Representations as Conceptual Tools: Process and Structural Perspectives

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Much has been written about the apparent duality of many mathematical concepts as processes and objects. Further, the importance of the representational forms in which mathematical concepts are presented has been analysed. In one of these representations, the symbolic form, the idea of a procept linking process and object forms, has proven a valuable concept. However, this analysis has been lacking from other representations. In this paper we look at the characteristics of the process and object view of some other representations with a view to beginning a classification of interaction with them and give examples to indicate the validity of the classes.

Process and Object

Many mathematical concepts take the form of objects, and these may arise in a number of different ways. Tall, Thomas, Davis, Gray & Simpson (1999, p. 239) have distinguished three types of object construction: perceived objects; procepts; and axiomatic objects. The second of these has been given considerable attention over a number of recent years, with researchers describing in detail both the distinction between the dynamic process and static object view of mathematical concepts, which Sfard (1991), calls an operational and structural duality, as well as the manner in which the former is transformed into the latter in the mind of the learner. Sfard (*ibid*) proposes that processes are *interiorised* and then *reified* into objects, while Dubinsky and his colleagues (Dubinsky & Lewin, 1986; Dubinsky, 1991) talk about processes being *encapsulated* as objects and have imbedded this in an Action-Process-Object-Schema or APOS theory, for the construction of conceptual mathematical schemas. The term *procept,* as used above, arose in the work of Gray & Tall (1994) to describe the use of mathematical symbols to represent a process (which the symbols may invoke) or a concept (which they may represent), or, depending on the context, the viewpoint, and the cognitive aim of the individual.

These theoretical ideas have proven useful, with widespread applications in algebra, calculus, and advanced mathematical thinking, as described by Tall (2000), and others (e.g., Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Clark, Cordero, Cottrill, Czarnocha, Devries, St. John, Tolias, & Vidakovic, 1997).

Representations of Mathematical Objects

We note that in the discussion on the nature of mathematical concepts and their description in terms of process and object, and especially in the case of procept, there has been a firm emphasis on the symbolic representation. This is understandable of course, because of the tremendous power of symbolism, and algebraic symbolism in particular. However, a key component of schemas is the representation of conceptual processes and objects in a number of associated but different ways (Kaput, 1987, 1998). How then do representations other than the algebraic symbolic relate to the process-object conceptualisation of much of mathematics? If this is a valid view of the underlying mathematical concepts then there should be corresponding perspectives for these. In particular, the graphical and tabular representations which are so common in secondary mathematics should be amenable to such an analysis. In the symbolic representation it has been proposed that the object view enables one to take the object and perform an action on it, using it for example in a further process. An example of this is the construction of groups of functions. But this is due to the power of the symbolic notation, mentioned above. What is the corresponding position with tables and graphs, and other representations, and how would we recognise this shift in perspective for them?

We argue here that a representation can be seen as a multi-faceted construction which assumes different roles depending on the way that students interact with it. When an image is on a computer screen, for example, Mason (1992) has suggested that students can be *looking at* the images or *looking through* them depending on the focus of their attention. In this sense we say that students can interact with a representation in at least two different ways, by observing it or acting on it. The observation can be at a surface level, looking at, or at a deeper level, looking through. For example, looking at a representation a student may comment on a property of the representation itself but by *looking through* it students may use it to assist them to notice properties of the conceptual processes or object(s) represented. This is in line with the *property noticing* of Pirie & Kieren (1989), who talk about how images can be examined for specific or relevant properties. An example occurs in the paper by Ainley, Barton, Jones, Pfannkuch, and Thomas (2001), where, looking at a graphical representation on a spreadsheet of five data points, two of the students comment on surface features of the representation, saying, for example, that “It’s a hill” and “It’s like a mountain there” (p. 3). In contrast, if they had been looking through the graphical representation they may have commented on properties of the function, such as its maximum or minimum values.

