# DEEP STRUCTURES OF ALGEBRA WORD PROBLEMS: IS IT APPROACH (IN)DEPENDENT?

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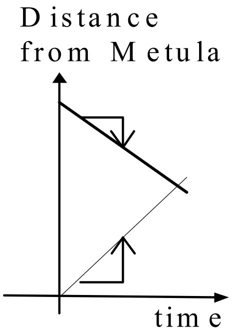
*Research of word problems in arithmetic as well as in algebra has long stated that meaningful categorization that should lead to problem solving with understanding takes into account the deep structure of the problem. But is the ~~-~~"deep structure" approach independent? In this article we discuss the possible impacts of the domain of functions on the ability and the difficulty to solve motion problems in algebra. We studied students who participate in the VisualMath curriculum (a function approach to algebra in grades seven to nine). What might be a pedagogical obstacle was encountered in work of students learning by this functions-based approach. It suggests that construction of equations as a comparison of two functions is harder when the equation cannot explicitly describe the situation model unless another variable or a parameter is introduced. Thus, the emphasis on algebraic symbols being meaningful modeling language rather than solely the objects of manipulations may set a new cognitive sequence that removes previously known obstacles and introduces new ones. The finding that the categorization of word problems seems to be approach-dependent represents a more general view about emergent research of curricular change.*

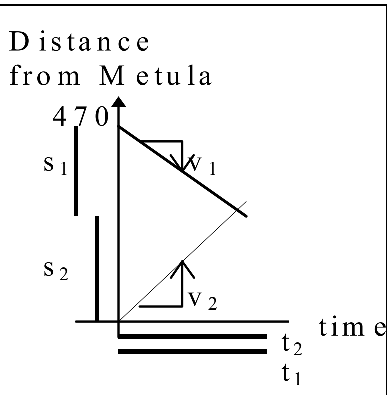
## DEEP STRUCTURES OF WORD PROBLEMS

A word problem is examined at two levels of abstraction: the *quantitative structure*, which describes arithmetic operations and relations among symbolic or numerical entities, and the *situational structure*, which describes relations among physical properties of the entities within a story problem (Bednarz and Janvier,~~1998~~1996; Hall et al., 1989; Shalin and Bee, ~~1985~~1987; Yerushalmy and al, 1999). Each one of these two structures determines the deep structure of a problem and therefore might be responsible for the problem's difficulty. Several studies have shown that performance on solving word problems is a result of an interaction between an individual and a problem, so it needs to be understood in light of both, the knowledge and skills the individual brings to the solution process and the nature of demands imposed by the problem (Sebrecht, 1996; Bednarz and Janvier, ~~1998~~1996; Nesher and Hershcovitz, 1994).

The traditional approach to algebra centered on symbolic manipulations, solving for unknowns and structures of algebraic expressions. This strand usually navigated the solving of algebra word problems to concentrate on the quantitative structure, by assigning symbols to unknown quantities and arranging them in an algebraic relation to answer the questions posed (Nathan and Koedinger, 2000). On the other hand, studies demonstrated that pre-algebra students often think of a problem purely in situational terms (Baranes, Perry and Stigler, 1989). Yet another finding is that students representing story problems in formal problem model terms tend to disregard the meaning associated with the equation of the values and thus may provide solutions which are physically or situational implausible (Silver, 1988). Observations of highly competent solvers show their skills in using the situation model and the quantitative model within a problem (Hall et al., 1989) in an integrated fashion. The coordinated use of situation model with one's formal problem model appears to be fundamental to problem solving with understanding, in a variety of domains (Koedinger and Anderson, 1990).

**Problem 1**
Metula and Eilat are 470 km apart. A truck and a cab started traveling at the same time towards each other. The cab traveled from Metula to Sdom at an average speed of 80 km per hour. The truck traveled from Eilat to Sdom average speed of 56 km per hour. Both drivers reached Sdom at the same time.
a. How long after starting their journey did the two vehicles reach Sdom?
b. How many kilometers did each of them travel until they arrived to Sdom?




