CHILDREN'S GRAPHICAL CONCEPTIONS: ASSESSMENT OF LEARNING FOR TEACHING

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We report a study of 14-15 year old children's graphical conceptions and misconceptions using a diagnostic instrument developed from the research literature to suit the UK National Curriculum. Rasch measurement methodology was used to develop the instrument with a pilot sample and the final instrument and resulting scale is here evaluated and reported based on the full sample of 425 children. The result is that a hierarchy of responses is confirmed, each level of which is described as a characteristic performance including key misconceptions. We compare this with previous work on graphs and functions, explore a small group of teachers' knowledge and discuss the applications of the work in schools in the present stage of the programme.

Introduction and background

This study builds on previous work on misconceptions in children's graphical thinking, and especially in their interpretations of graphs (Clement, 1985; Even, 1998; Janvier, 1981; Kerslake, 1993; Sharma, 1993). Unfortunately, as Leinhardt et al (1990) said, "of the many articles we reviewed almost 75% had an obligatory section at the end called something like 'Implications for teaching' but few dealt directly with research on the study of teaching these topics" (p. 45). We would add that the 'teaching implications' drawn from research on the psychology of learning mathematics are in any case in general problematic: for many reasons these implications rarely impact on practice. Williams and Ryan (2000) argued that research knowledge about students' misconceptions and learning generally needs to be located within the curriculum and associated with relevant teaching strategies if it is to be made useful for teachers. This involves a significant transformation and development of such knowledge into pedagogical content knowledge (Even, 1998) which requires its own study. In particular teachers need to know at which stages of their development pupils are likely to exhibit the researched misconceptions and errors and where in the curriculum they are relevant. Ryan and Williams (2000) produced such data for errors scattered across the curriculum. The present study develops this work by focussing in depth on the area of 'graphical understanding and interpretation' relevant to years 9 and 10 of the mathematics curriculum, and by connecting the research with teachers' knowledge of these misconceptions. In particular this study:

- developed an instrument from the research literature to assess children's learning and misconceptions on a scale related to their curriculum, which we claim is a prerequisite for transforming this knowledge into professional practice, and

- explored the development of this into an instrument for assessing teachers' pedagogical content knowledge.

The development of the assessment instrument involved the tuning of, or the development of, diagnostic items from the research literature on graphicacy to fit the school curriculum. This developed from an analysis of the key work in the field of children's thinking, identifying items which related appropriately to:

- slope-height confusion: the height is a distracting feature when interpreting the slope (Clement, 1985);
- linearity-smooth prototypes: pupils tend to sketch linear graphs and expect some form or reasonableness, such as 'smooth', 'symmetrical', 'continuous' (Leinhardt et al, 1990);
- the 'y=x' prototype: pupils' tendency to construct the y=x graph;
- the 'Origin' prototype: graphs are drawn through the origin;
- graph as 'picture': many pupils, unable to treat the graph as an abstract representation of relationships, appear to interpret it as a literal picture of the underlying situation (Clement, 1985);
- co-ordinates: pupils' tendency to reverse the x and the y co-ordinates and their inability to adjust their knowledge in unfamiliar situations (Kerslake, 1993);
- scale: pupils prototypically read a scale to a unit of one or ten (Williams and Ryan, 2000).

Originally it was also intended to incorporate the misconception related to misinterpreting time-dependent and time-independent graphs (Hitt, 1998), but the items did not work effectively with our sample in the pilot and this was dropped from the study.

The scaling of the test provides a measure on which pupils, item-difficulty and errors can be located (following the methodology described in Ryan and Williams, 2000 and Ryan et al 1998). The pilot study, previously reported in Hadjidemetriou and Williams (2000), involved:

- interviewing 4 teachers about these items to confirm their curriculum relevance;
- testing 50 pupils, identifying errors and development of a possible hierarchy into a measurement 'scale' using Rasch methodology, and
- interviewing the children who made the expected errors to validate the error categories as significant misconceptions in the children's reasoning which are in accord with the literature.

This paper reports the confirmation of the instrument in the main study of 450 pupils, and the beginnings of our research into teachers' related pedagogical content knowledge (the instrument and report of the pilot will be found at http://www.man.ac.uk/cme/ch).

