THE CONSTRUCTION OF ABSTRACT KNOWLEDGE IN INTERACTION

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We take abstraction to be an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. Abstraction is thus a context dependent process. During group work, peer interaction is an important component of the context. In a previous publication, we proposed a model for processes of abstraction. The model is operational in that its components are observable epistemic actions. Here we use the model to analyse an interview with a pair of grade seven boys collaboratively constructing an algebraic proof. The analysis of the interview reveals subtle links between the abstraction process and the peer interaction.

Abstraction is a central process in learning mathematics; however, it is notoriously difficult to observe. In a previous paper (Hershkowitz, Schwarz & Dreyfus, 2001, referred to below as HSD), we have proposed a model for abstraction that is operational in the sense that its components are three observable epistemic actions. The model considers abstraction as a process occurring in context. Although our outlook is theoretical, our thinking about abstraction has emerged from the analysis of experimental data from the CompuMath curriculum development project (Hershkowitz & Schwarz, 1997). Group work in a computer-rich environment is a prominent feature of CompuMath and thus part of the context for the processes of abstraction we observe. In this paper, we briefly review our definition and model of abstraction (see HSD for a more detailed description), and illustrate them by means of processes of abstraction by an interacting pair of students working on an algebra problem with a spreadsheet.

Mathematics educators have proposed that abstraction consist in focusing on some distinguished properties and relationships of a set of objects rather than on the objects themselves. Abstraction is thus a process of decontextualization. According to Davydov (1972/1990), on the other hand, abstraction starts from an initial, undeveloped form and ends with a consistent and elaborate final form. Similarly, Ohlsson and Lehtinen (1997) see the cognitive mechanism of abstraction as the assembly of existing ideas into more complex ones. Noss and Hoyles (1996) go even further. They situate abstraction in relation to the conceptual resources students have at their disposal and see it as attuning practices from previous contexts to new ones. Therefore, according to Noss and Hoyles, students do not detach from concrete referents at all. Leaning on ideas of these and other authors, we define abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. The use of the term activity in our definition of abstraction is intentional. The term is directly borrowed from Activity Theory (Leont'ev, 1981) and emphasises that actions occur in a social and historical context. The reorganisation of knowledge is achieved by means of actions on mental (or material) objects. Such reorganisation is called vertical (Treffers and Goffree, 1985), if new connections are established, thus integrating the knowledge and making it more profound.

According to this definition, abstraction is not an objective, universal process but depends strongly on context, on the history of the participants, on their interactions, and on artefacts available to them. As abstraction is an activity consisting of actions, our research included the identification of actions involved in abstraction. We focussed on epistemic actions, that is actions relating to the acquisition of knowledge (Pontecorvo & Girardet, 1993). In many social contexts, such as small group problem solving, participants' verbalisations may attest to epistemic actions thus making them observable. The three epistemic actions we identified as related to processes of abstraction are Recognising, Building-With and Constructing, or RBC.

Constructing is the central step of abstraction. It consists of assembling knowledge artefacts to produce a new structure to which the participants become acquainted. *Recognising* a familiar mathematical structure occurs when a student realises that the structure is inherent in a given mathematical situation. The process of recognising involves appeal to an outcome of a previous action and expressing that it is similar (by analogy), or that it fits (by specialisation). *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. The same task may thus lead to building-with by one student but to constructing by another, depending on the student's personal history, and more specifically on whether or not the required artefacts are at the student's disposal. Another important difference between constructing and building-with lies in the relationship of the action to the motive driving the activity: In building-with structures, the goal is attained by using knowledge that was previously acquired or constructed. In constructing, the process itself, namely the construction or restructuring of knowledge is often the goal of the activity; and even if it is not, it is indispensable for attaining the goal. The goals students have (or are given) thus strongly influence whether they build-with or construct.

