REALISTIC MATHEMATICS EDUCATION RESEARCH: LEEN STREEFLAND'S WORK CONTINUES[1]

Leen Streefland

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On this appropriate occasion of the 25th annual meeting of PME, at the Freudenthal Institute where Leen Streefland carried out his work, this Research Forum describes and commemorates some of the aspects of the Realistic Mathematics Education research in which he was engaged, and illustrates projects in which that work is being continued in The Netherlands and in the USA. Core constructs that Streefland's legacy addresses include the centrality of the learner, rooting of learning in prior or current experiences of the learner, mathematics education research as a developmental process, learning communities and interactive learning environments, learning from history, didactising and guided reinvention in developmental mathematics courses, argumentation and discourse in classroom mathematical practices, social learning trajectories and the role of teachers in guided reinvention.

Introduction: A focus on experiences – my experience with Leen. Willi Dörfler

It has been very good luck for me to have the opportunity for enjoying a close cooperation with Leen Streefland. We got to know and appreciate each other in the context of PME. When I was entrusted with editing Educational Studies of Mathematics (ESM) I decided to share this demanding task. It was absolutely clear to me that Leen had to be my first choice and I was very happy that he agreed. Thus we had six years from 1990 until 1995 jointly dedicated to the journal and to the enhancement of maths education research. Over this time, of course partly together with Gila Hanna as the third editor and with the members of the editorial board, Leen and I had a fruitful and productive exchange of ideas and opinions about questions like: main issues of mathematics education research, basic orientations, criteria for quality of research, role of theories in maths education, policy of the journal. Mostly these questions and issues were discussed with respect to submitted papers and thus they were not just esoteric deliberations but had a strong relevance for concrete action.

It is against this background of shared experiences that I will now turn to some traits of the personality of Leen which could be found in his personal life, in his academic life and in his so highly valuable work in mathematics education. Leen`s interest in and concern for people had a strong influence on what he did, what he said, what he wrote, in short on how he lived his life. Foremost there was the feeling that you really matter as a person and a human being which you had when sharing your time with Leen. This was never an abstract or detached interest, say, just in your academic work but it took into account the complex living conditions, the emotions, fears and wishes of the respective other. One can read this general attitude from many letters by Leen to authors of papers submitted to ESM. In those he tried to establish a communicative basis on which to negotiate about the content, style or quality of the paper. Leen never forgot that it always is a concrete person who writes and that a judgement on a paper implies a judgement on the author. This should not be misinterpreted as a tendency to lowering standards. Quite to the contrary, it was the struggle to guide authors in matching high standards.

The concern for people be they pupils, authors, colleagues or friends in my view most prominently resides in Leen`s own work in mathematics education. This research in the context of Realistic Mathematics Education (RME) genuinely and seriously puts the learner into the center. And again it is not an abstract or epistemological subject from which maths education is stipulated to start. The subjective, individual and personal experience of the learner is on the one hand the background and the basis for all (mathematical) learning. On the other hand learning mathematics according to RME is to be organized by making new experiences possible, by engaging students in reflections on their personal experiences in mathematical ways of thinking. This reflects a deep respect for the individual student and his or her faculty for building up mathematical meaning provided adequate experiential situations are offered. It also expresses a view on mathematics as originating out of human experiences and their description, organization, structuring and planning. That this nowadays is a broadly accepted stance can be attributed to the insistence with which Leen and other representatives of this school of thought have expounded their position: mathematics is relevant for the future lives of our students and can be experienced as being meaningful by them. But for this to occur the widespread separation and isolation of school mathematics have to be overcome. Leen`s work impressively shows us a way to attain that goal and convinces us that it really is attainable. The related demands Leen did not only impose on his own work, but he always tried to urge the authors of papers submitted to ESM to think about consequences of their work for the students in the classroom and how it relates to their struggles with making their mathematics meaningful to themselves. When admitting that there is a multitude of possibilities for approaching that goal this could and should mean a basic guiding framework and orientation for future research in mathematics education: to make the mathematics experientially real to the learner.

A related feature of Leen`s thinking about research and scientific theories in general and specfically in maths education is the following one: As the learning of mathematics itself, also the development of theories about it has to be experientially grounded. This attitude of Leen`s showed itself in a kind of doubtfulness and suspicion of what he sometimes called a jargon. By this he labeled texts that used vague or opaque concepts too much detached from the concrete realities of the mathematical classrooms to have any sensible implications for the organization of the latter. In other words the basic tenets of RME in Leen`s view have also to be applied to maths education research. As the teaching of mathematics in school has to be rooted in prior or current experiences of the learner, a valuable piece of research and its presentation in a paper has to be related to the experiences of the readers. It must make sense by making clear the meaning of used notions and terms and should have the potential to change the experience of the reader. As the relevance of most mathematical concepts and methods resides in their potential to structure and organize the experience and activity of the learner, research and theories in mathematics education should have analogous implications for the practice of school mathematics. I remember various vivid discussions on this issue that showed us both that such general tenets have to be substantiated in each single case, such as a specific contribution to ESM. It also became clear that experience is not a given which passively is imposed on the individual but that it is something that is actively constructed and developed by the latter in his/her social context. This inherent indeterminacy of individual experience, be it by the pupil in the classroom or by the reader of a journal article, brings in notions such as affordances and constraints. Whatever a teacher does in the class or an author writes in a paper establishes affordances and constraints for the thinking and understanding on the part of the students or of the readers. This might make teaching and writing a daunting endeavour; but only if one believes in fixed and absolute meanings (in mathematics and mathematics education), which have to be acquired and transmittered adequately. Contrary to that, Leen`s conception of RME and of mathematics education research is that of a developmental process that leaves room for interpretations, negotiations, inventions, deviations and the like, which yet constantly and consciously is devoted to sense-making. And in this framework a jargon is a way of teaching, speaking or writing which inhibits or even prevents the above cognitive and communicative processes. And I take it as a kind of legacy from Leen to avoid jargon in this sense because it acts against the interest in people as learners, readers and researchers.

Social interaction as reflection: Leen Streefland as a teacher of primary school children. Ed Elbers

Leen Streefland did pioneer work in creating a community of inquiry in the mathematics classroom. He encouraged students to “do research” and to adopt the attitude of researchers. The task of the teacher was to guide and assist students who had been given considerable responsibility for their own learning. Streefland was convinced that creating a learning community in which students had ample opportunity to produce and discuss ideas would allow their mathematical creativity to blossom. Interaction and collaborative learning would stimulate children to make their own mathematical constructions and to discuss them in what amounted to a social process of reflection. Streefland was involved not only as a researcher but also as a teacher. He worked with the primary school teacher Rob Gertsen for about 15 years.

I shall present a case study of a lesson co-taught by Leen Streefland and Rob Gertsen. In this lesson Streefland alternated whole class discussions with individual or group work. I want to use this particular case as an illustration of Streefland's ideas about mathematics education. Moreover, I shall analyse the relationship between whole class discussions and learning processes of individual students and the tension between teacher's guidance and students' invention.

