Following students' development in a traditional university analysis course

Márcia Maria Fusaro Pinto

Departamento de Matemática
Universidade Federal de Minas Gerais
Brazil
e-mail: marciaxxss@mat.ufmg.br

David Tall

Mathematics Education Research Centre
University of Warwick
CV4 7AL, UK
e-mail: david.tall@warwick

This paper presents a framework to follow students' development through a formal mathematics course. Built on learning episodes of two groups of three students attending a traditional university course, it highlights different cognitive demands learners will have according to their personal learning strategy.

Introduction

In this article we trace the development of students through the learning of limit concepts in sequences, series, continuity and differentiation, using and refining the routes of learning identified as *formal* and *natural* (Pinto 1998, Pinto & Tall, 1999). In this study we focus on three levels of development in each route. Formal thinkers attempt to base their work on the definitions, but the handling of three nested quantifiers imposes great cognitive strain. Some fail to cope with the complexity of constructing meaning, often focusing on manipulating the symbols and inequalities rather than the logic. Natural thinkers reconstruct new knowledge from their concept image. Unsuccessful learners in this route attempt to interpret the definitions in a personal way that fits their imagery, rejecting the formal theory or leading to conflict. Students who get past this initial hurdle cope with the definitions and deductions in their own personalised ways. Formal learners essentially construct the theory by deduction, coping with the great cognitive strain as best they can, producing a deductive formal theory. Natural learners—working from their concept imagery—reconstruct it taking account of more general ideas met in the course. They must then develop the formal theory from their reconstructed imagery, producing a formal theory integrating both imagery and deduction. This summary of data leads to a concise framework of development (table 1).

	Formal learning	**Natural learning**
1 Initial obstacles	Routine (based on concept definition) (a) disjoint from images, partial procedures (b) attempt to link to images, weak links	Informal (based on concept image) (a) formalism rejected, maintaining images (b) formalism embedded in informal knowledge, with some conflict
2 Theory Building	Formal construction Constructing formal theory	Formal reconstruction (with some conflict) (a) Thought experiments, reconstructing images (b) Deductions reconstructing formal theory

3 Formal theory	Formal (deductive)	Formal (integrated)

Table 1: levels of development in natural and formal routes to learning formal mathematics

Methodology

This study builds on a cross-sectional analysis of three pairs of students selected to best approximate prototypical strategies of learning from eleven students who were studied in seven interviews at three-week intervals. At this stage of theory development we move from a study of a single limit concept (the convergence of a sequence) to take into account successive encounters with sequences, series, continuity, differentiation, and a final reflective interview with each student. Our purpose is to study the ways in which different students interpret the material presented to them by the lecturer.

The 'formal' route of learning

Three levels of development are identified for the 'formal' route, which we describe as:

1: Initial Obstacles (working with the concept definition).

1a: routine approach, disjoint from images, focusing on rules and procedures, partially achieved.

1b: routine approach, some link with images, causing conflict, unable to coordinate processes.

2: Theory building (with some evidence of routinising reflectively).

3: Formal deductive knowledge.

We consider three students, Rolf, Robin and Ross, who focus on the formal structure of the course. Rolf's constructions define level 1a; he withdraws from the course after the first ten weeks. Robin starts at level 1b, and then consolidates his work at level 2. Ross distrusts his limited visual imagery and concentrates on the formal theory at levels 2 and 3. On occasion he uses visual imagery unsatisfactorily and this part of his activity is also considered.

Rolf attempts to build from the formal definitions and proofs from the outset. DEFINITIONS are rote-learned inaccurately. However, in handling questions involving direct calculation, he is able to provide ARGUMENTS that seem to be based on the formal theory. For instance, he explains the constant sequence 1, 1, 1, … is convergent, saying:

> "Take the sequence 1, 1, 1, ... the N is just any greater than zero. Big N is just 1 basically."
> (Rolf, first interview)

He relies on the authority of the lecturer to decide what is important:

> "...I have been told that a formal proof is a proof involving epsilons. That's why I think that."
> (Rolf, second interview)

He focuses on rules, procedures and his IMAGES arise from his routine activities rather than from pictures. He is confused rather than in conflict (level 1a):

> "*I understand the reasoning, but I mean, I can do it.* And the reason I can do it is maybe because I understand it err I don't know. I don't know if I understand everything or I don't, because I don't know. If I am not understanding something that I don't know, then I don't know

basically, you see what I mean. I mean, as far as I feel, I understand everything about this, […] but I haven't a clue. You see, I don't know."
[Okay, you are saying you can work it out.]
"Yes, *I can work it out.*" (Rolf, third interview)

At the beginning of his second term, Rolf transferred to another university.

