RE-EVALUATING ASSESSMENT IN LIGHT OF AN INTEGRATED MODEL OF MATHEMATICS TEACHING AND LEARNING

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***Abstract**: This is a preliminary report on a study that examined, in practice, an integrated model of mathematics teaching and learning. The paper addresses a combination of two problems—how credible is a key theoretical distinction about learning a new conception and how does that distinction inform teacher's assessment of students' thinking. . I conducted the study as a whole-class teaching experiment in a 3rd grade classroom over a 4-month period. The analysis indicates that the distinction between a participatory and an anticipatory stage is theoretically sound and practically useful in setting the teacher's goals for and activities of assessing students' thinking. The study highlights how the model provides a new way of thinking about the role for and organization of assessment.*

In this paper I present part of a comprehensive study that examined an integrated model of mathematics learning and teaching in a real classroom setting. Examining the model was a multi-faceted task and reporting on it goes beyond the scope of a single article. Therefore, in this paper I address a combination of two problems—how credible is a key theoretical distinction about learning a new conception and how does that distinction inform teacher's assessment of students' conceptual understanding. This focus highlights the interplay between teacher goals of precisely assessing students' thinking and advancing their understanding. I begin with a brief description of the model and what its examination in practice might consist of, then I present the methods used for the study and an analysis of data regarding the combined focus, and finally I discuss the significance of the study.

Conceptual Framework

The integrated model of conceptual teaching and learning of mathematics (cf., Simon et al., 2000; Tzur & Simon, 1999) is an elaboration of the psychological aspect of the emergent perspective (Cobb and Yackel, 1996). The model evolved as a response to a theoretical problem known as the "learning paradox" (Pascaul-Leone, 1976). This paradox is implied by Piaget's fundamental notion of assimilation. If one can only recognize and respond to aspects of reality by assimilating them into existing conceptions, how is one to ever construct new conceptions? The model untangles the paradox by addressing three questions: what is a conception, how is a new conception formed (mechanism and stages), and how can teaching promote formation of intended conceptions. I briefly describe each of the three below.

The primitive unit of knowing in the model is a dynamic compound, a relationship. The relationship is between an activity and its effect(s) (abbreviated as A-E relationship). The unit is not mainly the activity nor mainly the effect(s) but the dynamic compound consisting of both. Terms such as scheme, concept, mental object, cognitive process, and the like are various notions used by different scholars to refer to A-E relationships. In this paper I will use the term conception and A-E relationship interchangeably.

The primitive unit of learning in the model is the mental mechanism of reflection on A-E relationship (abbreviated as Ref*A-E relationship). The term reflection refers to the ceaseless mental comparison between one's goals for and effects of her activities (von Glasersfeld, 1995). Note that from the learner's point of view, Ref*A-E relationship does not need to be directed toward making specific conceptual advances (i.e., it does not imply awareness).

Through Ref*A-E relationship a learner might make a distinction between desired effects she anticipates of an activity and the effects she notices during or after carrying out the activity. This type of distinction, termed the initial phase, is thought of as a critical precursor for forming a new conception. However, the initial phase is not considered a stage because no new A-E relationship has been abstracted, yet. If goal and effects do differ, the cognitive system searches for some new recurrence, that is, a new regularity in A-E relationship other than the regularity—existing conception—that triggered the activity.

Forming new, regular A-E relationships is postulated to occur in three stages. First is the participatory stage. Knowing at this stage is marked by the learner's ability to relate effects and activity only when one is somehow oriented to focus on the activity. Thus, at this stage it is assumed that the learner does not know, spontaneously, to call up the activity for the particular goal. The second stage is the anticipatory stage. In contrast to the participatory stage, the learner, upon setting her goal in a situation, can spontaneously select the activity she newly related to that goal(s) and figure out its effects. Though the effects cannot yet be known immediately, the learner can spontaneously initiate the activity from within the A-E relationship, generating and reflecting upon its effects. The third stage is the reified stage. Unlike in the anticipatory stage, the learner can immediately identify the anticipated effects whereas the activity that generated the A-E relationship fades to the background. To an observer it appears as if the learner uses a type of "idle" knowing, knowing without activity. This misleading appearance is probably one reason why "knowing" is often mistaken with quick recognition of facts. However, an activity of a reified A-E relationship is always implicit and it reappears when, for example, the learner is asked to justify her solution. I use the term reified because the third stage seems consistent with Sfard's (1991) notion of reification. The formation of a reified conception completes a learning cycle such that it can then afford a new distinction among effects, and so on.

