symbols and reflect on the decimal number properties and relationships. Efforts to connect decimal numbers and decimal measures are insufficient. As a consequence, children do learn to carry out the required computations, but they have difficulties in mastering the meaning of decimal notation and between fractional and decimal representations, and finally in ordering sequences of decimals.

Measuring activities as an alternative introduction to decimal numbers
According to innovative instructional approaches, we mantain that children's decimal number understanding can be fostered in rich classroom environments, where learners can transfer their out-of-school knowledge and utilize familiar tools (such as the ruler) to accomplish a recurrent set of mathematical activities, and where they can share some minimal presuppositions about the problem definitions and the goals. "*The roots in the student's reality are expected to foster the meaningfulness and usefulness of the so-developed mathematical knowledge*" (Gravemeijer, 1997).

We propose that a set of measuring activities that require an extensive use of the ruler can offer the children good opportunities to move toward the construction of an encompassing numerical structure, which integrates in a consistent whole both the natural and the decimal number systems.

The ruler is a cultural artifact which can offer the children a first approach to the decimal number as the result of a given measurement. On the ruler, "mathematical facts" are represented through its signs: the natural number sequence is visible, and some fractional parts are marked. Therefore, the ruler can offer a "situation-specific imagery" of the additive structure of the written decimal number notation, which supports the child's progressive understanding. For example in order to draw a 3.15dm segment, the child first draws a 3dm line and marks the final extreme, then she/he adds a 1cm line to it, and finally a 5mm line, and expresses each affixion as 'plus', or 'and'. The child can understand that if there are two decimal digits after the decimal point, then there are units, plus tenths plus hundredths, and that each digit specifies how many parts of a given magnitude are included in the addition. The learner is expected to form images out of her/his actions through the use of the ruler, and to visualize relevant properties. The child can map this visualization onto the decimal number representation to attribute a meaning to the decimal digits after the decimal point; her/his ability to solve ordering problems is enhanced.

As to regard the basic characteristics of the teaching/learning environment that have been designed and implemented in the classroom, a set of activities based on suitable cultural artifacts, in particular the ruler, on interactive teaching methods and on introduction of new sociomathematical, in the sense of Yackel & Cobb, 1996, were combined in an attempt to create a substantially modified teaching/learning environment. This environment focused on fostering a mindful approach toward realistic mathematical modeling, i.e. both real-world based and quantitatively constrained sense-making, see Reusser & Stebler, 1997.

The research

A previous exploratory study (see Basso, Bonotto, Sorzio, op. cit.) concerned the introduction of the concept of decimal numbers, in the normal classroom curriculum, with third-grade children. In order to achieve this objective, we have designed a classroom practice that engaged students in a sustained mathematical activity which requires an extensive use of the ruler to accomplish different functions (measuring, drawing segments, ordering and approximating decimal numbers). The ruler has been a mediational role in their understanding of the additive structure underlying the standard written decimal notation. The results obtained showed how they correctly measured and expressed lengths with numbers containing only one digit after the decimal point; where numbers with a second digit following the decimal point were concerned, the children had difficulties in distinguishing between the decimal digit value – which represents how many parts of a given magnitude there are – and the meaning to be attributed to each decimal digit position – which represents its magnitude. For example the whole group of digits after the decimal point referred to the decimal unit directly following the main unit that was being measured. The fact of measuring with a ruler, however, did offer the children a concrete anchor and clearly illustrated their errors, allowing them to autocorrect themselves.

Later we decided to carry the study further, to evaluate the influence that the preceding activities with the ruler had had on the children. The same children were thus given poblems, in both the fourth and fifth grades, which involved comparing, ordering and approximating decimal numbers. The idea was also to start the children thinking about the structural properties of a line of numbers.