$v_1$=56 km per hour
$v_2$=80 km per hour
$s_1+s_2$=470 km
$t_1=t_2$ hours

Figure1a: An example of a typical rate problem to which we refer in this paper

Figure1b: Qualitative graph of the situational structure of problem1

Figure1c: Qualitative graph of the situational structure and the quantitative structure within the same model

The concept of function provides a set of terms for mathematical modeling that can turn routine symbolic work into model construction to describe the problem's situation in numerical, graphical, and symbolic terms (Heid, 1995; Yerushalmy, 1997; Nemirovsky 1996). More specifically, situations involving a single variable can be described in a two-dimensional Cartesian system by use of a triad of quantities, two quantities representing the independent and dependent variables respectively, and the third representing the rate of change of the second quantity as a function of the first. Constant-rate problems are frequently present in algebra texts. Such problems (as in figure 1a) are usually solved by means of DeRT tables as an organizing structure to the construction of an equation. When experiencing a new function-based algebra curriculum based on intensive use of functions' graphing tools (Yerushalmy and Gilead, 1999), we learned that functions can turn solving such common algebra rate problems into a meaningful activity that emphasizes both the modeling and manipulation skills. The situation can be modeled graphically by two linear functions of time. Semi-quantitative sketches of functions (figure 1b) or graphs that accurately describe the given quantities form a visual presentation of the situational structure and may help students to express correctly in algebraic terms the quantitative relations. A set of given quantities, unknowns, and some constraints, formed by arithmetic operations between pairs of unknowns, form the quantitative structure of the problem. Thus the graph (as in figure1c) models the situational structure of a problem and the quantitative structure (Chazan, 1993).

Studying our students' problem-solving attempts, we observed unexpected achievements and unexpected obstacles. Often we were surprised to see how successful was the solving of problems, considered complex by the traditional approach, and at other times how very similar problems were far harder. A first attempt to investigate further what seemed to be a phenomenon, was a systematic analysis of the domain of constant rate problems according to the range of possible interactions between quantitative structures and situational structures (Gilead, 1998; Yerushalmy et al., 1999; Yerushalmy and Gilead, 1999). We suspected that differences between constant rate problems would not be fully explained by differences in the quantitative structures or by the differences in situational structures, but by the interaction between the two. We also wondered whether this complexity is an epistemological or a didactic obstacle and whether it is an outcome of the function approach. An empirical study was designed to test our conjecture on the effect of the situational structure, the quantitative structure, and their mutual correspondence on students' performance in solving constant rate problems. The work presented in this paper is part of this research.

## METHOD

Following the analytic categorization we prepared for this study 21 different rate problems, each representing one of the types of problem as defined in Yerushalmy et al. (1999). A questionnaire containing four problems, randomly selected for each participant, was administered in 17 different classes of ninth graders, who had already learned to solve rate problems, by a function approach within the VisualMath[1] curriculum.

These same problems were administered in another 17 classes of ninth ~~graders~~graders who had learned to solve rate problems by the equation-unknown strand, (which below we call the non-functional approach). About 100 students of the two populations solved each problem. In this paper we refer only to four combinations of two situational structures and two quantitative structures (Table 1).
The pair of problems 3 and 4 have the same quantitative structure of the type: $v_1$=given, $v_2$=given $t_1=t_2$, $s_1-s_2$=given, while the pair of problems 1 and 2 share Another quantitative structure of the type: $v_1$=given, $v_2$=given $t_1=t_2$, $s_1+s_2$=given.

[1] Visual Mathematics is an intensive technology function approach algebra curriculum developed by the Center of Educational Technology, Israel.

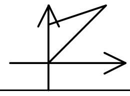
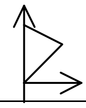
| Situational structure / Quantitative structure |  |  |
|---|---|---|
| $v_1$=given , $v_2$=given, $t_1=t_2$<br>$s_1-s_2$=given | Problem 3 | Problem 4 |
| $v_1$=given , $v_2$=given, $t_1=t_2$<br>$s_1+s_2$=given | Problem 2 | Problem 1 |

Table 1: The four combinations of the two situational structures with the two quantitative structures and the number of the problem that presents each combination

The pair of problems 2 and 3 share the same situational structure:  (two vehicles driving in the same direction at the same time). Problems 1 and 4 both share another situational structure:  (two vehicles driving towards each at the same time). Students' performance was scored as correct or incorrect. A correct model was either formed symbolically (a correct equation), graphically (reading the solution from accurate graphs), or numerically (reading the solution from a table of values).

