Jennifer's Journey: Seeing and Remembering Mathematical Connections in a Pre-service Elementary Teachers Course

Gary E. Davis
University of Southampton
UK
plr199@soton.ac.uk

Mercedes A. McGowen
William Rainey Harper College
Palatine, Illinois, USA
mmcgowen@harper.cc.il.us

We detail one young woman's journey in a 16 week mathematics content course for pre-service teachers. We analyse her initial efforts at problem solving in groups, and her ability to see and recall explicitly connections that were inherent in the instructor's conceptions of the problems. Using her writing during the course, we compare her memories with those of other students of differing achievement, and infer that she made remarkable strides in coming to terms with the language of and ideas of mathematics, and understanding how engaging with those can help her be a better teacher.

Correct answers and mathematical connections

Pre-service elementary teachers want to get mathematical answers right. They want to know which formulas to use, and how to get the correct answer. A typical comment about their perceptions of mathematics on entering a mathematics content class is: "Coming into this class, I was under the impression that finding a formula to solve a problem was, in reality, the answer to the problem."

This is where many of them stop in their understanding of mathematics. A strict utilitarian perspective often limits their mathematical vision. We are concerned when pre-service teachers figure the answer to $9+3\frac{5}{8}$ by reducing the problem to one of "how many eighths?" We are frustrated when they justify their answer that there are 16 ways of building towers of height 4 using blocks of 2 colors with the statement: "I know I have found all possible towers because every other group in the classroom got the exact same answer as us."

Changing what students *value* in mathematics is frequently a much harder challenge than teaching them mathematical procedures and application of formulas. We need an antidote to a severely procedural orientation to mathematics focused on 'correct answers' that prospective teachers have learned to value above all. How can we explicitly emphasize connections, and assist students to construct relationships between parts of mathematics that they see as different?

We addressed this issue with a class of pre-service elementary teachers at Harper College, Illinois (www.harper.cc.il.us), during the Fall 2000 semester. The two authors co-taught the first three weeks of a 16 week course on mathematical content for pre-service elementary teachers. Differences between this and previous classes of the second author are in the explicit and intensive focus on building connections in the early part of the course. The emphasis in the first three weeks of the course was on making connections between different combinatorial problems and on multiple ways of interpreting answers.

Theoretical background

We wanted to know whether the problems we set could promote the formation of useful long-term mathematical memories. We followed the model of Davis, Hill and Smith (2000) in assisting students to make their implicit, procedural memories declarative (Squire, 1994). The latter are memories we are capable of expressing in words, drawings or gestures. They are to be distinguished from *implicit*, or non-declarative, memories that assist us to carry out routine procedures and habits. There are three major types of declarative memories relevant to mathematics. Two of these are familiar from everyday memory, whilst the third is more commonly seen in its full form in mathematics and science.

Episodic memory is the system of memory that allows us to explicitly recall events in time or place in which we were personally involved. (Tulving, 1983; Tulving & Craik, 2000, and references). *Semantic memory* is the memory system that deals with our knowledge of facts and concepts, including names and terms of language. (Tulving, 1972, p. 386; Tulving, 1983; Tulving & Craik, 2000, and references). *Explanative memory* is that part of declarative memory dealing with explanations for facts. Davis, Hill, Simpson, & Smith (2000) present a case that explanative memory is a separate memory system, linked to, but different from episodic or semantic memory.

Most psychological studies of memory are oriented to memory for language. Studies in memory for mathematics are much less common. A semantic memory such as '*Paris is the capital of France*' has quite different content to one such as '*the number of prime numbers less than n is asymptotically n/log(n)*'. The first is a linguistic convention, the other expresses a deep, non-obvious fact. Our experience with student mathematical writing and verbal recall suggests that there are, at least, the following distinctions in memory for mathematical facts:

(1) Memories of labels, customs, and conventions. For example: *A prime number is a whole number with exactly 2 factors.* This sets up '*prime number*" as a conventional term. We refer to these memories as *semantic labels*.

(2) Factual memories of things sensed, or done. For example: *The proportion of prime numbers less than 500 is 19%.* One might recall this as a fact from having done a series of calculations: the recollection is of the fact, not the episode of calculation. We refer to these as *semantic actions*.

