SOME COGNITIVE ASPECTS OF THE RELATIONSHIP BETWEEN ARGUMENTATION AND PROOF IN MATHEMATICS.

Bettina Pedemonte

Laboratoire LEIBNIZ - Université Joseph Fourier – Grenoble I

bettyped@tin.it

Abstract

The purpose of this paper is to analyze some aspects of the relationships between argumentation and proof. Our assumption is that argumentation and proof can be compared from two points of view: content and structure. Toulmin's model (Toulmin, 1958) can be a tool to compare the two structures. The paper shows how Toulmin's model can be used to detect some structural analogies and changes between argumentation and proof during the solution of geometric problems needing the production of conjectures and related proofs.

Introduction

We will consider the solving process of geometric problems in which students interact with dynamic environments that are represented by Cabri-Geometry software. We consider a situation in which the student produces an argumentation during the production of the conjecture and then constructs a proof of this statement. The purpose of this paper is to analyse the relationships between argumentation and proof. Our research aim is to analyse similarities and differences between the structures of the two processes. In this paper we will consider situations in which the cognitive unity (Boero, Garuti, Mariotti, 1996) works.

In general, in dealing with problems asking for a conjecture, the solution is not immediate. Then the production of an argumentation during the construction of a conjecture is expected. We gave some open-ended problems to 12th-grade students in Italy and in France. The students worked in pairs on a computer running the Cabri-Geometry software. In order to favour an argumentation activity between the students, we decided to gather them in pairs. Cabri-Geometry was chosen because our hypothesis was that the software could help the students to identify the geometrical proprieties which are beneath the figure construction and which are necessary to the production of proof.

1. Cognitive unity between argumentation and proof

The relationships between the production of a conjecture and the construction of proof has been an objet of study from a cognitive perspective. Actually, research studies showed the possibility that some kinds of continuity exists between the two processes. In particular, continuity can take the following shape:

"*During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of previously produced arguments according to a logical chain*" (Boero, Garuti, Mariotti, 1996).

This phenomenon is referred to by the authors as **cognitive unity**[1].

During the solving process, which leads to a theorem, we may suppose that an argumentation activity is developed in order to produce a conjecture. When the statement expressing the conjecture is made valid in a mathematical theory, we can say that a proof is produced. This proof is a particular argumentation based on a mathematical theory. We want to compare the argumentation process in producing a conjecture and proof.

2. From the relationships between conjecture and valid statement to the relationships between argumentation and proof.

The relationship between argumentation and proof is strictly connected to the relationship between conjecture and valid statement We might say that argumentation is to a conjecture what mathematical proof is to a valid statement (Balacheff, 1999).

Really, a conjecture could be provided without any argumentation. A conjecture can be a "**fact**", derived directly from a drawing, from an intuition and the like. In this case there is not an explicit argumentation justifying this fact. But, we are interested to the following kind of conjecture:

Let us define *a **conjecture** a statement strictly connected with an argumentation and a set of conceptions* (Balacheff, 1994) *where the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it.*

The conjecture can be transformed into a valid statement if a proof justifying it, is produced.

Let us define a ***valid statement** as a statement which is provided with a proof referring to a mathematical theory. The statement is valid because a mathematical theory allow the construction of a proof that justifies it*.

We are interested to compare the processes used to construct the conjecture and its validation: argumentation and proof.

The analysis of the solution process from the perspective of cognitive unity needs tools that allow the comparison between an argumentation process and a proof. Our purpose is to find these tools.

3. Cognitive unity in content and in structure

The previous research studies about cognitive unity considered the conditions for its existence (see Boero & al., 1996; Garuti & al., 1998). We are rather interested in its working. Our assumption is that the argumentative process in producing a conjecture and a related proof can be compared from two points of view: content and structure.

The "cognitive unity" considered by Boero & al. (1996) concerns the content. It is possible to observe whether there are analogies or differences between argumentation content and proof content. During the production of several theorems, there are many similar content elements in the argumentation and proof, therefore we can say that it is frequent to find cognitive unity (Pedemonte, 1998).

[1] The word « argument » refers to a reason given to support or disprove something. In this paper the word « argumentation » refers to a discursive activity (cf. Grize, 1996) based on arguments.

We think that it is interesting to analyse and compare the two processes also from the structural point of view. We wish to become able to detect analogies and differences between argumentation structure and proof structure.

