A STUDY OF CHILDREN'S VISUAL IMAGERY IN SOLVING PROBLEMS WITH FRACTIONS [1]

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We describe some contrasting aspects of the internal, visual imagistic representations of two elementary-school children, inferred from fine-grained analysis of their problem solving during a pair of carefully scripted, videotaped task-based clinical interviews 16 months apart. The problems involve conceptual interpretation and use of fractions. We look for and find concrete/pictorial, pattern, and dynamic imagery, based on set (discrete quantity) and spatial extent (continuous quantity) models, and memory images of formal notation. We also infer internal operations on and transformations of imagery, and discuss our interpretations.

This study considers in some detail the internal, visual imagistic representations of two children as they solve some problems involving rational numbers (fractions). We acknowledge at the outset important limitations to the methodology of inferring internal representational configurations from observed external statements, behaviors, or productions. Internal representation involves ambiguity, and inferences about it entail context-dependent interpretations. In this exploratory and descriptive "status study," our goal is not (yet) to commit to a definitive, reliable or generalizable coding scheme, but to make as explicit as possible the bases for our inferences, improving task-based interview methodology as we explore individual children's imagery.

Various researchers investigate and discuss imagistic representation generally, and visual imagery in particular, as a fundamental system of cognitive representation for mathematical problem solving (Bishop, 1989; English, 1997; Goldin, 1982, 1987, 1998; Goldin & Kaput, 1996; Owens, 1993; Presmeg, 1985, 1986, 1998; Thomas & Mulligan, 1995). We shall focus here on concrete/pictorial imagery, pattern imagery, memory images of symbolic notation, dynamic imagery, and mental operations on or transformations of images. These characteristics are not mutually exclusive; e.g., pattern imagery may be dynamic, and transformed *via* mental operations.

We use definitions (consistent with those of Owens and of Presmeg) as follows. *Concrete/pictorial imagery* is spoken of or gestured at as if it were a picture or a

[1] Research partially supported by a U.S. National Science Foundation (NSF) grant, "A Three-Year Longitudinal Study of Children's Development of Mathematical Knowledge," directed by Robert Davis and Carolyn Maher. Conclusions expressed are not necessarily those of the NSF. We thank Valerie DeBellis, Claire Passantino, and Richard Zang, who developed and administered interviews #2 and #5 with us, and Alice Alston, Lynda Deming, and Amy Martino for technical assistance.

physical object. The image may be given a name referencing what it resembles in real life. Presmeg (1985) found pictorial imagery to be the predominant form during problem solving; Owens also found it to occur frequently, especially during the orientation phase of problem solving and again as a kind of retrospective, global check. In *pattern imagery* the image abstracts and/or generalizes mathematical relationships. Presmeg (1985, p.175) notes "such imagery may be vague or vivid but its essential feature is that it is pattern-like and stripped of concrete details." Krutetskii (1976) calls pattern imagery a graphic scheme where the problem's terms and relations are represented in a visually schematic way. *Memory images of symbolic notation* refer to the visualization of formal mathematical expressions, as Presmeg reports students who describe "seeing" a mental picture of a formula and "reading" the image. In *dynamic imagery* there is active movement or change in all or part of the image (Thomas & Mulligan, 1995). *Mental operations or transformations* refer here to visualized purposeful acts by the imager that modify or transform the image, suggesting but not requiring dynamic imagery (as they can also be visualized as a succession of static images). Presmeg also uses the term "kinaesthetic imagery" because of associated physical actions; Owens' term is "action imagery."

As our study of imagery is centered in the domain of fractions (Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuth, & Post, 1988), we consider the children's imagery in relation to various possible conceptual models: e.g. discrete (set) *vs.* continuous quantity or spatial extent (area or volume, sometimes termed "region") models; part/whole *vs.* comparison or ratio relationships; mental operations such as partition. We also focus on understandings of fractional symbolic notation through visual imagery.

