ADAPTING PIRIE AND KIEREN'S MODEL OF MATHEMATICAL UNDERSTANDING TO TEACHER PREPARATION

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Abstract
This theoretical paper examines a conceptual framework for studying prospective teachers' understanding of what and how to teach high school mathematics. The framework draws from Pirie and Kieren's model of mathematical understanding and incorporates ideas from Schoenfeld, Ma, and Lave. Several conjectures are made for future study in relation to folding back and primitive knowledge.

1. 0 Focus. The focus of this theoretical paper is to examine the model for mathematical understanding proposed by Pirie and Kieren (1994) with respect to the preparation of high school mathematics teachers. We adapted the model for teacher preparation to accommodate for prospective teachers' understanding of school mathematics and their understanding of strategies to teach school mathematics.

2. 0 Conceptual Frameworks. Among three distinct types of research frameworks, Eisenhart (1991) proposed that conceptual frameworks bring more flexibility to the research process than those based on theory or practice. She argued that the adoption of frameworks built exclusively from a single theory excluded the reporting of a broad range of results whereas conceptual frameworks provided a more comprehensive approach to the intended inquiry. The conceptual framework described in this paper scaffolds several broad ideas related to teaching and learning. They include what prospective teachers plan to teach [school mathematics], and how they plan to teach [teaching strategies]. The framework is intended to capture the dynamic process of learning over time and task.

Schoenfeld (1999) expressed the need to bridge the two dominant theoretical frameworks in education - cognitive and social. He proposed that theories or models of content competencies and acting-in-context have potential to span the divide between the two current research traditions. Further, Schoenfeld proposed that researchers consider how to maximize their contributions to fundamental understanding and, at the same time, contribute to practice. Studies of teachers in context, conducted by Schoenfeld's Berkeley team, are cited as examples of research directed toward the development of a theory of thought and action. This study draws from this proposition with respect to studying prospective teachers' understanding of teaching high school mathematics in the context of teaching tasks. Related to Schoenfeld's work is the Ma (1999) investigation of elementary teachers' content knowledge [thought] and strategies [action]. Ma chose to examine what Chinese and American elementary teachers' knew about topics in school mathematics and the strategies they selected to teach those topics. This approach gave more depth to her investigation of teachers' understanding of teaching mathematics.

The model of mathematical understanding proposed by Pirie and Kieren (1994) is a keystone of this conceptual framework. Modifications were made to accommodate for the cognitive and social dimensions of what the prospective teachers understand about school mathematics and teaching strategies. From the model of mathematical understanding we drew heavily upon the first five activities in the model: 1) primitive knowledge, 2) making an image, 3) having an image, 4) property noticing, and 5) formalizing. The meanings of these activities, as defined by Pirie and Kieren, were maintained for this framework. Their model was used to examine processes related to mathematical understanding, but we are proposing to adapt that model to study prospective teachers' understanding of what and how to teach. We expanded the Pirie-Kieren model to accommodate for the tasks of teacher preparation. Tasks such as lesson planning, teaching, and assessment all require an understanding of what and how to teach. A prospective teacher may have formalized the school mathematics related to rate of change, but is making an image of what teaching strategies to use to teach rate of change. The conceptual framework considers the understanding of school mathematics to be a function of mathematical understanding in which prospective teachers access their primitive knowledge of mathematics to make connections with what is taught in high school. Depending on the amount of mathematical study, the prospective teachers' primitive mathematical knowledge can be extensive. The primitive knowledge of teaching strategies, on the other hand, may be quite limited for the prospective teachers. This dichotomy will be discussed later in this paper. The process of folding back, taken from the Pirie-Kieren model, is a critical feature in changing the prospective teachers' understanding of teaching.

3.0 Related Literature. In this section we will briefly review two areas of the literature that Schoenfeld (1999) suggested as having potential to bridge the cognitive versus social divide. We consider those who have used frameworks built around content competencies and those who have approached their inquiry from an acting in context perspective.

