HOW DO MATHEMATICS TEACHERS (INSERVICE AND PRESERVICE) PERCIEVE THE CONCEPT OF PARABOLA?

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The study described in this paper examined mathematics teachers' (inservice and preservice) knowledge regarding the concept of parabola. The participants (33 preservice and 21 inservice) were asked to perform two tasks: in the first one they were given four different verbal definitions of sets of curves, and for each definition they were asked to sketch a curve, which they believed, compatible with the definition, and to describe its properties. In the second task the participants were asked to sketch a Venn-Diagram in order to describe the logical connections between the four sets of curves, which were formed by the four definitions that appeared in the first task. All the definitions concerned the parabola.

The results show that only a few possess a full concept image concerning the parabola and thus a few of them are capable of perceiving the parabola in its algebraic as well as in its geometrical contexts or to identify links between them.

INTRODUCTION

In many countries students in ninth or tenth grade learn how to solve quadratic equations, and become familiar with quadratic functions and their graphic representations. In other words they get acquainted with the parabola as the graph of **quadratic function** i.e., as an **algebraic entity** (though they are not given any formal definition of the parabola). Later on, the parabola appears as a **geometrical entity** in analytic geometry. Although the concept is exposed in its algebraic as well as in its geometrical contexts, teachers often neglect the connections between the two, and do not initiate a discussion about the difference between the concepts in the two contexts.

THEORETICAL BACKGROUND

During the process of learning a certain concept one builds in mind a **concept image** and a **concept definition** (Tall & Vinner, 1981). A concept image is the "*total cognitive structure that is associated with a concept*" and a concept definition is the "*form of words used to specify that concept*". One might hold a concept definition that does not coincide with its mathematical definition or a concept definition that is not necessarily linked to his or her concept image. As a consequence, there is a gap between the mathematical definition of the concept and the way one perceives it. According to Hershkowitz (1990) individuals who possess poor concept images use a few prototypical examples of the concept while considering that concept. They tend to reject as examples figures that do not coincide with those prototypes, because they base their judgment upon visual properties. Individuals who possess somewhat richer concept images base their judgment upon more prototypical examples plus their mathematical properties. They try to apply the properties to the figures they are dealing with, and reject those that seem not to match them. Individuals who possess full concept images hold a wide variety of examples connected to the concept together with their properties. These individuals are able to make correct judgments based upon the analysis of the properties.

Vinner (1991) emphasized the fact that a good learning process is one that integrates concept images and concept definitions, and thus enables to distinguish between examples, counterexamples, and nonexamples of that concept.

Regarding our study, representing the parabola in both contexts induce the creation of two separate concept images and two separate concept definitions, which are different from one another. It is essential to create links between the perception of the parabola as an algebraic entity and its perception as a geometrical entity in order to create a full concept image. Otherwise students may not unify the two into one concept image and never get the complete one.

Teachers should strive to help their students to create those links. Vinner (ibid.) points out the importance of students' experience and the examples of a concept they are requested to deal with. These experiences are crucial for the formation of concept images. Since teachers' instructional foci are constrained by their own mathematical conceptions (Lloyd & Wilson, 1998, Guitiérrez & Jaime,1999), then only if the teachers themselves possess a full concept image of a certain concept, they would be able to convey it to their students. Research (e.g. Stump, 1999) had revealed the existence of a wide gap between the implementation of the recommendations for making changes in mathematics education (NCTM, 1989,1991) and the actual practice. Part of it can be referred to the use of traditional textbooks, but there is no doubt that the implementation of the intended curriculum depends mostly on the teachers' knowledge. This view motivated our study.

This paper describes the results of a study, focused on the ability of preservice and inservice teachers of mathematics to interpret various definitions that are connected with the concept of parabola and on their ability to identify links between them.

RESEARCH DESIGN

Twenty-one inservice mathematics teachers, each having at least ten years of experience in teaching high-school mathematics, and thirty-three preservice teachers who are in their third year of learning towards B.Sc. in mathematics education, participated in the study. The participants were divided into six separate groups (three of the inservice teachers and three of the preservice teachers).

