STUDENTS' CONCEPTIONS OF CUBIC FUNCTIONS

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This paper explores some aspects of students' conceptions of cubic functions of the form $f(x) = a(x-h)^3 + k$ *. This was done by analyzing students' responses to a problem dealing with family of functions. On the basis of these answers, a series of hypothesis was formulated. These hypotheses were corroborated with a second group of students who solved the problem and were asked to correct and comment a pre–arranged solution to it. It was found that an important proportion of students develops a consolidated and invalid conception of the cubic function with special characteristics concerning its domain.*

Introduction

In this study we wanted to explore some aspects of students' conceptions of cubic functions of the form $f(x) = a(x-h)^3 + k$. We did so by considering the performance of a group of university students solving a problem dealing with graphics of cubic functions. Our purpose was to describe some characteristics of the students' graphics and to explore, on the basis of those graphics, some aspects of the students' conceptions of the cubic function. The study was done in two phases. During the first phase, we constructed a set of categories that enabled us to characterize the students' graphics and we formulated some conjectures concerning the possible conceptions that were behind that behavior. During the second phase, working with a different group of students, but the same problem, we confirmed the previously found characterization for the graphics and were able to test the proposed conjectures. We formulated as well a partial description of the students' conceptions of this type of cubic function.

In what follows we discuss some conceptual aspects concerning the understanding of the notion of function, in general, and of the cubic function, in particular. We then describe in detail the instruments we used to collect, codify and analyze the students' performance. Finally, we present the results found and draw some conclusions.

Understanding of the notion of function

The understanding of the notion of function has drawn some attention recently (Harel & Dubinsky, 1992; Tall, 1991; Romberg et al., 1993; Leindhardt et al., 1990). In particular, Sierpinska (1992) has produced a list of acts of understanding and epistemological obstacles related to the notion of function. For example, an act of understanding concerning the representation of functions is the discrimination between different means of representing functions and the functions themselves. An epistemological obstacle concerning the graph of a function is that the graph of a func-

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tion is a geometrical model of the functional relationship. It does not have to be exact and can contain pairs (x,y) such that the function is not defined for x.

The notion of *representative* (Schwarz & Dreyfus, 1995) has become important as technology is used more widely in the teaching of functions. This notion has to do with the fact that a given function can have multiple representatives within a given representation system. For example, $f(x) = (x-2)^2 + 3$ and $f(x) = x^2 - 4x + 7$ are two representatives of the same function. Likewise, a function can have an infinite number of representatives in the graphical representation system, depending on the range and scale used for the axis.

The process of understanding can be seen as proceeding by "states". Each state corresponds to a certain partial knowledge (a *conception*) that has worked with previous experience and that allows the student to feel comfortable while solving tasks. For a given type of problems a conception can correspond to either a valid or invalid version of the mathematical knowledge at stake. We will say that a conception (valid or invalid) is consolidated whenever the student having it "feels comfortable" putting it into play while solving problems. As it will be explained later, this sense of "comfort" can be observed through the coherence of the answers of the student to a series of related questions. On the other hand, a conception can be in an unconsolidated state. When this happens, partial knowledge has not been established and the answers of the student do not follow a coherent pattern.

Understanding of the cubic function

There is little research made on the understanding of the cubic function. Curran found that students exhibit links between their understanding of the graph of a cubic function and their understandings of the graphs of linear and quadratic functions (Curran, 1995).

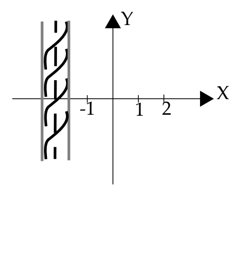
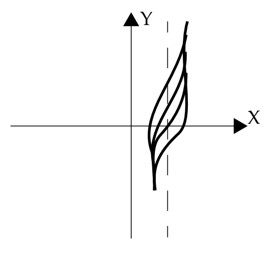
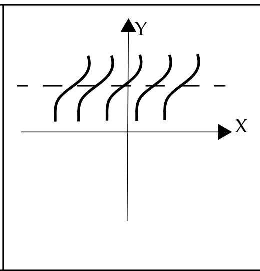
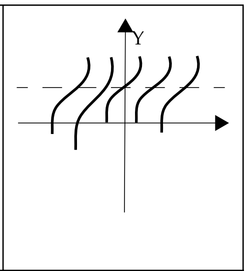
We considered a specific form of the cubic function: $f(x) = a(x-h)^3 + k$. We did so, because the precalculus course under study followed a strategy of teaching translations and dilatations in the construction of graphs of functions. The general form of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is introduced later on.

