THE CONCEPT OF DEFINITE INTEGRAL: COORDINATION OF TWO SCHEMAS

Bronislaw Czarnocha, Hostos Community College, CUNY
Sergio Loch, Grand View College
Vrunda Prabhu, William Woods University
Draga Vidakovic, Georgia State University

This is a report of a study examining students' understanding of the concept of definite integral. Using APOS, a specific framework for research and curriculum development in collegiate mathematics education, as a guide in investigation, we analyze and interpret students' responses to the interview questions. The analyses of the interviews and the results of other studies indicate that the coordination between the visual schema of the Riemann sum and the schema of the limit of the numerical sequence is necessary for developing a good understanding of the concept of definite integral. Consequently, we give suggestions for didactic and curricular changes when teaching the concept.

In communicating mathematics to students, the professor presents concepts as a combination of (a) his or her understanding (b) the understanding established by the mathematical community, and (c) the pedagogical values the professor incorporates into his or her teaching. Communication however is a two-way process, and hence an important question arises: does the communication initiated by the professor achieve its objective? In particular, in the realm of the definite integral, do students indeed think along the lines defined by the classroom instruction? Is there a lack of understanding of some key concepts? Do modifications need to be made to our communication, our sequencing of topics that lead to the definite integral? What can we incorporate into our teaching that we learn from our current students thinking?

The definite integral is the composition of two distinct constructions: (1) the geometrical one, ultimately based on[1]e principle of exhaustion, and (2) the numerical one of the infinite converging sequences and their limits. The problems encountered in the understanding of the sequences and their limits are quite widely investigated, the understanding of the definite integral has received much less attention, while the combination of sequences and limits leading upto the definite integral has received the least attention. Because of the dependence of the definite integral on the notion of the limit of the sequence, one can suspect that the problems in understanding the limit of a sequence will create difficulties in the understanding of the definite integral as well. One of the main goals of our paper is to demonstrate how this suspicion bears out. Thus, we report in this article, our findings of student understanding of the Riemann sum as revealed in the interviews, the need for a more intensive treatment of sequences and limits of

[1] Our appreciation goes to Ed Dubinsky for his helpful suggestions during numerous conversations.

sequences prior to the treatment of Riemann sums and the justification for this suggestion.

As stated earlier, inspite of being one of the essential concepts developed during the first two semesters of college mathematics, there are comparatively few articles investigating the understanding of the definite integral by students. One of the most interesting works is the article by Orton who points out students' difficulties in using the limit process for their understanding of the concept of the definite integral. The analysis of our data expands and builds upon Orton's findings, presents the evidence of students' successes and difficulties and offers suggestions concerning significant changes in the instruction design.

Literature review

Two articles that deal with pedagogy as relates to the definite integral are by Orton [1983], and Davis and Vinner [1986]. Articles that shed light on student understanding of sequences and limits of sequences and functions are by Sierpinska [1987], Williams [1991], and Cornu[1992].

One of the interesting tools of analysis in mathematics education is that of a cognitive obstacle. According to Cornu [1992] there are the following types of cognitive obstacles: genetic, didactical and epistemological. A cognitive obstacle helps to identify the difficulties encountered by students in the learning process, and to determine appropriate strategies for teaching. Genetic and psychological obstacles arise as a result of the personal development of the student, didactical obstacles arise as a result of the nature of teaching and epistemological obstacles arise as a result of the nature of mathematical concepts themselves. It is equally interesting that according to Cornu, the concept of the limit has been fraught with several major epistemological obstacles in the course of history of which we mention two that are of significance to our work: a) the failure to link the geometry with numbers; and b) The difficulties with the last term, can one reach it or not?

Davis and Vinner [1986] report on a special 2-year Calculus course, in which they included a brief treatment of sequences and limits of sequences. Sierpinska's [1987] study was intended to explore the possibilities of elaborating didactical situations that would help students overcome epistemological obstacles related to limits. She lists several epistemological obstacles encountered by students, related to limits, obstacles that maybe due to a lack of rigor, due to incomplete induction, due to notions that limits only amount to approximating, etc. Orton [1983] reports on a study investigating students' understanding of integration and differentiation. He thinks it would be very unlikely that the introduction of integration can be made easy. He feels that the topic of limits is one of the most neglected topics at the school level, which in turn does not help the introduction of integration.

