The Role of Mathematical Beliefs in the Problem Solving Actions of College Algebra Students

Victor V. Cifarelli
Department of Mathematics
University of North Carolina at Charlotte

Tracy Goodson-Espy
School of Education
University of Alabama at Huntsville

This paper reports on the results of a study of the beliefs of College Algebra students. Subjects came from College Algebra classes at two universities in the southern United States. A total of 115 subjects participated in the study. Data sources included both a mathematical beliefs and attitudes survey instrument and on-going individual interviews conducted with 25 of the students. Drawing from the episodes of two students, Brad and Carrie, the analysis demonstrates and explains how the students' mathematical beliefs about formal algebraic concepts influence and sustain their problem solving actions.

Introduction. Among four-year universities, the number of students required to enroll in College Algebra classes forms a critical mass that presents unique challenges for the mathematics faculty whose mission it is to provide quality instruction. The large numbers of College Algebra students enrolled in these classes can be traced back to the 1970s, when remedial enrollments increased 72% as non-science academic programs such as Nursing and Business looked to mathematics departments to provide the necessary mathematics preparation for their students (Leitzel, 1987). In the most recent survey of the mathematics preparation of incoming students, the National Center for Education Statistics reported that 34% of entering freshmen at two-year public colleges were required to take remedial coursework in mathematics as were 18% of the freshmen enrolling at public four-year institutions (NCES, 1995). Other sources have placed similar figures much higher (Watkins, 1993). Because College Algebra serves as the core mathematics requirement for many majors, both universities and community colleges are looking for innovative ways to address the needs of College Algebra students.

The research conducted on the College Algebra population includes studies which surveyed the mathematical beliefs of these students (see for example the work of Peskoff, 1998), studies which have documented the fragmented conceptual understandings that many of these students possess (Carlson (1997), and studies which examined the effectiveness of specific instructional strategies (Underwood Gregg and Yackel, 2000; Yackel and Underwood, 1998). However, there have been no studies examining how the mathematical beliefs and conceptions of these students influence the ways they conceptualize mathematical situations and pose mathematics problems to solve. The role of mathematical beliefs in the evolution of mathematical activity needs to be documented and explored.

Purpose and Theory. The purpose of the study was to examine the beliefs and conceptions of College Algebra students, with the view that their mathematical conceptions and beliefs interact to influence their cognitive actions in mathematical learning situations. Drawing from the work of Cooney, Shealy, and

Arvold (1998), we focus on the learner's beliefs as mental structures which aid his/her interpretations in mathematical situations. According to Schoenfeld (1985), the mathematical beliefs of students help constitute their "mathematical world view" (Schoenfeld, 1985, p. 157), and hence play a crucial role in the ways they "see" the mathematical problems they face. This view is compatible with Vergnaud's (1984) notion that exists a formal connection between the learner's mathematical beliefs and conceptual actions; he asserted that problem solvers often demonstrate their "mathematical beliefs-in-action" as they solve problems, and that these beliefs serve them as conceptual models upon which they can develop successful solution strategies (1984, p.7). Given these theoretical underpinnings, the study examined the role played by the students' mathematical beliefs in the evolution of their mathematical problem solving activity.

Methods. Subjects came from College Algebra classes at two universities in the southern United States. A total of 115 subjects completed a mathematics beliefs and attitudes survey developed by Yackel (1984). While the survey would yield a snapshot of sorts of the students' mathematical beliefs, we wanted to observe the reasoning activity of individual students as they solved mathematics problems. Hence, 25 of the students participated in a series of individual teaching interviews, which occurred bi-weekly and lasted about 40 minutes each. Each interview included approximately 20 minutes where the students solved algebra tasks given by the researchers; during the remaining time, the students introduced their own problems and questions.

Analysis. The analysis proceeded as follows. First, the survey of beliefs and attitudes was compared with students' activities in the interviews. Next, the interview data were examined through protocol analysis. The video-taped recordings were examined to identify instances where significant conceptual structuring activity appeared to occur. This enabled the researchers to focus on episodes of novel activity, and make inferences about the constructive role played by the subject's mathematical beliefs in the evolution of their conceptual knowledge. In addition to the video protocols, transcripts of the videos, paper-and-pencil records, the researchers' field notes, and the subjects' written tests were examined and used to develop case studies.

