

TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE: GRAPHS, FROM A COGNITIVIST TO A SITUATED PERSPECTIVE

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This report builds on our previous work on graphical conceptions using a diagnostic tool specially constructed to elicit graphical misconceptions, but also designed to function as a questionnaire for assessing teachers' pedagogical content knowledge (PCK). In this study we investigated 12 teachers' judgements of the difficulty of the items, their proposed learning sequences and their awareness of errors and misconceptions. We present a teachers' perception of difficulty hierarchy and contrast it with the learners' difficulty hierarchy. Results showed that teachers' judgement of what is difficult is structured by their curriculum and also their knowledge to be highly sensitive to the methodology adopted to collect it. This provokes us to develop a situated, social practice perspective on teacher knowledge, in which tools and instruments mediate teacher knowledge and its impact on practice.

INTRODUCTION AND BACKGROUND

This paper extends previous work on teachers' awareness of their pupils' errors and misconceptions, in the context of graphs, which Shulman (1986) classified as Pedagogical Content Knowledge (PCK). Shulman refers to PCK as knowledge 'which goes beyond knowledge of subject matter per se to the dimension of subject-matter knowledge for teaching' (p.9), which includes 'the ways of representing and formatting the subject that make it comprehensible to others' (p.9) and hence relies on an appreciation of learners difficulties and misconceptions. This categorisation was designed to draw attention to the traditional (but not historic!) underemphasis on PCK as opposed to other forms of teacher knowledge such as subject matter knowledge. He proposes that such knowledge is required in the triple forms of propositional, case-based and strategic knowledge. These might include research knowledge transformed for teaching, e.g. empirically-based *propositions* organised theoretically or conceptually, but rich in examples of memorable, prototypical *cases* (the analogy with case study seems applicable) and the *strategic* judgment to use the knowledge effectively in practice. In this paper we discuss teachers propositional and case knowledge of their learners' graphicacy.

Leinhardt et al (1990) reviewed the literature on functions and graphs and said, "*of the many articles we reviewed almost 75% had an obligatory section at the end called something like 'Implications for teaching' but few dealt directly with research on the study of teaching these topics*" (p. 45). A little later, Norman (1993) characterised research on teachers' knowledge on functions and graphs as insufficient or even non existent. He also stresses that 'there is little in the research literature documenting either what teachers know or the nature of their knowledge' (p.180). Williams (1993) also argues that 'the study of functions and graphs with an eye

towards informing teaching and learning is in its infancy' (p.314).

We would add that the 'teaching implications' drawn from research on the psychology of learning mathematics are in any case in general problematic: for many reasons these rarely impact on practice. In particular teachers need to know at which stages of their development pupils are likely to exhibit the researched misconceptions and errors and where in the curriculum they are relevant. Williams and Ryan (2000, 2001) produced such data for errors scattered across the curriculum. Hadjidemetriou & Williams (2001) developed similar work by focussing in depth on the area of 'graphical understanding and interpretation' relevant to years 9 and 10 of the mathematics curriculum, using a diagnostic instrument based on previous cognitive research in the field. In this study we have developed this instrument to function as a questionnaire for assessing teachers' awareness of the difficulties and errors this diagnostic instrument reveals.

METHOD

The development of the original pupils' diagnostic assessment instrument is described in Hadjidemetriou and Williams (2001). Briefly, we mention that it involved the tuning of, or the development of, diagnostic items from the research literature on graphicacy which related appropriately to the following errors: slope-height confusion; linearity-smooth prototypes (Leinhardt et al, 1990); ' $y=x$ ' prototype; the 'origin' prototype; graph as 'picture' (Clement, 1985); reversing or misreading co-ordinates; misreading the scale (Williams and Ryan, 2000). The test can be seen in full on the web at <http://www.education.man.ac.uk/Ita/ch/>.