Two main research questions were addressed:

1. What kind of concept image do inservice and preservice teachers of mathematics hold regarding the parabola?
2. What are the differences, if any, between the way in which the preservice and inservice teachers perceive that concept?

In order to answer these questions the participants were asked to perform two tasks. In the first one they were given four different verbal definitions of sets of curves. The first definition was the geometrical definition of the parabola, and thus compatible for every parabola. The others were algebraic definitions of subsets of the parabolas. For each definition the participants were asked to sketch a curve, which they believed, compatible with the definition, and to describe its properties. In the second task the participants were asked to sketch a Venn-Diagram in order to describe the logical connections between the four sets of curves, which were formed by the four definitions that appeared in the first task.

The definitions given in the first task were:

1. **Set no. 1:** λ_1 is an element of set no. 1 iff: Given a line *l* and a point *F* not on the line, λ_1 is the locus of points in the plane that their distance from the point *F* equals their distance from the line *l*.

2. **Set no. 2:** λ_2 is an element of set no. 2 iff: λ_2 is a graph of a quadratic function of the form: $y = ax^2 + bx + c$, where $a \neq 0 ; a,b,c \in R$.

3. **Set no. 3:** λ_3 is an element of set no. 3 iff: λ_3 is the graph of an implicit function of the form $y^2 = 2px$, where $p \neq 0$, $p \in R$.

4. **Set no. 4:** λ_4 is an element of set no. 4 iff: λ_4 is a graph of a function which its pattern is a product of two non constant linear patterns.

All the participants worked individually on the two tasks. Later on an instructed discussion was held, guided by the researchers. During that session the participants could raise questions, wonders and thoughts in a form of a dialogue or a conversation. Each discussion was tape-recorded and used for further analysis.

RESULTS

The first task

The participants' responses to the first task were first classified and then ranked as poor/non concept images, partial concept images and full concept images, based upon Hershkowitz's (1990) research. Table 1 shows the types of answers obtained, their classification and ranking, and the distribution of the answers in each group (inservice/preservice). The average level of the inservice teachers' performance was slightly higher then that of the preservice teachers' performance. Regarding the first task, it seemed like 38.64% of the preservice teachers possess full concept images as compared to 48.81% of the inservice teachers. Concerning the category that was ranked as "partial concept images" it was found that both groups possess the same concept images.

	Poor/non Concept images	Partial concept images	Full concept images
	Definition no. 1		
	A geometrical shape different from parabola (line, circle, ellipse)	Related to parabola but at the same time related to a function and depended on an axis	A parabola that is not depended on an axis
PS (N=33)	18 (54.54%)	7 (21.21%)	8 (24.24%)
IS (N=21)	0 (0%)	9 (42.85%)	12 (57.14%)
	Definition no. 2		
	—	—	A graph of a quadratic function
PS (N=33)	0 (0%)	0 (0%)	33 (100%)
IS (N=21)	0 (0%)	0 (0%)	21 (100%)
	Definition no. 3		
	A graph of non-quadratic function	Referring only to p>0	Referring to p>0 and p<0
PS (N=33)	8 (24.24%)	23 (69.69%)	2 (6.06%)
IS (N=21)	0 (0%)	14 (66.66%)	7 (33.33%)
	Definition no. 4		
	Two lines	Parabola without any constrains regarding the number of intersection points with x-axis	Parabola with at least one intersection point with x-axis
PS (N=33)	8 (24.24%)	23 (69.69%)	2 (6.06%)
IS (N=21)	0 (0%)	14 (66.66%)	7 (33.33%)
	Averages		
PS (N=33)	30.30%	31.06%	38.64%
IS (N=21)	0%	51.19%	48.81%

Table 1: Types of answers, their classification and ranking (preservice = PS, inservice = IS)

The second task

The most striking result was the fact that a high percent of the participants (39.39% of the preservice teachers and 57.15% of the inservice teachers) did not sketch any diagram. The most common mistake among the inservice teachers who tried to sketch a suitable Venn-Diagram was the misplacement of set no. 4 (23.81% of them had such a difficulty). The most common mistake of the preservice teachers (48.48%) stem from their inability to interpret correctly definition no. 1 or to identify links between that definition and the others. The results are summarized in table 2.