Research objectives: data are gathered and analyzed concerning:

- the children's understanding of the signs and intervals on the ruler;
- the children's understanding of the additive structure underlying the standard written decimal notation;
- the children's process of detachment from the representation on the ruler and from the presence of a given unit of measure;
- the children's understanding of the relation between different units of measurement;
- the possibility for the children to grasp that the approximation is a limit of the physical instrument;
- the children's understanding of the density property of the enriched decimal number line ;
- the passage from the number as a measure to the number as a mental object.

Subjects: 21 fifth-grade children (aged 10-11 years) in a small school in a village (northeast Italy) participated in this explorative study; they had first started measurement activities in the second grade. Data were gathered from participant observations and children's written works.

Procedure: Each student was given a sheet that listed three tasks:

1. *Write down at least two measurements between 1dm and 2dm.*
2. *Write down at least two measurements between 1.2dm and 1.3dm.*

3. Write down at least two measurements between 1.9dm and 2dm. Explain how you found these numbers.

Discussion

We briefly present some significant extracts from the written work that show

i) the role of concrete model (characterized by a particular symbolic code-system), as well as the conceptual model of the ruler, in passing from the number as a measure to the number as a mental object;

ii) the emergence, in terms of prospective learning, of some of the properties of the decimal number and, in particular, of the property of density on the real number line.

N.1 (Moreno): 1st answer*: 11cm – 12cm – 13cm – 14cm – 15cm – 16cm – 17cm – 18cm – 19cm*

The measurements I got without looking at the ruler were all between the 10cm and the 20cm marks.

2nd answer*: 1,21dm – 1,23dm*

I got them by adding anotherdigit after the decimal point: the millimeters.

N.2 (Daniele): 1st answer*: 1,1 – 1,2 – 1,3 – 1,4 – 1,5dm*

To get these numbers I thought that between 1dm and 2dm there are the little pieces that are smaller that are centimeters.

2nd answer: *1,21 – 1,22 – 1,23 – 1,24 – 1,25dm*

I have to go from 1.2 to 1.3 and all I have to do is add the smaller pieces that is the millimeters.

In these first two protocols (like in about 30% of the class's total), one sees that the students still reason very much along manipulative and procedural lines linked to the image of the ruler.

In fact, numbers/measurements that fit within the required interval, arrived at via mental operations, were clearly thought out placed in the physical spaces of the artifact/instrument. This kind of reasoning highlights some of the characteristics of the ruler as model. Mentally the children refer to the physical act and manipulation of the ruler involved in measuring a segment of, say, 1.21dm: first they draw a 1dm-long segment and mark off the end point; there they add a 2cm segment and then a 1mm one. In this case the physical manipulation they had previously carried out helped them to understand the meaning of a number written with a decimal point and expressing units of measure, and it helped them to understand the additive structure underlying the standard written decimal notation.

In other protocols (about 50% of the total), the model offered by the ruler proved to be more flexible. Here the ruler no longer served just as a visual and tactile model (as it had in the activities of the preceding years), but as a model that induces thinking in terms of relations between numbers and quantities involved, in a certain sense more "conceptual". A process had been set off from thinking on the basis of concrete, material objects to thinking on the basis of mental, mathematical objects.

Let us look at the examples given by the following two protocols.

N.3 (Simone): 1st answer: *1,3dm – 1,4dm – 1,5dm – 1,6dm – 1,7dm*

I went like this: in 1dm there are 10cm. Therefore 14cm is more than 1dm and less than 2dm. Since we're doing marks in decimeters, I did 1.4dm.

2nd answer*: 1,21dm – 1,23dm – 1,24dm – 1,25dm*

I went like this: what's smaller than centimeters is millimeters.

In this case we have to use millimeters because if you add or take away centimeters you don't get a number between 1.2dm and 1.3dm.

N. 4 (Pamela):1st answer: *1,3dm – 1,5dm – 1,7dm – 1,9dm – 1,06dm*

1,32dm – 1,54dm – 1,73dm – 1,99dm – 1,05dm

To get these measurements I didn't make my brain work very hard, I thought of them, first, thinking of all the possible ones, and then out of these, taking the ones that seemed better to me. For example, 1.3dm equals thirteen centimeters while 1.32dm equals thirteen centimeters plus two-tenths of a centimeter. Therefore one decimeter plus three centimeters makes thirteen centimeters.