Building on the learning process as postulated above and on Simon's (1995) and Tzur's (1999) works, the component of teaching in the integrated model is cyclic in nature and consists of four principal phases:

1. assessing (inferring) learners' current conceptions based on their actions and language;
2. hypothesizing (trying to predict) a learning trajectory for the learners on the basis of what they know, that is, identifying a higher stage and a potential process of change toward that stage;
3. selecting and engaging learners in tasks (problem situations) they are likely to understand and use in service of the intended advance, that is, to assimilate into available conceptions, to set appropriate goal(s), and to initiate appropriate activities;

4. using probing questions and/or comments both to foster reflection on patterns of A-E relationships and to re-assess learners' current conceptions, and so on.

Regarding phase 3 in the cycle, four types of tasks that correspond to the four postulated transformations from an available conception into a new, intended conception are proposed. (Note: The name refers to the intended phase or stage.) The teacher uses initial tasks to foster the use of activities available in current conceptions to make new distinctions among effects of that activity. For example, consider children who already constructed whole numbers at least as an anticipatory relationship between the activity of iterating the unit of one and the effect of having established a composite unit of a certain amount (e.g., 3=1+1+1). Then, using paper strips, the teacher can engage learners in using the repeat strategy (see next) to share a paper strip among a given number of people. The repeat strategy consists of four activities that learners can already initiate and use in sequence: estimating the size of one piece, iterating that piece the desired number of times, comparing the whole produced in iteration to the given one, adjusting (re-estimating) the size of the piece, etc. The repeat strategy is likely to foster learners' initial distinction between a piece that is too short, too long, or exactly the size needed. The teacher uses participatory tasks to orient learners' to reflect on and identify a new relationship between the activity and the newly distinguished effects. In the repeat strategy example, a participatory task can be the question, "Why did you make that piece shorter than the previous one?" Such a question might orient learners' reflection on the magnitude relationship between the activity of adjusting the size of the estimated piece and the uniqueness of that size relative to the whole. Moreover, the learners may notice a new (inverse) regularity between the number of times the piece is iterated and its size (e.g., 1/6 is larger than 1/7 because each of the 6 pieces has to occupy more space). The teacher uses anticipatory tasks to foster learners' abstraction of the participatory A-E relationship into an anticipatory A-E relationship. If learners are to advance from the participatory stage, it is critical that anticipatory tasks do not indicate what activity the learners should use. In the repeat strategy example, an anticipatory task might be, "You received 1/7 of a pizza; can you figure out a way to cut another piece of the same-size pizza so that your friend gets a smaller piece than yours? What fraction of the pizza is your friend's piece?" It is the learner who translates the question into another one, "How could I share a pizza among 7 people?" which triggers the activity (repeat strategy) she was already using prior to the initial phase, which fosters mental reprocessing of the activity and its effects. The learner realizes that she has to iterate the piece more than 7 times and regenerates the magnitude relationships in her thinking. The teacher uses reified tasks to foster learners' abstraction of anticipatory A-E relationship into a new, reified A-E relationship. Unlike at the participatory stage, at the anticipatory stage the learner can assimilate abstract symbols and mentally reprocess the anticipatory A-E relationship. Through further reflection, the anticipatory A-E relationship becomes the signified "object" encapsulated within the symbolic, signifying entity and the construction of the new conception is established at the reified stage. In the repeat strategy example, a reified task might be, "What is bigger, 1/4 or 1/7?" because it calls upon the anticipatory inverse relationship between size and number of pieces and fosters further reflection on and reification of that relationship.

