PREREQUISITES FOR THE UNDERSTANDING OF PROOFS IN THE GEOMETRY CLASSROOM

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Geometry is a field, which is a good starting point to teach and learn mathematical argumentation, to explore mathematical concepts, to fill the gap between every day life and mathematics, and to value mathematics as a part of human culture. Accordingly, geometrical competence has to be regarded as an important prerequisite of understanding mathematics. Our research aims at identifying aspects of geometrical competence. Based on empirical data from upper secondary school, we argue that high level geometrical competence is specifically influenced by spatial ability, declarative knowledge, and methodological knowledge.

1 Theoretical framework

The role of geometry in the German mathematics classrooms has changed considerably in the last decades. Despite the fact that geometry is part of the regular curriculum, it is often regarded by teachers as a less important topic. While this is well known for primary schools there is little research concerning secondary schools. In contrast, mathematicians as well as mathematics educators agree that geometry should be an important part of mathematics education (Lehrer & Kazan, 1998).

1.1 Understanding of proof and geometrical competence

Proofs and mathematical argumentations play an important role in the geometry classroom. Particularly in recent years, many researchers contributed to the description of the role of proofs for the development of mathematical competence. Authors like Hanna and Jahnke (1993), Hersh (1993), Moore (1994), Hoyles, (1997), Harel and Sowder (1998) have pointed out that in both, mathematical research and school instruction, proving spans a broad range of formal and informal arguments and that being able to understand or generate such proofs is an essential component of mathematical competence. In constructivist-oriented mathematics instruction, the critical exchange of arguments and elements of proof is accorded new significance.

Some empirical surveys of North American high school students (Senk, 1985; Usiskin, 1987) and pre-service teachers (Martin & Harel, 1989) have revealed wide gaps in respondents' understanding of proofs. Healy and Hoyles (1998) made a significant contribution to the field with their recent systematic investigation of students' understanding of proofs, ability to construct proofs, and views on the role of proof. Their empirical study was conducted in various types of schools spread across England and Wales. Almost 2,500 tenth grade students, nearly all of them from the top mathematics set, participated in the study. The results show that even these highattaining students had great difficulties in generating proofs. The students were far from proficient in constructing mathematical proofs, and were more likely to rely on empirical verification. However, most of them were well aware that once a statement has been proved it holds for all cases within its domain of validity. Moreover, they were frequently able to recognise a correct proof, though their choices were influenced by factors other than correctness, such as perceived teacher preference. Students considered that their teachers would be more likely to accept formally-presented proofs, though they were personally more likely to construct proofs which they deemed to have an explanatory character. In all domains, students with higher levels of mathematical competence outperformed less able students.

Recent research in the field of cognitive psychology has focused on the cognitive processes and specific knowledge structures needed to solve geometry problems. Geometrical reasoning has been investigated in detail by setting various types of test items, observing students and experts working on the items by means of think-aloud protocols, and computer simulation of thought processes. Koedinger and Anderson (1990) emphasize that when experts construct geometrical proofs, they do not merely retrieve definitions, axioms and theorems from the memory and combine these to make logical deductions. On the contrary, they skip details of the proving process, and outline their argumentation in broad terms, taking a constructivist approach. They use visual models, in which they are able to "see" properties and connections, and "pragmatic reasoning schemas" such as set patterns for individual steps in the proving process. As stated by Koedinger (1998), this indicates that geometrical competence is not merely a question of talent, but of specific skills and knowledge.

Geometrical competence does require specific knowledge; it is based on general psychological mechanisms that are central to other domains of mathematics as well as to thinking and problem solving in general. Where geometrical knowledge is concerned – as shown by the cognitive psychological research cited above – a distinction must be drawn between declarative knowledge and methodological knowledge. Moreover, metacognition can be identified as a general mechanism. Various components of general intelligence are relevant; according to Clements and Battista (1992), spatial reasoning is of particular importance for geometrical competence. They suggest that geometrical competence is largely dependent on spatial visualisation skills, but that spatial ability can also be enhanced by exposure to geometry.

1.2 Research Questions

The present study integrates the lines of research described above. Geometrical competence and its cognitive prerequisites are investigated by reference to TIMSS items, with a particular focus on respondents' understanding of proof. The research questions to be addressed in this paper are as follows:

- Is geometrical competence dependent on students' declarative knowledge, methodological knowledge, metacognitive competences and spatial abilities? What is the relative importance of each of these factors in explaining interindividual differences in geometrical competence?

