DIFFICULTIES WITH NEGATIVE SOLUTIONS IN KINEMATICS PROBLEMS

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***ABSTRACT**: The article reports a Case Study (Heidi) described in a wider research work on Problem Solving in Kinematics, involving 28 high school students. The study provides evidence that Heidi's proficient academic performance leads her to the correct answers and verbal description of the physical phenomena involved. However, when explaining the answer in algebraic languaje. Heidi cannot interpret it because of her lack of acquaintance with negative numbers as final results, although she has dealt with them as part of the intermediate procedures.*

***RESUMEN**: Este artículo reporta un Estudio de Caso (Heidi) descrito en una investigación más amplia sobre Resolución de Problemas de Cinemática, realizado con 28 estudiantes a Nivel Secundaria. Los resultados de este Estudio muestran que el alto desempeño académico de Heidi le permite llegar a las respuestas y dar descripciones verbales de los fenómenos físicos involucrados en los problemas. Sin embargo, al expresar las respuestas en lenguaje algebraico, no logra interpretarlas debido a que Heidi desconoce a los negativos como resultados finales a pesar de trabajar con ellos en procesos intermedios de resolución.*

INTRODUCTION

A research project was recently carried out whose central problem was the study of negative numbers in their interaction with the languages and methods used to solve equations and problems (Gallardo, 1994). The general methodology of the project dealt with the interaction of these categories on two levels, the historical and the didactic. The historical-critical analysis carried out in this work allows us to conclude that the presence of subtractive terms and the laws of the signs appear in remote times, as do the elements necessary for the operativity of signed numbers. It can be said that a crucial step in the recognition of these numbers is the acceptance of negative solutions. The empirical analysis is based in the preceding historical study of negative numbers in the resolution of algebraic equations. In this research, we identified different stages of conceptualization of negative numbers which appear both in the historical and didactical spheres.

THE STUDY

Once research on negative numbers in the arithmetical-algebraic domain has been concluded, a second stage is initiated where the incidence of these numbers in high school physics is analyzed. The methodology used in this particular stage is the same methodology applied in the project as a whole, that is, using the historical-critical method in order to search for elements of analysis that may explain students' difficulties in interpreting negative magnitudes in elementary kinematics problems. Galileo and Newton's physics were revisited. In Galileo's text, motion laws are described and interpreted through dialogues that lead to the birth of the New Science (Crew, 1914). Newton sets up kinematics' axioms which, as their name indicates, are expressed mathematically (Whiteside, 1972). In relation to our purpose, we must mention that neither Galileo nor Newton present any occurrence of negative numbers as solutions.[1]

This article reports a first experience in the teaching of kinematics. The traditional teaching-learning process involving the motion laws is analyzed. The problems dealt with herein are those applied in the high school context and were picked out of the Lima, Perú High School Physics Program (Encalada, 1999). Three out of 20 considered "Typical Problems" (Laglois et al, 1995) were selected. An exploratory questionnaire including these three problems was developed and applied to 28 students from four different groups of a same teacher. The researcher picked out the brighter students for the clinical videotaped interview purpose. Such selection was based on the fact that a good performance in physics and mathematics is necessary condition to do well at solving physics problems (Lang Da Silveira et al, 1992).

The case of one of the students, Heidi, is featured in the paper. A Formula Table containing conventions regarding the plus and minus signs[2] is handed out to the students together with the problems (Fig. 1).

Uniformly accelerated motion	Free fall	Vertical thrust upward
The positive sign (+) is used when the mobile is moving. The negative sign (–) is used when the mobile slow down or stops. $V_f = V_0 \pm at$; $V_f^2 = V_0^2 \pm 2ae$; $e = V_o^t \pm \frac{at^2}{2}$; $t = \frac{V_f - V_0}{a}$	The positive sign (+) is used when the mobile drops. The negative sign (–) is used when the mobile moves upward. $V_f = V_0 \pm gt$; $V_f^2 = V_0^2 \pm 2gh$ $h = V_0 t \pm \frac{gt^2}{2}$; $t = \frac{V_f - V_0}{g}$	

Fig. 1: FORMULA TABLE handed out to the students.

As shown hereafter, the use of the Formula Table leads to positive solutions on both problems.[3] In fact, the general formulas of the Uniformly Accelerated line Motion are the following: $V_f = V_0 + at$; $V_f^2 = V_0^2 + 2ae$ where letters may have any value, including negative solutions.

SOLUTION PROCEDURES OF TWO KINEMATICS PROBLEMS.

