Connecting Partitioning and Iterating: A Path to Improper Fractions

John Olive

The University of Georgia

In this paper I illustrate how an approach to establishing fractions based on a child's multiplicative relations among whole numbers provided one child (Joe) with both affordances and constraints for constructing meaning for both common and improper fractions. Joe established "one third" of a composite number as that number which, when multiplied by three, will give the composite number. In the computer microworld TIMA: Sticks, Joe modeled this multiplicative approach by finding a stick that he could repeat three times to equal the given "number-stick." The Sticks environment provided a venue for Joe to construct unit fractions as "repeatable connected numbers." In order to construct non-unit (common) fractions and (eventually) improper fractions, Joe had to connect this multiplicative approach to fractions with his part-of-whole view of a fraction.

Understanding fractions greater than the whole (improper fractions) has been a stumbling block for many children (Olive, 1999, Tzur, 1999). A major obstacle to their understanding stems from the over-emphasis in curricular materials on representing fractional quantities as the number of parts shaded of a partitioned figure. Interpreting 3/5 solely as "three parts shaded out of five parts total" provides no means for making sense of 7/5. A junior in college once complained to me "How can you have seven fifths of something?" when faced with the following problem (referring to a long thin rectangle on the test page): *This bar is 7/5 of another bar. Draw the other bar.* She was blocked by her part-in-whole interpretation of a fraction. She could not imagine the given bar as seven of five equal parts of another bar.

The research reported in this paper is part of an on-going retrospective analysis of videotaped data from a three-year constructivist teaching experiment with 12 children (Steffe & Olive, 1990; Steffe, 1998). The major hypothesis to be tested in the teaching experiment was that children could (and should) reorganize their whole number knowledge in order to build schemes for working with fractional quantities and numbers (the rational numbers of arithmetic) in meaningful ways. This reorganization hypothesis (Olive, 1999) contrasted with the prevailing assumption that whole number knowledge is a "barrier" or "interferes" with rational number knowledge (Behr et al., 1984; Streefland, 1993; Lamon, 1999).

Description of the Teaching Experiment

A team of teacher/researchers, led by Steffe and Olive, worked with the children, primarily in pairs, outside of the classroom once a week for approximately 20 weeks a year for three years. All teaching episodes were videotaped. Computer microworlds called *Tools for Interactive Mathematical Activity* (TIMA) (Biddlecomb, 1994; Olive & Steffe, 1994; Olive, 2000) were specifically designed for the teaching experiment. The TIMA provide young children with possibilities for enacting their mathematical operations with whole numbers and fractions. They also provide the teacher/researcher with opportunities to provoke perturbations in the children's mathematical schemes and observe the children's mathematical thinking in action.

We began working with Joe during his third grade year. Joe is an African-American student who was often in trouble with his classroom teacher. Our emphasis during the first year was on investigating the children's whole number multiplicative operations. We determined that, by the end of the first year, Joe had constructed an *explicitly nested number sequence* (Steffe & Cobb, 1988; Steffe, 1992). A hallmark of the explicitly nested number sequence (ENS) is the construction of *composite units* (units of units) that are necessary for multiplicative operations (Steffe, 1992). In a series of activities that we designed on the computer using the TIMA: Toys microworld, Joe was able to find how many (hidden) cookies would result from baking three trays of cookies, each tray having 7 hidden cookies, using his multiplication facts. He was also able to determine, however, that when he had 7 trays of cookies (a total of 49 hidden cookies), he would need to bake three more trays in order to have a total of 70 cookies. He was able to work with the difference of 70 and 49 as three units of seven because he knew that 70 was ten sevens.

The Construction of Fractions as Repeated Connected Numbers

For the first half of the second year of the project Joe worked without a student partner with a teacher/researcher, Azita (a doctoral student working with the project). The focus during this period was on using the children's whole number structures and operations in situations that we would regard as involving *continuous quantities.* Most of the activities took place in the computer microworld TIMA: Sticks. The manipulatives in this microworld were horizontal line segments that we called "sticks." Sticks were created by the child dragging the computer mouse across the screen. Thus the extent of a stick was the result of a deliberate motion on the part of the child. Sticks could be copied, joined together to form longer sticks (with vertical marks indicating each of the sticks that were joined), marked with vertical tick marks, partitioned into equal parts using a numeric indicator for the number of parts, and broken into separate, smaller sticks determined by the marks or parts created in the stick. Pieces could also be cut off of a stick using a mouse click. Any stick could be measured with reference to any other stick that had been designated as a measuring unit. The measure would appear as a ratio fraction or whole number. Sticks on the computer screen could be hidden from view by rectangular regions called "covers."