(3) Memories of things believed. For example: *There are infinitely many prime numbers*: *one recalls this from a book on number theory*. We refer to these as *semantic beliefs*.

(4) Memories of explanations. For example: *A proof that there are infinitely many prime numbers: one recalls an explanation*. We refer to these as *explanative memories*.

Method

We set a number of connected problems in the first three weeks of the course. These were specifically designed to set up strong episodic memories as a result of students discussing their solutions in class. For example, after students had attempted the problem of finding how many towers of heights 4 and 5 they could build using blocks of 2 colors, they were shown, and discussed, a video clip of three grade 4 students attempting the same problem. This problem and its connections with algebra, which we utilized, has been reported on by Maher & Speiser (1997). For a detailed description of these, and other problems set in the course, see: www.soton.ac.uk/~plr199/algebra.html

The combinatorial problems we set for the students in the first three weeks were connected in *our* minds: they all deal with different aspects and representations of a systematic counting problem related to binary choices. We focused on students' written expressions of memories of the course. The reason for this is that long-term declarative memories are mediated by protein formation, following gene expression, to stimulate novel neuronal connections (Squire and Kandell, 2000). The relevance of this neurological fact is that long-term memory formation is an energetic, committed process for an individual. Long-term memories - certainly those sustained over two months - are therefore a good indicator of what a student values.

Students worked on the problems in groups. After completing the sequence of problems, they explained connections as a homework exercise. We asked them to write reflectively after each of the combinatorial problem sessions, and re-writes were encouraged. Opportunities for making connections with their earlier work were provided during the semester in questions on three group and two individual exams students hadn't seen previously. Students also wrote mid-term and end-of course self-evaluations. Twenty-two students began the course and nineteen completed. Their writings provided us with a great deal of data for analysis. We present a preliminary analysis of some of that data by focussing on the development of one student: Jennifer. For fuller details of these and other student's written statements see: www.soton.ac.uk/~plr199/algebra.html

Results and analysis

We begin by placing Jennifer in the class in terms of her initial and final test scores. Table 1 shows the initial and final test results for Jennifer and two other selected students as well as the *shift* statistic, defined, as follows:

$$\textit{shift} = (\textit{final test }\% - \textit{initial test }\%)/(100\textit{-initial test }\%)$$

We interpret *shift* as how much a student has moved form their initial test result to their final test result. Of course, a student who has a relatively high initial test score does not have as much room for improvement as a student with a low initial test score. The other two students - Allison and Rebecca - we use to compare with Jennifer were chosen to be representative of students with a middle and low shift value, respectively.

(a)

Student	Initial	Final	Shift
Jennifer	63	92	0.78
Allison	13	64	0.59
Rebecca	63	75	0.32

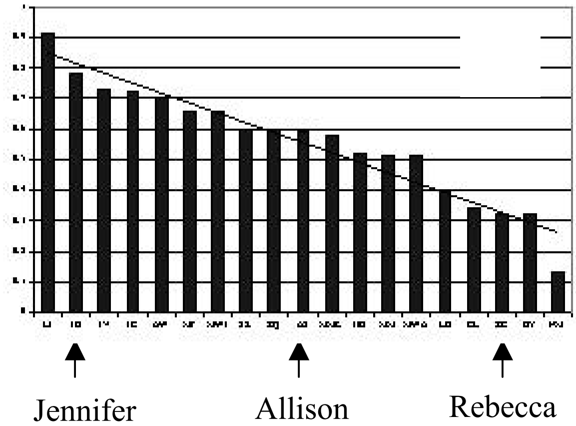

Table 1: (a) Initial and final test scores for the 3 students. (b) Distribution of the *shift* statistic for the whole class. Note that *shift* scales linearly with rank: $r^2 = 0.93$

<u>Building towers and the grade 4 video clip</u>

Jennifer focussed on building towers of height 4 by swapping colors, a strategy used commonly by pre-service elementary teachers. They refer to this as “being systematic”: when challenged as to why they have all possible towers they commonly reply that they systematically swapped colors. Every group in the current study used this strategy.

“We began with combinations of 3 red and 1 blue. We then altered the placing of the one blue cube which resulted in 4 different combinations. Second, we alternated the colors and made 4 combinations of 3 blue and 1 red cube, which gave us 4 combinations. Third, we made two groups of cubes, alternating between red and blue with a total of 4 in each tower. … After building several towers, we realized that each tower had an opposite.”