It is possible to compare the argumentation and proof according to relevant structural aspects like deduction, abduction and induction. In a deductive argumentation, the statement is deduced from the data by means of a principle (which permit the inference) allowing its assertion from the data. In an abductive argumentation the statement is deducted before the data is identified (Arzarello, 1998). In this case a principle allows the assertion of a statement even if all the data are not available. In an inductive argumentation the statement is deduced as a generic case after research from specific cases.

Only the deductive argumentation can be easily and directly transposed into a deductive proof. In order to transform an abductive argumentation into a proof the structure needs to be reversed. The inductive argumentation has a structure far away from the structure of a deductive proof; in this case, a link between argumentation and proof can be found only when the argumentation contains the "generic case".

According to the previous analysis we can expect that even in the case of "cognitive unity" (which concerns content) the transition from argumentation to proof may demand relevant (and sometimes difficult to perform) changes concerning structures – in particular those from abductive or inductive argumentation to deductive proof.

4. The Duval's answer

According to Duval (1991), deductive thinking does not work like argumentation: there is a "gap" between the two processes even if they use very similar linguistic forms and proposition's connectives. The structure of a proof may be described by a ternary diagram: data, claim, and inference rules (axioms, theorems, or definitions). Within proofs, the steps are connected by a "recycling process" (Duval, 1992–1993): the conclusion of a step serves as input condition to the next step. On the contrary, in argumentation, inferences are based on the contents of the statement. In other words the connection between two propositions is an intrinsic connection (Duval, 1992–1993): the statement is considered and re-interpreted from different points of view. For these reasons (according to Duval) the distance between proof and argumentation is not only logic but also cognitive: in a proof, the epistemic value[2] depends on the theoretical status whereas in argumentation it depends on the content. Then it is easy to observe the cognitive distance between the two processes.

"*... Pour passer d'un mode de fonctionnement à l'autre* (argumentation and proof)*, une décentration à l'égard du contenu d'une part, et une prise de conscience de l'existence d'une autre valeur épistémique d'autre part, sont donc nécessaires. On peut donc parler aussi de "distance cognitive" entre le fonctionnement d'un raisonnement valide et celui d'une argumentation*". (Duval, 1992-1993).

[2] The epistemic value is the degree of certitude or convinction associated with a proposition (Duval,1991).

According to Duval, the distance between these two processes can explain why most of the students don't understand the necessity of a mathematical proof: if there is an argumentation that justifies the statement the proof can be unnecessary.

Some doubts are currently expressed about the nature and the educational relevance of the gap between argumentation and proof, as described by Duval (in particular, see Douek, 1999). We share these doubts. We think that there are some very similar elements between argumentation and proof. In particular, assuming that a proof is a particular argumentation, both argumentation and proof structures can be described by a ternary diagram.

This is the reason why we need a tool to compare the structure of the two processes.

5. How to analyse or compare the structure of the argumentation process and the proof process ?

We have built up a theoretical framework to analyse argumentation structure and proof structure. Toulmin propose a model for the argumentation structure (1958). We use this model as a tool to compare the structures relating to the two processes: argumentation and proof.

In any argumentation the first step is expressed by a standpoint (an assertion, an opinion or the like). In Toulmin's terminology the standpoint is called the claim. The second step consists of the production of data supporting it. It is important to provide the justification or warrant for using the data concerned as support for the data-claim relationships. The warrant can be expressed as a principle, a rule, and the like. The warrant acts as a bridge between the data and the claim. This is the base structure of argumentation, but auxiliary steps may be necessary to describe an argumentation. Toulmin describes three of them: the qualifier, the rebuttal and the backing. The force of the warrant would be weakened if there were exceptions to the rule, in such a case conditions of exceptions or rebuttal should be inserted. The claim must then be weakened by means of a qualifier. A backing is required if the authority of the warrant is not accepted straight away.

Therefore, Toulmin's model of argumentation contains six related elements as showed in the following figure.

Q : qualifier

D : data ⟶ **C : claim**

since **W : warrant** unless **R : rebuttal**

on account of **B : backing**

Fig.1. The Toulmin's model of argumentation[3].

