The Struggle Towards Algebraic Generalization And its Consolidation

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The activity of abstraction is central to mathematization. In the past, it has been discussed but generally not studied experimentally. The following study exemplifies a way for tracing processes of mathematization. We extend the nested model of abstraction elaborated by Hershkowitz, Schwarz, & Dreyfus (in press), to study successive activities of two Grade 7 students collaborating to solve algebra tasks in technological learning environment. The analysis demonstrates the consolidation of abstractions among the activities.

Theoretical framework

Mathematical activity, like any other human activity, is embedded in a socio-cultural environment (e.g., Voigt, 1995). This view is increasingly accounted for by the mathematics education community, which sees learning as a culture of mathematization in practice. Such an enculturation gains from alternating collective with individual activities, analytic with reflective stages, and integrating intra with inter-processes that are at the root of mathematical development (Hershkowitz & Schwarz, 1999a, see related ideas in scientific enculturation in classrooms in Woodruff & Meyer, 1997). A crucial issue concerns the relationship between construction of shared knowledge and the contribution of the individual (Hershkowitz, 1999). This issue is "hot", especially when the analysis does not focus on sole activities but on a series of activities.

Like in several other studies conducted in the CompuMath project (e.g., Hershkowitz & Schwarz, 1999a, 1999b; Schwarz & Hershkowitz, in press), we adopted the *activity theory* perspective (Leonte'ev, 1981), as a framework for studying different forms of practice of the individual and of the group within an between activities. The *activity theory* is a descriptive tool appropriate for socio-cultural analysis. The unit of analysis is not the individual human action but the activity as a whole, that is, "the minimal meaningful context for understanding individual action" (Kuutti, 1996, p. 28). An activity is a chain of actions done (cooperatively or individually) on the same object. Participant's motives determine their actions. Artifacts mediate actions on objects. Activities are under continuous change and development, where parts of previous activities are often embedded in following ones.

Mathematisation is central in mathematical enculturation. Abstraction is at the heart of mathematisation (Freudenthal, 1991; Gravenmeyer, 1995). To study abstraction experimentally, Hershkowitz, Schwarz, and Dreyfus (2001) gave an operational definition of abstraction: an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. The suggested model is based on three observable epistemic actions, which are nested in each other: *Constructing* (*C*) is the central action of abstraction. It consists of assembling knowledge artefacts to produce a new structure with which the participants become acquainted. The action of *Recognising* (*R*) a familiar mathematical structure, occurs when a student realises that the structure is relevant to the problem situation on which participants are engaged. The *Building-With* (*B*) action consists of combining existing artefacts in order to comply with a goal such as exploiting a strategy or justifying a statement. The *RBC* model of abstraction will be used in this article to trace the construction of new mathematical knowledge between different activities.

Collaborative problem solving in an interactive setting takes many forms. Kieran and Dreyfus (1998) recognised different types of peer interaction. The types designate the interaction itself and the same pair may adopt various types of interaction during the same setting (Kieran, 1999).

The study

We focus here on the work of two Grade 7 students who participated in a one-year algebra course. The basis for the selection of these two students was that they were used to talk to each other. Five activities were chosen out of an algebra course. The algebra course, an introductory course in the Compu-Math project (Hershkowitz et al, in press) consists of a sequence of activities organized around problem situations. Students had a spreadsheet program (Excel) at their disposal. The tasks in the algebra course were designed to give opportunities to students' construction of structures of mathematical concepts (algebraic variables and models) and of various mathematical processes (hypothesizing, making generalizations, testing hypotheses, interpreting representational information, solving and justifying). In the present study, we examine one socio-cultural setting in which these constructions took place. That is, we observe (a) the types of interactions while collaborative work is taking place and (b) the construction of a shared knowledge of the pair and, the contribution of each participant, as well as what is left of it in the individual. All these aspects are examined within and between five activities as one continuum along the academic year.

All activities were open - no guidance for solution was provided, neither instruction for making use of the Excel. The tasks in each activity were of increasing difficulty.

The work of the pair in the class was videotaped and written work was collected. The videotapes were transcribed. Following Chi (1997), the protocols were divided into "cognitive segments". The dimension of interaction was considered and analyzed as well (Resnick et al., 1993; Hershkowitz, 1999), so that the chronological flow of the interaction and its logical flow might be seen clearly, including the underlying assumptions and motives of the students.