3.1 Content competencies. Skemp (1976) argued that for mathematics teachers to make a reasoned choice between instrumental and relational approaches to instruction (how to teach) they require a relational understanding of the mathematics (what to teach) itself. His ideas laid the groundwork for many mathematics education research efforts related to content competencies. Ball's (1990) research illustrated the importance of relational or conceptual understanding when prospective elementary teachers attempted to develop application problems of division of fractions. In a study of Chinese and US elementary teachers, Ma (1999) noted that "a teacher's subject matter knowledge may not automatically produce promising teaching methods or new teaching conceptions" (p. 38). She suggested that new ideas about teaching (how to teach) cannot be realized without strong support from subject matter knowledge (what to teach). We assumed that prospective teachers' knowledge of what and how to teach is enriched with both instrumental and relational understanding of mathematics.

3.2 Acting in context. Lave (1991) claimed that the decontextualized practices of schooling contradicted her theories of learning in context but suggested that a view of peripheral participation may prove valuable in analyzing learning in school settings. She defined peripheral participation as actions within a community of practice over time and discussed changes in the apprentices' thoughts and actions. A number of researchers in mathematics education have situated their studies within the classroom to analyze students' or teachers' thoughts and actions within specific contexts over time. Maher and Martino (1996) reported on a five-year study of the thoughts and actions of one student's developing understanding of mathematical proof. An international study of prospective teachers' understanding of area within the context of a lesson planning activity reported a variety of thoughts that apprentices as peripheral participants bring to the community of practice (Berenson, et al. 1997). In a much larger study, Eisenhart et al. (1993) examined the many contexts of the educational communities affecting practice to better understand the field experiences of one prospective teacher. Decontextualized practice, the key term in Lave's description, is of great importance to the conceptual framework proposed here. We contend that to maximize prospective teachers' experiences as apprentices in the community of teaching high school mathematics, their preparation must engage them in the tasks of teaching. This does not imply that all tasks occur in the classroom with students; teachers can engage in a variety of tasks outside the classroom, such as developing lesson plans and creating assessments. These tasks are repeated in one form or another and occur over time as the prospective teachers participate peripherally within the community of mathematics teaching.

4.0 Activities of Understanding. In this section, descriptions of the first five Pirie - Kieren levels of understanding are given with some data from previous research. Together they are intended to clarify the application of these levels to our conceptual framework. In the previous study, prospective teachers were asked to plan a lesson to introduce the concept of rate of change to an Algebra 1 class and to relate the concept to ratio and proportion. The lesson plan was viewed as acting in context while content competencies were needed to complete the task. Individual interviews were conducted before and after the planning task. Subjects were 19-21 year old undergraduates with Grade Point Averages between 3.2 – 3.75 [out of 4] who had completed 6-8 university mathematics courses.

4.1 Primitive knowledge. *Primitive knowledge* was defined by Pirie and Kieren as all that is known mathematically before coming to the new learning task. There were several aspects of primitive knowledge accessed during the interviews including college mathematics, college physics, college chemistry, school mathematics [what], and teaching strategies [how]. We defined school mathematics of this task to refer to the specific mathematics (rate of change) and the related mathematics, such as, slope, fraction, ratio, proportion, and rate.

4.2 Making an image. At Pirie and Kieren's second level, *image making*, one uses primitive knowledge in new ways. In their model, one's image-making activity is an important step toward the understanding of a mathematical concept.

Amy's Lesson: An Example of Image Making

What to teach: Interpreting changing rates

How to teach: Collecting and representing distance/time data

The following excerpt is an example of Amy's image-making process in terms of what to teach [school mathematics]. She began tentatively to explain the relationship between the circumference of a circle to pi as an example of a ratio. The interviewer's prompt caused Amy to revise her formalized definition of a circle's circumference, folding back to make an image of the relationship between the circumference and the radius.

A: … what I was thinking about is the way that pi is related to … kind of makes a ratio out of the circumference and the radius, maybe.

I: What happens if you solve algebraically for *pi*?

A: [Considering the formula $c=2\pi r$,she divides both sides of the equation by $2r$ to solve for *pi*] Yeah, the circumference over $2r$. I think that is what I was thinking about.

One teaching strategy developed by Amy involved students in collecting distance/time data using toy cars and stopwatches. Making an image of how to teach, she tried to explain her ideas to the interviewer. In the process, she remade her image of how to teach, deciding that a table of data was better than recording the data directly onto the graph.

