AN INVESTIGATION OF PRESERVICE ELEMENTARY TEACHERS' SOLUTION PROCESSES TO PROBLEMATIC STORY PROBLEMS INVOLVING DIVISION OF FRACTIONS AND THEIR INTERPRETATIONS OF SOLUTIONS

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In this study, we examine 68 preservice elementary teachers' solution processes to a problematic story problem involving division of fractions. The problem was problematic in the sense that the numerical answer to the division does not necessarily provides the appropriate solution to the problem, at least if one takes into consideration the realities of the situation embedded in the context of the word problem. It was found that the sample of prospective elementary teachers did not always base their responses on realistic considerations of the context situation. In fact, only 19 (28%) of the participants' responses contained a realistic solution to the given problem. In addition, only 4 (6%) participants provided an explanation for their solutions. None of the participants made any comments about the problematic nature of the problem. It was also found that other factors contributed, at least in part, to students' unrealistic solutions: inappropriate mathematical models and mistakes on the execution of the mathematical procedures.

Story problems play a fundamental role in school mathematics because of several reasons. First, they intent to offer examples of where mathematics can be applied in "real-life" contexts. Second, they offer students opportunities to make connections between mathematics procedures or formulas and real-world situations. Third, they offer opportunities for students to practice mathematical procedures and formulas. However, in many cases the "stereotyped" nature (Nesher, 1980) of word problems leads students to apply mathematical procedures in an instrumental way, without realizing the inappropriateness of their actions. As an example, consider the following problem: What will be the temperature of water in a container if you pour 1 jug of water at 80˚F and 1 jug of water at 40˚F into it? (Nesher, 1980, p. 46). Verschaffel, De Corte, and Lasure (1994) used this item with 75 fifth graders in Flanders. They reported that only 13 (17%) students provided a realistic response to the problem. Other studies (e.g., Contreras, 2000; Reusser & Stebler, 1997; Silver, Shapiro, & Deutsch, 1993) further illustrate that many students fail to activate their real-world knowledge or to use realistic considerations to solve problematic word problems.

This study is part of a larger project whose main purpose is to understand prospective teachers' knowledge of the meaning of quotients and reminders when solving story problems involving division. The purpose of the present paper is threefold. First, we examine prospective elementary teachers' solution processes and their use (or lack of) of realistic considerations to solve problematic story problems involving division of fractions. Second, we examine the extent to which prospective elementary teachers explicitly interpret the solution to mathematical procedures. Third, we also examine the extent to which the participants' solutions support or refute a referential-and-semantic-processing model proposed by previous research (Silver et al., 1992, 1993).

THEORETICAL AND EMPIRICAL BACKGROUND

Word problems also provide students with experiences about the process of mathematization, especially mathematical modeling. We define mathematization as the representation of aspects of reality by means of mathematical procedures. The process of mathematization involves the application of mathematical ideas and procedures to solve real-world problems. We will adopt Silver et al.'s (1992, 1993) model of mathematization for the story problems presented to the participants of the present study. Figure 1 displays Silver and colleagues' model.

