ON CONVERGENCE OF A SERIES: THE UNBEARABLE INCONCLUSIVENESS OF THE LIMIT-COMPARISON TEST

Elena Nardi and Paola Iannone
School of Education and School of Mathematics,
University of East Anglia, UK

Formal mathematical reasoning is often studied in terms of the students' conceptualisation of the necessity for proof, as opposed to empirical ways of reasoning; or in terms of the mechanics of the students' reasoning regarding specific proving techniques. Here, in the context of testing the convergence of a series in Calculus, we address one issue regarding the latter. In this, statements about convergence, for instance the Limit Comparison Test, are transformed by the students into statements about divergence in particularly problematic ways; in fact in ways that suggest a multiplicity of difficulties with mathematical logic and a resistance to the idea that, in certain occasions, convergence tests are inconclusive.

Transition to advanced mathematical thinking is often described as acknowledgement of and fluency with the abstract nature of mathematical objects and with formal mathematical reasoning (e.g. Tall 1991). Within upper secondary and university mathematics the latter has been studied mostly in the context of Proof: either in terms of the students' conceptualisation of a necessity for proof (as opposed, for instance, to intuitive, informal or empirical ways of reasoning (e.g. Coe and Ruthven 1994)); or it terms of the students' enactment of these conceptualisations (namely the mechanics of their reasoning, for instance regarding specific proving techniques such as Mathematical Induction (e.g. Movshovitz-Hadar 1993)). Studying the students' formal mathematical reasoning with the foci suggested in the latter studies is a particularly multi-layered, complex task as it involves a consideration of the conceptual difficulties within the specific mathematical topics, students' problem solving skills (a map of this complexity can be found in (Moore 1994)).

In this paper we wish to address one issue regarding the students' enactment of proving techniques. This is in the context of testing the convergence of a series in Calculus, a task encountered by most students in the beginning of the first year of their undergraduate studies. For this we will draw on data collected in a study of the transition from school to university mathematics currently in progress in one Mathematics Department in the UK. First however we outline the study and its methodology.

Methodology. This project is funded by the Nuffield Foundation and its initial phase (Phase 1: Calculus and Linear Algebra, October - December 2000) lasted 3 months (Phase 2: Probability will be January - March 2001). It is located within a series of projects that the first author has been involved in for several years (see Note 1) and its title is *The First-Year Mathematics Undergraduate's Problematic Transition from Informal to Formal Mathematical Writing: Foci of Caution and Action for the Teacher of Mathematics at Undergraduate Level*. It is an Action Research project (Elliott 1991) and can be seen as a natural descendant of its predecessors (see Note 2).

The aims of the study are: identifying the major problematic aspects of the students' mathematical writing in their drafts submitted to tutors on a fortnightly basis; increasing awareness of the students' difficulties for the tutors at this University's School of Mathematics; providing a set of foci of caution, action and possibly immediate reform of practice; and, setting foundations for a further larger-scale research project.

The study is carried out as a collaboration between the School of Education (where the first author is a Lecturer) and the School of Mathematics (where the second author teaches the first-year undergraduates) at UEA. The focus of the research, examining the students' *written* expression, has been identified as a worthy domain of investigation in Projects 1-3: these studies examined the students' development of mathematical reasoning in the wider context of both *oral* and *written* expression - the latter merits further elaboration and refinement and has also been highlighted by teachers of mathematics at university level as an aspect of the students' learning that calls for rather urgent pedagogical action (e.g. Nardi 1999).

This is a small, exploratory data-grounded study (Glaser and Strauss 1967) of the mathematical writing of the students in Year 1 (60 students in total, 16 in the second author's tutorial class). Phase 1 was conducted in 6 cycles of Data Collection and Processing following the fortnightly submission of written work by the students during a 12-week term. Phase 2 will be conducted in two such cycles. Each 2-week cycle consists of the following stages:

- Beginning of Week 1: Students attend lectures and problem sheets are handed out.
- Middle of Week 1: Students participate in a Question Clinic, a forum for questions from the students to the lecturers.
- End of Week 1: Students submit written work on aforementioned problem sheets.

- Beginning of Week 2: Students attend tutorials in groups of six and discuss the now marked work with their tutor.
- End of Week 2: Data Analysis Version 1, towards Data Analysis Version 2.

The second author, who is also a tutor and is responsible for collecting and marking the students' work, carries out an initial scrutiny of the students' scripts and composes Data Analysis Version 1 (see Note 3): this consists of a Question/Student table where each student's responses to (a selection of) the problem sheet's questions are summarised and commented upon. The focus of her comments is quite open and covers a large ground of regarding the content and format of the students' writing. In an appendix to this table she produces rough frequency tables that reflect patterns in the students' writing and informal commentary by the tutors who teach the rest of the 60 students. Following a detailed discussion of Data Analysis Version 1, the first author produces Data Analysis Version 2, a question by question table where the major issues are summarised, characteristic examples of the students' work are referred to and links with current literature are made. A large part of these discussions revolve around the exchange of ideas and expertise. Examples of this exchange include: the communication of the second author's experiences as a tutor and a mathematician as well as her observations of the lectures and the Question Clinic, her consultation of other tutors and lecturers involved with teaching the students in Year 1; also her introduction to relevant findings from mathematics education research and educational research methodology.