Given the space limitations here, we will only mention highlights of the analysis of the survey data, and then devote the remainder of the paper to episodes from the student interviews. Briefly, our survey results are consistent with what other researchers have found regarding the nature of mathematical beliefs and its impact on performance (Frank, 1986; Sackur and Drouhard, 1997; Schoenfeld, 1985). For example, 90% of the students demonstrating high-level achievement in the classes demonstrated flexible mathematical beliefs, viewing mathematics as a tool of their reasoning that is supposed to make sense to them, and that the teacher's way of solving math problems represents only one of many possible solutions. In contrast, 88% of the students demonstrating low-level achievement in the classes appeared to have more rigid beliefs about mathematics, viewing mathematics as a collection of rules and tricks, where the teacher determines what is correct and the student's goal is to imitate the actions of the teacher.

In this paper we focus on the mathematical activity of two subjects by examining episodes that illustrate the significant interplay between the students' beliefs, their conceptions, and their demonstration of mathematical structure through their problem-solving activities. We will discuss these episodes in terms of 1) how the students conceived of and interpreted their problems initially; 2) the complexities of their mathematical ideas; and 3) how they worked through the dilemmas and difficulties they faced as they solved their problems.

<u>An Interview with Brad</u>: Brad was a first-year Business major who had taken College Algebra the previous semester and earned a grade of D. He was repeating the class, a practice common among College Algebra students, because he needed a grade of C to satisfy his academic major.

During the first interview, Brad worked a series of tasks that involved simplifying radicals and applying the laws of exponents. After completing the tasks, Brad asked a question from the current homework on radicals. Brad had tried to simplify the expression $2\sqrt{50}+12\sqrt{8}$ using the laws of radicals, and he was concerned that his answer, **34**, did not agree with the answer given in the book, $34\sqrt{2}$. The interviewer asked Brad to re-work the problem at the blackboard.

> **Brad**: *I've worked it out twice but I didn't get the answer that's in the back of the book. First thing I do is look at radicals and see if I can simplify anything, just to drop one of the radicals. And um ..., you can't break these down into terms that can't so ... 50 will break down into 2 and 25, which are both perfect squares (sic), so that's what I went ahead and did.* ***(Writes*** $2\sqrt{25*2}$ ***)*** *Some people like to break them up, I'll keep them together. And 8 is not a perfect square either, but I know that 2 and 4 are (sic), which are factors of 8, so I went ahead and wrote that down* ***(Writes*** $12\sqrt{2*4}$ ***) (Re-writes entire expression)***

$$2\sqrt{50}+12\sqrt{8}=2\sqrt{25*2}+12\sqrt{2*4}$$

Brad's retrospective reporting of how he tried to solve the problem demonstrated an overall understanding of the task -- that he could both re-capitulate and monitor his prior activity in an objective manner. In addition, while Brad invoked an appropriate strategy, his problem involved making sense of a discrepancy between his answer and that given in the back of the book.

> **Brad**: *And from here I just go ahead and take the square root of this,* ***25*** *, which would bring the* ***5*** *out front, which would leave me ...* ***2·5*** *and go ahead and bring the* ***2*** *out which would be* ***1****, right ? ... or now it'd just be a* ***2****, right ?...****(stares in space, rolls eyes)*** *... and then plus* ***12*** *then ... that that'll just be* ***1*** *and bring out a* ***4****, which will be* ***2****, and we multiply by* ***2****, ...* ***2 x 5*** *will be* ***10****, and* ***12 x 2*** *will be* ***24****, which will leave you with* ***34****, but that's not what the book got.*

Brad's solution is summarized below (#1-4). Step #3 includes Brad's erroneous action, $\sqrt{2}=1$*)*.

Brad's Solution

1. $2\sqrt{50}+12\sqrt{8}$
2. $2\sqrt{25*2}+12\sqrt{2*4}$
3. $2x5x1+12x1x2$

Brad's hesitation in asserting that he could simplify $\sqrt{2}=1$ ("go ahead and bring the **2** out which would be **1**, right ?") indicated that he was becoming aware of the probable source of his problem, that $\sqrt{2}$ may not simplify to 1 as he had previously thought.

