THE FORMULATION OF A CONJECTURE: THE ROLE OF DRAWINGS

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Abstract. *The study presented in this paper arises from a research project dealing with the process of solving open-ended problems in a geometrical context. The specific moment of conjecture formulating is taken into consideration and namely a particular behaviour concerning the process of drawing producing is analysed. The theoretical framework of figural concepts provides us with the starting point from which developing our considerations.*

Introduction

Many investigations on the role drawings play within geometrical activity have been carried out from different points of view. Duval (1995) focuses on the "traitements spécifiques au registre des figures et à celui d'un discours théorique en langue naturelle" (p.173) and highlights the differences between them. Other studies point out and inquire into the intimate relation that exists among drawings, figures and concepts (Fischbein 1993, Laborde – Capponi 1994).

Notwithstanding many questions remain to be answered and in particular the role drawings play in solving geometrical problems has to be deepened.

In this paper we present a study developed from a research project, still in progress, which intends to investigate on the solving strategies of open-ended problems (Maracci 1998). With reference to the theoretical framework of figural concepts (Fischbein 1993) we focus on the role that producing drawings plays in solving construction-problems.

Figural concepts and *satisfactory* drawings

According to Fischbein's theory of figural concepts (1993) when dealing with geometrical activity we are concerned with a mental construct which simultaneously and intrinsically possesses both figural and conceptual properties.

The perfect fusion between the two components of a figural concept seems to be only an ideal and extreme situation, indeed "what happens is that the conceptual and the figural properties remain under the influence of the respective systems, the conceptual and figural ones" (ibidem, p.150).

According to this theory a drawing is the material concrete representation of a figural concept, reflecting the tension between the figural and the conceptual component.

In previous studies (Maracci 2001) the hypothesis has been formulated that when producing a geometrical drawing students try to harmonize figural and conceptual aspects. The search for such harmony does not seem to be conscious, one could speak of a generic feeling of satisfation from students' point of view. Three factors have been pointed out which might characterize this generic feeling and make a drawing *satisfactory*:

- a drawing should correctly represent the geometrical situation into consideration, students' interpretation of the given situation and of the produced drawing should be consistent;

- a drawing should be recognized as sufficiently generic;
- a drawing should possess a good gestalt, it should satisfy the fundamental laws which control the basic processes of perception.

These conditions may appear and combine in many different ways.

Our research

As mentioned above our study derives from a more extensive research concerning the processes of solving open-ended problems in a geometrical context (Maracci 1998). Seventeen students (11^{th} and 12^{th} grades) selected from different scientific high school were involved, all of them evaluated by their teachers as medium - high achievers. These students were presented with 4 open-ended problems to be solved in individual videotaped interviews during which they were asked to think aloud.

The study presented in this report focuses on the specific moment of the formulation of a conjecture. The analysis of the transcripts of the interviews shows some different behaviours on students' part. In this report we will focus on the following one:

- after a period of investigation conducted with the aid of drawings, students get the correct conjecture and face the task of formulating it precisely;
- the conjecture is achieved with clear and explicit reference to one or more specific drawings;
- students verbally formulate their conjecture producing at the same time a new drawing very similar to those which they previously referred to.

With respect to such a behaviour the following question may be posed:

> when they seem to have elaborated the correct conjecture with reference to a satisfactory drawing, why do some students feel the need of producing a new drawing so similar, if not identical, to the previous one?

A first answer has been proposed by formulating the hypothesis that students may interpret drawings diachronically, i.e. they may consider each drawing with its history as representing a specific process of the problem solving session (Maracci, 2001).

Here we will concentrate on the particular case of construction-problems and try to face the question from a different point of view.

The only problem we will refer to in the following is:

Problem: *Given two equal segments, construct two equal triangles with a common vertex and having the two given segments as homologous sides.*

This problem presents some difficulties in the identification of a solution. In order to solve it one could reformulate the problem shifting her attention from the construction of the two equal triangles to the construction of the common vertex.

Conjecture formulating and drawing producing in construction-problems

A strategy to successfully approach a construction-problem may be assuming the required objects as given and then proceeding backward in the search for some characterizing properties from which getting the needed construction. In this case when formulating her conjecture one needs, first of all, to restore the correct logical order of the relations.