In this presentation we will analyze selected parts from the last activity (The *Sequences of Dots* activity) in which the creation of new structures of knowledge takes place while students collaborate together. Conclusions concerning knowledge constructing and ways of interacting will be drawn.

The *Sequences of Dots'* activity.

This activity took place at the last month of the year while students were quite familiar with the spreadsheet, and were accustomed to work pair with dedicated peers. The activity consists of 6 tasks, two of which we focus on here. In each of the first two tasks students observed a sequence of dots (see Figure **1a** & **1b**), were asked to discover a pattern for the number of dots in the shapes of the sequence, and to express it algebraically. Such a kind of activity was quite new to students.

Figure 1 – a. the first three shapes in Task 1; b. the third and fifth shapes in Task 2.

Possible path for solving the tasks.

Note: Students may obtain a sequence of numbers describe a given pattern on a spreadsheet by using one of the following generalizations methods: (a) Relating recursively to the previous number in the sequence (usually appearing in the previous cell of the same column) (b) The explicit generalization - using the position numbers (usually appearing in the same row in an adjacent column). Whenever possible, most students tend to use the recursive method, in which they consider locally the difference between two consecutive numbers of the sequence. With the spreadsheet tool, the recursive strategy, which is primarily local, may turn to be global when the dragging operation is used, thus leading to a recursive generalization. It is obvious that educators valorize more explicit generalization - the position number method - in an algebra course, because it articulates the algebraic method of modeling which is general, and can be used in learning equations and functions.

Each of the first two tasks in the Dots activity was designed to promote generalizing of the dot pattern into an explicit algebraic expression, using the position number method (b). The design of the task supports the connection between a specific *counting method* of the dots number in the specific elements of a given dotted sequence and the corresponding algebraic expression. For example, in the sequence in Figure 1b one may see a central dot and four "arms" each of which containing n dots, leading then to the expression 1 + 4n. Alternatively, one may count the horizontal "arm" with 2n + 1 dots, and the vertical "arm" with 2n + 1 dots, and subtracting one dot counted twice. This counting method is reflected in the expression 2*(2n + 1) - 1. In short, various *counting methods* lead to different expressions, which are all equivalent. Linking together the counting method and the algebraic expression may support and help student at an early stage of instruction to generate symbolic generalizations.

In the first sequence (Figure **1a**), the three first shapes are presented, to help students to familiarize with the idea of sequence of shapes. In the second task (Figure **1b**), two non-consecutive shapes are presented. Our experience from experimental classes, showed that presenting a sequence of consecutive elements leads students to express the pattern they could generalize through a recursive method.

Analysis of the collaborative work of a dyad on the first and second task

In this part we will describe shortly the work done by a pair of students Avi&Ben on the task. We will present selected parts from the full transcript of their work in this activity.

Avi (A) & Ben (B) started working on the first task of the activity (Figure **1a**). They made several trials to generalize the given sequence of dots. Some of the trials were wrong. Finally they counted the first sequence of dots systematically in a correct way, generalized it through the use of verbal informal explanations, but failed to generate an algebraic expression. They moved on to the second sequence, and initiated a correct counting method. Yet, as Ben himself mentioned it, they did not generate an algebraic expression. Following the intervention of the teacher (T), they succeeded in finding an expression using the position number method. Then, they voluntary went back to the first task, counted it globally, using a correct method -- in a sense an application of the method they used in the second task, to generate also an explicit algebraic expression, using the position number method.

We go through some episodes to clarify the above general description:

In the first episode Avi is about to find a correct way to count the number of dots in each shape of the first sequence (Figure 1a) using the position number method.

A52 *I think I found something here*
B53 *What?*
A54 *In the first we add 4, in the second we add 6, in the third we add 8, in the fourth we add 10. Its place A…*
A55 *I don't know how to write this expression. I found the pattern: the first add 4, the second, you take the second and add to it 6, the third, you take the third and add 8, the fourth, you take the four that is its place and add 10.*
…
B87 *I know what can we do*
A88 *What?*
B89 *=A1+, NO, it will not do it. We need something that each time what we are adding will grow by 2.*

We see that Avi generates a systematic counting method (A54-A55), and expresses it by using the positing number. But, he does not know how to formulate it as an algebraic expression (A55).

After a while (B87-B89), Ben tries to write Avi's idea on the spreadsheet and fails as well.

The two boys move to the second task (Figure **1b**). So far, Avi was dominant. We will see in the next episode that this dominance will vanish. Both students observe the shapes in Task 2, and try to find how many dots there are in the 20th shape.

