Composition of Geometric Figures[1]

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The purpose of this research is to chart the mathematical actions-on-objects young children use to compose geometric shapes. We designed a hypothesized learning trajectory based on previous research and an instrument to assess levels of this trajectory. We tested both the trajectory and the instrument with extensive case studies of 60 children, ages 3 to 7. Our research reveals that children move through levels of thinking in the composition and decomposition of 2-D figures. From lack of competence in composing geometric shapes, they gain abilities to combine shapes into pictures, then synthesize combinations of shapes into new shapes (composite shapes), eventually operating on and iterating those composite shapes.

The ability to define, use, and visualize the effects of composing (putting together) and decomposing (taking apart) geometric forms is a major conceptual field and set of competencies in the domain of geometry. This domain is significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis (Clements, Battista, Sarama, & Swaminathan, 1997; Reynolds & Wheatley, 1996; Steffe & Cobb, 1988). There is empirical support that this type of composition corresponds with, and supports, children's ability to compose and decompose numbers (Clements, Sarama, Battista, & Swaminathan, 1996). Although there is limited research on children's thinking about geometric composition, there is a lack of research detailing specific learning trajectories. The purpose of this research is to chart the mathematical actions-on-objects young children use to compose geometric shapes.

The genesis of the study was in observations we made of children using *Shapes* software (Sarama, Clements, & Vukelic, 1996) to compose shapes. *Shapes* is a computer manipulative, a software version of pattern blocks, that extends what children can do with these shapes. Children create as many copies of each shape

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as they want and use computer tools to move, combine (compose and decompose) and duplicate these shapes to make pictures and designs and to solve problems. We noticed that several of our case-study students followed a similar progression in choosing and combining shapes (e.g., rhombi or equilateral triangles) to make another shape (e.g., to fill a hexagonal frame). At first, children merely appreciated the relationship between pattern blocks, how one pattern block could be made using other pattern blocks, but their efforts to fill a hexagonal frame with other pattern blocks was by trial-and-error. Second, they could fill the hexagon with 2 trapezoids. Then they followed a sequence of filling the hexagon with 6 triangles, the trapezoid with 3 triangles, the trapezoid with 1 rhombus and 1 triangle, and the hexagon with 3 rhombi. To ascertain whether this sequence was a valid indicator of developing competencies in composing shapes, we conducted a series of studies; we report on one here.

Theoretical Framework

Our theoretical assumption is that to solve composition tasks such as ours effectively and efficiently, children must build an image of a shape, then match that image to the goal shape by superposition (of both components and shapes), performing mental rotations as necessary to match these images. We wished to ascertain what features made certain composition tasks more or less difficult. In brief, the literature indicates that possibilities include: a horizontal or vertical side in the frame or component shape (Fisher, 1978; Hemphill, 1987; Ibbotson & Bryant, 1976), few component shapes (Vurpillot, 1976), symmetric components (Bremner & Taylor, 1982; Vurpillot, 1976), components which are symmetric halves of the frame (Clements & Battista, 1992), components that match a maximum number of the frames components (sides and angles), and presence or lack of mental rotation (Kail, Pellegrino, & Carter, 1980; Presmeg, 1991; Rosser, 1994; Shepard & Metzler, 1971). It is important to note that we are drawing indirect inferences from most of these studies in which the actual tasks were related to Piagetian horizontality and verticality tasks or disembedding tasks, not composition tasks in which we were interested.

We created a hypothetical learning trajectory from the existing research on shape composition, including our own research and intuitions generated from our work with children (Clements, in press). Hypothesized learning trajectories (Cobb & McClain, in press; Gravemeijer, 1999; Simon, 1995) ideally include "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133). Unlike other approaches (Gravemeijer, 1994), we believe that existing research should be a primary means of constructing the first draft of these learning trajectories (which may, in turn, ameliorate the difficulty many development teams appear to have incorporating the research of others). The following levels constitute our hypothesized learning trajectory for the composition of shapes.

