SECONDARY SCHOOL PUPILS' IMPROPER PROPORTIONAL REASONING: AN IN-DEPTH STUDY OF THE NATURE AND PERSISTENCE OF PUPILS' ERRORS

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Abstract. *This paper describes an in-depth investigation (through individual interviews) of the problem-solving processes of 12–16-year old pupils who improperly apply linear models when solving problems involving lengths, areas and volumes of similar plane figures and solids. The results showed that the linear model was used in a spontaneous and self-confident way by almost all pupils, and that these pupils were almost insensible to the confrontation with conflicting data. Furthermore, it was shown that the poor results were due to a bad understanding of the principles governing the enlargement (or reduction) of geometrical figures, to pupils' inadaptive beliefs about and attitudes towards mathematical problem-solving, and to their poor use of heuristics and of metacognitive strategies.*

1. Theoretical and empirical background

Pupils' tendency to apply proportional or linear reasoning in non-proportional problem situations was already exemplarily described in several mathematical domains, such as elementary arithmetic, algebra, probability and geometry. Best-known in the domain of geometry and measurement is pupils' improper application of linearity in problems about the relationships between the lengths and the area and/or volume of similarly enlarged or reduced figures. In the NCTM Standards, for instance, it is stated that "… most students in grades 5–8 incorrectly believe that if the sides of a figure are doubled to produce a similar figure, the area and volume also will be doubled." (NCTM, 1989, pp. 114–115). Gaining insight in the quadratic, respectively, cubic growth rates of areas and volumes, appears to be a slow and difficult process, and, therefore, it deserves our close attention, both from a phenomenological and a didactical point of view. According to Freudenthal (1983, p. 401), "this principle deserves (…) priority above algorithmic computations and applications of formulae because it deepens the insight and the rich context in the naive, scientific and social reality where it operates."

Recently, several studies have shown that – in the context of enlargements or reductions of plane figures and solids – the "illusion of proportionality" (or linearity) is a widespread and almost irresistible tendency among pupils (see, e.g., De Bock, Verschaffel & Janssens, 1998; De Bock, Verschaffel, Janssens & Claes, 2000). In these studies, large groups of 12–16-year old pupils were administered (under different experimental conditions) a written test consisting of proportional and non-proportional word problems about lengths, areas and/or volumes of different types of regular and irregular figures. The majority of the pupils in these studies failed on the non-proportional problems because of their alarmingly strong tendency to apply proportional reasoning "everywhere". Even with considerable support (such as the provision of drawings, of metacognitive stimuli in the form of an introductory item accompanied with both a correct and an incorrect solution, or embedding the problems in an authentic problem context), only very few pupils appeared to make the shift to the correct non-proportional reasoning.

Despite our rather extensive knowledge about the phenomenon of the "illusion of linearity" in this domain, the research method used so far, namely administering a collective test of large groups of pupils under different experimental conditions, did not yield adequate information on the *problem-solving processes* underlying improper proportional responses. This is one of the main reasons that the data could not provide a satisfying answer to the question *why and how* so many pupils fell into the "proportionality trap". Therefore, we made a shift in our methodology by having exploratory in-depth interviews with individual pupils who fall into the "proportionality trap".

2. Method

To obtain in-depth information about pupils' problem-solving processes, semi-standardized individual interviews were performed with eighteen 12–13-year olds and twenty-three 15–16-year olds. During **Phase 1** of the interview, each pupil had to solve one non-proportional word problem from a set of problems involving irregular plane figures or solids. Previous research had shown that the vast majority of pupils from these age groups solves these problems in a proportional way. Below, we give two examples of such problems.

Problem with irregular plane figure	**Problem with irregular solid**
A publicity painter needs 5 ml paint to make a drawing of a 40 cm high Santa Claus on a store window. How much paint does he need to make a drawing of a Santa Claus with the same shape, but a height of 120 cm?	In a perfume store, bottles of "Eau Fraîche" are sold. The bottles have a height of 8 cm and contain 10 cl perfume. In the store window, a publicity bottle is shown with the same shape, but enlarged, and also filled with "Eau Fraîche". This bottle has a height of 24 cm. How much perfume will this large bottle contain?

