Visualisation in Geometry : Multiple Linked Representations?

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At a first glance, geometry seems to be the most appropriate field for visualising problems and supporting its solutions. The paper looks into this statement by analysing the role of multiple (sometimes: linked) representations, especially in computer environments like Dynamical Geometry Software (DGS). Using a prototypic example, the essay throws some doubts on the above optimism and gives reasons for a more sceptical evaluation of the role of visualisation in geometry and its learning.

1 The field of interest and related literature

Visualisation is a continuous field of interest within the PME community (see the plenary session by Dreyfus 1991, the plenary panel in 1992, pp.3-191ff of the PME16-proceedings, or the papers of the research forum in the 1999-proceedings, pp. I-197ff). "Imagery and Visualization" since long is a category in the "index of presentations by research domain" in the PME-proceedings. It is also a constant field of interest in the broad community of mathematics education research (see for instance the handbook edited by Zimmermann&Cunningham 1991).

At a first glance, geometry is an easy domain for studying visualisation because, traditionally, geometry is THE mathematical domain where imagery abounds. It is often described as the field where icons, imagery and their inherent relations are studied - but: research in geometry education seems not to have an accepted description of the role of visualisation in geometry. Even historically we find times when geometry relies on visualisation via numerous drawings which alternate with periods or authors widely refusing the use of diagrams (see e.g. the well-known case of Lagrange's introduction to the "Mécanique analytique").

2 The focus of the study and its framework

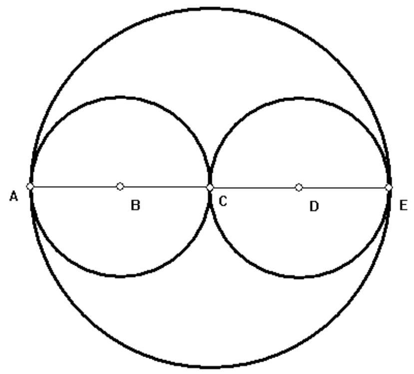
Within the field of visualisation and geometry, the paper tries to better understand the role of the variety of (sometimes, especially in computer environments: linked) representations of a geometrical problem in order to better understand visualisation in the field of geometry. The (non-surprising !) guess that this also throws some light on a more general idea of visualisation will not be treated here (for this, see Kadunz 2000).

The framework of the essay is the idea *that* -for the purpose of the study- *it helps* to distinguish between a human being (sometimes: a learner) and mathematical (in the essay normally) geometrical knowledge which are linked by external and/or internal representations (for this concept of representations see e.g. Goldin 1992). Consequently, the reader should not infer any special position of the authors with respect to epistemological questions (like for instance the adherence or degree of adherence to constructivism) from this study.

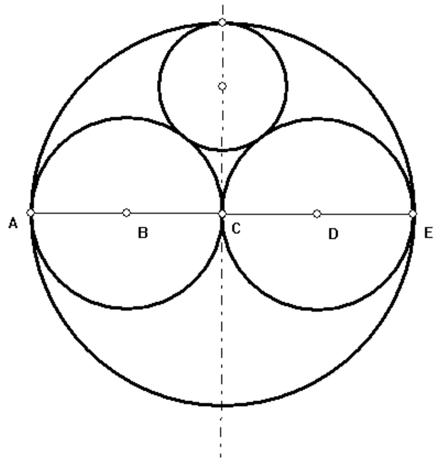
3 A prototypic example

3.1 The problem

Segment AE is the diameter of a circle with centre C, B and D are midpoints of the segments AC and CE. With respective circles around B and D through A and E, we come to a configuration represented in drawing 1a. How to construct a circle which only touches the circles around B, C and D (for a "solution" see drawing 1b).



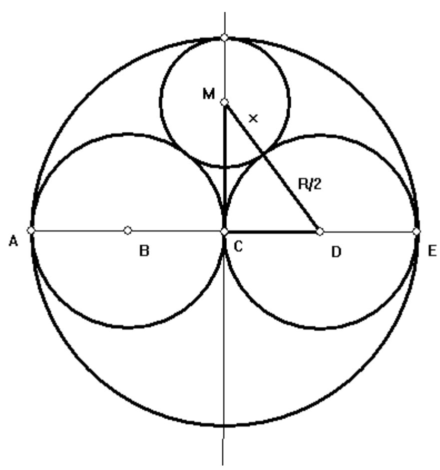
drawing 1a



drawing 1b

3.2 The drawing

Drawing 1a is an easy construction task with points, segments, circles and their names - the signs of standard elementary geometry. The fourth circle touching the first three circles may be arranged using ruler and compass or appropriate geometry software (like Dynamic Geometry Systems -"DGS"). A DGS-solution could place the centre M on the mid-perpendicular of segment AE (as a variable point on the perpendicular) and an additional point H in the intersection of the circle around C and the mid-perpendicular. Dragging M then arranges an appropriate circle with centre M through H to touch the circles around B and D.