Methodology

I conducted the comprehensive study as a classroom teaching experiment (Cobb, 2000) in the context of fractions. I chose fractions because research (cf. Tzur, 1999) identified a developmental sequence of fraction conceptions that could guide my teaching. In a public elementary school in Israel, I selected a 3rd grade classroom that had not yet received any instruction on fractions. Thus, for all 28 students the following two-part description applied. First, they could reason about whole numbers. Some were also able to engage in part-whole numerical reasoning. For example, they could think of 12 as composed of 5 "ones" and 7 "ones," and, when told that a dozen-egg carton already contains 7 eggs, could figure out how many more eggs would fill the carton. Second, as students' responses to a pre-program, problem-solving questionnaire indicated, none used any fraction conception beyond, maybe, a primitive, generic notion of half. Thus, I could begin fostering in these students a sense of magnitude of, and operation on, unit fractions (1/3, 1/4, etc.).

As a researcher-teacher I collected data by conducting two videotaped lessons a week, for a total of 26 lessons. I taught the lessons consecutively, on Friday and Sunday, because I wanted to have a sense of continuation while allotting the time needed between lessons for analysis and planning. The ongoing analysis after each lesson focused on students' conceptions and whether or not a conceptual change took place. As part of that analysis I conducted audio-recorded conversations with the classroom teacher and 1-2 other teachers who observed every Sunday lesson. On the basis of my reflections on each lesson I created and/or adjusted tasks for the next lesson. Besides the reflective sessions with the teachers, I systematically documented (audio recorded and/or wrote) my reflections about students' evolving understandings and my past and future teaching.

Upon completion of data collection, I will conduct a retrospective analysis of the teaching-learning process. However, teaching is not over, yet. Thus, the analysis in the next section represents a work in progress—my first screening of data from three lessons that are relevant to the combined foci of this paper. This screening included careful reflection on lesson segments (videotapes) of the three lessons and on chunks of transcripts of my notes (written or recorded) regarding my teaching activities and decisions.

Analysis

I did <u>not</u> set the goal to specifically examine the aspect of assessment from the outset. Rather, assessing students' conceptions became a perturbation for me in the practice of teaching. In this section I present how I resolved the perturbation. In the first take at the perturbation my goal of promoting students' learning overruled the goal of making subtle distinctions among students' ways of thinking. In the second take the order was reversed. Through paying attention to considerations of both ways of organizing assessment, a pattern emerges that highlights both the credibility and power (in terms of assessment) of the theoretical distinction between the participatory and anticipatory stages.

<u>First Take at the Perturbation: Promoting Learning Overrules Subtle Assessment</u>. After a few lessons in which the students were engaged in using the repeat strategy, I had a

rough sense that some already formed a participatory sense of the inverse relationship between size and number of pieces while others did not. However, I made several notes that indicate that I became frustrated because of my initial inability to get to know how each student was thinking. For example, I wrote in reflection on the lesson of November 11, 2000:

> I thought, intensively, about the problem: How to overcome the enormous difficulty I have to get to each and every student and observe what they do, to "interview" them on the spot … so I can understand how they think. I realized that I need to give them some task that requires written responses that will give me a more detailed information than I currently am able to gather, information that is structured by the model I use.

Consequently, my planning for the next lesson turned to creating a task that would delineate students' thinking. After several hours of thinking and jotting down ideas for different tasks, I designed a set of 7 questions that I hoped would provide insight into students' current thinking about the conceptual aspect at issue—direction of adjustment in size of the estimated piece (make it shorter or longer). I considered this aspect as the conceptual root of the uniqueness of the size of a unit fraction and of the inverse relationship between size and number of unit fractions in a given whole (cf. Tzur, 1999). For example, questions #1 & #2, and #5 & #6, read as follows (figures are not presented):

> Question #1: In Figure 1 you see a paper strip that Pat tried to share among 3 people and underneath it is the piece Pat used to mark the strip [Note: the piece was a bit short]. Please draw, under Pat's piece, another piece that you think will fit better for sharing the strip among 3 people.
> Question #2: In the second attempt to share the strip among 3 people, Pat used a piece that was longer than the piece in Figure 1. Do you think it was smart to do so? _____ Why? _____
>
> Question #5: In Figure 4 you see a paper strip that Danielle shared equally among 3 people and underneath it is the piece she used to mark the strip. Please draw, under Danielle's piece, a piece that you think will be appropriate to share the strip among 4 people.
> Question #6: In the previous question, did you draw a piece that is shorter or longer than Danielle's piece? _____ Why? _____

On the basis of my rough assessment that many students did not yet form the intended relationship, I planned a two-part lesson: administering the questionnaire and engaging the students in using the repeat strategy some more (share paper strips among 7 and 11 people). Once the questionnaire was ready, I turned to think about ordering the two parts. My notes indicate how the model informed my decision to do the activity before the questionnaire:

> I was aware that this order will not allow me to evaluate anticipatory conceptions, but decided to do it because I thought that otherwise many students would not be able to participate at all.