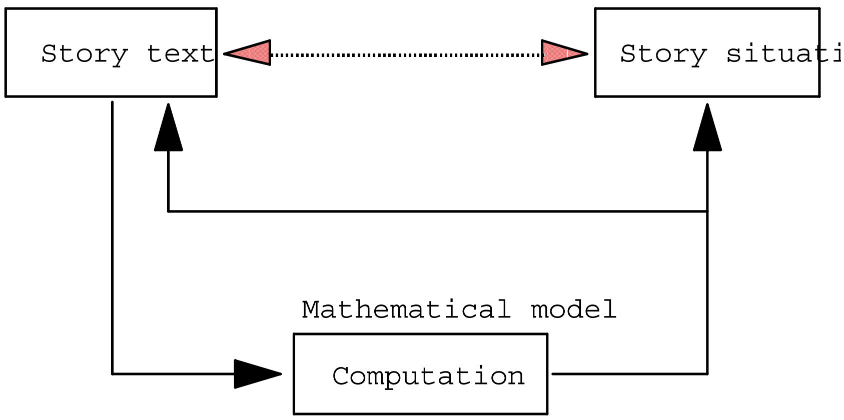

Figure 1: Silver et al.'s (1992, 1993) referential-and-semantic-processing model

According to this model, students' understanding of the structure of the problem might enhance their abilities to represent its solution through a mathematical operation or procedure. That is, students would map from the story text (the problem) to an appropriate mathematical model. Students then perform the required computations. Next, students should interpret the results of the computation using their real-world knowledge about the story text or story situation in the "real world." In other words, students need to map the computational result back to the story problem or to the implied "real-world" situation. If students fail to perform any of those three activities of the process of mathematization, they may not be able to provide a realistic response to the problem posed. While Silver et al. (1992, 1993) focused on students' lack of mapping from the numerical answer of the mathematical model to either the story problem or story situation, we hypothesized that students' range of solutions to the problem could be explained more completely by paying attention not only to the semantic processing feature of the model (mapping from the computation of the mathematical model to the story text or story situation) but also to the mapping from the story text to the mathematical model and to the execution of the mathematical procedures. Our hypothesis was based mainly on our teaching experience that suggests that problems involving division fractions are cognitively more complex than division of whole numbers. Our hypothesis turn out to be reasonable. We will refer to the Silver et al.'s model as *referential-and-semantic-processing model* to distinguish it from their emphasis on the semantic feature of the model.

The story problem examined in the present study involve division of fractions in which the quotient does not represent the solution to the problem, at least if one takes into account some realistic considerations of the story context or story situation in the "real world". Such problems are called problematic division problems. One of the first sources to document students' solutions to problematic problems involving division is the Third National Assessment of Educational Progress (Carpenter, Lindquist, Matthews, & Silver, 1983). It was reported that only about 24% of the students who took such test gave a correct solution to the problem: "An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?" Others studies (e.g., Cai & Silver, 1995; Ruwisch, 1999; Silver, Mukhopadhyay, & Gabriele, 1992; Silver, Shapiro, and Deutsch, 1993) have investigated students' solutions processes and responses to problematic division problems. In their study, Silver, Shapiro, and Deutsch (1993) examined 195 middle school students' solution processes and their interpretation of solutions to a division problem involving remainders. The problem read as follows

> The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game?

The researchers administered three versions of this problem. The different versions differed in the amount of people going to the game (532, 540, and 554) so that the decimal part of the numerical quotient were less than .5, .5, and greater than .5. The researchers had hypothesized that the size of the remainder could influence students' responses. They reported that over 60% of the students who used an appropriate procedure to find the solution executed their procedures correctly, but only 43% of the total number of participants provided 14 as their solution to the problem. The researchers also found that the size of the reminder did not influence students' responses. The authors concluded that the semantic feature of their model (Fig. 1) provides a solid explanation about students' lack of sense making or "suspension of sense-making" (Schoenfeld, 1991) when solving problematic story problems. In this paper we extend Silver and colleagues' research to a new content domain (fractions) and to a different population (preservice elementary teachers).

METHOD

Sixty eight prospective teachers participated in the study. A paper-and-pencil test was constructed consisting of five items involving division of fractions and decimals. As stated above, our original purpose was to examine prospective elementary teachers' understanding of the meaning of quotients and remainders of division with fractions or division with decimals. In this paper we report the results of the first problem which was stated as follows, Lida is making muffins that require $\frac{3}{8}$ of a cup of flour each. If she has 10 cups of flour, how many muffins can Lida make? (*The muffins problem*)

The prospective elementary teachers were enrolled in two sections of a mathematics course for elementary majors at a southeastern state university in the USA. The test was given to students during regular class and they were told orally that they would have enough time to complete it. Students were not allowed to use calculators. Written instructions asked students to explain each of their solutions and to write down any questions, comments, or concerns that they might have about each problem.

ANALYSIS AND RESULTS

Students' written responses to the problem were examined to detect four features of their solution processes (a) mathematical model, (b) execution of the procedures, (c) solution to the problem, and (d) explicit interpretation of the solution to the mathematical model. The mathematical model refers to the mathematical operation or procedures that students used to represent and obtain the solution to the problem. Execution of the procedure referred to the set of steps or actions that students took to obtain the solution represented with a mathematical operation or procedure. The solution to the problem referred to the solution that students provided to the story problem presented to them. Finally, the explicit interpretation of the solution provided by the mathematical model referred to the explanation that students provided to justify their solution to the word problem. Table 1 presents the distribution of the mathematical operations or procedures (mathematical model) that students used to represent and obtain the solution to the word problem.

