MATHEMATICS SUBJECT KNOWLEDGE REVISITED.

Stephanie Prestage and Pat Perks

School of Education, University of Birmingham

Our research over the last few years about teachers' mathematics subject knowledge, has led to a model for thinking about subject knowledge which distinguishes between knowledge needed to pass examinations and that needed to help others to come to know. This paper explores this model in depth and uses interviews with pre and in-service teachers responding to questions on graphs and percentages, to exemplify the model.

Introduction

Teachers hold many professional knowledges, knowledge about pupils, systems and structures; about styles of teaching and learning; about management, resources and assessment as well as knowledge about the subject. Research offers definitions of professional knowledge and the different forms of knowledge that a teacher holds, (Brown & McIntyre 1993, Cooper & McIntyre 1996, Desforges & McNamara 1979, Ernest 1989, Marks 1990, Calderhead & Shorrock 1997, Banks et al 1999).

> *It is clear that learning to teach involves more than a mastery of a limited set of competencies. It is a complex process. It is also a lengthy process, extending for most teachers well after their initial training. (Calderhead & Shorrock 1997, p.194)*

This paper considers mathematics teachers' subject knowledge and describes a model for different aspects of a teacher's knowledge about mathematics. Subject knowledge is an aspect of teachers' professional knowledge known to be problematic, but one which 'has provoked more controversy than study' (Grossman et al 1989).

> *While one can infer from studies of teacher thinking that teachers have knowledge of their students, of their curriculum, of the learning process that is used to make decisions, it remains unclear what teachers know about their subject matter. (Wilson et al., 1987 p.108)*

Research in the particular area of subject knowledge and pedagogical content knowledge (Shulman, 1986, Wilson et al. 1987, Tamir 1988, Aubrey 1997) explores the transformation of subject matter knowledge for the classroom; teachers' knowledge about explanations, tasks and activities, about styles of teaching and learning. But it does not include explicit detail of how such subject knowledge is held in an intellectual way by teachers, other than is shown by activities or explanations given. The mathematics is rarely explicit. Whilst Shulman and others have categorised the different components of subject knowledge and discussed its transformation through classroom events, our research data was used to investigate the ways in which teachers' subject knowledge in mathematics is held and transformed.

We agree with Buchmann (1984) that teachers need a rich and deep understanding of their subject in order to respond to all aspects of pupils' needs: 'Content knowledge of this kind encourages the mobility of teacher conceptions and yields knowledge in the form of multiple and fluid conceptions' (ibid. p.46). Evidence from our earlier research led to a hypothesis that teachers' subject knowledge in mathematics is held in two forms either as *learner-knowledge* or as *teacher-knowledge* in mathematics; the former is the knowledge needed to pass examinations; the latter is the knowledge needed to plan for others to come to learn the mathematics. Auditing the former might be necessary (currently demanded on Initial Teacher Education (ITE) courses, DfEE 1998) but is not sufficient for the developing teacher. Such audits offer limited list-like perspectives of knowledge to be held by teachers, a view shared by others.

> *The shared assumption underlying [such] research is that a teacher's knowledge of the subject matter can be treated as a list-like collection of individual propositions readily sampled and measured by standardised tests …[these] researchers ask what a teacher knows and not how that knowledge is organised, justified or validated … [such research] has failed to provide insight into the character of the knowledge held by students and teachers and the ways in which that knowledge is developed, enriched and used in classrooms. (Wilson et al 1987p. 107)*

It is the very 'character of the knowledge' in mathematics that we sought to provide insight by interviewing experienced and pre-service teachers.

A model for describing subject knowledge

The nature of the initial research has been reported elsewhere (Prestage and Perks 1999a). Analysis of teachers' responses to discussions of subject matter gave categories of phases in which aspects of their mathematics knowledge was held. This offered ways to explain the differences in the types of responses and how transitions might be described, leading to a model. Whilst the roots of the model lie in a variety of research, an explanation of its components is more easily given via our ITE students.

Graduate mathematician students to our secondary pre-service course arrive with personal subject knowledge (*learner-knowledge*) that enables them to answer mathematical questions. When asked to calculate the division of one fraction by another or to differentiate a function, all respond correctly. When asked why the answers are correct they do not know. They can do mathematics but they do not necessarily hold 'multiple and fluid conceptions'. ITE students also bring with them their personal beliefs about 'being a teacher'. Their view of teaching is to replicate the learner-knowledge they hold for others to learn, a view sometimes held by experienced teachers (Prestage 1999). Their subject knowledge is often ill-connected and they have to work on this when planning for teaching (Perks & Prestage 1994).

