WHAT CONSTITUTES A (GOOD) DEFINITION? THE CASE OF A SQUARE

Karni Shir and Orit Zaslavsky
Technion – Israel Institute of Technology, Haifa

The notion of definition is central in the study of mathematics. Several researchers discuss the significant role definitions play in understanding mathematical concepts, in problem solving and in proving. This work deals with mathematics teachers' conceptions of a mathematical definition. Their conceptions of definition were revealed through individual and group activities in which they were asked to consider a number of possible definitions of a square. Data were collected from written questionnaires and recorded observations. The findings point to a number of perspectives underlying teachers' conceptions of an acceptable mathematical definition.

Theoretical Background

Mathematical definitions play a central role in mathematics and in mathematics education. According to Pimm (1993) "The mathematical term *definition* is one of a meta-mathematical marker terms (others include axiom, theorem, proof, lemma, proposition, corollary), terms which serve to indicate the purported status and function of various elements of written mathematics" (p. 261-262 ibid).

The following citation from Wilson (1990) expresses the motivation for the current study:

> "Although we frequently use definitions, we rarely focus on the nature of definitions. There is little agreement on what constitutes a good definition" (p. 33, ibid).

A definition is a way to create uniformity in the meaning of concepts, it is a tool for communication among human beings, and it is a foundation for proving and problem solving. Pimm (1993) brings the following citation of Ludwig Wittgenstein when speaking about definitions "… in order to communicate, people must agree with one another about the meaning of words" (p. 272, ibid). Borasi (1992) refers to the uniformity aspect of definitions through students' thoughts regarding the use of definitions in geometry: "So we bring unity, to make things uniform…" (p. 14, ibid). Moore (1994) discusses the connections between definitions and proving. According to Moore, there are three possible ways of operating with definitions in doing proofs: (a) using definitions for generating examples; (b) using definitions for justifying steps in a proof; and (c) using definitions for planning an overall structure of a proof.

There is a body of research dealing with the role definitions play in mathematical concept formation and concept understanding. Feldman

(1972) reports on three experiments done to determine the effect of several instructional variables on concept attainment. According to Feldman, providing a rational set of positive and negative instances with a definition was significantly more facilitative in promoting concept learning than a rational set alone. According to Wilson (1990) definitions, examples and non-examples are the building blocks needed to construct mathematical concepts. Klausemeier & Feldman (1975) and Sowder (1980) suggest a model of concept learning, which includes the following steps: recognizing examples, classifying examples and non-examples, and stating a definition of the concept. Vinner (1991) presents the notions of concept image and concept definition as two cells in which the knowledge about the concept is located. He adds, "the ability to construct a formal definition is a possible indication of deep understanding" (p. 79, ibid). Moore (1994), based on Vinner's and others work, suggests a 'concept understanding scheme', which consists of a third aspect - *concept usage* - in addition to concept images and concept definitions.

In spite of the significant roles definitions play in learning and doing mathematics, many students have difficulties in understanding and using definitions. In a study dealing with classification of students' mathematical errors, Movshovitz-Hadar, Zaslavsky, and Inbar (1987) found that many of the errors students perform are related to distortion of definitions. Moore (1994) found that mathematics and mathematics education undergraduate students, who either lack the knowledge of certain mathematical definitions or do not know how definitions may be used, have difficulties in constructing mathematical proofs. Many difficulties students have in constructing meaning of a mathematical concept are related to the compartmentalization between the formal definition of a concept and the (personal) concept image (Tall & Vinner, 1981; Vinner & Dreyfus, 1989; Vinner, 1991).