Three principles of realistic mathematics education form the basis of this lesson. Firstly, the starting point is the problem instead of the mathematical strategy or solution. The teacher introduces a meaningful problem, which the students use as a source for constructing mathematical understanding. Secondly, a basic element in realistic mathematics education is to motivate students to 'mathematize', to turn everyday issues into mathematical problems and use the mathematics resulting from these activities to solve other problems. Thirdly, the students do not depend on the teacher to find out whether their ideas are correct. Part of their task is to develop good arguments to support their approach and solutions.

Streefland's lessons had a basic format (Elbers & Streefland, 2000a, b). They started with a statement of the principle of a community of inquiry: "We are researchers, let us do research." The students were given a topic or problem as the subject of their research (often with reference to some example from everyday life, a newspaper clipping, a photograph, etc.). After the introduction of the topic, the students were invited to formulate research questions and develop answers to these questions. Work in small groups of 4 or 5 students, and sometimes individual work, alternated with class discussions in which the results were made available for discussion in the whole class. Leen Streefland and Rob Gertsen worked as a team. They introduced themselves not as teachers, but as senior researchers. This role allowed them to participate in the discussion themselves. The students knew, of course, that they could expect assistance and guidance by their teachers. However, by acting as they did the teachers gave the students responsibility for their work and made it clear that the validation of their solutions comes from mathematical argument and not from the teacher's authority.

The case study

The lesson was taught in a combined seventh and eighth grade class at a primary school in the Netherlands (children between 11 and 13 years of age). It was part of a short series of lessons in June, at the end of the school term, when the students had already completed their regular mathematics curriculum for that year. An activity sheet was used with the problem and its variations printed on it, leaving space for the students to write down their own solutions. My presentation is based on an analysis of the video recording of this lesson, a transcript and a short Dutch article Leen Streefland wrote about this lesson in 1997. The case is divided into a number of Episodes. The problem introduced in the classroom was set in a pharmacy and involved calculating the number of tablets prescribed by a physician.

"Elisa works at a pharmacist's. She is preparing medicines prescribed by Doctor Sterk for Mrs. Jansen.

For Mrs. Jansen: 40 falderal tablets.

6 tablets a day for 2 days;

then 5 tablets a day for 2 days;

then 4 tablets a day for 2 days;

then 3 tablets a day for 2 days;

then 2 tablets a day for 2 days;

then 1 tablet a day for 2 days.

Elisa thinks: 'This isn't right! The doctor has made a mistake.'

What do you think? Is Elisa right?"

During the lesson variations on the original problem were given which amounted to changes in the number of tablets. The students, first, had to solve the original version of the problem (starting with 6 tablets), next they had to calculate the number of tablets in a prescription starting with 8 tablets (8 tablets for the first two days, 7 for the next two days, etc.) and then the number of tablets in a prescription starting with 10 tablets. For reasons of space, I shall restrict my presentation here to Episodes 5, 6 and 7. In the preceding Episodes 1 to 4, the students, in order to solve the problem in its original form, had developed two approaches: (1) multiplying the various numbers of tablets (2x6; 2x5; etc.) and adding them up (12 + 10 + etc.), and (2) adding up the numbers (6 + 5 + etc) before multiplying them (2 x 21). In Episode 4, which consisted of individual and group work, students worked on the version of the problem starting with 8 tablets. Episode 5 is a class discussion immediately following Episode 4.

Episode 5. One of the children showed his solution to the whole class. He used the solution to the original version (starting with 6 tablets) as a starting point for solving the new problem (starting with 8 tablets): 42 + 2x7 + 2x8. After this, the teachers asked who had applied a different approach. One of the children showed that she had made combinations of tens (8+2=10, 7+3=10 etc.) in order to add up quickly: “I took out the tens”. In the ensuing discussion about this approach, Streefland asked: “What shall we call this approach? Can we invent a name for it?” The children proposed expressions such as “making tens”, “jumping to tens” and “bridges of ten”. Leen Streefland concluded this discussion by suggesting they call this method: “making combinations” and “making clever combinations”.

The third variant of the problem was then introduced, starting with 10 tablets. But before the students started their work on the activity sheets, the teachers presented a challenge:

Sequence 1.

Gertsen: You can solve the next problem. Just try to solve it: if I start with 10 tablets, how many do I need? The children who have found the answer quickly, then think about this extra question: can I find the answer without making the calculation? I’ll give you a hint: compare the three numbers. When I start with 6 tablets..., when I start with 8..., when I start with 10... When you compare, you may come to a conclusion, a discovery. After all, you are researchers, aren’t you?

Streefland: I would like to add that it may be fun to try out other combinations. Try to experiment with combinations. Maybe you’ll discover something surprising. If you discover that, the problem is a piece of cake.

Episode 6. In this Episode, some students invented a new solution to the problem. Their invention was the result of a discussion in a group of four boys.

Episode 7. In the subsequent class discussion, these boys’ solution was presented by one of them in the following way:

Sequence 2.

Gertsen: Researcher Tom, show your solution on the blackboard.

Tom: If you start with 42, it is 6x7. If you look at 72, that is 9x8. Beginning with 10, that is 10x11. (He writes these numbers on the blackboard).(...)

Streefland: Be consistent: 6x7, 8x9, 10x11.

Gertsen: Put the smaller number first.

Streefland: It is very nice to do it this way.

Gertsen: I can know the next one, too, because, look, (pointing at the numbers on the blackboard) here is 6, 8, 10, and the next one should be: 12x13. (...)

Streefland: That is very good, but I think that he should show it by writing it out in full. Because it does not appear out of the blue, of course!

Gertsen (addressing Tom): Show it, prove it.

With some help Tom succeeded in showing on the blackboard that 6+5+4+3+2+1 = 3x7 (6+1, 5+2, etc.), and because the numbers should be doubled, the result is 6x7. Next, Gertsen demonstrated this way of calculating for other variants of the problem. The prescription starting with 8 tablets can be solved by making combinations of 9. With many students participating, the teachers demonstrated the outcome of the problem starting with 12 and with 14 tablets.

Analysis of the activity sheets demonstrates a clear and direct influence of the solutions discussed in the whole class on the individual work. The majority of students adjusted their solutions on their activity sheets and adopted the strategies developed just before in the whole class. The sheets show that, in three episodes of individual work, 26, 19 and 16 (of 28) students appropriated the strategies discussed in the whole class. The activity sheets also show that students did not stick to one strategy once they had invented it and found it to be correct. They acted in line with the teachers' encouragement to continue finding other, more efficient and clever, solutions.

Discussion.

At first sight the results would seem to fit into a two level approach (suggested by Inagaki et al., 1998): understandings are first constructed collectively, and then appropriated by individual students. This theory would seem to apply here, since the case study showed that the majority of students accommodated their solutions on the worksheets to the previous collective argumentation in the whole classroom. The difficulty with an account in terms of a two level approach is that it is not so easy to tell where the collective work ends and individual learning begins. Both collective argumentation and individual work took place within a discursive structure with rules such as: find out for yourself, choose a practical solution, present it understandably. These rules structured the students' work, both in the whole class discussions and during their work on the activity sheets. In their individual work, students applied the same kind of arguments which they had to use in the class discussions. Students' achievements are best understood by referring to this discursive structure. Individual work is to be considered as an anticipation of a class discussion or a reconstruction of it. Given this discursive structure, there is no priority for individual or collective work; they are two sides of the same coin.