By asking each pupil to "think aloud" (Ginsburg, Kossan, Schwartz & Swanson, 1982) while solving the problem, we could retrieve (parts of) the solution process. The interviewer also asked well-specified questions to make (parts of) the reasoning process more transparent. The pupil was asked how (s)he exactly calculated the answer, why (s)he thought his or her answer was correct, and how sure (s)he was about the correctness (using a five-point scale from "certainly wrong" to "certainly correct").

In **Phase 2**, we tried to raise a first, weak form of cognitive conflict by confronting the proportional reasoning pupils with a fictitious frequency table of the answers given by a group of peers. There were two major answer categories in this fictitious frequency table: 41% of the peers gave an incorrect, linear answer (i.e., respectively 15 ml and 30 cl in the examples given above), but another 41% gave the correct, non-linear answer (i.e., respectively, 45 ml and 270 cl). Then, the pupil was asked to re-evaluate his or her own initial answer.

If the pupil dit not change his or her answer, a stronger conflict was elicited in **Phase 3** by giving the argumentation of a fictitious peer from the 41% who answered the problem correctly (e.g. in the first example: "one pupil told me that if the Santa Claus becomes three times as high but keeps the same shape, not only his height is multiplied by 3, also the *width* has to be multiplied by 3, so that you have to multiply by 9"). Again, the pupil was asked to re-evaluate his or her own answer. Pupils who had exchanged their original linear answer for the correct non-linear one in Phase 2 or 3, were interrogated at the end of the interview about the origin of their initial wrong answer.

3. Results

The table below presents the (cumulative) number of pupils who chose the correct answer in each phase. The tendency to give a linear answer was strongly

Age group	*N*	Phase 1	Phase 2	Phase 3
12–13-year	18	0	0	3
15–16-year	23	0	1	7

present in both age groups. All pupils spontaneously gave the wrong linear answer. Even in Phase 2, *only one* pupil realised that his original answer was wrong. In Phase 3, finally, only another nine pupils changed their answer to the correct one, so that in sum only ten pupils (24%) accepted the non-linear answer as the correct one. Most of these pupils belonged to the older age group. We now will look at each phase in more detail.

Phase 1: Solving the word problem

The mean response time after the first confrontation with the word problem was one minute. During this minute, none of the pupils made a drawing or any other kind of external representation involving more than the three given numbers. They mainly read and re-read the problem, wrote down the numerical data, and performed calculations on these numbers to obtain the answer. Hereby, thirty-seven pupils calculated the ratio of the given lengths of the two figures, and thought that the ratios of the *areas* or the *volumes* (or their indirect parameters:

paint or content) should be the same. Four pupils applied formally the so-called "rule of three": first calculating "the amount of paint (or the content of the bottle) needed for 1 cm", and then multiplying the result with the height of the large figure. The use of one of these strategies led all pupils to an improper linear answer, as expected on the basis of earlier studies by De Bock et al. (1998, 2000).

Most pupils had great difficulties explaining why their method was the correct one. After insisting, most pupils (a) referred to the fact that their solution is the most logical one, (b) explained that they solved the problem as they had learned to do at school or (c) simply repeated how they carried out their computations. These superficial answers seem to indicate that pupils do not spontaneously check whether a model is applicable in a given situation or not. Pupils do not seem to have clear arguments justifying its use, or do not realise there are other possible models too. It seems that the linear model is used in an implicit, routine and mindless way.

Despite this difficulty in justifying the correctness of their answers, most pupils were very or quite sure they *were* correct. Twenty-one pupils said their answer was "certainly correct", fifteen said it was "probably correct" and only five "had no idea". The alternatives "probably wrong" and "certainly wrong" were never chosen. If pupils had reasons to doubt about their solutions, these reasons were mainly superficial and general (e.g., "word problems are difficult", "I am not good in math", "I might have made a calculation mistake"). None of the argumentations expressed any doubt about the correctness of the applied model.

