NEGATION IN MATHEMATICS: OBSTACLES EMERGING FROM AN EXPLORATORY STUDY

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ABSTRACT: In a previous research we have elaborated three schemes of behaviour to classify the difficulties of students in the interpretation and formulation of mathematical negation. This report provides a detailed analysis of one of these schemes, particularly its link with common practice, showing that some of the greatest obstacles in learning the meaning of mathematical negation are the difference between negation in mathematics and in natural language, and the tendency to classify into particular categories, that take into account the differences more than the analogies.

1.Introduction

In the works which involve the mathematical negation, the university students in the first year of the faculty of sciences often incur in errors. Such errors witness students' difficulties in understanding of mathematical negation and the presence of an obstacle to accept indirect proof.

There are currently few works in mathematics education directly concerned with negation. However, we would like to quote Thompson (1996), who report on the problems with proofs by contradiction due to the difficulties of negation. Thompson sees the correct formulation of negation as an important prerequisite for the use of indirect proof. Moreover, he draws up a list of some typical mistakes students make when negating a sentence. In Barnard (1995) there is a more detailed list of typical mistakes in recognising negations.

The theoretical framework which we refer to is Antonini (to appear). This framework, based on the notion of scheme of Piaget (1967) re-formulated by Vergnaud (1990), distinguishes three different schemes - scheme of the opposite, of the possibilities and of the properties - which guide the behaviour of the students in situations that involve the mathematical negation.

After a short description of the three schemes, we intend to widen the analysis of one of them (the scheme of the possibilities), by hypothesising about the existence of elements that, on the one hand, strengthen it and, on the other hand, can hinder the passage to the scheme more suitable to mathematics (scheme of the properties).

2. Method

The results we present in this report are related to a research study which, in this early stage, is essentially exploratory and based on a collection of data which can be furtherly refined. The goal of this first phase is to implement a framework of theoretical reference for a more detailed analysis. The author was support tutor in the first year of the calculus course, held at Pisa University in Italy. On this occasion, different types of students' behaviours were observed through questionnaires and discussions in the classroom. Furthermore, a number of interviews were conducted and then recorded with students of the fourth year of the degree course in physics at Pisa University.

3. Negation schemes

In this paragraph we shall try to classify the behaviour of students faced with situations concerning negation.

In this respect, we shall use Piaget's notion of scheme (Piaget, 1967), as re-formulated by Vergnaud (1990):

"On peut distinguer:

1) des classes de situations pour lesquelles le sujet dispose dans son répertoire, à un moment donné de son développement et sous certaines circonstances, des compétences nécessaires au traitement relativement immédiat de la situation;
2) des classes de situations pour lesquelles le sujet ne dispose pas de toutes les compétences nécessaires, ce qui l'oblige à un temps de réflexion et d'exploration, à des hésitations, à des tentatives avortées, et le conduit éventuellement à la réussite, éventuellement à l'échec.

[…]

Appelons 'schème' *l'organisation invariante de la conduite pour une classe de situations donnée*".

(Vergnaud, 1990)

A first classification of students' behaviour when facing with mathematics negation resulted in the definition of three mental schemes (Antonini, to appear), which can be briefly described as follows.

SCHEME A (of the opposite): The negation of "x is p(x)" is "x is q(x)" where q is "the opposite" of p. Examples of opposites are increasing-decreasing, even-odd, all-none, major-minor.

The subject is often aware that besides p and q there are other possibilities as well, but these are considered "exceptions", "extreme cases".

We observe that a concept and its opposite are strongly linked by analogies, symmetries, oppositions, and are often two aspects of the same concept (e.g. monotonicity: increasing-decreasing; order relation: major-minor).

The example which follows is designed to illustrate these features:

Interview to Vincenzo (4th year Physics student)

1.Int: […] I say: f is an increasing function. What is its negation?

2.Vinc: f is decreasing (he answers immediately, without thinking).

(…)

3.Int: Well, you must prove by contradiction a theorem whose thesis affirms that f is an increasing function.