Through the activities with TIMA: Sticks Joe was able to use his composite units to construct what Steffe and Wiegel (1994) have called a *connected number.* Connected numbers are constructed by a child mentally projecting his/her concept of a whole number (e.g. eight) into an unmarked line segment to establish an "interiorized continuous but segmented unit" as a situation of the child's number sequence (p.121). This connected number 8 (say) would contain within it an implicit nesting of the number sequence from 1 to 8, and the sense of those 8 units being united into one composite unit. Connected numbers carry with them a notion of number as indicating relative size (length). Children were able to instantiate connected numbers in TIMA: Sticks by creating what we refer to as *number-sticks* --that is sticks created from joined repetitions (iterations) of an established unit stick. Number-sticks were referred to by their number-name. That is, a stick created from 8 repetitions of a unit stick would be called an 8-stick.

The following protocol illustrates how Joe (J) and Azita (A) used activities with number-sticks to establish fractions as *repeated connected numbers*. For instance, one third of a 24-stick is that stick, which repeated three times, will make a 24-stick. This approach to fractions of composite wholes provided Joe with the opportunity of using his whole number multiplication scheme (*a composite units-coordinating scheme*) to find the appropriate stick. This teaching episode took place in the fall of Joe's fourth grade year, the second year of the Project. A set of number-sticks had been constructed at the top of the playground area of the computer screen, separated by a long thin cover that stretched across the full width of the screen (see Figure 1). Sticks created by repeating or joining copies of sticks from this set of number-sticks were also named as number-sticks (e.g. a stick created from 4 repetitions of the 6-stick was a 24-stick)

Figure 1: A set of Number-Sticks in TIMA: Sticks

Protocol I

While Joe has his eyes closed, Azita makes a 24-stick below the cover using repetitions of a copy of one of the sticks above the cover and erases the marks from her 24-stick.

A: *The stick that I used was one third of the length of the stick I have right here* (pointing to the unmarked 24-stick).

Joe measures the stick (24 appears in the number box). He then smiles to himself and counts down the set of the number sticks ending on the 8-stick. He copies this stick and repeats it 3 times to make a stick the same length as the 24-stick.

A: *That is right!*

J: *You said one third, so what will be…three times eight is 24.*

Azita suggests doing more problems with the 24-stick. Joe wants to use a 21-stick but Azita asks him to do one more with the 24-stick.

A: *Think of a stick you could use to make the 24-stick and tell me what fractional part of the 24-stick it would be, and I will try to tell you what size stick it is and how many times I should use it.*

J: *Close your eyes.*

Joe trashes the 3-part stick and looks at the set of number sticks.

J: *O.K. I didn't have to do nothing.*

J: *It's umm... It's one sixth.*

A: *The stick that you used is one sixth of the 24 stick?* (Joe nods his head.)

A: *So, I want something, I want a stick that when I repeat it six times would give me…*

J: *No!*

A: *Would give me the 24..*

J: *One fourth!* (at the same time as Azita is speaking).

A: *Oh! You used the one-fourth stick?* (Joe nods yes.)

A: *You used one-fourth, so I want a stick that when I repeat it 4 times will give me the 24, and I think that is the 6-stick! What do you think?* (Joe nods yes.)

Azita copies the 6-stick and repeats it 4 times to make a stick the same as the 24-stick.

Joe knew to use the 8-stick for 1/3 of the 24-stick because "three times eight is 24." I regard this as a modification of Joe's composite units-coordinating scheme (a multiplication scheme) whereby Joe could coordinate the elements of one composite unit (3) with the elements of another composite unit (8) to produce the target number 24. Repeating the 8-stick three times to match the 24-stick was an enactment of this coordination. Joe's interpretation of "one third" as something that when multiplied by 3 gave the total number, was a novel application of his units-coordinating scheme. Joe's strategy supports the major hypothesis of the project: that fractions could emerge from such modifications of the children's whole number schemes (Steffe and Olive, 1990; Olive, 1999; Olive, 2000).