Phase 2: Reactions to a weak form of cognitive conflict

For the majority of the pupils, the confrontation with the frequency table really induced a cognitive conflict: they started wondering where the alternative frequently-chosen answer could come from. However, the search for its origin was done – again – in a very superficial way. The single pupil who made a drawing of the small and large figure in this phase, chose the correct answer, but all other pupils limited themselves to "randomly" trying out several combinations with the basic arithmetical operations (+, –, ×, :) and the given numbers to obtain the other answer, regardless of their contextual meaning (e.g., some pupils added 40 cm and 5 cl to obtain the alternative answer 45). Only in very rare cases, we observed pupils re-evaluating their own strategy, searching for the meaning of the alternative answer or representing the problem. The superficial strategies used by most pupils were not very helpful. All pupils (except for one of the oldest group) persisted in their original linear solution.

A further deduction from the fact that the pupils did not immediately change their incorrect answer is that the mistake was not simply caused by an underestimation of the difficulty level of the word problem. In the latter case, the correct solution (strategy) would have been a sufficient scaffold to choose the correct answer.

Phase 3: Reactions to a strong form of cognitive conflict

In the third phase, another nine pupils (three of the youngest and six of the oldest age group) changed their incorrect answer into the correct one. Apparently, the given argumentation provided them the insight that, to maintain the same shape a figure has to be enlarged in all dimensions. The nine pupils who changed their answer were asked to explain why they originally gave the wrong linear solution. Their explanations referred to the fact that (1) they did not solve the problem in a reflective manner but immediately (a pupil called it "instinctively") started calculating or (2) they had made no real mental representation of the problem, but just were fixating on the formulation of the word problem (which only referred to the height).

The thirty-one pupils who still chose to withhold their original answer after the argumentation of the fictitious peer, made serious efforts to justify their choice. Their reactions were diverse, but can be grouped into three different categories (each covering about one third of the reactions).

A first group of pupils justified their answer by referring to the implicit rules for solving school mathematics word problems. Often a simplistic view was shown, assuming that all word problems can be solved using simple mathematical calculations, and that real-world knowledge should not be involved in the solution process. Some examples are: "I think you don't have to use such a complex solution to solve a word problem", "you have to calculate only with the data that are given", "if they wanted you to calculate the width too, they should have explicated that in the problem statement".

A second group of pupils violated the mathematical principles relevant to this problem. The first principle that is ignored or not understood is that *if a figure is enlarged (or reduced) but maintains its shape, all dimensions (height, width and depth) are enlarged (or reduced) by the same factor.* Some pupils reacted that "if you only know the height, you can't know the width", "height and with are not that much related to each other", or "the width and depth will change too, but you cannot know how much", and used these arguments to simply *ignore* that the width (and depth) also change and determine the solution. The second principle pupils seemed to struggle with is that *if the linear measurements of a figure are enlarged (or reduced) by a factor k, its area is enlarged (or reduced) by k^2 (and its volume by k^3).* Typical examples are: "the width is already incorporated in the small one, so it isn't necessary to calculate it in for the big one again", "the Santa Claus is not a spatial object where you have to calculate the volume, it is flat; consequently, only the height plays a role", "I think 270 cl actually is quite a lot".

After being confronted with the argumentation for the correct solution, a third group of pupils tried to give an alternative interpretation to the word problem. As already said earlier, most pupils did not construct any mental representation about the problem before the third stage of the interview. Once they were confronted with an argumentation rejecting their answer, many pupils looked for an alternative interpretation in which their incorrect solution still would make sense: “the figure is stretched, only the height changes and the rest remains the same”, “if you make it higher, that doesn’t mean it becomes wider”, “it says *with the same shape*, so it only is a higher one, not wider or deeper”. We cannot absolutely exclude that some pupils may have held this alternative interpretation of the problem situation already earlier during the interview, but our data indicate that they form only a small minority. Moreover, when the interviewer confronted pupils with the concrete consequences of their alternative interpretation (by means of a drawing or a description) most pupils admitted the strangeness of it (e.g. a very high but narrow Santa Claus, a copy of a perfume bottle that isn’t really a copy with the same design). Analogous defensive reactions of students, who try to withhold an original erroneous answer, even if they realise this answer is untenable, have also been observed in other studies. E.g. Verschaffel, De Corte and Vierstraete report that pupils “tirelessly came up with contextual considerations in which their unrealistic response would still hold. (…) These far-fetched context-based considerations were (…) only made during the whole class discussions by pupils who became aware their group had answered the problem in an uncritical, stereotyped manner” (Verschaffel, De Corte & Vierstraete, 1997, p. 595).