4.Vinc: by contradiction…

5.Int: Let us suppose by contradiction that …

6.Vinc: That f(x) is decreasing.

7.Int: That f is decreasing; o.k. you start**:** let us suppose that f is decreasing.

8.Vinc: I must show … (pause)

9.Int: Well, you start from decreasing f and then, what happens?

10.Vinc: I must show that I reach a contradiction … of the hypotheses.

11.Int: Ok, at a certain point we reach a contradiction of the hypotheses.
12.Vinc: Yes.
13.Int: And so? Have you finished the proof?
14.Vinc: So… the fact is that it is not decreasing … therefore it is increasing …
15.Int: Are you sure?
16.Vinc: Yes.

Vincenzo uses the idea of the opposite to construct the negation of “f is increasing”. In the last part of the protocol we notice how “f is decreasing” is for the subject the logical negation of “f is increasing”. As a matter of fact, in 14 Vincenzo says: f “is not decreasing … and therefore it is increasing”.

SCHEME B (of the possibilities): If x is not p(x) then it can be p1(x), p2(x) or p3(x), etc. In other words, the statement “x does not possess the property p” means that there are various possibilities. Whilst there were only two possibilities for scheme A, for scheme B the negation dissolves in a moltitude of different cases.
What follows are some examples in which the students formulate the negations proposing various possibilities rather than using an opposite.

1) Written questionnaire (freshman Science students). One question was: “What is the negation of a proposition? (Try to give also an example)”.

A reply:
“For example: $f(x)>0$, negation: $f(x)\leq 0$.
In this case negation can include both < and =, that is all the possibilities in which the proposition does not occur. For example, if we were to deny that an f is increasing we should say instead that it is decreasing or constant or increasing and decreasing …” (underlining is ours).

2) Test (freshman Science students)
“We know that a certain function g is not strictly decreasing in the interval (-2,+3). Is there anything that you could define as being certainly true? (Give a justification for your answer).”
A reply:
“The fact that g is not strictly decreasing does not say anything about the function because it leaves the possibility of g being both increasing and constant and also that it is decreasing but not strictly or even increasing and decreasing at the same time.
Therefore nothing absolutely true can be said about function g”. (underlining is ours).

SCHEME C (of the properties): If p is false we look for a property q common to all x for which p(x) is false.
This scheme is commonly used in mathematical reasoning. On many occasions it leads to a really efficient behaviour from the operative point of view , for example if we want to deny that “f is increasing”, we could say that “there exist x, y such that $x<y$ and $f(x)\geq f(y)$”: this property is common to every non-increasing function, whether they are constant, increasing, discontinuous, etc.
On the contrary this scheme is a very rarely used by students, who seem to prefer scheme B.

4. Dominance of scheme B

The proposed schemes represent a first attempt to classify the behaviours. We observe that, in keeping with Vergnaud (1990), the behaviour of a subject can be guided either by different schemes in various situations or by different schemes in the same situation.

The obtained results show that scheme B is very common. This scheme corresponds to some aspect of the everyday language. It is also easily extended to mathematical context, even though it leads to poor and wrong behaviours.

In this paragraph we intend to analyse more in detail the nature of scheme B. We think that this analysis allows to interpret and better understand the students' behaviours guided by this scheme.

We retain that two are the fundamental elements that strengthen scheme B and hinder the acquisition of scheme C:

1) the natural tendency to a certain type of classification;

2) the impossibility with the natural language, unlike that mathematical, to always express the negation in the affirmative form - for instance, in Italian, without the use of "non".

