THE THEORY OF LIMITS AS AN OBSTACLE TO INFINITESIMAL ANALYSIS

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ABSTRACT

This paper reports on a study aimed at determining how students would respond to an introduction to intuitive infinitesimal ideas once they were granted that these constitute legitimate mathematical knowledge. During six meetings a group of four freshmen in a calculus course for physics students worked on the basic ideas of calculus, including the second fundamental theorem, with the support of CorelDraw zoom. Following a method of data collection used by Sierpisnka [1987], we asked the students to make a demonstration to the whole class. We report on the outcome and discuss the theoretical implication in terms of Bachelard's concept of epistemological obstacle.

INTRODUCTION

The pioneer work of Abraham Robinson [1966], granting that infinitesimals are a legitimate mathematical notion, generated a new branch of mathematics, called *non standard analysis* [Stroyan and Luxemburg, 1976, Cutland, 1988, Nelson, 1977]. For the sake of the readers who need a briefing on these ideas, we reprint the following excerpt, whose range and conciseness we have not been able to improve.

> "The concept of infinitesimal, of an "infinitely small" quantity, has met a variable fate along history. Banished by some, used in heuristic but circumspect ways by others, the infinitesimals until very recently, did not have a right of citizenship in mathematics, moreover after the 19th century analysts introduced in the differential and integral calculus, via the ε-δ, the cannon of rigor that came up to our days. Of course, the physicists and engineers persisted in their intuitive usage of infinitesimals but the mathematician knew that all this could (and should!) be replaced by a rigorous discourse evacuating all notion of an actual infinitely small" [Hodgson, 1994:157].

The endeavor of 19th-20th century mathematicians to get rid of infinitesimals has created: 1) an abyss between mathematics and its uses in other sciences and techniques [Harthong, 1983]; 2) the need to supply information to non-specialist mathematicians, introducing them to elementary and intuitive ideas about infinitesimals [Kossak, 1996]; 3) a problem for mathematics education, timidly tackled so far, in spite of appeals to bring infinitesimals back to school [Harnik,1986, Grattan-Guinness, 1991]. Keisler [1986] made an attempt to produce an infinitesimal

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calculus textbook launching the idea of infinitely powerful microscopes and lunettes to "see" infinitesimal and infinite numbers. Stroyan [1998] renews such an attempt, in a recent software-supported textbook, using infinitesimals but without naming them so. Bell [1998] develops nilpotent infinitesimals, more suited to interpret some passages of Leibniz, where $(dx)^2$ is not just an infinitesimal of higher order, but it is actually zero. Some teaching experiments are reported by Tall [1980a, b, 1982].

The following example illustrates how strongly limit conceptions are taken as the only legitimate mathematical interpretation and how strongly infinitesimal conceptions in the students' culture refuse to die. Czarnocha et al [2001] investigate students' conceptions about the definite Cauchy-Riemann integral. First they made clear a genetic decomposition, making explicit their structured set of mental constructions about the integral. This genetic decomposition is infinitesimal-free. Then they interviewed students and finally they adjusted their genetic decomposition according to the data collected. The students' reported views are that "the limit of the Riemann sum is seen as the infinite sum of the rectangles of small width" or "of zero width" [p. 297]. However, the authors did not interpret this outcome as an emergence of an infinite-infinitesimal mental construction and insisted on a limit-conception interpretation: "rather than the limit of the sum of the areas of n rectangles the students state it as the sum of the limit of the areas of the rectangles" [ibid.].

Infinitesimal conceptions have been observed to emerge as *epistemological obstacles* in attempts to teach limits [Cornu, 1983, 1991, Sierpinska, 1987]. In these studies, infinitesimal conceptions were not asked for, but they emerged spontaneously from students' speeches. Our basic research question is then the following: what if, instead of waiting for the infinitesimal conceptions to emerge, we stimulated them? What if the students became aware of the abyss between mathematics and its applications produced by the, now unfounded, discrimination of infinitesimals? Will such an awareness stimulate the transposition of the obstacle towards the understanding of limits or will it create new obstacles? In what sense is there an obstacle?