Method

The two study samples (pilot: n=50 and main: N=425) were all of year 10 pupils in the North West of the UK, whose teachers were interviewed (n=4 and N=12) to check that the test was regarded as fair and valid. The test results were subjected to a Rasch analysis in the usual way, including the coding and analysis of errors on the same scale as the items. (For a summary of this method see Williams and Ryan, 2000: the Rasch scaling is the modern stochastic development of the Guttman scaling model used in the CSMS studies in Hart, 1981). The result is a single difficulty estimate for each item and an ability estimate for each child consistent with the Rasch measurement assumptions, (only 4 mark points fell outside a model 'infit' tolerance of mean square 0.7 to 1.3 in the pilot, and one in the main study). Several items were modified between the pilot and main study, as described in detail in the pilot report in Hadjidemetriou and Williams (2000). The main study data was used also to scale the 'common' errors on the same scale (using the average ability parameter of the children who made the error). We are aware of the debate in mathematics education about the nature of such hierarchies: we accept that there may be serious dangers in fixing such constructs which may make improvements in curriculum and methods difficult. On the other hand teaching in the UK is dominated by a national curriculum which is so structured, and the engagement of teachers in practice requires us to adapt to this.

In addition to the test analyses, we drew on interviews with groups of children about the test items to gain some insight into the cause of the errors and their relation to significant misconceptions. In these interviews children were selected who made the interesting errors and discussions were organised in groups which allowed us to confirm our diagnoses of the children's thinking (along the lines described in Williams and Ryan, 2000). Furthermore we interviewed teachers and asked them to complete a questionnaire in which they were asked to suggest how difficult their children would find the items on the pupils' test, and to suggest likely errors and misconceptions the children would make. This data is used here to explore the validity of the research data on misconceptions and also the state of knowledge of this small group of teachers.

Results

The table below (Table 1) shows the resulting hierarchy of children's performance and thinking. A comparison with CSMS results reveals a comparability between the CSMS levels 1 and 2 and our own levels 1 and 2. The underlined statements in the figure are common to the hierarchy that Kerslake assembled (Kerslake, 1981; Sharma, 1993). However the emphasis is different in our test, because we included many more items which involved interpretation and sketching of graphs within contexts involving understanding of rates of change and associated misconceptions. Therefore our hierarchy at level 3 branches from the relatively common levels 1 and 2.

Interviews with children to validate the instrument

The main purpose of the pupil interviews was validation of the test, in particular our interpretation that the errors in the test are symptomatic of the misconception discussed in the literature. In this section we illustrate with interviews of a couple

Level	Typical performance at each level (Performance descriptions underlined are descriptors of the parallel levels in Kerslake's heirarchy for comparison.)	Typical common errors (Scaling logits are in italics)
5 *(Logit: 1.6 – inf)*	Sketching complex graphs to tell a story, including non-linear, two part and interpreting discontinuous graphs. Understanding calculation of gradient of a graph (y=4x)	Gradient = x/y instead of y/x *(logit 1.2)* Linear prototype errors. (in drawing a graph where a curve is expected*: 0.5, 0.92, 1.2)*
4 *(0.4 to 1.6 logits)*	Harder interpretation of 'constant rate' graphs. Overcoming the 'graph as picture' misconception by pointwise interpretation. Interpreting the meaning of (0,0) in context. Harder interpolation on y=x-squared. Sketching linear graphs to tell a story.	Constant rate graphed as y=x prototype *(0.6)* Unit (or tens) prototype for scales *(0.2 and 0.4 logits)*
3 *(-0.5 to +0.4 logits)*	Parallel graphs have the same gradient, speeds interpreted as slopes: same speeds are drawn parallel on graphs. Understands varying slope of a curve (eg y=x-squared) and rate of change in an interval. Compares y-ordinates of two graphs in context. Distinguish slope from height. <u>(Graph and its algebraic expression… not in our test)</u>	y=x prototype (graph is linear and through the origin) *(logit 0, 0)* graph interpreted as picture *(-0.2)* slope-height confusion in context and out of context *(-0.2, -0.2, -0.3)*
2 *(-1.5 to –0.5 logits)*	<u>Reading coordinates off a graph by interpolation</u> and extrapolation. <u>Recognise the slope as rate of change</u> in interpretation of graphs of y on x: eg negative slope is decrease and steeper slope is greater rate of change than shallower. <u>Use of scales in graph reading, interpretation of simple travel graphs.</u>	Confusion of axes (*logit –0.5, also 0.1* a harder case where Sq-root 6 = 36) Misinterpretation of the origin (0,0) in context *(-.55)* Reversal of coordinates *(-0.7, -0.8)*
1 *(-Inf to –1.5)*	<u>Understanding of coordinates</u> (interpret in context), and <u>change or no change and 'steepness' of a graph</u>. Use of unfamiliar coords.	
	Performance of children at higher levels includes those indicated for lower levels.	**The errors listed are most likely to be made by children at the level adjacent or below**