	Venn-Diagram			A sketch which is not a Venn-Diagram	No Sketch
	Correct diagram	Difficulties with the placement of the set of curves, which satisfies definition no. 4	Difficulties with the set of curves, which satisfies definition no. 1		
PS (N=33)	1 (3.03%)	1 (3.03%)	16 (48.48%)	2 (6.06%)	13(39.39%)
IS (N=21)	2 (9.52%)	5 (23.81%)	1 (4.76%)	1 (4.76%)	12(57.15%)

Table 2: Distribution of the results obtained for the second task (preservice = PS, inservice = IS)

ANALYSIS AND DISCUSSION

Table 1 and table 2 show that the inservice teachers' performance was better then that of the preservice teachers' on both tasks and especially on the first one. It has also been found that both groups shared similar difficulties and misconceptions (except for the poor concept images related to the first task). These findings are consistent with other studies focused on preservice and inservice teachers' mathematical knowledge (e.g. Hershkowitz, 1989, Stump, 1999, Guitiérrez & Jaime, 1999).

The first task

The results reveal a strong and clear tendency towards conceiving the parabola as an algebraic entity. It seems that **definition no. 2** is the closest one to the participants' concept images. Hershkowitz (1989) refers to that as "the prototype phenomenon". Hershkowitz (ibid.) had found that when children build their concept image they often use a prototypical example they have in their mind. They base their judgment on that prototypical example and try to impose its properties on other examples of that concept. As was mentioned, the initial examples of the parabola the students are encountered with are graphs of quadratic functions. Such graphs are elements of set no. 2. It seems that this can explain the fact that the inservice as well as the preservice teachers used those examples as prototypes. The way the participants imposed these prototype examples' properties are expressed in their responses to all the other definitions: **Definition no. 3** should have been well known to all participants since it is an integrated part of high school analytic geometry curriculum. Yet, many preservice teachers (24.24%) could not identify the obtained curve or analyze its properties. They attributed the curve function properties. Among the inservice teachers we found a remarkable phenomenon – most of them referred only to the case in which $p>0$. This finding is standing in a contradiction to their responses to definition no. 2. Those responses included lists of properties designating the various possibilities for each of the coefficients that appeared in the pattern ax^2+bx+c. It seems like this can be attributed to the fact that the common algebra books, though deal with the two possibilities, refer only to the possibility of $p>0$ while exploring the properties of the obtained curve. Since the inservice teachers have been using those books for years, it is reasonable to assume that they have developed a prototypical view towards that curve. In their responses to **definition no. 4** most participants in both groups did not mention any constrains regarding the number of intersecting points between the obtained graph and the x-axis. This finding is consistent with former findings mentioned above regarding the perception of the parabola as a prototypical algebraic entity. According to that perception, if two linear patterns are being multiplied the result has to be a quadratic pattern and thus the obtained graph is a parabola, and vice versa. In the responses to **definition no. 1**, many participants (21.21% of the preservice teachers and 42.85% of the inservice teachers) tended to exhibit again their algebraic perception, and associated the parabola the properties of a function. Among the preservice teachers there exists a fairly large group (54.54%) that was unable to identify that definition as a definition of a set of parabolas, and as a consequence they sketched different curves. The main evident difficulty is the participants' inability to identify links between the geometrical and the algebraic representation of the parabola. The mental formation of those links is not obvious since, as was mentioned, each representation bears its own concept definition and concept images. If the prototypical example is an algebraic one, there is a low probability that a full concept image would be acquired without any deliberate intervention. Evidence to the absence of mental links was also obtained from the discussions that were held. The vast majority of the inservice teachers designated that they use both definitions (in accordance to the grade levels). But at the same breath they admit that "*the geometrical definition is embarrassing since it enables us to get a parabola that is not a graph of a function*". Others express their conflict by saying that "*the parabola seems like an algebraic entity that sometimes makes problems*" or "*it is time for us to recognize the fact that we, as teachers, also perceive the parabola as a graph of a quadratic function*". The preservice teachers justified their poor performances by saying "*we didn't learn that subject*" or "*our teachers didn't discussed the subject with us*". Those "excuses" are supporting the research findings, since teachers that do not posses a full concept image cannot convey it to their students.