Robin begins in the same way as Rolf, routinising mechanically, but unable to coordinate processes (level 1a). DEFINITIONS and ARGUMENTS may be categorised in the same manner as Rolf. However, Robin attempts to link IMAGES built from the formal theory to previous experiences, which generates conflict (level 1b).

> "I mean, the making ... I could understand it said: by finding the value of N ... it actually ... is the proof that the ... sequence is tending to zero. That was a bit awkward, I couldn't get that to start with. I could understand what the definition was saying, but, to me, it didn't seem to be proving ... that it went to zero. ... I'm not ... not fully happy with this. There's still things I've got to get sorted out. But ... I get ... I have made some progress since I started."
> (Robin, first interview)

We understand that time pressure and weakness in his elementary mathematical background hampered his attempts to construct meaning for the formal theory.

Robin sometimes evokes visual images; he explained that imagery was helpful for him to get a sense of what was being proved.

> "... I was saying that in Analysis II, we did a lot ... from the theorems, he (the lecturer) would say: here is a graph for the theorem; well, he is showing what is going on, and that, to see it pictorially, helps to understand incredibly, really." (Robin, seventh interview)

However, he is unable to translate such analogical representations into formal language, and continued to focus on the formal route while failing to cope with the full complexity of the definitions.

Ross—the most successful formal learner—starts by constructing formal knowledge through routinising reflectively on the theory presented in the lectures (level 2). DEFINITIONS are formal, correct with occasional slips. ARGUMENTS are meaningful, based on the formal theory. IMAGES are generally built from the formal theory, usually compartmentalised from previous experiences. For instance, contrasting old and new experiences does not cause him conflict as in the limit of a constant sequence.

> "Umm 1, 1, 1, ... umm yes, because it's a set of ones, so it would tend to 1, because ... it's not changing. But ... now again when she [the lecturer] first said that like 1, 1, 1, ... tends to 1, then I thought it was bit strange, because you tend to think of the sequence going up and then gradually getting closer and closer to a value so that if you go sufficiently far out then it's reached a value, whereas 1, 1, 1, ..., I mean, you can take the first term it's already at the limit umm but I mean ... this ... definition works for that, so ... so it must tend to value ... so..."
> (Ross, first interview)

Ross forms strong conceptual links between successive notions of limit. For example, he perceives the link between series and the previous formal theory of sequences:

> "Umm ... well, the way, I mean, in the order of the things we've done umm it makes sense the feature of relating the series back to sequences, which was what we started off doing so ... if you can convert a series into a sequence ... then you just deal with a sequence ... ... I suppose ...

> umm by ... relating a convergent series with convergence of sequences then ... you are building up all of this umm area of maths from a very small ... ... each new step that you do up series you will be relating to something you have already done so ...” (Ross, third interview)

This aspect is a characteristic of Ross’s performance during the course.

Ross maintains his strategy of practising routines and familiarising himself with the formal theory (level 2). There are times when he is unable to solve problems, such as writing a proof that a real function defined on $\boldsymbol{Z}$ is continuous, even though he could write down a formal definition of continuity for a function $f: \boldsymbol{R} \to \boldsymbol{R}$. His DEFINITIONS are always formal, correct or nearly correct. He always tries to relate ARGUMENTS to the formal theory. He realises the weakness of his own visual imagery and seeks the safety of formal definitions and deductions. On the few occasions where he evokes visual IMAGES they are often linked to previous experiences, with conflicts or inconsistencies. For instance when discussing the function $f(x) = x\sin(1/x)$ he argues:

> “...I think there is sort of ... the oscillations are getting closer and closer together and ... so I suppose a tiny change in x produces a huge change in the gradient and so at nought itself there is no way to find the gradient ... I suppose ... it’s going up and down (??)”
>
> (Ross, sixth interview)

He seems to be talking about $f'(x)$ rather than $\frac{f(x)-f(0)}{x}$ as x nears zero.

In this last example we see a student we have classified as a formal learner using visual links. We do not claim that formal learners *never* use imagery. Taking the formal route means relying on formal definitions and proof to generate the theory. Meaning arises from the formalism itself rather than from any underlying pictorial images.

Following the ‘natural’ route of learning

The three levels proposed for the natural learner are as follows:

1: Initial obstacles (based on the concept image).
- .1a: formal theory is rejected, not assimilated,
- 1b: formal theory is embedded within the old.

2: Theory building (in conflict with formal theory).
- 2a: thought experiments reconstructing images,
- 2b: Deductions reconstructing formal theory.

3: Formal theory (integrated with imagery).