We were interested in exploring some aspects of the students' conceptions of the cubic function. In particular, we wanted to see if it was possible that students develop consolidated but invalid conceptions concerning the domain and range of the function. This interest came from a first phase of the study in which students were asked to solve a problem concerning family of cubic functions (the problem is shown later). We found that many students draw graphs similar to those shown in table 1.

The proportion of answers of these types made us think that they could be a consequence of an invalid and consolidated conception of the cubic function, instead of simply being a consequence of drawing mistakes or specific circumstances related to the specific problem at hand. This conception could be expressed as follows:

> *The domain and range of the cubic function is a proper subset of the real numbers and can be seen as an interval around the inflection point of the function.*

We felt that if a student has this kind of conception, then he/she could agree with situ-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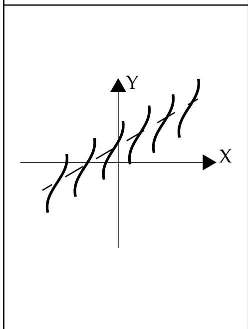
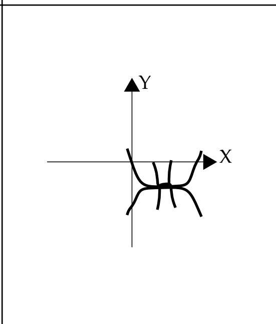
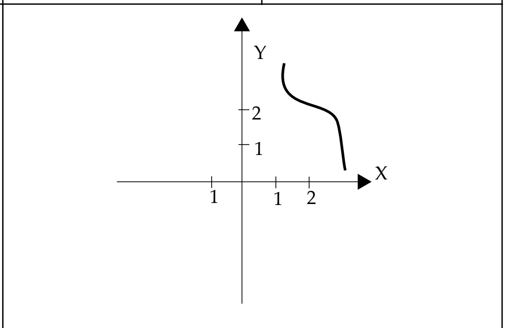
			
5	6	7	

Table 1: Types of graphs produced by students

ations as the following:

- ▲ If the inflection point of the function is on the y-axis or close enough to it, then the graph crosses that axis (Graph types 3, 4, 5, 6).
- ▲ If the inflection point of the function is not close to the y-axis, but it is not too far away, then the graph of the function has the y-axis as one of its asymptotes (Graph type 2).
- ▲ If the inflection point is far enough from the y-axis, then the graph of the function has other asymptotes (Graph type 1).
- ▲ If the inflection point of the function is far enough from the x-axis, then it does not cross that axis (Graph type 3).

In order to test these hypotheses we decided to work with a different group of students and the same problem. This was the second phase of the study.

Context and data collection

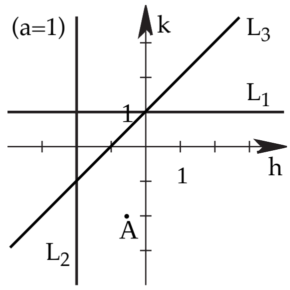
The study was done with first–semester students of a Precalculus course in a private university in Bogotá, Colombia. This course was taught on the basis of a curriculum innovation that involved graphic calculators (Gómez et al., 1996). The course concerns an introduction to the study of functions with special emphasis in the relationship between the symbolic and graphical representation systems and problem solving. One fourth of the course deals with linear functions, followed by the study of cuadratic, cubic, polynomial, rational and radical functions. Special attention is given to the graphical role of the parameters in the different possible symbolic representations of a function.

Some results are already known concerning this curriculum innovation. Mesa and Gómez (1996) found no differences in some aspects of understanding between the students who took the traditional course and those who took the curriculum innovation. Gómez and Rico (1995) found that the students of this group participated more actively in social interaction and in the construction of the mathematical discourse, changes that can partially be attributed to a different behavior of the teacher. Even though she changed her behavior, Valero and Gómez (1996) found that the teacher could not change completely her beliefs system. Carulla and Gómez (1996) found that the teachers and researchers who participated in the curriculum innovation underwent significant changes on their visions about mathematics, its learning and teaching. Gómez (In press) found that the effects of technology use on achievement depends on the way it is integrated into the curriculum.

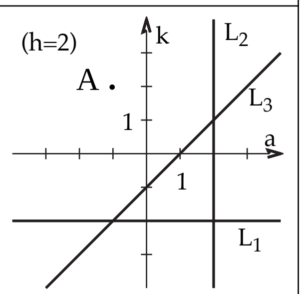
The problem used was one of the problems scheduled to be done at the time cubic functions were taught in the course. We wanted to explore whether students of this new group produced graphs similar to those found in the first group; and whether those graphs were a consequence of a consolidated but invalid conception of the cubic function. In order to perform this exploration, we collected and analyzed three different types of information: 1) the answers of the students to the problem; 2) the way students corrected and commented a solution to the problem produced by us that contained most of the mistakes corresponding to the consolidated and invalid conception; 3) the comments made by the students to a series of statements related to the above solution and to the hypotheses presented earlier.