The note above is important because questions #1 and #5 above, if asked prior to being engaged in the activity, could be used to make a subtler distinction between students who used only a participatory conception of the inverse relationship and those who already used it in anticipation. Although I was not aware of this at the time, the short note indicates that promoting all students' learning of the conception at least at the participatory stage overruled the desire to make a subtler distinction among the more knowledgeable students. In this sense, the model informed my teaching, and the first take at the perturbation, in that it highlighted a local trade-off between two phases of the

teaching cycle, assessing students' thinking and engaging students in tasks that promote their learning. In the case where my rough assessment of the class suggested the lack of even a participatory way of thinking on the part of many students, promoting learning overruled subtler assessment.

Second Take at the Perturbation: Subtle Assessment Overrules Promoting Learning. After the two-part lesson, I read the students' responses to the questionnaire and found that 14 of them knew the direction of adjustment. In my post-lesson conversation with the teachers, I emphasized that due to the order chosen the most I could claim about these students was the formation of a participatory conception. I was aware of and said that there was no way to distinguish if they had formed a higher stage. To promote all students' understanding of the direction of adjustment, we planned to engage pairs of students in using the repeat strategy some more. We paired each of the 14 students whose responses to the questionnaire indicated a participatory conception with another student who seemed not to understand the direction of adjustment. The idea was that both students in each pair could benefit from reflecting on direction of adjustment offered by the more knowledgeable one.

After the next lesson (Friday, 11-17-2000), a short conversation between the teacher and me indicated that both of us assessed, roughly, that all but maybe 2-3 students knew the direction of adjustment at least in a participatory sense. But I was still perturbed:

> The lesson was difficult in terms of my ability to follow and analyze what each child does and thinks. While I was trying to help a pair I had to focus mainly on their execution of the repeat strategy and all the time other children would come and pull my sleeves to get my attention. I trusted that the videotape will give me the data needed for retrospective analysis, but in terms of monitoring and documenting while interacting with the students I was still unable to follow what's going on.

This time, the ongoing perturbation regarding the lack of precision in assessing students' thinking, along with my indefinite sense that most students advanced at least to the participatory stage, turned into a new, model-rooted perturbation. I wanted to devise a tool and organize the lesson so that I could distinguish students who could use the inverse relationship in anticipation from those that could not. Interestingly, this goal was accomplished semi-accidentally, while I was relating my research proposal of conducting occasional interviews with the idea to let the class do silent reading while the teacher and I work with pairs of students. Below are my notes from Saturday, 11-18-2000.

> Last night I went to a concert. During the concert I was intensively thinking about the continual difficulty I have to assess students' thinking during the lesson and about how to use the organization of the lesson for that purpose. [Initially] I was thinking about my research proposal to conduct occasional interviews as part of the data collection methods. Then, something "clicked"—I realized that the idea to work with one pair at a time actually created the possibility to conduct mini-interviews with students … Thus, I immediately began to think what questions will I ask as well as who among the students I would like to interview. I started with a question that seemed promising in terms of making a distinction between participatory and anticipatory thinking regarding the direction of adjustment: "You have a piece that fits, exactly, for sharing a whole among 6 people and now you have to share the whole among 7 people. Show me what will you do."

It took only a short, focused reflection to create a set of 3 questions that would enable me to make the intended distinction and to design a form that would enable me to document

how each student responded to the question. Moreover, I immediately chose a different order than in the previous assessment event. Because students would be engaged in reading and not in the repeat strategy, I would first ask the "hard" question where students are not prompted what activity to use and thus I would be able to distinguish those who used the conception in anticipation from those who did not. Then, I would present the questions that give some orientation. The three questions were:

> Question #1: (The teacher presents a drawing of a paper strip marked into 6 equal parts and the piece "he used" to create the sharing underneath it. He gives the students another piece of paper, and says): Can you mark (or cut) a piece that could serve in equally sharing the paper strip among 7 people? (Pending students' drawing he asks): Any reason why you made it shorter/longer than the "sixth"?
> Question #2: (The teacher puts, underneath the "sixth," two pieces that he prepared ahead of time, one shorter and one longer than the "sixth" and says): Which of the two pieces would fit better to share the strip among 7 people? Why?
> Question #3: (The teacher asks the students to observe how he accurately iterates each of those two pieces and asks): Which piece helped more in sharing the strip into 7 equal pieces? Why?