Research questions and design of the study

The following questions motivated our analysis. (1) *Inferring internal imagery.* Can we create explicit criteria for inferring from hand gestures, body movements, paper and pen or marker diagrams, pictures, and charts, and physical manipulations of concrete materials, the five characteristics of visual imagery discussed above? Do we find consistency among these in the individual child? (2) *Relation between internal and external imagistic representation.* What relationship can be discerned between the child's internal visual representational capability and her facility in constructing external imagistic representations? How does the use of paper and pencil or marker to draw pictures, diagrams, and/or create charts, or the use of concrete manipulative materials, assist in or influence the visual imagery used in the problem solving? (3) *Relation between internal visual imagery and strategy.* Heuristic strategies of interest include drawing diagrams, working backward, solving simpler or similar problems, guess and check, subproblem decomposition, etc. How does the child's internal imagery interact with or influence strategy use? (4) *Relation between internal imagery and inference of mathematical patterns.* How is the child's facility in constructing or recognizing a mathematical pattern influenced by her imagery. Which imagery characteristics are important here? (5) *Internal imagistic representation and* *mathematical conceptual development.* Is there a relation between the child's internal imagery and the development of her conception of a fraction over time?

In a longitudinal study of individual children's mathematical development, whose design has been described in greater detail elsewhere (Goldin, DeBellis, DeWindt-King, Passantino, & Zang, 1993; Goldin, 1997), 22 children ages 8 to 10 years at the outset were selected to participated in five highly structured task-based interviews over three school years. Interviews #2 and #5 involved non-routine problems about and with fractions. A broad, cognitive analysis with respect to fraction representations and strategies of all 20 children who participated in these two interviews has been completed (Passantino, 1997; see also Goldin & Passantino, 1996). The present report is part of a study focusing in greater depth on the visual imagery of a cross-sectional subset of four children. Below we discuss brief excerpts from interviews with two of them, Marcia and Londa, inferring some contrasting representational characteristics.

The interview scripts were developed at Rutgers by a team of experienced teachers that included the clinicians, and revised after rehearsal and pilot testing to incorporate many different contingencies. Free problem solving is encouraged throughout--after posing a question, the clinician allows for *spontaneous* response before questioning to explore the child's answer or request an external representation. If the child is not engaged or seems at impasse, planned suggestions/hints are offered. The clinicians are not to impose their methods or correct or confirm the child's solution, as it is not the goal here to teach rational number concepts. At certain points, decided in advance and described in the interview script, the children are guided toward particular understandings essential for subsequent questions to be meaningful. Two video-cameras operated simultaneously during each interview, one focusing on the child's work and the other showing the interaction of the clinician with the child. Outlined below are elements of the two interview scripts needed for the discussion; the complete scripts are in Passantino (1997), or available from the authors.

In task-based interview #2, materials placed before the child include a pad, pencil, markers, and red and black chips (checkers). Some preliminary, non-mathematical questions precede a sequence of mathematical questions posed by the clinician. For *each* one, the follow-up includes, as appropriate, "Can you help me understand that better?" "Why?" and/or "Are there any other ways to take one half (one third)?" [The first such questions are:] ♦ "When you think of one half, what comes to mind?" ♦ "When you think of one third, what comes to mind?" ♦ "Suppose you had twelve apples. How would you take one half?" ♦ "... one third?" ♦ [Cutouts are presented in succession: a square, a circle, a 6-petal flower. For each the child is asked:] "Here is a shape. How would you take one half?" ♦ "... one third?" ♦ ... ♦ [Part way through:] "What was on your mind when you were answering the questions up to this point?" and if no image is described, "Did you have a picture in your mind while you were answering any of the questions?" with follow-up. ♦ [The interview continues:] "Can you write the fraction one half?" "What does this fraction mean to you?" (similarly for one third) ♦ ... ♦ The clinician then goes on with more complicated exploratory activities that involve taking one half and one third of an array, and visualizing the cutting of a cube into fractional parts. At the end of the interview retrospective questions again address the child's visual imagery.

