MOVING SYMBOLS AROUND OR DEVELOPING UNDERSTANDING: THE CASE OF ALGEBRAIC EXPRESSIONS

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Abstract

This paper deals with the complex relationship between the meanings and symbols of algebraic expressions. The study reported here uses a theoretical model to interpret some basic dynamics of algebraic thinking in the analysis of students' protocols. The strong difference noticed in protocols by students who have been given very different instructional approaches, suggests further investigation on the sources, mainly in view of educational implications.

Introduction

The complex relationship between the meaning of algebraic expressions and their symbolic representation is a major issue in research. Many authors have pointed out the incapability of relating symbolic expressions to their meaning (see, for example, Linchevsky and Sfard, 1991, Mac Gregor and Stacey, 1996). In the inadequacy of such a relationship are the roots of many misunderstandings, wrong performances and blind manipulations with algebraic symbolism. A consequence of this is that many secondary school students do not master the sense of those symbols which they have learned to handle formally. On the other hand, some students, even if clever "algebraic calculators", seem to be unable to see and use algebra as a means suitable for understanding generalisations, to grasp structural connections, and to argue in mathematics (Boero, 1994).

Our study is originated by noticing strong differences in protocols provided by students: on the one hand "moving symbols around", on the other "good mastery and understanding". This "surface element" has been recognised in need of further investigation, by means of a theoretical model apt to interpret some basic dynamics of algebraic thinking. This application has revealed to be very useful to guess hypotheses about the origin of such great differences in the students' behaviour: as a result we assume that such differences are strongly related to differences in the teaching styles. This report aims at describing the work we have been carrying out, in view of further development as far as school practice is concerned.

The theoretical framework

The relationship between thought and language is a key point in didactic research. Algebraic thinking is generally recognized to be inseparable from the formalized language with which it expresses itself. However, it is "reductionist" to believe that algebraic thinking exists only at this level. In such a case everything would be reduced to a manipulative mechanism, which often does not work in students' hands.

Vygotsky's theory, with reference to verbal language, suggests a functional interaction between thought and language. They are considered as intertwined and mutually dependent aspects of the same process.

In the specific case of algebraic language, a vygostkyan point of view is taken by Radford (1999). According to Radford, instead of seeing signs as the reflecting mirror of internal cognitive processes, we consider them as tools or prostheses of the mind to accomplish actions as required by the contextual demands in which individuals find themselves located.

Furthermore, signs bear a kind of "embodied intelligence", in that they were historically built for some purpose and, as such, carry patterns of previous reasoning. Signs, tools and other cultural artefacts, do not speak for themselves, but acquire life in the life of individuals through acts of communication and interaction where they become endowed with meaning.

So, *becoming endowed with meaning* is at the very core of our investigation.

In the following, we will adopt the theoretical model by Arzarello, Bazzini and Chiappini (1993, 1994, 1995), aiming at interpreting the very nature of algebraic thinking.

This model has taken its inspiration from the epistemological triangle by Frege, which specifies the distinction between sense and denotation of a given expression.

In particular, an algebraic expression (E) incorporates in its writing the mathematical object involved (the denotation) and the way in which such an object is expressed (the sense). For example the expression $y = x^2-2x-3$ denotes a set of couples which satisfies the given relation. This expression activates, for example, the sense (S) of finding y by starting from x, squaring x, subtracting 2x and finally subtracting 3. A different sense (S') could be that of thinking of x and y as coordinates in a Cartesian plane (hence the graphic of a parabola))

Furthermore, if we transform $y = x^2-2x-3$ into $y = (x-1)^2-4$, this new expression (E') activates a new sense (S"), which points out the coordinates of the vertex of the parabola.

Thus, algebraic transformations are closely related to the activation of senses: doing algebra means interpreting expressions and relating them with senses, coherently with the given denotation.

The theoretical model outlined here provides us the means to approach our investigation of the students' protocols, when they face algebraic formulas.

Aims and methodology of the study

In this report we are mainly interested in the differences that students show when required to face algebraic expressions and to relate them to meaning.

Additionally, we are also concerned about the level of awareness students have when dealing with algebraic language and its relationship to instruction.

For this purpose, we addressed our attention to students coming from different instructional treatments and we tried to relate their typical behaviours to the didactic interventions they had received previously.

Our assumption is that different instructional approaches highly influence the mastering of algebraic expressions, especially as far as the triangle sign-sense-denotation is concerned.

In the following, we report the description of four paradigmatic protocols: the first two show a limited and incorrect reading of algebraic expressions, while the others witness to understanding.