Cliff, Colin and Chris are natural learners developing through the course at different levels. Cliff, the least successful, reveals some characteristics of level 1, Colin’s conflict characterises level 2a and the most successful, Chris, works at levels 2 and 3.

Cliff interprets new knowledge in terms of old, at levels 1a and 1b, ignoring the new experience if it does not fit his previous experiences, or rote-learning, to respond as he perceives being required by the course assessment. ARGUMENTS are image-based, or eventually pragmatic, and IMAGES are not reconstructed to fit the formal theory. For instance, when asked to consider the convergence of a constant sequence, he says it does not tend to a limit because

> "1, 1, 1, 1, 1 ... no, it's just 1, 1, 1, 1 continuously." (Cliff, first interview).

As the course progresses, the formal theory is never assimilated (level 1a). DEFINITIONS are not reproduced formally and are given in terms of imagery:

> "Err ... continuous means what ... I've got my definition of continuous function as ... I can draw the graph and you don't lift the pencil if it's continuous..." (Cliff, fifth interview)

or restricted to particular prototypes:

> "Umm ... well, I think it's any function ... any function can be differentiated … any normal function."
> [What is a normal function for you?]
> "Umm ... like $y = 3x$ or $y = 3x + x^2$." (Cliff, sixth interview)

This instance shows Cliff's evoked images of functions as prototypes restricted not to continuous functions, but to polynomial ones (Vinner, 1983; Vinner & Dreyfus, 1989). In the seventh interview, he is back to the same strategies as in the very beginning (level 1b), rote-learning definitions in a way that is not suitable for formal deduction.

In a middle trajectory, Colin starts in the same way as Cliff and interprets new knowledge in terms of old. Sensing results and theorems are true, he does not understand *why* he is being asked to prove results that he believes to be obvious.

> "It seemed to be a silly question that ... if a_n tends to 1 then if you question when a_n is greater than $\frac{3}{4}$ ... this is a bound, it seems ... I don't know why... ..." (Colin, second interview)

As the course progresses, he gradually perceives his IMAGES in conflict with the formal theory (level 2a); allowing him to attempt to begin attempting to tackle proof:

> "... now I am sort of getting into it (into the course) ... what I am supposed to be doing ... how I am supposed to be proving things ... so it's got a bit easier." (Colin fifth interview)

DEFINITIONS remain distorted or eventually rote-learned and ARGUMENTS are image based such as:

> "... $\sin x$ is differentiable because I can always find the gradient of $\sin x$."
> (Colin, sixth interview)

Conflict between old and new images makes him move a little:

> "How do I prove this ... umm well, you just show that the umm you can work out the gradient for all points on the curve."
>
> [Okay, but how do you prove that ...]
>
> "How do I prove it ... umm ...
>
> $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$
>
> umm ... so like that ... (Colin, sixth interview)

In this sense the natural learner's difficulty with formalism corresponds to the formal learner's confusions with the underlying concept image. However, Robin's formal trajectory discussed earlier contrasts with Colin's conflicting 'natural' route. Robin does not know *what* he is proving so he builds a theory with weak conceptual links. Colin does not know *why* he is being asked to prove statements but he senses their truth.

Colin's old images deeply interfere with any reconstruction and he appears to be in a state of conflict until the end of the course. For example, after presenting a proof that 0.999...(recurring) is 1, using partial sums and the definition of convergence of a series, he comments:

> "... ... It's sort of ... I understand it should be 1 ... and that the limit of the sequence is actually 1 just ... 1 down as notation. It just it's a bit hard to let go of 0.9999 recurring ..."
>
> (Colin, seventh interview)

Chris—the most successful natural learner—constantly seeks reconstruction of old knowledge to build new. (level 2) and finally develops an integrated formalism (level 3) in which he can write both the quantified form of the definition of limit and also verbalise his own image for it:

> A sequence has a limit if~~f~~ and only if as the sequence progresses, eventually, all values of the sequence gather around a certain value.
>
> (Chris, seventh interview)

His DEFINITIONS are always correct, or nearly correct. ARGUMENTS are based on thought experiments, most of the time articulating images reconstructed with the new theory. For example, a classical question resolved in classroom by handling inequalities:

> If $a_n \rightarrow 1$, prove that there exists $N \in \boldsymbol{N}$ such that $a_n > \frac{3}{4}$ for all $n>N$.

has its solution verbalised by Chris as an experiment

> "I chose epsilon as 0.1 ... and showed that a_n lies between 0.9 and 1.1; so it must be greater than $\frac{3}{4}$." (Chris, second interview)