The three epistemic actions are the elements of a model, called the dynamically nested RBC model of abstraction. According to this model, constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions. The genesis of an abstraction passes through (a) a need for a new structure; (b) the construction of a new abstract entity; (c) the consolidation of the abstract entity through repeated recognition of the new structure and building-with it in further activities with increasing ease. We have argued in HSD that this model fits the genesis of abstract scientific concepts acquired in activities designed for the special purpose of learning. In such activities the participants create a new structure that gives a different perspective on previous knowledge. The model is operational: It allows the researchers to identify processes of abstraction by observing the epistemic actions and the manner in which they are nested within each other.

In the remainder of this paper, we will illustrate the model and its use for studying processes of abstraction by an interacting pair of students. For this purpose, we focus on a pair of grade 7 boys who will be identified as Yo and Ra, or collectively as Yo&Ra. These students' CompuMath algebra curriculum consisted of a sequence of activities, most of them with a spreadsheet, in which they learned to use algebra and the spreadsheet to express generality. On the other hand, they were not asked to justify general properties by using algebraic manipulation. The students usually worked in pairs. In an interview situation, Yo&Ra were presented with an activity that presented a definite potential for abstraction to them. The activity was designed for students from whom the use of algebra for proving properties could possibly be expected but who had never actually done it. The activity was intended to lead students into a situation, in which they felt the need to justify a property whose proof requires algebraic manipulation. Students were asked to investigate properties of rectangles of same type as the following ones:

7	13
9	15

3	9
5	11

After creating (in the spreadsheet) a 'seal' that generates such rectangles upon input of any number into the upper left cell, and after discovering and investigating properties of such rectangles, students' attention was drawn to the difference between the products of the diagonals. When they claimed that this difference equals 12 for all such rectangles (the diagonal product property or DPP), they were asked to justify their claim. The easiest way of justifying the DPP is to use algebraic manipulation and compare X(X+8) (the expression for the main diagonal) to (X+6)(X+2) (the expression for the secondary diagonal). While reorganising their knowledge so as to arrive at a proof of the DPP, this activity presented two opportunities for abstraction to the students. The first such opportunity is the construction of the extended distributive law (a+b)(c+d)=ac+ad+bc+bd: The students had never used that law yet but needed it in the present activity to transform the expression for the secondary diagonal. The second opportunity for abstraction more global in that it concerns the entire proof task, namely the establishment of the general perspective that algebra can serve as a tool for the justification of general properties.

In order to study abstraction by interacting pairs of students one needs to study simultaneous cognitive and social processes. For this purpose, we videotaped and transcribed Yo&Ra's work on the seals activity and then carried out two analyses of the interview protocols, one that analyses the cognition and one that analyses the interaction. Our aim was not to give precedence to either of these analyses, but to carry them out as independently from each other as feasible, and then to compare the resulting patterns.

We first produced a coarse segmentation of the protocol into cognitive segments. Next, we proceeded to two independent analyses. On the one hand, we identified interaction patterns between the peers. We categorised the students' conversational moves according to their function into six categories to be discussed below. On the other hand, we closely followed the methodology we used in HSD for the identification of the students' epistemic actions. For pairs of students, this identification poses problems, which were not present in the previous study. Because of the subjectivity of the epistemic actions, what is constructing for one student may be building-with or recognising for the other. We classified such cases as constructing for the pair.

Yo&Ra began by generalising the pattern of the seal so that the spreadsheet produces the entire seal from the left upper cell. When asked to find properties of the seal, they found many simple ones and some more complex ones. In particular, they found the DPP and verified it inductively by substituting numbers in the left upper cell of the generalised seal. When asked to justify the DPP, they produced, after some difficulties, the algebraic expressions X(X+8) and (X+2)(X+6) for the diagonal products. They were unable to progress further because they faced the algebraic obstacle of comparing these expressions. The interviewer's suggestion to end the interview triggered the segment Y263-R290, transcribed below, and three more segments, which together constitute a justification of the DPP.

