STUDENTS' CHOICE OF TOOLS IN SOLVING EQUATIONS IN A TECHNOLOGICAL LEARNING ENVIRONMENT

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This paper presents several findings on 13-14 year old students' ability to choose, employ and integrate various tools in their solution of four equations. We analyzed the ability and preferences of six pairs of students to use paper and pencil and four computer tools in their solutions. The interviewed students were able to produce various solution methods, make transitions between them, and find a complete solution of each equation. With regard to their choice of a preferred tool, the students displayed in their actual work a preference for manual, algebraic algorithms. In their comments, however, they frequently expressed a preference for a technological tool, and could provide an explanation for their opinion.

Introduction

Many researchers in mathematics education recommend the use of computerized tools in algebra (Dörfler, 1993; Balacheff & Kaput, 1996; Yerushalmy & Schwartz, 1993). These claims are backed by research on students' work on algebraic tasks with specific computerized tools – such as spreadsheets, graph plotters and algebraic symbol manipulators. On the other side, research on students' ability to choose, employ and integrate various tools in the solution of an algebraic task is scant. Our limited knowledge on this issue is also related to the fact that regular classroom tasks in algebra frequently recommend explicitly the representations and tools that should be employed.

The goal of this paper is to present several findings on (a) students' solutions of algebraic equations, in an environment that provides a variety of tools, (b) students' ability to choose, employ and integrate various representations and tools in their solution process, and (c) students' view of their solution tools.

The study was conducted within the learning environment of the *CompuMath* project (Hershkowitz et al., in press). The project developed a junior high school mathematics curriculum, integrating an interactive computerized learning environment. The *CompuMath* students work one to two (out of five) weekly lessons, in a computer lab. The topics of the algebra course include algebraic generalizations in the first year (with spreadsheets as a technological tool), solutions of equations in the second, and functions in the third year (with a graph plotter and a symbol manipulator as technological tools). Learning is based on a cyclic sequence of investigating open problem situations in small heterogeneous groups, followed by consolidations of mathematical concepts and processes and reflective actions. The curriculum supports a broad understanding of the equation concept, both as an algebraic or numerical equality, and as a particular point in a continuous variation.

Procedure

Six pairs of 13-14 year-old higher, and average ability students were interviewed. The students belonged to a selective, but not mathematically oriented urban school. They were in the first month of their last year of a three-year algebra course. The findings reported here are based on written protocols taken during the interviews and on students' written work from these interviews. Each interview lasted about 90 minutes.

The students were presented a sequence of two single equations and two systems of equations (one linear and one quadratic in each case). The equations are presented in the leftmost column of Table 1. Before presenting the first equation, the interviewer explained to the students, that they will receive a sequence of four algebraic equations and will be required to solve them in *as many different ways* as they can. The students were also shown four possible tools, to accomplish the task: paper and pencil, and three computerized tools -- graph plotter, algebraic symbol manipulator and spreadsheet. During their work, the students received no instructions or hints with regard to their choice of tools or solution methods. At the conclusion of each of the four tasks, each student was asked separately (but in the presence of his/her peer) about his preferences towards the employed tools.

Solution methods

During the interview, the students used a variety of solution methods (an average of 3.8 methods per pair, per equation) and employed all four available tools. All pairs found the solution of each equation, even when they were unable to provide a paper and pencil solution. Table 1 presents the students' solutions of the four equations. Due to limitation of space, some secondary variations in solution processes were not included.

Remarks: (a) During their course work, the students did not encounter the algebraic algorithm for solving quadratic equations or systems**.** (b) During their course work, spreadsheets were used to describe and investigate variations, but were not employed as a tool for solving algebraically presented equations. (c) A spreadsheet solution of equations with two variables is very complex and inconvenient. (d) The students had no previous experience with parallel work on several computer tools and with the need to choose a tool according to their considerations.

The first remark explains the students' relatively low level of success in the use of paper and pencil to solve quadratic equations, whereas (b) and (c) may explain the relatively low frequency of their use of spreadsheets, as compared to other tools. Another fact that should be mentioned is that (d) did not seem to have a limiting

Table 1. Solution methods employed by the interviewed students (number of pairs using each method*).