drawing 2

3.3 An algebraic solution

The problem solver can use drawing 1b as a plan for an exact construction: Because of symmetry, the centre M of the fourth circle must be on the mid-perpendicular to segment AE - giving rise to a numerical solution if the radius of the circle is known. As the circles around D and M have only one point in common, the intersection and points D and M have to be collinear - offering a right-angled triangle DCM (see drawing 2). With R as radius of the circle around C and x as radius of the circle with centre M, we come to the equation (*) below and its easy and simple solution of x=R/3.

$$\text{(*)}\quad (x+R/2)^2 = (R-x)^2+(R/2)^2$$

The algorithmic transposition of the formula could even be handed over to a Computer Algebra System (CAS) - and drawing 3 gives the commands for the CAS-"Derive".

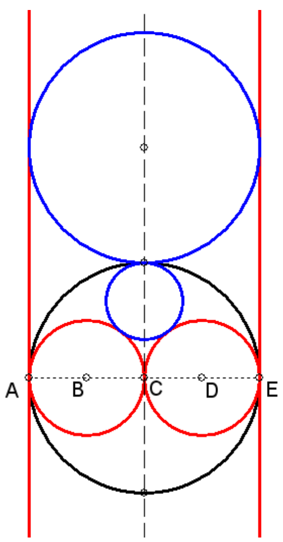
$$\left(x + \frac{r}{2}\right)^2 = (r - x)^2 + \left(\frac{r}{2}\right)^2$$

$$\text{SOLVE}\left(\left(x + \frac{r}{2}\right)^2 = (r - x)^2 + \left(\frac{r}{2}\right)^2,\ x,\ \text{Real}\right)$$

$$x = \frac{r}{3}$$

drawing 3

By concentration onto the algebraic solution of the equation (*), the original problem disappears. The equation does not tell its genesis and could be given as a drill-and-practice problem on equations without showing its geometrical origin. This is the price to pay for the use of the algorithmic solution - and only the unexpected result x=R/3 (only one solution of a quadratic equation - and a “simple” one!) may motivate the question why this special result had to be calculated. The algebraic solution does not offer a clue to an answer of this question and may come as a surprise because it can hardly be anticipated from drawing 1a/b. On the other hand, the algebraic solution by means of the Pythagorean theorem offers no clue or interpretation of this rather "simple" solution. Even the effective construction does not offer a hint to embed the solution x : R = 1 : 3 into a broader geometrical context. Nevertheless, it is easy to end the construction by using the mid-perpendicular of AE and a circle around A with radius 2R/3 for finding the centre and then the radius of the inscribed circle.



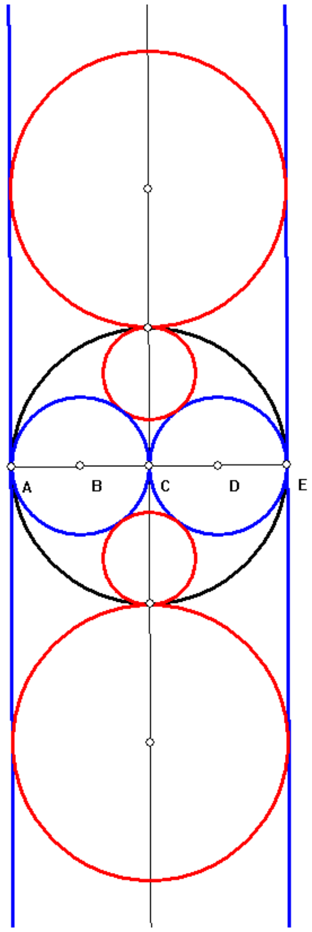
drawing 4

3.4 A geometric solution

The solution using the algebraic information x=R/3 did not offer a geometrical interpretation for the solution. So we look for a way to cope with the construction of a circle tangent to three other circles. Within the range of elementary geometry, a productive method to cope with problems like these is the inversion of circles and we will use it to explore the problem - even if inversion is no more part of the elementary school geometry. With DGS and its power to group chains of constructions by means of a "macro"-definition, by means of modularising a construction, we have a chance to solve the problem (we will not elaborate on inversion because of the abundance of appropriate literature on this topic; see for instance Lindner, 1999, pp.336-341; Coxeter, 1989, pp.104-117).

Let us take the "large" circle with centre C as the circle of inversion. The circle around M and tangent to the other three circles is transformed into a circle tangent to the images of the other three circles. The images of the these three circles are the circle of inversion itself and two tangents (straight lines !) to the circle of inversion (!!). In short: as images of the circles around B and D we have perpendiculars f(B) and f(D) to segment AE through A and E because they touch the circle of inversion in A and E and go through the centre C of inversion.

