STUDENTS' SHIFTING CONCEPTIONS OF THE INFINITE THROUGH COMPUTER EXPLORATIONS OF FRACTALS AND OTHER VISUAL MODELS

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This paper presents some results from a study whose main concern was to investigate students' developing conceptions of infinity and infinite processes as mediated by a Logo-based environment or microworld. The general findings of the study indicate that the environment and its tools shaped these students' understandings of the infinite in rich ways, allowing them to discriminate subtle process-oriented features of infinite processes; it also permitted the students to deal with the complexity of the infinite by assisting them in coordinating the different elements present. The corpus of data is based on case studies of 8 individuals, whose ages ranged form 14 to mid-thirties, interacting with the microworld as pairs of the same age group.

The concept of infinity is central to calculus: infinite processes form the basis for the concept of limit; it is also present in other important areas of mathematics. This concept, however, has always been recognised as difficult and has historically been the origin of paradoxes and confusions. Fischbein et al. (1979) claim that the concept of infinity (and specifically of infinite divisibility) is intuitively contradictory. As explained by these and other authors (e.g. Waldegg, 1988), contradictory situations arise because "finitist" interpretations tend to prevail (such as the idea that the whole is always bigger than the parts), a fact recognised more than 350 years ago by Galileo[1]. Another finding (see also Nuñez, 1993 and Hauchart & Rouche, 1987) is that the context and form of representation are very influential in the type of responses the students give: if a geometric set is bounded, this may become an obstacle for its infinite quantification. It has also been argued (e.g. Waldegg, 1988; Sierpinska & Viwegier, 1989; Sacristán, 1991; Cornu, 1991) that the spontaneous conceptions and intuitions that people have of infinite processes and of infinite (mathematical) objects can become obstacles for the adequate construction of formalised versions of these

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[1] Galileo, in his in his *Dialogues Concerning Two New Sciences* (1638), wrote: "difficulties arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited".

concepts. However, both Fischbein et al. and Waldegg have found that formal mathematics teaching does not modify students' conceptions and intuitions of infinity. Furthermore, the areas of mathematics where infinity occurs are those that have traditionally been presented to students mainly from an algebraic/symbolic perspective, which has tended to make it difficult to link formal and intuitive knowledge. However, in a context that combines numerical and geometrical contexts through the use of algebraic language, Waldegg claims that some of the obstacles observed in previous cases seemed to have disappeared. This is an important finding which supports the idea that by building connections between different types of representations (in this case through algebraic language) some of the difficulties that arise when working in a single context can be diminished. We thus took it upon ourselves to create situations in which the learning of the infinite infinity could be facilitated by incorporating the use of the computer and the representational systems it provides, even though we are aware that attempts to use the computer for the learning of the concept of limits have pointed to difficulties (e.g. Monaghan, Sun & Tall, 1994).

Our theoretical approach follows the 'constructionist' paradigm (see Harel & Papert, 1991) adopting the position that the construction of meanings involves the use of representations; that representations are tools for understanding; and that the learning of a concept is facilitated when there are more opportunities of constructing and interacting with, as diverse as possible, external representations of a concept. We thus postulated that at least some of the infinite processes found in mathematics, could become more accessible if studied in an environment that facilitates the construction and articulation of diverse types of representations, including visual ones.

A computer microworld to study thinking in change

Based on this premise, we built a computational set of open tools (diSessa, 1997) — a microworld —, using the Logo programming language, which could simultaneously provide its users with insights into a range of infinity-related ideas, and offer the researcher a window (see Noss & Hoyles, 1996) into the users' thinking about the infinite. The main research issue of the study was *to investigate students' developing conceptions of infinity and infinite processes as mediated by this microworld.*

The microworld provided a means for students to actively construct and explore different types of representations — symbolic, graphical and numerical — of infinite processes (infinite sequences and the construction of fractals, as explained below) via programming activities. In general, the computer setting provided an opportunity to analyse and discuss in conceptual (and concrete) terms the meaning of a mathematical situation. For example, drawing a geometric figure using the computer necessitated an analysis of the geometric structure under study and an analysis of the relationship between the visual and analytic representations.

There were many reasons for choosing Logo, which will not all be outlined here, but in particular Logo is easily accessible for programming activities (recursive programming, specially) by the students. The Logo procedures used in the microworld were generally not given in written form (except for the initial activities). They were usually programmed by the students using suggestions given by the researcher. It is important to emphasise that for all the above activities, "measurement" procedures were used for computing numerical data which could complement the visual models. Most of the time, tables were used for structuring this numerical (and algebraic) data, becoming an additional representation of the sequences under study.