Students did not just internalize or incorporate the outcomes of the class discussion, they had to reconstruct them (cf. Elbers, Maier, Hoekstra & Hoogsteder, 1992). Even the use of the outcomes of class discussions for writing an answer on their sheets was not a reproduction, but demanded creativity. Students' creation of novel solutions, as in Episodes 6 and 7, can illustrate this. The discursive structure of the interaction was the outcome of the teachers' transformation of the mathematics class into a community of inquiry. At the beginning of the lesson Streefland told the students: "I am convinced that, if you have the courage to figure out something, you can do much better than you thought you could!" Addressing the class as a learning and researching community created roles and responsibilities for the teachers and the students which differed from a conventional classroom (Ben-Chaim et al, 1998; Elbers and Streefland, 2000b).

Because of the students' contribution to the class, the teachers faced a problem originating from their double role. On the one hand, they were in charge and responsible for the students' activities. They decided what topics would be worked on and they had their ideas of what knowledge students should acquire during the lessons. On the other hand, they wanted the students to find out for themselves: to invent solutions to problems and to prove their validity. They did not want to frustrate children's creativity by using their authority for supporting certain answers instead of others. For solving the problem originating from this double role, the teachers used three strategies to channell the discussion. Firstly, the teachers selected students to give a presentation in front of the whole class. During the parts of the lesson in which students worked on their worksheets the teachers walked around and sometimes asked individual students to show and explain their work. During these episodes, the teachers observed what solutions the students were trying to work out and they singled out students with novel solutions rather than with familiar ones to present their work to the whole class. Secondly, the teachers stimulated variation in solutions. Students trying to discover a different solution from one already found were rewarded with compliments and enthusiasm by the teachers. Thirdly, the teachers made general suggestions to help students to view the problem from a different perspective. An example of this can be seen in Episode 5 above. The children, who at this stage had only made combinations of ten, proposed calling this procedure: taking out tens, etc. Streefland taught them to use the term: making combinations. In this way he paved the way for students to find out that they could make combinations which add up to numbers other than ten. Using these strategies, the teachers could direct the discussion and at the same time leave the students ample freedom to find out and make inventions. After having worked out a correct solution, there was no reason for students to stop, since there was always an even more efficient solution to be found.

The case demonstrates how students in an atmosphere of collaboration and interaction contributed to their learning and how the teachers exploited the various productions and constructions made by the students to structure the learning process.

Learning from history to solve equations. Barbara van Amerom

Several research projects of recent years report on learning difficulties related to algebraic equation solving (Kieran 1989, 1992; Filloy & Rojano 1989; Sfard 1991, 1996; Herscovics and Linchevski,1994, 1996; Bednarz et al. 1996). These difficulties include constructing equations from arithmetical word problems, as well as interpreting, rewriting and simplifying algebraic expressions. According to some researchers part of the problem is caused by fundamental differences between arithmetic and algebra. A good starting point for an investigation into this issue could be a return to the roots. By looking into the past we shall try to gain insight into the differences and similarities between arithmetic and algebra and learn from the experiences of others. Streefland emphasized the value of ‘reciprocal shifting’: changing one’s point of view, looking back at the origins in order to anticipate (Streefland 1996). Such a change of perspective can propel the learning process of the researcher, the teacher and the student alike.

Algebra and arithmetic

A closer look at the similarities and differences between algebra and arithmetic can help us understand some of the problems that students have with the early learning of algebra. Arithmetic deals with straightforward calculations with known numbers, while algebra requires reasoning about unknown or variable quantities and recognizing the difference between specific and general situations. There are differences regarding the interpretation of letters, symbols, expressions and the concept of equality. For instance, in arithmetic letters are usually abbreviations or units, whereas algebraic letters are stand-ins for variable or unknown numbers. According to Freudenthal (1962), the difficulty of algebraic language is often underestimated and certainly not self-explanatory: 'Its syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and of the language of arithmetic, and which are even mutually contradictory' (p. 35). He continues:

> The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic 3 + 4 means a problem. It has to be interpreted as a command: add 4 to 3. In algebra 3 + 4 means a number, viz. 7. This is a switch which proves essential as letters occur in the formulae. a + b cannot easily be interpreted as a problem" (Freudenthal 1962, p. 35).

Several researchers (Kieran 1989; Sfard 1991) have studied problems related to the recognition of mathematical structures in algebraic expressions. Kieran speaks of two conceptions of mathematical expressions: procedural (concerned with operations on numbers, working towards an outcome) and structural (concerned with operations on mathematical objects). But despite the contrasting natures of algebra and arithmetic, they also have definite interfaces. For example, algebra relies heavily on arithmetical operations and arithmetical expressions are sometimes treated algebraically. Arithmetical activities like solving open sentences and inverting chains of operations prepare the studying of linear relations. Furthermore, the historical development of algebra shows that word problems have always been a part of mathematics that brings together algebraic and arithmetical reasoning.

Cognitive obstacles of learning algebra

An enormous increase in research during the last decade has produced an abundance of new conjectures on the difficult transition from arithmetic to algebra. For instance, with regard to equation solving there is claimed to be a discrepancy known as cognitive gap (Herscovics & Linchevski 1994) or didactic cut (Filloy & Rojano 1989). They point out a break in the development of operating on the unknown in an equation. Operating on an unknown requires a new notion of equality. In the transfer from a word problem (arithmetic) to an equation (algebraic), the meaning of the equal sign changes from announcing a result to stating equivalence. And when the unknown appears on both sides of the equality sign instead of one side, the equation can no longer be solved arithmetically (by inverting the operations one by one). Sfard (1996) has compared discontinuities in student conceptions of algebra with the historical development of algebra. She writes that rhetoric (in words) and syncopated algebra (involving abbreviated notations) is linked to an operational (or procedural) conception of algebra, whereas symbolic algebra corresponds with a structural conception of algebra. Da Rocha Falcaõ (1995) suggests that the disruption between arithmetic and algebra is contained in the approach to problem-solving. Arithmetical problems can be solved directly, possibly with intermediate answers if necessary. Algebraic problems, on the other hand, need to be translated and written in formal representations first, after which they can be solved. Mason (1996, p.23) formulates the problem as follows: 'Arithmetic proceeds directly from the known to the unknown using known computations; algebra proceeds indirectly from the unknown, via the known, to equations and inequalities which can then be solved using established techniques.'