During ITE courses, students gain different knowledge and understandings of other professional traditions - some national like the National Curriculum and examination systems and some local traditions from particular schools such as schemes and textbooks. Learner-knowledge and professional traditions merge in the first instance to create classroom events for others to engage with learning mathematics (figure 1).

This combination is also evident from the experienced teachers. High on the list when justifying decisions about the curriculum were text books and other departmental resources, experiences of learning mathematics, ideas related to teaching practices (Baturo & Nason 1996; Ball 1988,1990) and from the new legislative curriculum.

There comes a moment for many teachers (often early in their professional education) when they realise that giving their learner-knowledge directly to pupils does not work:

if you present a problem to the class and there is a need for them to know something about a particular shape and the area of it, and therefore they would set the agenda ... things are encompassed in a problem and the pupils are setting the agenda.

Reflection upon these classroom events, with the integration of learner-knowledge and professional traditions, leads to the beginnings of practical wisdom that enables teachers to adapt activities from the professional traditions to suit their particular circumstances (figure 2).

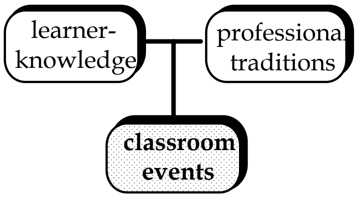

Figure 1

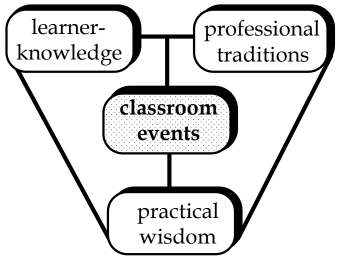

Figure 2

Pre-service teachers developed practical wisdom during their teaching experiences as shown in their evaluation of lessons. Many of the experienced teachers talked in the interviews about the consequences of diverse classroom interactions, of altering the teaching decisions in order to respond pupils' needs. Classroom experiences give rise to practical wisdom based on perceived learner needs and new explanations and contexts are found to support learner-knowledge. However, the learner-knowledge is not necessarily questioned, the teacher may still offer the rule ''change the divide sign to a multiply and turn the second fraction upside down" but previous lessons may have included "how many quarters in one whole, how many in two etc." and other starting points to support the learning of the algorithm. Existing research makes assumptions that teachers have full access to subject matter knowledge and that it is transformed by activities developed for teaching. We argue that for both experienced and novice teachers much of this subject matter knowledge remains as learner-knowledge and is not transformed into teacher-knowledge, as Aubrey (1997 found):

There is however, little evidence to suggest that the development of project teachers' subject matter through teaching occurred. The capacity to transform personal understanding, thus, depends on what teachers bring to the classroom. Whilst knowledge of learning and teaching and classrooms increases with experience, knowledge of subject content does not. (pp.159-160)

In certain topics there was evidence from the research data that some teachers had thought more about the subject matter, beyond reacting to pupils. For one primary co-ordinator curriculum decisions were made in response to the pupils in her Y2 class 'depending on where the conversation goes'. There was evidence of her deliberate decisions about progression through the mathematics. This aspect of deliberate reflection towards teacher-knowledge was also partially evident in other interviews and also emerges occasionally in the data from the student teachers (Perks 1997).

We believe that 'good' teachers need to reflect upon classroom events not simply to consider their success or failure for the students but to reconsider their own personal understandings of mathematics, to reflect upon the 'why' not only of teaching but also of mathematics. We would argue that it is in this way that they come to own a better personal knowledge of mathematics (teacher-knowledge), that learner-knowledge (the only explicit content knowledge) requires transforming through deliberate reflection.

This analytic process requires a synthesis of the reflection on the three elements of figure 2. The model is completed in the form of a tetrahedron, where the struts represent the reflective/analytic process.

This research explores each of these vertices to provide exemplification.