> ***PROBLEM ENUNCIATION 1:*** *The brakes of a car travelling at 200m/s are applied and the car comes to a complete stop after 80 meters. Estimate the car's acceleration and the time it takes to stop.*

Analysis of the first question: Estimate the acceleration

- The student writes problem's data and unknown.

[1] It is important to mention that historical analysis of this second stage of the project in still under way.
[2] All of the letters take on positive values or zero only.
[3] Two problems with negative solutions were selected. The third one has a positive solution and is not presented in this article.

$V = 200$ m/s; $V_f = 0$; e=80 m; a=? ; t=?

- She explains that "V=200m/s is start off velocity" [Note that initial velocity is designated by V and not by Vo]

- She recognizes that final velocity is zero when she asserts that "velocity decreases and the car comes to a stop".

- She checks her Formula Table (See Fig.1) and writes: $V_f^2 = V_0 - 2ae$. (Mistaken because initial velocity is not squared).

- She inserts her data and obtains: $(0)^2 = 200 - 2a(80)$. She continues the algebraic procedure and states the following: "This two $(0)^2 = 200 - 2a(80)$, comes

here $0 = (200)^2 - 2a(80)$". The error in formula $V_f^2 = V_0 - 2ae$, is compensated when placing the exponent 2 on the numerical term that corresponds to Vo. She adds:

"zero $0 = (200)^2 - 2a(80)$, you don't put anything else. $= 40000 - 160a$, because it's zero". For a moment, the first member of the expression disappears. This indicates that she does not have full understanding of the equation concept. The process continues correctly: 160a=40000; a= 40000/160; a=250m/s (Incorrect unit). Deficiencies in the acceleration concept can be perceived in the following dialogue:

Interviewer asks: "what does acceleration mean?"

Heidi answers: "Acceleration appears to be the relationship between the distance and… and the distance. Well, that is to say, the distance it covers during the time it's moving[4]. That is veloc…acceleration. Distance over square time". She adds exponent 2 to her previous answer: $a = \frac{250m}{s^2}$

When obtaining a positive solution, she does not retrieve the interpretation she herself verbalized before she looked at her Formula Table when she stated: "velocity is decreasing and the car stops".

If the student had used the general formula: $V_f^2 = V_i^2 + 2ae$, she would have arrived at a negative solution $a = \frac{250m}{s^2}$.

The minus sign could have helped her provide a physical interpretation of the problem instead of repeating mechanically that acceleration is "the distance divided by the square time", as occurred when she used the "positive expression" that appeared on the Formula Table. The previous fact identifies a difficulty in the teaching of kinematics at high school level.

***<u>Analysis of the second question</u>*: Estimate the time it takes to stop.**

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[4] She's not clear about the acceleration. She mistakes it for velocity. The correct relationship established by the aceleration is between velocity and time.

- Once the acceleration is known, the student realices she must find out the time. She then writes: $V_f = V_o - at$. She explains this by indicating "because I have all those data and only the time is missing".

She inserts her data on the previous formula, obtaining: $0 = 200 - 250\ t$. She then writes: $200 - 250t$ and states " Here 200-250t is zero, so you don't put it". Again, the equation disappears in front of zero. She now adds the zero and the equal sign in order to do an operation. She obtains $0+250t=200$; $250t=200$; $t=0.8s$ (correct answer).

- The researcher poses her an additional question to analyze the velocity magnitude.

ADDITIONAL QUESTION TO PROBLEM 1: *A car starts from rest. Its velocity is 80 m/s after running 100 meters. Estimate the acceleration.*

- She asserts that "the car is at absolute rest so it doesn't move and the velocity is zero". She writes the problem's data and unknown:

$V_o=0$; $e = 100$; $V = 80$ m/s; $a = ?$; $V_f = 0$ (error).

Because of the error on V_f [final velocity] and because Heidi writes down three velocities, the researcher asks her "what does V=80m/s mean?" She replies: "It starts from rest, so $V_0=0$ and $V = 80$ m/s is the velocity it goes at; it is just the velocity. It cannot be the final velocity because…" (silence).

She is mistaking uniform straight line motion, where velocity V is constant, with uniformly accelerated motion where the velocity changes at each time unit at the so called acceleration rate. The researcher uses a scheme to elucidate the difficulty to the student.

V_0 80 m/s

0 ———————— 0

e = 100m

She adds the velocity's notation V_f

The diagram now appears like:

V_0 $V_f = 80$m/s

0 ———————— 0

e = 100m

This way, the student recognizes that V is the final velocity V_f.