THE SETTING AND METHOD

On order to boost the political dimensions of our research we chose to do the experiment in a regular calculus classroom. We chose a freshmen 2001-course for physics students[2] for whom the language of infinitesimals would be largely used in later courses given by the Physics Department. Since infinitesimals are not part of the calculus courses syllabuses, we expected that this choice would assure the legitimacy of the taught object so as to minimize the criticism from the colleagues from the Mathematics Department. We selected a group of four students willing to participate in the experiment and followed the method used by Sierpinska [1987]: during six videotaped encounters of two hours during class time but in a separate room, we

[2] We received stimulus and support from the responsible teacher, Prof. Dr. Miriam Godoy Penteado, Graduate Program on Mathematics Education, UNESP, Rio Claro, SP, Brazil.

introduced the students to the intuitive basic ideas and a brief history of infinitesimals. In the last meeting, we asked them to prepare a demonstration for the whole class. We did not interfere in their preparation. In our country, students do not generally come in contact with limits in high school, and by the time of our first meeting, the teacher had not introduced them. Since these physics students do not take analysis courses, we made no attempts at formalization, neither defining hyper-real numbers via classes of sequences of real numbers [Lindstrom, 1988], nor providing axioms to decide the truth of propositions [Tall, 1982, Keisler, 1986]. The four participants received bonus in the course and the written mid-term had a question that could be answered optionally by limits or infinitesimals.

Our presentations used a CorelDraw zoom in the following instances: to visualize the merging together of a curve and its tangent line at the tangency point P, to visualize that curve and tangent appear as parallel straight lines at $P+dP$, to visualize the merging together of the segments representing an infinitesimal arch, dx, and sin (dx), to visualize the difference between cos (dx) and 1, and to visualize the trapezium with parallel sides $f(x)$ and $f(x+dx)$ in integration. In all these cases we asked the students to foresee what would come out, before we showed them the zoom on the computer.

We introduced the sing $\approx$ to indicate "*infinitely close to*". The infinitesimal increment of the dependent variable was defined according to Leibniz and Robinson as $df = f(x+dx) - f(x)$ and called as *almost differential*. We maintained $f'(x)\ dx$ to the standard differential. The derivative was defined as what one gets from the infinitesimal quotient df / dx by neglecting the infinitesimal excess and retaining only the real part; it was interpreted as the constant of almost proportionality between df and dx, so that $f'(x)$ was equivalently defined by $df \approx f'(x)\, dx$. Several calculations were carried out. Here is an example:

$y = x^3 = f(x)$	$dy = dx(3x^2 + 3xdx + dx^2)$
$dy = (x+dx)^3 - x^3$	$\frac{dy}{dx} = 3x^2 + 3xdx + dx^2$
$dy = x^3 + 3x^2dx + 3xdx^2 + dx^3 - x^3$	
$dy = 3x^2dx + 3xdx^2 + dx^3$	$f'(x) = y' = re\left[\frac{dy}{dx}\right] =$
	$re[3x^2 + 3xdx + dx^2] = 3x^2$

The chain rule was proved as an immediate consequence of the definition (box 1). The definite integral was introduced as the *expression* of the area under a curve *in terms of* an infinite sum of infinitesimal parcels. Once the students believed that the sum of the infinitesimal variations dF due to an infinitesimal partition of the domain interval $[a, b]$ equals the variation $F(b)–F(a)$ in the whole interval, the second fundamental theorem of calculus was proved in one line (box 2) and expressed in the statement: *the area is the variation of a primitive.*

$h(x) = f(g(x)) = f(u)$ $u = g(x)$ $dh \approx f'(u)du \approx f'(g(x))g'(x)\, dx$ $QED.$ $\quad$ $Box\ 1$	$dF \approx F'(x)dx = f(x)dx$ $F(b) - F(a) = \int_a^b dF \approx \int_a^b f(x)\, dx$ $QED.$ $\quad$ $Box\ 2$