Task-based interview #5 re-explores specific topics from interview #2, and extends to further rational number problems. Materials at the outset include red and white chips, paper circles, squares, and triangles, markers, pencil and paper, a calculator, a ruler, ribbon, scissors, and other items specific to later questions. The follow-up for each question includes, as appropriate, "Why?" "Why not?" "Can you show me what you mean?" "Can you show me [using] the materials?" [The main questions begin:] ♦ "When you think of a fraction, what comes to mind?" ♦ [Bold-face printed expressions in vertical format for five fractions, 1/2, 1/3, 2/3, 3/4, and 4/6, are presented on a sheet of paper:] "What fractions do you see here?" ♦ "Can you explain ... what one of these fractions means?" ♦ "Why is it written this way?" ♦ ... ♦ [Several main questions later, the clinician asks:] "Imagine a big birthday cake shaped like a rectangle. Can you imagine what it looks like?" ♦ "Describe what it looks like." ♦ "Now imagine that there are 12 people coming to the birthday party and they each want a piece of cake. Your job is to cut the cake so that each person gets the same size piece. How will you cut the cake?" ♦ ... ♦ "Are there any other ways to cut it?" ♦ "Now think about the icing. Suppose the cake has icing on the top and side." ♦ ... ♦ At the end of the interview further retrospective questions address visual imagery.

Observations and inferences: Londa

At the time of interview #2 Londa was 9 yrs. 8 mos. old, in the 4th grade of a school in a low-income, urban district. Her spontaneous representations of "one half" and "one third" are verbal: *"a cookie, split in half" "... and then when you split it in half, it's two ... Two smaller pieces,"* and *"Half of our classroom, kids in our classroom," "... we have 19 kids in our classroom so half of that would be 9 ... no, we have 18, half of that would be 9,"* followed by *"box cut in threes ... because I think a box would be easy to picture," "one-third of an orange,"* and *"one-third of an apple. Really one-third of anything."* All these phrases suggest 3-dimensional, concrete/ pictorial imagistic representation, drawn from familiar real-life contexts, with interpretation of the fraction based on a mental operation of partition of a whole into equal parts. All but the second provide a "spatial extent" or continuous quantity model. The "kids in our classroom" can be characterized as a "set" or discrete quantity model. Asked to take one-half of 12 apples, she replies, *"Split all of the apples in half."* Showing with checkers, *"Well, take 12 of these"* [takes 6 red, 6 black] *"I would split them all in half. So then there'd be 24"* [she counts the 12 checkers, assigning a value of two to each] *"'Cause you cut this in half this would be two ... four, six, eight"* [she continues to 24]. When asked to take one-third, *"Cut them all into threes"* [she counts by threes with the 12 checkers, to 36]. Londa never explicitly states what would be one-third. Her external representations to this point suggest mental operations visualized through a succession of static, 3-dimensional concrete/pictorial image-configurations, using exclusively a continuous quantity model for "fraction."

She does not come back to a "set" model; having acted to transform her image from 12 apples into 24 pieces, she does not seem to treat the latter as equivalent elements but retains the pairing derived from the apple at which they originated.

Asked to write the fractions one-half and one-third, Londa does so correctly in a vertical format, symbolic notation. Interpreting what she wrote, she explains, *"This* [the 2] *means that you have two parts, and this* [the 1] *being one whole. One whole and two parts."* She explains one-third as *"one whole, three parts."* [Clinician:] "Is there any other way you can think of what those fractions mean?" [Londa:] *"Either that or three wholes and one part."* She explains, *"if I had one cookie, two cookies, three cookies"* [draws three circles], *"and I take one cut, part out of each of them"* [makes a small wedge at the top of each circle]. Londa translates consistently but with some instability among verbal expressions, pictorial images with mental operations, and memory images of symbolic notation. Her interpretation of the numerator as representing the whole and the denominator as the number of parts is consistent with her first answers, and with her method of taking one-half and one-third of twelve apples. She does not show evidence of a ratio or comparison model for fractions.

