OPERATING ON THE UNKNOWNS: WHAT DOES IT REALLY MEAN?

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Carraher, Schliemann, and Brizuela raise in their paper a fundamental question: when is a decision to teach a subject at a certain age or in a certain sequence soundly based on some developmental prerequisites and when it is simply a result of a long unquestioned educational tradition? For them this question is crucial not because of time wasted or students' potential not fully developed but because a delayed introduction, they believe, in and of itself might be the very reason for future conceptual difficulties. They claim that sometimes we postpone the introduction of a mathematical topic until the emergence of some theoretically required ability while the late emergence of this ability is a direct result of the late introduction.

The mathematical topic Carraher *et al* deal with in this paper is the transition from arithmetic to algebra and more specifically the ability of beginning algebra students to operate on the unknowns. They believe that the research reported on "cut-points" separating arithmetical from algebraic thought (Filloy and Rojano, 1989), on "cognitive-gaps" between arithmetic and algebra (Herscovics and Linchevski, 1994), and other well-documented difficulties in early algebra (e.g. Collis, 1975; Kieran, 1985, 1989; Steinberg, Sleeman and Ktorza, 1990; Sfard and Linchevski, 1994; Sfard, 1995) has played a major role in educators' decision to introduce algebra only at higher grades. They question this universally accepted decision claiming that these well-documented cognitive-gaps are so widely detected among beginning algebra students *because* algebra enters the mathematics curriculum too late and with odds with students' knowledge and intuitions about arithmetic.

The lesson

The children's performance while negotiating the event of the story is interesting and it, indeed, challenges the idea of a "gap". The problem presented to the children consisted of several parts where the departure information is an unknown number. According to the paper some of the children treated this unknown initial value as a "known" and were even able to use the algebraic notation – the N – to represent it. Moreover, the later parts of the story – which contain only specific numbers – were translated into the mathematical language and added sequentially to the initial value (e.g. $N + 3$), the obtained expressions were treated as numbers albeit not yet known. Intermediate manipulations were carried out by the children (e.g. $N + 3 - 3 = N$, or even the more impressive example $N + 3 - 5 = N - 2$) leading to equivalent expressions.

The authors report that their teaching interventions included an explicit introduction of the letter as representing an unknown (or any) quantity and the "N-number line" as the model to be used for representing operations on letters, numbers, or combinations of the two. It can be concluded from the paper that the researchers, who served also as teachers/interviewers, also brought the mathematical voice into the class. For example, sentences like: “N, it’s for any number” were initially introduced by them.

Some Reflections

It will be helpful to further elaborate on some of the theoretical ideas this paper challenges and to reexamine these ideas in the light of the reported teaching experiment.

Using letters to represent unknown numbers
It is widely accepted that letters can be used meaningfully within children’s arithmetic experience. Carraher*et al* takes this idea far beyond what is considered the norm in many classes. The debate, however, is whether the presence of letters in and of itself guarantees that algebra has been introduced. Some people tend to see algebra everywhere and claim that whenever letters or missing addends are present the children are “doing algebra”. Others claim that the presence or absence of letters cannot be considered as the indicator for algebraic thinking, claiming that the criterion is not to be found in the displayed task (whether it contains letters or not) but rather in the solver strategy. If letters are part of the expression at hand, one of the requirements the solver has to meet is the ability to perceive “letters” as numbers. This ability is a combination of several aspects.

I. The Lack of Closure
Collis (1974) observed that children at the age of 7 require that two elements connected by an operation be actually replaced by a third element. From the age of 10 onwards, they do not find it necessary. This observation lead to the conclusion that algebraic expressions cannot be introduced to children before the age of 10 (generally speaking) since the operations performed on letters cannot be closed as in arithmetic. The research conducted by Carraher *et al* seriously challenges these observations; their young pupils referred to N + 3 (for example) as a number and did not rejected it as uncompleted or unclosed process. From this perspective these children perceived the unknown as a number. However, Herscovics and Linchevski (1994) note that Collis’ age levels have to be taken in some caution since the algebraic expressions used in his work were formal and detached from any context.

