ON THE RELUCTANCE TO VISUALIZE IN MATHEMATICS: IS THE PICTURE CHANGING?

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This study examined the perceived and actual role of visual representation use as a possible heuristic in advanced mathematics problem solving by experts (mathematicians) and novices (undergraduate students). While both groups perceived visual representations as a useful tool and frequently attempt to use them, further analysis showed that students have little training associated with the use of visual representations.

The role of visualization in mathematical problem solving remains an active question in educational research. For centuries, visual tools such as diagrams, graphs, and sketches were considered to be indispensable in the work of mathematicians (Rival, 1987). Reports on the work of expert mathematicians have provided anecdotal evidence of the use and value of diagrams and other visual tools in the research work. Pólya (1945) for example, argued that the use of visual representations is an essential element in problem solving and offer specific advises to novices as to how to use visual representations in their own problem solving. But, despite the obvious importance of visualization in mathematical activity, deceiving visual clues often led mathematicians to false beliefs, resulting to the tendency to consider visual representation use as only an informal part of mathematicians' work. Little empirical work has been done towards the better understanding of the processes related to the use of visual representations by experienced problem solvers. Detailed accounts of individual practices are still relatively rare with the personal accounts of a few mathematicians like Pólya and Hadamard being notable exceptions and bearing the limitations of self-reports. Only recently have mathematics educators begun to study empirically and systematically the nature of mathematicians' practices.

In the realm of education, research studies indicated that advanced students are often reluctant to use visualization to process mathematical information (Eisenberg & Dreyfus, 1986, 1991; Vinner, 1989) and that, whenever possible, students choose a symbolic framework to process information and to approach problems rather than a visual one. Eisenberg and Dreyfus present a review of the literature with several cases in which college calculus students repeatedly resist the use of visual representations in solving problems. Yet, more recent studies in advanced mathematical problems suggest that the picture may have been changing in the past decade; Partly due to the changing curricula and attitudes toward the use of diagrams, mathematics students now appear to be interested in using visual representations (George, 1999; Gibson, 1998; Stylianou & Dubinsky, 1999).

Further research regarding use of visual representations is warranted. This study, whose purpose includes an investigation of the ways that both expert mathematicians and students reasoned with visual representations in solving advanced mathematical problems, aims to provide further insight into this issue. Specifically, this study investigated (i) the perceived role of visual representations as a possible heuristic in advanced mathematical problem solving by experts and novices, and (ii) the frequency of visual representation use during the actual problem solving (iii) the relationship between these two aspects of problem solving behavior. The study participants were 10 mathematics professors (experts) and 10 college mathematics students (novices).

I. Perception of visual representation utility

In the first part of the study, the role of visual representation use as a problem-solving heuristic by experts and novices was ascertained. Experts and novices were asked to categorize a set of 24 mathematics problems according to their similarity in their solution process. The study aimed at finding out whether "draw-a-figure" is viewed as a viable strategy when solving a problem.

Experts produced a total of 13 categories. These included some well-established strategies such as use of induction and contradiction, or use of similar problems; they also produced three categories which were closely related to the use of visual representations: “geometry/analytic geometry”, “algebra and analytic geometry” and “draw-a-figure”. The first category, “geometry and analytic geometry”, was used by 8 experts primarily to group together problems which were geometric in nature (i.e., they used topic as their classification criterion). The second and third categories, “analytic geometry” and “draw-a-figure”, were used to describe problems which relied strongly on the use of visual representations (that is, problems whose solution can be facilitated by the use of a visual representation). The 5 experts who produced these categories argued that the problems they classified as “draw-a-figure” are different from problems they classified as “geometry” in that the problems in the draw-a-figure category were not necessarily geometry problems. These are problems which reside in other topics or areas of mathematics (such as algebra and calculus) but visual representations can be helpful in the solution process.