Yo263	*Oh, I know: You can do the distributive law here.*
Int264	*Yes.*
Yo265	*X times 8 plus X times X ...*
Int266	*Yes.*
Yo267	*Here one can do ...*
Int268	*Here I'll write X times 8 ...*
Yo269	*Yes, plus ...*
Int270	*Plus X times X, yes?*
Yo271	*Now, here one can do X times X ...*
Ra272	*...plus X times*
Yo273	*... plus X times 6 plus 2 times X plus 2 times 6.*
Ra274	*Why? You are using too many factors!*
Yo275	*No, it's OK.*
Int276	*Yes* [writes]*, don't you agree?*
Ra277	*No, there are too many factors!*
Yo278	*It's correct!*
Ra279	*We already used the ..., I don't understand what the 8 is for, first of all*
Int280	*This?*
Ra281	*Yes!*
Int282	*Ah, yes!*
Ra283	*We have X times X ...*
Int284	*Yes!*
Ra285	*plus X times 6, and then the normal continuation will be X times 2.*

Int286 *X times 2, OK!*

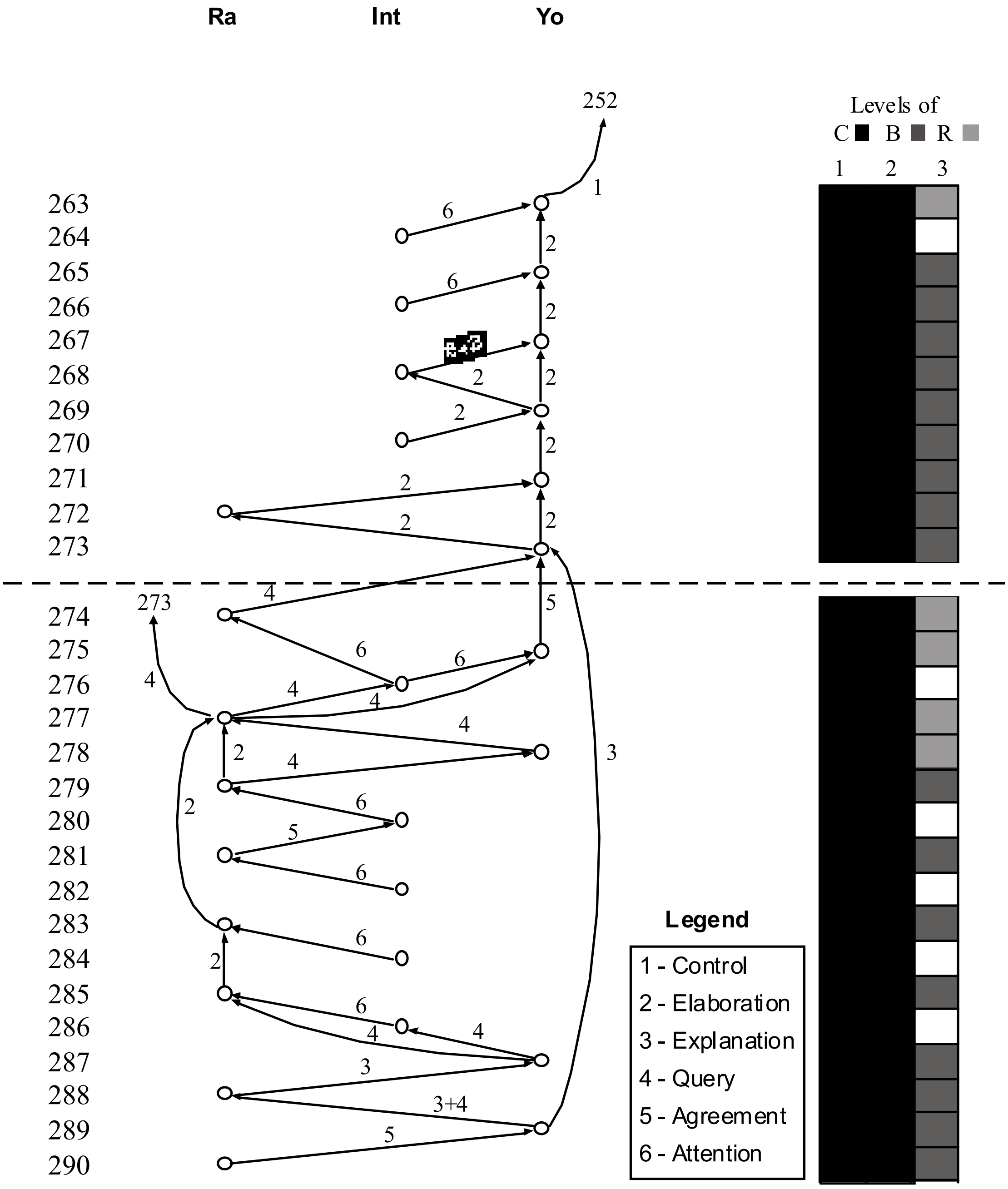
Yo287 *Why X times 2?*

Ra288 *Because I go according to how you do it, then this is X times X, X times 6, X times 2...*

Yo289 *No, this is X plus 2. Look, you do, this is the distributive law, and you do X times X plus X times 6; now you pass to the 2; 2 times X and 2 times 6.*

Ra290 *Ah, logical! I got it.*

Because of space limitations, we analyse only this segment in the present paper, although some of the conclusions we will draw will be partially based on other segments as well. Our main tools of analysis are presented in the following figure:



The analysis will be presented in three stages: Interaction patterns, epistemic actions, and relationships between them.

Interaction patterns: At the beginning of the interview Yo and Ra both have the mathematical curiosity and the drive to complete the mathematical activity. In addition, each of them is conscious about his mathematical potential, and likes it to be seen by the other and by the interviewer and even the future video observers. At the same time each of them is quite aware of his friend's mathematical ability. Both students are assertive and try to convince the other from time to time.

Details of the interaction patterns in this segment are coded in the central part of Figure 1. They show an interaction pattern that is very typical for Yo&Ra. Although the "seen interaction" indicates mostly individual work done by each student, the conclusions are at the end agreed on by both of them. A thorough analysis suggests a quite intensive type of interaction between them: In 263, Yo makes a proposal (category 1) that he elaborates step by step (category 2), while Ra is silent but follows very carefully the proposal and the elaboration. Then (274) Ra starts his elaboration of the same proposal by opposing it (category 4). This opposition expresses his struggle to understand the elaboration, which Yo just completed. Only then Ra elaborates it at his own speed. He