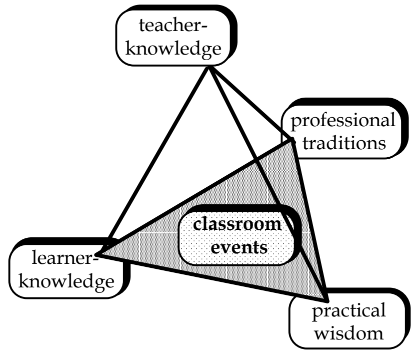

Figure 3

The current project: methods and analysis

Four experienced teachers (ET) and four pre-service teachers (NT) (at the end of their course) were interviewed to gather data about subject knowledge. Two different areas of mathematics were used, one from number (percentages), as previous research (Prestage 1999) had shown more evidence of teacher-knowledge in the area of number and one from data handling, to compare data from another project (Soares & Prestage 2000. The teachers were asked to complete several questions (Appendix) and to talk about the solutions in construction as well as their understanding of the pedagogical issues. The interviews were recorded.

Analysis of the data looked for evidence for describing the vertices of the tetrahedron (Figure 3). Whilst certain prompts during the interviews led to particular descriptions of subject knowledge, aspects of all the phrases of the tetrahedron were interwoven through the data.

Learner-knowledge: this was the data taken as they solved each question with some interim prompts such as "Why are you doing that?"; analysed by strategy.

Professional traditions: aspects of this were found in response to 'why' a particular strategy had been chosen; evidence of the impact of their own schooling, texts, schemes and government policies.

Practical wisdom: this was evidenced in response to 'how would you teach this in the classroom' and from any pupil difficulties and errors described.

We have taken teacher-knowledge to be an amalgam of all of these categories. We looked for evidence of wider connections in the mathematics, reflections upon the integration of the three categories, and when the interviewee was aware of the consequences of choices made for teaching with a definite purpose for doing the mathematics beyond getting the right answer.

Findings

Leaner-Knowledge

This was evident throughout the interviews as the teachers described their solutions. There was more variety in methods offered for the percentages questions than the graphs questions. The approach to the solutions for the graph questions focused on scale, type of graph, axes and care in plotting. There was some uncertainty on the difference between discrete and continuous data. For percentages the methods varied from mental calculations to use of calculator but the interchange of, e.g 73% to 0.73 to 73/100 was common but there was surprising variety of solutions in a small group. There were few errors in calculation which were not corrected immediately, but for the graphs two of the novice teachers tried to use time on wrong axis.

Professional Traditions.

For percentages when asked why they had done the calculation in that way, the teachers gave three types of replies:

1. using algorithms which depend on what they had done at school - the fraction and ‘10% is divide by 10’;
2. mental methods which they were aware of having adapted from what they had done at school, or methods thy felt they had devised to make it more efficient to do it mentally;
3. the use of the calculator.

For ET1 who was using the National Numeracy Strategy (NNS, DfEE 1999) in year 7, the influence of the NNS was clear. She mentioned the examples and the ways they had make her think about methods. She felt that she had adapted her own learner-knowledge in working with this new professional tradition (our language).

For graphs, the novice teachers had done similar work at school or at university and the error about the axis for time was attributed to a learned response from school “time is the dependent variable - so it’s across”. For the experienced teachers the influence was more likely to be what they had taught - “the textbook always uses bar charts for shoe sizes” - “we have just been using spike graphs in our new scheme”.

Practical Wisdom

For the work on graphs any description of classroom activities tended to focus on children’s errors. The teachers described how to help pupils to choose the correct graph, which way to use the paper, the problem of scale, where to draw the axes, labels such as the axes and title and accuracy in general. The experienced teachers were very similar in the speed and manner in which they described what they would work on. The only noticeable difference in response lay between the two sets of novice teachers. The good novices were as good at identifying errors and activities as the experienced teachers, whereas the others were slower and had a much more limited range of possible activities which they described. The major teaching approach relied on telling the students what to do before they made errors. There was talk of such things as “discussion of scale”, but this tended to mean question and answer. It was suggested that pupils needed to work on different data sets. Only ET1 and NT1&2 talked about activities such as offering tables and ready-drawn graphs with errors for the pupils to identify or having several graphs on cards for the pupils to choose the ‘best’ and then justify their decisions.

Percentages saw a difference between the two good novices and the rest of the interviewees. Only NT1&2 felt that teaching pupils to change a percentage to a fraction and then do ‘fraction of’ was unnecessary, describing the algorithm as “usually unhelpful, it confuses rather than helps”. All of the others felt that they had to teach this algorithm - their justification lying in the professional traditions, “it’s in the school scheme” or their own learner-knowledge “that’s the way I learned it and it’s the way to understand it”. For other written methods, some teachers described using 10% as the base for teaching another routine and the others used 1%. These written methods were described by means of examples.

