Encapsulation of a Process in Geometry

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Abstract: *The theory of encapsulating a process concentrates on "procepts" in arithmetic, algebra, and calculus (DAVIS, SFARD, TALL & GRAY et al, DUBINSKY et al). In this paper we will discuss the existence of procepts in geometry and we will give examples.*

1. Encapsulation of a Process

VINNER (in TALL 1991, p. 65ff) discussed the role of definitions in learning mathematics and how children may overcome the "conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition". Mathematical definitions mainly describe objects or a static view while the process of acquiring new insight often runs in parallel with activities or procedures or mental processes in time. Thus there are divergent roots to develop individual mathematical concepts and TALL & VINNER (1981) use the term "evoked concept image" to describe the part of the memory evoked in a given context.

These inconsistent views, an object on the one hand and procedures on the other, must grow together to form an appropriate mathematical concept (or rich and powerful "concept image" with the words of Vinner). PIAGET (1985, p. 49) has already pointed out that "actions and operations become thematized objects of thought or assimilation". This idea has become very powerful today to understand the development of certain concept images in mathematics education as a process of "interiorization" or "reification" or "encapsulation ".

"When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practiced, the procedure itself becomes an entity - it becomes a *thing*. It, itself, is an input or object of scrutiny. All of the full range of perception, analysis, pattern recognition and other information processing capabilities that can be used on any input data can be brought to bear on this particular procedure. Its similarities to some other procedure can be noted, and also its key points of difference. The procedure, formerly only a thing to be done - a verb - has now become an object of scrutiny and analysis; it is now, in this sense, a noun" (DAVIS 1984, p.29f).

Especially guess-and-test procedures - we think - are a valuable tool to develop such a "full range of perception", a rich "concept image", an efficient "relational understanding" (SKEMP 1978, MEISSNER 1979, 1983). In our case studies using the ONE-WAY-PRINCIPLE with calculators or computers we observed exactly that mental development as described by Davis. The children first learnt to press the correct sequence of buttons to solve the given problem. Then, when the problems change, the input must be guessed now to use the meanwhile familiar sequence of buttons to get a given output. An intuitive feeling developed how to select better inputs (MEISSNER 1987). By the use of the ONE-WAY-PRINCIPLE the children developed a relational understanding for multiplicative structures, for percentages, for growth and decay, and for the concept of function[1].

SFARD (1987) distinguishes also two kinds of mathematical definition, referring to abstract concepts as if they were real objects or speaking about processes and actions. "The structural descriptions seem to be more abstract. ... To speak about mathematical objects, we must be able to deal with products of some processes without bothering about the processes themselves." She claims (1987, p. 168) that the operational conceptions develop at an early stage of learning even if they are not deliberately fostered at school. In SFARD (1992, p. 64f) she identified a constant three-step pattern

[1] MUELLER-PHILIPP describes in her doctoral dissertation (1994) how the dynamic view ($y = f(x)$) and the static view (graph) grow together by guess-and-test and how children become able to switch from one view to the other.

in the successive transitions from operational to structural conceptions: "First there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired."

SFARD calls these three components of concept development "interiorization", "condensation", and "reification", respectively. "Condensation means a rather technical change of approach, which expresses itself in an ability to deal with a given process in terms of input/output without necessarily considering its component steps. Reification is the next step: in the mind of the learner, it converts the already condensed process into an object-like entity."

Since 1986 DUBINSKY and his colleagues (see TALL 1991, COTTRILL et al 1996) also studied the encapsulation phenomenon and they developed the APOS theory. They see three steps (A - P - O) to get mental objects (which then become part of a **S**chema **S**):

A **A**ctions - usually step-by-step activities - are necessary at the beginning,

P By controlling and reflecting the action step-by-step consciously (interiorizing) the action becomes a **P**rocess.

O The process becomes an **O**bject when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations".