Table 1: Distribution of the mathematical models

Procedure	Frequency		Percentage	
Appropriate procedures (Total)		47		69%
Division	38		56%	
Proportion	4		6%	
Multiplication	2		3%	
Unit rate and multiplication	2		3%	
Others	1		1.5%	
Inappropriate procedures (Total)		20		29.5%
Multiplication	12		17.5%	
Reverse division	3		4.5%	
Others	5		7.5%	
No responses provided		1		1.5%

A procedure was judged appropriate if it could potentially produce the correct solution to the problem. As indicated in Table 1, less than 70% of the students used an appropriate procedure. In particular, nearly 6% used a proportion, about 3% used multiplication, and nearly 3% used the concept of unit rate and then multiplied by 10. Table 1 also shows that about 29.5% of the students used procedures that were judged as inappropriate. Specifically, about 17.5% multiplied the two given number, 4.5% used reverse division (3/8 ÷10), and nearly 7.5% used other procedures such as estimation, incorrect concrete models, etc. Only the appropriate procedures were examined for correctness of execution. Out of the 47 students who set up an appropriate procedure, 40 executed the procedure correctly, 5 incorrectly, and 2 did not show work. As we can see, only forty students were able to set up an appropriate procedure, execute it correctly, and showed work.

Table 2 exhibits a categorization of students' solutions and the percentage of the students providing each solution. A solution was coded as realistic (RS) if it was 26, 27 or any other solution with an appropriate justification. Nearly 28% of the solutions were categorized as RSs. A solution was categorized as reasonable unrealistic solution (RUS) if it was the result of an appropriate procedure executed correctly and it was different from 26. About 37% of the solutions were categorized as RUSs. A solution was categorized as incorrect reasonable unrealistic solution (nearly 6%) if it was the result of an appropriate procedure executed incorrectly. Finally, underestimated solutions close to 3 (3 and 3 3/4) or 23 (23.1 and 23) and overestimated solutions (28, 35, 375) were classified as unreasonable unrealistic responses (about 28%).

Table 2: Distribution of students' solutions

Category and numerical solution	Number of students		Percentage of students	
Realistic solutions		19		28%
26	17		25%	
27	1		1.5%	
26 2/3 or 26	1		1.5%	
Reasonable unrealistic solutions		25		37%
26 2/3	15		22%	
26.6_	3		4.5%	
26.6‾	3		4.5%	
Other solutions	4		6%	
Incorrect reasonable unrealistic solutions		4		6%
Unreasonable unrealistic responses		19		28%
No response		1		1.5%
Total		68		100%

We also examined students' interpretations of the solutions to their procedures. An interpretation was defined as the written explanations provided by the students to justify the solution to the problem based on the execution of the procedure. An appropriate interpretation was one in which the student gave a meaningful interpretation to both the integer and fractional parts of the solution produced by the mathematical model. Only two students provided appropriate interpretations. An interpretation was judged inappropriate when the student's interpretation of either the integer or fractional part of the solution to the mathematical model was incorrect. Two students gave inappropriate interpretations. For example, one of them said that "Lida can make 26 muffins with 2/3 cup of flour left over." The other student wrote a similar statement. Sixty three (about 93%) students did not provide an interpretation to the solution produced by the mathematical model. In particular, 10 students wrote procedural explanations that described the procedure followed to get the answer to the problem, 36 reported a solution to the problem equal to the solution produced by the mathematical model (and hence they did not probably see the need to justify their solutions), and 17 did not explain why their solution to the problem was different from the solution produced by the mathematical model.

Finally, we also gather evidence to support or refute aspects of the referential-and-semantic-processing model proposed by Silver et al. (1992, 1993). About 78% of the responses provided direct evidence to support the model. Specifically, nearly 6% of the responses provided an appropriate justification to support a realistic solution and 72% contained information that explained, at least in part, why some students failed to obtain a realistic solution. The 6% of the realistic solutions supporting the model contained the numerical solution of 26 and also an appropriate interpretation of 26. Out of the 72% of the unrealistic solutions, about 26.5% contained an inappropriate mathematical model, 4.5% a procedure executed incorrectly, and about 39.5% contained a reasonable unrealistic solution that was the result of an appropriate procedure executed correctly but there was not an interpretation of the result produced by the mathematical model.