As far as construction-problems are concerned students are required to explicitly give a procedure, a list of operations to perform in order to actually carry out the construction. The conjecture students have to elaborate and formulate consists indeed of a series of instructions which may hardly be put in the form “if… then…” or in any case in a more concise form.

Correctly describing a construction is a really demanding activity: it requires both to control the global organization of the procedure and to assure that each step may be actually performed. Performing it on the paper might provide a very useful support and in fact many students actually carry out their construction.

Thus students are engaged both in the verbal description of the construction and in the production of a new drawing.

On the other hand the production of a drawing is a demanding activity too. In previous researches students' difficulties in producing and managing drawings in solving geometrical problems have been highlighted (Maracci 1998, Maracci 2001). When producing a drawing students try to reach a harmony between figural and conceptual aspects. Even if it is not consciously stated, the need of balancing correctness, generality and good gestalt permeates the whole process of production of a drawing.

Thus when formulating their conjecture students are involved in at least two really demanding processes: the verbal formulation of the construction itself and the production of a new drawing.

Silver (1987) pointed out that the “overwhelming number of processes to control” is one common difficulty students face in solving mathematics problems. In order to overcome such difficulty students might try to activate some *form of control* over the processes to be carried out.

Production of drawings as a means of control

We remarked that when verbally describing their construction some students produce a new drawing estabilishing a, more or less explicit, correspondence with those to which they referred in the search for the construction itself.

Our main hypothesis is that *the production of a drawing similar to previous ones is an attempt to activate a form of control over the processes of description of a construction and of production of a drawing.*

According to our hypothesis estabilishing such correspondence might provide a control at least at two different levels:

- in order to be a satisfactory representation a drawing has to combine and balance needs for correctness, generality and good gestalt. When producing a new drawing, making reference to an already made satisfactory one may assure students that they are really producing a satisfactory drawing.
- Moreover, with concern to construction-problems, producing a new drawing similar to previous ones may play a more specific role.

  The drawing, with respect to which the correct construction has been grasped, might represent the final configuration, the outcome itself of the construction.

Estabilishing a connection, a correspondence between that drawing and the new one might allow students to interpret the former as a preview of the result of the construction they are performing and describing.

If the construction is correct and it is correctly performed the estabilished correspondence has to persist up to the end of the construction itself and the resulting drawing has to be consistent.

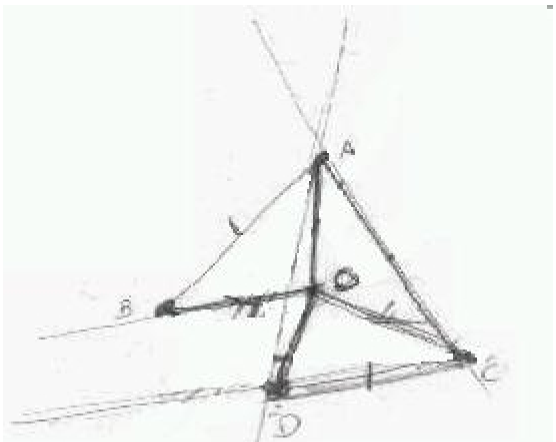
Extracts from problem solving sessions

In the following we present some excerpts from two protocols relative to the activity of formulation of a construction[1]. We shall analyse the excerpts in the light of the previous considerations.

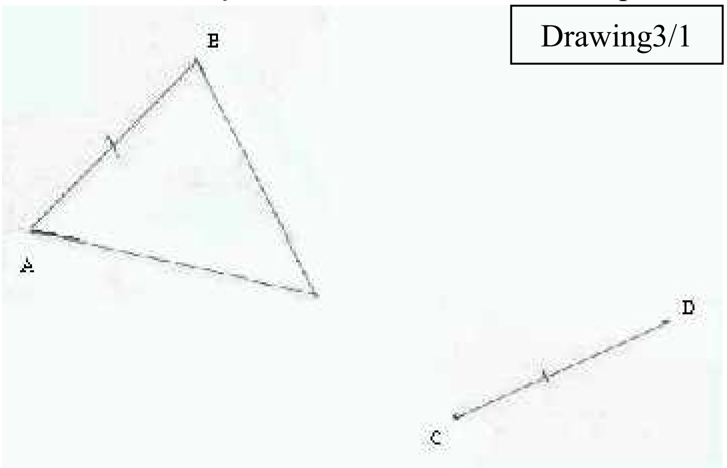
Barbara (12th grade, scientific high school)