A: … they want to go another 50 and maybe do it slower or faster. And then plot the next point, how long it took them, for the second time, and then do it a third time for another 50 centimeters. … And you know, now that I think about it, it might be even easier to have a little chart of first time, second time, third time and the distance for each time already given.

4.3 Having an image. According to Pirie and Kieren, *image having* occurs when a person uses a mental construct to assist his/her understanding without having to repeat related activities. One has an image when he or she can mentally manipulate ideas to consider different aspects of the learning tasks.

Chris's Lesson: An Examples of Image Having

What to teach: Comparing distance/time ratios and calculating rates of speed

How to teach: Collecting and calculating distance/time data

Throughout the pre-planning interview Chris had an incorrect image of ratio as fraction.

C: I remember from the early years doing shading in squares and parts of the circle. And how much of the squares this covered. That's the majority of what I remember.

Then she drew a representation to explain her meaning of ratio. She drew a 3x3 square, shaded 4 out of the 9 squares, and explained that the ratio of shaded parts was 4/9. The lesson planning process helped Chris to fold back, changing her image of ratio as fraction as is shown below.

Chris had an image of how to teach rate of change that involved her students going outside to collect walking and running data using meter sticks and stopwatches. She was able to mentally manipulate her image of data collection to describe how those data would be used for teaching back in the classroom.

C: And then when you come back inside, [we will] talk about what we did and put up an example on the board of how we can use these numbers to find a rate of how fast we were traveling. … if Jane ran 12 meters in 3 seconds what is her rate in meters per second? So that is like two ratios.

4.4 Noticing properties. Pirie and Kieren explained that *property noticing* occurs when one manipulates or combines aspects of an image to identify related properties or contexts. This level of understanding is closely related to the images that the prospective teachers made and had about their lesson plans.

Sam's Lesson: An Example of Property Noticing
What to teach: Solving missing value problems
How to teach: Students working problems and teacher demonstration

Sam planned for her students to use hands-on materials while working in small groups to solve a missing value problem. Explaining her solution, she noted that one way to solve the problem was to use the simplest whole number ratio and that another option was to find the unit rate. This demonstrated her noticing properties about the ways proportions can be solved and ratios can be simplified.

S: I would give them the trophy problem that was in the book and then each group would have blocks… and fake dollars…If they struggled with this, I would … help all of them see the ratio of 3 blocks to 2 dollars and then go through my solution of the tall trophy…Our rate of cost here is $2 per 3 blocks. We could also break this down into … I can show them how we could call it $2 for 3 blocks or 67 cents per block.

Sam had noticed properties of teaching with concrete materials and their usefulness in teaching mathematical ideas to high school students. She based these properties from her lab experience in a high school classroom.

S: [The lab teacher] didn't use hands-on manipulatives, and I could see a lot of students struggle with that. I would use hands-on manipulatives.

4.5 Formalizing. The activity of *formalizing* occurs when a person identifies common features of the image he or she has made. Examples include finding patterns, applying an algorithm, or creating a formula.

George's Lesson:	An Example of Formalizing
What to teach:	Unit conversions of rates
How to teach:	Calculations, data collection, graphing

George formalized his meaning of rate of change to be the slope of a line, a derivative, and speed, but when describing what he would teach, he focused narrowly on only one aspect of rate of change, unit conversions.

G: I looked at [the lesson from] more of a physics standpoint than a trig-math standpoint. I figured the most important thing about proportions and rate of change is the units because if you have the units wrong the answer is wrong. I would start with some basic unit measurement and conversion type deals.

George planned to ask students to collect time/distance data using toy cars, stopwatches, and tape measures and then work through a series of unit conversions. The interviewer's comment below encouraged George to fold back to make a new image of how to teach.

I: Would you have them do any graphs along the way to look at representations like the one you drew earlier?

G: I think that after we did all the units and stuff, we'd graph the speed of the car. I think I would do one on the board or overhead and say OK, so this is time and draw a straight line like I did before. Depending on whether or not they had slope yet, I would introduce that to the class.