GRAY & TALL analyzed the duality between process and concept and came to a similar view. They consider (1991, p. 72ff) "the duality between process and concept in mathematics, in particular using the same symbolism to present both a process (such as the addition of two numbers 3+2) and the product of that process (the sum 3+2). The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema." They hypothesized that the successful mathematical thinker uses a mental structure "which is an amalgam of process and concept". TALL (1991, p. 251ff), reflecting the dual roles of several symbols and notations: "Given the importance of a concept which is both process and product, I find it somewhat amazing that it has no name. So I coined the portmanteau term "procept". In 1994 GRAY & TALL proposed the following definitions: "An ***elementary procept*** is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object. A ***procept*** consists of a collection of elementary procepts which have the same object." In TALL et al (2000a) we find a table of examples for symbols as process and concept.

Especially when discussing advanced mathematical thinking we can discover a lot of "procepts". DUBINSKY (2000, p. 43) lists such concepts, that have been treated on the use of APOS theory: "functions, binary operations, groups, subgroups, cosets, normality, quotient groups, induction, permutations, symmetries, existential and universal quantifiers, limits, chain rule, derivatives, infinite sequences, mean, standard deviation, central limit theorem, place value, base conversion and fractions"[2].

2. Procepts in Geometry ?

Studying the above mentioned references we miss geometry, at least "concrete", visual geometry. Are there no procepts in geometry? Is the process of learning geometry that much different from the process of learning arithmetic and algebra and calculus? Are there no procedures or processes in geometry to become objects on a procept level? Most of the work on the "encapsulation of a process to an object" concentrates on examples in arithmetic, algebra, and calculus. We do not know papers on examples in geometry.

One of the reasons might be that in many countries geometry is not in the center of teaching mathematics and therefore there is not much research on how children learn geometry. In German

[2] For more details see http://www.cs.gsu.edu/~rumec/index.htm

primary school books for example we have only about 5% of the pages with geometry topics. (And even less than 5% of the time spent on mathematics education in German primary school classes then really are used for teaching geometry).

Another reason might be that there still is a method of teaching and learning geometry which is similar to an axiomatic approach: We start with "definitions" and properties (line, point, circle, square, ...), discover relations and prove statements. Of course, it will be difficult then to discover (like in arithmetic) "processes which may produce a mental mathematical object". Then there also is no necessity in geometry for getting symbols which are used to represent either a process or an object.

TALL et al (2000b) formulate the hypothesis that there are three types of mathematics (space & shape, symbolic mathematics, axiomatic mathematics) and that each of them is accompanied by a different type of cognitive development. They consider - before focusing on the growth of symbolic thinking - "briefly ... the very different cognitive development in geometry". There are perceptions of real objects initially recognized as whole gestalts and classifications of prototypes. Reconstructions are necessary to give hierarchies of shapes and to see a shape not as a physical object, but as a mental object.

STRUVE (1987) also analyses how concept images in geometry develop. He summarizes, that children in primary and lower secondary classes learn geometry like a natural science, they describe, explain, and generalize phenomena. Thus, for them, geometry becomes an empirical theory.

For the author of this paper it is a miracle that we in physics can use mathematical formulae and even complex mathematical theories to predict future events. We trust - but we cannot *prove* like in mathematics - that events will occur tomorrow in the same mode as yesterday when there will be the same conditions. There are big similarities between physics and empirical geometry: Given certain assumptions we can predict events - by the use of mathematical theories.

What does that mean for the theory of procepts? When we analyze in "3+2" possible step-by-step procedures of the children we also observe "empirical mathematics" with real objects. And like in geometry the children generalize and learn to predict future events. We trust, but we cannot prove, that "3+2" always "will be the same", but we (as mathematics educators or researchers) avoid speaking of a miracle by introducing "counting principles" (like axioms in geometry). In this view an elementary procept in the meaning of TALL et al (2000a) just is the shift from the empirical stage to the theoretical stage. Consequently, following these ideas there should be no fundamental obstacle to find procepts in geometry, too.