What follows is the exact moment in which Barbara gets the properties characterizing the common vertex of the two unknown triangles. Drawing 4 is that with respect to which she has conducted the core of her exploration and it is that to which she is referring when grasping the correct construction (item 128)

128. **Bar:** there! I have to consider the perpendicular bisector of the segment... oh, there... I found it [...]. So I have to trace, to consider the segment AD - *she indicates the segment AD in drawing 4* - and to find... wait, - *she indicates segments AO and DO in drawing 4* - then segment AD, I found the perpendicular bisector of the segment by cutting it in two equal parts and tracing the perpendicular, then I know that point O has necessarily to belong to the perpendicular bisector of AD, wait - *she looks at drawing 3/1 and renames the endpoints of the two segments (drawing 3/2)* - I trace, - *she traces AD* **-** I trace... I consider the midpoint and trough it I trace - *she traces the perpendicular bisector of AD* - then, now I know that point O has necessarily to belong to the perpendicular bisector of this segment, but it has to belong to the perpendicular bisector of segment BC - *she indicates BC in drawing 4* - isn't it?... I trace BC and - *in drawing 3 she traces BC and its perpendicular bisector* - there ...



Drawing 4



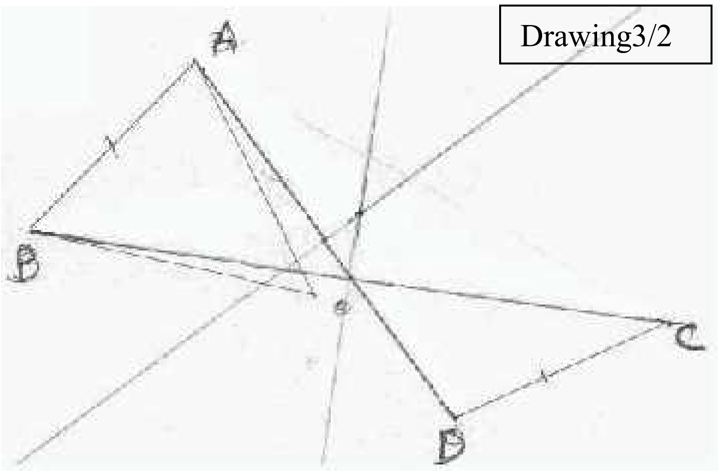
Drawing3/1

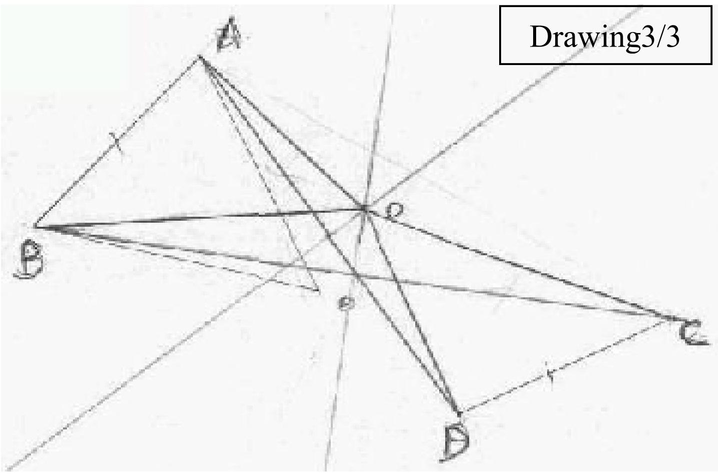
130. **Bar:** no, it does not work [...]

134. **Bar:** not, since... in this drawing - *she is referring to drawing 4* - I assumed these two segments equals - *she marks BO and CO* - but in this drawing - *she refers to drawing 3/2* - I'd actually get this one equal to this one - *she indicates (drawing 3/2) the segments AO and CO* [2]

136. **Bar:** point O should be here theorecally - *she labels by O the intersection point of AD and BC* - but it is here actually - *she indicates the intersection point between the perpendicular bisectors of AD and BC*

140. **Bar:** no, since... I'm really fool! - *she marks the correct point O and traces AO, BO, CO and DO (drawing 3/3)*





After having caught the correct construction, Barbara does not formulate it referring directly to drawing 4. She chooses to refer to another one. Her choice is indeed quite unusual: Barbara does not produce a new drawing from the beginning, she prefers to refer to an already made one (drawing 3/1). Drawing 3/1 provides her with a satisfactory representation of the initial configuration (i.e. the two equal segments); the presence of the triangle, related to previous investigations, seems not to influence the formulation of the conjecture.