Framework

The study reported in this article, used APOS [Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K., 1996], a specific framework for research and curriculum development in collegiate mathematics education, as a guide to investigating students' understanding of the definite integral. The premise of this framework is that educators can develop knowledge about students' learning of mathematical concepts by going through a cycle of theoretical analysis, instructional treatment, and observations and assessment of student learning. The initial step in our approach, which we refer to as theoretical analysis, is to hypothesize about mental constructions that a student might make when learning a specific mathematical concept. We refer to this structured set of mental constructions as a *genetic decomposition*. The researcher's own understanding of mathematics along with her or his learning and teaching experiences are the most important components for this step of the framework. Subsequent iterations of the framework lead to an evolving genetic decomposition of the concept and an instructional treatment.

The present study is at the point of analyzing data collected during and after the instructional treatment in the first iteration of the teaching cycle. In the process of analyzing data and answering our research questions, we describe our observations in terms of actions, processes and objects [Asiala, M. et. al., 1996]. Figure 1 shown below represents the initial genetic decomposition of definite integral that was hypothesized by the instructor/researcher and used in developing instructional treatment and interview questions. This genetic decomposition assumed that students would have an object level understanding of functions, partitions, and would develop their understanding of Riemann sums by the usual approach of an action, process and object level progression. It ended by assuming that students would apply a limit schema to obtain a number. It was not clear at that point how this particular limit schema would enfold, and how it would affect the development of the entire Riemann sum schema. Thus, it seems that this particular development in the instruction would have to be determined by the enfolding illustrated by the students.

1. Object level of function
2. Object level of partition
3. Action on a function and a partition.
   Construct one Riemann sum of one function with one partition.
4. Process conception of Riemann sum.
   Coordinate the process of a function and the process of a partition via the Riemann sum formula
5. Object conception of Riemann sum
   Encapsulate 4.
   Variations of the sum (Left, Right, Trap, Mid)

Dependence on n.

6. Action on Riemann sum

   Compare with an area or a solution to a differential equation.

   This is done on a vague, pictorial, intuitive level.

   Improve the approximation.

7. Process on Riemann sum

   Interiorization of 6.

8. Apply limit schema to obtain a number.

   At this point, very few students will have a strong limit schema so it is unclear how the

   concept of definite integral will grow for them. It needs study.

Figure 1: Initial genetic decomposition of definite integral

The setting and data gathering

Data for this research were collected during the fall semester of 1992. The participants were 32 engineering, science and mathematics students who had, during the previous year, taken two semesters of single variable calculus at a large mid-western university. The interview consisted of 10 questions about the concept of integral. On the average, each interview lasted for approximately one hour. The interviews were audio taped and tapes were transcribed by paid student aides.

In this paper we attempt to answer the following three research questions: What is the relationship between the preliminary genetic decomposition and the students' mental constructions of the definite integral? What are the mental constructions that were not made by students? What should the modified genetic decomposition be to accommodate for the possibility of making the required mental constructions?

To answer the above research questions, we will analyze the responses to Questions 4, 6, 7, 9, and 10 from the interviews. Analysis of the remaining items will be presented in other studies. Below are the interview questions on which this study is based.

Interview Question 4. What is the mathematical meaning of $\int_{-3}^{-1} \frac{1}{x^2} dx$.

Interview Question 6. Suppose that an object moves in a straight line at a velocity which is a function of time, $v = V(t)$. Write a formula for the net distance which the object moves starting at time $t = t_0$ and ending at time $t = t_1$.

Interview Question 7. Explain why your formula gives the distance.

Interview Question 9. Suppose now that you have a region S in space which is a body of density ρ which has a different value at different points in the region. Write a formula for the mass of this body.

Interview Question 10. Explain why your formula gives the mass.

