EXPLORING MULTIPLICATIVE AND ADDITIVE STRUCTURE OF ARITHMETIC SEQUENCES

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Arithmetic sequence is used in this study as a means to explore preservice elementary school teachers' connections between additive and multiplicative structures as well as several concepts related to introductory number theory. Vergnaud's theory of conceptual fields is used and refined to analyze students' attempts to test membership of given numbers and to generate elements that are members of a given infinite arithmetic sequence. Our results indicate that participants made a strong distinction between two types of arithmetic sequences, sequences of multiples (e.g. 7, 14, 21, 28, ...) and sequences of "non-multiples," (e.g. 8, 15, 22, 29, ...). Students were more successful in recognizing the underlying structure of elements in sequences of multiples, whereas for sequences of non-multiples students often preferred algebraic computations and were mostly unaware of the invariant structure linking the two types.

An arithmetic sequence is a sequence of numbers with a common difference. The topic of arithmetic sequence, along with other sequences, is usually introduced in high school and the standard approach utilizes algebraic representation and manipulation. Despite being a part of a high-school, rather than elementary school, curriculum the topic of arithmetic sequence is frequently approached in mathematics courses for preservice elementary school teachers. This is mainly because arithmetic sequences surface in the discussions of pattern recognition and understanding relations, generalization, and problem posing techniques (Brown and Walter, 1990; Principles and Standards for School Mathematics, 2000). However, little research has been done on students' learning of this topic. In this study we explore preservice elementary school teachers' understanding of the structure underlying arithmetic sequences.

THEORETICAL FRAMEWORK

Vergnaud's Theory of Conceptual Fields

In several publications over the last two decades Vergnaud developed, proposed and elaborated on the theory of conceptual fields (Vergnaud, 1988, 1994, 1996, 1997). The development of the theory of conceptual fields was motivated by the need to establish connections between explicit mathematical concepts, relations and theorems and between students', at times implicit, dynamic conceptions and competencies related to these mathematical concepts, relations and theorems. The following are among the established terms of reference:

• *Conceptual field* (1996, p. 225) is a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situation.

• A *scheme* (1996, p. 222; 1997, p. 12) is the invariant organization of behavior for a certain class of situations.

• A *theorem-in-action* is a proposition that is held to be true by the individual subject for a certain range of the situation variables (1996, p. 225).

Cognitive Development in a Conceptual Field - Theory Extension

Vergnaud claims that the theory of conceptual field is "a theory of representation and cognitive development" (1996, p. 220). Problem situations serve as triggers in generating and promoting cognitive development. When students are faced with a new situation, "they use the knowledge which has been shaped by their experience with simpler and more familiar situation and try to adapt it to this new situation" (Vergnaud, 1988, p. 141). This description is similar to Piagetian accommodation and assimilation. We elaborate further on the mechanism for the development of a particular scheme.

Theorems-in-action are identifiers of students' knowledge, as they describe mathematical relationships, either correct or incorrect, that are taken into account by students when they choose a path to solve a problem. Vergnaud suggests that "theorems-in-action have the potential to be the links among situations in the conceptual fields" (1988, p. 145). We add that theorems-in-action may also serve as separators, rather than links, and recognizing either could serve as a stepping-stone to learning.

Scheme development may occur in two ways: A student may recognize differences in two seemingly similar classes of situations. As an outcome, different theorems-in-action will be invoked, and, as a result, different routes will be taken by a student in dealing with each of the two classes of situations. A student may also recognize invariant structure in two classes of situations that were formerly perceived as "different". This may lead to an adaptation of two previously used theorems-in-action into one more general theorem-in-action that is applicable for both classes of situations. Furthermore, identifying invariant structure in situations may serve as a bridge that takes a student from one conceptual field to another.

METHODOLOGY

Participants

Participants in this study were preservice elementary school teachers enrolled in a core mathematics course. Students' work with arithmetic sequences in this course included pattern recognition, generating sequences given the first element and the difference, and developing and implementing formulas for calculating the n-th element as well as the sum of the first n elements of the sequence. Students also worked with elementary number theory topics, including divisibility, factors and multiples, and the division algorithm . Twenty out of 64 students enrolled in the course volunteered to participate in a clinical interview, which was the main source of the data collection.

Situations

Situation is a key feature in Vergnaud's definition of a scheme and a conceptual field. In this research we take a broad interpretation of situations, and classify as situations not only contextualized word-problems, but also mathematical problems and questions that are "abstract" or "decontextualized"; that is, not rooted in "real world" context. The following interview questions represent the core of the situations that were presented to participants.

1. Describing and exemplifying. Give several examples of arithmetic sequences. Can you think of an example that is different from others?