Because of the mutual position of the two segments, drawing 3/1 seems suitable to be put in correspondence with drawing 4 and Barbara makes this correspondence complete by renaming the endpoints of the segments so as to conform them with drawing 4 (item 128).

While formulating her conjecture Barbara actually performs the construction (drawing 3/2), but suddenly she stops (item 130). At this point no one could tell what is wrong. In the following minutes Barbara continually shifts her attention from one drawing to the other.

In items 134 and 136 Barbara finally explains what causes her uncertainity. Trying to mentally anticipate the result of her construction, Barbara failed in positioning point O in her mind (it is only in item 136 that she labels point O in her drawing). Because of this mistake, the correspondence between the two drawing does not seem to be suitable any longer (item 130).

As it clearly appears (item 134), Barbara expected to get the equality between the sides BO and CO as a result of her construction. But what she gets, or better she thinks she is getting is the equality between AO and CO. Barbara's expectation derives from the fact that she is performing her construction making explicit reference to drawing 4. This allows her to interpret that drawing as a prevue of the outcome of the construction itself.

As soon as Barbara thinks that the estabilished correspondence fails, she has no doubt that some mistake occured, even if she is not immediately able to specifies which one.

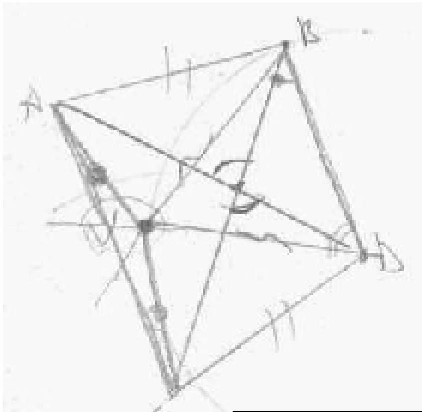
It takes some minutes before Barbara realizes what really happened (item 140) and can, so, successfully conclude her problem solving session.

This protocol gives an interesting example how estabilishing a correspondence among drawings might provide one with a means of control over both the production of a new drawing and the correctness of the construction itself.

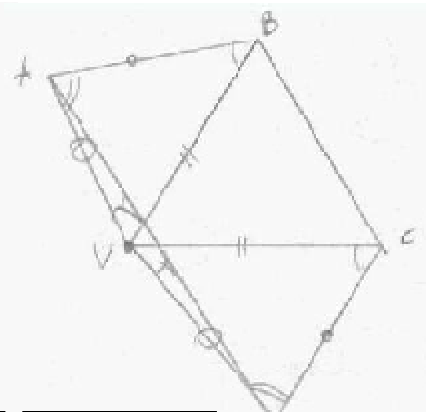
Let us remark a further aspect, at the end of the construction the resulting drawing (drawing 3/3) is quite similar to drawing 4: they share characteristics as the mutual position of the triangles and their shape. One might wonder whether such perceptive similarity plays the fondamental role in the correspondence between the two drawings. We could better discuss this question after having considered the next protocol.

Davide (12th grade, scientific high school)

Davide conducted his search for a conjecture mainly with the aid of two drawings (drawing 4 and drawing 7), and it is exactly with respect to drawing 7 that he grasped it. As Barbara did, he chooses not to refer to that drawing; he decides to produce a new one.