## FINDINGS

Following the traditional observations of algebra word problems, we started by analyzing what effect the quantitative structure had on students' performance in each population. For the function approach students, no significant difference was found between the two quantitative structures of problems 3 and 4, and problems 1and 2 (Table 2).

| Quantitative structure | **Function approach** Percentage of students who built a correct model | **Non-functional approach** Percentage of students who built a correct model |
|---|---|---|
| $v_1$=given, $v_2$=given, $t_1=t_2$<br>$s_1-s_2$=given | 64%<br>n=187 | 44%<br>n=208 |
| $v_1$=given, $v_2$=given, $t_1=t_2$<br>$s_1+s_2$=given | 65%<br>n=206 | 63%<br>n=208 |

Table 2: Percentage of correct solutions in the two quantitative structures

This was not the case for the non-functional approach population (correctness × quantitative structure significant: $p\{\chi^2(13.948;1)\}=0.001<0.01$). Among these students the problems with the quantitative structure $v_1$=given, $v_2$=given, $t_1=t_2$, $s_1+s_2$=given seemed easier (with 63% success) than the problems with the quantitative structure $v_1$=given, $v_2$=given, $t_1=t_2$, $s_1-s_2$=given (with only 44% success). The latter structure is reported as problematic in other studies (Mayer, 1982; Clement, 1982) because of the relational proposition involved in it (one traveled 45 km more than the other). The higher success (64% against 44%) with this problematic structure within the function approach population might be explained by their use of graphs of functions to describe the situation. This visual representation of the processes might have helped to state correctly the algebraic relations among quantities and form a right algebraic model (Hall et al., 1989).

A similar analysis was conducted for the situational structures of the pairs of problems 2,3 and 1,4 (Table 3). No significant differences were found in the two populations.

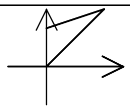
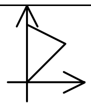
| Situational structure | **Function approach** Percentage of students who built a correct model | **Non-functional approach** Percentage of students who built a correct model |
|---|---|---|
|  | 63% n=205 | 54% n=208 |
|  | 64% n=188 | 53% n=208 |

Table 3: Percentage of correct solutions in the two situational structures

However, Table 4, which presents the correct solutions to the four possible combinations of the two situational structures and the two quantitative structures, reveals that some combinations were easier than others for the function approach population. Problems 1 and 3 seemed to be of the same level of complexity (89%) and significantly easier than problems 2 (39% success) and 4 (33% success).

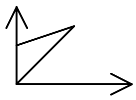
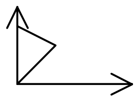
| Situational structure / Quantitative structure |  |  |
|---|---|---|
| $v_1$=given ,$v_2$=given, $t_1=t_2$, $s_1-s_2$=given | 89% n=103 | 33% n=84 |
| $v_1$=given,$v_2$=given, $t_1=t_2$, $s_1+s_2$=given | 39% n=102 | 89% n=104 |

Table 4: Percentage of correct solutions to the four problems by the function approach students

We assume that the combination of the specific correspondence of the quantitative structure and the situational structure of problems 1and 3 were responsible for the results. For each function in this case, a point and a slope are given, which make it possible both accurately to graph and explicitly to describe the linear function symbolically. We termed such combinations between the situational structure and the quantitative structure *canonical* (Yerushalmy and Gilead, 1999). The situational model of problems 2 and 4 can only be sketched, not drawn by accurate graphs. Thus the symbolic model (the equation) cannot be fully described by the given quantities when phrased according to the situational model. A symbolic representation would have to use a parameter or another independent variable in addition to the time-independent variable, and the solution cannot be determined by straightforward comparison of two functions G(x)=F(x) as it can be determined in the canonical problems. This quality of the four rate problems is detailed in Figure 2.

