ALGEBRAIC SYNTAX AND WORD PROBLEMS SOLUTION: FIRST STEPS

Eugenio Filloy and Teresa Rojano
Department of Matematica Educativa
Center of Research and Advanced Studies, Mexico

In Filloy and Rojano (1989) we introduced the use of concrete models for the teaching of solving linear equations and studied the abstraction processes that take place when such models are put in work by pupils of 12-13 years of age. In this paper we discuss how concrete modeling in algebra influences the way children in the arithmetic/algebra transition can learn to solve word problems. We discuss the stage in which the algebraic syntax developed to solve equations has just been acquired and then used to translate the word problems into the algebraic code.

Previous research has been undertaken to probe cognitive processes that take place in solving word problems in the transition from arithmetic to algebraic thinking. Bednarz and Janvier (1996) have substantially contributed to this aspect of research in problem solving in algebra. Puig and Cerdan (1990) have formulated criteria to determine when a word problem can be considered as algebraic. From a different perspective, A. Bell (1996) has approached this matter by showing through examples how generic problems can provide algebraic experiences that develop manipulative algebraic abilities. Sutherland and Rojano (1993) have studied how a technological environment can help students to represent and solve word problems without having to take on board with the algebra symbolic code, from the very beginning. The present paper addresses the theme of *extending the use of basic algebraic syntax to the context of the solution of arithmetic/algebraic word problems*. Pupils of 12-13 years of age were interviewed just after they had learned how to solve linear equations with one unknown appearing on both sides. In a first stage of the interview, a teaching approach using "concrete models" for solving equations was used (items in sequence I) and in a second stage, children were given two sequences of word problems. Sequence A included problems of the type: “find a number that…”; and Sequence P, included problems posed in a variety of contexts and in which a progressive symbolization process is required to make up the equation that solves the problem. The purpose of facing children with problems of sequence P was to study their potential in transferring basic algebraic syntax, recently learned, to other contexts, different from those used for the teaching of solving equations. Items of sequences I, A and P of the interview constitute a teaching strategy that aims to provide with senses the learning of algebraic syntax (Filloy, 1991). With this strategy, it is possible to complete the teaching cycle:

I.-Meaningful introduction to algebraic syntax (through "concrete models" for solving linear equations);

II.-Immediate use of the elements of manipulative algebra, just acquired, in the resolution of word problems (progress towards a semantic use of algebraic syntax);

III.-Meaningful progress towards a more complex level of algebraic syntax.

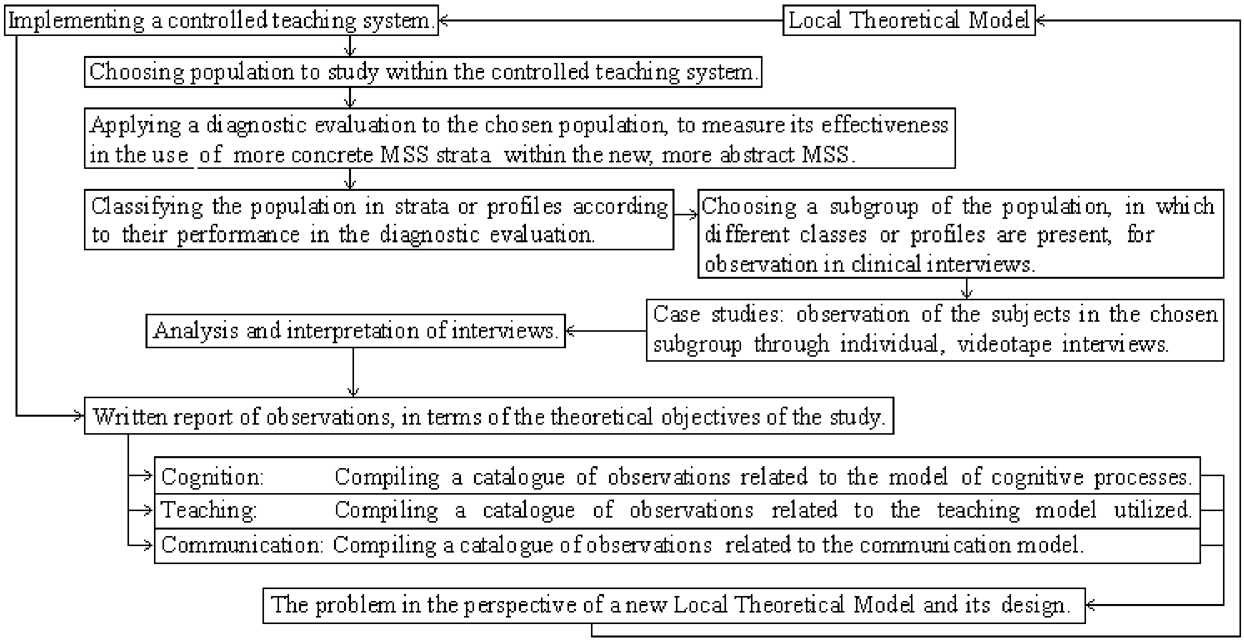
In this paper we report results from a case study with 12-13 year olds, who were introduced to the algebraic realm, using the teaching cycle described above. Among the twelve participants in the study, Mariana, a girl with a history of being successful in mathematics, got to complete the teaching cycle during a 1:45 hr session of clinical interview. In the following sections, we discuss some relevant issues from Mariana's interview, with regards the extension of syntactic skills through problem solving activity.