'Reinvention of algebra'

Recent research on the advantages and possibilities of using and implementing history of mathematics in the classroom has led to a growing interest in the role of history of mathematics in the learning and teaching of mathematics. Inspired by Streefland's work as well as the HIMED (History in Mathematics Education) movement, a developmental research project called 'Reinvention of Algebra' was started at the Freudenthal Institute in 1995 to investigate which didactical means enable students to make a smooth transition from arithmetic to early algebra. Specifically, the 'invention' of algebra from a historical perspective will be compared with possibilities of 're-invention' by the students. The historical development of algebra indicates certain courses of evolution that the individual learner can reinvent. Word or story-problems offer ample opportunity for mathematizing activities. Babylonian, Egyptian, Chinese and early Western algebra was primarily concerned with the solving problems situated in every day life, although they also showed interest in mathematical riddles and recreational problems. Fair exchange, money, mathematical riddles and recreational puzzles have shown to be rich contexts for developing handy solution methods and notation systems and are also appealing and meaningful for students. Another possible access is based on notation use, for instance comparing the historical progress in symbolization and schematization with the contemporary one.

The learning strand: pre-algebra as a link between arithmetic and equation solving

The barter context in particular appears to be a natural, suitable setting to develop (pre-)algebraic notations and tools such as a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (un)known quantities. The following Chinese barter problem from the 'Nine Chapters on the Mathematical Art' has inspired Streefland (1995a, 1996a) and the author to use the context of barter as a natural and historically-founded starting point for the teaching of linear equations:

> *By selling 2 buffaloes and 5 wethers and buying 13 pigs, 1000 qian remains. One can buy 9 wethers by selling 3 buffaloes and 3 pigs. By selling 6 wethers and 8 pigs one can buy 5 buffaloes and is short of 600 qian. How much do a buffalo, a wether and a pig cost?*

In modern notation we can write the following system:

$$2b + 5w = 13p + 1000 \qquad (1)$$
$$3b + 3p = 9w \qquad (2)$$
$$6w + 8p + 600 = 5\mathrm{b} \qquad (3)$$

where the unknowns b, w and p stand for the price of a buffalo, a whether and a pig respectively. The example is interesting especially when looking at the second equation, where no number of 'qian' is present. In this 'barter' equation the unknowns b, w and p can also represent the animals themselves, instead of their money value. The introduction of an isolated number in the equations (1) and (3) therefore changes not only the medium of the equation (from number of animals to money) but also the meaning of the unknowns (from object-related to quality-of-object-related). Streefland (1995) has found in his teaching experiment on candy that the meaning of literal symbols is an important constituent of the vertical mathematizing process (progressive formalization) of the pupils. "The changes of meaning that letters undergo, need to be observed and made aware very carefully during the learning process. In this way the children's level of mathematical thinking evolves." (Streefland 1995, p 36, transl.).

We also intend to investigate how notation and mathematical abstraction are related. The categorization rhetoric – syncopated – symbolic is the result of our modern conception of how algebra developed, and for this reason it is often mistaken for a gradation of mathematical abstraction (Radford 1997). When the development of algebra is seen from a socio-cultural perspective, instead, syncopated algebra was not an intermediate stage of maturation but it was merely a technical matter. As Radford explains, the limitations of writing and lack of book printing quite naturally led to abbreviations and contractions of words. Perhaps our students will reveal similar needs for efficiency when they develop their own notations (from context-bound notation to an independent, general mathematical language), but this may or may not coincide with the conceptual development of letter use.

Classroom examples

The first version of the experimental learning strand for primary level was tried out in 1997-1998 in two primary school classes grade 5-6 pupils (combined). The general topic of the primary school lesson series is recognizing and describing relations between quantities using different representations: tables, sums, rhetoric descriptions and (word) equations. No prior knowledge was required apart from the basic operations and ratio tables. The study is based on data collected through the observation and analysis of classroom work and the evaluation of a written assessment test taken by the students after the last lesson.

Shortened notations form one of the spear points in the learning strand. One of the units for grade 5-6 is centered on the context of a game of cards. In one of the lessons children suggested what could be the meaning of the expression ‘pA = 3 × pJ’. Our decision to use this kind of symbolism is based on other pupils’ free productions in a preliminary try-out. The letter combination maintains the link with the context: the letter p stands for ‘number of points belonging to’ and the capital letter stands for the person in question, in this case Ann and Jerry. In the expression, such a letter combination behaves like a variable for which numbers can be substituted. When the score of one of the players is given, the expression becomes an equation which can be solved. The teacher asked the children for an example that will illustrate that the relation between the variables pA and pJ is ‘3 times as much’:

Figure 1: inconsistent symbolizing

Figure 1 illustrates three samples of inconvenient symbolizing: Yvette’s choice to write a capital letter A and then a small letter j, her use of the letter p as a unit even though it is already part of the variable, and Sanne’s suggestion at the end. Apparently it was not a problem to the children that letters mean different things at the same time. As long as the pupils and the teacher are all conscious of this fact, the development and refinement of notations is a natural process. On the other hand, it is not our intention to cause unnecessary confusion regarding the meaning of letters. It was decided that if children have a natural tendency to use the letter p as a unit, p should not be included in the expressions and formulas.

The lesson materials were adjusted and tested again in 1999. The dual character of the learning strand – to develop reasoning and symbolizing skills in the study of relations – was maintained but placed in a more problem-oriented setting and with a more explicit historical component. We have selected two examples of student work from the final classroom experiment to demonstrate that (pre-)algebraic symbolizing tends to be more difficult for students than reasoning.

Symbolizing versus reasoning

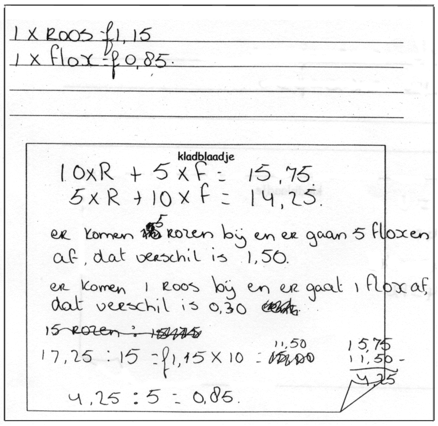
The project's final experiment was conducted in three primary schools (grade 6) and two secondary schools (grade 7). Encouraged by the ideas and results of the classroom experiment on candy by Streefland (1995), a grade 7 unit on equation solving was designed based on the mathematization of fancy fair attractions into equations. One of the tasks in the written test was:

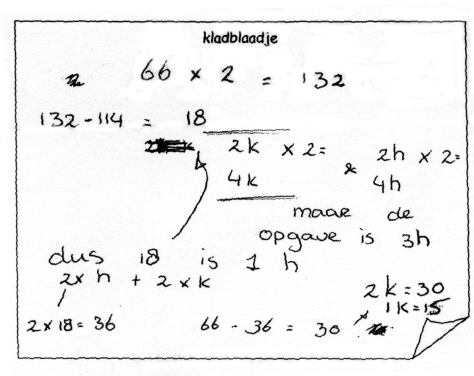
> *Sacha wants to make two bouquets using roses and phloxes. The florist replies: 'Uhm ... 10 rozes and 5 phloxes for f15,75, and 5 roses and 10 phloxes for 14,25; that will be 30 guilders altogether please.'*
>
> *What is the price of one rose? And one phlox? Show your calculations.*