Now the image of the circle C_M with centre M (a circle again!) has to touch these two perpendiculars and the circle of inversion. Its centre f(M) must be on the mid-parallel between f(B) and f(D). Consequently, the image of the fourth circle has the same radius R as the circle of inversion. The construction of it is very easy and the inversion of it immediately gives the solution C_M of our problem. Symmetry adds the other solution (as in drawing 5).



drawing 5

The only information lacking now is the value x = R/3 of the radius of the circle C_M - and one could even doubt the necessity of knowing the numerical value because the construction is already finished. But inversion can inform us about this radius with the following argument: The image f(C_M) has the same radius as the circle of inversion around C. If one additionally constructs a circle around C with radius 3R, this circle will (obviously!) touch the image f(C_M) of the solution of our problem. The image of this larger circle around C also has to touch the circle with centre M and has the same centre as the circle of inversion. If we consult the definition of the inversion, this image has to have a radius of R/3 - hence our solution as well.

With this, we solved the problem by means of geometry in a double sense: We found, we geometrically constructed the circle around M we looked for AND we deduced its radius from the characteristics of the inversion. Inversion was the key to our geometrical solution.

4 On visualisation in geometry

If we look back to the solution process, we can clearly distinguish three phases: At first we constructed an "empirical" solution using the drag-mode of the DGS, this offered a clue for the second phase, namely an algebraic calculation for the radius of the inscribed circle (and a consecutive construction of it). The third phase deliberately put aside this numerical/algebraic solution while inversion offered a purely iconic, geometrical solution with the length of the radius as an additional information from a careful analysis of the inversion.

4.1 Traditional representations

The succession of the different phases is clearly marked by the use of different types of representations: at first, the solution is sought in the "register" of geometrical, iconic representations (for the concept of "register" see DUVAL 2000), we "visualised" the problem. Unfortunately, this only leads to an empirical solution (the drag-mode solution), but also produces the drawing, the graphical representation which opens the door for the second representation: the algebraic, symbolic description of the problem. In some sense, the algebraic description hides the initial problem, but offers powerful means to solve it - namely: the transposition of equations - and a surprisingly and numerically simple solution. The effective construction in the iconic representation "degenerates" to a mere technical process. It was exactly the change of register, the change of representation which made possible a solution of the problem. The interplay of the iconic representation of the empirical "solution" and the symbolic representation of the situation (via equations) and its rule governed manipulation produced the exact and elementary solution of the problem.

But this change of representations (from iconic to symbolic registers), the algebraic solution of the problem and its iconic realisation have a severe deficit: The symbolic representation put aside and masked the initial problem, offered a general and effective solution - but no clue to understand this solution in terms of (synthetic, iconic) geometry. Having found a symbolic solution, we now looked for an iconic one, i.e. after the change from iconic to symbolic representation, we turned back to the iconic representation to "understand" our solution in terms of geometry. The driving force of this return to geometry may have been the simplicity of our algebraic solution (or the love of geometry and consequently a search for a geometrical solution). In all, the symbols of the algebraic result made us go back to icons and forced us back into geometry. And with the use of a somewhat more advanced tool like inversion, we came across a "purely" iconic solution and could even give a geometrical reason for the simple algebraic result.

4.2 Representations with DGS

In both "icons phases", the first as well as the third phase of the solution process, the special features of DGS were of special help: In phase 1 the drag-mode helped to produce the empirical solution. If we additionally go into technical details of Dynamic Geometry Software, we come across a representation behind the iconic representation on screen: invisible for the user, but nevertheless essential for DGS, there is a “second” representation inside every DGS “behind” the visible representation on screen. Parallel to the construction of the user, DGS save all input (like position, shape and measures of the constructed objects), including information on relations between parts of the geometrical construction. This representation is kept hidden from the “normal” user – but makes possible the drag-mode by the quick and iterated recalculation of the whole construction depending on the position of basic objects. The high calculation power of modern hardware enables the computer to produce a quasi-continuous movement on screen when varying basic objects of a construction. Following the functional dependencies of the other objects, the DGS produces an actualisation of the representation on screen following the new position of basic elements. The visible iconic representation is internally controlled by an invisible algebraic representation of relations. This algebraic representation in return obviously depends on the position values of the user input for basic objects. It is this power which – on the other hand – in some sense prevents the problem solver from searching for more general, geometrical means to solve the problem because the power of the algorithms of analytic geometry make the drag-mode so simple and comfortable. As a consequence, the user is kept inside the icons of geometry – the concepts simulated by the drag-mode remain blind, whereas the solution in phase 2 is conceptually void. Nevertheless, the surprisingly simple solution is a good reason to continue working on the problem.