4. Conclusions and discussion

The interviews provided a lot of information about the actual process of problem-solving from pupils falling into the “proportionality trap” and the mechanisms behind it.

First of all, several possible causes were *rejected* by the research data. From the collective tests (De Bock et al., 1998, 2000), it was impossible to find out (1) whether pupils gave the wrong linear answer reluctantly by lack of a better alternative or (2) whether pupils gave the wrong linear answer because of the expectation that the test would contain routine tasks only. We think both possible explanations can be refuted, the first one because most pupils declared to be sure about their initial incorrect answer, the second one since the confrontation with the correct solution (even with an accompanying explanation) was not sufficient to make them change their answer.

Second, there is a parallel between the problem-solving processes observed in the first phase of the interview and the “intuitive rules” theory developed by Tirosh and Stavy (1999). These authors claim that there are some common, intuitive rules that come in action when students solve problems in mathematics and sciences. These rules appear to be self-evident (i.e. true without a need for further justification), receive great confidence, and are persistent despite formal learning. All these characteristics seem to apply to the incorrect reasonings of the interviewed pupils too. More specifically, Tirosh and Stavy have distinguished two schemes that (whether correct or not) frequently come in action in an intuitive way: "Same A – same B" (while in fact $A_1 = A_2$, but sometimes $B_1 \neq B_2$) and "More A – more B" (while in fact $A_1 < A_2$, but sometimes $B_1 \geq B_2$). In our case, pupils seem to apply first the "More A – more B" rule (which is correct for this problem: the *more* height, the *more* area/volume). The mistake happens, however, during the intuitive quantification when applying the "Same A – same B" rule: pupils reason that the figures share the *same* shape, so all measures (length, area and/or volume) enlarge by the *same* factor. This is illustrated in the following quotations: "I knew it was enlarged, but not how much, so I calculated 180 : 60 and then I knew the multiplier", "because the picture becomes larger, you need more paint, so you have to multiply by 3", "it has the same shape, but is enlarged, so you have to multiply the content by the same number". The specific connection between the "illusion of linearity" and the "intuitive rules" theory certainly needs further investigation.

Third, we found that many pupils (as well the younger as the older ones) struggled with the principles behind the enlargement of figures/objects and the relationship between length and area/volume. They had already learned these principles in school, but nevertheless they seemed to have a bad or weak understanding of them, or at least they were not able to apply them correctly. Further research should determine whether the struggling really is a *cause* of the mistakes or rather that pupils *post hoc* violate the principles in a self-defensive attempt to save their original answer. The same goes for the alternative interpretations some pupils gave to the word problem after they heard the interpretation for the correct answer. Some pupils may have had it in advance, but most of them had made no real representation of the problem until the third phase.

We want to argue that the described findings are also related to the fact (supported by a vast amount of research, see, e.g., Verschaffel, Greer & De Corte, 2000; Wyndhamn & Säljö, 1997) that many pupils have inadaptive beliefs and attitudes towards mathematical problem-solving, and have a poor use of heuristics and metacognitive strategies. The intuitive reasoning in the first phase and the small impact of the conflict in the second and third phase only could occur because the pupils approached the word problem in a superficial way, only looking at the numbers without making a clear and realistic problem representation, assuming that all application problems can be solved with some simple mathematical operations on the given numbers, and without any control of the correctness of their answer afterwards. Further research will have to evidence if stimulating realistic modelling in pupils has a beneficial impact on overcoming the "illusion of linearity'[1].

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[1] We found some evidence for this supposition since we asked several pupils – after the interviews were finished – to make a drawing for the word problem. At that moment, nearly all of them then really "discovered" that a figure with the same shape must enlarge in all dimensions, so that the area/volume enlarge by a larger factor.

A final remark concerns the educational value of the cognitive conflict to enhance pupils' metacognitive awareness or to provoke cognitive change (see, e.g., Forman & Cazden, 1985). Our experiences show that it is very difficult to induce an effective cognitive conflict in pupils if they have no minimum metacognitive awareness about their problem solving process.

Note

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