Novices produced a total of 23 categorizations for their groupings and 6 of these are closely related to the use of visual representations: “geometry/analytic geometry”, “algebra and analytic geometry”, “draw-a-figure”, “circles”, “area”, and “physical constructions”. The “geometry and analytic geometry” category was used by 9 novices primarily to group together problems which were geometric in nature. Three novices used the “geometry and analytic geometry” and “draw-a-figure” categories to classify a small number of problems which they perceived as problems which are not geometry problems but which required the use of visual representations. Finally, 3 of the novices produced categories which focused on contextual features of the problems such as “circles”, “area” and “constructions”.

Data was analyzed using a cluster analysis – a process which allows for the arrangement of objects into clusters. The problems within a cluster are more homogeneous than they would be if they were compared to problems that belong to other clusters. The information provided by the clustering process was transformed into dendograms to facilitate the interpretation of the data. Figures 1 and 2 show the dendograms that were constructed using the experts’ and novices’ classification data

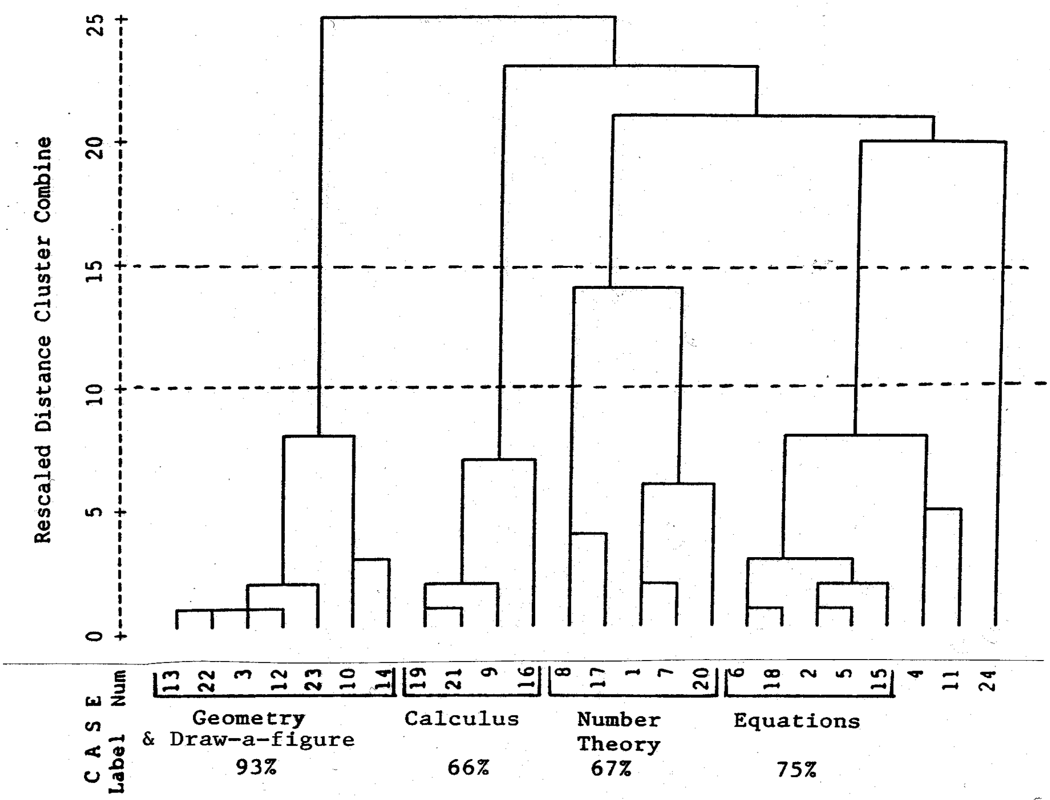

Figure 1: Experts' classification data

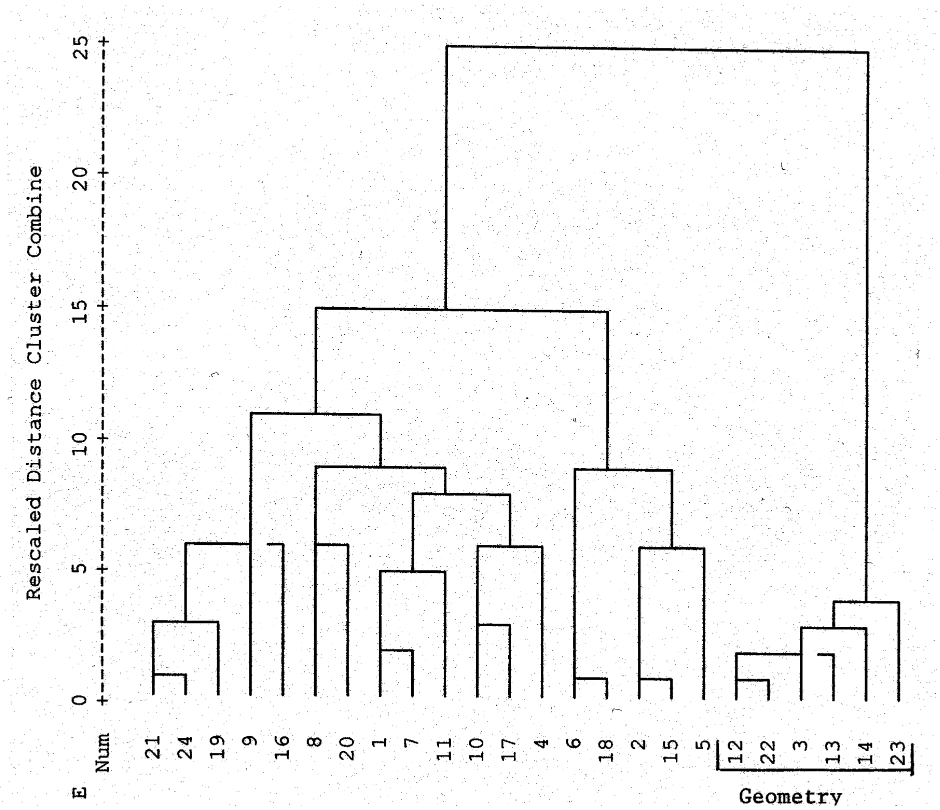

Figure 2: Novices' classification data

respectively. The 24 mathematics problems are represented as equidistant points along the horizontal axis. The measure of similarity, which is plotted as an ordinate, is the distance between problems belonging to different categories.

Problems in the dendogram with greater similarity are connected at lower ordinal values.

The expert dendogram in Figure1 contains a large cluster which was labeled as "Geometry and Draw-a-figure." This cluster contains both geometry problem and problems for which the use of visual representations would facilitate their solutions even though they may not be geometry problems. The novice dendogram in Figure 2 also includes a "Geometry and Draw-a-figure" cluster. In fact, this was the only cluster on which novices agreed to a large extent on the labeling of the category.