- Can Healy and Hoyles' (1998) findings on the connection between students' ability to construct proofs and their views of the role of proof be replicated in another instructional culture, namely the German mathematics classroom? Is is possible to identify prerequisites for the correct understanding of proof?

2. Design of the Study

In the present study, geometrical competence was assessed using nine TIMSS items from the so-called *advanced mathematics* domain in the upper secondary level. The sample consisted of 81 students from German schools (48 female), 59 of them attending a regular mathematics course and 22 an advanced course, who tackled selected TIMSS items as well as additional tests (metacognitive assessment, declarative knowledge, understanding of proof, spatial reasoning), and were videotaped as they worked on the geometry items using the think-aloud method. Based on analyses of the entire set of TIMSS items, the nine items were allocated to two proficiency levels (Klieme, 2000). The lower level items were answered correctly by more than half of the students in the international TIMSS population, the three higher level items by one third or less of the students.

The first prerequisite of geometrical competence to be measured independent of the TIMSS items was declarative geometrical knowledge. Linking up with earlier work on the conceptual knowledge required for mathematical problem-solving tasks (Klieme, 1989; Reiss, 1999), we chose a central concept of school geometry, namely "congruence", for the evaluation of students' declarative knowledge. Students were asked to give a definition, an example, a visual or graphic portrayal of the word "congruent", and to name a mathematical theorem in which the concept features. The students' open-ended answers were coded according to a specially developed category system; one point could be earned for each of the four aspects.

Methodological knowledge was assessed using an item from Healy and Hoyles' (1998) proof questionnaire. The item dealt with the question of whether a given triangle could be proved to be isosceles. Students were presented with a correct formal proof, a correct narrative proof and two incorrect arguments. They were then asked to assess the correctness and generality of each of the four arguments.

As a measure of general intellectual abilities, particularly of spatial ability, an instrument which is well-known in Germany – Stumpf and Fay's so-called *Schlauchfiguren-Test* – was administered. *Schlauchfiguren* presents different views of complex tubular figures, which have to be judged with respect to the specific point of view. This kind of task has been shown to predict mathematical problem-solving competence (Klieme, 1989). Validiation studies have shown that the test calls for both spatial ability and deductive reasoning. It is therefore a suitable instrument to capture those aspects of general intellectual ability which are cognitive prerequisites of geometrical competence.

Scale	Number of items	Theoretical maximum	Mean	Standard deviation	Reliability (Cronbach`s α)
Geometrical competence	9	14	6,90	2,60	.53
Level I/II	6	8	5,67	1,77	.49
Level III/IV	3	6	1,23	1,49	.35
Methodological knowledge	8	8	3,52	2,14	.68
Declarative knowl-edge	4	4	1,49	0,95	.47
Spatial reasoning	21	21	13,17	3,95	.76

Table 1: Distributional parameters and reliability of scales

Table 1 provides an overview of the various scales, the number of items in each, and the most important distributional parameters. The estimated reliability (Cronbach's α) for our sample is also shown. Because of the limited test time, we were only able to administer short tests, particularly for conceptual knowledge and the two levels of geometrical competence, the scales for which consisted of only three to six items. The estimated reliability for these scales is correspondingly low. If extrapolated to tests of the standard 20-item length, however, an acceptable α of between .71 and .84 emerges in all cases. This suggests that the constructs behind the indicators represent dimensions of ability which may be regarded as reliable. If significant correlations are found in spite of the technical limitations and the associated lack of reliability of our instruments, it can be assumed that these are valid findings with relevant effect sizes.

3. Results

3.1 Descriptive findings

In the following we will report on certain aspects of our findings. Figure 1 shows the percentage of students providing correct solutions for each of the nine geometry items administered in our study, along with the corresponding results for the international TIMSS sample and the German national TIMSS sample.

The results show a remarkably high level of correspondence across the three samples, both in the average achievement level and in the performance in each of the nine questions. Averaged out across the nine items, 53% of the students in our sample provided correct solutions, compared to 51% in the international sample and 47% in the German sample. Across the nine items, the correlations between the performance in our sample on the one hand and in the representative German and international TIMSS samples on the other amount to .97 and .89 respectively; both of these correlations are highly significant. The relative strengths and weaknesses of the German students are thus also reflected in our small sample.