2nd answer: *1,23dm – 1,24dm – 1,25dm – 1,26dm but also 1,2321dm*

For example 1.23dm consists of one decimeter, two centimeters and 3 tenths of a centimeter. And the same goes for 1.2321dm which means 1dm, 2 centimeters, 3 tenths of a centimeter, 2 hundredths of a centimeter and 1 thousandth of a centimeter, that is 12 centimeters and 321 thousandths of a centimeter.

Pamela carries out transformations going from one unit of measure to another in a meaningful way using fractional expressions. On can see the distinction starting between decimal digit value – which represents how many parts of a given magnitude there are – and the meaning to be attributed to each decimal digit position – which represents its magnitude.

In the following answers, the fact that the subdivisions on the ruler are decimals becomes evident, as does the notion of subdivisions that can go on infinitely, even if one cannot actually see them on the ruler. Thus, the density property of decimal numbers in the number line intuitively emerges. Here are some examples.

N.5 (Selenia): *One could go on writing numbers infinitely, because there are always littler spaces that you don't see on the ruler because it would be impossible to see them all because they're infinite. Because you only see centimeters and millimeters on the ruler.*

N.6 (Veronica): *You can have 1,22dm – 1,23dm – 1,24dm – 1,25dm – 1,26dm – 1,27dm – 1,28dm – 1,29dm …*

I found them like this. First of all between 1.2dm and 1.3dm there are ten spaces and therefore I can go like this. There aren't only these measurements, but there are infinite ones because every space continues to be divided in 10 parts.

N.7 (Sara): 1st answer: *1,2dm – 1,4dm – 1,6dm – 1,9dm – 1,5dm …*

It was very simple to find them I thought what numbers there are between 1dm and 2dm and among so many numbers I chose these, because the *interval between one decimeter and two decimeters is decimal and you always divide by ten.*

2nd answer: *1,21dm – 1,22dm – 1,263dm – 1,25dm – 1,299dm …*

I used the same method as before and if the teachers give me another task like this to do, I'll use the same method again. It's like a cycle that repeats itself … because the spaces inside a centimeter are infinite.

Conclusion and open problems

In this paper we presented an explorative study in the upper elementary school about the role of ruler in passing from the number as a measure to the number as a mental object; the emergence, in terms of prospective learning, of some of the properties of the decimal number, and in particular of the property of density on the real number line is also investigated.

Concerning the first point of our analysis, the results show the passage from mathematics understood as an instrument and incorporated in certain cultural artifacts, like the ruler, and mathematics as an object of study. The use of this cultural artifact first gave meaning to the operations of measuring, to numbers-measures and to decimal number notation in general (cf. Basso, Bonotto, Sorzio, op. cit.); then it favoured integration between acting and thinking in mathematics.

As to regard the second point, and that is the emergence of prospective learning, we hold, in agreement with Freudenthal, that anticipatory intuitions must be encouraged and not stymied: "*Prospective learning should not only be allowed but also stimulated, just as retrospective learning should not only be organized by teaching but also activated as a learning habit*" (Freudenthal, op. cit.). Mathematics teaching should try to exploit both forms of learning: "*prospective*" learning (which makes the fullest possible use of the student's intuitions, and to the greatest extent possible encourages anticipation of results), and "*retrospective*" learning (which makes use of "old" or previously acquired knowledge, revisits it, and re-composes it in new contexts). "*Just as prospective and retrospective learning aims at an integration of past and future learning processes, so does intertwining learning strands locally, yet with a view on the involved learning processes as a whole*".

There remain some problems that Freudenthal highlighted which still need to be confronted and solved, e.g.,

- how do mental objects develop into concepts?
- what criteria are there by which to judge if the process has taken place?

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