4.6 Folding Back. As defined by Pirie and Kieran, *folding back* is a revision process that occurs when a new issue is raised with respect to one's current understanding. It is a recursive process in which a person folds back to an inner level of understanding in such a way as to inform and change that inner level. For example, Amy folded back to her primitive knowledge to remake her image of how to teach rate of change to include a table-making activity for her students. She also folded back from her formalized knowledge of school mathematics to remake her image of the relationship between circumference and radius. We consider folding back to be critical to the development of prospective teachers' understanding of what and how to teach high school mathematics. In George's example, the interviewers were unable to precipitate his folding back from the formalized level of unit conversions to remake his image of what to teach about rate of change.

5.0 Conjectures and Implications. In this paper we were prepared to discuss and apply Pirie and Kieren's first five levels of understanding activity. As our research expands to involve tasks of teaching in the classroom, results may inform the last three levels of prospective teachers' understanding: observing, structuring, and inventising. From another perspective, perhaps only expert teachers attain these outer levels and that they do not pertain to apprentices. We have made several conjectures for future discussion among ourselves and other researchers. The first conjecture involves the critical nature of the process of folding back to extend and expand prospective teachers' understanding of what and how to teach. The second and third conjectures consider the retrieval and storage functions of primitive knowledge of school mathematics and teaching strategies.

Conjecture 1. Folding back is critical to the development of prospective teachers' understanding of what and how to teach and can be prompted by conversations with others. This function of understanding appears to be critical to the preparation of teachers and one that we will explore more fully in future research. The interviewing processes of the lesson plan task opened a dialogue between the mathematics educators and the prospective teachers in such a way as to encourage the apprentices to rethink their knowledge of school mathematics and approaches to teaching. An important feature of the dialogues was that they were focused on the lesson planning task, thereby situating the conversations in context. The implications of this conjecture to teacher preparation is that extended dialogues between educators and apprentices can increase understanding if situated within apprentice tasks involving what and how to teach.

Conjecture 2: Deep stacks of primitive mathematical knowledge may hinder the process of folding back to revise images of what to teach in school mathematics. The difficulty may stem from one's memory search functions or the ability to unpack one's knowledge base of college mathematics to find the connections to school mathematics. We noted that the prospective teachers in our study were readily able to access what they had learned in college mathematics, college physics, and college chemistry, but were unable to recall what they learned in Algebra 1. Several described rate of change to be the derivative but were unable to see the relationship to Algebra 1 topics. Some were surprised to learn that the slope of a linear equation was a ratio. Perhaps the depth and/or the lack of connections of their accumulated mathematical knowledge made it difficult for them to unpack their stacks of university learning to retrieve their understanding of school mathematics. A number of studies, primarily at the elementary level, have noted the lack of mathematical understanding of prospective and practicing teachers. Ma (1999) redefined mathematical understanding as profound understanding of fundamental mathematics [PUFM]. This definition may have important implications for the practices and programs of high school teacher preparation.

Conjecture 3. Shallow stacks of teaching strategies are to be expected among prospective teachers. The lack of primitive knowledge allows the prospective teachers to search through their strategy stacks quickly. Additionally, they appear to be more receptive to interviewer's probes so as to revise their images of how to teach and add to or pack these stacks with new ideas. The interviewers' prompts promoted folding back by the prospective teachers to their primitive knowledge of teaching strategies. We concluded that their stacks of primitive knowledge were thin because of the two basic strategies used by the prospective teachers. They either planned to give information directly or to have students collect data, reserving the generalizations from the data as the teachers' roles. This lack of strategy knowledge on the part of the prospective teachers, provided opportunities for folding back to pack their understanding of how to teach. For example, one interviewer asked if the prospective teacher would consider using a graph to represent the data. Another asked what would happen if the prospective teacher placed the student activity at the beginning of the lesson rather than the end. In all instances, the prospective teachers incorporated these strategies into their lessons. A goal of teacher preparation is to pack more understanding of how to teach into the prospective teachers' primitive knowledge. We suggest that instruction in teaching strategies be associated with the tasks of teaching and the content of high school mathematics to increase prospective teachers' understanding of how to teach.

We plan to ask Susan Pirie and Tom Kieren to provide us with critical comments on our proposed conceptual framework. We will apply the revised framework to analyze data from another study on prospective teachers. Finally, we plan to revise our own practice: <u>what</u> to teach prospective high school mathematics teacher about teaching and <u>how</u> to teach them about teaching.

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