completes this process by expressing his agreement with the process (290). During Ra's elaboration Yo is silent. Each of them has the need to elaborate a given idea in his own way. By remaining silent for a while, each student respects his peer's need. But both of them reach a point where they refer explicitly to the other's elaboration and criticise it, until they reach an agreement (category 5). This agreement appears within a collaborative explanation (category 3) of the original claim (Y263, as elaborated in Y273).

Epistemic actions of abstraction: In HSD we showed that constructing is a combination of the three epistemic actions where recognising actions are nested in the two others, and building-with actions are nested in constructing actions. Here, we show that the nested model is even more intricate. The constructing action may be quite long and contain shorter segments, which themselves are constructing actions. For example, in the case of Yo&Ra the main constructing action (which we will name C_1) occurs along the entire justification process, and includes the construction of the extended distributive law (which we will name C_2, see the right side of the Figure). C_2 is nested in C_1. Building-with and recognising actions are nested in C_1 as well as in C_2. In other words, constructing actions, like building-with and recognising actions, may be nested in a more global constructing action.

The Figure (right hand side) describes the model for our segment. Level 1 represents the whole process of C_1 (only part of it can be seen in this segment), in which the epistemic actions of level 2 are nested. In the present segment, all level 2 actions are constructing actions (C_2). In other segments, building-with and recognising actions also occur at level 2.

The construction of the extended distributive law starts with Yo's breakthrough suggestion to use the distributive law. The students both recognise the structure of the expression X(X+8) as appropriate to apply the (simple) distributive law. Yo then builds-with the elements of (X+6)(X+2) and obtains the correct expansion (Y273); in other words, he constructs the expanded distributive law. Although he makes no explicit reference to the simple law, we surmise that the immediately preceding use of the simple law has guided him in building the more complex expression. Now Ra takes over. After being momentarily confused by the many addends, he goes through the same constructing process Yo went through, step by step building up the expanded law with the elements of the expression (X+6)(X+2). This lower level (C_2) construction does not occur in the void but as a crucial part of the justification of the DPP. It is therefore nested in the higher-level (C_1) construction of the algebraic justification of the DPP.

