ANALOGY IN THE EXPLORATION OF REPEATING PATTERNS

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Problems that require examination of repeating patterns were presented to a group of 106 preservice elementary school teachers. A theoretical framework of reasoning by analogy was used to analyze students' responses. It was found that not only relational structure of problems but also their relative computational complexity influence students' ability to successfully map between familiar and unfamiliar tasks. Consideration of multiples and division with remainder were the main mathematical tools utilized in students' solutions, but in many cases students didn't demonstrate their awareness of the relationship between these tools. It is suggested that pedagogical approach should help students connect their understanding of multiples with their understanding of division with remainder and contribute to a more complete understanding of repeating patterns.

Patterns are the heart of mathematics. Students' ability to recognize or develop patterns is related to their ability to reason mathematically in general and to develop reasoning by analogy in specific (White, Alexander & Daugherty, 1998). In this study we examine students' strategies as they engage in tasks that invite exploration of repeating patterns in familiar and unfamiliar situations. What mathematical tools are utilized? Is there consistency in students' approaches? What influences their choice of a strategy? -- These questions are of interest in this investigation.

THEORETICAL FRAMEWORK: REASONING BY ANALOGY.

Analogical reasoning was explored generally in the context of science and cognitive science. Despite apparent relationship between analogical reasoning and mathematical learning (White, Alexander & Daugherty, 1998), only recently researchers in mathematics education became interested in reasoning by analogy in specific mathematical contexts. Reasoning by analogy in mathematics was explored mostly in considering word-problems in arithmetic and beginning algebra (Novick, 1995, English, 1998).

According to English (1998) reasoning by analogy in problem solving involves mapping the relational structure of a known problem (that has been solved previously, referred to as “source”) onto a new problem (referred to as “target”) and using this known structure to help solve the new problem. Analogy can be recognized in isomorphic or “almost isomorphic” problems. In our interpretation, isomorphic problems, also denoted by English as “completely isomorphic”, are those possessing the same structure not only from a mathematical, but also from a linguistic perspective. They can be presented in (i) the same situational setting, or in (ii) different situational settings. We refer to problems as “almost isomorphic” when they have similar components, but are not “completely isomorphic”, that is, (i) they possess the same mathematical, but different linguistic structure, or (ii) a source problem can be seen as isomorphic to a part of a target problem or vice versa.

Mapping the structure of the source problem to the target problem is the key aspect of reasoning by analogy. In order for a mapping to take place a relational structure of the source problem must be identified, as well as similar relational structure of the target problem. English (1997) suggests that failure to identify the relational structure of the source problem is a significant cause of children's impaired mathematical reasoning. Furthermore, successful solution of a target problem is not an immediate and natural consequence of a successful mapping (Novick, 1995). The solution procedure may need to be adapted or extended, especially in case of problems that are not completely isomorphic. Successful mapping is essential for recognizing the need for such adaptation. In summary, solution by analogy requires recognition/identification of a similar relational structure, mapping between source and target, and adaptation or extension. Each step presents its own challenges.

METHOD

Tasks

The participants were presented with three problems and in each case they were asked to explain and justify their solution.

1. What is the 177th digit to the right of the decimal point in the decimal representation of $0.\overline{76543}$?

2. A toy train has 100 cars. The first car is red, the second is blue , the third is yellow, the fourth is red, the fifth is blue and sixth is yellow and so on.

(a) What is the color of the 80th car?

(b) What is the number of the last blue car?

3. Imagine a toy train with 1000 cars, following the 7 colour repeating pattern

1- red, 2 - orange, 3 - yellow, 4 - green, 5 - blue, 6 - purple 7 - white

(a) What is the color of the 800th car?

(b) What is the number of the last blue car?

Each of these problems presents, either explicitly or implicitly, a repeating pattern (of length 5, 3 and 7 respectively) and asks to determine the attribute (digit or color) of this repeating pattern that corresponds to a particular place. A possible approach to the solution of these problems is to identify the *remainder in division* of “place” by “length” and assign the attribute that corresponds to the place of this remainder. For example, in question 1, the *remainder in division* of 177 by 5 is 2, and therefore the attribute corresponding to the 177th place is the same as the attribute corresponding to the second place, in this case digit 6. Another approach is to notice that every 5th digit is 3, therefore digit 3 appears in every place that is a multiple of 5; 175 in particular. Then the attribute of the 177th place is determined by *counting up from a multiple* . Mathematically, the two approaches described above are equivalent, however, as suggested by our data, they present a different cognitive difficulty and have a different explanatory power.