Table 1: Hierarchy of performance and errors

of children's slope-height confusion in relation to their errors in interpreting a graph of the growth of girls and boys in their teenage years. From the graph presented it can be concluded that the girls at age 14 are bigger (graph height) but the boys are growing faster (graph slope). The item (question 7 of the test) was developed from one of from Janvier's, and is designed to probe for slope-height confusion:

INT:	So 'which group is growing faster at the age of 14'. You have 'girls' and why did you choose girls?
Nicole:	Went up 14 and it's more than the boys.
INT:	It is more…
Nicole:	It is more than the boys at the age of 14.
INT:	Right, and what do you understand when I am asking you 'which group is growing faster at the age of 14'?
Nicole:	Which one is growing faster, which one is heavier.
INT:	Which one is heavier. All right, is that what you understand Sara?
Sara:	Yeah yeah
INT:	Which group is growing faster? You went for the one which is growing, who is…
Sara:	Heavier
INT:	Heavier. So you put girls.
Sara:	Yeah

Later when asked to interpret the curve of the graph for boys' growth, Sara effectively explained that their growth was slow up to the age of 12 and then grew fast, then stopped:

Sara:	They grow quite quickly
INT:	Quickly? Till which age do they grow quickly?
Sara:	'Till about 12
INT:	12. And then what happens after 12?
Sara:	They are growing even quicker.
INT:	Even quicker. And then?
Sara:	The line just like … stops.

The interviewer subsequently confirmed that both girls Sarah and Nicole thought that the 'girls' were growing faster and would not change their minds. It seems clear that the classic 'slope height' confusion operated, in that the height of the graph serves as a powerful distractor in interpretation of the graph, leading to the error we sought to confirm.

Interviews and questionnaires with teachers

In the main study the test was given as a questionnaire to the teachers with instructions that they should record their perception of the difficulties of the items on a Likert scale, and suggest misconceptions students might have that would cause difficulty. We built a rating scale from these data and the item-perception-difficulty measures that resulted were correlated with the children's actual difficulty as estimated by the test analysis (rho = 0.395). In addition we sought to confirm the teachers' responses through informal interviews, where we also began to explore their teaching practices.

