ANALYTIC-SYNTHETIC ACTIVITIES IN THE LEARNING OF MATHEMATICS

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We consider the well-known in psychological literature, but not yet sufficiently investigated in application to mathematics education mechanisms of thinking – the analytic-synthetic activities. Examples of using such means of mental activity as "synthesis" and "the analysis through synthesis" in solving geometric and algebraic problems are given. From these examples, one can deduce that various problems require various aspects of analytic-synthetic activities, and the most complicated means of such activity – "analysis through synthesis" can be developed only as a result of serious and specially designed teaching. The important task of mathematics education is to develop an effective system of teaching pupils to use all means of analytic-synthetic activity.

In this theoretical essay we will consider the well-known in psychological literature, but not yet sufficiently investigated in application to mathematics education mechanisms of thinking – the analytic-synthetic activities. The thoretical framework of our paper is Soviet activity approach in psychology and, particularly, S. L. Rubinshtein's conception of means of mental activity, or *mental operations* (S. L. Rubinshtein, 1989, p. 377). S. L. Rubinshtein (1958, p. 28) wrote : "The process of thinking is first of all the analysing and the synthesising of what is resulted in the analysis, then processes of abstraction and generalisation which are derivatives of analysis and synthesis. Regularities of these processes and their interrelations with each other are essentially basic interior regularities of thinking".

Analysis (in the ancient Greek *decomposition, partition, dissection*) is a procedure of mental, and frequently as well real dissection of a subject (phenomenon, process), of a property of a subject (subjects) on its components; extraction from a subject of aspects of its study; drawing out of a subjects their sides, properties, relations between them.

The analysis frequently is represented as multistage process. Something, reached as a result of the initial analysis, becomes then subject of the deeper analysis in the next stage. This passage from one level of the analysis to other, deeper level, is determined by the requirements and character of new tasks arising during the process of cognition.

Synthesis (in the ancient Greek *junction, compiling, integration*) is a mental combination of parts and sides, extracted by the analysis, in some new mental unity, in which the typical features of the analysed subject are fixed. The synthesis is connected with the simplification of what is analysed, with the detection in a subject of essential internal connections constructing the mental unity, with obtaining new piece of knowledge.

The synthetic activity of generalisation and meaning-making of the results of the analysis together with the analogy and other means of mental activity is a powerful tool of the discovering new truths, of obtaining principally new scientific results; of constructing revolutionary ideas creating new landmarks and directions in the development of sciences.

The synthesis reproduces the analysed subject, but is connected with the improvement, enrichment, deepening of that knowledge about a subject as a whole, which we had before the analysis. Synthesis already uses methods of that scientific theory, within the framework of which the analysis is produced, and, hence, those idealisations and abstraction, on the basis of which that theory is constructed. As a result of the analysis and synthesis the subject is reproduced in its essential and necessary relations and with that degree of the precision and adequacy, which is determined, on the one hand, by the contemporary state of the science and experimental technique and, on the other hand, by the character of the problem of research.

Thus, the synthesis is a procedure, inverse with respect to the analysis, but with which the analysis is frequently combined and intertwined in practical or cognitive activities. The analysis and synthesis are studied by such sciences as psychology, epistemology, logic, pedagogy and didactics of various disciplines.

Historically, the analysis was considered as a path (method of thinking) from the whole to the parts of the whole, and synthesis as a path from the parts to the whole. Therefore, analysis and synthesis are practically inseparable from each other. They accompany each other, complete each other, constituting the united analytic-synthetic method. The analysis assumes synthesis, and the synthesis is impossible without the analysis.

The analysis and synthesis as the methods of scientific knowledge play the important role in mathematical research. Similarly, extremely great is their role in mathematics teaching, in which they appear in the different forms: as methods of the solution of problems, of the proof of the theorems, of the study of properties of mathematical concepts etc.

Some of synthetic methods of solving problems involving geometrical constructions (methods of geometric loci, of similarity) were known still by ancient Greek geometers, and the distinction between analytical and synthetic methods was introduced to mathematics by Euclid. He considered the analysis and synthesis as two kinds of the "syllogistic" method of proof. In the thirteenth book of the "Elements" Euclid wrote: "In synthesis we begin with what is already proved, and come to the inference or to the knowledge of what is necessary to prove" (Euclid, 1956).

One can also find the definitions of the analysis and synthesis in the works of Francois Viete, who remarked that “there is a way of investigating the truth in mathematics, and the invention of that way is assigned to Plato; Theo has named it “analysis” and has defined as follows: “we consider the required as known and pass from a corollary to a corollary until we are convinced in the truth of the required; the synthesis consists in the following: proceeding from known, we, from a corollary to a corollary, come to the discovery of the required” (Yushkevich, 1970).

In the didactics of mathematics the terms “analysis” and “synthesis” traditionally meant two oppositely directed courses of reasoning, usually used in problem solving and in proving of theorems; the analysis is the reasoning directed from what is required to find or to prove to what is given or established earlier; the synthesis is the reasoning in the opposite direction.