At the time of interview #5 Londa was 11 yrs. 0 mos. old, in the 5th grade in the same school. Her initial statement of what comes to mind for a fraction is, *"I think about how the denominator means that it's the whole and the numerator means how many parts are out of it." ... "Like two thirds, the three is the denominator, that means the whole thing; and two means how many, um, two pieces out of the three whole."* [Clinician:] "Could you show me using some of these materials perhaps?" [Londa:] *"Okay, this is two,* [picks up two yellow cutout circles] *this is the two-thirds, and then you have three on the bottom, you have three all together* [puts a third circle with the others] *and then you have this is two* [indicates the two circles] *and then you have one left over* [indicates the third circle]. *So, well, like, if you say two-thirds minus, um, one-third you have two-third, no, two-thirds minus one-third you have one third and that's it."* [without further action with the circles] Londa's first words suggest a memory image of symbolic notation related directly to part-whole imagery. Her use of the circles makes no further reference to "pieces of the whole," but suggests a possible relation to a set model, even the genesis of a ratio model, but with additive imagery. She evidences a memory image of an algorithmic procedure for fractions.

Later, asked to visualize the rectangular birthday cake, Londa describes it, explains that *"Since it's a rectangle you can divide it into equal parts,"* shows its height with the ruler perpendicular to the table, *"It's about two inches thick,"* and draws a rectangle with two rows and six columns to show exactly how to cut it in 12 equal pieces. Asked if there are other ways, she replies, *"I don't think so because that's the way we cut my birthday cake. I,* [shakes her head] *I don't think so."* Asked how many, and then what fraction, of her pieces would have icing on exactly two sides and the top, she (correctly) explains *"Just the four, because ... you can't put icing on the inside of a cake ..."* and *"one-third, well, four-twelfths you can reduce it, four-twelfths can be reduced by four and it'd be one-third."* Her verbal and external pictorial representations of how she cut the cake, and her way of determining the number of pieces with icing on exactly two sides and the top, suggest static, internal 3-dimensional, real-life pictorial imagery, with operations of partition and counting closely embedded in the context. Londa uses her imagistic part-whole model and an internal algorithm to obtain the desired fraction of the pieces.

Observations and inferences: Marcia

At the time of interview #2 Marcia was 10 yrs. 4 mos. old, in the 5th grade in a small, lower middle income school district. Her spontaneous, verbal representations of "one half" and "one third" are *"a half of a circle," "a half of a triangle,"* and *"a third of a circle."* These suggest possible internal pattern or pictorial imagery with a region model for the fraction. Asked about the 12 apples, Marcia replies, *"Count 'em. Well if you know there's 12, and 6 ... and since 6 and 6 is 12, take 6 of them if you want half."* [arranging 12 red checkers in two rows of 6, and counting them] *"... you know that 6 and 6 is 12 you could just take half of them away and you would have half."* [She pushes the 6 checkers from the top row into a casual irregular group. Clinician:] "Why is it one half?" [Marcia:] *"Because, well, if you place it like this and you add the 12 to your 6 here"* [she arranges the two rows of 6 again, indicating that taking the bottom row of checkers is equivalent to taking the top row, and counts] *"... it's the same amount the other way."* For one-third of twelve apples Marcia replies, *"if you knew 4 times 3 was 12, you could take 4 away and you would know it was a third ... because 4 and 4 is 8 and then another 4 is 12."* [pushes 4 checkers to the right, then separates the remaining 8 into two groups of 4, maintaining the row and column structure] Marcia's external representations suggest memory images of notation, and additive and multiplicative structures for part-whole relation in a "set" model. She focuses on the numbers of apples, doing some mental computation; importantly, her physical manipulation of the checkers suggests an internal pattern image that includes operational one-to-one correspondence between items in parallel rows.