DATA OUTCOME AND COMMENTS

In a pilot study, we had asked the class of the year 2000 for a written report on the naught point nine classical question. We got 16 equalities, 16 "less than" and 5 "less than or equal to" answers, but only one evoked a limit conception, though in imprecise terms: "this number may be written as limit, then we should have $\lim_{x \to 1} x = 1$. So, among the properties of limits, this number may be considered as being 1, because it gets closer and closer to 1". The remaining justifications, either of = or <, did not differ qualitatively. They indicate that the notation has a strong appeal to concept images [Tall & Vinner, 1981] through categories such as: 1) **process**: "closer and closer to", "the number 0.999... tends to infinity and it becomes so close to 1 that..."; 2) **negligible difference**: "...becomes so close to 1 that one may say that this number has the same value as 1", "because experimentally, this number is rounded off"; 3) **the unthinkable**: "they are so, so close that... they may be considered as equal"; 4) **physical sense**: (from the 2001 study:) "depending on where you are going to use it, you may consider them as equal, but in fact they are not", "(used) in a rounding off, in a measure, you put 1 to facilitate understanding"; 5) **infinitesimal**: "because there is a gap of $1x10^{(-\text{infinity})}$ " ; "it will always be less than 1 because in spite of tending to 1, a number will always fail to spot it as equal to 1"; (from the 2001 study:) "be as close as they can be and the difference as small as it can be, one is always greater". Every calculus teacher should, be aware that such are the spontaneous conceptions of their students if s/he is committed to teaching limits.

In the first 2001 meeting we asked whether the students had ever heard about infinitesimals and whether they recalled any word, phrase or figure related to them. We got answers such as: "I have heard of infinitesimals related to very small points. They are negligible points. Infinitely small points. They are related to infinite decimals. Small points that can be neglected in some calculations. I heard infinitesimal related to fractal, in the sense of something that tends to the uni-dimensional. It diminishes so much that it loses dimension. You may neglect them depending on your point of view, depending from where you look at them. (Pilot-2000 study:) It is something very small indeed, so small that you cannot imagine it, it tends to zero so small it is". There is nothing new here, except the notion of fractal that looks like a nice teaching device.

We then continuation we introduced the hyper-real line through the usual diagram and gave a provisional definition: An infinitesimal is a number smaller than any real positive number. We asked for examples. Here are discussion excerpts: "If you imagine a very small number, 0.000…1 it is always possible to put one more zero before the 1 and it will become smaller. It can be very close to zero but is not zero, so you can say that there are numbers that are even smaller than it". Nobody thought of zero nor of negative numbers. We asked whether zero would be an infinitesimal. Three students agreed, but one argued: "If an infinitesimal is positive, than it is greater than zero, not equal to zero. So I think that zero does not belong in this classification". The novelty here is the emergence of a concept image of an infinitesimal as a small amount of matter, like a particle, which would not make sense to think of as negative. Along these lines, Hanna & Jahnke ask: "What is the possible role of arguments from physics within mathematical proof, and how should this role be reflected in the classroom?" Hanna & Jahnke [1999:74].

After some calculations of derivatives, we asked the students how they justified dropping off infinitesimals. We got answers based on concept images such as: "The infinitesimal part is so small that it can be neglected. The infinitesimal in this case are not going to make much of a difference". But we also got answers indicating retention of the formal concept definition [Tall & Vinner, 1981]: "The infinitesimals do no exist in the reals. When you calculate the derivative you take only the real part". This kind of answer (the derivative is the real part *by definition*) was only received by the mathematics of 20th century and could not have been given by Newton and Leibniz to Berkeley, were they still alive by the time The Analyst was published. In the face of these two last answers, can we say that "we have the impression that students do not ascribe cognitive values to definitions, they seem to perceive them only as labels which are not relevant to mathematical work" [Furinghetti & Paola, 2000:345] ? Or is this conclusion only valid when definitions and images differ considerably, which *is the case* of limits, but *not* of infinitesimals?

Here are some excerpts from the students' discussion at the end of one of the meetings: "In physics we have to imagine the situation, we have to imagine what happens in a very small space, with tiny dimensions. The zoom helps. In the physics class the teacher also spoke about the zoom, taking an infinitely close view, but it was not clear for all the students". They seemed to feel privileged for being able to understand the teacher. "In the infinitesimals you see the reality of the thing; you keep the sense of the approximation. In the limits you approach the thing; you do not see what is behind. You have to believe. (In the infinitesimals) we understand the part we are neglecting, we see it. In the limit, you don't. In our calculus course, or even in any area, it would be more coherent to work with these ideas. It is interesting to see how this has developed, to show both sides. By studying the history you understand". We think that this remark requires no additional comments. We intend to check these students next year to see if their esteem for infinitesimals persists.