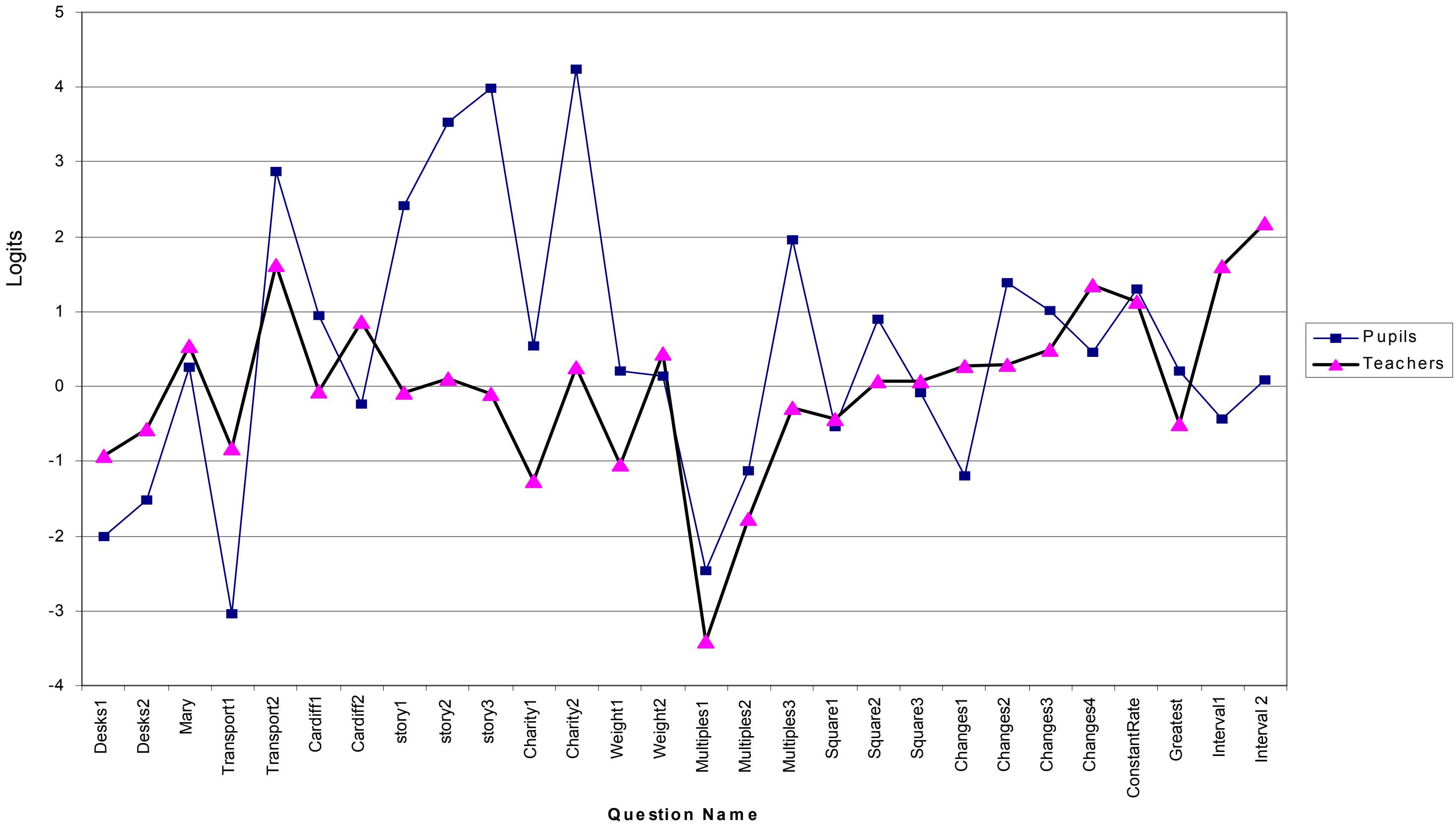

Figure: Teacher estimate and actual pupil difficulty

However, the teachers' estimates were significantly awry on a number of items (see figure, in which teachers' ratings of difficulty were scaled on a rating scale analysis, and plotted against 'actual' scaled values of the pupils difficulties).

The 'discrepant' items were examined for face validity and found perfectly acceptable as test items. However, the teachers' mis-estimation of their (relative) difficulty could be explained by one of two reasons:

(a) in at least three items the teachers underestimated the difficulty for the children because they apparently misunderstood the actual question themselves, i.e. they had the misconception the item was designed to elicit, or they had a limited understanding that did not receive full credit; or

(b) on two items the teachers' overestimated the difficulty because they did not realise the children could answer the question without a sophisticated understanding of gradient.

In the questionnaire and interviews, the teachers were encouraged to list the misconceptions that children might exhibit. Here we summarise the misconceptions mentioned by the 12 teachers we worked with:

Teacher Misconception	1	2	3	4	5	6	7	8	9	10	11	12
Slope height	✓			✓	✓	✓			✓			
Linearity											✓	
Y=X prototype					✓				✓			
Origin prototype					✓				✓			
Picture as graph	✓		✓		✓	✓			✓	✓		✓
Co-ordinates			✓		✓	✓		✓	✓			✓
Scale	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		

Table 2: Misconceptions mentioned by 12 teachers in interview or in the questionnaire

Conclusions and discussion

We have developed an instrument and a hierarchy describing children's graphical thinking and misconceptions which respects their curriculum and is regarded as valid by the, admittedly small, sample of teachers involved. The hierarchy summarised in the chart in this paper suggests how more and less sophisticated pupils behave with graphs and what their main misconceptions are. This is linked approximately at the lowest levels with the hierarchy for graphs described by Kerslake, but covers the literature on graphical interpretation.