One of the outcomes of the experiment is that algebraic equation solving need not necessarily develop synchronously with algebraic symbolization. For instance, we have observed student work where a correct symbolic system of equations was followed by an incorrect or lower order strategy, or where the student proceeded with the solution process rhetorically. The student in figure 2 mathematizes the problem by constructing a system of equations, and then applies an informal, pre-algebraic exchange strategy which is developed in the unit. Below the equations she writes: 'We get 5 roses more and 5 phloxes less, the difference is 1.50. We get 1 rose more and 1 phlox less, the difference is 0.30.' The calculations show that she continues the pattern to get 15 roses for the price of 17.25 guilders, and then she determines the price of 1 rose and 1 phlox. The level of symbolizing may appear to be high at first due to the presence of symbolic equations, but the student does not operate on the equations. The equations may have helped her structurize the problem but they are not a part of the solution process. And even though the unknown numbers of flowers are an integral part of her reasoning, the letters representing them are not needed in the calculations. There is a parallel here with the historical development of symbolizing the solution. In the rhetoric and syncopated stages of algebra the unknown was mentioned only at the start and at the end of the problem; the calculations were done using only the coefficients.

Figure 2: symbolic notations and informal reasoning

Figure 3: informal notations and algebraic





reasoning

Alternatively the solution in figure 3 illustrates how the level of reasoning can be higher than the level of symbolizing. This student solves the system of equations

$$2 \times h + 2 \times k = 66$$

$$3 \times h + 4 \times k = 114$$

by doubling the first equation and then subtracting the second from it. First he deals with the right hand sides of the equations (66×2 and $132 - 114$). In between the two horizontal lines we observe how he multiplies the terms with the unknowns. Then he writes ‘but the task says 3h so 18 is 1 h’. Finally he substitutes the value 18 to solve for k. A remarkable contrast presents itself. This student successfully applies a formal algebraic strategy of eliminating one unknown by operating on the equations, while his symbolizing is still at a very informal level. Again the unknown is only partially included in the solution process; it appears only where necessary. In other words, both examples of equation solving illustrate that competence of reasoning and symbolizing are separate issues.

Conclusion

Difficulties in the learning of algebra can be partially ascribed to ontological differences between arithmetic and algebra. The project 'Reinvention of algebra' uses informal, pre-algebraic methods of reasoning and symbolizing as a way to facilitate the transition from an arithmetical to an algebraic mode of problem solving. We have shown some examples where informal notations deviate from conventional algebra syntax, such as inconsequent symbolizing and the pseudo-absence of the unknown in solving systems of equations. These side effects bring new considerations for teaching: how can we bridge the gap between students' intuitive, meaningful notations and the more formal level of conventional symbolism. The observation that symbolizing and reasoning competencies are not necessarily developed at the same pace – neither in ancient nor in modern times – also has pedagogical implications. It appears that equation solving does not depend on a structural perception of equations, nor does it rely on correct manipulations of the equation. In retrospect we can say that knowledge of the historical development of algebra has led to a sharper analysis of student work and the discovery of certain parallels between contemporary and historical methods of symbolizing. Streefland's notice to look back at the origins in order to anticipate has turned out to be a valuable piece of his legacy.

Didactising: Continuing the work of Leen Streefland

Erna Yackel, Diana Underwood, Michelle Stephan, & Chris Rasmussen

When we think of the work of Leen Streefland we think of his seminal work in developing prototypical courses and instructional sequences (fractions, negative numbers and algebraic expressions and equations). In developing these courses and sequences Streefland was not only putting into practice the general principles of Realistic Mathematics Education (RME) that had been set forth by Freudenthal and Treffers but he was demonstrating how these principles might be realized in practice over an extended period of instructional time. In doing so, he went beyond earlier work that demonstrated one or more of the principles for individual problems, such as the van Gogh sunflower problem (Treffers, 1993). However, Streefland viewed his work as much more than the development of prototypical courses and sequences. In the abstract of his paper, *The Design of a Mathematics Course, A Theoretical Reflection,* Streefland (1993) pointed to what he saw as the major contribution of this work, namely operationalizing RME instructional design theory and thereby raising it to a higher level. As the title of the paper indicates, Streefland's purpose was to reflect on the development process and identify strategies used in the design of the exemplary materials. To this end, he analyzed his fractions course and other examples of prototypical instructional sequences that were developed following the principles of RME. As Streefland noted,

> In consequence an important theoretical change of perspective looms ahead. Where the theory was first an *after*-image, it can now act as a *pre*-image, i.e., as a *model for* realistic mathematics education in advance (p. 109).

The activity of developing such after-images that then can be used as pre-images in future work is what Streefland called *didactising.*

In one sense, the work of our research group might be thought of as applying the model that Streefland has articulated since we are developing prototypical courses in mathematics for various audiences, including university students. However, the intention of our work extends beyond applying Streefland's model. We prefer to view our work as having the same character as Streefland's in that as we engage in the process of the developmental research that is required to develop prototypical courses, we are continually analyzing aspects of our own activity for potential after-images that might be useful as pre-images in other situations. In this sense, we, too, are engaged in didactising.

Each of the researchers in this session will describe their didactising activities within the context of their respective research. First, Underwood will discuss designing instructional sequences so that students' mathematical understanding grows out of their development of symbolic representations while at the same time contributes to the development of those representations. Next, Stephan argues that argumentation analyses are useful not only for analyzing students' learning as they engage in prototypical courses, but also as a tool for the designer in her attempts to anticipate the quality of the social interaction and discourse associated with the instructional sequences under development. Stephan's work is a form of didactising in the sense that she is using argumenation theory as a tool for describing how the conditions for learning the desired mathematics can be created and sustained in social interaction (Streefland, 1993). Rasmussen uses different modes of "listening" as a conceptual tool for describing aspects of the activity of analysing the vast amounts of data collected from developmental research for the purposes of informing and revising the development of instructional sequences. As an after-image, these different modes of listening have the potential to be useful pre-images for others engaged in RME-based instructional design.

Thus, each of the three researchers demonstrates a form of didactising. In doing so, each goes beyond treating the development of prototypical courses for various mathematical content areas and various audiences as a simple matter of applying Streefland's model. In each case, the researcher gives explicit attention to reflecting on critical aspects and strategies of the design process which includes: developing means of recording and notating that can describe informal activity and that have the potential to lead to formal and/or conventional mathematical notation, anticipating the classroom discourse that can emerge as students solve problems, and selecting or designing "realistic" contexts that have the potential to lead to formal mathematizing.