In this study the diagnostic test was then given as a questionnaire to the teachers (N=12) with instructions that they should answer all the items and:

- predict how difficult their children would find the items (on a five-point scale starting from Very Easy, Easy, Moderate, Difficult, Very Difficult)
- suggest likely errors and misconceptions the children would make and
- suggest methods/ideas they would use to help pupils overcome these difficulties

Teacher's predictions of the difficulty of the items were subjected to a rating scale analysis (Wright and Masters, 1982). This provides an item-perception difficulty measure and consequently items can then be ranked according to their difficulty estimate. Their pupils' test results (N=425) were analysed using the Rasch model in order to get a pupil difficulty estimate for each item.

This data is used to explore the state of the subject matter and pedagogical content knowledge of this small group of teachers. The teachers were chosen as being thought to be knowledgeable, leading teachers involved in training and management. We ask: 'What do teachers know and what can they recall about their students' problems/difficulties in graphicacy'? The teachers were also interviewed using a semi-structured format based on the way they introduce graphs to their classrooms, problems and difficulties students have in graphicacy, and how they teach graphs. These themes were used also in categorisation during transcript analysis.

RESULTS

The questionnaire data were subjected to a rating scale analysis and the item-perception-difficulty measures that resulted were correlated with the pupils 'actual' difficulty as estimated by the test analysis ($\rho=0.395$). However, the teachers' estimates were significantly awry on seven items. When the seven worse items (having an absolute difference more than 2) were excluded from the analysis the correlation increased to $\rho=0.65$. Their mis-estimation of their (relative) difficulty could be explained by one of the following reasons:

1. in at least three items the teachers underestimated the difficulty for the children because they apparently misunderstood the actual question themselves, i.e. they had the misconception the item was designed to elicit, or they had a limited understanding that did not receive full credit. We interpret this as subject knowledge
2. on two items the teachers' overestimated the difficulty because they did not realise the children could answer the question without a sophisticated understanding of gradient. This was interpreted as pedagogical content knowledge.

'Story 3', was the name of the most discrepant item with teachers underestimating its difficulty. It required pupils to draw the 'Height of a person' from Birth up to late thirties. A closer look at some of the teachers' graphs below illustrates the problem. Prototypes such as the 'Origin' (pupils), and here teachers', tendency to draw all their graphs through the origin and 'Linearity' are dominant.

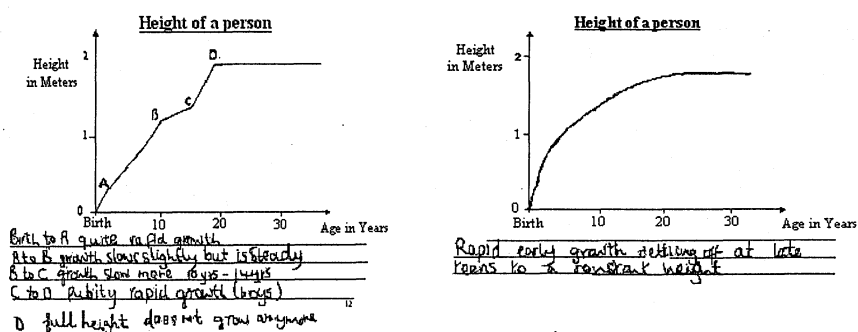


Figure 1: Two teachers' graphs for the 'Story 3' Item

'Transport 1' (shown below) was an easy question according to pupils' answers but some teachers seem to have given quite high difficulty ratings. These teachers believed that pupils had to be aware that the slope of the distance-time graph represents the speed of each transport. Pupils' transcripts verify that they could find the answer by looking at the time taken for each transport to travel to school:

INT: How can you see that it (A) is quick and that D is slower?

Sara: Because ...

Andrew: It takes more time.

Sara: Yes it takes more time. It takes more time to get to the same part. It takes 40 minutes to get to school and the others it takes 15, 10...