The last two instruments were designed in order to make sure that the mistakes found in the graphs were not a consequence of drawing problems and to induce the students to put into play their conceptions under different circumstances concerning the same problem. The problem proposed to the students was the following.

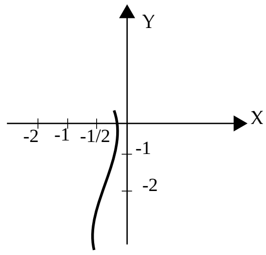
1) Assume that a = 1. The figure shows the plane **h–k** plane, where the point (h, k) represents the cubic function $y = (x - h)^3 + k$. Draw in different cartesian planes **x–y** the functions or family of functions corresponding to a) A; b) L_1; c) L_2; and d) L_3.



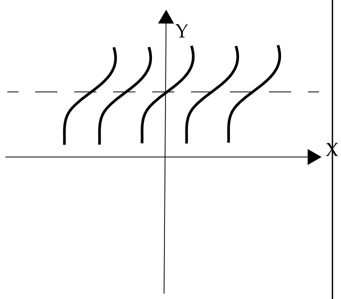
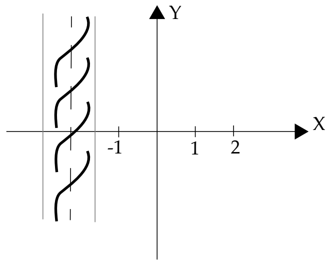
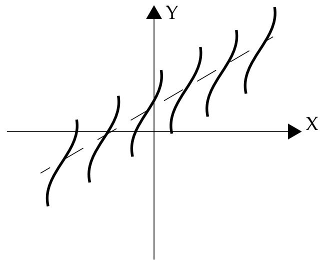
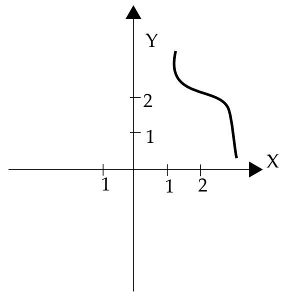
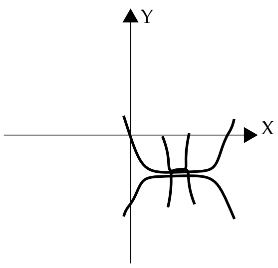
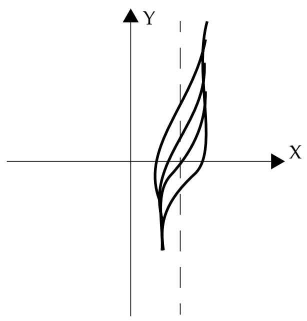
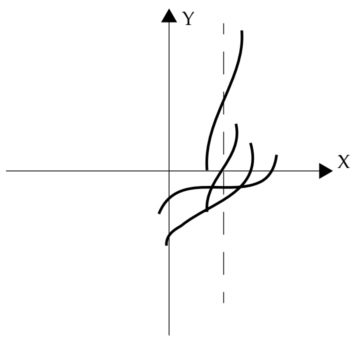
2) Assume that h = -2. The figure shows the **a–k** plane**,** where a point (a, k) represents the cubic function $y = a(x - 2)^3 + k$. Draw in different planes **x–y** the functions or family of functions corresponding to e) A; f) L_1; g) L_2 and h) L_3.



The following is an example of the solution to the problem that the students were asked to correct and comment and the statement they were expected to comment.

1) a. The point A corresponds to a graph whose inflection point is $(-1/2, -2)$. Its symbolic representation is $y = (x + 1/2)^3 - 2$.		The graph does not cross the y-axis.

The following are the rest of the graphs and statements that were proposed to the students for correcting and commenting on.

			
1) b.The graphs do not cross the x-axis.	1) c. The graphs seem to have two asymptotes.	1) d. Some graphs cross the y-axis and others do not.	2) e. The graph has restricted domain and range.
			
2) f. Some graphs do not cross the y-axis. The domain is the same for all the functions.	2) g. For big values of y the graphs join each other.	2) h. Those graphs with big enough value of k do not cross the x-axis.	