Emergent Models in a Context of Linear Equations

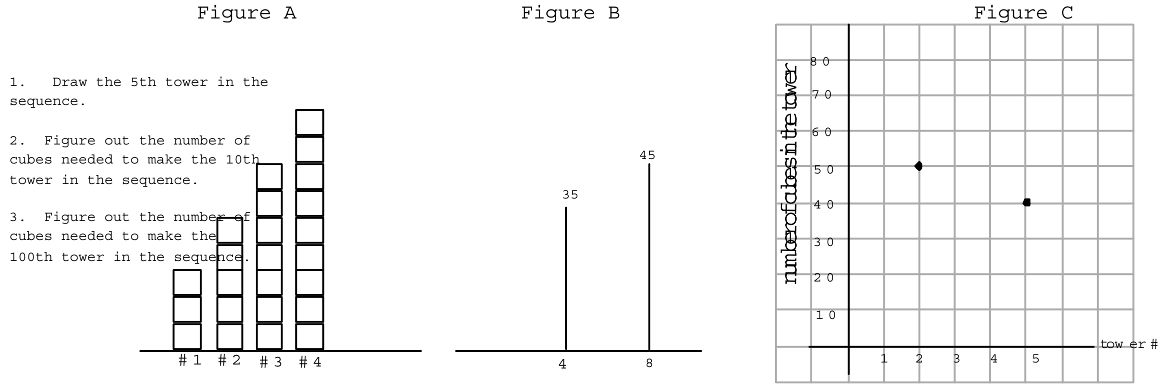
The purpose of this section is to illustrate how mathematics instruction might be designed to facilitate the emergence of conventional symbolism for linear equations from students' ways of representing and notating their reasoning in situations of linear change. This approach is in contrast to much of the recent reform curricula concerning linear functions that focus on facilitating students' ability to move flexibly within and across tabular, symbolic, and graphical representations. While it is important that students are able to interpret linear functions in a variety of ways, a problem with this approach is that the student still needs to integrate them. For example, even when students are able to describe slope graphically as "rise over run" and are provided opportunities to "discover" that the number representing the slope is the coefficient of x in the equation for a line, they usually cannot explain a basis for this relationship.

One explanation for this difficulty is that students are asked to create and use the graphs of functions on a Cartesian plane as a *model for* reasoning about quantities without first facilitating development of the plane as a *model of* anything. The Cartesian plane is a symbol system used in creating a visual (dynamic) representation of the relationship among quantities. According to the principles of RME, this symbolism should emerge from students' mathematical activity (Gravemeijer, 1994) rather than be given to them prepackaged.

The Stacking Cubes instructional sequence attempts to promote students' understanding of a coordinate system while simultaneously facilitating their understanding of linear relationships. Our inspiration for creating this sequence grew out of noting students' solutions to a data recording and graphing activity. In this activity from the Connected Mathematics series (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), students were asked to flip a coin for 90 seconds, record the cumulative number of flips at 10-second intervals, and make a graph of the data. Many students generated bar graphs to represent their data. In an earlier instructional sequence adapted from the Mathematics in Context series (Wijers, Roodhardt, van Reeuwijk, Burrill, Cole, & Pligge, 1997), students had successfully developed formulas for growing spatial patterns. By combining the idea of writing formulas for growing patterns with the idea of bar graphs, our goal was to design an instructional sequence that would help students develop a conceptually-based understanding of linear relationships, thereby enabling the symbolic representation of a linear function to grow naturally out of graphical representations.

The instructional design was based on the tenets of RME emphasizing that through engaging in realistic tasks, students create models of/for their mathematical activity (Gravemeijer, 1994). In the initial activity students were presented with a sequence of “towers,” asked to draw the next few towers in the sequence, and figure out the number of cubes that would be in the 100th tower in the sequence (see Figure A). The spatial arrangement of the towers resembled the bar graphs that students had drawn in their prior data recording and graphing activity. As the instructional sequence progressed, students were asked similar questions, but were given less information to answer them. At the same time, the notation used to represent the stacking cubes evolved to look more like ordered pairs. For example, towers were first replaced by "sticks" and the sticks were ultimately replaced by points on the Cartesian grid:

Figure A   Figure B   Figure C



The students' descriptions of the spatial patterns also evolved. Initially they gave detailed verbal descriptions such as, *I figured out that the change between the towers was 2 and then I counted back to the zero building, which is one. So I know that the 100th building is 2 times 100 plus 1, or 201 cubes high*. Later, this type of description was generalized to *height of the nth building = zero building + pattern number * change***.** Eventually students developed the symbolic notation H = a + bP to represent the relationship that was illustrated in the graph.

While the bar graphs/towers initially served as models of students' thinking in the graphing activity, the sticks/points became models for their reasoning about the relationship between height and pattern number. This symbolic progression can be thought of as *a chain of signification* (Cobb et. al, 1997; Gravemeijer, 1999):

$$\{\text{pictures of cubes}_{\text{signified 1}}\text{-pictures of sticks}_{\text{signifier 1}}\}_{\text{signified 2}} \ - \ \text{graphs of points}_{\text{signifier 2}}$$

In a similar way, the verbal/symbolic descriptions of the patterns that the students developed were models of their reasoning about the graphical representations of the patterns. Here again, the progression from an extended verbal description to a verbal formula to a symbolic formula can also be viewed as a second chain of signification.

The two chains of signification are intertwined in the sense that the symbolic representation of the relationship grew out of the students' thinking about the different graphical representations (towers of cubes, sticks, points) while at the same time the graphical representations enabled students to develop meaning for the different components of the equation $H = a + bP$. In contrast to the multi-representational approach, the two chains of signification are not two separate ways to represent a linear relationship. Instead, the instructional goal is that students' concept of a linear relationship will grow out of their development of symbolic representations while at the same time contributing to the development of those representations. That is, they evolved together as a dynamic, interactive system.

Under the guidance of the RME emergent models heuristic, we designed the Stacking Cubes sequence to support students' moving beyond using symbolic descriptions as models of patterns in towers to using them as models for reasoning about linear relationships. We believe that the approach that we take can be viewed more broadly as a *pre*-image for designing instruction that provides students with opportunities to create and reason with conventional symbols.

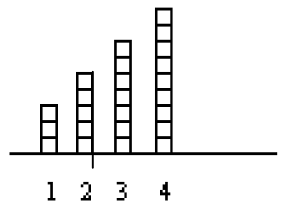
Argumentation as a Tool for Didactising

The purpose of this section is to show how analyzing students' argumentations impacts the design of instructional activities within a broader instructional sequence. Generally, argumentation has been used to analyze students' learning (e.g., Krummheuer, 1995; Yackel, 1997). In addition to this function, we will argue that argumentation analyses can serve to provide feedback to the RME designer by informing her of the nature of the justifications that students provide as they engage in the designed activities. The justifications may not be those that are anticipated by the designer and thus, she can revise the sequence by constructing tasks that better provide students the opportunity to construct mathematical justifications that are more in keeping with the overall mathematical goals for the instructional sequence. To begin this conversation, we first describe Toulmin's (1969) scheme for analyzing argumentation.

For Toulmin, an argument consists of at least four parts: the data, claim, warrant and backing. In any argument, the speaker makes a claim and, usually presents evidence or data to support that claim. Even so, a listener may not understand what the data presented has to do with the conclusion that was drawn and, therefore, challenges the presenter to clarify the role of the data in making a claim, a *warrant*. Perhaps the listener understands why the data supports the conclusion but does not agree with the mathematical content of the warrant used. The authority of the warrant can be challenged and the presenter must provide a *backing* to justify why the warrant, and therefore the entire argument, is valid mathematically.