II. Operating on and with the unknown
In Herscovics and Linchevski (1994) we write: “…the idea (of Collis) of a pronumeral evolving into a generalized number is quite enlightening. However, it is not sufficient to endow it with “the same properties as any number”, for this can be interpreted quite passively, as for example “let *n* be an even number”. (*let N be the initial amount of money in the piggy bank.*) In fact, the pronumeral must also be endowed with the operational properties of number; the unknown must be perceived as a generalized

number that can be subjected to all operations performed with or on the numbers. Perhaps the expression “operational generalized number” describes this necessary evaluation…”.

What does it mean? Operating on and with the unknown implies understanding that the letter is a number. It does not only symbolize a number, stand for a number, and it does not only a tag/label/sign for an unknown number; it is a number. And from this understanding the ability to operate on and with the unknown is emerged. The ability to perform operations on the letters is derived from this perception. Thus, student that has constructed this concept has the ability to add, subtract, bracket…, unknown numbers exactly as he or she has in the context of numbers. To transform, for example, X + 3X, into 4X realizing that while doing it he or she were adding numbers and not just executing formal rules. Moreover, these pupils are expected to be able to use, for example, inverse operations on variables as naturally and as spontaneously as they do it on numbers. For example, the solution of an equation like 32 + X = 3X should trigger the use of inverse operation on the X on the left hand-side of the equation thus transforming this equation to 32 = 3X – X, exactly as it occurs with an equation like X + 15 = 31 where they intuitively say that X equals to 31-15. (This last sentence puts Filloy and Rojano’s notion of the didactic-cut in context). Thus, transforming an expression like N + 3 – 3 to N (as appears in the current paper), does not satisfies this criterion since these transformations do not involve the unknown. It is what we labeled as “static view of the literal symbol” (Linchevski and Herscovics, 1996) or “working around the literal symbol”. The ability of the children in Carraher, Schliemann, and Brizuela’s research to manipulate the numbers in an expression with one occurrence of the unknown and a string of numbers, is described in details in Herscovics and Linchevski (1994), and Linchevski and Herscovics (1994, 1996). Moreover, in these papers it was reported that sixth and seventh graders operated on the numbers in algebraic expressions of this sort spontaneously, without any prior instruction in algebra. The fact that the research population of these studies was sixth and seventh graders does not imply that younger students would not react in the same way. However, it certainly implies that the choice of the target population was heavily influenced by the current curriculum where algebra is usually introduced in the seventh grade. From this perspective Carraher*et al’s* research is a major step forward. Nevertheless, we have to bear in mind that in their study an intensive and direct teaching interventions took place while the other papers reported on spontaneous development.

III. The cognitive gap

The existence of a cognitive gap (as defined in Herscovics and Linchevski, 1994) implies that the students were encouraged to proceed on their own as far as they wish and as long as their own procedures and approaches satisfied them. And a teaching intervention took place only at the point where their intuitive methods, drawn from their existence knowledge and their interaction with the new material, reached an upper limit and they became aware of the limits of their methods thus looking for new points of view. The notion cognitive gap is reserved to these steps in the pupil's learning experience where without a teaching intervention (to our best judgment and research methodology) he or she would not make a certain step. Some of these junctures might be different for different people and some are shared by many. Operating on and with the unknown as discussed in the previous paragraph is one of this junctures. Thus, after identifying a cognitive gap it is trivial to find it in (almost) every age if teaching has not taken place and it is less surprising not to find it after teaching took place.

However, this explanation still leaves the question with regard to the "optimal" age in which the teaching should take place, what are the desired interventions and what differences would be traced in the future is still unanswered. Carraher*et al's* research is a trial to start answering these questions. They do not reject the need for an explicit teaching intervention that provides the students with new tools and new mathematical language: thus they are actually accepting the existence of the cognitive gap.

In fact their study does not refute the existence of the cognitive gap. It explores the possibilities of crossing it earlier. It does not prove that indeed the gap has been crossed but it definitely shows that to a certain extent it might be crossed earlier and that young children can speak in the "algebraic voice".

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