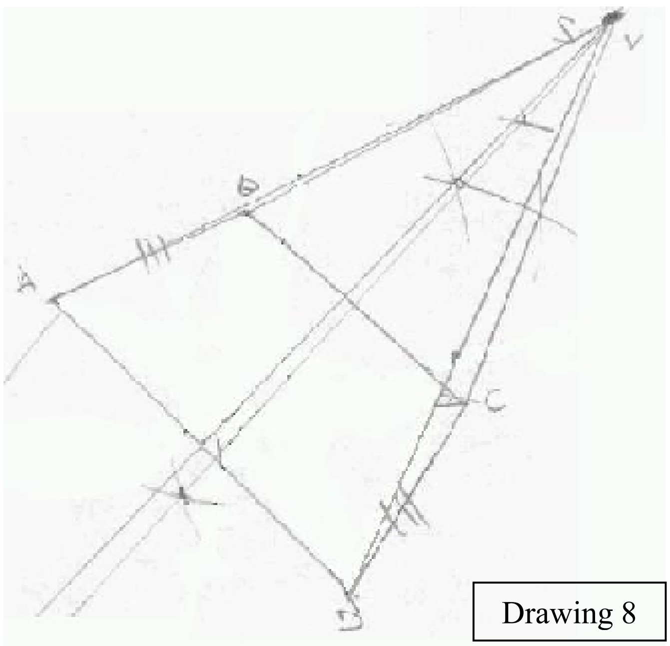


Drawing 4



Drawing 7

80. **Dav:** *he draws two equal segments (drawing 8)* - I know that these two angles… these two sides are equal – *he marks VD and AV in drawing 7* – in the same way these two ones – *he marks BV and CV* – so here, one can… however, I'm constructing it below – *he labels the endpoints of the previously drawing segments in conformity with drawing 7* - I mark the midpoint of the segments AD and BC – *he constructs the midpoints of the two segments* – these are... – *he indicates the perpendicular bisector of AD* – the geometrical locus of the points with equal distance from the endpoints A and D, […]



Drawing 8

I construct another straight line which is… the geometrical locus of point with the same distance from the endpoints B and C – *he constructs the perpendicular bisector of BC and labels the intersection point by V*

We can notice that drawing 8 shares the same initial configuration with drawings 4 and 7, that is the mutual position of the segment AB. The core of the search for the construction has been conducted with reference to such configuration, it is evidently satisfactory to Davide, so that he decides to use it as the starting point for his new drawing. Moreover he makes the correspondence between drawing 8 and drawing 7 stronger by naming the endpoints of the two initial segments in the same way (we observed the same behaviour on Barbara's part (item 128)).

While verbally making his conjecture explicit Davide constructs the vertex V.

As we can see, the final configuration appears completely different from previous ones, notwithstanding this does not affect Davide's confidence about the correctness of his conjecture nor of his drawing.

We want to point out that despite the undeniable perceptive differences, the correspondence between drawings 8 and 7 persists in a deeper sense. Drawing 8 represents the segment BV equal to CV and the segment AV equal to DV, as they were represented in drawing 7. Even if they are "globally" different the two drawings represent the same "analytic" relations among the same elements.

On the other hand if we look back at Barbara's protocol we can observe that she never refers to how drawings appear globally. She explicitly refers to analytic relations among the drawn elements.

Consistently with the theory of figural concepts (Fischbein, 1993), students interpret drawings as reflecting figural and conceptual aspects, the correspondence students estabilish among drawings is not limited to perceptive aspects, it deeply involves analytic relations too.

Conclusions

Solving a construction-problem requires one to explicitly give a procedure, a list of constructing operations. Producing a drawing might provide a useful support for the verbal description of the construction. But verbally formulating a construction and contemporary producing a new drawing are two really demanding activities, in order to successfully manage them some means of control may be needed.

The analysis of the experimental data suggests that such a control may be found in relating the new drawing to that with reference to which the construction has been caught.

Estabilishing such a correspondence might assure that one is really producing a satisfactory drawing.

Furthermore previously produced might be interpreted as a models of the outcome of the construction itself.

Many key points remain to be clarified. Far from being exahustive, our research poses new questions to be answered. We should like to point out the followings ones:

- when problems different from construction ones are involved, is any correspondence among different drawings estabilished? Which eventually is its role?

- might estabilishing correspondences among drawing play roles of control over other activities in the process of problem solving?

Students' difficulties in managing drawings have been highlighted in previous researches (Maracci 1998, Maracci 2001). We think that deepening which role drawings play in the process of solving geometrical problems might contribute to clarify such difficulties. In order to be able to plan specific didactical activities a further development of this research is anyway needed.

Notes

[1] The original drawings were scanned and processed by means of a computer in order to obtain, on the basis of the analysis of the videotapes, the way they appeared at each moment of the problem solving sessions. Numbers which designate drawings refer to the whole session. One of the drawings realtive to the protocol of Barbara (drawing 3) was processed in order to be allowed to follow its production step by step.

[2] Let us remark that at this point Barbara has not still labeled point O on her drawing (cfr item 136)

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