Teaching the limit concept in a CAS environment: Students' dynamic perceptions and reasoning

Ivy Kidron, Nurit Zehavi, and Esther Openhaim

Weizmann Institute of Science, Israel

By applying symbolic computation and graphics, we tried to enhance students' ability to go from visual interpretation of the limit concept to formal reasoning. While being taught the topic "approximations of functions by Taylor polynomials", the students analyzed the remainder and performed animations that illustrated its convergence. They used the Mathematica software for manipulating algebraic expressions and for generating a wide variety of dynamic graphics. We observed that the graphics produced by the animations were in a sense present in the students' minds even when the computer was turned off. Here we describe a situation in which the interaction with computer graphics helped the students overcome confusion caused by misleading images of the limit concept.

Introduction

This paper deals with the conceptual understanding of the convergence process obtained by approximating a function by means of Taylor polynomials. Central concepts in analysis such as limit and infinite sum are very much related to approximation theory. Therefore, by means of polynomial approximations, we tried to clarify the limit concept. We analyzed students' perceptions of the limit concept in the context of a computer-based mathematics laboratory program. For this purpose, we used Mathematica software (Wolfram Research), which permits symbolic computation, graphics, and animation. Special attention was given to using animation in order to visualize and analyze the dynamic process of convergence. Our research focused on the question to what extent did the use of symbolic computation and dynamic graphics actually help the students in the transition from their visual intuitive interpretation of the limit concept to formal reasoning.

This paper describes some research that examined an approach to teaching analysis at the high school level. The main topics taught were Approximation and Interpolation, from which we explored the issue of Approximation theory in connection to the limit concept. High school students learning at the highest level (Age 16-17, N=84) were involved in the research.

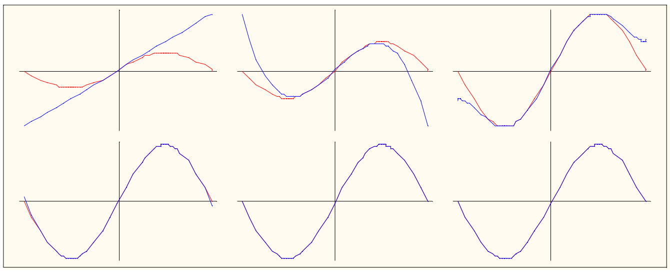
Theoretical Background

In studying students' perceptions of the limit concept, it is important to take into consideration the intuition of infinity. Our logical schemes are naturally adapted to finite realities. As Fischbein, Tirosh, and Hess (1979) observed, the natural concept of infinity is the concept of potential infinity. Openhaim (1986) noticed students' difficulties in grasping that the behavior of a sequence with regard to convergence is unaltered if we omit a finite number of the terms. Davis & Vinner (1986) noticed some unavoidable misconception stages in understanding the limit concept. In trying to understand the difficulties in learning the limit concept, Cornu (1981) described "spontaneous models" that pre-exist before learning the limit notion. Moreover, the definition of limit is formulated in terms of an unencapsulated *process* (given ε, an N can be found such that…) rather than being described explicitly as an *object* (Cornu, 1991).

In attempting to overcome such difficulties, Dubinsky & Tall (1991) proposed using computers in order to enable the students to make constructions on the computer screen leading to corresponding constructions in their minds. Li & Tall (1993) discussed three approaches to teach the limit concept: (1) a (formula-bound) dynamic limit approach, (2) a functional/numeric computer approach, and (3) the formal $\varepsilon - N$ approach. Monaghan et al. (1994) added a key stroke computer algebra approach. We suggest an additional approach: the use of animation to visualize the processes of convergence and to interact with the dynamic graphics. The "Calculus & Mathematica" course (Brown et al., 1991) and Devitt's "Calculus with Maple" course (1993) helped us in preparing the chapters on approximation by expansions. The reference to Euler analysis (Brown et al., 1990) was especially helpful. We used Mathematica for animating the remainder. For analyzing the results we were aided by Verillon & Rabarbel's article (1995) on cognition and artifacts. Assuming that cognition evolves through interaction with the environment, the authors studied the effect of accommodating to artifacts on cognitive development, knowledge construction and processing, and on the nature of the knowledge generated. They stressed the difference between the artifact, as a man-made material object, and the instrument, as a psychological construct.