Participants and Procedure

Written responses were collected from a group of 106 preservice elementary school teachers, enrolled in a core mathematics course. It should also be noted that tasks similar to question 1, that is, consideration of digits in a repeating decimal pattern were discussed in class. Consideration of the remainder was among the possible solutions presented and examined by the participants. Therefore, question 1 presented a familiar situations, while questions 2 and 3 presented situations that can be describe as unfamiliar. Strategies used by participants to solve each problem were identified and recorded. The partial data analysis presented below focuses on inconsistencies in students’ approaches when dealing with analogous situations. In analyzing students' solutions the researchers were interested in students' mapping between isomorphic problems as well as mapping and adaptation in cases that were not completely isomorphic.

RESULTS AND ANALYSIS

Isomorphic Mapping

(i) Same situational setting.

This mapping is represented within our instrument in the structural relationship between questions 2a and 3a (or 2b and 3b), where the only differences are in the numbers. Both the target attribute and the pattern length are larger in problem 3 than in problem 2. Most participants were successful in this isomorphic mapping. In most cases (79 of our of 106) the strategies presented by participants in problem 2 were exactly the same as the strategies chosen in problem 3. However, simply varying the magnitude of the numbers in the situation was enough to prompt 27 of the participants to change their strategies when going from one problem to the next. Carlo, for example, in dealing with problem 2a chose to attend to the *remainder in division* by 3 (the pattern length).

> *Q2a. When 80 is divided by 3 the remainder is 2 that means the 80th car will be blue because that is the colour of the 2nd car.*

However, when faced with the same situation in 3a he changed his strategy and *counted up from a multiple* of 7, the pattern length.

> *Q3a. Every 7th car is white, so car 798 is white because 7x114=798. Then 799 is red and 800 is orange.*

Why wasn't his strategy in 2a successfully mapped to 3a? A possible explanation is in the relative computational complexity of the task. While the *remainder in division* by numbers 3 can be easily identified mentally considering divisibility tests, the case of division by 7 is different. Therefore, Carlo's work can be seen as an attempt to relate the problem to pieces of easily identified components of prior knowledge, considering multiples of 7 as a point of reference and counting up from that point.

It is interesting to note that although division of whole numbers results in quotient and remainder, only the remainder provides useful information for these questions. Although many students felt comfortable discussing the *remainder in division* without pointing to the quotient, there were 12 participants who changed from this very consistent strategy. They felt compelled to explicitly state both outcomes of the division statement only when dealing with the larger numbers in question 3. This is further evidence supporting the claim that computational complexity of isomorphic problems is enough to prompt students to change their strategies.

(ii) Different situational setting.

Although question 1, and 2a have identical mathematical and linguistic structure, they present the student with a different situation. The variation of the problem from finding the 177th digit to finding the colour of the 80th car was enough to cause 21 of our students to change their strategy for dealing with the new situation.

Amanda displayed good use of the *counting up from a multiple* in question 1, but opts for *remainder in division* in problem 2a.

> *Q1. Since the repeating portion is 5 digits long, then every 5th digit will be a 3. So, 175 [digit] will be 3 and 177 [digit] will be a 6.*

> *Q2a. 80/3 = 26r2. Since the remainder is 2 then it will be the same colour as the second car… blue.*

Both of these strategies are equally effective. So effective, in fact, that it is surprising that Amanda would change from her initial one. Perhaps this is due to a refinement of her strategy as she maps from one problem to the next, as *remainder in division* can be seen as an improvement of the strategy of *counting up from a multiple.* Of the 21 participants that showed an inconsistent mapping in the face of a situational change, only six could be attributed to a refining of strategies. The other 15 produced blaring inconsistencies in their failure to map their procedures from one situation to the next. This is demonstrated in Jason's solution.

Q1. 177/5 has a remainder of 2. Therefore, the 177^{th} digit will be 6.

Q2a. 80 is divisible by 2, and since the 2^{nd} car is blue, the 80^{th} car is blue.

Jason's strategy in 2a referred to as *multiple of the initial position*, will be discussed in a later section.

<u>*Almost Isomorphic Mapping*</u>

(i) Different linguistic structure.

Our instrument did not provide data representative of this type of mapping.

(ii) Target and source are partially isomorphic.

Problems 2b and 3b, are isomorphic to each other, but present a slight variation on 2a and 3a respectively. Problems 2a and 3a ask for an attribute that corresponds to a given place, while 2b and 3b ask to determine the largest place that has a given attribute. We consider the pairs 2a and 2b, 3a and 3b as almost isomorphic. An attempt to map a new problem (3b) to a familiar one (3a), is demonstrated in Cindy's work.

Q3a. 800/7 = 114 with a remainder of 2. So, the 800^{th} car will have the same colour as the 2^{nd} car. Orange.

Q3b. 1000/7 = 142 with a remainder of 6. Blue cars have a remainder of 5, so it must be car 999.