- Heidi indicates she will use the formula: $V_f^2=V_0^2-2ae$, but with the positive sign, namely, $V_f^2=V_0^2+2ae$, "because the mobile is moving".

- She inserts her data$(80)^2=(0)^2+2a(100)$. She goes on with the procedure:
$6400 = 0+200a$; $6400 = 200a$; $a = 6400/200$ $a = 32$ m/s^2 (Correct result. She spontaneously adds the measurement unit).

SOLUTION PROCEDURE FOR PROBLEM 2

PROBLEM 2 ENUNCIATION: *A balloon*[5] *rises at a constant velocity of 5m/s. When it is 30 m. away from the ground, a stone is let down from it. At what velocity and after how long will the stone reach the ground?*

- The student begins with the second question (time). She indicates: "This is the height reached by the stone" and writes h =30 m (She does not consider the balloon). She writes the problem's unknowns V=?, t=?; she mentions that "when a body is at height, there is gravity". She writes g = 9.8 (She omits the measurement unit). Note that she recognizes an implicit datum. Gravity does not appear in the problem's enunciation.

- Heidi now goes to the Formula Table and writes $h = at^2/2$ (She does not consider gravity as g in this expression). The researcher asks her about this: "This formula $h = at^2/2$ Why?". The student answer:

"They ask for the velocity, they ask for the time, but I have none. There are two unknowns here ($h = at^2/2$) and I must find one".

She then points out at the data on the paper (V=?, t=?, g= 9.8) and recovers the gravity's acceleration. She substitutes acceleration (a) for (g) on the formula, obtaining $h= gt^2 / 2$

She inserts the height as well (h=30m) and carries out the inverse operations correctly: $30=\frac{9.8(t)^2}{2}$; $60=9.8t^2$, $\frac{60}{9.8}=t^2; 2.47=t$ (She omits the measurement unit)[6]

- She now goes to the first question of the enunciation: what is the stone's velocity when reaching the ground? She writes: $V_f{}^2=V_o{}^2+2gh$, and explains "This is (V_f) what they are asking me for ". She places the value of V_o on the expression and obtains

$V_f{}^2=(0)+ 2gh$, and explains "because initial velocity is zero" (error).

- She inserts her other data in the formula: $V_f^2 = (0)^2 +2(9.8)(2.47)$. She mistakenly justifies initial velocity zero stating that: "it starts from rest and then it goes up". She also makes an error when substituting time for height (she writes 2.47 where it should be 30).

Observe a contradictory situation on the following dialogue:

E: "What is the stone's velocity when you let go of it?"
H: "Well, the same. If you go up at that velocity and come down at the same velocity, it is going to be the same. When going up, the balloon and the stone go at the same velocity V_o=5m/s, but when reaching a 30 m. height, the stone gets there at V_f=0 because it is going to come back down".

Note that taking the balloon into consideration helps her conceive $V_o \neq 0$. When she omits the balloon and only takes the stone into consideration, she obtains $V_o =$

[5] Hidrostatic balloon containing the stone.
[6] Only the positive square root is considered in teaching. So time is always not-negative. This solution is incorrect because the stone's initial velocity is different from zero, namely V_o = 5m/s.

0. After she provides the answer V_o = 5m/s, the researcher has her check her previous formula to compare both results.

E: "If V_o=5m/s, so why did you put zero in here $V_f^2 = (0)^2 + 2(9.8)(2.47)$?"

The student remains silent. She now goes on to the first question of the problem. She does not notice that she made a mistake when placing time instead of height and goes on with the procedure: $V_f^2 = 0 + 48.4$; $V_f = 6.95$ (Omits the measurement unit). Her final result is wrong, it should be 25 m/s (stone's final velocity when reaching the ground)[7]

- The researcher poses one more question to elucidate student's confusion in relation to the initial velocity.

ADDITIONAL QUESTION TO PROBLEM 2.

> *A stone falls from a 100 m. height. Estimate the time it takes for it to reach the ground and the velocity when reaching it.*

- Heidi writes the problem's data and unknown: h = 100 m; t =? ; V = ? ; g = 9.8 (Omits the measurement unit).