In a summary, when asked to classify a set of problems using mathematical similarity (that is, similarity in the way problems would be approached, and in the strategies that would be utilized during the solution process), both experts and novices produced categories which related to visual representation use. Therefore, the results showed strong evidence of both experts' and novices' perception of visual representation use as a viable strategy in mathematical problem solving. We may conclude then that visual representation use is perceived by expert and novices to be a viable heuristic in advanced mathematical problem-solving.

II. Frequency of visual representation use

In the second part of the study, the frequency in the use of visual representations was determined. Subjects were asked to solve five of the 24 problems and their written solutions were coded with respect to evidence of diagram use. The coded data were converted into numerical scores for analysis purposes by assigning a 0 for the solution of a problem for which there was no evident use of a visual representation and a 1 for each solution that contained evidence of visual representation use. For each participant a visual representation use score was then computed, by summing scores across the five problems. Thus, visual representation scores ranged from 0 to 5.

The frequency of visual representation use by the two groups was determined by examining experts' and novices' visual representation scores. A summary is shown in Table 1. In general, most experts used visual representations when solving the five

Table 1: Distribution of Visual Representation Use Scores

	Visual Representation Score					
	5	4	3	2	1	0
Number of Experts	6	3	--	1	--	--
Percent	60%	30%		10%		
Number of Novices	--	5	4	1	--	--
Percent		50%	40%	10%		

problems. Ninety percent of the experts used a visual representations in solving all, or all-but-one of the 5 problems, and no expert used a visual representation on fewer than 2 problems. Similarly, novices also used visual representations to a great extent, though somewhat less frequently than experts. Only 50 percent of them used visual representations in solving all-but-one of the 5 problems.