While this idea of observation of representations is important, in order to build rich cognitive structures more is needed. When he goes beyond such acts of observation of a representation and performs an action on it, *doing* in the sense of Mason (1992), in order to obtain further information or understanding from it, then we maintain that the representation becomes a *conceptual tool* for that student. The metaphor of a tool is appropriate here since it is not the shape of a spade, or its properties that make it a tool, but the actions of a person using it. Such *learning from* activity has been described as *construing* by Mason (1992) and it is this type of activity which gives rise to a *conceptual representation tool*. However, thinking back to the process/object views of concepts, the ways in which a student observes or acts on a particular representation will depend on whether they have a process or an object view (or both) of the concept(s) it represents. For example a student may use a table or a graph and perform linear interpolation on values obtained in order to approximate an intermediate value of the function. Whether the student sees the function as the sum of the discrete results of an input-output process or as a function object may not be clear. In contrast, to be given the graph (or table of values) of a function *f(x)* and being asked to draw the graph of, say, the function *f(x+1),* when there is no specific function given, may require a structural or object view of the function.

Table 1 *Possible Modes of Interaction Between Student and Representation*

Interaction	Concept View	
	Process	Structural/Object
Surface Observation	Process Surface Observation (PSO)	Structural Surface Observation (SSO)
Observation of Conceptual Properties	Process Property Observation (PPO)	Structural Property Observation (SPO)
Action on the Representation	Conceptual Process Representation Tool (CPRT)	Conceptual Object Representation Tool (CORT)

This gives a matrix of six different possible modes of interaction with a conceptual representation, as delineated in Table 1. We have tried to exemplify each of these modes below, choosing to talk about a structural rather than an object view of concepts in this context, since it seems to convey the idea better. One should not get the impression from the discussion so far that because it is the primary source of examples that these concepts are only applicable to the learning of function. An example from group theory may help support this contention. Figure 1 contains a representation of the klein four-group, namely the multiplication table of the set {1, 3, 5, 7} under the operation of multiplication modulo 8.

× mod 8	**1**	**3**	**5**	**7**
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Figure 1. A multiplication table representation of the klein four-group.

A student with a process view of a group as the sum of the combinations of elements under the law of composition may interact with this representation by (for example): spotting that combining elements in the set gives rise to only elements of the set, since these appear in the body of the table (with no reference to closure) (PSO); noticing that every element is self-inverse (PPO); using the table to verify that $ab=ba$ for all $a, b \in \{1, 3, 5, 7\}$ (CPRT). Alternatively the student who has a structural view of a group may interact in quite different ways by (for example): noticing that there is a leading diagonal of ones in the table (SSO); observing that there is a subgroup {1, 3} or that the symmetry of the table means that the group is Abelian (SPO); demonstrating by rearranging the columns of a table (we assume here the necessity of this) that the group is isomorphic to the group represented by a multiplication table of the matrices:

$\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ (CORT, action to compare two object structures).

It needs to be said at this point that it is often difficult, even with student interviews, to be sure whether a particular student has a process or object view of a given concept. However, we can still say whether he/she is acting on the representation and using it as a conceptual tool rather than simply observing it or even for abstracting properties. In cases like this we may call such usages *surface observation (SO), property observation (PO)* and *conceptual representation tool (CRT),* leaving out the process or object distinction.

Experimental Exemplars

Space does not allow us to exemplify from our research each one of the six categories here, but we will try to present sufficient examples to allow the reader to see how they arise. What we have found in our recent research is that students often interact with function graphs and tables of values in a *process-oriented* manner (Thomas, 1994), as series of discrete pairs of values or points. In Figure 2 we see an example of a student who, in trying to solve the equation $4–2x = 3x–1$ has a process perspective of the two given tables, stating (translated from Korean) "because the values have (–) on the table". This is a surface property of the representation and so the interaction is a PSO.

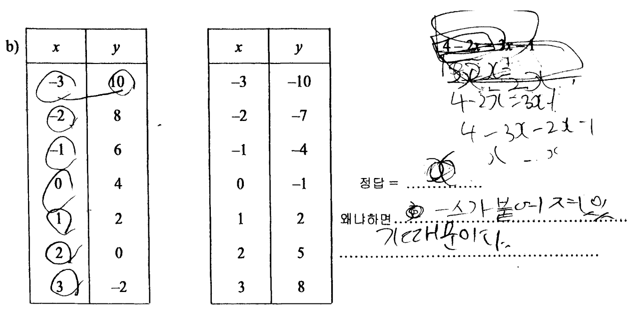

Figure 2. Student A's process surface observation (PSO) use of tables.