[3] Let us illustrate this model with the same example used by Toulmin (1958). *Claim :* Harry is a British subject. *Data :* Harry was born in Bermuda. *Warrant :* A man born in Bermuda will generally be a British subject. *Rebuttal :* No, but it

It is interesting to compare the idea of epistemic value (Duval, 1991) and the idea of the qualifier. The epistemic value of the claim is inherited by the epistemic value of the data. The claim's force is inherited by the data's force. On the contrary, the qualifier is given by the data and also by the warrant's force. The warrant's force is important because the warrant plays a basic role in the argumentation.

If we consider a proof as a particular argumentation, the warrant is an axiom, or a definition, or a theorem, in a specific theory.

Toulmin's model reveals a very powerful tool to compare the process of argumentation and the proof subsequently produced. We can compare the argumentation warrant and the proof warrant. For example if the warrant in an argumentation is based on an intuitive conception, we can see whether in the proof the warrant becomes a theorem of a theory or on the contrary if it remains at the level of conception.

In the following section, we illustrate the use of this model in analysing the resolution process of an open-ended problem.

Interview

The experiment was carried out in four 12^{th}-grade classes in Italy, and in one 12^{th}-grade class in France. The students worked in pairs on a computer running the Cabri-Geometry software. We will transcribe a part of a solution protocol related to the proposed problem. This part is based on the transcriptions of the audio recordings and the written productions of the students. The experiment lasted an hour and a half. The problem proposed was the following:

Problem. ABC is a triangle. Three exteriors squares are constructed on the triangle's sides. The free points of the squares are connected defining three other triangles. Compare the areas of these triangles with the area of triangle ABC (see figure pg. 6).

According to the classification given in the previous section the following types of argumentation can be found in the students' resolutions.

A typical <u>deductive</u> argumentation could be the following. Suppose the student compares the lengths of the base and the height between triangle ABC and one of the external triangles in order to compare the two areas (see figure pg. 6). It's possible to consider the sides of the same square as bases for some triangles and compare the heights considering the small triangles constructed on the heights. The view that the small triangles have two equal angles and an equal side, allows the conclusion that the two triangles are equal under the SAA congruence criterion. Then the large triangles have equal areas.

A typical <u>abductive</u> argumentation could be the following. The student, who wants to compare the two areas, sees that the two bases of the triangles have the same length. It's possible to prove that the heights have the same length in order to prove that the areas are equal. The view that the small triangles constructed on the heights

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generally is. If his parents are foreigners or if he has become a naturalised American, then the rule doesn't apply. *Qualifier :* True : its only presumably so. *Backing :* It's embodied in the following legislation :….

are equal can encourage the search for a theorem to prove this fact. The congruence criterions are remembered and the data to apply one of them is sought out

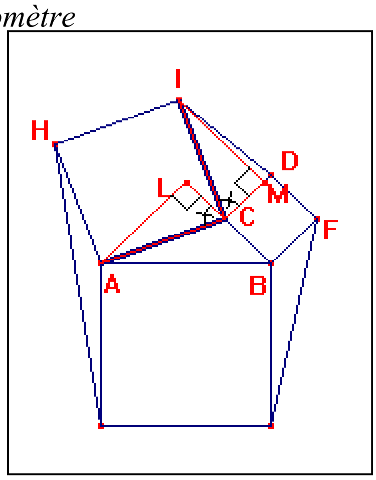
A typical <u>inductive</u> argumentation could be the following. The student may consider some particular types of the triangles ABC: rectangle, equilateral and the like; or he may consider limit cases, for example, when the points A, B, and C are on the same line. This is an "inductive search" moving from particular to general laws. One of these examples can evolve into a particular example (*exemple générique*: N. Balacheff, 1988) which can lead to the proof.

Example

Using the model described above, we analyze an excerpt of the argumentation and the proof produced by students. Our purpose is to show how the analysis works in order to prove the efficacy of Toulmin's model.

In order to analyse the argumentation, we select the assertions produced by students and we reconstruct the structure of the argumentative step: claim C, data D and warrant W. The indices identify each argumentative step. The student's text is in the left column, and our comments and analyses are reported in the right column. The text has been translated from Italian into English. We start the analysis at claim C_7; at this point students are comparing the area of the triangle ABC and the area of the triangle ICD. So far the students spoke about the construction of the heights of the two triangles. They decided to construct the heights in order to compare the areas of the triangles ABC and ICD.