Kolyagin et al. (1975, p. 52) wrote that “nowadays one understands the analysis as the means of thinking, leading from the corollary to the reason generating that corollary, and the synthesis is understood as the means of thinking, leading from the reason to the corollary generated by this reason”.

Giving much attention to the means of thinking “analysis” and “synthesis”, we certainly understand, that the mental activities are not reduced to these means only. The importance of such means as abstraction, concretisation, generalisation, analogy etc. for thinking in general and for mathematical thinking in particular is well known and widely admitted.

The analysis and synthesis can be combined with each other.

S. L. Rubinshtein distinguished the important form of the analysis – one which is carried out through synthesis. The essence of such analysis is the following: “the object of thinking is being repeatedly included in new connections and thus it arises in new appearances, with new qualities fixed in new concepts; thus, new contents are repeatedly taken out of the object, it turns repeatedly to new sides; new properties of the object come to light”(Rubinshtein, 1958, p. 98-99).

Thus, the important means of thinking arises: “the analysis through synthesis”. Its role in psychology is connected with the detection of new qualities, sides and properties of objects. Therefore, this means is connected to the creative processes. S. L. Rubinshtein (1976) named this means “a quintessence of thinking”. In our view, the possession of this means of mental activity is the highest level of person’s development in general and her/his mathematical development in particular.

As a result of the using of means of mental activities “synthesis”, “analysis” and “the analysis through synthesis” the special kind of intellectual activity – analytic-synthetic activity is born. This new kind of activity, in its turn, generates various analytic-synthetic methods of reasoning and problem solving in teaching/learning mathematics.

We will illustrate the essence of means “synthesis” and “the analysis through synthesis” by examples of problem solving.

The first example is a problem which can be solved by “pure” synthesis.

Problem 1. The perimeter of an isosceles triangle is equal to 1 meter, and the base of the triangle is equal to 0,4 m. What is the length of the lateral side (fig. 1)?

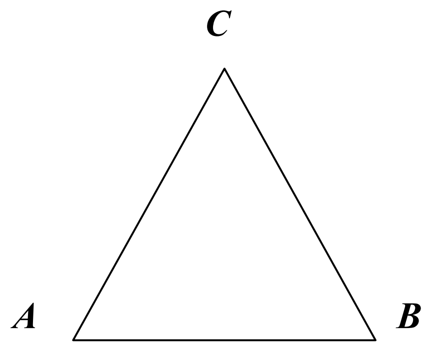

Fig. 1

Solution:

We will denote properties by the letter P with appropriate numbers. We have:

P1: the triangle ABC is isosceles (given).

P2: the perimeter of a triangle ABC is equal to $AB + AC + BC = 1$ m (given).

P3: $AB = 0,4$ m (given).

P4: $AC = BC$ (from P1).

P5: $AC = BC =$? (it is required to find).

Compare the properties.

P6: 0,4 m + 2 AC = 1 m.

P7: $AC = 0,3$ m (P6).

Thus, in this problem we, deducing corollaries from the condition, come to the answer. This is the most simple example of the use of *synthesis*.

In the next problem the synthesis is accompanied by the analysis.

Problem 2. FOX: OX = 5. Find numeric values of the letters in this equality.

Solution:

We begin with synthesis, i.e. with deriving corollaries from the given equality.

P1: FOX: OX = 5 (given).

P2: FOX = 5 OX (P1).

The further reasoning is based on the possibility of the representation of a number as a sum of digit summands.

P3: 100 F + OX = 5 OX (P2).

Here we have distinguished OX, i.e. have used the analysis in addition to the synthesis. We remark also, that OX is a two-digit number. Further:

P4: 100 F = 4 OX (P3).

P5: 25 F = OX (P4).

P6: 25 F is a two-digit number (P5).

Now the analysis will be used once more. It is necessary to answer a question: for what numbers F, the number 25 F is two-digit? It is easy to find, that there are only three possibilities: F = 1, F = 2, F = 3. Hence the answer is: FOX can be deciphered in three ways: 125, 250, 375.

In this example synthesis was accompanied by the simple, but very essential for finding a solution analysis. Therefore, it is possible to name the mental activity used in this solution “synthesis through analysis”.

Finally, we will consider two examples (geometric and algebraic) of application of the most advanced and complicated means of mental activity – “analysis through synthesis”.

Problem 3. Prove that the triangle ABC, in which bisector coincides with a median, is isosceles (fig. 2).

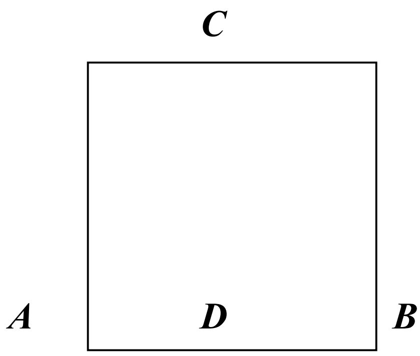

Fig. 2

Solution:

P1: ∠ ACD = ∠ BCD (i.e. CD is a bisector).

P2: $BD = AD$ (i. e. AD is a median).

P3: The triangle ABC is isosceles (required to prove).