Legend: I for interviewer, S for student.

Student A (**Mario**, 16 y.o., medium scores, High School, humanistically oriented course, Liceo classico, traditional teaching).

I - $-8x-3x^2+11=0$, *what is this?*

S -It is a second grade equation; it could be the equation of a parabola.

I- Which kind of equations do parabolas have?

S- $ax+bx^2+c$.

I - What?

S- Equals 0.

Comment: there are conflicting ideas about the word *equation.* The writing $-8x-3x^2+11=0$ is not connected to its denotation. The equation of a parabola is evoked, due to similarity in writing. When requested to make it explicit, the student is not able to link the denotation (i.e. the couples of numbers) of the parabola to its algebraic representation.

Student B (**Stefania**, 16 y.o., high scores, High School, scientifically oriented course, Liceo Scientifico, traditional teaching).

I writes

$y=x^3+6x$

$y=x^3+3x^2$

and says*;" Compare these two functions, try to say when a function is "greater than the other" (questo è un modo per dire: quando i punti del grafico di una funzione stanno sopra i punti del grafico dell'altra).*

S.: I would make a system (and she puts {)

{ $y=x(x^2+6)$

$y=x^2(x+3)$

I: Solving a system means finding the common solution of the two equations, I have asked you just to compare, that is saying when, for example, $x^3+6x>x^3+3x^2$.

S: and y, where does it go?, Ah, it is the solution.

I: It doesn't matter

S: So, why is there a y and then it disappears?

I: If you consider the system, you have y=y, thus also $x^3+6x= x^3 +3x^2$, do you agree?

S: Yes

I: So, go on.

S: I should solve the inequality $x^3+6x> x^3 +3x^2$, so

$-3x^2+6>0$ and $-3x(x-2)>0$ $-3x>0 \rightarrow x<0$ $x-2 >0\rightarrow x>2$

and this is what she wrote

	0		2	
+		-		-
-		-		+
-		+		-

$0<x<2$

I: What does that mean?

S; Perhaps, if I substitute a value which is between 0 and 2, the equality is true. But I don't know if it is true, or whatever. That is, maybe the inequality is true.

Comment: Stefania feels lost in front of symbols. This task is not a standard task for her, however, she has all the knowledge needed to solve the problem.

As for student A, this student evokes senses starting from a very superficial reading of the formula. The writing of the functions evokes the sense of solving a system, which is totally out of place here. Also, in following the procedure, no connection between signs, senses and denotations seems to exist.

Student C (**Davide**, 14 y.o., medium scores, Junior Secondary School, teaching strongly oriented to understanding and verbalizing)

I: Take the function $y = x^2+3$: what does it remind you of? Do you first think of numbers which change or how the graphic might be?

S: $y = x^2+3$ I first think of the graphic, which is a raised parabola with the x axis of 3 cm. This helps me to understand how it works. In fact, with positive numbers, x^2 increases y of 0, 1, 4, 16… and with the negatives it increases on the opposite side. Then +3 does not allow the values to reach zero $0^2+3=3$…

At the beginning, I think of the formula without doing any calculation, because I know how it works in each part…

I: Namely?

S: $y = x^2+3$ is like taking two equal numbers, multiplying them and adding 3 to the value: 3 increases the value. Within the negatives, the two numbers which have been multiplied become a positive in any case, because we know that + x+=+ and -x-=+.

This is the reasoning when facing a formula, after having seen the formula I see the graphic which represents the formula itself.

Finally Davide concludes:

The function is a unique thing. It is important to consider it as a whole, however it is important to understand each part, in order to grasp it globally.

Comment: Davide clearly shows a holistic view of the function $y = x^2+3$. The numbers which change and the related graphic are closely related and Davide is able to pass from one sense to another easily. This student also shows a global approach to the notion of function in general (*the function is a unique thing…)*.

Student D (**Federico**, 14 y.o., high scores, Junior Secondary School. Same class and same teacher as Davide)

Federico has the same task as Stefania (student B): He is required to compare these two formulas from an algebraic and graphical point of view, then to guess how the graphic is and finally he has to draw a sketch.