Working in her group she came up with a formula based on opposites that did not, however, extend to the case of towers of height 5

“… if you multiplied the number of blocks in a tower by the number of options, you would have the number of combinations possible. Then you would double that number because each combination has an opposite: Ex: 4 high x 2 = 8; 8 x 2 = 16. However, this formula does not hold up for 5-high towers.”

Jennifer realized that the number of towers had something to do with doubling. She gave no reason or explanation, other than empirical evidence, as to why this might be so:

"It appears that you are doubling the possibilities when adding a cube to each tower. The formula we discussed in class appears to make sense where when you add a level to each tower, the possibilities double. 1 level = 2 possible; 2 levels = 4 possible; 3 high = 8 possible; 4 high = 16 possible; 5 high = 32 possible, etc."

She recognized some system in the grade 4 students' explanations, but she did not relate this precisely to her group's approach and she was – mistakenly – under the impression that she solved the problem as Stephanie did in the video clip:

"… I now realize that there are several patterns and options to solving this type of problem. Each of us in class recognized patterns, but not one formula could clearly explain or define our cases. When building my towers, I looked for patterns similar to the way Stephanie did. My pattern differed in that I grouped my towers by building towers with one blue, and then built those with one red. I continually built towers and followed them with their opposites."

After building towers and watching the grade 4 students argue why they found all towers of a given height Jennifer stuck to her belief that building opposites is a key to systematically building towers. From our perspective the pre-service teachers were uniformly unsystematic in their attempts to build towers and to explain why they had built all possible and not repeated any. Jennifer was not alone in expressing the sentiment that since they did not know how to tackle this problem "mathematically" they would approach it through common sense:

"Instead of looking at it as a math problem, I was looking at it as a building exercise. I first attempted the problem by guessing and testing. Tony and I first attempted the problem of four high by creating combinations of four that would design an obvious pattern."

Seeing and valuing connections

Jennifer did not immediately see connections between the problems set in the first 3 weeks of the course. Some of those connections she learned about through class discussions, following insights of other students. At the time of writing these reflections, however, she was able to articulate a common vision of "algebra" in all the problems. Not the algebra she initially imagined, namely 2^n as the formula for the number of towers of height n, but algebra based on multiple interpretations. The algebraic expansion worksheet showed $(a+b)^2 = a^2 + 2ab + b^2$ and asked students to similarly expand $(a+b)^3$ and $(a+b)^4$. Only one student in the class (not Jennifer) could do this problem. The tunnels problem was to figure how many ways there are to run through a series of 4 tunnels if each could be black or white.

"The algebraic expansion worksheet threw everyone off at first. We really were not sure how it related to the first three exercises. What we did not see was that the "towers" were actually algebraic expansion. If two different color cubes can make 16 different towers four high, how do you mathematically write this out? Answer: 2^4. Let's say that the cubes are the colors black and white...then the formula would be written $(w + b)^2$. This is how exercise one and exercise four relate. Tunnel travel led us to a new discovery. A student can look at the problem and sketch the different possibilities just as he/she did with the tower building exercise or they can apply the algebraic expansion $(w + b)^4$ where w = white; b = black, 4 = number of tunnels and $(w+b)^4 = w^4 + 4w^3b + 6w^2b^2 + 4wb^3 + b^4$."

Jennifer was able to use her insights to help her solve two further problems: (a) how many pizzas can be made from 8 toppings, and (b) how many towers of a given height can be built using at most 3 colors?

(a) "This situation is similar to the former exercises of building towers, the committee vote exercise, the grid walk problem, and the tunnel exercise with Mork. The "'with" or "without" question resembles the two color combination for the tower building exercises, the "yes" or "no" vote of the committee members, the "up" or "right" direction for the grid walk and the alternating pattern of the tunnels. ... The "with" or "without" strongly indicated powers of two as in the tower exercise. We extrapolated this to apply to the Pascalini's dilemma, so we figured that $2^8 = 256$, therefore, there are 256 combinations for pizza made of 8 toppings."