Joe's realization that he did not have to do anything in order to pose the problem for Azita (as the unmarked 24-stick was still visible on the screen) indicates that he was able to *imagine* himself acting within the microworld. According to von Glasersfeld (1981), such imagined actions are critical, as it is through re-presenting mentally their actions that children construct their numerical operations. Joe hesitated in naming the fraction when posing his problem for Azita. I suggest that he was trying to hold both the imagined stick in his head and the number of times he would have to use it. Joe ended up using the stick-size to generate a name for the fraction rather than the number of times he would have to use that stick. Joe realized his mistake as soon as Azita voiced her interpretation of 1/6. Azita's explicit use of fractional language to describe her imitation of Joe's actions in TIMA: Sticks provoked a reflection by Joe on what he had imagined, and provided scaffolding for making sense of the fractional terms. He corrected his error rather than going with it and accepting Azita's actions. This indicates that Joe could generate his result prior to action through both numerical calculation and visualized action. It is in this sense that Joe was constructing meaning for unit fractions as *repeated connected numbers.*

Connecting Partitioning, Segmenting and Numerical Operations

While this approach to unit fractions of composite wholes was productive for Joe, it had limitations that stemmed from its attachment to known multiplication facts.

In the continuation of the teaching episode in Protocol I, Joe successfully identified the 4-stick as 1/7 of the 28-stick, but when asked by Azita what fraction of the 28-stick two 4-sticks joined together would be, Joe responded with "One fourteenth…because you add one seventh and another seventh it makes 14." Streefland (1993) refers to such errors as N-distractors (miss-application of whole-number arithmetic to a fraction situation). Unit fractions were results of numerical operations at this point for Joe; they were not yet quantities that could be united.

Joe had difficulty with making equal portions of a stick in sharing situations. His visual estimates for sharing a stick into three or four equal portions were not very accurate. Rather than making a mental partition of the stick into 4 equal parts, and indicating this partition by placing three marks on a stick, he would draw an estimate for 1/4 of the stick and repeat this estimate 4 times to see if it matched the stick. In the teaching episode following the one above, we attempted to provoke Joe's partitioning operations by asking him to make fractions of a number-stick for which there was no available number-stick (e.g. make a stick that is 1/4 of a 27-stick). Our hypothesis was that, without a known multiplication fact to solve the problem numerically, Joe would need to mentally partition the 27-stick in order to make a reasonable estimate for 1/4 of the stick.

Joe had created the 27-stick by repeating an unmarked 9-stick three times. The screen display consisted of a set of number sticks as in Figure 1 (above). the 3-part 27-stick and an unmarked copy of this 27-stick. Joe's first estimate was approximately half the length of a 9-stick. He repeated this estimate four times to create a stick approximately the same as 2/3 of the 27-stick. His next estimate was approximately the same as a 9-stick. He only repeated this stick three times and then trashed it. He adjusted these estimates, gradually getting closer to a visual approximation for 1/4 of the 27-stick. He did not use his whole number multiplication facts to help with choosing an initial estimate (e.g. a 7-stick might work as 4 times 7 is 28 -- a fact he had used in the previous episode to find 1/7 of 28). There was little indication that Joe was mentally partitioning the target stick (unmarked 27-stick) to gauge his estimates for 1/4 of the stick. He appeared to be attempting to segment the unmarked stick with his estimate and adjusting his estimate until the segmentation worked. In the continuation of this teaching episode, however, Joe did use his multiplicative relations to make an estimate for a stick that would be 1/7 of a mystery stick that had been drawn freehand on the screen. He saw that a 2-stick, repeated 7 times was approximately 2/3 of the unknown stick. He made a second guess that the 3-stick would be 1/7 of the mystery stick because “It's about 27. 21, I mean.” Azita continued using the 2-stick Joe had originally chosen, creating 10 repetitions to make a stick that was about one unit stick short of the target stick. Joe confirmed that he was thinking of the mystery stick as a 21-stick when he then exclaimed “21! That's what I said!” and that the 3-stick would be 1/7 of the 21-stick because "If you use 3 seven times you might get 21."