Brad: *The book got* $34\sqrt{2}$. *I can't figure where* $\sqrt{2}$ *is ?*

Interviewer: *It looks awfully close, only thing is the* $\sqrt{2}$ *there in the answer. Why don't you look back at an earlier step and see if there's some place where there could've been a* $\sqrt{2}$ *and maybe it got lost in the shuffle when you reduced things. Where do you think the* $\sqrt{2}$ *might be ?*

Brad: ***(reflects)*** *... There and there.* ***(points to*** $\sqrt{25*2}$ ***and*** $12\sqrt{2*4}$ ***)***

Interviewer: *So, what did you do at that point in the process ?*

Brad: *I just took the square root of* ***25****, which was* **5** *and the square root of* ***2****, ...* ***(long reflection here)*** *... that's not perfect! ..., yes, it's perfect, ... yeah for some reason, I cannot ...* ***(realizes he has a problem here, but tries to work it out)***

Interviewer: *So the question is, is* $\sqrt{2}$ *perfect ?*

Brad: *... no, it's not, is it. You know what I was getting confused with ? ... is because that* ***(writes*** $\sqrt[2]{2}$ ***)*** *and for some reason I thought I could cancel* ***(cancels 2s in expression*** $\sqrt[2]{2}$ ***,e.g.,*** $\sqrt[\cancel{2}]{\cancel{2}}$ ***)****. Maybe* <u>*that's*</u> *where I got lost, that has to be it, because there's no other place.*

Interviewer: *So why don't you fix it from this point on ?*

Brad: *OK, so just go from this line ?* ***(goes back to his board work and starts with*** $2\sqrt{25*2}+12\sqrt{2*4}$ ***)*** *Um.* $2*5\sqrt{2}$. *Still gonna keep the* $\sqrt{2}$ *and it's gonna be plus* ***12****, um* $\sqrt{4}$ ***is 2****, still have* $\sqrt{2}$ *there. Then we go ahead and multiply that to be* $\sqrt{2}$, $5*2\sqrt{2}$ *plus* $12*2\sqrt{2}$*, that stays there.* ***(writes*** $10\sqrt{2}+24\sqrt{2}$ ***).***

Brad: (**several seconds of reflection**) *I guess this is just like* ***10X*** *+* ***24X****, the* ***X*** *stays the same and you just go ahead and bring down the radical. Then* ***24*** *and* ***10*** *is* ***34 (writes*** $34\sqrt{2}$ ***)*** *O.K. That's what I was doing. That's the kind of mental lapse I'll have, that right there. ... that's crazy, for some reason it didn't register with me on the homework And that's the kind of crazy thing I do ... crazy little careless mistakes like that. It kills me on the test. I usually catch it on the homework, I checked it twice.*

We believe that Brad's episode is noteworthy for the following reasons. First, Brad was able to distance himself from his prior activity and objectively review, monitor, and then report results to the interviewer. College Algebra students are seldom able to engage in such retrospective analysis of their actions. That Brad was able to demonstrate such a grasp over his actions indicates both the robust nature of his conceptions (his knowledge of what he needed to do) and the strength of his convictions about how these types of problems are to be solved. He systematically set about to simplify the radicals (#2-3) and never wavered from his belief that his overall reasoning was sound -- he knew what he needed to do to solve the problem with the radicals and could carry out and evaluate the efficacy of his actions. Second, Brad's inability to self-diagnose and correct his erroneous idea about cancellation of radicals ($\sqrt[\cancel{2}]{\cancel{\mathbf{2}}} = \mathbf{1}$), suggests that his misunderstandings were deep-rooted within his flow of continuous action. While Brad could "see" an overall structure of appropriate solution activity to carry out, he had great difficulty isolating the source of error even after repeated attempts. It was only with the intervention of the interviewer's questions that Brad became aware of the error and set about to correct his solution accordingly.