The mean score for the entire sample was 3.90, while experts' and novices' mean scores were 4.4 and 3.4 respectively. A t-test comparison with an alpha level of 0.05 revealed a statistically significant difference in the frequency of visual representation use by the two groups ($t=2.65$, $p<0.05$). Thus, this analysis also suggests that experts were more likely than novices to use visual representations.

These results gave prevalent evidence that both experts and novices frequently attempt to use visual representations in the form of diagrams, figures and graphs when solving advanced mathematics problems. The frequency in which experts utilized visual representations in their written solutions of mathematics problems in this study provides empirical evidence for the anecdotal reports of expert mathematicians claiming that visual representation use is an essential element in their mathematical problem solving. With regard to novices' use of visual representation, the results showed that advanced undergraduates, similar to experts, frequently utilized visual representations within their written solutions to advanced problems.

This result seemingly contradicts earlier findings by educators (Eisenberg & Dreyfus, 1986, 1991; Vinner, 1989) who documented a reluctance on the part of advanced students to use visual representations in mathematics. One way of interpreting the results of this group of studies is, as Vinner commented, to see that they reflect the current situation in mathematics learning (especially at early levels) where success is essentially measured by routine problems which do not require visual ability; students give up meaningful learning and prefer to memorize fomulae and algebraic techniques since experience has shown them that this is an effective prescription for success in standard tests. This interpretation is consistent with Lean and Clements (1981) who admit that "in [their] study the mathematical variables were measured by tests which did not require the solution of difficult, unfamiliar word problems" (p.294). The problems used in this study are different in that respect; they were chosen to not resemble standard textbook tasks, so that both novices and experts would have to make an effort to first understand and then solve the problems.

Finally, the frequency in visual representation use by novices may also be explained, in part, by recognizing the difference in time and curricular trends between the empirical work conducted in the early eighties and this study. A report by the Mathematical Association of America (Tucker & Leitzel, 1995) assessing the reform efforts in American higher education institutions showed that more than half mathematics departments were engaged in some sort of reform efforts and concluded that calculus reform, in particular, is gaining widespread acceptance. This suggested that undergraduate students receive different instruction using different curricula than students did approximately one decade ago. Reform calculus textbooks present a large number of visual representations. Further, they often encourage or expect students to use graphing technology in the form of handheld calculators, or computers. In short, undergraduate students in the past few years have been exposed to the use of visual representations to a larger extent than their counterparts a decade ago. The novice in this study were undergraduate students who received “reform” calculus instruction. Therefore, the contradiction between the findings of studies conducted a few decades ago and this study may be explained in part by differences in experiences students have been having with visual representations as part of their mathematics curricula.

III. Perception versus use

This study focused on experts' and novices' perception of visual representation viability in problem solving, and the two groups' actual use of visual representations during problem solving. Yet, the "obvious" question of the relationship between the two has not been addressed: How consistent are subjects’ perceptions of the usefulness or viability of visual representation use in advanced mathematical problem, with their actual use of visual representations during problem solving?

The results from the first part of this study showed that both experts and novices perceive visual representation use to be a viable strategy in problem solving. Both groups' classifications indicated that they perceive visual representations to be useful not only when solving geometry problems, but also for problems from other areas of mathematics. This result indicated that visual representation use is part of the declarative knowledge of both experts and novices.

The results from the second part of this study suggested that the two groups' perception and actual use of visual representations was consistent; Both groups constructed visual representations relatively frequently. Based on these results we could argue that the "picture" of particularly novices with respect to visual representation use has been changing; Novices are no longer reluctant to visualize. On the contrary, novices are well-aware of the utility of visual representations in mathematical problem solving and are, in fact, eager to use visual representations in their own problem solving. Since the problems that were given to subjects to solve were given previously to them in the sorting task, we can compare the number of experts and novices who initially perceived visual representations as a potentially useful tool for these problems, and the number of experts and novices who actually used visual representations during problem solving.