1. Pre-Composer. Manipulates shapes as individuals, but is unable to combine them to compose a larger shape.
2. Piece Assembler. Similar to step 1, but can concatenate shapes to form pictures. In free-form "make a picture" tasks, for example, each shape used

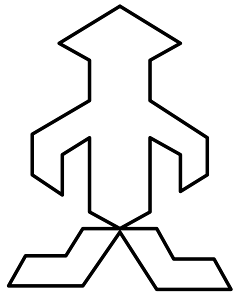
represents a unique role, or function in the picture. Can fill simple frames using trial and error (Mansfield & Scott, 1990; Sales, 1994). Uses turns or flips to do so, but again by trial and error; cannot use motions to see shapes from different perspectives (Sarama et al., 1996). Thus, children at steps 1 and 2 view shapes only as wholes and see no geometric relationship between shapes or between parts of shapes (i.e., a property of the shape).

3. Picture Maker. Can concatenate shapes to form pictures in which several shapes play a single role, but uses trial and error and does not anticipate creation of a new *geometric shape*. Chooses shapes using gestalt configuration or one component such as side length (Sarama et al., 1996). If several sides of the existing arrangement form a partial boundary of a shape (instantiating a schema for it), the child can find and place that shape. If such cues are not present, the child matches by a side length. The child may attempt to match corners, but does not possess angle as a quantitative entity, so will try to match shapes into corners of existing arrangements in which their angles do not fit. Rotating and flipping are used, usually by trial-and-error, to try different arrangements (a "picking and discarding" strategy). Thus, can complete a frame that suggests that placement of the individual shapes but in which several shapes together may play a single semantic role in the picture.
4. Shape Composer. Combines shapes to make new shapes or fill frames, with growing intentionality and anticipation ("I know what will fit"). Chooses shapes using angles as well as side lengths. Eventually considers several alternative shapes with angles equal to the existing arrangement. Rotation and flipping are used intentionally (and mentally, i.e., with anticipation) to select and place shapes (Sarama et al., 1996). Can fill complex frames (Sales, 1994) or cover regions (Mansfield & Scott, 1990). Imagery and systematicity grow within this and the next levels. In summary, there is intentionality and anticipation, based on shapes' attributes, and thus, the child has imagery of the component shapes, although imagery of the composite shape develops within this level (and throughout the next levels).
5. Substitution Composer. Deliberately forms composite units of shapes (Clements et al., 1997) and recognizes and uses substitution relationships among these shapes (e.g., two pattern block trapezoids can make a hexagon).
6. Shape Composite Iterator. Constructs and operates on composite units intentionally. Can continue a pattern of shapes that leads to a "good covering," but without coordinating units of units.

We had two research goals, to evaluate (a) the geometric composing instrument and (b) the validity of the hypothesized levels of thinking in the domain of composing geometric figures.

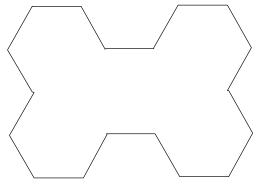
Method

Based on this model of students' learning, we created a instrument to measure each of the first five levels of thinking. The following are examples of two items on the instrument.



On the first, children are given pattern blocks and a frame of a "man" and asked to "Use pattern blocks to fill this puzzle." We categorized children as follows. *Pre-Composer:* Cannot match even well-defined, simple frame, such as the "feet." *Piece Assembler:* Can fill simple frames (e.g., "feet" only) using trial and error. *Picture Maker:* Fills frames with trial-and-error, matching shapes by boundary or matching side lengths.

Shape Composer: Completes entire frame with deliberate choices of shapes; to do so, matches configurations, sides, or angles. *Substitution Composer:* Deliberately replaces a group of shapes (e.g., two triangles) with one shape (e.g., blue rhombus) or vice versa. *Shape Composite Iterator.* Deliberately, systematically iterates a composite group of shapes to fill a region.



A second example asks children, in three separate questions, to determine how many yellow hexagons, red trapezoids, and green triangles they would need the cover the puzzle(given a limited number of the latter shapes).