The programming and exploratory activities included:

- *Explorations of infinite sequences, such as* $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$, *and* $\{1/n\}$, $\{1/n^{1.1}\}$,..., $\{1/n^2\}$, *and their corresponding series*, through geometric models such as spirals, bar graphs, staircases, and straight lines, and the corresponding Logo procedures, with a complementary analysis of the numerical values. These models were chosen since they constitute a straightforward way of translating arithmetic series into geometric form (e.g. in the 'spiral' type of representation each term of the sequence is translated into a length, visually separated by a turn, so that the total length of the spiral corresponds to that of the sum of the terms, i.e. the corresponding series.) Through the observation of the visual (and numeric) behaviour of the models, students were able to explore the convergence, and the type of convergence, or divergence, of a sequence and its corresponding series. The different geometric models for the same sequence provided different perspectives of the same process.

- *Exploration of fractal figures.* These included the Koch curve and snowflake (formed by putting together three Koch 'segments'), and the Sierpinski triangle. The explorations involved the study of their recursive structures (apparent both visually and in the programming code), and dealing with apparent paradoxes at infinity, such as a finite area bounded by an infinite perimeter.

The study

The main phase of the study was carried out with 4 pairs[2] of students of varying ages and backgrounds (in total there were 4 female and 4 male students). Two of these students were as young as fourteen years of age. On the one hand we were interested in introducing younger students to infinity-related mathematical ideas. We were also interested in observing the conceptions of younger students and the ways in which they worked in the environment, in comparison to older students. The pairs were as follows: Pair 1: two 14-year old girls; Pair 2: two high-school boys aged 16 (M) and 17 (J); Pair 3: Two college students (1 male, 1 female) in their twenties in non-mathematical areas; Pair 4: Math teachers in their thirties (1 male, 1 female). Pairs 1 & 3 were not mathematically inclined, as opposed to Pairs 2 & 4. Each pair of students worked for 5 sessions of 3 hours with the microworld. The analysis of the data was carried out through detailed case studies of the interactions of each of these 4 pairs with the microworld.

The role of the researcher was that of a participant observer, suggesting the field of work (the initial procedures and activities), as well as new ideas for exploration when needed, yet allowing students to be in control of the explorations, giving them freedom to explore and express their ideas. Students were informally interviewed throughout the sessions but formal interviews were conducted at the beginning and end of the study.

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[2]. Students worked in pairs with one computer to facilitate the sharing and discussion of ideas (simultaneously providing the researcher with insights into their thinking processes), and give them independence from the instructor. To facilitate the analysis of students' experiences, we worked with only one pair of students at a time, using a clinical interview style.

Students' shifting conceptions of the infinite

One of the advantages of the microworld was that the *behaviour* of the process could be observed, rather than the end result as is usually the case in traditional school mathematics. Observing the behaviour, such as the rate of convergence, played a very important role for giving meaning and finding explanations as to why in a particular instance a process converged or diverged. The exploration of the behaviour was done in several ways which included the observation of the process through its unfolding visual and numerical behaviour, the possibility to compare different sequences and models, and in the case of series, coordinating the behaviour of the series with that of the corresponding sequences.

Students discovered and explored limiting (or divergent) behaviours first through the graphical representations and then carrying out a back and forth process between these representations and numeric values; only in the case of Pairs 2 & 4 was there some degree of a more traditional "mathematical" analysis of the formula. The graphical element played a role in indicating the existence of a limit when there was *visual invariance* through several stages. For example, in the fractal explorations of the Koch snowflake, the apparently invariant visual image conveyed the boundaries of the area, highlighting its independent behaviour from the infinite perimeter that delineates it. At a second level, students would use numeric values, organised into tables, to complement and confirm the observed visual behaviour and give an indication of the value of the limit or divergence of the sequences.

Below, I briefly describe some examples illustrating a few of the conceptions of the infinite shown by students during the study, and how these evolved through the explorations.

- *Intuition that if process is infinite, then it will diverge*: Throughout the study, a common intuition arose, particularly among the less mathematically oriented students (Pairs 1 & 3): the confusion that if a process is infinite then it will diverge. This intuition has been found by other researchers. Nuñez (1993) in particular, explains that the problem arises when there are several competing components (processes) present; that is, when two types of iterations of perhaps different nature (cardinality vs. measure) are confused: the process itself and the divergent process of adding terms to the sequence. Thus, in the case where infinite sums are involved the intuition appears as: "if an infinite number of terms or elements (*cardinality*) is added then the *measure* of the sum must be infinite, it must pass any preset value". For example, in an early part of the study, Pair 1 expected the line model representing the series $\sum \frac{1}{2^n}$ to grow without bounds, since an infinite number of segments were being added. She and her partner were quite surprised to see that the line got "stuck" at a length twice the initial value. Because they were convinced that the line would grow indefinitely, they attempted increasing the scale, but always got a line that eventually "got stuck". By working with the microworld in different ways, as outlined above, these students gradually found ways to make sense of how a process could continue infinitely and not grow to be infinite. In particular, they focused extensively (as did Pair 3) on the decimal structure of the real numbers, realising that in the decimal expansion of the values under study, the number of digits would increase more and more as the sequence progressed. Thus, the infinite nature of the process was reflected in the decimal structure of the numeric values. On the one hand the infinite process was seen to take place in the "infinitely small"; also, seeing the process from the point of view of the numeric, temporarily disassociated from the geometry, allowed students to cope with the visual boundaries.