Figure 3 shows the contrast between perception and actual use of visual representation for each of the two groups. For experts, it is clear that their actual use of the visual representations during problem solving was higher than their perception of the possible utility of visual representation. For novices, on the other hand, the pattern was reversed; more novices thought a visual representation would be useful, than those who actually used visual representations during problem solving.

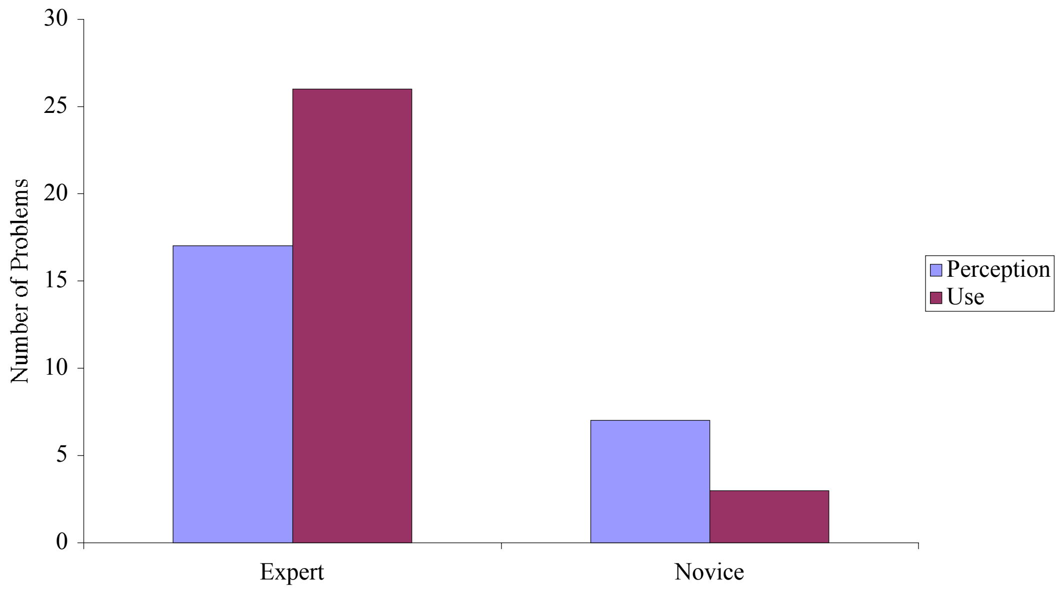

Figure 3: Perception versus use of visual representations

With respect to experts' visual representation use, the above comparison confirms the earlier conclusion that experts perceive visual representation use as a very useful tool in problem solving (declarative knowledge). Additionally, experts have strong procedural knowledge attached to this heuristic; once an expert decides that a visual representation would be a useful tool in solving a problem, even if visual representation use was not the first tool that came to the expert's mind, the expert is very likely to pursue the use of this visual representation, and, as the results for the second research question showed, experts know how to make use of visual representations as problem solving tools (procedural knowledge). Novices, though, appear to be very different; even though they appear to value visual representation use, and to have developed an ability to foresee the potential viability of visual representation use for a certain type of problem (relatively strong declarative knowledge), novices still lack the skill to use visual representations (lack of procedural knowledge).

Closing comments

Pólya (1945) and Schoenfeld (1985) argued that visual representation use is an essential element in problem solving and offered specific advises to students of as to how to use visual representations in their own problem solving. Pólya introduced the use of visual representations as one of the main problem-solving “heuristics”, and Schoenfeld’s subsequent work supported and extended Pólya’s discussion of visual representation use as a problem-solving strategy.

The results of this study gave prevalent evidence that both experts and novices perceive visual representations as a useful tool and frequently attempt to use them when solving problems, suggesting that the “picture” in advanced mathematics instruction may be changing. However, further analysis clearly showed that the changes may only be covering the surface; students may be willing to use visual representations but have little training associated with this skill. Recognition of the willingness and at the same time difficulties identified in this study can lead mathematics educators to make more explicit and informed decisions about visual representation use in curricular materials and instruction, providing opportunities for students to become successful mathematical problem solvers.

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