In general we have found that students' warrants consist of further elaboration of their methods for solving a problem and that backings involve justifying why their method or interpretation should be mathematically acceptable in the classroom. In this section we would like to explore the usefulness of Toulmin's model of argumentation from a design perspective. In other words, what kinds of reasoning might the designer/teacher find useful to capitalize on in whole class discussions and what warrants and backings for a particular type of task are productive for learning? Do the instructional tasks she has designed provide the opportunity for such justifications to arise? Anticipating the nature of the warrants and backings that we think could be useful for supporting the classroom argumentation can aid in the development of mathematically productive instructional tasks. We will illustrate this with an example from the Stacking Cubes sequence. On the first day that students engaged in the Stacking Cubes sequence, the diagram shown in Figure D was drawn.

Figure D



The teacher explained that the picture showed a series of buildings constructed by a company and asked, "How many more little blocks do I need to make the 13th building?" While some students attempted to count how many increases of two there would be to get to the 13^{th} building, one student explained that she found a formula for finding the number of floors in any building.

Abby: I tried to figure out how many blocks would zero have. And it would be 1. Each building as you go down is decreasing by 2, so I just subtract the 2 [to get the 1]…2P + 1.

Teacher: How did you get the 2P + 1 part of it though? That's the part I don't understand.

Abby: They are each increasing by 2. So I just figured we're going to multiply by 2. And I know there is just 1, because the zero. That one has 1. So, 1 is already there. So you would be adding 1.

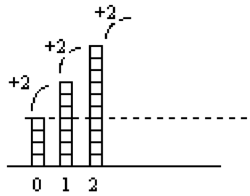
Teacher: She's going to be adding 1. Do you know what she means when she says that she is going to be adding 1? Do you know? Do you want to say something about that, Terry?

Terry: She is adding the 1 because it is…[inaudible]…you still have to add the 1 that was like the odd man out…1 times 2 is 2, but you have to add that 1 man in building 2 because of the one block in the zero building. 2 times 2 is four, plus 1 which is 5 in building 2.

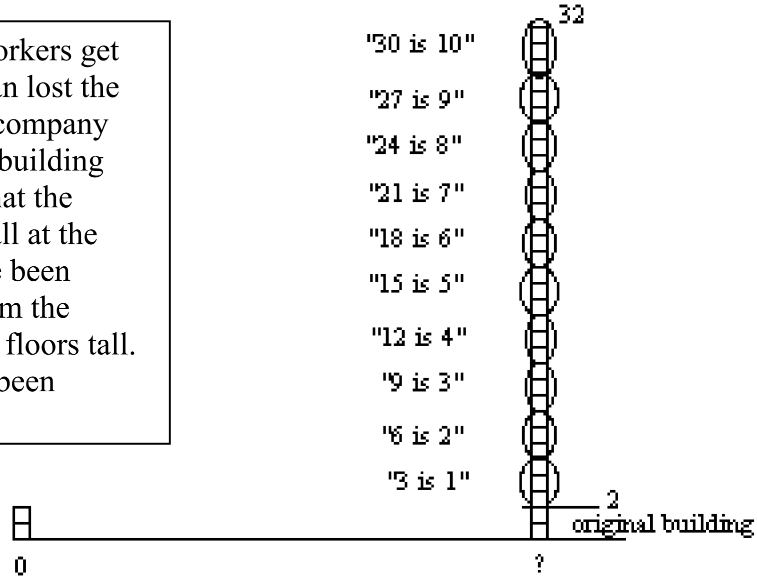
Analyzing the structure of the argumentation above, we see that Abby provided a claim that consisted of her formula, 2P + 1. She provided data for her claim when she explained, "I tried to figure out how many blocks would zero have." Once Abby explained her claim, the teacher challenged Abby to explain how finding the plus one led her to the formula 2P + 1. In other words, she was asking for Abby to make the warrant, how the data "+1 leads to the formula 2P + 1," more explicit. Abby responded by explaining the origin and necessity of each term in the formula. In Toulmin's terms, Terry provided a backing regarding why multiplying by 2 and adding the "odd man out" each time led to the desired results in each pattern ("2 times 2 is four, plus 1 which is 5 in building 2").

This type of argumentation for generating a formula was typical early in the instructional activity. The backing provided justification for the origin of the formula, but it was grounded in the specific example of the first and second buildings. As students' understanding becomes more general, we would expect that the backings to become more general in nature, i.e., the buildings are increasing by 2 floors per pattern number (*a rate*) from the original pattern (the "zero building"). However, this type of backing never arose in the course of instruction. As a consequence, we created two new tasks designed to provide students the opportunity to construct justifications for the formula that were based more explicitly on rates of change.

Task 1: Show students a picture of a building pattern that increases by 2 floors per day and starts at 4 floors. Only show the first three days. On which day will you have 14 more floors than the original? The teacher can draw a horizontal line on top of the 0 building and also notate each jump of two as pictured:



Task 2: Some construction workers get paid by the day but the foreman lost the record of how many days the company has worked since they started building onto the original. He knows that the current building is 32 floors tall at the end of the day and that they've been putting 3 floors on per day from the original building that was two floors tall. How many days has the crew been working?



In new Task 1, students may either draw subsequent buildings, counting the extra floors each time until they have 14 more floors than the original, or they might double count, keeping track of the number of twos and the number of days increased until they have counted 14. Double counting in this manner might allow students to draw attention to the 2 floors per day justification. The dashed line might support students constructing the backing described above: a rate of change of 2 floors per day added onto the original 4 floors. The dashed line might also support students seeing the original building as nested in each subsequent building.

In Task 2, some students might actually draw every building between the original and the last day while others may simply double count again to find how many counts of size three can be found between the last day and the original. The teacher might even symbolize those double counts by circling 10 groups of three blocks on the last day only (see figure accompanying Task 1). She might also ask students to explain what each number means as they double count. In this case, the intent again is to bring attention to the constant rate of change as it goes on from the original number of floors.

The point of this short excerpt is to bring attention to the role of argumentation in the designer's activities. By examining students' argumentations as they engage in tasks, we can discover the nature of the students' current justifications *and the range of potentially productive justifications*. This type of examination can lead to revisions in the instructional activities within a sequence.

Listening as a Conceptual Tool

The purpose of this section is to describe how the construct of "listening" to data can be used as a conceptual tool for the purposes of instructional design. As Streefland's work demonstrated, paying close attention to students' reasoning has always been a fundamental characteristic of RME based instructional design work. Using our developmental research efforts in differential equations as an example, we use different modes of "listening" as a lens to reflect on our retrospective analysis of the data collected during a semester-long classroom teaching experiment. In particular, a mode of listening that we call *generative* listening helps shape and clarify our thinking about realistic starting points for instruction that are mathematical in nature.