Several aspects of mathematical definitions are discussed by a number of mathematics educators (Leikin & Winicki-Landmean, 2000; van Dormolen & Zaslavsky, 1999; Pimm, 1993; Borasi, 1987, 1992; Vinner, 1991; Or-Bach, 1991; Leron, 1988). Most of the aspects discussed are considered critical requirements for a mathematical definition. Thus, a mathematical definition must be: *hierarchical* (i.e., based on previously basic or defined terms), *existent* (i.e., having at least one existing instance), *noncircular*, *non-contradicting* (i.e., all conditions of the definitions may co-exist), *unambiguous*, and *independent of the representation used*. In addition, two definitions of the same concept must be *logically equivalent*. There are two aspects on which there is no consensus regarding their ultimate need, that is, it is not commonly agreed whether or not a mathematical definition must be *minimal* (i.e., economical, with no superfluous conditions or information), and *elegant* (this is the most subjective aspect that is hard to articulate). Leron (1988) and Pimm (1993) discuss another relevant feature that distinguishes between definitions: a definition can be either *procedural* or *structural* (according to Leron). In Pimm's terms, it can be either *by genesis* or *by property*. Generally, the discussions on features of mathematical definitions distinguish between mathematical requirements and pedagogical choices.

A number of recent studies, which relate to the features mentioned above, propose ways to facilitate the understanding of definitions in mathematics. Leikin & Winicki-Landman (2000) presented teachers with a number of equivalent definitions of a certain concept (e.g., *absolute value*). Each definition used a different term for the defined concept. The teachers, who were not aware that the definitions were equivalent and that all define the same concept, were asked to investigate the mutual logical relationships between every two definitions. Through these activities they developed an understanding of equivalent definitions and discussed the freedom to choose a definition from a collection of equivalent statements.

In another study Furinghetti and Paola (2000) asked students to consider two alternative non-equivalent definitions of a trapezoid. The authors point to the value of the group discussions focusing on the advantages and disadvantages of each definition. In this study, similar to the work of Leikin and Winicki-Landman, students became aware of the issue of arbitrariness of a definition and the underlying considerations in determining what definition to accept.

De Villies (1998) asked students to define quadrangles. Following their responses, activities and classroom discussions were conducted focusing on the advantages of economical definitions. Through these activities students' tendency to suggest more economical definitions increased.

The current study focuses on ways in which secondary mathematics teachers view a mathematical definition, particularly, the aspects of a mathematical definition they consider critical, from both mathematical and pedagogical points of view.

The Study

The aim of the current study was to investigate mathematics teachers' conceptions of a mathematical definition, through their justifications for accepting or rejecting specific statements as possible definitions of a certain mathematical concept.

In order to focus on the notion of definition rather than on the defined concept, it was important to use a simple and familiar concept. Thus, the square was chosen as the focal concept for the study. The research instrument consisted of a questionnaire with eight equivalent statements, all of which describe a square (see Table 1). For each statement, the participating teachers were asked to determine whether they would accept it as a definition of a square. All along they were prompted to provide justifications for their responses.

In constructing the different statements for the research instrument, attention was given to several features. The statements differed from each other with respect to whether it is minimal or not, whether it is procedural or structural, and the degree of hierarchy in the statement (see below).

As mentioned above, one of the requirements of a definition is heirarchy. However, the kind of heirarchy for a given concept may vary. We distinguish between different levels of heirarchy. For example, we can define an isosceles triangle as *a triangle that has two equal sides*. We can go one step back and define it as *a polygon with three sides, two of which are equal.* The further back we go, the degree of heirarchy decreases. The focal concept of the present study – a square – can be defined based on a rectangle or on a rhombus (this is the 1st and highest level of heirarchy), on a parallelogram (the 2nd level), on a quadrangle (the 3rd level), on a polygon (the 4th level), and so on. Figure 1 illustrates the different levels of heirarchy that are associated with the notion of a square.