Theoretical and Methodological Framework

In Filloy (1990) we introduced the methodological framework of **local theoretical models** in which the object of study is brought into focus through four inter-related components:

(1) **Teaching models** together with (2) **models of cognitive processes**, both related to (3) **models of formal competence** that simulate competent performance of the ideal user of a Mathematical Sign System (MSS) and (4) **communication models** to describe rules of communicative competence, production of texts, texts decoding, and contextual clarification.

The following scheme describes the rationale of the case study:


Implementing a controlled teaching system.
Local Theoretical Model
Choosing population to study within the controlled teaching system.
Applying a diagnostic evaluation to the chosen population, to measure its effectiveness in the use of more concrete MSS strata within the new, more abstract MSS.
Classifying the population in strata or profiles according to their performance in the diagnostic evaluation.
Choosing a subgroup of the population, in which different classes or profiles are present, for observation in clinical interviews.
Case studies: observation of the subjects in the chosen subgroup through individual, videotape interviews.
Analysis and interpretation of interviews.
Written report of observations, in terms of the theoretical objectives of the study.
Cognition: Compiling a catalogue of observations related to the model of cognitive processes.
Teaching: Compiling a catalogue of observations related to the teaching model utilized.
Communication: Compiling a catalogue of observations related to the communication model.
The problem in the perspective of a new Local Theoretical Model and its design.

Results from the diagnostic test located Mariana in the category of students with high proficiency in a) solving "arithmetic" linear equations; b) solving arithmetic word problems; and c) numeric skills. Before the interview, Mariana had not had been taught any algebra.

THE INTERVIEW ITEMS (Sequences I, A and P).

We will write **I Mr.n** for the **nth** item of Sequence I Series in Mariana's interview.

Sequence I

I Mr.1	$x+2=2x$	I Mr.9	$5x+8=8x$	I Mr.17	$15x+1590=71x-202$
I Mr.2	$x+5=2x$	I Mr.10	$7x+468=19x$	I Mr.18	$11x+687=45x-27$
I Mr.3	$2x+4=4x$	I Mr.11	$113x+332=321x$	I Mr.19	$10x-18=4x$
I Mr.4	$2x+3=5x$	I Mr.12	$7x+23x+6$	I Mr.20	$8x-7=4x+13$
I Mr.5	$6x+15=9x$	I Mr.13	$13x+20=x+164$	I Mr.21	$7x-1114=3x+1001$
I Mr.6	$4x+12=x$	I Mr.14	$8x+12=4x+52$	I Mr.22	$7x-1114=3x-1001$
I Mr.7	$4x+12=6x$	I Mr.15	$28x+348=52x+12$	I Mr.23	$7x+1114=3x-1001$
I Mr.8	$5x+8=3x$	I Mr.16	$5x-3=2x+6$	I Mr.24	$113x-70=22x+1022$

Sequence A (a sample of items)

Item

A Mr.1 If you add five to a number, then subtract thirteen and the result is forty-five, what is the number?