Equation \ Tool	Paper and Pencil	Graph Plotter	Alg. Symbol Manipulator	Spreadsheet
1.2 (x – 0.5)= 8.4	• Expanding the expression and solving for x. (6) • Dividing first by 1.2 and then solving for x. (2)	• First, an unsuccessful attempt to graph the equation by entering it as a whole, and then a) give up. (1) or b) graph each side separately and trace the intersection point (eventually changing step size or scaling). (2) or c) graph the left side, trace and monitor the y-coordinate. (1) • Direct performance of stage b) or c). (2)	• Entering the equation and pressing the "Solve" key. (6)	• Entering a sequence of numbers in Column A (e.g., from 1 to 10 in steps of 0.1), possibly changing the step size. Then, entering in Column B the left side or the whole equation, as a formula, copying it downwards and looking for the appearance of the right-side value (8.4), or the True value. (4) • Unsuccessful attempts (confusing the role of the independent and dependent variable, omitting the sequence of independent variable). (2)
$x^2 - 5x + 6 = 0$	• Dividing by x (correctly) and unable to continue.(2) • Dividing by x (incorrectly) and receiving an incorrect solution. (3) • No attempt (awareness of an unknown method). (1)	• Graphing each side separately and tracing the intersection point (possibly changing step size or scaling). (2) • Graphing the left side, tracing and monitoring the y-coordinate. (3) • Mentioning graphing, but no attempt. (1)	• Entering the equation and pressing the "Solve" key. (6)	• Entering a sequence of numbers in Column A (e.g., from 1 to 10 in steps of 0.1), possibly changing the step size. Then, entering in Column B the left side or the whole equation as a formula, copying it downwards and looking for the appearance of the right-side value, or the True value. (3) • No attempt. (3)

* Occasionally, the same pair produced more than one written (paper and pencil) solution for the same equation.

Table 1 (continued). Solution methods employed by the interviewed students (numbers of pairs using each method).

<table>
<tr><th>Tool / Equation</th><th>Paper and Pencil</th><th>Graph Plotter</th><th>Alg. Symbol Manipulator</th><th>Spreadsheet</th></tr>
<tr><td>$x + y = -4$
$x - y = 8$</td><td>• Adding the two equations and solving. (3)
• Subtracting the two equations and solving. (2)
• Using one equation to express one of the variables and substituting the obtained expression in the second equation. (5)</td><td>• Graphing each equation separately and tracing the intersection point (possibly changing step size or scaling). (6)</td><td>• Entering the two equations as a system and pressing the “Solve” key. (6)</td><td>• Entering sequences of numbers in Columns A and B. Then entering the sequence of A+B in column C. Realizing the dependence between the x values (in A) and the y values (in B) and giving up. (3)
• Attempting to search in the “Functions” menu. (1)
• No attempt. (2)</td></tr>
<tr><td>$y = x + 1$
$y = x^2 + x$</td><td>• Substituting for y (as expressed in the first equation) in the second equation and then
a) complete solution (2) or
b) partial solution (2) or
c) incorrect solution (1)
• Substituting for x (as expressed in the first equation) in the second equation and unable to obtain a solution. (2)
• Considering any pair (x, y) satisfying the first equation as a solution. (1)</td><td>• Graphing each equation separately and tracing the intersection point (possibly changing step size or scaling). (6)</td><td>• Entering the two equations as a system and pressing the “Solve” key. (6)</td><td>• No attempt. (6)</td></tr>
</table>

on students’ facility in their transition from one tool to another and on their ability to make connections between their work on different tools. Thus, for example there were at least ten cases of going back from a technological tool to previously performed paper work, in order to find out “what went wrong”. Instances of students making connections between outcomes obtained on different technological tools were also observed. For example, pairs E and F reported that they used the results obtained on the algebraic manipulator, in order to verify their previous work on the graph plotter.

Choice of solution tools.

We will report here several findings on students' preference for solution tools. The analysis was based on several sources: (a) observations of their actual work, (b) students’ comments made after the solution of each equation and (c) spontaneous comments made by students during their work.

All six pairs started each task with paper and pencil. The fact that the solution algorithm for solving a quadratic equation was unknown did not deter them from employing an algebraic approach as a first attempt. For example, pair A attempted to solve Equation 2 as follows:

A1: *Let’s do –6* [Writes $x^2 - 5x = -6$] *Can we divide by x?*

A2: *May be [square] root of x?*

A1: *Let’s do +5x and then the root* [Writes $x^2 = 5x - 6$ / $\sqrt{x}$]

A2: *We were not taught to take this apart [separate x].* [Writes $x = -6 + \sqrt{5x}$]

A1: *We’ll put it together, take it apart, put it together, take it apart, until we get [the solution].*

Table 2 presents the interviewed students’ preference of tools, as expressed explicitly at the end of solving each of the four equations.

Table 2. Students’ preference of tools for each of the four equations (N = 12).