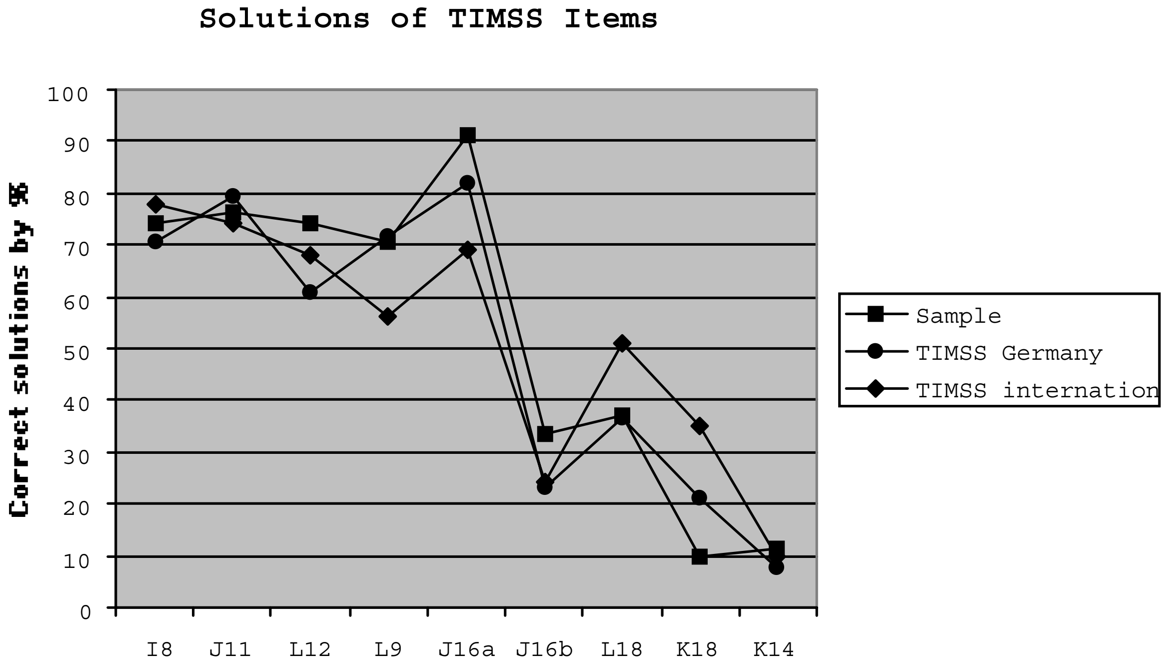

Figure 1: Solutions of TIMSS Items

In view of the observation that very few students (20% and 35% of the representative German and international TIMSS samples respectively) were able to construct correct Euclidean geometry proofs, we also expected the levels of performance to be rather unsatisfactory in students' understanding of proof and their views of the role of proof (taken from Healy and Hoyles, 1998). Interestingly, our students also found it much easier to judge given proofs than to construct their own proofs. This confirms the results of Healy and Hoyles (1998).

Proof / feature	Relative frequency (in percent)	Corrected item total-correlation
Correct formal proof		
/ correct	57	.49
/ general	57	.45
Correct narrative proof		
/ correct	42	.45
/ general	30	.38
Empirical argument		
/ incorrect	46	.13
/ not generalizable	60	.26
Formal, circular argument		
/ incorrect	33	.39
/ not generalizable	27	.40

Table 2: Components of methodological knowledge (understanding proof)

As shown in Table 2, 57% of our respondents recognised the correct formal proof (using congruence) to be correct, and the same proportion of participants correctly appreciated its generality. A similar proportion of respondents recognised a purely exemplary, empirical argumentation to be incorrect: 46% said that the argument was incorrect, and 60% recognised that it was not generalisable. However, the low item-total correlations of these two answers (see right-hand column of Table 2) showed that even students with a low general understanding of proof were aware the purely empirical argument was incorrect and not generalisable. Our findings on the respondents' declarative knowledge, are also less than satisfactory from the standpoint of mathematics education. When asked to describe the concept of "congruence", 82% of respondents were able to illustrate the concept in a sketch, most of them drawing congruent triangles. Less than half of the respondents were able to give an example of congruence, however. Only about one in ten of the students mastered the mathematically central components of the concept, i.e., were able to provide a definition of the concept and name a mathematical theorem in which it features (e.g., a theorem of triangular congruence).