A Mr.3 If you add two to a number and the result is double that number, what is the number?

A Mr. 5 If to forty-eight you add three times a number and the result is nineteen times that number, what is the number?

A Mr.6 Five times a number plus three is equal to double that number plus twelve. What is the number?

Sequence P (a sample of items)

Item

P Mr.1 There are chickens and rabbits in a yard. There are sixteen heads and fifty-two feet. How many chickens and rabbits are here in the yard?

P Mr.2 There are chickens and rabbits in a yard. There are sixteen heads and fifty-two feet. How many chickens and rabbits are there in the yard?

P Mr.4 Mariana is thirteen; Eugenio is forty. When will Eugenio be twice Mariana's age?

P Mr.5 Mariana is thirteen; Roberto is twenty-eight. When will Roberto be twice Mariana's age?

The Interview-Some issues.

MARIANA´S PERFORMANCE BEFORE THE INSTRUCTION PHASE

Mr. approaches the first items of sequence **I** with the trial and error method. She previously reads the equation: "I need to find a number that multiplied by 2, plus 4, equals …". Like all the children in the same category as Mariana, she, very quickly, stops trying to use this method. Thus, in item I Mr. 4 $2x+3=5x$, which could be easily solved by trial and error, she demands being taught a (school) method. At this stage the instruction phase begins on how to operate with unknown quantities by using a "concrete" (geometric) model (see Filloy/Rojano, 1989, and Filloy/Sutherland, 1996, for a detailed description of this model).

MARIANA’S PERFORMANCE AFTER THE INSTRUCTION PHASE

Mariana’s work with the concrete model to operate with unknown quantities is characterized by an immediate abstraction of the actions performed in the model to the level of algebraic syntax. This could be clearly observed in item I Mr.6. A factor that may have influenced this abstraction process is the parallel register that Mariana carried out using the algebraic code, while she was dealing with areas comparison in the model. In this way, the abstraction process towards the algebraic language was reduced to an immediate translation of her actions in the model to this language. This immediate abstraction of the model’s actions to algebraic language is not uniform throughout Mr.’s performance. The presence of more complex modes of equation, or equations with a non-trivial numerical structure ($C<A$, in $Ax+B=Cx$), hindered the way operations were abstracted. Items I Mr. 8 and I Mr. 12 showed this phenomenon: If certain syntactic elements are locally generated (let’s say, using a certain mode of equation) they are not transferred automatically to new modes. It is necessary to return to work with the semantics of the model to reconstruct the actions carried out and recover them at the level of algebraic syntax.

PROGRESS TOWARDS SEMANTICS (Solving word problems and the role of algebraic syntax)

Finally, in sequence P, Mariana was asked to solve the word problems that were presented in pairs. The second problem, of each pair, is obtained by changing only the data of the first one. The pairs in the sequence P are P Mr. 1 and P Mr. 2; P Mr. 4 and P Mr. 5. The interviewer intervened to help Mariana to build up the equation in the first of each pair, while in the second, Mariana showed to be able to produce the equation and to solve it, using the recently-acquired syntactic skills to solve equations with more than one occurrence of the unknown.

PMr1 and PMr2 (fragments of Mariana's interview)

There are chickens and rabbits in a yard. There are sixteen heads and fifty-two feet. How many chickens and rabbits are here in the yard?

The interviewer (**I**) suggests Mariana (**M**) to use letters to represent the number of rabbits and chickens. She writes down ***x*** for the number of rabbits and **16-*x*** for the number of chickens.

I: How many rabbit's feet I can see?

M: eight

I: If there is one rabbit, how many feet?

M: four

I: If there are 2 rabbits?