The Teaching Experiment

The first author taught the students mathematics six hours a week, two of the six hours in the PC lab. The laboratory consists of 20 PCs, each equipped with Mathematica and a hardware system (called classnet) that permits transmitting the content of the screen of each computer to all the computers in the classroom. A pedagogical strategy in the experiment was to use the technology to follow great mathematicians' thought processes. For example, two different approaches were used to approximate a given function by polynomials: analytical and algebraic. In the analytical approach the notion of *order of contact* was introduced, and as an application the students were required to find the polynomials of degree 2, 3, 4,…. that have the highest possible order of contact with a given function at x=0. Mathematica helped the students to solve the relevant systems of equations. In the algebraic approach Taylor polynomials were introduced by using the intuitive idea of Euler: to express non-polynomial functions as polynomials with "an infinite number of terms". The students used Mathematica to *follow the original text of Euler* described in Euler (1988). Following Euler's "experimentalist" thinking, the students used his algebraic approach to represent infinite sums: they used Mathematica syntax in order to expand functions as power series, applying the method of undetermined coefficients exactly as Euler did. Both approaches converged to the coefficients of the Taylor series but each one has its own characteristics: the analytical approach describes the *process* of the different polynomials approaching a given function; the algebraic approach represents the polynomials with "an infinite number of terms" as an *object*. The students made a graphical representation of the results. By means of animation (Kidron, 2000 - Example 1), they were asked to "encapsulate" the process into an object. For example, Figure 1 shows a "dynamic" plot that illustrates the fact that in a given interval, the higher the degree of the approximating polynomials, the function f(x)=Sin(x) and the approximating polynomial are closer.

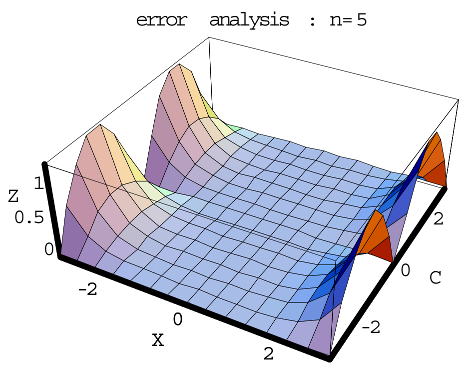


f (x) and $P_n(x)$ for n=1,3,5

f (x) and $P_n(x)$ for n=7,9,11

Figure 1 a "dynamic" plot of sin (x) and the approximating polynomials for $-\pi \le x \le \pi$

The animation permitted the students to see the dynamic process in one picture: they were also requested to stop the animation and observe the different steps of the dynamic picture. In the laboratory, the teacher demonstrated a full process, by means of animation, followed by a group discussion using the classnet. The students noticed that for x values nearer to 0, the function f(x) and the approximating polynomial $P_n(x)$ are closer. In order to clarify the meaning of "closer", the teacher had the students analyze the remainder. The students were given the proof of Taylor's theorem at x = 0 and they computed the expansion of sin (x) around x=0 up to exponent 5. The error ($f(x) - P_n(x)$) - the remainder of Lagrange - is $\frac{f^{(6)}(c)\,x^6}{6!}$ for some c value between 0 and the current x value. The absolute value of the error as a function of x and c with $-\pi \le x \le \pi$, $-\pi \le c \le \pi$ was plotted. Because



the c value in $-\pi \le c \le \pi$ that corresponds to the exact error is an unknown number, the students were requested to look at all pairs (x, c) such that $-\pi \le x \le \pi$, $-\pi \le c \le \pi$.

The following 3-dimension graphics (see Figure 2) represents the error (in fact, an upper estimate on the error) as a function of the two variables x and c.

