TOWARDS A COMPARATIVE ANALYSIS OF PROOF TEACHING

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Abstract: *In this paper, the first results of a comparative study on proof and proving in geometry teaching are represented. Twelve eighth grade classes (approximately 14 year-old students) were observed in France and Germany, in order to analyse the impact of culturally embedded classroom practices on the teaching and learning of proof. Also, the differences in functions of proofs based on the observed French and German teaching practices are presented here. In particular two ideal types of mathematical cultures in classroom practice have been singled out.*

Results of international studies which compare mathematics teaching in different countries suggest that cultural diversity in mathematics education goes far beyond instructional methods. This research has outlined qualitative differences between the topics that were the focus of lessons and what students were expected to do as they studied these (Cogan and Schmidt 1999). Hitherto there have not been carried out any comparative studies concerning the teaching of proof, one main topic of mathematics teaching. However, Balacheff has been reflecting upon the idea that ”ethnomathematical” questioning on the proof is just as necessary as its commonly accepted epistemological inquiry (Balacheff 1999).

In my comparative case study on proof and proving in French and German geometry lessons, differences in classroom practice and their implications for the learning and teaching of proof will be analysed. This study questions: *How do culturally embedded classroom practices influence the teaching and learning of proof?* For this I focus on the differences within proof forms and functions. The hope is that this will enrich the research in the proof and proving field from an inter-cultural as well as a classroom-cultural perspective.

Research in the Field of Proof and its Teaching

One main problem of the teaching of proofs is that students, once convinced of a statement's validity, experience so much difficulty understanding the importance of the statement's proving. In the following different approaches to this problem will be briefly represented and discussed.

On the one hand a series of learning situations, including proving exercises, have been created in order to stimulate among students processes of formulating conjectures and refutations (Balacheff 1987). Therefore, problems were designed to lead to controversy among students which should involve them into a process of proving (see further Boero, Garuti, and Mariotti 1996). In these approaches the social dimension of proof and proving is seen as essential for didactical concerns.

Moreover teaching practices of proof and forms of proofs being taught have been analysed from an epistemological perspective. Hanna and others criticised the over emphasis of formal proofs in everyday teaching (Hanna 1989; Wittmann and Müller 1990), which does not give students an adequate understanding of the meaning and function of proofs. Research in this field has made it clear that proofs have a variety of functions (de Villiers 1990), for example an epistemic function that is to understand why; a systematic function which is to relate different mathematical concepts already studied in class.

Further, proof conceptions of teachers and students as well as students' aptitudes in proving have been examined in several empirical studies (Healy and Hoyles 1998; Reiss and Heinze 2000). The latter having a focus rather on cognitive than on social aspects, even though including proving processes. These studies have given interesting insights into implicit proof conceptions of students, but they give little evidence in how far these conceptions are caused by the teaching of proof in mathematics lessons.

Research on argumentation processes gives a clue of argumentation formats in classroom situations, but is limited at the same time to the level of primary school (Krummheuer 1993; Schwarzkopf 2000). It might be interesting to see how argumentation formats in class might be linked to proving processes; analyses which have not been done yet.

All in all, in proof research few empirical studies exist on proofs and proving in everyday classroom situations (Herbst 1998). This means that we have little evidence showing in how far students’ difficulties are due to everyday mathematics teaching practice.

Theoretical framework and methodological considerations

The present study refers to sociological and didactical research theorising everyday practices. These practices can be characterised as mostly determined by pragmatic aims and a general concern of effectiveness in action (Krummheuer and Naujok 1999). In these situations interacting partners typically presume common understanding and references, which allows to structure learning processes on the basis of habitual practices. Discourses are further marked by a high degree of implicity and indexicality, which means that statements often can be understood only within the context they have been stated. From a sociological perspective this has been described as a ”pragmatic cognitive style” of everyday practices (Soeffner 1989).

Focussing on everyday classroom practices from this theoretical approach helps to analyse from students’ and teachers’ point of view what we might consider as learning or teaching obstacles. A focus on their prospect is inevitable as didactic research not only aims at offering alternatives to habitual practices but hopes changing misleading practices.

In order to analyse everyday classroom practices comparing these becomes an insightful tool. A comparative approach allows to single out characteristics of teaching practices in different contexts which could not have been reconstructed from single cases (Kelle 1994).