All but two interviewees, ET 2&3, mentioned mental work. Listening to children’s methods and to develop activities was given importance. ET1 highlighted the stress placed on this in the NNS. For the novices, this was an expected inclusion as the place of mental methods had been a major strand of the course they were just completing.

Only ET3 and NT4 failed to mention calculator work, but only ET1 and NT1&2 talked in any detail. NT1&2 offered justification for decimals, “the calculator and the way percentage keys work”. They talked about the importance of linking percentages to decimals directly, almost as pattern recognition: 76% = 0.76, 123% = 1.23

Discussion

The purpose of research was to find teachers’ mathematical descriptions from to exemplify the phases in figure 3. Evidence for triad at the base of the tetrahedron is easy to identify from teachers talking about mathematics. However, much teacher-knowledge appears implicit rather than explicit in the data and has to be inferred. More questions or a longer time frame would have been useful. This raises the challenge of identifying what would help teachers to articulate their subject knowledge in this way, when they have not been expected to articulate it in this form. Whereas, for us as tutors on a pre-service course, we must be able to in order to help our students come to refine their subject knowledge for better mathematics teaching.

Teacher-Knowledge

To identify aspects of teacher-knowledge, we were looking for evidence of the teachers knowing the connections between the areas of mathematics and having a purpose for the activity. From the data, it appears that the implicit teacher-knowledge for percentages is more developed than that for graphs. More variety of connections and approaches were given. All of the methods of calculation would be recognised by others, whereas some of the data questions might be unfamiliar even as learner-knowledge to some.

The answers from ET1 showed her using a variety of methods, including instant recall. She used connections to other methods - “10% of works the same way as divide by 10 so we shift the numbers one place”, as did NT1 “10% is like dividing by 10 or multiplying by 0.1, so 10% of means a shift of the numbers one place”. There is a strong connection to other aspects of place value and a shifting image. NT4 also appears to be connecting yet when asked why he used “10% of is divide by 10” he replied “That’s what you do”.

For the graphs questions, there is less variety; there is evidence of different learner-knowledge related to the mathematics of the data. The teacher-knowledge related to grouped data and choice of appropriate diagram is not very explicit. There is, however, some sense of purpose for the tasks ET 1 and NT 1&2, other than practising drawing, to develop pupils’ decision making skills, in choosing appropriate diagrams.

Discussion

Shulman argued that pedagogical content knowledge was the missing paradigm for teaching and we agree with its importance. We believe that designing activities for the classroom is necessary but not sufficient to develop the type of subject knowledge, the *teacher-knowledge*, we feel essential for the development of strong mathematics teaching. Nor is it acceptable to demand more learner-knowledge. More of this kind of knowledge is not sufficient for the development of teacher-knowledge. The analytic process has been long recognised and is essential for the intellectual professional.

> *Ideally we may wish to have teachers who are not only competent actors in the classroom but also who are practitioners capable of understanding what they are doing, why they are doing it and how they might change their practice to suit changing curricula, contexts or circumstances. This produces a tension between the need for teachers to understand teaching and the need to be able to perform teaching. (Calderhead & Shorrock 1997, p.195)*

Teacher-knowledge in mathematics allows teachers to not only answer the questions correctly but also help to build a variety of connections and routes through knowledge, that provides answers to ‘why’ something is so (Prestage, 1999). It is our contention that only when such *teacher-knowledge* is informing classroom practice that the real needs of learners and the challenges of mathematics are addressed. There are threads of teacher-knowledge emerging from the data, but the interviews did not sufficiently challenge the teachers to articulate this aspect. There is still the need to delve more deeply if the model is to be strongly exemplified.

References

Aubrey, C: 1997, Mathematics Teaching in the Early Years: An investigation into teachers’ subject knowledge, Falmer Press, London.

Ball, D.L.: 1988, Unlearning to teach mathematics, For the Learning of Mathematics, 8(1), 40-42.

Ball, D.L.: 1990, Breaking with experience to teach mathematics: the role of a preservice methods course, For the Learning of Mathematics, 10(2), 10-16.

Banks, F, Leach J &Moon B: 1999, ‘New Understandings of teachers’ Pedagogic Knowledge’ in: J Leach & B Moon (eds.), Learners and Pedagogy, PCP, London.