Within the theory of RME, realistic starting points for instruction refer to situational contexts that can serve as a building block for students' mathematical development. For example, Streefland (1990) discussed how distribution situations, like sharing 3 candy bars among 4 friends, can serve as a realistic starting point for students' learning the concept of fraction. Although this example of a realistic starting point is characterized by a real-world situation, the term "realistic" is intended to be broad enough to include mathematical situations that are experientially real for students. Relatively few examples, however, of realistic starting points that are themselves mathematical currently exist.

As researchers begin to explore ways in which the theory of RME can inform instructional design at the university level, it will be useful to have images of strategies that others find useful in locating realistic starting points that are mathematical. To begin to address this need, we use the notion of generative listening as a means to bring to light aspects of our developmental research activity that has yielded mathematical starting points that are experientially real for students.

Generative listening is intended to reflect the negotiated and participatory nature of interacting with data. This type of listening, which Davis (1997) called hermeneutic, is "an imaginative participation in the formation and transformation of experience" (p. 369). The notion of generative listening can be clarified by comparing it with what Davis calls interpretive listening and evaluative listening. In comparison with generative listening, where the purpose is to learn something new about one's own thinking, interpretive listening is to decipher the sense that students appear to be making of the mathematics under discussion. Davis posits that within interpretive listening, mathematics is still about constructing conventional associations between signifiers. Finally, evaluative listening is characterized by the fact that the listener is listening for something in particular. The motivation for evaluative listening is to evaluate the correctness of the contribution by judging it against a preconceived standard (Davis, 1997).

During the spring of 2000, we conducted a 15-week classroom teaching experiment in differential equations. At the commencement of the teaching experiment, we conjectured that population situations would serve as an experientially-real starting point for the development of students' concept of the solution space for differential equations, where solutions to these differential equations are functions of time. Note that the nature of this starting point has the same real-world character as Streefland's distribution situations for fractions. After engaging in extensive retrospective listening to the data collected during this teaching experiment, what students had to say transformed our thinking about what we could take as an experientially-real starting point.

To illustrate the notion of listening generatively to data, we use an excerpt from an end-of-the-semester interview with Marta, one of the students in the class. In the excerpt that follows, we asked Marta if she now thinks about the concept of function differently than she did before taking the differential equations course. We asked her this question because we take the viewpoint that solutions to differential equations are functions and thus the study of differential equations may provide opportunities for students' to deepen their notions of function. At the time of the interview, we were curious about students' evolving notions of function through their study of differential equations. That is, our listening was more interpretive and evaluative. Only later, upon retrospective analysis, did we listen generatively to this piece of data.

> *Marta: I can think of it more as, when you say this function, I can think of it more as instead of three x squared, I can think of it more as a motion, more as some kind of change. More as something that's actually going on opposed to, yeah, these are some numbers and this is what it looks like on a piece of paper…when I say 3 x squared, what I'm really talking about is, I'm talking about this marble moving from here to here and how it got there, you know?*

Although we think it is useful for developmental researchers to listen evaluatively and interpretively to data, we restrict the discussion to generative listening because when we listened generatively to this piece of data, we began to think differently about the possibilities for starting points in differential equations. In particular, our own thinking was transformed by engaging imaginatively in Marta’s description of motion and of a “marble moving from here to here.” We began to think about how the movement from “here to here” stems from conceptualizing rate of change and how solution functions can, for students, grow out of their mental and bodily experiences with rate of change. That is, the mathematical construct of rate of change, when coupled with population situations, can serve as an experientially-real starting point.

To take rate of change as an integral component of an experientially-real starting point is not to say that all students have a full conception of rate of change that is in line with expert notions. It is to say, however, that students’ at this level have some way to conceptualize rate of change as a mathematical construct so that they can proceed with a problem situation. For example, students might conceptualize rate of change as an intensive quantity by which a different quantity changes over time or they might view rate of change as a ratio of two co-varying quantities that gives rise to motion or movement and involves directionality. Although beyond the scope of this paper, we should note that the transformation in our thinking about taking rate of change as an experientially-real starting point has also led to revisions in the sequence of instructional activities.

The intention of this short example was to describe how we can use listening as a conceptual tool for reflecting on our developmental research efforts at the university level. Although we used the construct of listening to crystallize our efforts at locating experientially-real starting points that are mathematical in nature, the different modes of listening, generative, interpretive, and evaluative, may serve as a broader image for others engaged in developmental research.

Conclusion

As these three examples show, didactising can take a variety of forms depending on the mathematical content, on the student audience for the prototypical courses or instructional sequences, and on the interests of the researcher. At the same time, the researchers' interests evolve as the developmental research progresses. In this way there is an evolution of the nature of the after-images that researchers develop that then become pre-images for future work. Thus, in a sense Leen Streefland's work has set in motion a cyclic process that has the potential to move the field of mathematics education forward in substantive ways.

Reaction Koeno Gravemeijer

The work on RME at the Freudenthal Institute constitutes the heart of what people at the institute call, "educational development". The term educational development has been introduced to indicate an all-embracing process of educational innovation, encompassing both the actual enactment of the innovation in the classroom and an open dialogue between researchers and practitioners. Developmental research is conceived of as a catalyst of innovation. The results of developmental research are meant to function as a source of inspiration for practitioners. What we aim for is re-enactment of instructional sequences in various situations, adapted to those situations, and shaped according to the insights and preferences of the practitioners involved. This then is seen as an extension of the original developmental research, which produces feedback that will contribute to an enrichment of the original findings. Leen Streefland would be pleased to see how efforts to explicate the RME design theory with help of exemplary materials have led to a similar process in the mathematics education research community. To see that other researchers are inspired, experiment with, and expand RME theory. in this respect, the Purdue-Calumet research group has much to offer. Moreover, they address issues that were near to Leen Streefland's heart: - the perspective of a reflexive relation between the development of symbols/models and meaning, which fits so well with the quotation of Leen Streefland on page 1; - the focus on the role of argumentation within whole-class discussion, which is in line with Leen Streefland's research activities on the basis of the idea of a community of researchers; - the proposal to integrate generative listening as a conceptual tool in developmental research, which he would have welcomed as a valuable contribution to his effort to bring the RME design theory to a higher level. In conclusion, we may truly say: Leen Streefland's work continues.

Epilogue Marja van den Heuvel-Panhuizen

As Freudenthal stressed once, it was Leen Streefland who opened our eyes to the anticipatory learning of concepts that will develop in full at a later time. From the very beginning of his work in the field of mathematics education Leen was focused on where and how education can anticipate the learning process that is coming into view in the distance. This anticipatory perspective was not only true for Leen's ideas about how to teach mathematics to students, but is as true for Leen's role within our research group at Utrecht University and the international community of researchers of mathematics education. The concepts, the language, the way of thinking with which Leen provided us, turned out to be strong and continuing guides for deepening our understanding of the learning and teaching of mathematics. The contributions to this PME Research Forum prove this.

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[1] This Research Forum was initiated by Marja van den Heuvel-Panhuizen, who worked together with Leen Streefland on many projects.