$y= x^3 + 6x$

$y= x^3 + 3x^2$

Here is Federico's solution

Algebraic comparison

- *the first part is the same in both formulas*
- *they are two curves, because the increment from one value to another is greater every time*
- *if x is positive, then y is positive too, because there are no "minus signs" in either formula*
- *if x is negative, then in the first formula (the right side) is negative too, because* $x^3=x\cdot x\cdot x$ *and* $((-x)\cdot(-x)\cdot(-x))$ *and also* $6(-x)=$ *neg. In the second formula y is positive from –3 to 0, because* $(-3)^3+3\,(-3)^2= -27+ 3\cdot 9= -27+27 = 0$. *From –3 down, i-e. –4, -5, etc, y becomes negative because the first piece becomes greater and greater, after 3, than the second one.*
- *the first formula is greater than the second one from 0 to 2 and smaller from 0 down and from 2 up.*
- *If x=0, then y=0 in both formulas, because* $0^3+6\cdot 0=0 \quad 0^3+3\cdot 0^2=0$

Graph comparison

- *they are two curves, because the increment from one value to another is greater every time*

and he does the following calculation

$y= 2^3+ 6\cdot 2=20$ $\qquad$ $y= 3^3+6\cdot 3=27+18=45$

$y= 3^3+ 3\cdot 2^2=20$ $\qquad$ $y= 3^3+3\cdot 3^2=27+27=54$

- *they pass through the origin, because there are not + with fixed numbers which are not multiplied by x*
- *the first curve passes through the first and third quadrant, because when x is positive, then y is positive too; when x is negative, y is negative too. The second curve passes through the first, second and third quadrant, because if x is positive, then y is positive. If x is negative from 0 to –3, then y is positive because the first term is negative and the second positive, but the second term is greater because it is multiplied by a number greater than x, then it goes to the negatives, because the first term becomes greater than the second, in terms of numbers, not of signs.*

<u>*Predicting the curves*</u>

- *they are two curves, because the increment from one value to another is greater every time*
- *they pass through the origin*
- *the first curve passes through the first and third quadrant*
- *the second passes through the first, second and third quadrant.*

Finally, he draws the graphs correctly

Comment:

This student has a great mastery of algebraic symbols and he relates them to the graphical representation. Different senses are activated and they work together.

As in the case of Davide, Federico is able to manage the triangle sign-sense-denotation. This allows him to approach and solve the problem easily, notwithstanding the lack of advanced mathematical techniques (this task was given at the end of junior secondary school).

Provisional results and implications for teaching

The protocols show quite different behaviours: students A and B seem to move symbols around, while students C and D are much more concerned with understanding. The theoretical model allows us to read such behaviours in terms of the sign-sense-denotation relationship. Starting from an algebraic formula, different senses can be activated: some of them are coherent with denotation, some others not.

Students A and B evoke senses starting from a very superficial reading of the formula. Mario evokes the equation of a parabola incorrectly, due to similarity in writing. Similarly, Stefania evokes the sense of solving a system, when facing the writing of two functions. For both students, no connection between signs, senses and denotations seems to exist.

Things go differently for students C and D, who approach and solve the task from a very global point of view.

Davide approaches the function $y = x^2+3$ by considering the changing numbers and the graphical representation. He is also able to pass from one sense to another easily. A similar behaviour is noticeable in Federico's protocol: both students master algebraic symbols and are able to relate them to the graphical representation of the functions. In short, the triangle sign-sense-denotation is handled fruitfully.

Let us remind that the NCTM Principles and Standards 2000 suggest "…Being able to operate with algebraic symbols is also important because the ability to rewrite algebraic expressions enables students to re-express functions in ways that reveal different types of information about them" (p.301).

As already pointed out, the analysis of the protocols has suggested a closer investigation on the origin of such great differences

Our main assumption now is that previous instruction plays a major role. In fact students A and B come from a traditional approach: they are able to solve standard tasks (for example that of solving equations) but feel lost when required to handle symbols and meaning. On the other side, students C and D (who are two years younger) have been provided with innovative teaching, strongly oriented towards understanding and verbalising. From early algebra, these students have been asked to compare different expressions and different senses, with special attention to link geometrical investigation to the representation by numbers and letters. Furthermore, the use of metaphors have been highly encouraged: teaching and learning algebra has been framed in an *embodied cognition* perspective (Bazzini, Boero and Garuti, 2001). Further investigation on long term research (Malara, 1999) is needed to confirm our assumption: specific teaching experiments have to be designed to suit this purpose.

Finally, focussing on implications for teaching, we remind that, according to the NCTM Principles and Standards 2000, algebra is much more than moving symbols around. Doing algebra is not just formal manipulation, but rather a competence which deeply involves understanding. In this perspective, new emphasis should be given to research studies on the mutual relationship between algebraic expressions and their meaning and, consequently, on related educational choices.

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