Figure 2 the error as a function of x and c

In this plot the upper estimate of the error is obtained, for example for $x = \pi$ and $c = \frac{\pi}{2}$. In the laboratory, the teacher demonstrated that in the example f (x)= sin(x):

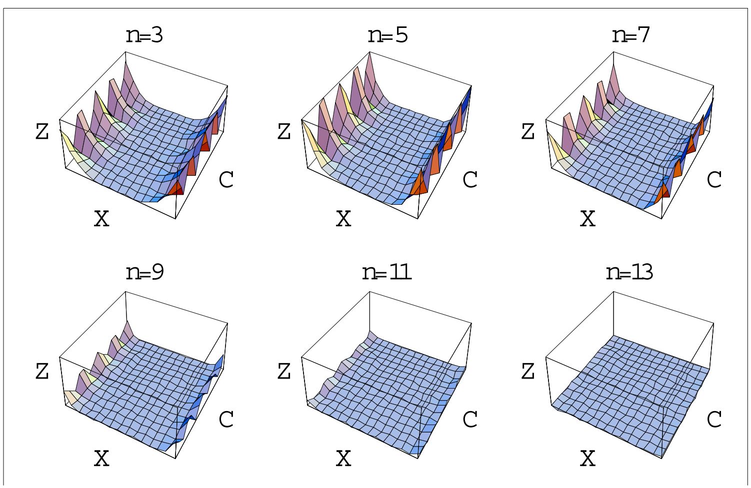
$$\lim_{n\to\infty} R_n(x) = 0, \; R_n(x) = f(x) - (a_0 + a_1x + a_2x^2 + ... + a_nx^n).$$

Using animations of 3-dimensional plots in a fixed domain, the students saw with f(x)=sin(x), how the upper estimate of the error gets smaller when the degree of the approximating polynomial is increased (Kidron, 2000 - Example 2). One student raised an interesting question: Suppose that the degree of the approximating polynomial is fixed; could we obtain the animation with the domain as a variable? At this stage of the course the students could use Mathematica as a programming language to obtain the dynamic graphical output. The teacher encouraged them to construct a visual representation of $\lim_{x\to 0} R_n(x) = 0$ by animation, where the domain of $R_n(x)$ is variable (Kidron, 2000 - Example 3).

Research methodology and data analysis

The methodology adopted for evaluating the students' work and for research purposes was as follows: the teacher demonstrated an idea in the PC lab. The students were then asked to explore the idea by applying their examples with Mathematica. The teacher collected three types of data: (1) students' questions and remarks during the sessions, (2) the Mathematica files of the students' examples, and (3) written tests without the use of Mathematica. We present here a class discussion and some findings from a written test. The following class discussion demonstrates the way the students used Mathematica to interact with the dynamic graphics produced, in order to re-construct their knowledge of the limit concept.

The class discussion The task given to the students in the lab was as follows: Select a function f (x) and illustrate $R_n(x) \underset{(n\to\infty)}{\to} 0$. Most of the students dealt with functions similar to the example given in the class. Here we describe the class discussion that followed the presentation of an example by one of the students, Matan. His example was: f(x)=



cos(2x) for $-\pi \le x \le \pi$ (see Figure 3). He animated (Kidron, 2000 – Example 4) the plots of the absolute value of the error $\frac{f^{(n+1)}(c)\ x^{n+1}}{(n+1)!}$ as a function of two variables x and c,

$-\pi \le x \le \pi$, $-\pi \le c \le \pi$

where n grows from 3 to 13

with step = +2

Figure 3 "animation" illustrating $R_n(x) \underset{(n\to\infty)}{\to} 0$ for f(x)=cos(2x)

In the example that was demonstrated in the laboratory, (f(x)=sin(x)), when n was increasing, the upper estimate of the error was steadily decreasing for every n. This was not the case in Matan's example, f(x)=cos (2x), as is seen for n = 5. We quote students' reactions:

Nimrod: *When the degree n of the approximating polynomial is increasing, the approximation must be better.*

Nimrod tried to explain what he meant by "the approximation must be better".

Nimrod: *In some place in the infinite they will be the same. I mean by "better" that when n is increasing the error is decreasing. It could not be that the error is getting bigger! Maybe the error is not getting smaller. I mean that maybe we cannot see it in the graph but the error is getting smaller all the time when n is increasing*.

We noticed some confusion in Nimrod's reactions. He did not expect that the error would suddenly increase for n=5.He attributed the surprising effect to some limitations of the graphics.