3.2 Explaining Geometrical Competence

We will now explore the relations between the scales for geometrical competence, methodological knowledge and declarative knowledge. Table 3 shows the intercorrelations, calculated as rank-correlation coefficients (Kendall's tau), on which this discussion is based. In addition to the mathematical dimensions of competence and knowledge, the two general psychological predictors – metacognition and spatial reasoning – are also included in the table.

Scale	(2)	(3)	(4)	(5)	(6)	(7)
(1) Geometrical Competence	.76**	.62***	.20*	.24**	.24**	.33***
(2) Geometry, level I/II		.27**	.10	.18*	.21*	.23**
(3) Geometry, level III/IV			.22**	.23**	.21*	.37***
(4) Methodological knowledge				-.01	.05	.12
(5) Declarative knowledge					-.02	.09
(6) Metacognition						.09
(7) Spatial reasoning						

Table 3: Intercorrelations of scales (Kendall's tau – b) *) $p< .05$ **) $p< .01$ ***) $p< .001$

The most important finding is that all four predictors exhibit significant correlations with geometrical competence. This lends support to our basic hypothesis that geometrical competence is dependent on methodological knowledge, declarative knowledge, metacognition, and spatial reasoning. The correlation matrix does not actually allow such causal interpretations to be made; but interpreting the results in the light of other research on geometrical knowledge (Reiss & Abel, 1999; Reiss & Thomas, to appear) makes it plausible to assume that scales (4) to (7) tap the *prerequisites*, and scales (1) to (3), the *results* of development of geometrical competence.

As expected, stronger correlations with the predictors emerge at the higher levels of geometrical competence (items on TIMSS proficiency levels III and IV) than at the lower levels of geometrical competence (levels I and II). Understanding of proof is a vital ingredient at the higher levels of competence, but is irrelevant to performance in the easier TIMSS geometry items. This confirms our assumption that the TIMSS proficiency levels really do reflect different standards of (geometrical) competence.

4. Discussion

In our study, students were presented not only with TIMSS geometry items, but with a number of additional test components. This enabled us to assess various types of mathematics-related skills and general psychological competencies that could possibly be prerequisites of geometrical competence. Where the geometry items are concerned, the performance of the students in our sample was well in line with the profile of results obtained for the national and international TIMSS samples. It is thus possible to assume that our findings can be generalised to these study populations.

Our findings provide evidence for the validity of the TIMSS advanced mathematics tests. In particular, we were able to demonstrate that items from the higher levels on the TIMSS proficiency scale really do make more complex demands on the problem solvers, calling for a broader base of declarative knowledge (e.g., comprehension of geometrical concepts such as congruence) and methodological knowledge (e.g., an understanding of proofs, their generality, etc.).

Investigation of individual items has shown that the demands made by each item vary greatly. In some cases, only spatial and deductive reasoning ability is required, in other cases conceptual and/or methodological knowledge is also essential. In other words, TIMSS items make different demands, and cover many different facets of geometrical (or general) mathematical competence. The test is nonetheless one-dimensional, as model testing indicated. This means that the various facets of mathematical competence are not independent of one another, but rather that they are highly correlated. In the German school system at least, high levels of geometrical competence are accompanied by high levels of overall mathematical competence, a good understanding of proof, and differentiated conceptual knowledge. In other words, geometrical competence does not develop in isolation.

These findings, revealing the students' inadequate understanding of proof, can be regarded as an important indication of where the problem areas in mathematics instruction lie. Indeed, this was the basic approach taken by Healy and Hoyles (1998). In the context of a theory of situated cognition, however, the discrepancy between abstract knowledge about the correct construction of proofs and (at least partly) erroneous personal preferences is easy to understand and can be positively evaluated: students bear the context in mind when evaluating differing formulations of mathematical arguments. This is precisely the sort of approach encouraged in modern, reform-oriented conceptions of mathematics instruction. After all, students should not only experience mathematics as a set of fixed rules. On the contrary, they should be able to construct appropriate mathematical arguments both in school and in applied contexts. Our findings indicate that the topic of "proof in mathematics instruction" is particularly well suited as an introduction to mathematical argumentation – precisely because of this juxtaposition of views and preferences. "The goal is to help students refine their own conception of what constitutes justification in mathematics from a conception that is largely dominated by surface perceptions, symbol manipulations, and proof rituals, to a conception that is based on intuition, internal conviction, and necessity" (Harel & Sowder 1998, p. 237).

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