Equation \ Tool	Paper	Graph plotter	Symbol manip.	Undecided
$1.2 (x – 0.5) = 8.4$	10		2	
$x^2 – 5x + 6 = 0$	2	5	3	2
$x + y = -4$; $x – y = 8$	2	2	7	1
$y = x + 1$; $y = x^2 + x$	2	5	4	1

Although in their actual work, all six pairs chose to start each of the four solutions on paper, the students were less committed to this tool in their comments made at the end of each task. Table 2 indicates that in the case of the linear equation, most students preferred a solution on paper. In the other three tasks, however, most students chose one of the two computerized tools as their first preference.

The preference for paper and pencil was backed up by several reasons. We cite selectively because of space limitations.

- Level of involvement in the solution process.

My knowledge is better on paper. You don't make efforts on the computer. That's good for the lazy ones. I like to use the paper, but sometimes it gets complicated … I like challenges. (E2, Eq. 1)

If I use Derive, it does not mean that I did it. The computer did it. (F2, Eq. 2)

- Availability.

Chances are that I may be without a computer – on the matriculation exam there are no computers. (B1, Eq. 1)

- Need for understanding and for transparency.

Derive did not show anything ...just gave the answer and did not show anything. (B2, Eq. 2)

The paper seems to me safer, although Derive solves everything. [Working with Derive] on complicated equations, it may get complicated and you don't understand what did you get … Derive is more convenient, but if [the equation] is complicated, I don't understand how did I get the x. (D2, Eq. 1)

- Teacher/Interviewer expectations.

Derive is the best and is quick, but in exercises the teacher wants us to show the way. (B2, Eq. 4)

The computer is better, easier, but this is not a way. On a test they won't let us use it. If we were allowed, of course. (F2, Eq. 3)

- Personal attitude.

I don't want to deal with graphs. (C1,C2 Eq. 2)

I like the paper. It's comfortable. (E1, Eq. 1)

Most students could recognize, at least at a declarative level, the advantages of computer tools. In view of the objective difficulties raised by solving an equation in a spreadsheet (*Excel*) environment, we will relate here to students' attitude towards the graph plotter (*Mathemati-X*) and the symbol manipulator (*Derive*). Students mentioned (mainly when their knowledge did not allow them to solve the given equation manually) the computer's ability to solve any equation.

The computer can find every possible result and I can't (C1, Eq. 4)

Students mentioned as the main advantage of the graph plotter its transparency, whereas the symbol manipulator's main strength, according to them, lies in its speed and operational easiness.

The solution [on paper] is a little difficult. I prefer the graph. It gives the intersection point. By substituting numbers [i.e., trial and error] we could have thought that there is only one solution x = 2. We could have stopped at 2. Here [on Mathemati-X] we see the graph going up and down and we can find both solutions. (A1, Eq. 2)

A preference for the graph plotter was frequently attributed to the fact, that the manipulator did not satisfy the need to understand the solution process.

Between Derive and Mathemati-X, I prefer Mathemati-X, because it's more concrete. When we see the graph, we understand, we see it concretely, clearly. (D1,D2, Eq. 2)

Derive was the best and the quickest. (B2, Eq. 4)

In some cases, the legitimacy of a symbol manipulator as a solution tool was questioned altogether.

Derive is not really a way [for solving equations]. (F1, Eq. 1)

[Derive] is easier – but it is not a way. (F1, Eq. 3)

Conclusion

The findings clearly indicate that the interviewed students were able to employ a variety of solution methods. They made connections between various meanings of the equation concept and of the solution process employed with each tool.

Some researches showed that even when students were able to solve standard problems in both symbolic and graphical representations, their actual understanding of connections between representations was often superficial and vague (Yerushalmy & Schwartz, 1993; Knuth, 2000). This difficulty was not observed in our case. The interviewed students were able to present the concept of equation in various representations, move between tools and between representations and connect the outcomes. They were able to produce various solution methods, make transitions between them, and find a complete solution, even when a standard algebraic, paper and pencil method could not be produced.

With regard to their preferences of the four available solution tools (paper and pencil, graph plotter, algebraic symbol manipulator and spreadsheet), the picture is less clear. For affective, cognitive and external reasons, students displayed in their actual work a preference for manual, algebraic algorithms. In their comments, however, they expressed a less univocal stand. They frequently chose a technological tool as their first preference, and explained the reason for their expressed preference. The main criteria that influenced the students' choice of tools were a potential to display the solution process, a potential to allow a higher extent of student involvement in the solution process and compliance with accepted norms of work.

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