2. Testing membership.

Consider the following sequence of numbers

(a) 2, 5, 8, 11, 14, ...

Is it arithmetic? Is the number 360 (or 440) an element in this sequence (assuming it is infinite)? Why? Is there another way to verify this?

(b) The same question with respect to sequence 3, 6, 9, 12, ...

(c) The same question with respect to sequence 17, 34, 51, ... and number 204

(d) The same question with respect to sequence 8, 15, 22, 29, ... and number 704.

3. Generating examples of members. Can you think of a "large" number that is an element in sequence 2, 5, 8, ... (If necessary, "large" was described as a 3- or 4-digit number). Can you think of a large number that is definitely not an element in this sequence?

The same questions were posed with respect to sequences listed in 2(b), (c) and (d) above.

Objectives

We explore students' attempts to deal with the situations, specifically aiming to:

(1) identify and describe strategies (rules of action) used as participants encounter problem situations related to arithmetic sequences,

(2) analyze students' strategies and uncover underlying theorems-in-action,

(3) suggest a path for a development of individual's scheme within the context of presented situations, and

(4) test empirically the (above) extension of the theory.

STUDENTS' SCHEMES: RESULTS AND ANALYSIS

Listing the elements in a given arithmetic sequence by adding the common difference will eventually generate "large" elements and determine whether a given number is the element in the sequence. In light of Vergnaud's theory the common difference can be seen as an invariant identifying a class of situations. For participants in this study listing the elements was not the preferred choice, however, this strategy was mentioned by 8 participants as a verifying strategy or as a default for not being able to generate a better strategy.

Using the formula $a_n = a_1 + (n-1)d$ was a popular choice of strategy. The formula is applied in routine questions to find the n-th element when the first element and the common difference are known. Furthermore, it can be used to calculate any one of the four variables when the other three are known. Application of formulas was the exclusive strategy suggested by only one of the participants. However, 17 participants used formulas for situations similar to 2a and 2d, whereas they applied considerations of form and pattern for sequences similar to 2b and 2c. This fact, taken together with the significant amount of prompting and invitation to think of "another way" during the interview, suggests that participants preferred formulas when the pattern in the sequence was not obvious to them; that is, when they weren't aware of the multiplicative invariants in the structure of the elements. Therefore, participants invoked a scheme previously established to deal with arithmetic sequence related questions, the scheme of plugging numbers into the formula.

Multiples and non-multiples. It became apparent from participants' responses to question 1 (request to provide examples of several arithmetic sequences) that there is a class of arithmetic sequences preferred by students. Each participant provided between 4 and 8 examples of arithmetic sequences. Most of these examples were sequences of multiples of a small natural number, such as 3, 6, 9, 12 or 5, 10, 15, 20, ... with a possible exception of a sequence of odd numbers. When the interviewer explicitly asked for "something different", the usual reaction was to provide sequences of multiples of "large" numbers, such as 50, 100, 150, 200, ... etc. or list multiples in a descending order. Though participants readily accepted other sequences, such as 2, 5, 8, 11, ..., as "arithmetic", they were not a part of their immediate repertoire of examples.

Realizing that some arithmetic sequences are "sequences of multiples" provides a tool for testing membership or generating "large" elements without relying on formulas. However, 8 participants overgeneralized the observation that every element in an arithmetic sequence is a multiple of the common difference *d* to hold for any arithmetic sequence. This tendency is exemplified in the following excerpt from the interview with Leah.

Interviewer:	Would you please consider the following sequence: 8, 15, 22, 29, so far it's an arithmetic sequence, how would you continue?
Leah:	36, 43?
Interviewer:	Okay. And how about the number 704? Is it an element in this sequence?
Leah:	I'm going to check and see if 7 is a factor of 704, (pause) no ...
Interviewer:	No for what?
Leah:	Um, 704 is not going to be in this sequence because 7 is not a factor of 704.
Interviewer:	Okay. How about 700?
Leah:	Yeah, um, 7 is a factor of 700, so I think it's going to be in the sequence. 7 x 100 is 700.

Leah claims that the number 700 is an element in a sequence 8, 15, 22, ... because 7 is a factor of 700. In such cases the student's theorem-in-action was challenged by the interviewer by pointing out contradictory evidence. As a result of these types of challenges some participants refined their scheme by limiting it to certain kind of situations. Once the difference between the two classes – multiples and non-multiples – was realized, it became apparent that the same scheme cannot be used to accommodate both. As a result the scheme of considering multiples of *d* was restricted to sequences of multiples only.

In the following excerpt Sally considers the sequence 8, 15, 22, ... and the number 704.