A105 How do you know that there are 81?
B106 *What? Because always on each line there are three at each side, or one, it depends on n, it depends on the number, it depends on the order.*
A107 *If it is 3 you add 1?*
B108 *No, here, this is the third place, so I have here 3, 3, 3, 3 and here another one.* [Ben waves his hand successively to the left, right, up, down, and finally points to the central dot.]
…..
B114 *Did you find the expression?*
A115 *No.*
T118 *Can you describe how the fourth shape is going to look like?*
B119 *Yes, four, that's like, four from every side, and here, an extra dot.* [pointing to the central point]
T120 *Four on every side and…*
B121 *And one point in the middle.*
T122 *And at the 200th place?*
B123 *Then it will be, how much is 199 divided by four?*
T124 *Wait, how the shape is going to look like at the 200th place, not the one that is made of 200 dots.*
A125 *Yes, a dot and another 200 on each side.* [draws in the air, lines at every side. Ben acquiesces]
T126 *A point in the middle and…*
A127 *200 on the right, 200 on the left, 200 up and 200 down.*
B128 *No*
T129 *Do you agree? Yes or not?*
B130 *Oh, yes, yes.*
T131 *So how would the n shape look like?*
AB132 *Oh, one plus n times 4.*
A133 *No, a dot and n here, n here, n here and n here* [show with his hand]

As we can see here, Ben is aware that he is invited to *construct* an algebraic model of the pattern. He fully understands of the pattern governing the sequence of shapes (B106, B108). He explains it to Avi, and yet, cannot find the algebraic expression (B114). We can see that Ben *Recognizes* the structure of the given shape in the sequence as being symmetric, consisting of four equal sides with one dot at its middle. Ben can *Build-With* it numerical solutions for various elements (B119, B123). The intervention of the teacher (T124) pushed the students to work on the shape at the 200th place. The students cannot use direct counting strategies anymore and are led to imagine the 200th shape, which is a significant step towards generalization. In this intervention, the teacher helps them to consolidate the pattern constructed before. Ben is even able to answer a question requiring "backwards" numerical thinking (B123), but he needs the support of the teacher in order to express the algebraic expression explicitly with the position number method. Both peers show understanding of the expression (AB132). The mediation of the teacher makes it possible for them to make the necessary *Construction* that leads from the verbal model to the algebraic model based on the position number method.

In the next episode, the two boys are going back voluntarily to the first task, in which they failed generalizing the given pattern algebraically:

B155 *So here, that's what I'm saying, each time it's like, here we have to add four, here six and here eight.*
…
T173 *The counting method that was used there didn't give you any idea? What was the idea there?*
B174 *Oh, I get it. I know what. It is like one of the lines, and we can double ,*
A175 *Oh, I found it, here we add one, one here and one here. =A1 + 3*
[writes the expression in the computer and drags down]*, oh, no!*
B176 *OK, I found it*
A177 *Yes?*
B178 *N + (N + 1) * 2*

We can see that Ben capitalizes on Avi's idea in their last attempt of generalizing the sequence (B155). Following an additional intervention from the part of the teacher (T173), Avi phrases a wrong generalization, combining together generalization by recursion and generalization by position number (A175). Ben implements successfully the *Construction* he made by himself in the second task, and generates a correct explicit algebraic expression, based on the position number. The knowledge, which was constructed in the second task, was *recognized* here, in a similar mathematical task, and was used for *Building-With* the needed generalization.

Conclusion remarks

We described here a pair of students working in collaboration in order to solve a problem. Both of them were determined to solve it, listened to each other, and tried to explain their ideas to each other. Kieran and Dreyfus (1998) designate this interacting style as *inhomogeneous,* each of the student trying to make an effort to understand his partner's thought (Trognon, 1993).

The analysis of the collaborative work of Avi&Ben is an additional example in which the RBC nested model of abstraction (Hershkowitz, Schwarz, & Dreyfus, in press) can be used. Here as well we can see that the first two actions (*Recognizing & Building-with*) are nested at the *Construction* action. Hershkowitz, et al. did not show that newly created knowledge structures are consolidated, as they are used as artifacts in further activities. In the present we succeeded to show how such a transition may happens; where the previous *Constructed* knowledge is becoming *Building-with* in the second cycle of the pair work on Task 1. The importance of this example is beyond the specific knowledge that was constructed and used later on in the activity. It shows that the suggested RBC model can be used as a methodological tool, by which consolidation of abstracted knowledge can be observed and investigated.

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