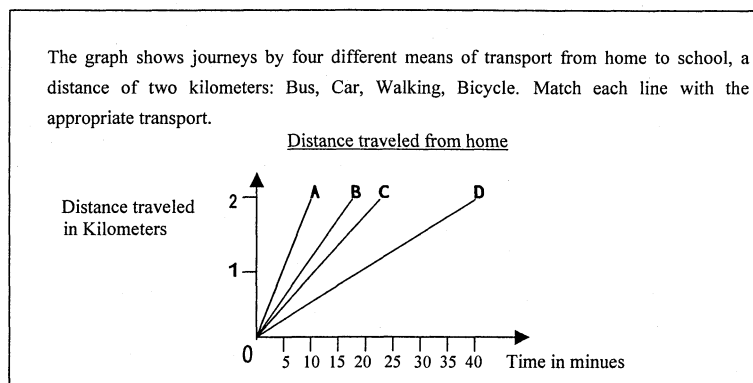


Table 1 shows teachers' proposed difficulty sequence described in 5 levels. Compared to pupils' hierarchy (Hadidemetriou and Williams, 2001) it yields significantly different results.

The table suggests that teachers have ranked the items in a 'curriculum' hierarchy. The bottom of the teachers' difficulty sequence is centred around pointwise reading. Further on teachers rate as slightly harder, items involving scales, parallel graphs and calculating the gradient. Interpretation of simple travel graphs is also included in this level.

In addition, this hierarchy matches the sequence of teaching evident in 6 teachers' descriptions of their curriculum during the interviews. They usually neglected the qualitative/interpretative perspective of graphs at the beginning of their teaching sequences and were preoccupied with abstract and algebraic aspects of graphs. For example:

SW: Starts of co-ordinates, study of co-ordinates. Exercising and using co-ordinates as positioning on a grid relative to a given plane, the origin. This is lower down the school, Year 7 Year 8. And then from there taken on to ordered pairs connections between x and y, mapping diagrams giving ordered pairs and then from that trying to take it on to equations. Straight lines and then on to curves.

INT: Straight line and then curves.

SW: And then when they've done that the use of obviously, em, apply to everyday sort of situations as well.

Levels	Brief Description	Description of Teachers' Difficulty Sequence
5	The idea of slope in the context of the 'rate of change'	Understand rate of change in an interval and instantaneous rate of change. Harder interpretation of 'constant rate' graphs.
4	Slope is adequately mastered and applied to situations involving linear graphs or curves.	Parallel graphs have the same gradient, speeds interpreted as slopes: same speeds are drawn parallel on graphs [Hard] Distinguish slope from height. Understanding no change or steady change. Understanding the 'covariance' of a graph. Interpreting discontinuous graphs.
3	Introduction of curves. Harder interpretation of linear graphs.	Understand varying slope of a curve (e.g. $y=x$ -squared). Harder interpolation on $y=x$ -squared. Overcoming the 'graph as picture' misconception by pointwise interpretation.
2	More complex reading and the introduction of slope (calculation and simple interpretation).	Understanding calculation of gradient of a graph ($y=4x$) [Easy] Use of scales in graph reading. Understanding of varying slope (linear graphs). Interpretation of simple travel graphs [Hard]
1	Pointwise reading of graphs, extracting information from points.	Use of unfamiliar co-ordinates. Compares y-ordinates of two graphs in context. Interpreting the meaning of (0,0) in context [Easy] Reading co-ordinates off a graph by interpolation and extrapolation. Reading co-ordinates off a graph.

Table 1: Teachers' Perception of Scale of the items of the Diagnostic Questionnaire
(in bold are major teachers' misjudgements i.e. relative misordering compared to pupils' hierarchy and in brackets, whether the items became Harder or Easier)

In the first two levels then, construction/algebraic related items are dominant. Teachers have only included here one contextualized task (the item called 'Transport 1' described above) and this resulted in the algebraically related items shifting downwards from their actual difficulty. For example calculating the gradient was an item located at the top of the pupils' difficulty sequence whereas teachers have awarded it a rather moderate difficulty.

Another underestimated item (Interpreting the meaning of (0,0) in context), located at the bottom of the teachers' difficulty scale (together with the rest of the coordinate-related items) was an item asking 'Why does the graph pass through the origin?'

Teachers' failure to consider possible terminology problems (pupils who were not aware of the meaning of the word 'origin') shifted this item downwards in difficulty.