Matan*: f(x) = cos(2x) is an even function. An even function is expanded in a power series with even exponents. I should have given values to n that go from 2 to 14 with step +2 instead of taking n from 3 to 13.*

Matan connected this surprising effect to an irrelevant fact. He used Mathematica to check his conjecture. He chose even values for n but the surprising effect remained unchanged.

Hannah: *Let us look at the different graphs that produced the animation. They do fill the requirement that* $R_n(x)$ *approach 0 as n* $\to \infty$*. The problem is with the degree n = 3 and not with n = 5. From the fifth degree and all the degrees onwards we got exactly what we expected: the error becomes smaller as n increases. We say*

$R_n(x) \to 0$ *if* $R_n(x) > R_{n+1}(x)$ *and this happened from a certain value of n onward. In Matan's example n>3.*

Tomer: *How could we know from which n=N the process begins?*

Nili: *Could it be that from a certain N the error will get smaller for a few steps and afterwards the error will get bigger? We could not find the N graphically. How could we know from which N the error becomes smaller all the time?*

Hannah: *Something disturbs me - if the accuracy (*ε*) is 0.8, for example, and you find N, for example, N=10 such that for all* $n \geq N$, $|R_n(x)| < 0.8$, *then you will not see the phenomenon we described for n = 3. This means that for every* ε *there will be the N that belongs to it. Maybe we will find N if we will compare* $\frac{f^{(n+1)}(c)\ x^{n+1}}{(n+1)!}$ *with* ε *and we will look for the first n for which this expression is smaller than* ε.

The students gave other examples that demonstrated that $|R_n(x)|$ was not always decreasing. Motivated by these examples, the students searched for the N from which onwards, the absolute value of the remainder decreased.

The written test We were interested in two aspects:

(1) The students' ability to visualize the process described by the formal definition of limit, and (2) their ability to express the formal definition correctly. One of the written tests dealt with the notion of the limit, $\lim_{x\to 0} R_n(x) = 0$. The test checked the students' ability to connect the visual and the analytical aspects of the limit concept. The students (N=84) worked on the test without using Mathematica. We identified different ability levels of connecting the visual and analytical aspects of the limit concept.

Most of the students (81%) were able to visualize the process described by the formal definition of the limit and to translate visual pictures to analytical language: "We are given $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots.$ find the appropriate $\delta_1, \delta_2, \delta_3, \ldots.$". They had no difficulty in proceeding step by step through a discrete sequence of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots.$ finding the appropriate $\delta_1, \delta_2, \delta_3, \ldots.$ and were aware that this process is infinite, probably in the sense of "potential infinity". "*You can always find a number which is smaller than the previous one, and so on infinitely"…. "To every* ε_n *there is* δ_n". A smaller number of the students (68%) were able to express the formal definition: "*to every positive number* ε, *there is a positive number* δ *such that…*". Some of the students who failed to express the formal definition wrote: "δ *is not dependent on* ε. ε *is dependent on* δ". "δ *is not dependent on the error, since* δ *is fixing the error: the nearer we approach the point x=0 about which the function was expanded, the smaller is the error*". These students remembered the order in which they worked in the laboratory -

beginning with domain and finding the error. The Cauchy's definition begins with ε ... It was difficult for them to reverse the order!

Discussion and Conclusions

The class discussion around Matan's example related to the concept of the limit, $\lim_{n\to\infty} R_n(x) = 0$. The surprising effect of the dynamic graphical feedback that Mathematica provided was very important for the students' learning experience. The students expected, as in the example demonstrated in the lab, that $R_n(x)$ will approach 0, steadily decreasing for every n. The unexpected effect of the little jump back when n=5 in Matan's example, f (x) = cos(2x), was stronger while observing the animation than in the static plots (Kidron, 2000 - Example 4). The contribution of such feedback to the learning process is particularly effective if it is surprising (Dreyfus & Hillel, 1998). The result could be a re-construction of the meaning of some mathematical notions. Mathematica helped the students to identify "that something is not going as they expected". They had to understand by themselves the cause of the confusion. The way the students used the dynamic graphical feedback enabled them to realize that the behavior of the sequence with regard to convergence is unaltered if we omit a finite number of the terms R_n.