- She then writes the following formula: $h = gt^2/2$

- She goes on with the correct insertion of her data and the inverse operations:

$100 = 9.8t^2/2$; $200 = 9.8t^2$; $t^2 = 200/9.8$; $t^2 = 20.4$; t= 4.5 (Omits the measurement unit)

- She writes $V_f^2 = V_o^2 + 2gh$. She asserts that "velocity is zero because I consider the stone only when it is let down" (she refers to the initial velocity). She inserts her data and obtains: $V_f^2 = (0)^2 + 2(9.8)(100)$; $V_f^2 = 1950$

$V_f = 44.2$ (Omits the measurement unit). This solution is correct.

- When the researcher asks her to compare problem 2 with the additional question, Heidi asserts "they are the same because you use the same formulas to solve them; but they follow a different procedure, because the balloon and the velocity 5m/s, which I never used, have a part in problem 2. On the other case (she refers to the additional question to problem 2), the velocity is null and the balloon does not appear". It is plain to see student's misconception that at free fall initial velocity is always zero. She forgets that an object may be hurled with an initial velocity different from zero.

STUDY CONCLUSIONS.

The article shows the difficulties that a highly qualified student faced in order to interpret the solution of kinematics problems. Such difficulties derived from the previous teaching on the subject. Such teaching encourages the use of the Algebraic Formula Table leading always to positive solutions. It is worth noting that the convention of the plus and minus signs in formulas implicitly contains a Cartesian reference system that is not mentioned to the students.

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[7] See Appendix where correct solving procedure is shown.

Teachers resort to the use of signs instead of referring to signed magnitudes which, at a higher level, would be considered as vectorial magnitudes. Also, given the fact that difficulties to these problems lie in the relativity of time and space where physical phenomena take place, defining the participating objects and their motion reference point or the time interval where they take place is necessary. Again, elucidating the reference system in use is very important.

Another problem in teaching is the omission of the dimensional analysis. The measurement units associated to magnitudes should be kept in mind all throughout the solving procedure. This consideration would help verifying if the answer is correct and might bring to light possible errors on the designated formulas and on the operations carried out with the corresponding magnitudes. The most generalized tendency among students is to fit in the measurement units once the final result has been obtained.

As far as the student's performance is concerned, our conclusions are the following:
She shows better understanding of the physical phenomena after reading the problem and before consulting the Formula Table. She is not able to put in algebraic language what she expresses in natural language, since the former is mediated by a convention of signs that appears unintelligible to the her, and which cover up the coordinates system in use. So in problem 2, she mistakes the velocity rate as null because she was not aware of the reference system; had she been aware, she would have placed the origin at 30 m. from the ground, and would have considered the balloon containing the stone, and would have noted that both move at initial velocity of 5m/s (See Appendix).

It is worth mentioning that during the Problem 1 solution procedure and Problem 1 Additional Question, arises what we have called an error compensation, that is, Heidi mistakenly omits exponent two in the algebraic expression but then puts it back in when carrying out an erroneous transposition of the members of the equation. So the second error makes up for the first error. On the other hand, we know the student has not consolidated the equation concept because she does not write the complete equation, and integrates it only when carrying out the operations. As far as the physical context is concerned, Heidi seems confused at the types of motion she has been exposed to, since she is not aware of the different velocities involved in each of them. This, in turn, manifests her not having full understanding of the acceleration concept.

APPENDIX: CORRECT ANSWER TO PROBLEM 2

ENUNCIATION OF PROBLEM 2: *A balloon rises at constant velocity 5m/s. When it is 30 m. away from the ground, a stone is let down from it. At what velocity and after how long will the stone reach the ground?*

DATA: $g = -9.81\ m/s^2 \cong -10$; $V_o = 5m/s$; $h = -30m$; $t = ?$; $V_f = ?$

Note that the problem involves the use of vectorial magnitudes, since it introduces a coordinates system. The origin is considered at 30 m above the ground, that is, at the point where the stone is let down.

SOLUTION:

- ***Time it takes the stone to reach the ground (Second question).***

$h = V_0 t + \frac{gt^2}{2}$. Replacing the data we have: $-30 = 5t - \frac{10t^2}{2}$

Solving the quadratic equation we arrive at answers: t= 3 y t =–2

Result t = −2 is discarded because that would mean the stone should have been let down two seconds before. If it were so, it would have not spanned the mentioned distance but more meters. So, the time it takes the balloon to reach the ground is 3 seconds.

- ***Velocity at which stone reaches the ground (First question).***

$V_f = V_o + gt$. Inserting the data, we have: $V_f = 5 + (-10)(3) = -25$

Thus, the stone will reach the ground at a velocity of 25m/s.

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