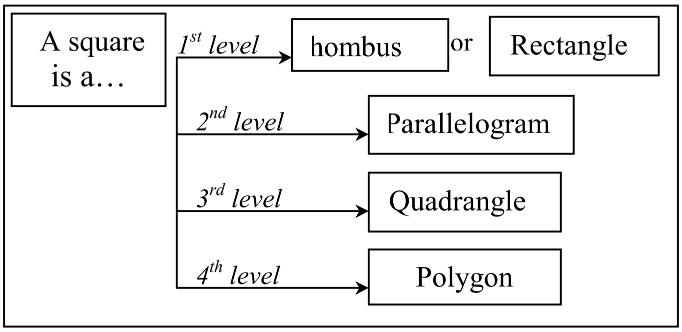

Figure 1

The statements in the research instrument appear in Table 1. The level of hierarchy in statements 2, 3, and 6 is 1, the level of hierarchy in statements 1, 5, and 8 is 3, and the level of hierarchy of statement 4 is lower than 4. However, the level of hierarchy of statement 7 is not decisive, since it relies also on the location of the notion of locus in the hierarchy of geometric concepts.

All except statement 1 are minimal statements in terms of a definition of a square. Statement 4 is the only procedural statement.

Twenty-four secondary mathematics teachers participated in the study. The participants took part in a 90-minute workshop dealing with alternative ways for defining a square. At the beginning, each teacher received a written questionnaire that contained the eight equivalent statements, and was asked to reply to it individually. Then, the teachers were divided into groups of 3-5, and were requested to discuss their answers to the written questionnaire and to try to reach an agreement. The third and last stage was a full classroom discussion, based on reports from the small groups.

Findings

For each statement, there were 4 kinds of responses: (1) accept the statement as a definition of a square; (2) do not accept the statement as a definition of a square; (3) not decisive; (4) no reply. Table 1 presents the distribution of responses to the eight statements in the questionnaire.

The Statement	Accept as Definition of a Square	Do Not Accept as Definition of a Square	Not Decisive	No Reply
1. A square is a quadrangle in which all sides are equal and all angles are 90º.	22 (92%)	1 (4%)	1 (4%)	- -
2. Of all the rectangles with a fixed perimeter, the square is the rectangle with the maximum area.	5 (21%)	15 (62%)	4 (17%)	- -
3. A square is a rhombus with a right angle.	20 (83%)	3 (13%)	1 (4%)	- -
4. A square is an object that can be constructed as follows: Sketch a segment, from both edges erect a perpendicular to the segment, each equal in length to the segment. Sketch the segment connecting the other 2 edges of the perpendiculars. The 4 segments form a quadrangle that is a square.	7 (29%)	14 (58%)	1 (4%)	2 (9%)
5. A square is a quadrangle with diagonals that are equal, perpendicular, and bisect each other.	14 (58%)	7 (29%)	- -	3 (13%)
6. A square is a rectangle with perpendicular diagonals.	18 (75%)	6 (25%)	- -	- -
7. A square is the locus of points for which the sum of their distances from two given perpendicular lines is constant.	10 (42%)	12 (50%)	1 (4%)	1 (4%)
8. A square is a regular quadrangle.	19 (78%)	2 (9%)	1 (4%)	2 (9%)

Table 1: Distribution of Teachers' Responses to the Statements

Although all the given statements are equivalent to a well-known and commonly accepted definition of a square, and all constitute a necessary and sufficient condition for a square, only 5 teachers accepted all 8 statements as possible definitions. Moreover, there was no unanimous agreement among the teachers about acceptance or rejection of any of the given statements. The percent of agreement on acceptance varied from the minimum of 21% for statement 2 to the maximum of 92% for statement 1.

There was also little agreement on the reasons for acceptance or rejection of the statements as possible definitions of a square. The written responses included 65 arguments justifying the acceptance and 54 arguments justifying the rejection of a statement as a possible definition. These arguments were classified into 7 reasons for acceptance and 7 reasons for rejection (Table 2) (it was mere coincidence that in both cases there was the same number of types of arguments). A further analysis grouped the different kinds of arguments according to their underlying perspective: Mathematical, pedagogical, both - mathematical and pedagogical, and embodied cognition. Table 2 presents the distribution of types of arguments that teachers used to support their decisions.