The methodological frame of the present study refers to the concept of ”ideal types” developed by Max Weber (*Webersche Idealtypen)*, and describes idealised types of mathematics teaching reconstructed from classroom observations in France and Germany (Weber 1904 /1988). This means that typical aspects of teaching practices are reconstructed on the basis of the whole qualitative studies rather than on an existing empirical case.

Methodical design and data analysis

The study uses methods of qualitative social sciences mainly participating classroom observation – about 30 lessons were seen at 6 different classes of eighth grade students (14 year olds) in France, compared to 30 lessons at 6 German schools (same age group). Within the observed classes, two were from a French-German college, the others from ordinary classes of collèges in Paris and German Gymnasium and higher level classes of comprehensive schools in Hamburg.

With respect to proving on the one hand and variation of contents on the other, six teaching units concerning Pythagoras’ Theorem and six further units dealing with similarity and special lines in triangles were chosen. Each lesson has been recorded, so that analyses of transcripts and blackboard drawings and writings could be possible later. Classroom observations have further been expanded by protocols. As a tool to support the data analyses the software Atlas-ti is used.

Analyses of the observed proofs are done from two perspectives which are supposed as complementary: firstly, content related analyses, secondly reconstruction of argumentations. This means that forms and functions of proofs are analysed with respect to their mathematical substance and not separate from it. For I suppose as Granger that form and content of proofs cannot be separated (Granger 1994).

Research on Pythagoras’ Theorem by Fraedrich and on geometrical settings by Parzysz and Douady is referred to as a theoretical framework for the content related analyses (Douady and Parzysz 1998; Fraedrich 1995). Analyses of the structures of argumentation are guided by the theoretical work of Duval and Toulmin (Duval 1995; Toulmin 1958). Duval's theoretical analyses allow for a distinction between argumentation and proof by formal aspects, whereas Toulmins' scheme helps to work out different argumentation structures. Results of the latter analyses will be published elsewhere.

First results of content related analyses

Sorting proofs of Pythagoras' Theorem which have been observed in classroom situations in France and Germany showed four different forms of proofs: 1) Proofs based on comparisons of areas 2) Proofs based on calculations of areas 3) Proofs where applying theorems on similarities, 4) Proofs using visualisations of the theorem of Euclid, meaning $a^2 = pc$ and $b^2=qc$. Analysing these proofs makes evident two different interpretations of Pythagoras' Theorem: a statement about comparing areas as well as an assertion about relations of lengths, which is evidently not the same.

Taking into account that every type of proof favours one or the other interpretation of the theorem, it is surprising that in our observations tackling the meaning of Pythagoras' Theorem is not necessarily coherent with the proof done in class. When comparing French and German teaching, one has to notice that no differences concerning such inconsistencies of teaching could be found. Whereas there had been differences in the meaning, which has been assigned to Pythagoras' theorem, both interpretations – about areas and about relations of lengths - have been found in German mathematics lessons, but not in French classes. Further in German mathematics teaching all different types of proofs as systemised above could be found, while in French mathematics lessons only proofs of type one and two have been identified which correspond with proofs in curricula and in textbooks used in class.

By analysing the role of proofs within teaching it became apparent that introductory phases are essential for the teaching of proof in the observed German lessons, whereas phases of exercises play an important role for French teaching of proofs.

All observed German classes start with two or three lessons where students and teachers are engaged in a process of discovery of the theorem before proving it. In contrast to this in French teaching Pythagoras' Theorem is directly introduced to the students in the very first lesson of the teaching unit and proved with guidance of the teacher.

Having proven the theorem in French lessons, complex and sophisticated problems have to be solved by students at home as well as in class. These exercises require an application of different theorems and concepts, which have been studied in former teaching units, including Pythagoras' Theorem. Whereas in German lessons all exercises have been analysed as typical routine tasks which request simple application of Pythagoras' Theorem.

Two cases of German (the case of Nissen) and French (the case of Pascal) teaching have been chosen and shall be presented in the following, in order to illustrate typical characteristics of different teaching patterns.

The Role of discovery of theorems - the case of Nissen

Teacher Nissen starts her teaching unit on Pythagoras' Theorem with a calculation problem: the length of the rafter of a rectangular saddle roof is to be figured out, the width of the house given. At this time Pythagoras' Theorem has not yet been introduced, so that the students have to find a way of solving the given problem. Completing the figure by squares on the sides of the triangle and calculating the areas of these squares leads to a relation between the sides of the given triangle: $2a^2 = c^2$.