For an effective realisation of the inversion in phase 3, the macro-functionality of the DGS is crucial. To state it in more general terms: The offer of a modularization of the solution by means of appropriate “macros” (like “inversion of a circle or straight line at a given circle of inversion”) is crucial for an effective and swift construction in phase 3. Traditional paper&pencil constructions would be very tedious and time consuming (and be definitely out of reach in a normal school context already because of a lack of exactness). With macros, more general: a cognitively appropriate modularization of the construction, the iconic register becomes more manageable and flexible to use it in as tentative, but rule-governed way as we are used to when transforming equations.

So the third phase of the solution is characterized by a constant change between the iconic representation of the problem and its solution, represented by the drawing, and the reflections and theoretical concepts (mainly from the geometrical inversion) to further the solution process. The drawing is (re-)structured by means of conceptual entities (“modules”) like “inversion of a line / a circle at a given circle of inversion”. These modules are used to guide the continuation of the construction by “offering” new concepts/modules to advance the solution process. On the one hand, they offer a possibility to an economical construction, hiding those objects which are only intermediate states of a given construction (like a mid-perpendicular if only a midpoint of a segment is to be constructed). In addition to that, they give way for a look onto the overall structure of a complex construction, hiding those elements which conceptually are not needed for it. In some sense, they do the same job in a geometrical construction as algebraic expressions ("terms") do in algebraic transformation of equations. So they deeply influence the conceptual understanding and making of a (complex) construction. In our example, the modules act as links between the actual drawing and the geometrical theory of inversion, sometimes even motivating the use of additional modules. They are "only" heuristical tools which - in contrast to algorithms - offer hints how to continue the solution, they do not prescribe the next step (as would have to do algorithms). Within this complex, heuristical process, an iconic solution is embedded into the geometrical theory.

4.3 On visualization

From a more or less phenomenological angle, the first two phases of the solution are clearly different: the first one is marked by iconic, whereas the second phase heavily relies on symbolic manipulation. Icons offer an elementary heuristic to prepare for the second symbolic phase. Using elementary algebra, symbols produce a numerical solution, which can easily be transformed into an (iconic) construction. Following a simple, traditional visualisation concept, the first phase "visualises" the problem, while the second phase brings into being a non-visual, algebraic solution. Such a traditional concept of visualisation is inappropriate to fully understand the third phase, but opens a way to describe the interaction of "external" representations as opposed to "internal" representations (for the distinction cf. Goldin 1992).

We now concentrate on the third phase to understand the solution. On the one hand, we find constructions in the sense of traditional geometry. On the other hand and at turnings points of this phase, we interpreted our constructions by means of concepts from the theory of transformation geometry, especially the theory of inversion. We condensed parts of the drawing into a distinct "Gestalt", a module which could be understood as an instance of a concept from inversion theory. So we linked the iconic representation with a conceptual one, the construction was seen as an external representation of the concepts of inversion, which were the respective internal representation. And DGS-macros could additionally represent these concepts as simple software commands. Software offered (or could offer) an additional external representation of the theoretical concepts. The solution process was a constant to-and-from between external representations (on paper and/or machine-based) and internal, conceptual representations. Concepts grounded the solution process because they linked the solution with a geometrical theory – and led to the development of new icons and images which had to be interpreted within the theory of inversion to further the solution process. Looking onto the external, iconic representation had a heuristical function to decide which concepts were appropriate to bring forward the solution process. With a decision on the "next" concept to use, an algorithm - with necessity – defined the next iconic representation which in turn grounded the next heuristical step. This process of constant move between iconic and conceptual representations lasted until a complete and satisfactory solution was reached.

The interaction between the two representations (internal/symbolic and external/iconic and linked by the problem solver) can be taken as an enlarged concept of visualisation. This view of visualisation is characterised by a continuous interaction of perspectives, a constant change to be decided on using geometrical knowledge (and – in learning, especially school contexts: the support of a an external mediator, for instance a teacher). Visualisation is constant interaction of iconic, external and other (external and/or internal) representations.

5 Consequences for research

For (research within the field of) geometry, especially school geometry, the above view on visualisation can "explain" some difficulties: If change between different representations is a, if not ***the*** key to progress in a problem solution, the only type of geometrical representation, the iconic one, and its continuous use will not advance the solution process. Geometry as such, inherently, has to overcome a specific difficulty: Already working in an iconic mode does not offer a chance to change representations to make available different registers to bring forward the solution process. Taken in a more constructive way, developmental research in geometry teaching and learning should deliberately further the change of representations, especially leaving the realm of geometrical, iconic representations and has to diligently analyse the consequences of such an effort.

More globally, research on visualization must look into both directions of change of representations - and not only analyze the change from other, especially symbolic representations to iconic ones. A perspective on visualization not only taking into account one direction of a necessarily two-way process to and from visual, iconic representations is needed to better understand the links between multiple representations, to better understand visualization.

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