We categorized children for the *Piece Assembler* to *Shape Composer* levels in ways similar to the first example, but this item was designed especially to target *Substitution Composer:*The deliberate recognition and use of the relationships between the hexagons, the trapezoids, and the triangles (e.g. 2 trapezoids = 1 hexagon).

Participants were 60 children from 4 classrooms selected at random from all children who completed a human subjects permission letter. All children were interviewed individually by one of two graduate research assistants following a protocol for administering the composition instrument. The researchers also asked questions of children, as in a clinical interview, whenever they believed that such questions would clarify the nature of children's thinking.

Each session was videotaped. These tapes were partially transcribed, coded, and analyzed, both to complete the scoring of the instrument for each child and to identify additional themes.

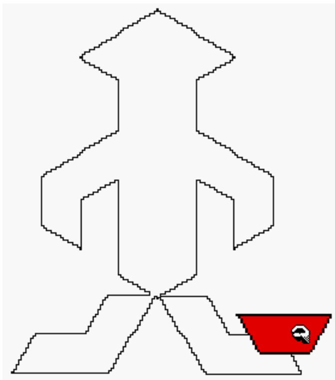
We compared the results of applying the scoring rubric to the qualitative analysis of each child's response to each item with the intent of determining, for each item, if (a) the item elicited the types of thinking we wished to observe and (b) the scoring rubric accurately encapsulated the type of thinking that the qualitative analysis revealed. We then used the results of both the scoring rubric and the qualitative analyses to determine whether (a) items designed to measure the same level of thinking elicited similar responses, providing information as to the reliability of the
items and the coherence of the hypothesized levels of thinking, (b) the levels form an invariant sequence. For example, each student's scores were entered into a

spreadsheet divided into categories based on the hypothesized trajectories; each item was classified according to the level it was designed to measure. The resultant spreadsheet was examined visually to answer the research questions. In addition, the proportion of items providing evidence of attainment of each level of thinking were computed. This allowed us to examine the percentage of children whose scores followed a pattern consistent with the hypothesized trajectory (e.g., if half of the items indicated thinking at level *n* , more than half should reliably indicate mastery of thinking at level $n – 1$, etc.). Qualitative analyses were also used to ascertain whether the levels evince "incorporation"; that is, if thinking and actions of an earlier level are incorporated in the next level. Thus, we were assessing the main developmental criteria of constancy and integration (across a period, there is a type of thinking that forms an integrated whole), invariant sequence, and incorporation (Steffe & Cobb, 1988).

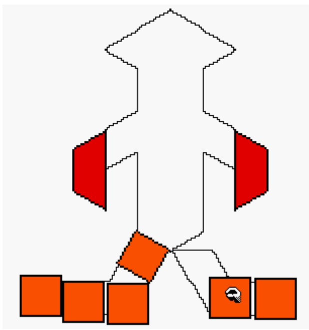
Findings and Discussion

Findings generally supported the hypothesis that children demonstrate the various levels of thinking when given tasks involving the composition and decomposition of 2-D figures, and that older children, and those with previous experience in geometry, tend to evince higher levels of thinking.

The most intensive work was in the qualitative analysis of children's responses. We found that the levels of thinking could be reliably differentiated, and that children could be reliably assigned to a level of development (including those in the process of developing the next level). Here we can discuss only a few examples.

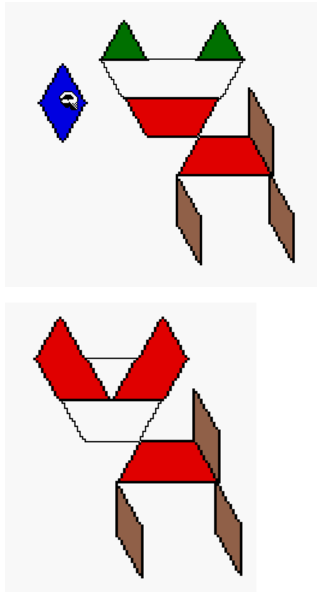


Mary, a preschool (4.3 years) child, exhibited actions that typified the behaviors of our early level of composition: the Piece Assembler. When Mary began working on the above described puzzle man, she tried (correctly) to place a trapezoid in the foot as shown. The trapezoid was 180° opposite of the orientation needed to fill the frame and Mary, through her minor rotations in each direction, was unable to arrive at the requisite orientation and thus rejected the piece. Moments later, she uses the trapezoid to fill the arm, this time successfully rotating the shape to match the frame.