(b) "$3^4 = 81$, n = # of cubes high; x = # of color choices; formula: x^n There are 81 towers that can be built. This is similar to white and black (2 colors) as we did in class. In class we built towers of four and five high with the combination of two colors (two choices).We also worked on committee votes of YES or NO (two choices). … This problem also relates to the pizza problem. Instead of 8 choices of toppings you would use 3.They differ in their number of choices."

Jennifer valued the insights she gained by seeing connections. At the conclusion of the course she articulated a different vision of mathematics:

"When I joined this class in August, I thought of math as a series of formulas, each of which should be followed in order to find an "answer". It was working on the tower building investigation and traveling through tunnels that I discovered how each relates… My original approach to the tower building revealed that instead of looking at the small picture (i.e., What do I really have in front of me? What is it I'm trying to solve?), I just dove in expecting multiple patterns. When our class finally concluded that the towers, tunnels, grids and Pascal's Triangle were all about "choices", everything seemed to fall into place. … my perspective of mathematics changed over this semester. The changes occurred due to learning that my mathematical understanding was instrumental and not relational. I had to re-learn basic math in order to eventually teach it to children."

Memory types

Table 2, below shows the number of different types of memory statements made by the 7 students for whom we currently have transcribed data. For these students the shift statistic correlates moderately well with the total number of semantic statements (semantic action + semantic belief + semantic label; $r^2 = 0.77$, $p < 0.0001$). Whether this correlation holds more generally we do not yet know.

Memory type	Jennifer	Jim	Amy	Shannon	Allison	Michelle	Rebecca
Episodic	80	14	26	17	49	25	36
Semantic action	90	78	127	102	85	83	46
Semantic belief	55	15	2	6	24	25	22
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

For this group of students the number of semantic labels correlates almost as well with the shift statistic: $r^2 = 0.71$, $p < 0.02$. Recall that semantic labels are memories of conventional facts: their mathematical depth is negligible. Some examples given by Jennifer are listed below. Bear in mind that these statements may also contain connotations of other types of memory (episodic, for example).

- "All problems assigned present two choices <u>or a binomial</u>." Jennifer illustrates here that she knows the meaning of the conventional term '*binomial*'.

- "...place value as we know it today is <u>also known as</u> the Hindu-Arabic numeration system." She shows that she knows another conventional name for the place value system.

- A number is considered <u>a factor</u> of another when it can divide that number without a remainder. This shows that Jennifer has a meaning for the term '*factor*s.

- We used <u>proof by exhaustion</u> when working with a finite set of numbers; <u>listing all of the possible cases</u>. Here she is able to explain '*proof by exhaustion*' in other terms.

Examples such as these are significant: they show that Jennifer is coming to terms with the language of mathematics, that she is able to interpret and use conventional mathematical terms. They show, we believe, that she has accepted her entry into the mathematical community and now feels part of it; perhaps a small part, but a part nonetheless. Compare this with part of her final written reflection, at the end of the course:

"One issue I have always had problems with in mathematics is definitions. I can physically work through a math problem, but to try to put my efforts into words is a challenge. *Definitions in mathematics play a vital role in building a solid base of one's knowledge and abilities*. It is the basis of your criteria. The mistakes our class made in defining even numbers were (a) we assumed that we were working in base ten and (b) we tried to define even numbers by using the word "even."! If definitions are the base of our mathematical foundation, then algorithms are the brick in the bridge of our mathematical path...an algorithm is a systematic procedure that one follows to find the answer to a computation."(Our italics).

Conclusion

Jennifer made a significant change in her understanding of mathematics. She began, as many pre-service elementary teachers do, expecting to apply formulas and get correct answers in order to be "mathematical". By the end of the 16 week course Jennifer expressed a different view of mathematics: one that she herself characterized as more relational. She established manifold long-term memories of mathematics: factual, episodic, and relating to the conventional use of mathematical language. Her tests score improved from not satisfactory to excellent.

How important were the experiences of the first 3 weeks in setting Jennifer on a path to seeing and valuing connections, and establishing lasting useful mathematical memories? In her words:

> "I feel this was the most productive experience I have ever had in my educational career. I deeply feel that I will be a better educator because of it."

The beautiful phrase: "If definitions are the base of our mathematical foundation, then algorithms are the brick in the bridge of our mathematical path," is a sharp illustration of how well she assimilated the mathematical experiences of the semester, and how these assisted in deepening her understanding of and competence in mathematics.

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