4.1. Classification

As far as classification process is concerned, previous studies have highlighted two different aspects which are fundamental. By referring to two different studies, we show two different aspects of the process of classification:

a) the necessity to differentiate (Mariotti-Fischbein, 1997);

b) the classification in *cognitive categories* (Lakoff, 1987).

a) In a study on definitions, Mariotti and Fischbein (1997) point out that the students tend to classify some solid figures on the basis of differences more than analogies:

"The figural <u>differences</u> between a parallelepiped and a hexagonal prism lead to a classification which aims to <u>separate</u> the two classes of objects, whilst, in the standard mathematical classification, the class of the parallelepipeds is included in the more general class of the prisms. In order to get such structural conceptualisation, differences between the particular objects should be overcome in favour of <u>analogies</u> between them." (Mariotti-Fischbein, 97) (underlining is ours).

The need to express the negation as a unique property therefore conflits with the natural tendency to classify different objects in different classes.

As a matter of fact, the objects which do not have a particular feature may represent even enormous differences.

Let us give an example: the set of non-continuous functions is made up of functions which are very different one from the other. There are functions discontinuous at a particular point but with right and left finite limits, functions unboundied at a point, functions discontinuous at every point, functions continuous only at one point, etc. Mathematicians have classified the various types of discontinuity in three species. In each species there are functions which have some features in common.

The subject using scheme B, even though he realizes the possibility of many different cases, does not manage to assemble all the elements for which a proposition is false in a unique whole. He gathers them in cases or possibilities; each possibility contains elements with some common characteristics. Describing "non-p" with a list of possibilities therefore helps to overcome the difficulties encountered by treating with the differences. The more the objects are different, the more the necessity to find a common property (to generalise) conflits with the necessity to differentiate:

"The process of generalisation requested by a theoretical definition conflicts with the need of differentiating. Difficulties arise when theoretical constraints state the equivalence between 'different' things, requiring to cancel the variety once for all." (Mariotti-Fischbein, 97).

b) There are different studies of the nature of human categorization and of the nature of the categories (see Lakoff , 1987):

"From the time of Aristotle to the later work of Wittgestein, categories were thought be well understood and unproblematic. They were assumed to be abstract containers, with things either inside or outside the category. Things were assumed to be in the same category if and only if they had certain properties in common. And the properties they had in common were taken as defining the category." (Lakoff, 1987, p. 6)

Nevertheless, the formulation of Aristotle finds a series of great limits in the explanation of some experimental data. More suitable to explain the collected experimental data is the *prototype theory* of Eleanor Rosch (see Lakoff, 1987, chapter 2).
From the cognitive point of view, one category is not so much determined by the common characteristics of all of its elements but by the similarity with a particular element, said *prototype*; if an object is too different from the prototype it cannot belong to that category. All this is particularly valid for the so-called *basic-level* categories, defined in Lakoff (1987, p. 46). The categories at the basic-level are the ones better differentiated and it is at this level that a great part of our knowledge is organised.

"The complements of basic-level categories are not basic level. They do not have the kinds of properties that basic-level categories have. For example, consider non-chairs, that is, those things that are not chairs. What do they look like? Do you have a mental image of a general or an abstract nonchair? People seem not to. How do you interact with a nonchair? Is there some general motor action one performs with nonchairs? Apparently not. What is nonchair used for? Do nonchairs have general functions? Apparently not.
In the classical theory, the complement of a set that is defined by necessary and sufficient conditions is another set that is defined by necessary and sufficient conditions. But the complement of a basic-level category is not itself a basic-level category." (Lakoff, 1987, p. 52)

In this theoretical framework we can explain the tendency of the subjects to behave by using scheme B in situations of negation: the objects that do not belong to a given category (ex. the non-increasing functions) do not set up a category, being composed of too much different elements and often deprived of a prototype, but are an union of categories, each of which has a good prototype (ex. decreasing, constant and periodic functions).

4.2 Language

The second element that strengthen scheme B and hinders C is the lack of adequacy of the natural language to express a negation in affirmative form. In the natural language, to express the negation, often we can only affirm “non-p” (for example, sentences like “I did not travel by train" or "it does not rain" can be unlikely formulated in affermative forms). Instead, in mathematics it is often possible to re-phrase “non-p” in affirmative way, removing any trace of negation (for example, “f is a non increasing function” is the same as saying that “there exist x, y such that $x<y$ and $f(x)\geq f(y)$”). This is therefore a newness, that requires a specific educational approach.