3. Pivotal: Tagging

When we look for procepts in geometry we first need activities or procedures. At the beginning they may be "experienced one step at a time". After practicing them for a while the user perceives "the overall patterns and continuity and flow of the entire activity, the procedure itself becomes an entity - it becomes a *thing*" (DAVIS).

TALL et al (2000a) distinguish procedure and process. For them procedure is like a specific algorithm. Using the example "4+2" there are lengthy procedures (as "count-all"), compressed into shorter procedures (like "count-on" or first "count-both" or "count-on-from-larger") or other techniques (i.e. "remembering known facts" or "deriving facts"). These different procedures all are used "to carry out essentially the same process in increasingly sophisticated ways".

We see the symbols used (4+2, 2+4, 6, etc.) like a tag to describe (or to evoke!) the according processes or objects mentioned. In general we think the symbols or tags play the same role as key words, they are tools to name or to recall a process or an object. They are used to abbreviate or to condense or to summarize the "evoked concept image" to get one single "sign" (for communication).

Symbols and tags serve like key words. In arithmetic/algebra/calculus we use letters a, b, c, d, ... and other symbols like +, %, dy/dx, ... to evoke concept images. But other key *words* like "six", "field", "parallel", ... or " ", "◊", ... or "⊕", "⊗", ... or "⊥" ... might do the same. Only important for mathematics education is the concept image evoked by that tag or symbol or key word.

TALL et al (2000b) point out that "symbols occupy a pivotal position between processes to be carried out and concepts to be thought about. They allow us both to do mathematical problems and to think about mathematical relationships". Important, there is only one symbol with a dual meaning. And we like to add, it is not important what type of symbol or tag or key word it is.

Thus we add explicitly the process of tagging or naming, that means communication is an essential part of developing procept images:

(a) carry out accurately the given one procedure/technique
(b) several procedures/techniques are possible, select one
(c) several procedures/techniques are possible, make an efficient choice ***and explain***
(d) carry out the process flexibly and efficiently, i.e. determine and select an appropriate procedure/technique ***and discuss possible alternatives***
(e) ***use an abbreviation (tags, symbols, key words)*** when discussing, arguing (the same "name" for both, the procedural and the conceptual aspect)

Table 1. Development of an (elementary) procept ***image***

4. The Spatial Procept "Net"

We will give an example from our classroom research (MEISSNER & PINKERNELL 2000):

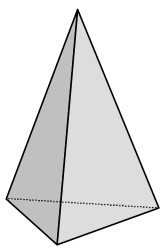


Fig. 1

A teacher showed a model of a three sided pyramid (Fig. 1) and asked the class: “What did the cardboard paper look like before I folded it to make this pyramid?” Friederike (age 8:2) drew a square and added three triangles to its sides (Fig. 2). She then showed with her hands how to fold the pyramid, pointed to the side of the square where there is no triangle, and said: “Then there's a hole, isn't there?”

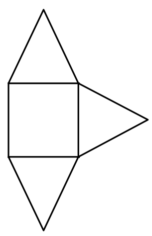


Fig. 2

Let us discuss and analyse this case in more detail. First, Friederike gets, probably without realising it, two contradictory stimuli at the same time. The key word "pyramid" leads to a square because all pyramids Friederike has known so far had a square as their base. However the given solid says that Friederike only needs three triangles. She compromises and gets the hole. Obviously she is familiar with a mental folding-up procedure, but she has not enough experiences to bridge the gap immediately.

But is it necessary at all to fold mentally before deciding if this drawing is a net of a pyramid? By looking at Fig. 2 experienced geometricians see that it does not represent the development of a solid. They can tell without actually folding the net. In their reasoning, the process of folding has been encapsulated to the static concept “development”. Friederike, however, has some notion of “development” in which she still needs to carry out the process of folding explicitly, as her hands indicate. Thus we consider “development” as a procept in spatial reasoning.

This episode was observed before we systematically introduced activities in the classroom to draw developments for solids (pyramids, rectangular solids, houses): The children learned to make the net of a pyramid by placing a wooden model onto a sheet of paper and then repeatedly tilting it from its base onto one side and back to the base again, each side being encircled with a pencil. The resulting figure would be a star shaped net. Next, we have asked them to make the net of a rectangular solid. What we have experienced many times is that in strictly following the learned procedure they forget the solid's upper side and produce a net that would fold to an open box.