Bassey, M.: 1995, Creating education through research, Kirklington Moor Press (in conjunction with BERA), England.

Baturo, A. & Nason, R.: 1996, Student Teachers' Subject Matter Knowledge within the Domain of Area Measurement, *Educational Studies in Mathematics*, 31, 235-268.
Brown, S. & McIntyre, D.: 1993, *Making Sense of Teaching*, Open University Press, Buckingham.
Buchmann, M.: 1984, The priority of knowledge and understanding in teaching in L Katz & J Roth (eds.), *Advances in teacher education (Vol 1),* Ablex, New Jersey.
Calderhead J. & Shorrock S.B.: 1997, *Understanding Teacher Education,* Falmer Press, London
Cooper, P. & McIntyre, D.: 1996*, Effective Teaching and Learning: Teachers' and Students' Perspective,* Open University Press, Buckingham.
DfEE.: 1998 Circular 4/98 *Teaching: High Status, High Standards*. Annex G and Annex B. DfEE, London.
DfEE: 1998 *The National Numeracy Strategy: Framework for Teaching Mathematics, Year 7, Draft 3/99*, DfEE, London.
Desforges, C. & McNamara, D.: 1979, Theory and practice: methodological procedures for the objectification of craft knowledge, *British Journal of Teacher Education,* 5(2), 145-152.
Eraut, M.: 1994, *Developing Professional Knowledge and Competence*, The Falmer Press, London.
Ernest, P.: 1989, The knowledge, beliefs and attitudes of the mathematics teacher: a model, *Journal of Education for Teaching*, 15(1), 13-33.
Grossman, PL, Wilson, SM., & Shulman, LS.: 1989, Teachers of substance: subject matter knowledge for teaching in MC Reynolds (ed*) Knowledge base for the beginning teacher*, Pergamon, Oxford.
Hopkins, D. & Stern, D.: 1996, Quality teachers, quality schools: international perspectives and policy implications, *Teaching and Teacher Education*, 12(5), 501-517.
Marks, P.R.: 1990, Pedagogical content knowledge: From mathematical case to a modified conception, *Journal of Teacher Education*, 41(4), 3-11.
Perks, P. & Prestage, S.: 1994, Planning for learning, B Jaworski, & A Watson, (eds.), *Mentoring in Mathematics Teaching*, Falmer Press, London.
Perks, P.A.: 1997, *Lesson Planning In Mathematics: a Study of PGCE Students*, unpublished Ph.D. Thesis, University of Birmingham.
Prestage, S. & Perks, P.: 1999a, Editorial: Extracting the generality, *Mathematics Education Review* **11**, pp.52-55.
Prestage S & Perks P.: 1999b, Questions about subject knowledge: learner-knowledge and teacher-knowledge in E Bills & A Harries (eds.) Proceedings from the 4th British Congress of Mathematics Education July 1999 University College of Northampton .
Prestage, S.: 1999, An Exploration into Teachers' Subject Knowledge, unpublished Ph.D. thesis, King's College, University of London.
Shulman, LS: 1986, Those who understand: Knowledge growth in teaching, *Educational Researcher*, 15(2), 4-14.
Tamir, P.: 1988, Subject matter and related pedagogical content knowledge in teacher education, *Teaching and Teacher Education*, 4(2), 99-110.
Wilson, S.M., Shulman, L.S. & Richert, A.E.: 1987, 150 Different Ways' of Knowing: Representations of Knowledge in Teaching in J Calderhead (ed) *Exploring Teacher's Thinking*, Cassell Education, London.

Appendix

Graph questions.

1. Shoe survey - a lesson on variation

Shoe size	1	2	3	4	5	6	7
Number of pupils	1	0	3	7	8	6	2

2. Stretching elastic

Number of marbles	1	2	3	4	5	6
Length of elastic (cm)	9	10	11	13	14	15

3. Falling Spinners

Some pupils made a paper spinner. They put a paper clip on the bottom and timed how long the spinner took to fall. *Their teacher asked them to investigate whether the number of paper clips affected the time it takes spinners to fall*. They added more paper clips, one at a time, timing each fall.

Number of paper clips	2	4	6	8	10
Time for spinner to fall (s)	4.5	3.0	2.0	1.5	1.5

Percentage questions.

1. 35% of £40 2. Increase £80 by 10% 3. 73% of £90