Later Marcia correctly writes the fractions one-half and one-third in vertical-format symbolic notation, explaining, *"well, I can tell that it's one-half because like this is a two* [points to denominator] *and it's a one* [points to numerator] *and if you add one and one it would equal two,"* and *"if you could times one times three or one and one and one is three so it would be one-third"* Her repeated addition of the numerator to reach the denominator value is suggestive of her earlier pattern imagery.

At the time of interview #5 Marcia was 11 yrs. 8 mos. old, in the 6th grade in the same community. She now describes a fraction in quite general terms, *"like a part of a whole of something"* [writes the fraction three-fifths] and explains, *"like three-fifths it's like the five-fifths is the whole, and it's like that's part of the whole of whatever the thing is."* To us this suggests not just internal memory images of notation, but also a level of abstraction that includes a pattern image of a fraction as an operator.

Marcia initially responds to the birthday cake question verbally, kinesthetically, and with a drawing: *"Well, um, it's like this* [tries to indicate with her hands how the cake would look, then draws a triangle] *like, a, uh, oh a rectangle"* [draws a long, thin vertical rectangle]. After some dialogue she measures and redraws a horizontal rectangle 4 inches long. *"... and like you would divide the four into half ... which is at two* [draws a vertical line at 2 inches, dividing the rectangle in half] *and then you would divide each of these into halves* [draws vertical lines dividing each section in half] *... so you'd have four pieces ... and then you divide these, which is at the half an inch, 'cause the other one was at the inch to divide ... and then you have to divide these which is at a fourth of an inch"* [counts 16 pieces]. ... *"But there's more than twelve so ..."* Prompted to redraw freehand, Marcia suggests, *"it would depend on like how big the cake is 'cause like if the cake is, like, um, 24 inches long then you would have to divide them each into two inches" ... "So, it depends how long the cake is."* Further encouraged to redraw freehand, Marcia again considers halving, *"... there would be 16 ... since there's 8 now ... so it would double it so like you ..."* [Clinician:] "Yeah, could you do it?" [Marcia:] *"... would have to divide the cake into thirds"* [draws a long horizontal rectangle, with vertical lines dividing it in thirds] "*... and then divide these so then there's 6* [draws vertical lines dividing each of the 3 sections in half] *and then you divide these into half* [draws additional vertical lines dividing each of the six sections in half] *which would make that 12 pieces."* Marcia does not think of other ways to cut the birthday cake into 12 equal pieces. Asked how many, and then what fraction, of her pieces would have icing on exactly two sides and the top, she answers (correctly), *"there would be 10, 'cause there, it would be on the top here* [points to interior of the rectangle] *and then it would be on this side and this side* [points to the upper and lower horizontal edges] *but you can't count these sides* [points to the right and left vertical edges] *'cause there would be three since there's the edges."* She continues, *"10 out of 12, or five-sixths," explaining, "you divide the 10 and the 12 by two and you get five-sixths."* Space permits no further detail, but from this and other evidence we infer internal visual and kinesthetic pattern imagery based on linear and region models, interacting with the "trial and evaluate" halving strategy. We think her linear model stems from her measurement process.

Comparison and conclusion

We infer a close relation between the children's internal imagery and their conceptual development of fraction, based not only on the presented excerpts but on our analyses of the full interviews. Londa's imagery is predominantly pictorial, centered in familiar concrete objects. She consistently connects her memory images of symbolic notation with just a part/whole conceptual model for fractions, relating each problem directly to a real-life embodiment. Her use of pattern imagery, which might help her abstract and connect different ideas or metaphors of fraction, is not well developed. In contrast, Marcia evidences predominantly pattern imagery, connected with memory images of symbolic notation. She uses part/whole, operator, additive, multiplicative, and linear conceptual models for fractions, and is not "stuck" in particular contexts. Her frequent pattern imagery and her wider range of conceptual models suggest that for her symbolic notation is more flexibly representational of her imagery. Both children show consistent visual imagery characteristics from the first interview to the second, with evidence of developing memory images of symbolic notation and school-taught algorithms between the two interviews.