Version 2 is then available to the other tutors for further informal commentary (we intend to introduce more formal strategies of evaluation in subsequent projects). An outcome of the discussion on Version 2 across the cycles will be a set of Macro and Micro Points of Action - a brief reference and examples of these can be found in (Nardi and Iannone 2000).

In Phase 1, by the end of the 12th week, 6 sets of data and analytical accounts as described above were produced. On completion of Phase 2 (Easter 2001), we intend to organise a Departmental Day Workshop to disseminate and discuss our results and also cultivate opportunities for extending the project towards an implementation of our Action Points.

Formal Mathematical Reasoning in the Context of Convergence of Series. The data we wish to draw on here originate in Cycles 5 and 6 of Phase 1 and concern the students' responses to Questions 5.4(2) and 6.1c(iii) below. The students answered by using the *Limit Comparison Test*: *Assume that for sequences* $a_n>0$ *and* $b_n>0$ $\lim_{n\to\infty} a_n / b_n= c\in\Re$. *Then: if* $\sum b_n$ *converges then* $\sum a_n$ *converges.*

Question 5.4:

(4) For each of the following series, decide whether or not it converges. State carefully any tests for convergence that you use. (1) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$. (2) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$. (3) $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$. (4) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$. (5) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

Question 6.1c:

(c) Decide whether each of the following series are convergent (justify your answers and make clear which results you are using):

(i) $\sum_{n=1}^{\infty} \frac{1}{n!}$;

(ii) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$;

(iii) $\sum_{n=1}^{\infty} \frac{n+2}{n^3-n^2+11}$.

In fact both series are convergent: 5.4(2) by the Comparison Test (compare, for example, with $2/n^2$) and 6.1c(iii) by the Limit Comparison Test for $b_n = 1/n^2$. In both occasions a substantial number of students used the Limit Comparison Test in particularly problematic ways. In the following we exemplify the students' responses and reflect on ensuing matters.

Regarding 5.4(2), out of 16 students only one concluded that it converges (by comparison with $\sqrt{n}/n^2$); one stated it converges but provided no justification; one stated it converges but provided an inscrutable (unintelligible scribble on the draft) justification based on the Ratio Test; and 5 did not attempt it at all. Here we are concerned with the remaining 8 responses, 4 of which involved the use of the Limit Comparison Test and 4 involved the use of the Comparison Test. In doing so we hope to illustrate one deep-seated difficulty with formal mathematical reasoning in the students' thinking.

Regarding 6.1c(iii), which was in the Problem Sheet of the following fortnight, results were better but still alarming: out of 16 students 5 did not submit any draft or did not attempt the particular question; 6 applied the Limit Comparison Test successfully and one used the Comparison Test successfully (comparison with $3/n^2$); one attempted but left incomplete and inscrutable use of the Comparison Test. Here we are concerned with the remaining three responses which involved the use of the Limit Comparison Test. As above the focus of our concern will be on the students' reasoning processes.

Here is one of the four responses to 5.4(2) that involved the use of the Limit Comparison Test, Hazel's - the other three were identical:

2) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$ by LCT, let $\sum b_n = \sum \frac{1}{n^2}$

$\frac{a_n}{b_n} = \frac{(\sqrt{n}+1)\cancel{n^2}}{\cancel{n^2}} = \sqrt{n}+1 \longrightarrow \infty$ $\therefore$ does not converge.

Hazel's interpretation of the Limit Comparison Test in this case seems to be the following: *if $\sum b_n$ converges but $\lim_{n\to\infty} a_n / b_n = \infty$, then $\sum a_n$ diverges*. In fact the test is inconclusive in this case.

Despite cautionary remarks in the Question Clinic and the following tutorial, Hazel's interpretation of the Limit Comparison Test is still problematic two weeks later. Here is her response to 6.1c(iii) - again identical to those of her peers:

iii) $\sum_{n=1}^{\infty} \frac{n+2}{n^3-n^2+11}$

by LCT: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ $\sum b_n$ does not converge

$\therefore$ $\frac{a_n}{b_n} = \frac{\cancel{n+2}}{(\cancel{n+2})(n^3-n^2+11)} = \frac{1}{n^3-n^2+11} \longrightarrow 0$

$\therefore$ a_n does not converge.

This time her choice of $\sum b_n$, the harmonic series, *does not converge and $\lim_{n\to\infty} a_n / b_n = 0$*. As a result she concludes that *$\sum a_n$ diverges*. Again the test is inconclusive in this case and the student ought to have pursued an answer via a different test.