The study of the special case of a right-angled isosceles triangle is used to introduce the central idea of Pythagoras' Theorem in two lessons. This approach allows for consideration of the theorem from two different points of view: the aspect of areas and the relation of length of the triangle's sides. At the same time one of the most important applications of Pythagoras' Theorem, the calculation of length, is already treated in the foregoing lessons. Throughout in the course of the teaching unit, the insights gained in the special case are prolonged to the general idea of Pythagoras' Theorem. In contrast, the proof strategy used in the special case is not taken up any more. This might be easily explained from a mathematical perspective for the proof in the special case cannot be generalised.

In the teaching the first two or three lessons do have a special role, they lead students to discover the theorem itself and to understand the theorem in a double way, as a theorem about areas and as a theorem on the relation of lengths of the triangle's sides. Further the special case makes it possible for the students to figure out a proof on their own, or with little help from the teacher, for the central idea of the proof is as simple that the students can do so as well.

Proofs and problems initiating justifications - the case of Pascal

Having prepared the technical part of the proof by working on the binomial in the preceding lesson, teacher Pascal works out Pythagoras' Theorem together with the students by its proof : the area of a square with lengths a+b is calculated in two different ways: first by using the binomial, then calculating the sum of the inner square's area, that is c^2, and the areas of four right triangles $4 \cdot \frac{a \cdot b}{2}$. This takes about half of the lesson and is completed by exercises applying Pythagoras' Theorem.

The problems given to the students in Pascal's class after the theorem has been introduced can be divided into two types of problems: routine versus complex exercises. Whereas in the routine tasks Pythagoras' Theorem only has to be simply applied for calculating length, for example the length of a triangle's side, the complex problems cannot be solved without analysing the geometrical configurations and the use of other theorems and concepts, as properties of the circumscribed circle or similarities . Further, analyses of these problems have shown that different ways of solutions are possible, among those very elementary ones, these are solutions which are based on concepts and properties that have already been introduced at grade 6 (12 year olds).

The teacher insists on complete justifications, why the way a student worked out a solution and her or his use of theorems and concepts is legitimate. The proof of Pythagoras' Theorem, which has been produced in collaboration between students and the teacher, offers in a way a model how justifications should be structured and marks the level of rigour which is expected by the teacher. Problems and proofs of theorems are functioning as an amalgam where the responsibility of truth is given from teacher to students and back. Everyone is asked to justify by good reasons the validity of mathematical statements she or he has claimed.

Teaching styles and functions of proofs

In the observed German lessons it seems typical that the discovery of theorems is based on special cases and applied problems, where proof and different interpretations of the theorem are merging into one another. In this pattern of teaching proofs gain the function of assigning meaning to the theorem. For meaning is here that students understand applications and different interpretations of a theorem. This explains why no problems occur when the central idea of the proof used in the special case is given up for a general proof for the same interpretation of the theorem is prolonged.

Whereas different interpretations and applications seem to be essential for German ways of teaching proof, successful defence of claims of validity of mathematical assertions can be described as typical for the observed French teaching.

Proving in French teaching is seen as an activity which characterises the whole teaching and not just phases when theorems as Pythagoras' Theorem are proven. Every exercise, even those where students are not explicitly asked to prove something, has to be edited so that the given solution is justified. It has to be made clear what is considered as assumptions, which theorems, concepts and properties are applied. For meaning is here to state the conditions of a statement’s or solution’s validity. The proof of Pythagoras’ Theorem, which has been done in collaboration but strongly guided by the teacher, serves as an exemplary scheme on how to organise one's thoughts. This pattern of argumentation, which is acquired through edited justifications for solutions of problems as well as through proofs of theorems, gives proofs the function of defining conditions of validity. The responsibility of justifying which of the already in class studied theorems and concepts are to be used and why shifts from teacher to students and back.

Conclusion

Comparing the role of proof in German and French teaching contexts has made aware of different teaching patterns and different functions of proofs. These functions of proof can hardly be interpreted as different levels of proofs, such as more formal and less formal proof types. It is the function of proofs which is different. In the observed German teaching the function of proof is to ”understand why”, whereas in French teaching it is important to ”defend why” a statement is true.

Differences in proof functions might explain why a plurality of proof forms could be found in German teaching. Different types of proofs can be beneficially used for working out distinct interpretations of theorems’ meanings. This might be regarded as a loss of time when proofs’ functions are to give a model as to how results should be legitimated. In mathematics instruction where processes of ”defending why” are typical for teaching and learning in general it is more important to allow time for students’ own argumentation and proving activity. We may presume that this characterises distinct relations to knowledge and rationality as ingrained in culture.