We then pointed onto the missing side of the given solid asking where this was drawn. Very often there was a laughter in the classroom and immediately the drawing was completed correctly. The procedure “development” they had acquired so far was based on an activity of what could be called “tilting from and back to the base”. With the rectangular solid this procedure of “tilting” had to be revised by extending it. This mental change is typical for the development of procepts. Proceptual thinking also includes the ability to revise an encapsulated procedure to meet new demands (GRAY 1994, p.2). We saw a similar expanding of the procedures when we used solids with concave sides.

To draw the developments the children got a card board and a wooden solid. Some of the children just started tilting and drawing. Others first took the solid to find an appropriate starting position on the card board (by tilting without drawing) to make sure that their drawing will fit on the paper. Here the activity already becomes a flexible and efficient process.

The last lesson of that teaching unit (details see MEISSNER & MUELLER-PHILLIP 1997) started with an exhibition of about 20 different (plane) developments of buildings fixed with tape on the black board. There were only lines drawn where to fold later on (but not distinguishing if to fold inside or outside). The children (grade 3, age about 8 - 9) had to describe which net might become what type of building before they could choose one of the developments to verify their guesses. We are sure some of the children just identified simple nets without any mental folding. They just *saw* "that is a tower" (Fig. 3) or "that is a garage" (Fig. 4) or "that is a house" (Fig. 5). We think that, for them, a simple net had become an elementary procept.

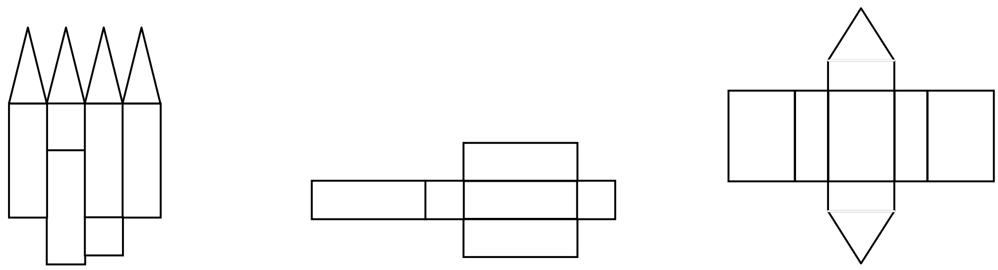


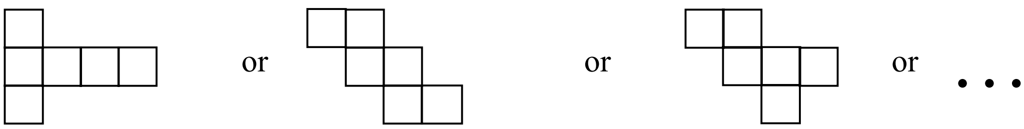
Fig. 3 Fig. 4 Fig. 5

But where is the symbol, one of the characteristics of a procept? We think the net itself is the symbol. The one interpretation of that symbol is a procedural one, "folding up". The other view is static, "this is ..." (an object).

Symbols of procepts follow syntactical rules. Also from this point of view there are reasons to take (at least simple nets) as a symbol. In the following we will demonstrate this view by comparing procepts from arithmetic or algebra with the procept "net".

A process is a set of procedures:

We can describe “6” by “4+2” or “5+1” or “3+3” or ... And we can describe “cube” by

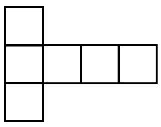
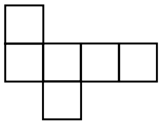
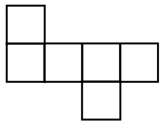


Each symbol belongs to a specific process:

This is true for “6” or “3^2” or “1/2” as well as for nets shown in figure 3, 4, or 5.