What we wish to bring attention to here is the students' ***resistance to the idea of a test's inconclusiveness***: what appears to be the case in Hazel's (and her peers') work is that, once a theorem has been selected for testing the convergence of a series, in this case the Limit Comparison Test, it *must* provide an answer. What escapes the students is that, for the Limit Comparison Test to provide an answer, that is the convergence of $\sum b_n$ to imply the convergence of $\sum a_n$, all conditions must apply: a_n and b_n must be positive AND $\lim_{n\to\infty} a_n / b_n$ must be real. If either of a_n or b_n is not positive, or the limit is not real, then the convergence of $\sum b_n$ cannot imply the convergence of $\sum a_n$. But not implying the convergence of $\sum a_n$ is not equivalent to implying its divergence (as Hazel's 5.4(2) response seems to suggest). Similarly, if either of a_n or b_n is not positive, or the limit is not real, then the divergence of Σb_n cannot imply the divergence of Σa_n (as Hazel's 6.1c(iii) response seems to suggest).

Resistance to the occasional inconclusiveness of the tests was evident in the responses to 5.4(2) of a substantial number of students who attempted to use the Comparison Test: *Assume that for sequences $a_n>0$ and $b_n>0$ there exist positive constants N and c such that $a_n<cb_n$ for $n>N$. Then: if Σb_n converges then Σa_n converges.* Here is a characteristic response, Nicolas':

(2) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$

Use comparison test with $\sum_{n=1}^{\infty} \frac{n+1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n}$ which does not conver.

and $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n}$ $\therefore$ $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$ does not converge

Nicolas' interpretation of the Comparison Test in this case seems to be the following: *if Σb_n diverges and $a_n<b_n$ then Σa_n diverges*. (Note also that in his draft it is $\Sigma a_n<\Sigma b_n$ that he has written, not $a_n<b_n$, but we leave this issue - dealt elsewhere (Nardi 1996) as tendency to handle series and Σ as finite sums - aside for the moment). The fact is that the test is inconclusive in this case. Again, what escapes the students is that, for the Comparison Test to provide an answer, that is the convergence of Σb_n to imply the convergence of Σa_n, all conditions must apply: a_n and b_n must be positive AND $a_n < b_n$. If either of a_n or b_n is not positive, or the inequality does not hold, then the convergence of Σb_n cannot imply the convergence of Σa_n. Also, if the conditions hold but Σb_n is divergent, then the divergence of Σb_n cannot imply the divergence of Σa_n (as Nicolas' 5.4(2) response seems to suggest).

Underlying the students' attitude seems to be a desire for closure and completeness: according to this, a convergence test *must* cover the ground of possible responses to a question; the contingency of absence of such an answer is unsettling and therefore students feel it needs to be avoided at all costs. Albeit, instead of seeking an answer via the employment of a different convergence test (lack of flexibility in switching modes of pursuing an answer is documented in the problem-solving literature, e.g. in (Schoenfeld and Hermann 1982)), they tend to 'mutate' the proposition at hand (here: the Limit Comparison Test) towards a convenient expression that serves their purpose. Here the 'mutants' include the versions of the Limit Comparison Test from which the alleged divergence of the series in question can be deduced and are ostensible reversals/negations of the proposition in the Test (difficulties with the action of negation have been documented e.g. in (Barnard 1995)).

This desire for closure and completeness is not uncommon at all: it seems to be located neatly side by side, for example, with the students' greater fluency and gravitation towards a sense of symmetry in reasoning in general. Examples: the students' far swifter handling of 'if and only if' statements as opposed to 'if' ones - as the literature on the students' difficulties with *modus ponens* suggests (dating back in the 1970s e.g. with O'Brien's work (1973)); of equalities as opposed to inequalities. The latter is documented e.g. in (Anderson 1994) and in parts of our data, sampled elsewhere (Nardi and Iannone, in preparation), where, for example, the students' handling of a proof by Mathematical Induction was hindered by the fact that the statement-to-be-proved was in the form of an inequality (hence the proof for P(n+1) could not be constructed from the assumption that P(n) is true as straightforwardedly as in the case of an equality).

NOTES

1. These projects are:
   Project 1: a doctorate (Nardi 1996) on the first-year undergraduates' learning difficulties in the encounter with the abstractions of advanced mathematics within a tutorial-based pedagogy
   Project 2: a study of the tutors' responses to and interpretations of the above mentioned difficulties (e.g. Nardi 1999), and,
   Project 3: UMTP, the *Undergraduate Mathematics Teaching Project* with Barbara Jaworski and Stephen Hegedus, a collaborative study between researchers and tutors on current conceptualisations of teaching as reflected in practice and their relations to mathematics as a discipline (e.g. Jaworski, Nardi and Hegedus 1999).
2. We tend to think of this study as Project 4, not only for its obvious thematic links with the previous projects but because it carries further the methodology of partnership (Wagner 1997) and materialises what was an underlying intention in Projects 2 and 3: the involvement of the mathematician as a reflective practitioner and her engagement with Action Research.
3. In the conference presentation we intend to demonstrate and discuss samples of the Problem Sheets, Versions of the Analysis (1 and 2), Extracts from the Data and the List of Micro and Macro Action Points.

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