BACK TO THE BASICS: RE-INTRODUCING ε - N DEFINITION OF THE THE LIMIT OF A SEQUENCE INTO CALCULUS 1 INSTRUCTION

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Abstract. The paper describes a teaching experiment whose goal has been re-introduction of ε -- N definition of the limit of a sequence into standard calc 1 instruction. It identifies the limit schema as the coordination of two processes, both involving epsilon. It is suggested that this coordination is the basic mental construction involved in the construction of the limit schema. The role of certain student misconceptions on the development of the understanding of the concept is also discussed.

Introduction. Why is it important to understand the fundamental concepts of mathematics from their basic definition? In particular, why is it important to develop a good understanding of the ε - N definition of the limit of a sequence among Calculus students, if, the majority of them are not math majors? It is so, in our opinion, because the understanding of the limit represents one of the more important achievements of modern mathematics, which helped to clarify its conceptual basis, and as such belongs to the area of general education. It is significant in acquainting students with the means of treating "infinity" with the help of *only finite processes* [7], and as such has a general educational and philosophical value.

Recent research [4], [5], [6], and not so recent [9] of students' understanding of the definite integral as the limit of Riemann Sums reveals serious absence of the knowledge of the relationship between the geometrical construction of Riemann Sums and the corresponding numerical sequence of partial sums, in the thinking of students of Calculus. It became clear that a mathematically correct understanding of the definite integral at the Calculus II level necessitates a precise understanding of the concept of sequences and their limits. The teaching experiment described in this article is first in the series devoted to the appropriate reorganization of the curriculum and instruction in that direction.

Whereas students' difficulties and treatment of this theme are quite well known (section below), yet at the same time there is a relative absence of reports about the successful incorporation of this knowledge into instruction, to advance its understanding among students of calculus. Our teaching experiment has been designed to bridge that gap.

Teaching experiment described in this was performed at Instituto Tecnologico de Monterey, Mexico, D.F. during the Fall 2000, with the specific goal of investigating the effectiveness of re-introducing ε - N definition of the limit of a sequence into standard Calculus 1 instruction. The discussion below will be presented in more of a conceptual manner rather then in a quantitative manner (although some broad assessment will be provided), due to the very short

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time which has elapsed since the experiment has been completed - yet some results are sufficiently important, and necessitate presentation in a preliminary fashion.

Literature review. Numerous researchers have found students to have seemingly inevitable cognitive and epistemological obstacles both with limits and the connected ideas of continuity, and differentiability [2],[7],[9], [10], [14]. In particular, Orton in [9], quotes some of the students he interviewed regarding the meaning of sequences in the context of Riemann sums, as saying that "the computation obtained in this way is approximate because the sequence never reaches the limit". This particular problem is related to the general issue of whether the sequence ever reaches the limit, and is discussed at length by [2],[7] and [10]. The authors of [7] discuss also another students' misconception, the neglect of the sequential order which in our experiment turned out to be quite essential. The students often approach the limit in the context of ε - N definition, saying that the larger N, the smaller the distance ε between the term and the limit - a description, in some sense, reciprocal to that incorporated by the definition of the limit of a sequence.

The difficulties with all these concepts seem to be so profound that Cottrill et al [3] remark, "We have not…found any reports of success in helping students to overcome these difficulties".

In this bleak landscape of the instruction and understanding of limits, two brighter results are worth mentioning. In her work, "The effects of writing assignments on second semester calculus students' understanding of limit concept", [13], Walhberg mentions a measurable improvement of student understanding under the influence of writing assignments. Orton [8], on the other hand, mentions that when students were introduced to limits of sequences with the help of the Wallis technique [12] and apart from their interpretation in the context of the Riemann sums, they were quite adept in finding the limit as well. The Wallis technique of finding the limit of sequences was used for the first time in the Aritmetica Infinitorium [12] on the occasion of finding the area under a parabola. It consists in estimating the limit L_e of the sequence given as a list of terms followed by the decomposition of each term a_n into the sum of L_e + "additional term". The additional term, can, in many cases, be written generally as $\frac{c}{f(n)}$ where f(n) is a linear function of n, so that this term can be shown to tend to 0 as $n \rightarrow \infty$, and that, consequently, the limit of the sequence is indeed L_e. In terms of the ε - N definition of the limit, the additional term is easily to be seen as the a_n - L, the difference between a term and the limit, whose absolute value can be made smaller, according to the definition, than arbitrary positive ε. This suggests that there is a natural relationship between the Wallis technique and the ε - N definition, which could be exploited to students' advantage.