<u>An Interview with Carrie</u>: Carrie was a 2nd year student whose performance in the class was consistently in the B to upper C range. Carrie's responses on the beliefs survey indicated that she believed mathematics to be difficult because in order for one to be successful solving problems, one must remember many rules and procedures. She indicated that she thought mathematics was important for many careers but that she personally took mathematics courses only because they were required. She also indicated that she thought some people were naturally better at mathematics than others but she strongly disagreed when asked if mathematical ability was determined by gender. During the latter part of the first interview, Carrie introduced a rather difficult complex fraction problem from homework that had puzzled her. Her solution is summarized below as a series of simplifications (#1-5) of the original problem.

Carrie's Solution

1. 2. 3. 4. 5.

$$\frac{m-\dfrac{1}{m^2-4}}{\dfrac{1}{m+2}} = \frac{m-\dfrac{1}{(m-2)(m+2}}{\dfrac{1}{m+2}} = \frac{\dfrac{m(m-2)(m+2)-1}{(m-2)(m+2)}}{m+2} = \frac{\dfrac{m^3-4m-1}{(m-2)((m+2)}}{\dfrac{1}{m+2}} = \frac{m^3-4m-1}{m-2}$$

<u>Summary of activity</u>. Carrie began her work on this problem by factoring $\mathbf{m^2-4}$ (#2)**.** Immediately thereafter, Carrie came to her first major decision - was this a division problem? Initially, Carrie stated that she did not know what to do with the denominator, **1/(m+2)**. In describing her source of indecision, it appeared that the numerator posed the more immediate problem for her. Carrie stated that she wanted to work on the numerator using the least common denominator which she identified very quickly as **(m+2)(m-2).** While Carrie had some difficulty describing what she wanted to do, her actions indicated that she understood the process for combining rational expressions. She correctly combined the terms in

the numerator (#3); however, she also altered the denominator from **1/(m+2)** to **m+2**. The interviewer intervened with a question and the subsequent episode served as a second major decision point for Carrie as she solved her problem.

> ***Interviewer:*** *How did you get this in the denominator (**points to m+2)**?*
>
> ***Carrie:*** *Do I apply the same LCD to this part or do I do it separately? Basically I get the LCD which is that [**points to (m-2)(m+2)**] and so all it is going to be is 1. (m+2), it's already there, so it's like..one.*
>
> ***Interviewer:*** *You keep saying one, I'm not sure what you mean.*
> ***[The interviewer inferred that Carrie was mentally dividing out (m+2)]**.*
>
> ***Carrie**: (**pause**) O.K. see my LCD for this part, 1/(m+2)?*
>
> ***Interviewer**: Yes, what are you going to do with it?*
>
> ***Carrie: (pause)** Here it is simplified?*
>
> ***Interviewer:*** *Yes.*
>
> ***Carrie:*** *Should I leave it as it is?*
>
> ***Interviewer**: Yes. What should you do now?*
>
> ***Carrie**: Well, you don't want to mark it out (**Indicates cancellation in the numerator**). So I want to multiply it out.*

The interviewer's interpretation of Carrie's activity was that she had confused the two common methods for simplifying complex fractions and was trying to apply both methods simultaneously. In her first attempt (#3), Carrie appeared to be trying to mentally multiply both the numerator and denominator with the LCD, **(m+2)(m-2)**. In the subsequent attempt, she considered the denominator separately and determined that her simplification of the denominator was incorrect. Carrie then returned to the numerator and addressed the issue of how to multiply **m(m-2)(m+2).** She noted that she understood how to apply the "**FOIL** method"[1] but she wasn't sure if she should multiply by the **m** first.

After the interviewer suggested that she could multiply in any order that she wished, Carrie wrote: **(m^3 - 4m - 1)/(m-2)(m+2)** (#4) and then mused as to how this answer should be written in relation to the rest of the problem. At this time, Carrie reached the third major decision point of her problem solving process, when she again reflected as to the kind of problem she was faced with. She immediately declared that it was a division problem and began to work on it using the invert and multiply method. Carrie hesitated for a moment as she considered whether or not she should try to factor **m^3 - 4m - 1**. After deciding against factoring, she divided out the common factor of **m+ 2** and wrote her final answer, **m^3- 4m - 1/m-2**.