Initially the students were given the opportunity to answer each question without prompting. Based on their response the interviewer asked additional questions or provided hints or clarification. The interviewer encouraged any kind of student's response (verbal, written, graphical) that might help to explain her or his ideas.

Data Analysis

In the excerpts of student interviews, we see instances of student thinking that follow the steps outlined in the genetic decomposition, viz., in achieving an object level of function and partition, an action and process level of Riemann sum. The next step students would need to make is to have an object level understanding of the Riemann sum. A student with an object level understanding of the Riemann sums should be able to talk and think about a Riemann sum on a partition of size 2, 3, 4…n and be able to realize that the size of the partition needs necessarily to be finite. However, it is in demonstrating an object level of the Riemann sum that we notice difficulties that are principally related to the limit concept in the following two ways:

1. The limit of the Riemann sum is seen as the infinite sum of the rectangles of small width.
2. The limit of the Riemann sum is seen as the sum of lines, i.e., as the infinite sum of rectangles of zero width. (i.e., rather than the limit of the sum of the areas of n rectangles, students state it as the sum of the limit of the areas of the rectangles.)

The student Kenard reveals the first difficulty. When questioned about what is being done to the Riemann sums in order to make it basically equivalent to the definite integral, he states:

S: *For instance if you are using Riemann sums, you take infinity number of rectangles, you will get almost exactly the same thing because there won't be as much error.*

Jernau when questioned about how he reconciles the two different meanings mentioned earlier, states

S: *Right. Um… well, the Riemann sum breaks this up into n, an infinite number of rectangles. And, it's difficult to use the theory behind it. It is difficult for me.*

Dadgaron reveals the second difficulty in response to the question of how one goes from the Riemann sum to make it equal 2/3, he replies

S: *By making these rectangles infinitesimally small…smaller and smaller, I mean almost until they are a line they are a unit… and then you are just adding up these units and like, the smaller this empty area is the more exact the estimation until you get to a point where there is no empty space to be accounted for and that will give you an exact number.*

In both of the above difficulties demonstrated by students, they (students) sense a correct need for fitting their intuitive tools, be it rectangles or lines, in the region under the curve, which can be pictorially seen as the area under the curve, however, they are unable to connect that area to the numerical sequence of partial sums. The best example of the absence of connection to the concept of the sequence is the following fragment of an A student (Jasarx) :

S: *Um, well the integral is, is um, basically the sum of $f(x_i)$ for i in this case $–3 < i < -1$ – not necessarily integers but every number, every single point between –3 and –1 is going to have an f(x) value…*

I: *Okay.*

S: *...and if you add all of these together…*

I: *Um-hum.*

S: *...you should get the area under the curve, …*

Clearly the student sees area as all the parallel lines contained in it, but in the next sentence where he notices his mistake in the formula he wrote, he immediately moves to the Riemann approach, without however anywhere indicating the presence of the sequence of which that limit is the limit of.

S: *… and the integral is just, um, actually it's $f(x_{i+1})$ - , okay it's f(x) times x(i+1)-x(i). Okay so now you have got, you've got a height which is f(x) times i, which is somewhere between x(i) and x(i+1).*

I: *Um-hum.*

S: *And so that would be your height and then your width would be, or your base would be x(i+1) – x(i), and so if you multiply those together you're gonna get some kind of little area…*

I: *Um-hum.*

S: *...within that section. So if you sum up all of the little areas between –3 and –1 you should get a certain value. Now when you take the integral of that same function, it's still f(x*) if you want to say that, and the x(i+1) - x(i) the integral makes it get --- takes the limit as, takes the limit of that sum as x(i+1) – x(i) goes to zero.*

One might then suspect that an integrated approach which from the start correlates the pictorial representation with the numerical sequence of partial Riemann sums, would provide an answer to students difficulties. Such an integrated approach was investigated by Orton. In his work, Orton identified several key ingredients of the conceptual (or structural) problems students have with the definite integral. One of them was the difficulty "students have with the power of the limiting process in mathematics" and in particular in calculus. The majority of students, who in one of Orton's investigational tasks, obtained initial 5 terms of the sequence Riemann Sums - which were understood as the approximations to the area under the curve $f(x) = x^2$, on the interval [0, 1], appeared to grasp that this sequence consisted of better and better approximations and that it was possible to continue improving them. They were asked whether that sequence could be used to obtain the exact area under the curve. However, he states that the students "quite rightly pointed out that such a procedure would never produce the correct answer, and were unable to state that the limit would provide the answer." We see then that such an integrated approach still does not solve the students' difficulty, it has the effect of allowing students to construct the first few terms of the sequence without being able to see how the sequence could ever converge to the number which is the exact area under the curve.