Interviewer:	So 704 is not divisible by 7, none of these elements in this sequence you believe will be divisible by 7, so can you draw conclusions from what you have now?
Sally:	It's, it's um very possibly in this set.
Interviewer:	Um hm. What, what will convince you?
Sally:	(laugh) Well just because it's not divisible by 7, doesn't mean it's in the set, right?
Interviewer:	Can you give me an example of a number that you know for sure that is not in this arithmetic sequence?
Sally:	Um hm, um 700. . .
Interviewer:	Another one. . .

Sally:	Um, 77.
Interviewer:	Okay. And how about 78?
Sally:	It may be in the set, but it’s not divisible by 7. . .
Interviewer:	(laugh) So 77 you’re sure is not, 78 you’re not sure.
Sally:	Right.
Interviewer:	79?
Sally:	Could be. . .
Interviewer:	Could be. 80?
Sally:	Could be. . .

Sally is confident that multiples of 7 are not elements in the given sequence, but she believes that any number that is not a multiple if 7 "could be" in the sequence. At this stage students are able to differentiate and note that previously generated theorems-in-action are not fruitful in a new situation. However, they have not yet revised their theorems-in-action to generate rules-of-action for the new class of situation. Whereas a number's property of "being a multiple" gives a clear indication of its belonging to a sequence of multiples and non-belonging to a sequence of non-multiples, the property of "being a non-multiple" identifies that a number doesn't belong to a sequence of multiples, but gives no explicit hint with respect to the number's membership in a given sequence of non-multiples.

It was a common observation that sequences of multiples have two identifying features (invariants), multiples of *d* and a common difference, whereas sequences of non-multiples have only one identifying feature of common difference. Identifying a class of situations as "multiples" often left participants without appropriate tools to deal with non-multiples. In 17 cases the participants took advantage of multiples in considering situations 2(b) and 2(c), but regressed to the use of formulas for 2(a) and 2(d).

A further development was to realize that any arithmetic sequence of whole numbers can be considered as a translation along the number line of a corresponding sequence of multiples. Therefore, the next important step in developing individual's scheme is to recognize the invariant multiplicative structure of elements in an arithmetic sequence of non-multiples, often referred to by participants as "multiples adjusted".

Interviewer:	Let’s take one more. 8, 15, 22, 29 another sequence.
Lily:	Okay, so this is a difference of 7. . .
Interviewer:	How about the number 704?
Lily:	(pause) 704, no (pause) this is, these numbers are plus 1 of multiples of 7, multiples of 7 plus 1. . .

Consideration of multiples and "adjustment" where necessary clearly equips Lily with a powerful scheme. Such an "adjustment" is expressed by Megan in a more mathematical way as she considers division with remainder.

> Interviewer: And what about 704?
> Megan: No, because that's got a remainder of (pause) 4, not 1. . . it needs to have a remainder of 1.
> Interviewer: So can you please describe for me your general strategy? How would you decide whether a number I give you does belong to this sequence or doesn't belong to it?
> Megan: Um, if it's divisible by 7, with the remainder of 1 then it does belong to the set.

Six participants eventually succeeded in suggesting some adjustment of either multiples or numbers divisible by d, and 2 explicitly mentioned the common reminder in division by d thereby identifying the multiplicative structure in a sequence of non-multiples.

Towards a Unified Scheme. Although the invariant of "multiples" in the sequences of multiples and the invariant of "multiples adjusted" in the sequences of "non-multiples" are analogous, the learner may still see multiples and non-multiples as two separate classes of situations having a different invariant structure. Therefore different schemes are invoked in dealing with these situations. The identification of a multiplicative invariant within non-multiples is essential for the development of a unified scheme. Identifying similarities, other than lexical, between the two classes, can be a next step in scheme development. It is a further sophistication to consider multiples with adjustment (that can be zero) or common remainder (that can be zero) in division by d as the invariant that unifies both classes of situations and allows an individual to invoke the same scheme for any arithmetic sequence. Recognizing invariant structure in two previously-treated-as-different multiplicative invariants supports the development of a unified more mature scheme. However, the interview situations provided little opportunity for participants to extend their scheme in a way that it could accommodate both classes.

CONCLUSION

Greer (1992) proposed that the analysis of the relationship between the conceptual fields of additive and multiplicative structures is a long-term objective on the agenda for further research in mathematics education. Our research is a step in this direction. Furthermore, this study contributes to prior research on preservice elementary school teachers understanding of elementary number theory (Zazkis & Campbell, 1996), specifically the concepts of multiples, divisibility and division with remainder. In addition, analysis of students' development of a specific problem related scheme leads to a more profound understanding of scheme development in general and in such provides an extension of Vergnaud's theory of conceptual fields. Furthermore, our research points at a possible direction for a pedagogical approach to the topic, an approach that capitalizes on the common structure of elements in an arithmetic sequence.

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