Design of instruction. The principles of instruction during the teaching experiment had three components, all suggested by either positive results indicated in the literature or by the critique of certain didactic practices:

1. The development of the intuition of the limit was based on the Wallis technique, which, as the time progressed, was joined by the standard technique of finding the limit from the general term by dividing it by the highest power of n and "sending" n in the new expression to infinity.
2. Using the positive and measurable results found by Walhberg [13] about the role of writing assignments in the promotion of understanding of limits, the process of written explanation and discussion of the results was introduced at the very beginning of the discussions of sequences. Every assignment and every limit related problem during the partial tests had both computational, and conceptual written components in order to encourage the dialectical relation between these two aspects of understanding.
3. **A substantial increase in the understanding of the limit concept and a coordination of all aspects involved therein, (more precisely, a substantial increase in the scope of the limit schema, explained in section below)** was effected by the introduction of
   a) the eps - N definition of convergence (sequences $a_n = \frac{f(n)}{g(n)}$, where f(n), g(n) are linear functions of n, constant sequences $a_n=c$ and the alternating versions of both: $(-1)^n c$, $(-1)^n \frac{f(n)}{g(n)}$), its negation as the basis for the discussion of non-convergence (alternating, not converging sequences i.e. $a_n = (-1)^n * 5$) and by the precise definition of divergence, and of
   b) the Heine definition of the limit of a function which states *that a function has a limit L at x= a if for every sequence x_n which has a limit at a, the sequence $f_n = f(x_n)$ has a limit L.* The strength of this definition (which is equivalent to ε-δ definition of Cauchy) consists in that it allows to unify the instruction of sequences, limits of functions, continuity and other concepts depending on the notion of the limit, around one central idea, that of the limit of a sequence. Its application (or of its negation) to the different types of discontinuities, to the limits at infinity (functions with horizontal asymptotes) and to infinite limits (functions with vertical asymptotes) allows for the simultaneous usage (thematization) of all three types of sequences discussed in a). Such a consequent use of the concept is a necessary condition, according to Piaget [17] as well as to other Piaget-based frameworks such as APOS [1], for the formation of an abstract object from a particular concept. Thus the main role of this definition was not so much to have students master its different applications, but rather to provide the ground where the definitions and usage of convergent, non-convergent and divergent sequences in a natural mathematics context might provide a nourishing soil for the construction of the final object from the schema under investigation.

Theoretical framework. There are several definitions of a schema in the math education literature [1], [17]. For the purpose of this research the definition of the schema contained in the APOS theory [1], will be used:

a schema is a collection of processes, objects, and other schema which can be organized in a structured manner, that is used to deal with a certain category of mathematical problem situations. The structure of a schema gives it coherence in the sense that the individual has some means of understanding what kinds of situations a particular schema can be used to deal with.

The schema of the limit of a sequence described in the definition of that limit:

The sequence a_n has a limit L, if for every $\varepsilon>0$, there exists N such that for all n > N,
$|a_n - L| < \varepsilon$

is the result of the composition of two processes, the process of choosing an ε which is understood as a distance, hence, is part of a topological schema, and the process of establishing the relationship between ε and N through the inequality contained in the definition, where ε is seen as an independent algebraic variable of the function which determines suitable terms a_n (with n>N) of the sequence. The student has to start the first process by choosing some particular ε understood as distance, then he or she has to change its meaning to that of an independent variable which can take on any positive real value, and then finally, to demonstrate the understanding of the values of n so determined, he or she has to again go back to the concept of the distance; hence again changing the meaning of ε from that of a variable to that of the measure of a distance. Such a coordination of mental processes has been described by Dubinsky [8] as one of the basic components of reflective abstraction.

We have taken the successful coordination of these two processes as the evidence that a foundation of the schema of the limit of a sequence had been constructed within the mental apparatus of the students. We have also taken the successful application of the schema to the non standard problem situation as evidence of the coherence of the schema foundation required by the definition.

As a tool of assessment to determine whether students have the required understanding of the discussed coordination of two processes we have used the following credo that many veterans of mathematics teaching profession say to students: You don't know whether you understand a concept until you can explain it in words. Indeed, anyone who has ever been in the position of learning a new mathematical concept and has been forced to explain it "in his/her own words" can recall the effort needed to follow, or even to formulate one's own thoughts and to give them the adequate verbal form of explanation while "thinking aloud" about a solution of a problem in question. This effort is precisely the effort of finding the meaning for the concept in question while the thought and the word are mutually accommodating to each other. This process of mutual accommodation between thought and word, under the name of a verbal thought, has been considered by Vygotsky [15] as the unit of meaning of a concept, which underlined his theory of concept formation. It makes sense therefore to take this relationship as the criterion for understanding of the ε-N definition of the limit of a sequence, especially since, explaining the meaning of procedures and discussing concepts in written words was one of the basic instructional strategies during the experiment. Thus for the purpose of this research, only a student, who, while presented with the problem:

Find the limit of the sequence $a_n = \frac{3n+2}{2n-1}$ *and prove using ε-N definition, that the number you found is indeed the limit.,*

was able to show the <u>necessary computational competence together with being able explain precisely why his or her calculations constituted the required proof</u>, was classified as understanding the definition of the limit.