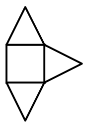
There are "syntactical rules" to transform symbols:

Replace “3+4” by “4+3” (3+4 = 4+3) or replace “2×8” by “8×2” (2×8 = 8×2) or replace “3×(4+2)” by “3×4 + 3×2” or ... We also can replace

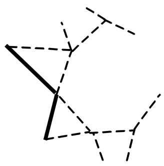
 by  or  or . . .

There are "syntax errors":

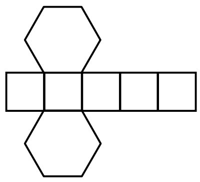
The notation of power does not allow symbols like “${}^2x$” or “ ${}_2x$” or “ x_2” or “ ${}_{2x}$” or “ ^{2x}”. Or the notation of addition does not allow “+2,4+” or “2,4,+” or ... The notation “net” does not allow



(missing side)



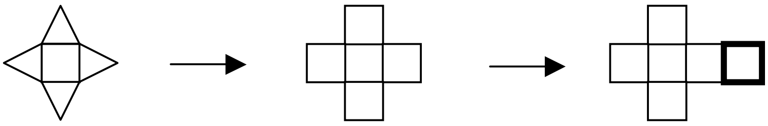
(neighbored lengths)



(number of side surfaces)

Procepts can be extended:

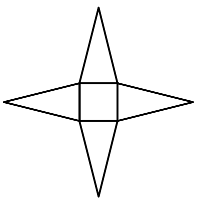
“3×4” (multiplication of integers) gets expanded to “3.5×6.9” (multiplication of decimals).



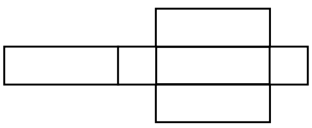
“tilting” gets expanded to a “conscious tilting”

Symbols can be variables:

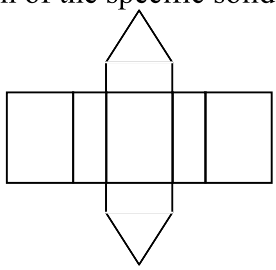
We use letters for variables in arithmetic or algebra. A “net” also may have the meaning only of a variable, i.e. by giving the spatial shape, but no geometric proportion of the specific solid:



“a” pyramid

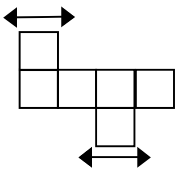


“a” rectangular solid

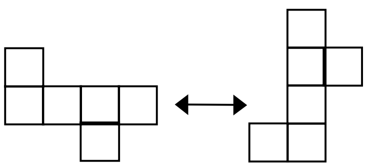


“a” house

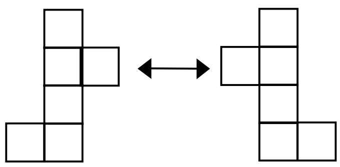
Symbols can be manipulated:



"shifting the base"



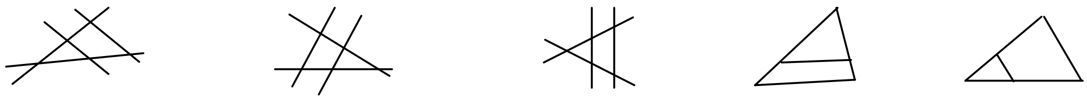
"rotation"



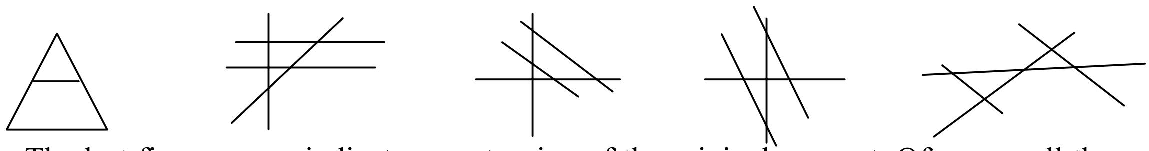
"reflecting"