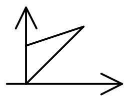
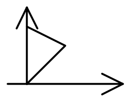
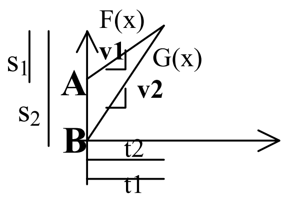
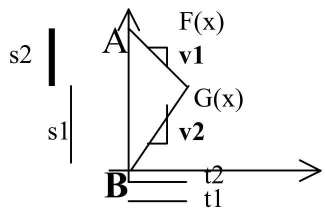
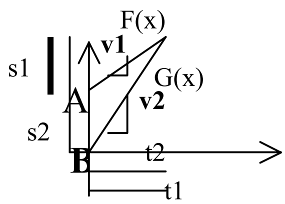
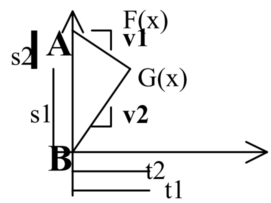
<table>
<tr><td>Situational structure<br>Quantitative structure</td><td></td><td></td></tr>
<tr><td>$v_1$=given<br>$v_2$=given<br>$t_1=t_2$<br>$s_1-s_2$=given</td><td>Problem 3<br><br><br><u>Canonical</u><br>F(x): determined by point A and slope $v_1$<br>G(x):determined by point B and slope v2</td><td>Problem 4<br><br><br><u>Non-canonical</u><br>F(x): point A is not determined<br>G(x):determined by point B and slope v2</td></tr>
<tr><td>$v_1$=given<br>$v_2$=given<br>$t_1=t_2$<br>$s_1+s_2$=given</td><td>Problem 2<br><br><br><u>Non-canonical</u><br>F(x): point A is not determined<br>G(x):determined by point B and slope v2</td><td>Problem1<br><br><br><u>Canonical</u><br>F(x): determined by point A and slope $v_1$<br>G(x): determined by point B and slope v2</td></tr>
</table>

Figure 2: The contribution of each quantitative structure to information about the two functions in each visual situational structure

Analyzing the performance of the function approach students according to the canonical and non-canonical terms, we found (table 5) significant differences ($p\{\chi^2(117.222;1)\}=0.001<0.01$).

| Type of combination | **Function approach** Percentage of students who built a correct model | **Non-functional approach** Percentage of students who built a correct model |
|---|---|---|
| Canonical | 89 % n=207 | 50% n=224 |
| Non canonical | 36% | 57% |

| | | |
|---|---|---|
| | n=186 | n=192 |

Table 5: Percentage of correct solutions to canonical and non-canonical problems.

Differences among the non-functional approach students (table 5) were not found significant (p $\{\chi^2(2.209;1)\}=0.137>0.05$).

Performance on canonical problems was significantly (p $\{\chi^2(73.473;1)\}=0.001<0.01$) better (89%) within the function approach population than within the population of the traditional solution approach (50%). However, performance on non-canonical problems was significantly (p $\{\chi^2(17.687; 1)\}=0.001<0.01$) better (57%) within the population of the traditional solution approach than within the function approach population (36%).

Thus, the multiple representations that ~~students were taught to use, as~~ is one of the foundations of the function's approach students were taught to use, was especially helpful in problems ~~that~~ in which the equation could be explicitly derived from the situational model. It was less helpful, and we conjecture that it was even an obstacle, where the symbolic model (equation and solution) could not be directly derived from the situational model. An informal review ~~that we conducted~~, of different "function approach to algebra " texts that we conducted~~texts~~, revealed that the majority of constant rate word problems are of the canonical type, while in the traditional approach texts ~~of~~ canonical and non-canonical types are interwoven. The findings have implications for the use of graphic tools (e.g; Graphic calculators) that most sequences of function approaches to algebra suggest, as an exploratory support for solving word problems. These tools might prove useful and helpful only for the canonical problems.

## SUMMARY

The paper discussed the possible impact of the domain of functions on the ability and the difficulty to solve rate problems in algebra. Some of the problems (the canonical ones) turned out to be significantly easier by the function solution approach than by the traditional solution approach. The non-canonical problems proved more difficult within the function solution approach. This finding that the categorization of word problems seems to be curriculum-dependent may represents a more general challenge for emergent research on curricular change. New curricula may help to eliminate known pedagogical obstacles, but may also generate new unexpected ones. And new curricula may have to revise the previously used set of tools for analyzing students' knowledge.

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