Carrie's activity indicated that she had possession of some basic mathematical tools that many students at this level have not yet mastered. Carrie could factor

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[1] The **FOIL** method refers to a memory device used by algebra students in the U.S. to remind them how to mutliply together a pair of binomials **--** **F**irst, **O**uter, **I**nner **L**ast.

polynomials, combine rational expressions, and simplify algebraic expressions. However, on the basis of her survey responses and interview data, we claim that her “mathematical world view” (Schoenfeld, 1985, p. 157) is procedurally based. For example, in utilizing her rules to simplify the complex fraction, she demonstrated solution activity that ultimately led to results that did not make sense to her. In order for her to resolve the confusion regarding what she perceived as similar solution paths, she was unable to mentally coordinate the two methods, one against the other, and determine which one to apply. Rather, she needed to choose one of the paths and physically carry out the process. She appeared unable to mentally carry out a process and evaluate the results it would yield. Finally, we noted that while Carrie immersed herself within the problem, she sometimes became lost inside the details of specific sub-tasks. We contend that one reason for this is that Carrie does not see mathematics as a world of connected mathematical ideas. Survey data revealed that she believes that “mathematics consists of many unrelated topics” (Yackel, 1984).

Conclusions. We posit that the experiences of Brad and Carrie are somewhat typical of College Algebra students. Such students enter college with a collection of mathematical rules, procedures and rigid expectations concerning what it means to do mathematics. As a result, the mental structures they invoke to help them organize and direct their mathematical actions are often fragmented. For example, memorizing the definition of a linear equation may help students recognize when they have a linear equation; however, it does not ensure that they will be able to mentally reflect upon, critically examine, and choose from among potential solution strategies. While memorizing definitions and rules is an important part of learning mathematics, it is not sufficient for the development of such reflective activity. For students such as Brad and Carrie, instructional practices that merely review and reinforce procedural tasks are not likely to benefit their mathematical development. Rather, these students need to face mathematical tasks that present dilemmas for them, the resolution of which contributes to their evolving awareness of algebraic concepts and, hence, to their evolving mathematical knowledge. Our continued work in this area is directed at developing instructional activities of this nature.

References

Cooney, Thomas J., Shealy, Barry E., and Arvold, Bridget (1998). Conceptualizing the belief structures of preservice secondary mathematics teachers. *Journal for research in Mathematics Education*, 29(3), 306-333.

Carlson, Marilyn P. (1997). Obstacles for college algebra students in understanding functions: What do high-performing students really know? *The AMATYC Review*, 19(1), p. 48-59.

Frank, Martha L. (1986). *Mathematical beliefs and problem solving*. Unpublished doctoral dissertation, Purdue University.

Leitzel, Joan (1987). Improving school-university articulation in Ohio, *Mathematics Teacher*, *79* (1), pp. 1-7.

Sackur, Catherine, and Drouhard, Jean-Philippe (1997). Algebraic expressions and equations: An example of the evolution of the notions. In Erkki Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 112-119). Lahti, Finland: Program Committee.

Peskoff, Fred (1998). Coping with mathematics anxiety in college: A statistical analysis. In Berenson, S.B., Dawkins, K.R., Blanton, M., Coulombe, W.N., Kolb, J., Norwood, K., & Stiff, L. (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 637). Columbus, Ohio: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Schoenfeld, Alan H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

U.S. Department of Education-National Center for Education Statistics (1995*). The condition of education 1997, indicator 26: Remedial education in higher education institutions*. URL: http://www.nces.ed.gov

Vergnaud, G. (1984, April). *Didactics as a content oriented approach to research on the learning of physics, mathematics and natural language*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans.

Watkins, Anne, (1993). A survey of two-year college mathematics programs: The boom continues. *The AMATYC Review*, 14(2), p. 57.

Underwood Gregg, Diana, and Yackel, Erna B. (2000). Supporting students' conceptualization of algebraic expressions and operations using composite units. In Maria L. Fernandez (Editor), *Proceedings of the Twenty Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol.1, 147-152). Columbus, Ohio: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Yackel, Erna B., and Underwood, Diana (1998). Developing an imagistic basis for understanding binomial multiplication. In Berenson, S.B., Dawkins, K.R., Blanton, M., Coulombe, W.N., Kolb, J., Norwood, K., & Stiff, L. (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 179-186). Columbus, Ohio: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Yackel, Erna B. (1984). Mathematical belief systems survey. Purdue University, West Lafayette, IN.