However, the misconception discussed here seems to be a deeply rooted one since it would often re-emerge, and was also observed with other students (e.g. Pair 3). It is also interesting that it particularly re-emerged after the explorations of the divergent (i.e. where the value of the infinite sum is infinite) Harmonic series $\sum \frac{1}{n}$, which is a case where the misconception would appear to be true. But generally, as the students gained more experience, this intuition would gradually lose strength and even though the intuition would often briefly re-emerge, it would be more easily dismissed than at the beginning of the study, as more meanings were constructed: e.g. the continuity of the process was found in the decimal expansion, not in the length.

Koch curve 'paradoxes': solving an indetermination by coordinating two simultaneous infinite processes. Above I described how coping with a bounded infinite process requires discerning and coordinating two simultaneous infinite processes: the infinite iterative process of adding or increasing the number of terms, with the behaviour of the process itself (which could be convergent). In the Koch curve explorations we find another example involving the coordination of several simultaneous infinite processes. For some students the idea of an infinite perimeter formed by an infinite number of "zero-length" segments caused anxiety. This was particularly the case of Pair 2. One of the boys in this pair (J) was aware of the problem of having two types of processes involved in the change of the perimeter: the increase in the number of segments, and the decrease in the size of those segments. He realised that *the behaviour of the numerical values* pointed towards the perimeter becoming very large, infinite. But when they considered that the segments *at infinity* measured zero, this seemed to indicate to them that *at infinity* the perimeter would measure zero! In fact, by focusing on the latter process student J would challenge the idea of the divergence of the perimeter: "The segments are getting smaller... The perimeter cannot be infinite...". His partner had a different perspective: he focused more on how the zero-sized segments would affect the *shape* of the figure first concluding that it would become a “smooth” curve with no segments, “an infinite sequence of points”. The students were of course dealing with what is formally defined as an *indetermination* (an infinite number of segments of size zero: $\infty \times 0$). The students realised it was necessary to carry out algebraic and numeric explorations to solve the paradoxes. A breakthrough came when student J became interested in how each of the factors (the rate of decrease in the size of each segment vs. the rate of increase of the perimeter in the number of segments) behaved in relationship to each other. He used numerical explorations (structured in a table) to explore the behaviour of the perimeter, verify his hypothesis, and become convinced of the divergence of the perimeter by observing that the perimeter's increase was faster than the segments' convergence to zero. Whereas limit indeterminations are traditionally solved through algebraic manipulation, in this case Jesús overcame the indetermination through analysis of the *behaviour* of each of the elements involved, observing specifically the difference in the *rate* of divergence or convergence of each of the elements and coordinating the two processes involved.

His partner (M), on the other hand, still had a conflict between what his intuitions told him, and his attempt to apply (finite) mathematics and "logical" principles ("a number multiplied by zero is zero" vs. "a number multiplied by infinity is infinite"), and his confusions would resurface during the explorations of the Koch Snowflake perimeter:

"if the number is infinite the perimeter is zero, and what will happen? That all of this will become a point!" He is considering that at infinity the segments forming the curve would measure zero implying a sort of "collapse" of the curve into a point. His difficulty is related to the epistemological obstacle described by Cornu (1986) of "the passage to the limit" where "that which happens at infinity" seems to be isolated from the dynamic limiting process. Also, this situation is analogous to Zeno's paradoxes[3]: as in that case, there are two components present: the *number* of segments, and the *measure* of the segments. This dilemma highlights the difficulties that emerge when thinking of the infinite with the schema of the finite and which are found historically and by mathematics education researchers (as discussed at the beginning of this paper). For this student it would take a long process of (particularly numeric and algebraic) explorations and reflections to become convinced of the divergence of the perimeter and some of his doubts may not have been clearly resolved. Interestingly, this student had no conflict with bounded area of the snowflake which he attributed to the *shape* of the figure being such that the perimeter simply folds up as it increases, not letting the area grow any further.

As the examples above show, the perspective adopted, and the context in which the infinite is presented, are likely to have a determinant role on how it is conceived. As David Tall (1980; p.281) points out: "our interpretation of infinity is relative to our schema of interpretation, rather than an absolute form of truth."

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[3] As explained by Struik's (1967, p.43) Zeno challenged the belief that "the sum of an infinite number of quantities can be made as large as we like, even if each quantity is extremely small ($\infty \times \varepsilon = \infty$), and also that the sum of a finite or infinite number of quantities of dimension zero is zero". His arguments highlighted the difficulty of saying that the line is formed by points.

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