About 16% of the responses contained indirect evidence to support the referential-and-semantic-processing model. These responses contained the solution of 26 to the problem but failed to include an appropriate interpretation or justification about why the solution to the problem was different from the result produced by the mathematical model. Nearly 6% of the responses were judged as containing counter-evidence for the model. These responses contained the realistic solution of 26 but also contained flaws either on the selection of an appropriate mathematical model or on the execution of the mathematical model. In other words, those students managed to obtain the realistic solution without a correct mathematization of the problem.

To examine the influence of students' lack of semantic processing on their final solutions to the given problem, we focused on the 48 responses that contained realistic solutions or reasonable unrealistic solutions. That is, responses that contained or potentially could have contained realistic solutions. We found that 29 (about 60.5%) of those responses were unrealistic because students failed to map from the computation produced by the mathematical model to the story problem or story situation in the "real world".

DISCUSSION

One of the major goals of this study was to better understand some of the contributing factors to prospective elementary teachers' solutions to a problematic story problem involving division of fractions. Silver et al.'s (1992, 1993) referential-and-semantic-processing model provided with some solid explanations of students' failure to offer a realistic solution to the problem. The model includes three important actions that successful problem solvers might take to assure a realistic solution: set up an appropriate mathematical model, correctly execute the procedural features or steps called upon by the mathematical model, and interpret the numerical solution produced by the mathematical model. About 72% of the solutions were categorized as unsuccessful solutions. In particular, nearly 1.5% of the responses contained no work, about 26.5% involved an inappropriate mathematical operation, nearly 4.5% had flaws on the execution of the procedure, and about 39.5% did not include a semantic interpretation of the result produced by the mathematical model. About 28% of the responses contained a successful (realistic) solution to the problem. In particular, only about 6% of the responses explicitly included a semantic interpretation to the numerical answer produced by the mathematical model, and over 16% reported a realistic solution without a written explanation or justification for their response. However, nearly 6% of the responses included a realistic response but contained either inappropriate mathematical models or flaws in the execution of procedures. This finding suggests that some students sometimes use non-mathematical means to solve mathematical problems. These responses were considered as evidence to refute the referential-and-semantic processing model. The other responses were judged as supporting either directly or indirectly the referential-and-semantic processing model.

Although the majority of prospective elementary teachers set up an appropriate procedure to solve the given problem, a high percentage of students failed to do so. In fact, about 30% of the students set up the problem incorrectly. The most common inappropriate procedure was multiplication (10 x 3/8). Given that the students had already posed and solved word problems involving multiplication and division of fractions, this finding seems somewhat unexpected and discouraging. On one side, instruction did not have the desirable effect; on the other, some of those prospective teachers might teach fractions without having a good understanding of modeling situations for which division or multiplication of fraction is an appropriate operation. As to execution of procedures, prospective elementary teachers had, in general, little difficulty in performing the computation procedures successfully.

We recognize that the muffins problem is problematic since it potentially admits multiple solutions or interpretations depending on the context but that was the whole point of the problem. The same division expression (in this case 10 ÷ 3/8) can represent different problem situations. We asked students to solve the muffins problem because we wanted to confront them with a problematic problem involving division of fractions for which the solution could be either the integer part of the quotient (26) or one more than the integer part of the quotient (27) depending on aspects of the situational context. In the muffins problem, we could argue that a realistic solution is 26 because each muffins requires 3/8 of a cup of flour and there is not enough flour to make another one. However, in another situational context some students might argue that Lida could make another muffin which would use 1/4 (2/3 of 3/8) of a cup of flour. In this case the answer would be 27 muffins. While we did not expect all students to provide realistic reactions or comments to the problematic situations, it was surprising to find that only 4 (about 6%) students provided an explicit explanation for their solution, even though they were asked to explain their answers and to write down any questions or concerns that they might have about each problem. We suspect that a factor that contributed to a great extent to students' failure to react realistically to the problems was students' impoverished experiences with standard problems in which the solution can be obtained by the straightforward application of one or more simple arithmetic operations with the given numbers. Further empirical research is needed to enhance our understanding of prospective elementary teachers' lack of semantic interpretation of solutions to problematic word problems.

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