This student has been asked to use the table as a tool to solve a linear equation, but because of his process-oriented perspective he was unable to relate the input-output values to the solution of an equation. In contrast, some students, after they had experienced the different related representations using calculators were able to solve these equations. Student B for example, who although unable to solve the equation correctly in the algebraic representation, making several errors, is able to use the tabular representation as a conceptual tool (CRT) to solve the equation (Figure 3), stating that the solution is x=1, y=2 because "when x=1 they are consistent with each other." We cannot be sure whether he has a process or a structural view of the tabular representation.

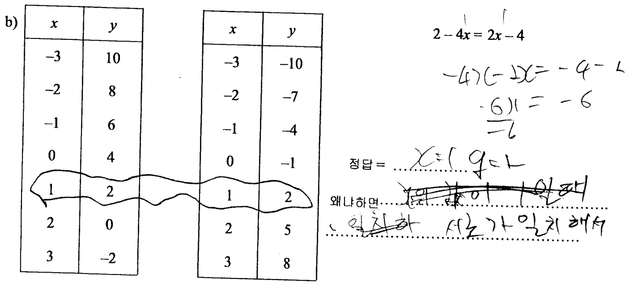

Figure 3. Student B's use of tables as a conceptual representation tool (CRT).

Other students were also using the tables as a CRT were able to give similar reasons for the same type of solution. We have also come across evidence that some students appear to see graphs in process terms, as a sequence of discrete points which happen to be joined with a curve. In Figure 4 we see an example of such thinking.

Figure 4. Student C's graph used as a conceptual process representation tool (CPRT)

Here student C, asked to solve $x^2 - 2x = 3$ using the graph, has acted on the graph in a process manner (CPRT), executing an inter-representational shift to produce a table of a discrete set of 6 integer-valued points. The answer happens to be incorrect in this case simply because of an error finding $f(-1)$.

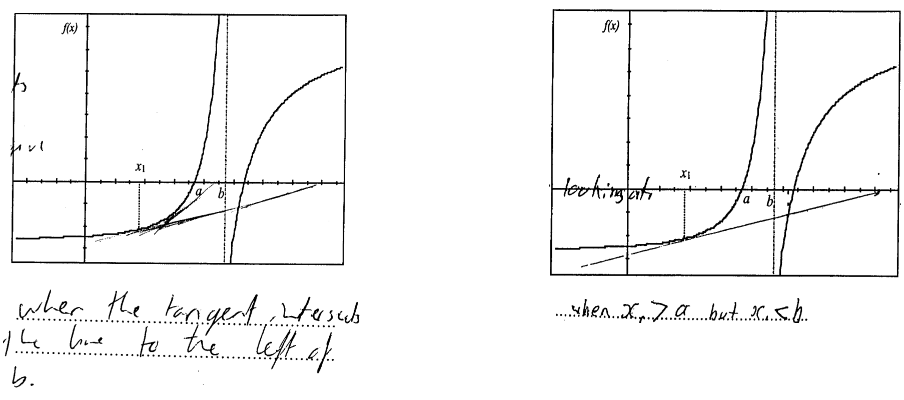

Figure 5. Two examples of using a given graphical representation as a CRT.

In Figure 5 students D and E employ the graphical representation of a function, whose symbolic formula is not given, as a CORT in order to explain conceptual ideas, namely, how the successive approximations in the Newton-Raphson method approach a root a, and when x_1 is a suitable first approximation for the root a of $f(x)$=0. These students are thinking conceptually using the graph and have no need to carry out a process. Students have also been able to operate conceptually on a graphical representation involving the relationship between the area under the graph of a function and its symbolic representation when the function undergoes a transformation which can be described as parallel to one of the axes, i.e., $x \to x \pm k$ or $f(x) \to f(x) \pm k$. Figure 6 shows two examples of students' work where they have interacted with a graphical representation they have constructed to answer a question with a symbolic presentation, namely:

If $\int_1^3 f(t)dt = 8.6$, then write down the value of $\int_2^4 f(t-1)dt$.

The first student has drawn separate graphical representations of the unknown function $f(t)$ (possibly using $y=t^2$), and the second has put both graphs on the same axes, but they have both acted on these by clearly marking the area represented by the symbolic definite integrals and operating on this area as a structural object, to answer successfully the conceptual question about the transformation. Once again These students are using the graph conceptually with no need to carry out a process. These are therefore examples of CORT interactions with the graphical representations.