The sequence .9, .99, .999,… sequence was taken as an instrument to assess the coherence of the schema. This type of a sequence had not been discussed in class during the course.

Methodology. The teaching experiment took place in a section of Calc 1 class with 32 students participating and taking the final exam. The sample of 10 students was chosen among them for the clinical interview which had lasted approximately an hour. The sample in this cycle of the experiment was representing better motivated students in the class. The criterion here was the judgment of the instructor; in the next cycle a well tested measure of motivation will be used.

The data for the study of the understanding of the ε -N definition of the limit of the sequence consists of transcribed clinical interviews at the end of the term, just before the final exam, and answers to two questions concerning the limit of sequences from the final exam. Whereas the full analysis of the data awaits the publication of the full report, here, the analysis of two questions from the clinical interviews will be presented and supported by a similar question from the final exam.

<u>Interview questions</u>

1. a) What does the statement "**the limit of the sequence** $\mathbf{a_n} = \frac{3n+1}{4n-3}$ **is** $\frac{3}{4}$ "mean to you?
   b) How one could prove that the number you found is indeed the limit of that sequence?
2. Find the limit of the sequence a_n={.9, .99, .999,…}. How would you prove it is, indeed, its limit?

<u>Final Exam question</u>

3. a) Find the limit of the sequence $a_n = \frac{3n+2}{2n-1}$
   b) Prove using eps-N definition of the sequence that the number you found is indeed the limit of that sequence (start your work by writing the definition of the limit of a sequence)
   c) Explain your work.

The questions 1 and 3 ***w***ere supposed give information about students' understanding of the limit in the context of ε -N definition, while the question 2 was designed to provide the information about their ability to apply that definition in the unfamiliar context, and to check their mastery of the Wallis technique in conjunction with ε-N definition.

Data analysis. As mentioned above, the concept described by the ε-N definition is the result of the composition of two process, the process of choosing epsilon and the process of establishing the relationship between ε and N. Clear articulation of the relationships between these two meanings of the epsilon, has been taken in this research as the indication that a certain conceptual whole has been mentally constructed by the students and that it can constitute the foundation of the construction of the schema of limits of sequences. Thus the student responses of the type (accompanied by the correct computations)*[6]:

In order for the value of ε to reduce itself, is necessary that N is larger than 7/4e + 1/2 . This will give me the value closer to the limit than the value of ε I desire.

Or of the type:

Well, epsilon served as the base for…, knowing the closeness which we want with epsilon, we can obtain that so great should be N to obtain that particular closeness

were considered as evidence of such a coordination.

On the other hand, the responses like:

- *With that we can see that when ε is smaller, N is greater.*
- *With this we know that this great have to be the values of N to approach the limit.*

were not considered such an evidence, even if they had elements of truth in them.

In the interviewed sample of students, 50% demonstrated such a construction while during the final exam about 25% demonstrated it.

These results seem to suggest that the **coordination of the two processes constitutes a dividing line between students who had constructed the foundation of the schema and those who had not completed such a construction.** Consequently, the coordination of mental processes should receive a special attention during the instruction.

Coordination of these two processes is most probably made still more difficult by the duality between the visual and algebraic approaches with which students are presented in the attempt to synthesize the concept of a converging sequence. The graphical representation of the limit and of the sequence seems to be clearly described by the statement that <u>as n is creasing to infinity, the distance between the</u>

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[6] Translated from Spanish by authors of the presentation.

term and the limit becomes smaller. This formulation, suggested by the graph of a sequence, and possibly, by the notation $\lim_{n \to \infty} a_n$, is reciprocal to that suggested by the algebraic definition, where epsilon is an independently chosen measure of the distance and N is its function. Thus students who are unable to transcend the "temporal"[7] view upon the sequence and its limit could not make the connection with the formal definition at all. Below are samples of thinking of these students who, while computationally able, could not transcend the difficulty and hence did not construct the required schema:

- *N should be sufficiently large for ε to be sufficiently small. We want to demonstrate that the distance between a_n and L is small and is decreasing when n is approaching infinity..*
- *N should be sufficiently large for ε to be sufficiently small.*