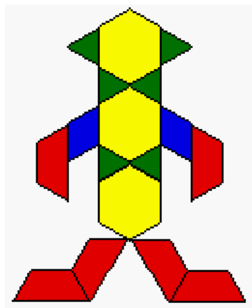
After similarly filling the other arm, Mary returned to the legs and concatenated 4 squares to (incorrectly) cover one leg, and two to cover the other before deciding to move on to another item. Using squares inappropriately shows that Mary was not attending to angle, a behavior typical of all of the children categorized as Piece Assemblers.

Kevin, a grade 1 student (6.0 years) demonstrated the actions used frequently by children at the Picture Maker level. The "picking and discarding" strategy that typifies this level can be observed repeatedly as Kevin attempts to fill the dog puzzle item. As Kevin tries to fill the puzzle, shapes are selected and "tried out" for a fit through placement and manipulation of the shape directly on the puzzle; there is a notable lack of the construction of a mental image of the shape and its relationship to the puzzle frame. The accompanying figure displays the puzzle nearly completed. Kevin attempted to fit a rhombus into the open space, then a square, and eventually, unable to fill the open frame, he rejected the arrangement and cleared away the shapes in the head. He then placed a trapezoid along the left side of the head and similarly on the right, thereby creating two simple frames that did allow him to complete the puzzle.



The Picture Maker level of thinking evinced by Kevin precedes the Shape Composer level which is exemplified in the work of Alice, a grade 2 (7.4 years) student. In this level we no longer observe the random selection of pieces, but rather deliberate selections are made as the child creates a mental image of how the shape may fill the frame. In addition, the process of completing a puzzle often becomes systematized.

As Alice worked on the puzzle man she first placed a trapezoid on one arm, then the other, followed by a rhombus on each arm. Similarly, Alice carefully considered how to fill the leg as she looked back and forth from the shapes to the puzzle prior to making her selection. Once she filled one leg with two trapezoids, she was able to think of the concatenated pieces as a whole and simply duplicated the process on the other leg. She filled the body of the puzzle man in a similarly systematic way with the finished puzzle reflecting this nicely.



Across all 60 children, examination of the items indicating attainment of each level similarly confirmed the hypotheses. If a child evinced a level of thinking on one item, they were more likely than not to attain it on the other items measuring that level. Most exceptions involved the highest level the child had attained; many children, unsurprisingly, were in the process of developing that level of thinking, so that scores were mixed. With few exceptions, once a higher level was reached, children had mastered the vast majority of items at each lower level. Quantitative summaries supported these conclusions. Computing the percentage of children whose scores followed a pattern consistent with the hypothesized trajectory, we found that 84% of the children followed the pattern exactly. Of the 16% that did not, all but one broke the pattern in the same way: they scored slightly higher on the Substitution Composer than the Shape Composer level. Older children, and those with previous experience in geometry, tend to evince higher levels of thinking. Total scores for PreK, K, 1, and 2 were 3.28, 9.91, 12.3, and 12.5.

Conclusions and Future Research

Our research reveals that children move through levels in the composition and decomposition of 2-D figures. From lack of competence in composing geometric shapes, they gain abilities to combine shapes into pictures, then synthesize combinations of shapes into new shapes (composite shapes), eventually operating on and iterating those composite shapes.

The next phase of this research is to evaluate the usefulness of the present findings for instruction and to assess children longitudinally in teaching experiments. We have created a sequence of activities aligned with the learning trajectory and will engage children from preschool to second grade in these activities, charting their development through the learning trajectory

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