<table>
<tr><td>…. Students construct the heights of the triangles ABC et ICD

31. L: I'm prolonging the straight line, yes, the straight line on the segment… what have I done?
32. G: The straight line by the points B and C
33. L: ah it's true !
34. G: now, we need to do the line perpendicular to this line
35. L: ah there that's it done but you know that it seems they are equal…</td><td>The figure as represented from the students using Cabri-géomètre

</td></tr>
<tr><td>36. G: almost equal !
37. L: no, more, it seems that they are perpendicular, I have observed this before
……. …</td><td>C_7: The heights seem to be equal.
C_8: The heights seem to be perpendiculars.

The statements are "facts" where the epistemic value is joined to perception of the figure in Cabri-Geometry.</td></tr>
<tr><td>44. Students together: hey, these are two triangles !
45. L: it's true, ALC and ICM these are two triangles…what do they have?</td><td>The Cabri-Geometry drag allows them to see the small triangles. The students realize that the heights are the heights of two equal triangles. The statement is now a fact.</td></tr>
<tr><td>46. G: we realized… then AC is equal to IC</td><td>C_9: The triangles ALC and ICM are equal.</td></tr>
</table>

because they are sides of the same square 47. L: wait! 48. G: AC is equal to IC because they are sides the same square, after 49. L: LC… 50. G: it's equal to CM, why ? 51. L: Then… Because it's equal to CM… in my opinion, it's better to prove … no wait this angle is right and this angle is right too.	*The structure of the speech of the students is:* The triangles are equal → to find sides and angles equal congruence criterion *The structure of the argumentative step is an abduction:* D_9= ? → C_9 W: congruence criterion

The structure of the argumentation is that of an abduction. The students see that the small triangles constructed on the height are equal and they search for a theorem to prove this fact. We can observe that during the proof, students make data explicit in order to affirm that triangles ALC and ICM are equal. The abductive structure of the argumentation is transformed into a deductive structure in the proof. Once obtained, claim C_9 is used to deduce that the heights of the triangles ABC and ICD are equal and consequently that their areas are equal.

The students write the proof:	*The proof structure is a deduction:*
I consider the triangle ABC and the triangle ICD. At once I consider the triangles ALC et ICM and I prove that they are equal triangles for the SAA congruence criterion because we have: • AC = IC because they are two sides of the same square • ALC = IMC because they are right angles (angles constructed as intersection between the sides and the heights) • ACL = ICM because they are complementary of the same right angle (-LCI) In particular IM = AL. Then the triangles ABC and ICD have the same base lengths (as sides of the same square) and the same heights, then they have the same area.	D_9: AC =IC ALC = IMC ACL = ICM → C_9: the triangles ALC and ICM are equal W: SAA congruence criterion *If the triangles are equal then it's possible to conclude that the heights are equal, and finally then the areas are equal because the bases are equal.* *The conclusion* C_9 *of the previous step is the date* D_{10} *to apply the inference to the second step.* D_{10}: C_9 → C_{10}: the heights are equal W: inheritance D_{11}: C_{10} → C_{11}: the areas of the triangles ABC et ICD are equal W: formula of area

The protocol is an example of cognitive unity (according to Boero, Garuti and Mariotti, 1996) Indeed, students use the "SAA congruence criterion" both in the argumentation and proof, in order to justify the statements. Words and expressions used in the two processes are often the same ("triangles ALC and ICM are equal", "heights are equal", and the like). But if we look more carefully, we can observe a change between the structures of the two processes: we find an abductive structure in the argumentation (from D_9 to C_9) that is transformed in a deductive structure in the proof. We cannot undervalue the importance of the structure in the comparison between argumentation and proof; it is not unusual that the student tries to transform abduction into a deduction during a resolution process (sometimes successfully, sometimes without getting an acceptable solution).

6. Conclusion

In this paper, we have analyzed some relationships between argumentation and proof; we have used Toulmin's model as a tool in order to compare the structure of the two processes.

In the student's protocols, it is easy to observe cognitive unity regarding the content; but even in this case, as far as structure is concerned, changes are frequently observed. The previous analysis carried out with Toulmin's model clearly reveals the structure of both argumentation and proof facilitating the comparison between them. In particular when students use abduction during argumentation (and this seems to be natural in the production of a conjecture), a structural change is needed and can be detected in students' protocols.

The study reported in this paper is still in progress. Further analysis will be carried out in order to clarify the nature of argumentation (particularly in the conjecturing phase) in order to find other analogies or differences with proof.

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