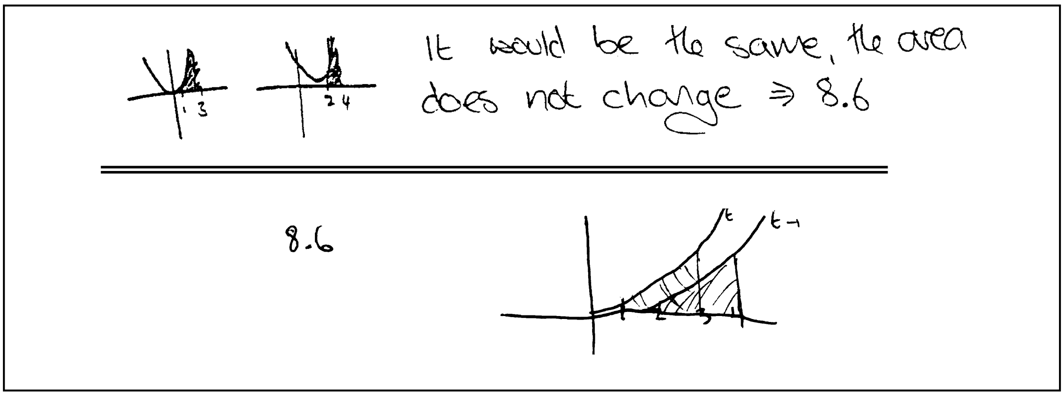

Figure 6. CORT use of graphs for area conservation under a transformation $t \to t-1$.

Discussion

We believe that the start we have made here on a classification of interaction with various mathematical representations of concepts has potential benefits for the teaching of mathematics. It is the teacher that is the key to benefits emanating from any theoretical position, and the types of interaction proposed above suggest possible ways in which teachers could address student learning. One approach they could be encouraged to consider is to construct lessons which build meaningful uses of different representations of concepts into modelling activities based on real world problems. Lesh (2000) has suggested that helping students to be able to construct conceptual tools that are models of complex systems in such a way that they can mathematise, interpret and analyse using these tools is a key goal of mathematics teaching. We strongly agree with his further statement that "… representational fluency is at the heart of what it means to "understand" many of the more important underlying mathematical constructs" (p. 74). Such 'fluency' includes the ability to interact with these representations, using them as conceptual tools, but doing so, as Kaput (1998, p. 273) suggests being aware of the potential "inadequacy of linked representations and the strong need to provide experiential anchors for function representations."

A second consideration for teachers of ways to approach the building of representational fluency is to consider use of the graphic and super-calculators, since the primary representations needed in schools arise naturally, in a dynamically related way, in the context of these machines (Kaput, 1992). For example, in many classrooms students may initially learn about functions only through the symbolic representation, becoming immersed in algebraic manipulations and equation solving. Only some time later will they approach the graphical solution of equations. By then it may be much harder for many students to build inter-representational links, if they are constrained to a process view of function. They may employ a surface, procedural method of solving equations graphically by drawing the graph of each function and reading off the x–value(s) of the point of intersection without engaging with deeper relationships of the four different representations: algebraic, tabular, ordered pairs and graphical. To build rich relational schemas based on these external representations, it seems a good idea that, where possible, students should interact with the sub–concepts of one–to–one, independent and dependent variable, etc. in each representation in close proximity, exploring the links between them.

The third implication for teachers is one of assessing diagnostically the type of thinking with which their students are approaching representational use. The term *versatile thinking* was used by Tall and Thomas (1991) to refer to the complementary combination of the sequential/verbal-symbolic mode of thinking and the more primitive holistic visuo-spatial mode, in which the individual is able to move freely and easily between them, as and when the mathematical situation renders it appropriate. However, with the theoretical stance we have presented here we can now enlarge this concept of *versatile thinking* and say that this would include the ability to move between the PSO, PPO, and CPRT modes of interaction with any representation and the SSO, SPO and CORT modes of interaction as and when each is considered applicable. Thus a useful goal for teachers would be to try assess the extent of their students' *versatile thinking* and aim to assist them to build it further so that they are not limited to a purely process approach to mathematics, important though that is.

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