From the beginning of their struggle to construct the DPP justification, the students are at the level of the C_1 construction. On this C_1 level their progress is controlled and monitored by their awareness and their need to accomplish the DPP justification. During this process, they face algebraic obstacles, which are quite unfamiliar to them. Overcoming these obstacles necessitates the construction of new mathematical structures, which are the C_2 level constructions. These C_2 constructions are controlled only indirectly by the motive of the C_1 construction. The students enter these C_2 level "adventures" without any knowledge about the needed mathematical structures, and they have to discover as well as to construct them. The C_2 constructions thus make the C_1 level into a deep holistic construction, which goes beyond the specific construction of the DPP justification, and in which the constructions of unfamiliar algebraic structures are nested. In this sense C_1 is an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure, which fits our definition of abstraction.

Relationships between epistemic actions and interaction patterns: The Yo&Ra interview is a case of collaboration between the two students. This collaboration finds its expression in the students' cognitive RBC actions on one hand and in their pattern of interaction on the other hand. The RBC flow and the flow of the interaction patterns are developing in parallel. There are no clear causal relationships between the two of them. It rather seems that both of them are different "indications" of the single collaborative process revealed in the interview. Our understanding of this process is dependent on our understanding of both, the RBC flow and the interaction patterns, as well as the relationships between them. In the following we will try to throw some light on these relationships.

Globally, Yo&Ra share the activity, because they share the motives of searching for the mathematical properties of the "seals" and of justifying these properties; they also share the justification processes themselves, as well as their conclusions. We claim that the students constructed a global structure of meaning for algebraic justification (the C_1 action, which can be seen only partially in this paper). From the interaction perspective, there are long-range control (category 1) and explanation (category 3) arrows that can be considered as the “glue” that ties this justification process together. The diagram of the entire process reveals that these long-range interaction arrows are connected to the beginning and/or end of the main cognitive segments. The part of the diagram in the figure of this paper shows a category 1 arrow emanating from the beginning of the segment to an earlier segment, and a category 3 arrow concluding this segment.

Hence, long-range interactions occur between statements that are milestones in the RBC flow. In this sense the interaction pattern has nesting characteristics similar to the RBC flow, where various patterns of interaction are nested in the overall global collaboration. And the cutting edges of the interaction patterns are those that at the same time define the different segments of the RBC flow. In other words, the cognitive segmentation we started out from fits the interaction as well.

REFERENCES

Davydov, V. V. (1972/1990). *Types of Generalisation in Instruction: Logical and Psychological Problems in the Structuring of School Curricula*. Volume 2 of J. Kilpatrick (Ed.), Soviet Studies in Mathematics. Reston, VA: NCTM.

Hershkowitz, R. & Schwarz, B. B. (1997). Unifying cognitive and socio-cultural aspects in research on learning the function concept. In E. Pehkonen (Ed.) *Proceedings of the 21st International Conference for the Psychology of Mathematics Education, Vol. I* (pp. 148-164). Lahti, Finland: University of Helsinki.

Hershkowitz, R., Schwarz, B. B. & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education.*

Leont’ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed*.), The Concept of Activity in Soviet Psychology* (pp. 37-71). Armonk, NY, USA: Sharpe.

Noss, R., & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht, The Netherlands: Kluwer.

Ohlsson, S., & Lehtinen, E. (1997). Abstraction and the acquisition of abstract ideas. *International Journal of Educational Psychology, 27*, 37-48.

Pontecorvo, C., & Girardet, H. (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction, 11*(3&4), 365-395.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education. In L. Streefland (Ed.), *Proceedings of the 9th International Conference for the Psychology of Mathematics Education, Vol. II* (pp. 97-123). Utrecht, The Netherlands: OW&OC.