In general Chris is able to present his arguments formally. However, as he builds on his imagery (level 2a), he sometimes formulates image-based thought-experiments, such as initially restricting his concept of divergence to plus or minus infinity he also visualises continuity as "drawing the graph without taking the pen off the paper". We hypothesise that solving conflicts between old and new ideas is central to the natural route (see also Vinner, 1991). For instance, the fact that a constant sequence converges according to the formal theory surprises Chris, because it conflicts with his old images:

> "(Laughter) I don't know really. It definitely it will ... it will always be one ... so I am not really sure (laughter) ... ... umm ... it's strange, because when something tends to a limit, you think of it as never reaching it ... so if it's ... 1 ... then by definition it has a limit but ... you don't really think of it as a limit (laughter) but just as a constant value." (Chris, first interview)

As in the case of Cliff, such images may have been evoked by the use of the word "tends to" in the formulation of the question (see Schwarzenberger & Tall, 1978; Cornu, 1991; Monaghan, 1991). Data shows Chris adding new facets as additional information, as a 'natural' strategy of coping with new experiences, closely related to *how* we function in our everyday world. In his last interview, he resolves his earlier conflicts with constant sequences:

> "Umm ... it's that ... where the values are the same, it doesn't deviate at all."
>
> [Doesn't deviate.]
>
> "Doesn't deviate on the line ... and then that must be the limit." (Chris, seventh interview)

Discussion: following students' development

Tables 2 and 3 summarise the development of the students over the twenty weeks of the course. (The shaded entries represent transitional movement into the given level.)

	Sequences	Series	Continuity	Derivative	Final Interview
1. Initial obstacles	Rolf (a) Robin (a& b)	Rolf (a) Robin (b)	[Rolf withdrew] Robin (b)		
2. Formal Construction				Robin	Robin
	Ross		Ross		
3. Formal (deductive)		Ross		Ross	
					Ross

Table 2: Students following an essentially formal route

	Sequences	Series	Continuity	Derivative	Final Interview
1. Initial obstacles	Cliff (a) Colin (b)	Cliff (a) Colin (b)	Cliff (a)	Cliff (a)	Cliff (a)
2. Formal Reconstruction			Colin (a)	Colin (b)	Colin (b)
		Chris (a&b)	Chris (a&b)	Chris (a&b)	
3. Formal (deductive)	Chris				
					Chris

Table 3: Students following an essentially natural route

Rolf, Robin and Ross are classified as 'formal learners' who build their schemas by routinising and familiarising themselves with the formal constructs. They attempt to handle quantified statements, constructing representations which are propositional (see Eysenck & Keane, 1997). Rolf cannot cope with the formal definition and only reproduces partial procedures (level 1a) before withdrawing from the course. Robin finds great difficulty initially, with his underlying images in conflict in sequences, series and continuity, (level 1b) but persists in his formal route and manages to start making formal constructions (level 2) in dealing with derivatives. Ross works mainly at level 2, memorising the definition and working with the formal proof. He shows an understanding that the topic 'series' has essentially already been formalised using ideas from the previous section ('sequences'). This is acknowledged by classifying his responses as being in transition to a full formal theory. In the final interview he is classified at level 3.

Cliff, Colin and Chris are classified as 'natural learners' because they use their own imagery as a starting point to build their theories. These learners support their construction with analogical representations (Eysenck & Keane, 1997), which they build using their old imagery. Cliff clings to his informal imagery, making no headway with the formal theory (level 1a) throughout the course. Colin makes an attempt at the formal theory building it first on imagery (level 1b) then modifying his imagery (level 2a) as he moves on to continuity, finally beginning to build formalism on his modified images (level 2b). Chris operates at level 2 and moves to level 3.

Drawing conclusions

This research unfolds aspects of individual construction of knowledge that differ from other proposed frameworks. For instance, the strategy of encapsulation of process into object (Dubinsky *et al*, 1988) does not appear to provide a model to explain the cognitive strategies of the 'natural' route of learning. Rather than constructing a concept image from a defined object, abstracting from 'actions on objects', the successful natural learner understands the defined object by reconstructing it from the concept image.

At the other end of the spectrum, we note that there are concept images not developed through formal deduction (Tall & Vinner, 1981) which nevertheless may seem coherent with the formal models. These may or not be capable of translation into formal language. The concern is that such concept images may give the learner an apparent grasp of the theory, though they may not be sufficient to guarantee long-term success.

In the teaching of advanced mathematics, this research shows there does not seem to be a single methodology or 'formula' that works for all students. Even first class students encounter times where they struggle badly. Learners have different cognitive demands, according to their own strategies of learning. Perhaps teachers may address the needs of both routes to learning. However, there is still a distinct possibility that natural learners may be confused by formal instructions just as formal learners may be confused by references to imagery, so that it may require more subtle treatments for different kinds of student.

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