The evidence causes us to doubt whether teachers are aware of the common misconceptions the instrument reveals in this field, and we believe that many teachers would benefit from using the instrument in their teaching. In our current work we are investigating this. In small numbers of observations of lessons, we have not yet seen teaching which takes account of the major misconceptions spontaneously.

We are aware of the criticism that the misconceptions children exhibit may arguably be strongly associated with the particular problems they are presented with and the tools they are given to handle them with. Indeed Roth's work (see http://www.educ.uvic.ca/faculty/mroth/Papers_Available) has shown that even expert scientists exhibit misconceptions when presented with tasks which demand interpretation in contexts outside their familiar experience. We agree also with Ainley (2000) that graphical work in general and interpretation skills in particular become transparent or fused when children embed them in their activity or social practice. This is entirely consistent with our other work (eg, Wake et al, 2000; Williams et al, in press) which suggests that experts interpret and use graphs effortlessly in their daily practice, when the graph as a semiotic tool is fused with its interpretant. Nevertheless we believe that learning to interpret graphs in new contexts is an important, if demanding skill, which requires its own practice. The difficulty of research on mathematics in use is that only rarely do we see mathematics actually being learnt rather than used.

We therefore persist in believing that work in academic settings which purports to represent unfamiliar situations is a valuable part of the curriculum, and that children's difficulties therein need charting and pedagogical attention. In the next stage of our work we will examine more closely teachers' use of such diagnostic instruments and how they might develop their practice in this respect.

References

Ainley, J. (2000). Exploring the transparency of graphs and graphing. In T. Nakahara & M. Koyama, (eds.) *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education,* PME-24 (2-9 to 2-16).

Bell, A., Brekke, G. and Swan, M. (1987). Diagnostic teaching: 5, graphical interpretation teaching styles and their effect. *Mathematics Teaching*, 120, 50-57.

Clement, J. (1985). Misconceptions in graphing, *Proceedings of the 9th Conference of the International Group for the Psychology of Mathematics Education*, 1, 369 - 375.

Even, R. (1998). Factors Involved in Linking Representations of Functions, *Journal of Mathematical Behavior*, 17(1), 105-121.

Even, R. (1993). Subject - Matter Knowledge and Pedagogical content Knowledge: prospective secondary teachers and the function concept, *Journal of Research in Mathematics Education,* 24(2), 94-116.

Hadjidemetriou, C. & Williams, J. S. (2000). *Assessing graphical literacy in year 10 Mathematics pupils*. British Educational Research Association student symposium. (http://www.man.ac.uk/cme/ch).

Hart, K. M. (1981). *Children's Understanding of Mathematics, 11-16,* London: John Murray.

Hitt, F. (1998). Difficulties in the Articulation of Different Representations Linked to the concept of Function, *Journal of Mathematical Behavior,* 17(1), 123-134.

Janvier, C. (1981). Use of Situations in Mathematics Education, *Educational Studies in Mathematics,* 12, 113-122.

Kerslake, D. (1981). Graphs, in K. Hart (ed.) *Children's understanding of mathematics 11-16,* London: John Murray.

Leinhardt, G., Zaslavsky, O. & Stein, M. S. (1990). Functions, Graphs and Graphing: Tasks, Learning, and Teaching, *Review of Educational Research*, 1, 1-64.

Ryan, J. T. & Williams, J. S. (2000). *Mathematical discussions with children: exploring methods and misconceptions as a teaching strategy*. Manchester: University of Manchester.

Ryan, J. T., Williams, J. S. and Doig, B. A. (1998). National tests: educating teachers about their children's mathematical thinking. In A. Oliver & K. Newstead, (eds.) *Proceedings of the 22nd conference of the International Group for the Psychology of Mathematics Education*, 4, 81-88. South Africa: University of Stellenbosch.

Sharma, S. S. (1993). *Graphs: Children's strategies and errors.* London: Kings College.

Wake, G. D., Williams, J. S. & Haighton, J. (2000). Spreadsheet mathematics in College and in the workplace: a mediating instrument? In T. Nakahara & M. Koyama, (eds.) *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education,* PME-24.

Williams, J. S & Ryan, J. T. (2000). National Testing and the improvement of Classroom Teaching: can they coexist?, *British Educational Research Journal,* 26(1), 49-73.

Williams, J. S., Wake, G. D. & Boreham, N. C. (in press). College mathematics and workplace practice: an activity theory perspective. *Research in Mathematics Education* (3).