5. Other Procepts in Geometry

One main theorem in geometry is, what we call in German the *Strahlensätze.* (At the PME conference in Hiroshima I was told that there is no key word for this theorem in English). A figure of four lines, where two intersecting lines cross two parallel lines, leads to three or four basic statements about the ratio of according lengths:

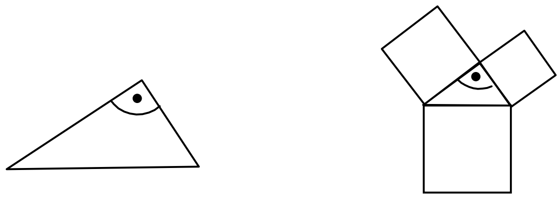


We think the symbol or tag of the Strahlensätze is one of the given figures. A proceptual thinking of "Strahlensätze" is only possible when we are able to regard the above figures as an entity (of related procedures). Then the procept "Strahlensätze" is encapsulated in each of these figures. The different types of figures can also be seen as manipulations of symbols according to syntactical rules. Some more manipulations may be the following:



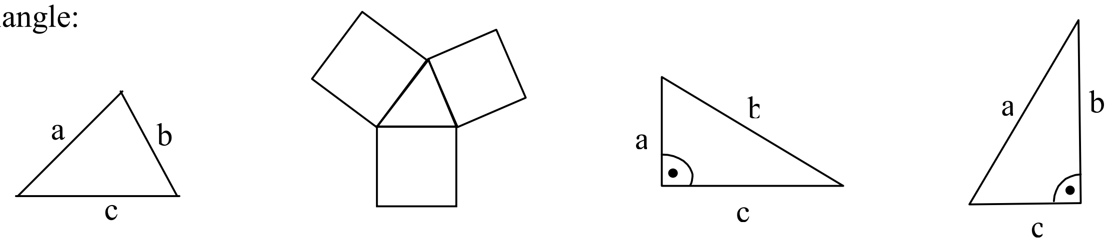
The last figures even indicate an extension of the original concept. Of course all these symbols also implicitly include variables: It is not important where the intersection point is in relation to the two parallels nor is the seize of the angle of the intersecting lines nor the width of the parallels.

Another example is Pythagoras' Theorem. There are several types of tags:



$\boxed{a^2 + b^2 = c^2}$

Often our students do not achieve a proceptual "pythagorean" thinking. They ignore or they do not see the property "perpendicular" or they have fixed mental images of how to name the sides of a triangle:



Also simple geometrical objects may lead to a procept. A keyword or a roughly drawn figure (of a triangle or a circle or a polygon) may provoke concept images of how to construct these figures, or of properties of these figures or of solids which these figures are part of. The roughly drawn figure might be the symbol to tag that procept with all possibilities of symbol manipulations or modifications as mentioned above. In this sense geometric drawings have a pivotal meaning. They may evoke a concept image of a single static geometric physical figure as well as of a complex procept.

6. References

COTTRILL, J., DUBINSKY, E., NICHOLS, D., SCHWINGENDORF, K., THOMAS, K., VIDAKOVIC, D.: Understanding the limit concept: Beginning with a co-ordinated process schema. In: Journal of Mathematical Behavior, vol. 15, pp. 167-192, 1996

DAVIS, R. B.: Learning mathematics: the cognitive science approach to mathematics education. Ablex, Norwood, NJ 1984

DUBINSKY, E.: Towards a Theory of Learning Advanced Mathematical Concepts. In: Abstracts of Plenary Lectures and Regular Lectures. ICME 9, Tokyo/Makuhari, Japan 2000

GRAY, E. M., TALL, D. O.: Duality, Ambiguity and Flexibility in Successful Mathematical Thinking. In: Proceedings of PME-XV, vol. II, pp. 72-79, Assisi, Italy 1991

GRAY, E. M.: Procepts and procedures: Traversing the Mathematical Landscape. Paper presented to the Collegium Fenomograficum, Kollegiet för Inlärning, Kognition Och Informationstechnologi, Goeteborg, Sweden 1994