Analysis

For the solution of the problem the students were divided in two groups: 13 students were asked to solve problem 1 and 9 students, problem 2. All students were asked to correct the solution proposed and to comment on the statements proposed. We calculated the following percentages:

▲ The percentage of students that, having produced a graph, drew a graph of one of the types expected, as described in table 1.

▲ For each answer corrected, the percentage of students that marked it as correct.

▲ For each statement commented, the percentage of students that accepted it as valid.

Furthermore, an analysis of each student's answers was made on the basis of his comments to the statements. The purpose of this analysis was to explore the coherence of the comments of each student to the series of statements, and be able to conclude about their conceptions. We considered only those students who commented at least three statements. We considered that a series of answers were coherent if at most one comment was contradictory with the other comments (from the point of view of their validity).

Results

Table 2 shows the percentage of students that, having produced a graph, draw a graph of one of the types expected, as described in table 1.

Graph type	1	2	3	4	5	6	7
Percentage	79	100	8	75	78	88	46
Total number of answers	19	1	12	12	9	8	13

Table 2: Graph types percentages

All students marked as correct all the answers presented in the solution proposed to them. Table 3 shows, for each statement commented, the percentage of students that accepted it as valid.

Statement	1.a	1.b	1.c	1.d	2.e	2.f	2.g	2.h
Percentage	32	54	57	38	56	75	11	75
Total number of answers	19	13	14	8	9	4	19	4

Table 3: Comments percentage

Table 4 shows the percentage of students that had a coherent (valid and invalid) series of comments to the statements, together with the percentage of students who proposed an incoherent series of comments. There were 9 students with less than 3 answers. For the other 10 students, the percentages were as follows.

	Coherent invalid	Coherent valid	Incoherent
Percentage	40	50	10

Table 4: Coherence percentages

Discussion

The results show that students of the second group continue drawing graphs of the types found with the first group of students. When asked to correct the pre–arranged solution, all students marked as correct all the invalid answers proposed to them. However, when asked to comment on the statements proposed, the reactions of the students differed. Many answers were found in which students did not agree with the statements. Nevertheless, this was not a consequence of a "random" reaction from the students to the statements. The analysis of each students' comments to the series of statements show that the students can be categorized into three groups: those with a coherent and invalid series of answers, those with coherent and valid answers, and those with incoherent answers. The relevant point here is that the group with coherent but invalid answers represents an important proportion of the total group. This leads us to think that many students can develop a consolidated and invalid conception of the cubic function of the form $f(x) = a(x-h)^3 + k$. However, our hypothesis concerning the range of the cubic function is not clear, given the results to questions 2g and 2h.

Conclusions

There might be many reasons why students develop this type of conception of the cubic function. From the mathematical point of view this seems to be a natural situation. Since cartesian planes are traditionally drawn with the same scale in both axis and the cubic function grows rapidly, the part of the domain of the function that can be "seen" in the graph is usually a subset of the real numbers. Furthermore, it seems that most textbooks and teachers tend to consider cubic functions for which $a \geq 1$.

Teaching and textbooks do not help either. We found that the textbook used in the course did not present graphs in which it could be seen that the domain were the real numbers. Furthermore, when checking some teaching materials drafts from one of the teachers, we found graphs similar to those drawn by the students.

It is very likely that this problem exists with cuadratic functions as well. As a matter of fact, when performing informal interviews with the students, one of them justified the "asymptotic" behavior of the cubic function as being the same as the one for the cuadratic function. In this sense, one can say that there is a link between the understanding of the cuadratic and the cubic function. However, even though we do not have data to justify, we think that this link is broken when cuadratic and cubic functions are compared to linear functions, a fact that would not corroborate Curran's results in this respect.

One may argue that the problem proposed to the students was about family of functions and not about the domain and range of the functions. Therefore, students could have been more concerned about answering the questions and correcting the answers proposed with respect to what they considered relevant in the problem and this might be the reason why all of them marked as correct all the answers proposed to them. However, this was not the case with the statements they had to comment on. Those statements referred to the graphs themselves and made no direct connections to the text of the problem.

Finally, technology might have played a role as well. The "window" problem identified by Schwarz and Dreyfus with their "representative" concept might induce students to construct their invalid conception. This is a somewhat paradoxical situation given that graphic calculators and computer software allow students to easily change the scale and range of the axis. However, this could be evidence of the fact that students do not take advantage of those features.

The learning difficulties found in this study would not be very important if they could be interpreted as a consequence of drawing problems related to the specific context of the task at hand. However, as we have found elsewhere (Carulla & Gómez, 1997), when technology is involved, students tend to construct their understanding based mainly on the graphical representation of the concepts. This might be the reason why we found these consolidated and invalid conceptions.

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