Underlying Perspective	**Reasons for Acceptance:** The statement is …	**N**	**Reasons for Rejection:** The statement is …	**N**
Mathematical	A necessary and sufficient condition for a square	24 (37%)	Not minimal	3 (5.5%)
	Equivalent to a known definition of a square	8 (12%)		
Pedagogical	Simple or clear	15 (23%)	Long or complicated	7 (13%)
	Based on students' previous knowledge	6 (9%)	Not based on students' previous knowledge	22 (41%)
	Familiar	5 (8%)	Not obvious – it requires more work in order to check	5 (9%)
Both - Mathematical & Pedagogical	A procedural description	3 (5%)	A procedural description	3 (5.5%)
Embodied			Based on properties of parts that are not integral parts of a square*	8 (15%)
Other	Other	4 (6%)	Other	6 (11%)
	Total	65 (100%)	Total	54 (100%)

Table 2: Arguments for Accepting or Rejecting a Statement as Definition of a Square

Note that about half the arguments (49%) for accepting a statement were based on mathematical arguments, while there was hardly any mathematical support (5.5%) for rejecting a statement. On the other hand, pedagogical considerations played a significant role both in accepting (40%) as well as in rejecting a statement (63%).

Discussion

We begin by pointing to the potential of the activity described in this paper as a vehicle for professional development, in addition to its power of eliciting teachers' conceptions of a mathematical definition and the role definitions play in teaching mathematical concepts. It is not surprising that many teachers drew on pedagogical considerations, which they are

* For example, some teachers, in reflecting on their ways of thinking about a square, referred to the diagonals of a square as non-integral parts of a square (opposed to the sides and angles of a square).

accustomed to take into account, even though they were asked to respond from their personal perspective, not necessarily as teachers.

We turn to a short discussion of the different perspectives that were identified, and offer some explanatory comments regarding each one.

The mathematical considerations teachers employed for accepting a statement as definition indicate their logical oriented view that there is a degree of arbitrariness in the choice of a definition. For them, an equivalent statement to a well-known definition, or a statement that constitutes a necessary and sufficient condition for a square, qualifies as a definition. Those who rejected a statement for mathematical reasons were convinced that a mathematical definition must be minimal (although, all of them probably teach their students the classical definition of congruent triangles that is not a minimal definition).

The pedagogical considerations that were given by the teachers indicate their expectation that a mathematical definition should be easily comprehended by students. For this reason a definition should be simple, clear, familiar, not complicated, and obvious. In addition, it should be based on students' previous knowledge. The requirement for previous knowledge may be seen as an extension and application of the mathematical hierarchy criterion to the mathematics curriculum, that is, to the order in which students learn (geometrical) concepts.

Procedural definitions seem to cause disagreement. Some teachers accepted the procedural statement from a mathematical standpoint and favored it from a pedagogical point of view, because it portrays the underlying structure of the object and lends itself well to construction of a mental image of the object. However, those who rejected the procedural statement rested mainly on mathematical grounds, and argued that a mathematical definition cannot be procedural. This view reflects the fact that procedural definitions are not very common in high school mathematic textbooks, and it is likely that many of them never came across a procedural definition before.

The last, but probably one of the more interesting considerations, is the embodied one. People are exposed to squares rather frequently in real life contexts, from early childhood. Thus, they probably conceptualize the technical mathematical concept of a square making use of their everyday concept of a square (Núñez, 2000), which appears without its diagonals. Statements 5 & 6 define a square through properties of its diagonals. For a number of teachers this was illegitimate, because the diagonals of a square are not perceived as integral parts of a square.

In this paper we reported findings of one part of a larger study dealing with what constitutes a (good) mathematical definition. Similar findings were obtained for other mathematical concepts in other groups of in-service and prospective mathematics teachers.

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