This is in direct opposition to our students who can see the limit without having access to the sequence that converges to the limit.

Both difficulties appear to be grounded in the same inability to separate the concept of a limit from the last term of the sequence. Our group influenced by the pictorial image of the area would like to take the infinite sum which fits under the given curve, Orton's group on the other hand which also sees the area under the curve as the last term to reach, cannot get to it due to the infinite number of steps required to reach there. Therefore it seems essential to take the precise e-n mathematical definition of the limit of a sequence as a base foundation on which to build the notion of the limit of a sequence and of the definite integral as the limit of partial sums, which bypasses the issue of reaching or not reaching the limit and instead focuses on a process of approach [Sierpinska].

Refining theory with pedagogical implications

Based on the data analysis we suggest that the preliminary genetic decomposition (Figure 1) should be modified as follows:

2. Object level of sequence
9. Schema of the limit of a sequence
   The distance between the term and the limit of the sequence;
   The notion of the measure of the distance
10. Schema of the Riemann sum
11. Coordination between the schema of the Riemann sum and the schema of the limit of the sequence

The numbers in front of the new items added to the initial genetic decomposition denote the position of the entries in the initial genetic decomposition illustrated in Figure 1. We wish to emphasize that the requirement of an 'object level of sequences' is a major change from the existing curriculum in the order of the concepts necessary to understand the definite integral as a limit of the Riemann sum. Currently, in a typical calculus course, the topic of sequences is studied in detail after the concept of the definite integral is studied extensively. Our proposed genetic decomposition requires a certain rearrangement of the order of the topics, with other emphasis as suggested above.

Our article emphasizes the pictorial understanding of the limit of the sequence of Riemann sums while Orton's emphasized understanding of the limit of a numerical sequence. The results of both studies point to difficulties by students when only one of them is seen separately. We conclude that there must be a coordination between the visual schema of the Riemann sum and the schema of the limit of the sequence. The source of students' difficulties can well be in the didactic and curriculum which a) doesn't develop the connection between the two in the design of the curriculum, and b) does not propose an alternative to the second by eliminating the precise defintion eps-N of the limit of the sequence. This definition, the Weierstarss definition of the limit of the sequence was created especially to bypass the problem of reaching or not reaching the limit, as Sierpinska points out. Hence the absence of the instruction of the formal definition is leaving our students on the pre-modern level.

Bibliography

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, e., Mathews, D., & Thomas, K. (1996). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. *Research in Collegiate Mathematics Education*, II (3), pp. 1--32.

Czarnocha, B., Dubinsky E., Loch, S., Prabhu, V., Vidakovic, D. (2001). Conceptions of Area: In Students and in History. College Mathematics Journal, MAA (to appear in May).

Cornu, B. (1992). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 153-166). Dordrecht, The Netherlands: Kluwer Academic.

Davis, R., Vinner, S. (1986). The Notion of Limit: Some Seemingly Unavoidable Misconception Stages. *Journal of Mathematical Behavior*, Vol 5, 281-303.

Dubinsky, E., Schwingendorf, K.E., & Mathews, D. M. (1995. *Calculus, Concepts and Computers*, New York: McGraw-Hill.

Orton, A. (1983). Students' Understanding of Integration. Educational Studies in Mathematics, 14.

Sierpinska, A. (1987). Humanities Students and Epistemological Obstacles Related to Limits. Educational Studies in Mathematics, Vol 18, pages 317-397.

Williams, S. (1991). Models of Limit Held By College Calculus Students. *Journal for Research in Mathematics Education*. Vol 22, No 3, 219-236.