This last task may have provoked a connection of Joe’s partitioning operations with his segmenting approach to unit fractions. He could imagine seven of something being equal to the unknown stick. This relation had to be imagined as a visual partition of the unknown stick, because he had no whole number to operate with. Thus, Joe’s *mental partitioning operations* were activated. Joe was then able to make a connection between his numerical strategy for finding a unit-fraction (find the missing factor) and his *equi-partitioning scheme* by estimating a numerical value for the length of the unknown stick.

Confirmation for this possible connection between Joe's numerical strategies and his partitioning operations came in the next teaching episode. Joe accurately chose a stick that was 1/5 of a target stick where no numerical values were known. Four sticks and the unknown target stick were arranged on the screen as shown in Figure 2.

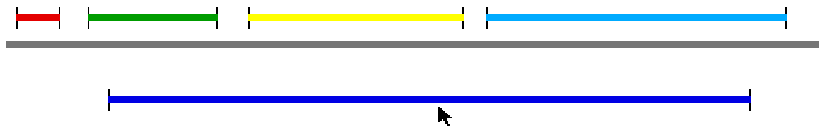

Figure 2: Which stick is 1/5 of the dark blue stick?

Joe copied the second of the four upper sticks below the lower target stick and repeated it 5 times to make a stick that matched the target. In order to have quickly chosen the correct stick, I hypothesize that he made a mental partition of the target stick. Later in this episode, Joe found 1/3 of the target stick (the third upper stick). Azita asked him to make a stick twice as long as this 1/3-stick. Joe did so by repeating the 1/3-stick. When asked what fraction of the target stick this repeated 1/3-stick would be, Joe at first compared it to the longest stick in the top row of sticks. He said this last stick was a bit shorter than his stick. When pressed for a fraction name he called his stick "two thirds" of the target stick. He also called three repetitions of the 1/3-stick a "whole stick" and "three thirds." He had established the inverse relation between a part and a whole that paralleled his numerical approach: What number multiplied by 3 will give me the whole? Further along in this same episode Joe was able to link a part-in-whole view of 5/11 with his multiplicative view of a stick that was "five times as long as the 1/11-stick." He justified calling his five repetitions of the 1/11-stick "Five elevenths!" "Because its 5 and its part of 11." He was able to project the stick that was the same as 5 parts out of the 11 (a disembedded stick) back into the 11/11-stick.. He was now able to combine unit fractions into composite parts of a whole through iteration. Joe was beginning to connect partitioning of a unit whole with the generation of both unit and non-unit (common) fractions. We refer to Joe's fractional scheme at this point in the teaching experiment as a *partitive unit fractional scheme.*

From Common Fractions to Improper Fractions through Iteration of Unit Fractions

The last teaching episode described above occurred just before the winter break in December of Joe's fourth grade year. In the next teaching episode (that took place after the winter break in February) Joe extended his fractional scheme into a *partitive fractional scheme* for common fractions. He was then able to generate improper fractions such as 6/5 and 9/7 through iteration of unit fractions. This was the beginning of Joe's *Iterative Unit Fractional Scheme.*

Protocol II begins after Joe had created 3/5 and 4/5 of a 5-part stick by drawing sticks that were the same length as 3 and 4 parts of the stick respectively. The 5-part unit stick, 3/5-stick and a unit stick with one mark 4/5 the way along the stick.

Protocol II

A: *That's really neat! Now I'm really hungry. I want you to make me another one. I want you to make me 6/5 of that candy* (meaning the unit stick).

J: *Can't!*

A: *Why not?*

J: *You only got 5 of them.*

A: *Five what?*

J: *Fifths.*

A: *You only got 5 fifths. So is there any way of making one, do you think?*

J: *Make a bigger stick.*

A: *Make a bigger stick. How much bigger do you think it should be?*

J: *One more fifth.*

A: *O.K. Do you want to show me?*

Joe pulls the end part out of the original stick that has a mark at the 4/5 position only, and joins this one piece to the original stick to make a stick one fifth larger than the original.