5. Analysis of a protocol

The following protocol represents a very good example.

Interview to Carlo (freshman engineering student)
1.Int: What is negation?
2.Carlo: It is the opposite.
3.Int: What do you mean?
4.Carlo: If I say “switched on”, the opposite is “switched off”. But if I say “it is raining”, there is no opposite, I can only say that “it isn’t raining”.
5.Int: Do you know what a proof by contradiction is?
6.Carlo: It means proving that the opposite of a thesis cannot be true. I don’t know whether this is always possible, I think it is possible only in those cases in which I only have two possibilities, like “switched on” and “switched off”.
7.Int: If I have a theorem whose hypothesis is that f is an increasing function. How would you begin a proof by contradiction?
8.Carlo: Let us suppose that it is decreasing … (pause) … no, because there are other cases which are not included …
9.Int: And so?
10.Carlo:…(pause)… Well, I should identify something in common … I mean a property which is common to all the non increasing f, then prove that the f of the theorem cannot have that property, and therefore is increasing. But the proofs by contradiction turn out better when I have only two cases, like “switched on” and “switched off”.

1-4: Carlo distinguishes two types of negation: the first one is a typical opposite ("switched on" - "switched off"), the second an asymmetrical case. Only in the first case the language makes it possible to describe the negation in affirmative terms (not switched on = switched off); in the second case "I can only say that it isn't raining."

5-6: Carlo retains that the proof by contradiction can be done only when there are two "possibilities", but it is important to underline that, while "switched on" and "switched off" are possibilities, "increasing" and "non increasing” are not considered as such, in accordance with that said above about the cognitive categories.

7-8: After having applied the opposite of increasing to build the negation, Carlo realizes that there are other "cases" (possibilities).

9-10: The subject builds by himself the idea to formulate the negation in terms of property. Nevertheless Carlo is not sure that this is really possible ("I should identify") and, actually, he does not try to do it. Our hypothesis is that the familiarity with the mathematical language could help to overcome this obstacle.
Finally, Carlo returns to the idea that the proof by contradiction "turns out better" in cases of opposites.
We can observe that the difference that Carlo underlines between the two types of negation is not considered from a logical point of view. In fact, given a proposition p, there are always only two cases: p and non-p (as "switched on" and "switched off"). The difference underlined by Carlo can be only explained with the fact that, while, in any case, we are in the presence of only two properties, we are not necessarily in the presence of only two possibilities (as “increasing” and “non increasing”).

6. Conclusions

We have described three schemes that guide students’ behaviour in situations which involve negation. These behaviours are sometimes guided by a single scheme, other times by the consecutive combination of various schemes.
These schemes are not correct or mistaken in themselves, since any of them can lead to results which can be both correct and incorrect; however they can be more or less adequate to the solution of the mathematical problem that we intend to deal with.
We can also observe that the difficulties that may lead to the use of schemes A or B are of a very different nature: while scheme A may lead to errors (see Vincenzo protocol), scheme B may lead to a block of mental processes (see Carlo protocol). From the didactic point of view, identifying the involved scheme can give important indications to differentiate the kinds of didactical approaches.
Moreover, scheme B is the most diffused scheme among the students, and for this, it requires an in-deepth study; in this article, we have suggested a first analysis of this scheme, and we have described two elements that strengthen it and hinder the passage to a scheme mathematically more refined.
Further investigation are required in order to fully highlight the complexity of the mental processes involved in mathematical negation.
The knowledge of elements like those we have described, of their origin and of their link with the common practice, is deemed very important as it provides the teacher with good indications for the construction of didactical approaches for the introduction to mathematical negation and to proof by contradiction. We retain extremely meaningful statement 6 of Carlo protocol, that individualises a narrow bond between the schemes of negation proposed and the problem of the learning of proof by contradiction.

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