We consider this refinement to be an adaptation consistent with the *remainder in division* by the pattern length mentioned earlier. Alternately, the strategy of *counting up from a multiple* can be adapted to target a specific attribute (in this case colour) rather than a specific position, as exemplified by Bill.

Q3a. 798 is a multiple of 7, so it will be white. 799 is red and 800 is orange.

Q3b. 994 is a multiple of 7 (994 = 142 x 7). So 995 is red, 996 is orange, 997 is yellow, 998 is green, 999 is blue.

Of our 106 participants, 29 of them didn't rely on the "almost isomorphic" structure of the two problems and thus did not map their strategies from problem 2a to problem 2b (or 3a to 3b). In 21 of these cases this involved going from a *remainder in division* process in 2a to a *counting up/down from a multiple* in 2b; an inconsistency that can be viewed as a hierarchical regression in the development of students' schemes (Zazkis & Liljedahl, 2000).

<u>*Errors in mapping*</u>

In this section we mention several errors, that were more than single occurrences and therefore require attention.

Recognition of a repeating pattern involves recognition of its length as well as its attributes. Five students consistently identified incorrect length of these patterns. In each case the identified length was one more than the actual length. It appeared that they considered the place where the digit (or the color) repeated for the first time, and this served as a reference. Writing out the repeating pattern several times may help clarify this confusion.

Another error was caused by incorrect correspondence when considering multiples. In question 1, for example, searching for the 177th digit to the right of the decimal point in the decimal representation of $0.\overline{76543}$, Sally correctly recorded that

> *Q1. Every fifth place had the same digit, and therefore a digit in a place that is a multiple of 5 is always the same.*

However, this digit was identified as 7, which is the first digit in the repeating pattern, rather than the last one. So her reasoning continued as follows:

> *Q1. So 175th digit is 7 and 177th digit is 5.*

Sally, as well as three of her classmates consistently applied this incorrect mapping between the place and the attribute. Requiring students to list the correspondence between digits/colors and places explicitly should help in identifying the correct correspondence.

While the above mentioned "bugs" appear not difficult to "debug", the reasoning Jason used (discussed earlier) hints at a more profound misunderstanding.

> *Q2a. 80 is divisible by 2, and since the 2^{nd} car is blue, the 80^{th} car is blue.*

In this reasoning, Jason is assuming that every multiple of the initial position of the blue car shares that attribute. Monica, expressed the same reasoning even more explicitly.

> *Q3b. Since the blue car is in the 5^{th} position all multiples of 5 will be blue.*

This logic remains a mystery to us. At first occurrence we were ready to dismiss such reasoning suggesting that these students memorized the ingredients that played a role in solving similar problems - multiples and remainders - and then mis-assembled them in their responses. It could be that students incorrectly generalized the observation that "every seventh car is white" (a statement which is true for our repeating pattern of length seven) to "every fifth car is blue" in the same repeating pattern. However, the fact that consideration of a *multiple of the initial position* occurred in responses of 15 different students, requires further investigation of this line of reasoning in future research.

CONCLUSION AND PEDAGOGICAL IMPLICATIONS

Reasoning by analogy follows the sequence of recognition–mapping–adaptation. We suggest that isomorphic mapping of a problem does not necessarily imply isomorphic mapping of a solution. Solution depends on computational complexity and often patterns with small and compatible numbers are not easily generalized. Therefore, adaptation may be required not only for problems that are not completely isomorphic, as suggested by English (1998) and Novik (1995), but also for completely isomorphic problems of different degree of computational complexity.

Although the claim that, for example, *996 leaves a remainder of 2 in division by 7* and the claim that *996 is larger by 2 than a multiple of 7* communicate the same mathematical relationship, these claims are not equivalent in terms of their explanatory power for learners. Consideration of *remainder in division* presents a unifying scheme that enables a student to deal with a large class of problems, including all of the problems in our instrument. However, students' ability to generalize this strategy and apply it for various questions is not well developed. When faced with unfamiliar situations students often prefer to invoke *counting up from a multiple* strategy. We found in these data, as well as in-class interactions among students, that a *distance from a multiple* is a reasoning that is applied by participants in a more spontaneous and natural way. This slip back towards the familiarity of multiples is an indicator of the lack of recognition of the invariant multiplicative structure inherent in division with remainder statements.

Our recommendation is that pedagogical approach to division with remainder should build on what appears as a natural tendency of students - consideration of multiples. Capitalizing on multiples, remainders can be considered as "adjustment" or "translation" along the number line. This approach would enforce the view of a remainder as indicator of partition of whole numbers, rather than simply an outcome of an operation, and empower the understanding of repeating patterns as well as the relationship between additive and multiplicative structures.

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