The presence of this type of confusion was already known the authors of [7]; however its significance for the construction of the limit schema was not as clear as it is now. **Summary of the results**

As mentioned above, the teaching experiment described above is the first in a cycle of experiments whose goal is to successfully reintroduce the formal definition of the limit (taking into account the research conducted on this topic in the past several years) into the standard calculus instruction and the assessment of the results has to be viewed from that point of view. Whereas a 25% success rate in the class is not very high in terms of what could be achieved with respect to understanding the concept, yet we have obtained several invaluable insights, which will help to modify the instruction in the Spring Semester 2001. It is clear that a serious emphasis has to be made on the coordination of two processes participating in the definition. Moreover, it seems that among the discussed misconceptions of students, the one which needs to be dealt with most urgently is the temporal versus sequential conflict [7] upon the relation between epsilon and N. Many students who have constructed the basic schema of the limit, were not clear about the meaning and significance of the sequential order. There are certain students' excerpts, which suggest that the emphasis on the development of input-output meaning of a function can be a help in this particular process of understanding.

The other commonly occurring misconception namely, the concern for the last term of the sequence, does not seem to impact students understanding of the definition very drastically at the Calculus 1 level, and in our opinion the strategies designed to deal with it should be dealt with in Calculus 2 when that misconception starts to impact seriously the understanding of the definite integral [6].

Let us add that the Wallis technique contributed to high rate of success on question 2 of the interview. Every student who had coordinated the two processes was also successful on the application of the schema to a_n=.9, .99, .999…

Bibliography

[1]Asiala, M., Brown, A., DeVries, D. J., Dubinsky, e., Mathews, D., & Thomas, K. (1996). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. *Research in Collegiate Mathematics Education*, II (3), pp. 1--32.

[2]Cornu, B., (1991) Limits. In *Advanced Mathematical Thinking*, (ed) Tall, D. Mathematics Education Library, Kluwer Academic Publishers,

[3]Cottril, J., Dubinsky, E., Nichols, D. Schwingendorf, K., Thomas, K., Vidakovic, D. (1996) Understanding the limit concept: Beginning with the coordinated process schema; *JMB 15, (2)*

[4]Czarnocha, B., Dubinsky, E., Loch, S., Prabhu, V., Vidakovic, D., (1997). Investigating the nature of students' understanding of definite integral: Wallis or Riemann's idea. *Proceedings of the 21st Annual Meeting of the International Group for the Psychology of Mathematics Education (PME21), Lahti, FI, July 14-19.*

[5]Czarnocha, B., Dubinsky E., Loch, S., Prabhu, V., Vidakovic, D. (2001). Conceptions of Area: In Students and in History. *College Mathematics Journal, MAA (to appear in May)*

[6]Czarnocha, B., Loch, S., Prabhu, V., Vidakovic, D. (2001). The Concept of the Definite Integral: Coordination of two schemas. *Submitted as a Research Report for PME 25 in Utrecht, Holland*

[7]Davis, R., Vinner, S. (1986). The Notion of Limit: Some Seemingly Unavoidable Misconception Stages. *Journal of Mathematical Behavior*, Vol 5, 281-303.

[8]Dubinsky, E. (1991) Reflective Abstraction in the Advanced Mathematical Thinking. In *Advanced Mathematical Thinking*, (ed) Tall, D. Mathematics Education Library, Kluwer Academic Publishers,

[9]Orton, A. (1983). Students' Understanding of Integration. *Educational Studies in Mathematics, 14.*

[10]Sierpinska, A. (1987). Humanities Students and Epistemological Obstacles Related to Limits. *Educational Studies in Mathematics, Vol 18, pages 317-397.*

[11]Sierpinska, A. (1994) *Understanding in Mathematics.* Studies in Mathematics Education Series 2. The Falmer Press

[12] Wallis, J. *Arithmetica Infinitorium,* Typis Leon Lichfield Academiae Typographi: Ozonii, 1656.

[13] Wahlberg, M. (1998) The effects of writing assignments on second-semester calculus students' understanding of the limit concept. Presentation, *3rd RUMEC International Conference in Advanced Mathematical Thinking.*

[14] Williams, S. (1991) Models of Limit Held by College Calculus Students, *JRME, v.22, #3, p.219-236*

[15] Vygotsky, L. S. (1986) *Thought and Language*. The MIT Press Cambridge, Massachussets.

[16] Piaget, J., Garcia, R. (1983) *Psychogenesis and The History of Science*, Columbia University Press

[17] Piaget, J. (1980*) Adaptation and intelligence: Organic selection and phenocopy*. Chicago-University of Chicago Press