The students used Mathematica to follow Euler's reasoning. In Euler's approach (Euler, 1988) infinite sums were represented as an *object*: the polynomials with an "infinite number of terms". The students used Mathematica also as a symbolic language to generate dynamic graphics, which enabled illustrating the convergence *process*. Using animation only to *visualize* the process of convergence was not enough in order for the *process* to become a *concept*, the concept of limit. In addition, the students had to *interact* with the dynamic graphics, to *have control over the dynamic representations*. Actions on the dynamic representations aided the students in developing their own reasoning. We could clearly see that the students' use of the artifact influenced the nature of the generated knowledge. In order to overcome their pre-conceptions of limit, the students were encouraged to further construct and re-construct their knowledge using the dynamic graphics approach to handle the limit concept explicitly.

We were interested in determining to what extent this re-construction of their knowledge helped the students in their transition from visual intuitive interpretation of the limit concept to formal reasoning. The class discussion around Matan's example enabled the students to modify the misleading idea that they could observe the approach $R_n(x) \to 0$ as n is increasing in the sense that $R_n(x)$ steadily decreases for every n. The class discussion paved the way to the formal definition of $\lim_{n\to\infty} R_n(x) = 0$ (beginning by ε, then finding N such that…). However, in the written test, only 2/3 of the students were able to write correctly the formal definition of

$\lim_{x\to 0} R_n(x) = 0$. The dynamic graphics produced by the animations were present in the students' minds even when the computer was turned off. Some even remembered the order in which they worked in the lab (beginning with domain and finding the error) and had difficulties in reversing the order. To overcome this difficulty, additional tasks are being prepared for use in further applications of the program.

References

Brown, D.P., Porta, H., Uhl, J., 1991, *Calculus & Mathematica*, Addison-Wesley Publishing Company.

Brown, D.P., Porta, H., Uhl, J., 1990, Calculus & Mathematica: Courseware for the Nineties, *the Mathematica Journal*, Volume 1(1), 43-50

Cornu, B., 1981, Apprentissage de la notion de limite: Modeles spontanes et modeles propres, *Proceedings of the 5th International Conference for the Psychology of Mathematics Education,* (pp. 322-329), Grenoble, France.

Cornu, B., 1991, Limits, in Tall, D., (ed) *Advanced Mathematical Thinking*, (pp. 153-166), Kluwer Academic Publishers, London.

Davis, R.B. & Vinner, S., 1986, The Notion of Limit: Some Seemingly Unavoidable Misconception Stages, *Journal of Mathematical Behavior*, 5, 281-303.

Devitt J.S., 1993, *Calculus with Maple* 5 Brooks/Cole Publishing Company,Wadsworth, California.

Dreyfus, T. & Hillel, J., 1998, Reconstruction of meanings for function approximation*, International Journal of Computers for Mathematical Learning* 3: 93-112.

Dubinsky, E. & Tall, D, 1991, Advanced mathematical thinking and the computer*,* in Tall, D. (ed) *Advanced Mathematical Thinking*, (pp.231-250), Kluwer Academic Publishers, London.

Euler, L., 1988, *Introduction to Analysis of the Infinite* , translated by John Blanton, (Vol.1,pp. 50-55), Springer-Verlag, New York.

Fischbein, E., Tirosh , D and Hess, P., 1979, The Intuition of Infinity, *Educational Studies in Mathematics,* 10, 3-40.

Kidron, I., 2000, http://stwww.weizmann.ac.il/g-math/limit-examples.html

Li, L. & Tall, D.O. ,1993, Constructing Different Concept Images of Sequences and Limits by Programming, *Proceedings of the 17th International Conference for the Psychology of Mathematics Education,* (Vol.2, pp. 41-48), Tsukuba, Japan.

Monaghan, J., Sun S., Tall, D, 1994, Construction of the Limit Concept with a Computer Algebra System, *Proceedings of the 8th International Conference for the Psychology of Mathematics Education*, (Vol.3, pp. 279-286), Lisbon, Portugal.

Openhaim, E., 1986, The effect of experience with numerical approximation on understanding the concept of limit, unpublished M.A. thesis, Tel-Aviv University, Israel.

Verillon, P. & Rabarbel, P., 1995, Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity, *European journal of Psychology of Education,* Vol. X (1), 77-101.