According to the model, if question #1 is asked after some time during which students did not use the activity that fostered the participatory stage of their thinking (here, the repeat strategy), one can answer it only if one already formed the inverse relationship at least at the anticipatory stage. The reason is that Question #1 does not call upon the size aspect of the relationship; it is the student who must attribute it to the situation. In contrast, Question #2 orients the student to the activity of comparing between sizes of pieces that are shorter or longer than a given piece and hence brings forth the next activities in the repeat strategy sequence, adjustment and iteration. Thus, students who already formed the participatory stage of the inverse relationship could solve Question #2. Question #3 allows an even finer distinction within the participatory stage, by bringing forth the entire activity sequence.

Due to scheduling and time limitations, I conducted the mini-interviews with students over 5 lessons. During those lessons, the class was engaged in non-related tasks while the classroom teacher worked with pairs I already interviewed on a specific way of executing the repeat strategy. Once they used it systematically and equally shared the paper strip among 3 people, she asked each pair: "Next, we have to share the strip among 4 people; should we make the piece shorter or longer than the share of one-out-of 3 people?" In response to her question, all students but one appropriately suggested to make it smaller.

Bearing in mind that only 12 students answered Question #1 in a way that indicated clear understanding of the reason to make the "seventh" shorter than the "sixth," a significant, twofold conclusion emerges. First, organizing the three questions that way resolved my perturbation about assessment. At last I was able to make fine distinctions in students' thinking that corresponded to the first two stages of the model. Second, the very distinction between the two stages was supported by the results of the assessment tool and process that I have designed. In part, this twofold conclusion was made possible because of my model-based choice to give the goal of precisely assessing students' thinking a priority over the goal of promoting students' learning. In turn, this precision allowed for better planning of the next lessons to fit the two sub-groups within the class.

Discussion

In this paper I examined two interrelated aspects of an integrated model of mathematics teaching and learning: the key theoretical distinction between participatory and anticipatory stages and teacher assessment of students' evolving conceptions. The analysis of the researcher-teacher activities to resolve a continual, model-based perturbation and of students' responses to several assessment tools that were devised accordingly supported the theoretical distinction. That is, if students appear to clearly understand a specific conception while using certain activities to solve mathematical problems, but on the next day they revert to lack of understanding, the reason may well be conceptual and not bad teaching or lack of effort on the students' part. For example, about a half of the class reverted to making the "seventh" larger than the given "sixth" (Q. #1, Second Take) even though they clearly knew, while engaged in the activity in the previous lessons, that it must have been smaller. The reason for that change in students' understanding can be explained in terms of competition between two, unevenly formed conceptions. The students have used at least an anticipatory and probably a reified conception of magnitude relationship among whole numbers, but only a participatory conception of inverse relationship between size and number of pieces in sharing situations. Thus, when not prompted for the activity to use, the problem situation was assimilated into the stronger conception of whole numbers and they made the "seventh" bigger than the "sixth" precisely because seven is bigger than six.

The study provided supporting evidence not only to the key theoretical distinction but also to the claim that teaching or assessing anticipatory conceptions is sensitive to orienting prompts. It demonstrated that order of teaching activities matters. Moreover, it provided a way of thinking about the relationship between what the teacher intends the assessment to accomplish and how she should organize it in practice. In particular, the study contributed to better understanding the conceptual goal underneath the potential organization of assessment tools from "hard" to "easy" questions and demonstrated specific ways in which the model can inform teaching (e.g., individual vs. pair work; mini-interviews vs. observation of the entire class, ordering questions). In this sense, the model both provides an articulated "map" of the conceptual terrain to be assessed and guides the teacher's organization and design of assessment tools and activities that fit to where, conceptually, students are.

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