The second task

Table 2 shows that both groups had great difficulties in sketching a proper Venn-Diagram. 57.15% of the inservice and 39.39% of the preservice teachers did not even sketch any diagram. We believe that coping with that task caused them a certain conflict or embracement, and therefore they have chosen not to expose their lack of knowledge, though the responses were all anonymous.

Sketching a Venn-Diagram requires the ability to identify logical links between concepts. Unless one possesses a full concept image he or she would not be able to sketch that diagram correctly. Relying solely on the analysis of the first task we could have wrongly deduced that a large portion of the preservice and inservice teachers (38.64%/48.81% respectively, as seen from table 1) held a full concept image regarding the concept of parabola. The results described in table 2 point out to the fact that only 1 (3%) preservice and 2 (9.5%) inservice teachers had succeeded in sketching a proper diagram. We can confidently declare that all the participants were acquainted with Venn-Diagrams, since while they were asked to sketch a diagram describing the set of quadrilaterals all the participants could easily do this. Thus it can be concluded that the demonstrated difficulties can be attributed only to their deficiencies regarding the concept image of the parabola. One of the inservice teachers summarized that difficulty as follows: "*it was hard to tell what drags what, since they all designate parabolas, they are all the same. It is just a matter of locating the axis correctly*".

How can the wide gap between the levels of the participants' performance in both tasks be interpreted? As we know, definition no. 1 is compatible for every parabola, while the other three definitions describe only subsets of the set of parabolas. In addition, definition no. 1 forms a geometrical condition while the others form algebraic ones. Naturally, while one deals with a specific concept, he or she may use the possessed concept image and concept definition regarding that concept. While considering the concept of parabola the situation is slightly more complicated. One can consider it either from a geometrical point of view or from the algebraic one without identifying links between them. Thus, it is not always possible to decide whether one possesses a full concept image or not only by examining the interpretation he or she gives to each definition separately. Doing that might produce insufficient evidence. The concept of parabola cannot be fully understood unless the learner identifies links between the two concept images and associate them under the same cognitive structure. Creating the links between the two concept images is an **action** one has to perform in his or her mind. This analysis leads us to complete our discussion using the A.P.O. theory (Dubinsky & Lewin 1986). According to that theory the acquisition of an insight regarding a certain mathematical concept can be characterized by three sequential levels -"action", "process" and "object". Conceiving a concept as an action enables one to refer to it only in associate with its definition. Conceiving a concept as a process makes it possible for one to slightly discharge from that conjunction. At the advanced level one perceives the concept as an object and thus he or she is capable of performing manipulations on it. Producing the mentioned links is such a manipulation. Its performance is possible only as a part of one's ability to perceive the concept of parabola in both contexts (algebraic and geometrical) as an object. Performing the second task successfully demands the possession of a wide concept image, one that integrates both aspects (the geometrical and the algebraic) of the parabola.

Finally, we would like to highlight the strength of using Venn-diagrams as a tool for identifying weather one conceives a certain mathematical concept in a high level or not. While sketching a Venn-Diagram it is necessary to relate to the concept images as objects and to perform manipulations upon them. The extent of which one can successfully build a Venn-Diagram can be used as an indication to the level in which he or she perceives the concept.

CONCLUSIONS

From our study it is clear that inservice and preservice teachers do not possess a full concept image regarding the concept of parabola. Other studies (e.g. Hershkowitz, 1989, Guitiérrez & Jaime, 1999, Stump, 1999) had shown that they also do not possess full concept images regarding other concepts such as the altitude of a triangle or a slope. It seems reasonable to believe that there are plenty of other concepts in which preservice as well as inservice teachers do not possess full concept images. Additionally, we could see from our study that it cannot be assumed that the teachers' experience would influence their ability to develop full concept images by their own.

As a consequence many doubts should be raised regarding the teachers' ability to implement the reform's recommendations.

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