GRAY, E. M., TALL, D. O.: Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic. In: The Journal for Research in Mathematics Education, vol. 26, pp. 115-141, 1994

MEISSNER, H.: Problem Solving with the One Way Principle. In: Proceedings of the Third International Conference for the Psychology of Mathematics Education, pp. 157-159, Warwick, England 1979

MEISSNER, H.: How to prove relational understanding. In: Proceedings of the Seventh International Conference for the Psychology of Mathematics Education, p. 76-81, Jerusalem, Israel 1983

MEISSNER, H.: Schuelerstrategien bei einem Taschenrechnerspiel. In: Journal fuer Mathematik-Didaktik, vol. 8, no.1-2, pp. 105-128, Schoeningh Verlag, Paderborn, Germany 1987

MEISSNER, H.: Procepts in Geometry. To Appear in: Proceedings of CERME 2, Marianske Lazne, Cech Republic 2001

MEISSNER, H., MUELLER-PHILIPP, S.: Wir bauen ein Dorf. In: Grundschulunterricht, no. 6, pp. 40-44, Paedagogischer Zeitschriftenverlag, Berlin, Germany 1997

MEISSNER, H., PINKERNELL, G.: Spatial Abilities in Primary Schools. In: Proceedings of PME-24, vol. III, pp. 287-294, Hiroshima, Japan 2000

MUELLER-PHILIPP, S.: Der Funktionsbegriff im Mathematikunterricht. Waxmann, Muenster/Germany, New York, USA 1994

PIAGET, J.: The Equilibrium of Cognitive Structures. Harvard University Press, Cambridge MA 1985

PINKERNELL, G.: Spatial Procepts. Manuscript. Muenster Nov. 2000

SFARD, A.: Two conceptions of mathematical notions: operational and structural. In: Proceedings of PME-XI, vol. III, pp. 162-169, Montréal, Canada 1987

SFARD, A.: Transition from Operational to Structural Conception: The notion of function revisited. In: Proceedings of PME-XIII, vol. 3, pp. 151-158, Paris, France 1989

SFARD, A.: Operational origins of mathematical objects and the quandary of reification - the case of function. In: Harel, G., Dubinsky, E. (Eds.): The Concept of Function: Aspects of Epistemology and Pedagogy, MAA Notes, vol 25, pp. 59-84, Mathematical Association of America, Washington DC 1992

SFARD, A.: Reification as the Birth of Metaphor. In: For the Learning of Mathematics, vol. 14, pp. 44-55, 1994

SKEMP, R. R.: Relational understanding and instrumental understanding. In: Arithmetic Teacher, vol. 26, pp. 9-25, NCTM, Reston, Virginia 1978

STRUVE, H.: Probleme der Begriffsbildung in der Schulgeometrie. In: Journal fuer Mathematik-Didaktik, vol. 8, pp. 257-276, Schoeningh Verlag, Paderborn, Germany 1987

TALL. D. O., VINNER, S.: Concept image and concept definition in mathematics, with special reference to limits and continuity. In: Educational Studies in Mathematics, vol. 12, pp. 151-169, 1981

TALL, D. O. (Ed.): Advanced Mathematical Thinking, Kluwer Academic Publishers, Dordrecht, NL 1991

TALL, D. O.: Cognitive Growth in Elementary and Advanced Mathematical Thinking. In: Proceedings of PME-XIX, vol. I, pp. 61-75, Recife, Brazil 1995

TALL, D. O., THOMAS, M., DAVIS, G., GRAY, E. M., SIMPSON, A.: What is the object of the encapsulation of a process? To appear in: Journal of Mathematical Behavior, 2000a

TALL, D. O., GRAY, E. M., BIN ALI, M., CROWLEY, L., DEMAROIS, P., MCGOWEN, M., PITTA, D., PINTO, M., THOMAS, M., YUSOF, Y.: Symbols and the Bifurcation between Procedural and Conceptual Thinking. To appear in: Canadian Journal of Mathematics Science & Technology Education, 2000b