From P1 and P2 nothing can be received, and other data is not present. What should one do in such case? There is a necessity in the analysis, and the analysis here is rather difficult.

It is necessary to prove, for example, the equality of sides AC and BC of the triangle ABC. What is necessary to know (to prove) for this purpose? The answer to this problem is rather difficult for the 7-th grade pupils of the Russian schools, where this problem is offered.

The more mathematically gifted pupils can use the sign of an isosceles triangle: the triangle is isosceles, if its base angles are equal. The idea is to construct, taking advantage of the equality of angles ACD and BCD, an isosceles triangle with the same angles and with a lateral side equal simultaneously to AC and BC. Thus, one can think about the inclusion of angles ACD and BCD and sides AC and BC into new connections, so that the equality of sides AC and BC becomes revealed. For this purpose it is possible to draw a segment DE on the prolongation of the segment CD, so that $DE=CD$, and, considering equal triangles EAD and CBD, one can show that the triangle CAE is isosceles and, hence, $AC = AE = BC$ (fig. 3).

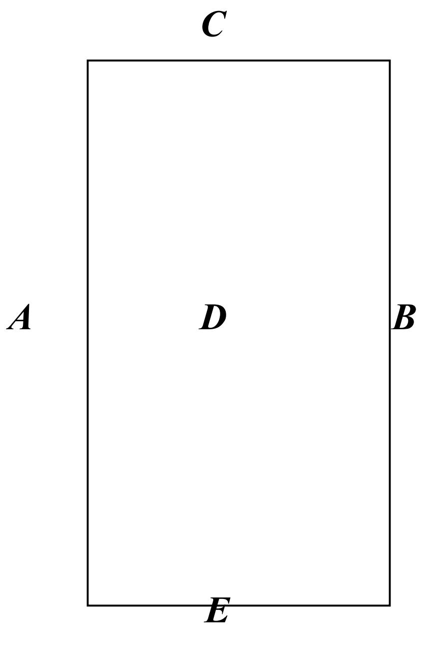

Fig. 3

The solution of this problem is a vivid example of the application of the *analysis through synthesis*. The analysis leads us here to the necessity of the rather complicated additional construction.

In the following problem the analysis through synthesis is applied to solving a non-standard algebraic problem .

Problem 4. Find natural solutions of the equation:

P1: $1 + x + x^2 + x^3 = 2^y$.

Solution:

We begin with synthesis, trying to transform this equation.

P2: $1 + x + x^2 (1 + x) = 2^y$ (P1).

P3: $(1 + x^2) (1 + x) = 2^y$ (P2).

What more can be deduced from this representation of the equation? One can read it as follows: the product of two natural numbers (x is a natural number, hence, $1 + x^2$ and $1 + x$ are also natural) is equal to the non-negative integer power of 2. In which case is it possible? Obviously, it is possible only when factors are also integer non-negative powers of 2. Thus, we have

P4: $1 + x^2$ and $1 + x$ are integer non-negative powers of 2, or, more concretely:

P5: $1 + x = 2^m$, m is a non-negative integer number.

P6: $1 + x^2 = 2^n$, n is a non-negative integer number.

P7: $x = 2^m - 1$ (P6).

P8: $(2^m - 1)^2 = 2^n$ (P6, P7).

P9: $2^{2m} - 2\text{x}2^m + 2 = 2^n$ (P8).

Note that if $n > 0$ then both parts of the equality can be simultaneously divided by 2. Consider separately the a case $n = 0$. In this case $x = 0$, but thus equality contradicts to the condition of naturality of number x. Therefore, $n > 0$. We have

P10: $2^{2m - 1} - 2^m + 1 = 2^{n - 1}$ (P9).

P11: $2^m (2^{m - 1} - 1) + 1 = 2^{n - 1}$ (P10).

All the transformations we accomplished are rather simple, however, it might seem unclear, for which purpose we obtained the equality P11. Here the analysis through synthesis has been accomplished, which has lead to the equality in which for all but one values of parameter n there appears an even number in the right part and odd one in the left part.

No we will consider, for which values of m and n this equality is possible. Obviously, n does not exceed 1 (if $n > 1$, we have an even number on the right and odd one on the left), and m is greater than 0 (if $m = 0$, the number on the left would not be integer). As we have established above, $n > 0$, therefore, $n = 1$, whence $2^m (2^{m-1} - 1) + 1 = 1$. Hence, $2^{m-1} - 1 = 0$, whence $2^{m-1} = 1$ and, hence, $m = 1$. Finally we obtain $m= 1, y = 2$.

In this non-standard and rather difficult example there are three subtle moments (ideas): 1) one should see the conclusion P4; 2) denotations P5 and P6; 3) when transforming the left part of equality P8, it was necessary to foresee the purpose, that on the right and on the left there should appear expressions with values of different parity for almost all values of parameters.

In these examples, one can see that various problems require various aspects of analytic-synthetic activities, and the most complicated means of such activity – "analysis through synthesis" can be elaborated only as a result of serious and specially designed teaching. The important task of the theory of mathematics education is to develop an effective system of teaching to all means of analytic-synthetic activity, including the most difficult one – analysis through synthesis.

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