Joe was using his partitive fractional scheme to make 3/5 and 4/5 of the unit candy stick. What is important about the above protocol is how Joe was able to interpret Azita's request for 6/5 of the candy as being one more fifth than the whole bar. This was a possibility for Joe because his partitive fractional scheme included unit fractions as repeatable units. Thus 6/5 was six of one fifth, which was one more fifth than the whole stick. Joe was not constrained by a *part-in-whole* restricted view of fractions as were some of my college students! In Protocol III (later in the same teaching episode), Joe confirms that he can now create fractions greater than the whole through iteration of a unit fraction. Joe had successfully estimated 1/7 of a candy stick and had used his estimate to mark off all seven 7ths on the original candy stick. He had then pulled out 4/7 of the candy stick to give to Dr. Steffe.

Protocol III

A: *Can you make a stick that is…say 9 times as long as the one…seventh stick* (points to the first part of the 7/7-stick)?

Joe cuts off the first part of the 4/7-stick. He accidentally cuts this part again so he joins it back together and erases the resulting mark. He then repeats this 1/7-stick 9 times to make a 9-part stick.

A: *How long is that stick?* (Joe thinks for 3 seconds.)

J: *Nine sevenths.*

A: *Why?* (Joe thinks for 15 seconds.)

A: *You are right. It is 9/7 but why do you think it is 9/7?*

J: *Because it was…you were making these the sevenths* (pointing to the parts of the 9-stick) *so each of these would be one seventh.*

A: *That's really nice!*

In this last episode, Joe was able to work with a fraction as both a part of a whole (the 3/5 and the 4/7) and a unit part *out of a whole*. In repeating a 1/7-stick 9 times to make a stick 9 times as long as 1/7 of the original whole stick, Joe was able to go beyond the whole, and name the resulting stick as nine sevenths "because …you were making the sevenths, so each of these would be one seventh". In this sense Joe had taken the 1/7-stick as an *iterable one* to generate a composite unit of 9, each one of the 9 units being 1/7 of his 7/7-stick. These operations suggest that Joe had not only constructed a *partitive fractional scheme* (for generating common fractions) but was on his way to developing an *iterative unit fractional scheme* for generating *improper fractions.*

Conclusions

The sequence of episodes with Joe, described in this paper, indicate how powerful a child's thinking about fractions can become when it builds from his numerical schemes and operations, assisted by appropriate language scaffolding by a sensitive teacher. They also indicate the role that both partitioning and iterating operations play in constructing fractions greater than the whole. The affordances created through Joe's use of TIMA: Sticks were also important factors in Joe's construction of improper fractions. Being able to make a stick that is "9 times as long as the 1/7-stick" through repetitions of a 1/7-stick provided Joe with an instantiation of his iterable unit fraction.

References

Behr, M., Wachsmuth, I., Post, T., &Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 14*, 323-341.

Biddlecomb, B. D. (1994). Theory-based development of computer microworlds. *Journal of Research in Childhood Education*. 8, 87-98.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding.* Mahwah, NJ: Lawrence Erlbaum Associates.

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning,* 1 (4): 279-314.

Olive, J. (2000). Computer tools for interactive mathematical activity in the elementary school. *International Journal of Computers for Mathematics Learning.5*(3), 241-262.

Olive, J. & Steffe, L. P. (1994). TIMA: Bars [Computer software]. Acton, MA: William K. Bradford Publishing Company.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences,* 4 (3), 259-309.

Steffe, L. P (1998). Construction & interaction: Children's fractional schemes. Proposal submitted to the National Science Foundation. The University of Georgia, Athens, GA: Author.

Steffe, L. P. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Steffe, L. P. & Olive, J. (1990). Children's Construction of the Rational Numbers of Arithmetic. Proposal submitted to the National Science Foundation. The University of Georgia, Athens, GA: Author.

Steffe, L. P. & Wiegel, H. (1994). Cognitive play and mathematical learning in computer microworlds. *Journal for Research in Childhood Education*, 8, 117-131.

Streefland, L. (1993). Fractions: a realistic approach. In T. P. Carpenter, E. Fennema & T A. Romberg (Eds.), *Rational numbers an integration of research.* Hillsdale, NJ: LEA.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting the learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

von Glasersfeld, E. (1981). An attentional model for the conceptual construction of units and number. *Journal for Research in Mathematics Education, 12* (2), 83-94.