It seems very interesting to see in how far these differences have an impact on the structures of argumentation in class. The argumentation analyses done so far point out that in general assumptions are made explicit in French discourse, whereas in German classes there might be more lenience for assumptions that rest implicit in argumentation. Formats of argumentation might remain implicit in German practices for sharing of meaning is more important than argumentation types.

Analysing ”mathematics classroom cultures” from a comparative perspective gives a way to single out different teaching practices of proof. These differences might help to understand student's difficulties with proof from new perspectives. What effects do classroom discourses have on the learning of proof? Which functions do proofs gain through teaching and what impact does this have on students’ conceptions of the proof and their aptitudes for proving?

References

Balacheff, N. (1987) Processus de preuve et situations de validation. In: *Educational Studies in Mathematics* 18 (2), 147-176.

Balacheff, N. (1999) For an ethnomathematical questioning on the teaching of proof. In: *International Newsletter on the Teaching and Learning of Mathematical Proof (Online)* (September/October 1999), .

Boero, P. / Garuti, R. / Mariotti, M.A. (1996) Some dynamic mental processes underlying producing and proving conjectures. In: Puig, L.; Gutierrez, A. (Ed.). *Proceeding of the 20th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, Valencia University, 121-128.

Cogan, L.S. / Schmidt, W.H. (1999) An Examination of Instructional Practices in Six Countries. In: Kaiser, G.; Luna, E.; Huntley, I. (Ed.). *International Comparisons in Mathematics Education*,Vol. 11. London, Philadelphia, Falmer Press, 68-85.

de Villiers, M. (1990) The Role and the Function of Proof in Mathematics. In: *Pythagoras* (24), 17-24.

Douady, R. / Parzysz, B. (1998) Geometry in the classroom. In: Mammana, C.; Villani, V. (Ed.). *Perspectives on the teaching of geometry for the 21st century. An ICMI study*,Vol. 5. Dordrecht, Kluwer, 159-192.

Duval (1995) Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels. In: Schneuwly, B.; Hameline, D. (Ed.) *Recherches en sciences de l'éducation*. Bern, Peter Lang.

Fraedrich, A.M. (1995) Die Satzgruppe des Pythagoras *Lehrbücher und Monographien zur Didaktik der Mathematik* Vol. 29. Mannheim, BI Wissenschaftsverlag.

Granger, G.-G. (1994) Formes, opérations, objets . Paris, VRIN.

Hanna, G. (1989) More than formal proof. In: *For the Learning of Mathematics* 9 (1), 20-25.

Healy, L. / Hoyles, C. (1998) Justifying and proving in school mathematics. Technical Report on the Nationwide Survey. Institute of Education, University of LondonPlace, Published, .

Herbst, P.G. (1998) What works as proof in the mathematics class. Ph.D. Dissertation, The University of Georgia, Athens GA. (USA).

Kelle, U. (1994) Empirisch begündete Theoriebildung. Zur Logik und Methodologie interpretativer Sozialforschung . Weinheim, Deutscher Studienverlag.

Krummheuer, G. (1993) The ethnography of argumentation. Bielefeld, University of Bielefeld.

Krummheuer, G. / Naujok, N. (1999) Grundlagen und Beispiele interpretativer Unterrichtsforschung. Opladen, Leske + Budrich.

Reiss, K. / Heinze, A. (2000) Begründen und Beweisen im Verständnis von Abiturienten. In: Neubrand, M. (Ed.). *Beiträge zum Mathematikunterricht 2000*. Hildesheim, Franzbecker.

Soeffner, H.-G. (1989) Auslegung des Alltags - der Alltag der Auslegung. Zur wissenssoziologischen Konzeption einer sozialwissenschaftlichen Hermeneutik . Frankfurt a.M., Suhrkamp.

Schwarzkopf, R. (2000) Argumentationsprozesse im Mathematikunterricht - Theoretische Grundlagen und Fallstudien. Hildesheim, Franzbecker.

Toulmin, S.E. (1958) The uses of argument. Cambridge, Cambridge University Press.

Weber, M. (1904 /1988) Die "Objektivität" sozialwissenschaftlicher und sozialpolitischer Erkenntnis. In: Winckelmann, J. (Ed.). *Gesammelte Aufsätze zur Wissenschaftslehre,* Vol. 7. Tübingen, Mohr.

Wittmann, E.C. / Müller, G. (1990) When is a Proof a Proof? In: *Bulletin Soc. Math. Belg.* 1, 15-40.