From the classroom demonstration, we collect the following excerpts, many of them enthusiastic. "This part of the subject matter is very interesting, a totally different universe. Sometimes you embark on a trip that leads nowhere. If you take a chronometer and record the time it takes for you to get home, you may notice that it will never leave zero". (Students from the class: Zeno? Zeno! Some laughed.) "You start perceiving that numbers are infinite, and this makes a big difference. You start thinking about this universe of which the infinitesimals are a part. There is even a special set, the hyper-reals, in which the infinitesimals are included". The students seemed to praise the existence of a "set" to lodge the infinitesimals; they felt that this made infinitesimals more legitimate. A student from the class asked: "Can you give an example of where you use such infinitesimals?" The answer: "You must be working with the hyper-reals. Consider extremely small particles in such a way that any variation could alter your result. If you are working with rays or with subatomic particles, maybe these infinitesimals will have to be included. If you are dealing with dilation, this infinitesimal could make a lot of difference. If you calculate the dilation coefficient of an iron bar, say the rails in a railroad, in the coupling of one bar to another, in the end it will make much of a difference in the dilation".

Here we can note the mingling of mathematical, continuous, infinitesimal conceptions with the physical, discrete, subatomic reality. The total dilation was conceived as the integration of infinitesimal dilations as if matter were continuous. It is reasonable to conclude that the whole research process led the students to spontaneously enlarge their concept images so as to incorporate elements from the microscopic physical world among infinitesimals. This inclusion has already been praised by a researcher in quantum mechanics:

> "We may regard the physicist who studies the macroscopic behavior of a phenomenon whose microscopic behavior is too complex for him, as a limited observer who cannot apprehend but the shadow of things. The microscopic behavior will be described by non-standard functions" [Harthong, 1983:1200].

On one hand, we may see that the inclusion may be praised as going in the direction recommended by Hanna & Jahnke [1999]. On the other hand, it may be criticized as leading to Bachelard's obstacle of quantitative knowledge: "magnitude as a property of [physical] extension" [Bachelard, 1980:211].

CONCLUDING REMARK: THE THEORETICAL FRAMEWORK

Bachelard's concept of epistemological obstacle has been evoked to describe the mode in which infinitesimal notions emerge in the learning of limits. The directive upon which we based our research is clearly indicated by the following "must": "The construction of pedagogical strategies for teaching students must then take such obstacles into account. It is not a question of avoiding them but, on the contrary, to lead the student to meet them and to overcome them" [Cornu, 1991:162]. There have been many misunderstandings about the concept of obstacle. Errors indicate obstacles but obstacles do not reduce to errors nor to mere difficulties. We rely on a thorough reading of Bachelard who states:

> "Errors are not only the effect of ignorance, of uncertainty, of chance, as espoused by empirist or behaviorist learning theories, but the effect of a previous piece of

knowledge which was interesting and successful, but which now is revealed as false or simply unadapted. Errors of this type are called obstacles" [Brousseau, 1997:82].

The difficulty resides in the "dialectical character of the process of overcoming an obstacle" [Brousseau, 1997:88] which stems from Bachelard: "In fact we know against a previous knowledge, destroying ill-formed knowledge, transcending what in the very spirit makes obstacle to spiritualization" [3]. [Bachelard, 1980:14]. The emphasis here should be on the verb "to transcend", in French "surmonter", which is generally used in philosophy to translate the German verb "aufheben", meaning to conserve-in-surpassing. Reading the excerpt from this dialectical perspective, the "destruction" acquires the meaning of transformation, of *becoming*, and the "ill-formed" refers to the surpassed basis of future knowledge.

It is falsifying the concept to say that the overcoming of an obstacle is a matter of making the original, insufficient, malformed knowledge disappear and replace it with new a concept which operates satisfactorily in the new domain. This distorted view of Bachelard's concept supports the commitment that pervaded the twentieth century teaching of advanced mathematics: erase infinitesimals and write limits. This uni-dimensional view is unable to think of limits and infinitesimals except in terms of "either…or…", whereas we say that both should be present. Robinson's non standard analysis is not a return to Leibniz; it is a theory posterior and subjected to Hilbert-Weierstrassian rigor.

If it is clear how infinitesimals play the role of obstacles in learning limits; it is not so clear how the theory of limits creates an obstacle, in the precise sense of Bachelard, to the learning of the now rigorous infinitesimal analysis and its use in Mathematics Education. We hope that this paper takes one step toward bringing this question to the fore.

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[3] "En fait, on connaît *contre* une connaissance antérieure, en détruisant des connaissances mal faites, en surmontant ce qui, dans l'esprit même, fait obstacle à la spiritualisation"

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