M: eight

I: If I see this number of rabbits (x) how many feet?

M: I don't know

I: How could you name the number of feet, using this …? (he points out the 'x')

M: Interrupting the interviewer writes down: $4x$

I: How could you find out the number of chickens' feet? Do you have a name for the number of chickens?

M: The name of the chickens' number times two

I: And, how is it? Two times…

M: writes down: 2´ (16-x)

I: If 4x is the number of rabbits' feet and 2´ (16-x) is the number of chickens' feet, How can I obtain the 52 which is the total number of feet?

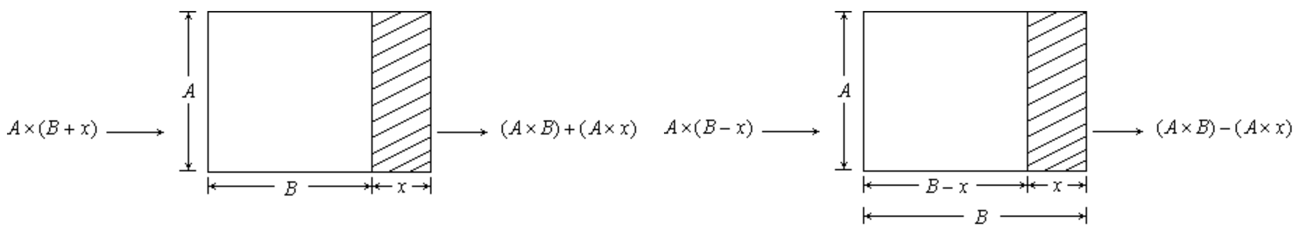
M: Adding, $4x$ to 2´ (16-x)

I: Write it down

M: writes: $4x$ + 2´ (16-x) = 52

I: This is an equation.

At this point, Mariana had not had instruction to remove the brackets in the left hand side expression. So, the interviewer intervenes with a piece of instruction to teach Mariana the distributive law in expressions that include ‘x’. He uses a geometric representation, which in general terms corresponds to the following figure:



The interviewer asks Mariana to generalize the law to expressions such as 3 ´ (16-x); 4 ´ (16-x); etc. Finally, she writes down the equation: $4x + 32 - 2x = 52$

I: Can you solve it?

M: writes down the simplified equation: $2x + 32 = 52$

At this point, she used the syntactic elements she learned in the sequence I of the interview.

Mariana inverts the operations present in this equation and finds the number of rabbits.

M: Ten, is the number of rabbits

I: And, how many chickens?

M: points out the expression 16-x, and says “six”.

When the item PMr.2 is presented to Mariana, she quickly recognizes that it is the same problem, but with different data and proceeds to solve it without any difficulty.

This strategy of using pairs of similar problems, is used in one part of sequence P. It could be observed that, once the greatest difficulty was overcome, that is the translation of the problem into an equation, the problem became a routine one for Mariana. This was because the syntactic elements that she had recently learned were applied immediately. This type of transfer allows the recently-acquired operational capacity to be considered as a potential tool to solve a larger family of problems, that is, a family that includes statements that lead to linear equations, which are here called ‘non-arithmetic’ (with more than one occurrence of the unknown).

FINAL REMARKS

The teaching strategy used to help Mariana to identify problems belonging to the same family consists in giving to the new type of expression, such as A´ (B+x) a treatment which is analogous to that given to the expressions of the type $Ax + B = Cx$. That is, translating these sorts of expressions into more 'concrete' terms in a context of areas and recovering the actions in the algebraic code. This strategy showed to be helpful to Mariana to extend the use of her syntactic skills to word problems, posed in a variety of contexts. In fact, Mariana's performance in the last items of sequence P, where she put into work new elements of algebraic syntax, provides evidence of a real progress towards the use of manipulative algebra in contexts different from those used in the teaching of such syntactic elements. Mariana's case illustrates how this type of return to the evolution of syntax can provide the elements of the manipulative algebra with *meanings* that enable them to be applied to the solution of entire families of word problems.

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