While initial verbal responses usually allow us to infer some imagery, the planned prompts for additional external representational forms yield evidence of different internal imagery, not easily discernible from the purely verbal descriptions. Throughout the interviews Londa's and Marcia's drawn pictures and physical manipulations of concrete materials suggest concrete/pictorial and/or pattern imagery; their hand gestures and body movements suggest dynamic imagery and/or mental operations; and their notations suggest memory images of symbols and algorithmic operations.

References

Behr, M.J., Lesh, R., Post, T.R., & Silver, E.A. (1983), Rational-number concepts, in R. Lesh & M. Landau (eds.), *Acquisition of mathematics concepts and processes*. NY: Academic Press, 91-126.

Behr, M.J., Wachsmuth, I., & Post, T. (1988), Rational number learning aids: Transfer from continuous models to discrete models. *Focus on Learning Problems in Mathematics, 10 (4),* 1-18.

Bishop, A.J. (1989), Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11 (1),* 7-15.

English, L., ed. (1997), *Mathematical reasoning: Analogies, metaphors, and images.* Mahwah, NJ: Erlbaum.

Goldin, G.A. (1982), Mathematical language and problem solving. *Visible Language 16 (3),* 221-238. --- (1987), Cognitive representational systems for mathematical problem solving, in C. Janvier (ed.), *Problems of representation in the teaching and learning of mathematics.* Hillsdale, NJ: Erlbaum, 125-145. --- (1997), Observing mathematical problem solving through task-based interviews, in A.R. Teppo (ed.), *Qualitative research methods in mathematics education, J. Res. Math. Educ. Monograph No. 9,* 40-62. --- (1998), Representational systems, learning, and problem solving in mathematics. *J. Math. Behavior 17 (2),* 137-165.

Goldin, G.A., DeBellis, V.A., DeWindt-King, A.M., Passantino, C.B., & Zang, R. (1993), Task-based interviews for a longitudinal study of children's mathematical development, in I. Hirabayashi *et al.* (eds.), *Procs. of the 17th annual PME conference, Tsukuba, Japan, v. 1,* 197-203.

Goldin, G.A. & Kaput, J.J. (1996), A joint perspective on the idea of representation in learning and doing mathematics, in L. Steffe *et al.* (eds.), Theories of mathematical learning. Mahwah, NJ: Erlbaum, 397-430.

Goldin, G.A. & Passantino, C.B. (1996), A longitudinal study of children's fraction representations and problem-solving behavior, in L. Puig & A. Gutierrez (eds.), *Procs. of the 20th annual PME conference, Valencia, Spain, v.3,* 3-10.

Krutetskii, V. A. (1976), *The psychology of mathematical abilities in schoolchildren*. Univ. of Chicago Press.

Owens, K. (1993), *Spatial thinking processes employed by primary school students engaged in mathematical problem solving*. Unpublished doctoral dissertation, Deakin University, Australia.

Passantino, C.B. (1997), *Making connections: A cognitive analysis of children's problem-solving behavior involving fraction representations and strategies* (2 vols.). Unpublished doctoral dissertation, Rutgers University, NJ. Ann Arbor, MI: UMI 57 (12), No. 9717241.

Presmeg, N.C. (1985), *The role of visually mediated processes in high school mathematics: A classroom investigation*. Unpublished doctoral dissertation, Cambridge University, England. --- (1986) Visualisation in high school mathematics. *For the Learning of Mathematics 6(3),* 42-46. --- (1998), Metaphoric and metonymic signification in mathematics. *J. Math. Behavior 17 (1),* 25-32.

Thomas, N. & Mulligan, J. (1995), Dynamic imagery in children's representations of number. *Math. Educ. Res. Journal 7 (1),* 5-25.