Teachers' attention to the abstract perspective of graphs also explains why the difficulty of several qualitative tasks involving interpretation (the slope height confusion) and sketching graphs to tell a story were overestimated.

This bias also had an impact on the errors and misconceptions they mentioned during the interview. The transcripts refer mainly to errors/difficulties such as: reversing the x and y, mapping an equation to the graph, substituting negative numbers, inaccuracy in plotting, calculating the gradient as 'x over y', generating points from equations and misreading the axes. However in the questionnaire the teachers were also encouraged to list the misconceptions that children might exhibit. Here we summarise the misconceptions they mentioned during the interview or in the questionnaire:

TEACHER MISCONCEPTION	1*	2*	3 i	4 i	5 q	6 i	7*	8*	9*	10*	11*	12*
Slope height					4q				4q	4q	4q	4q
Linearity						4i						
Y=X prototype											4q	4q
Origin prototype											4q	4q
Picture as graph			4i	4i			4q	4q	4q	4q	4q	4q
Co-ordinates			4i	4i			4iq	4i		4i	4iq	4q
Scale	4i	4i			4q	4i	4i	4i	4i	4iq	4iq	4iq

Table 2: Misconceptions mentioned by 12 teachers in interview or in the questionnaire
q/i/qi: indicate whether the misconception/error was mentioned in the Questionnaire (q), Interview (i) or both (iq)

*: indicates the teachers who were both interviewed and answered the questionnaire

In summary we found that the 12 teachers had (a) most difficulty in identifying the linearity (1/12 teachers) 'origin' and 'y=x' prototype (2/12 each) conceptions, (b) moderate difficulty in identifying the slope-height confusion, and the problem with order of co-ordinates (5/12 and 7/12 teachers), and (c) least difficulty in identifying the picture-as-graph and misreading of scale problems (8/12 and 10/12 teachers respectively). However, teacher knowledge of the seven different misconceptions varies dramatically, with half the teachers mentioning only one or two of them, and two of the teachers mentioning all but one of them.

Most strikingly, indications of teachers' knowledge seem highly sensitive to whether the data comes from the Questionnaire or the Interview. The different data sources suggest that much of the teacher's knowledge is tacit, and elicited when provoked by an example question. Only explicit propositional knowledge is usually suggested spontaneously in interview without the questionnaire prompt.

CONCLUSIONS AND DISCUSSION

A diagnostic test designed from the graphacy research literature and calibrated for

pupils diagnostic errors was here further developed as a tool to investigate teachers' knowledge about their learners. This was supported by semi-structured interviews.

The conclusion is that:

- (a) some teachers harbour misconceptions themselves, revealing some weaknesses in subject knowledge: eg the linearity prototype
- (b) very few of the common errors or misconceptions are called up spontaneously in questioning at interview, and these generally concern the technical and algebraic aspects, not the 'interpretative' misconceptions, but
- (c) a much wider range of errors are offered in response to the 'diagnostic' questionnaire, and
- (d) there is some mismatch of the teachers' perception of difficulty and the students actual difficulty, with teachers underestimating technical aspects of graphing and overestimating the difficulty of the interpretative.

Empirical evidence suggests that teachers' knowledge as elicited from the interviews was structured around their curriculum descriptions, was rich in algebraic and abstract elements of graphs but lacking in the interpretational. However, the questionnaire acted as a tool that bridged the gap, apparently bringing to the surface tacit awareness of their children. This is consistent with a theoretical approach which insists that knowledge is situated, even distributed (Hutchins, 1990), and confirms our belief that researchers can have an impact on teaching through the development of such pedagogical tools and instruments. As Engestrom (1987) puts it, one of the roles of R&D activity on an activity system is to develop more advanced tools and artifacts of various kinds which mediate the practice of the system. In our case we see a well-designed diagnostic assessment as just such an instrument for advancing practice.

However, we are less sanguine about the nature of the knowledge that these teachers were able to evidence with the diagnostic tool. We draw a distinction between an error, i.e. erroneous responses to a question, and a misconception which may be a faulty cognitive structure that lies behind, explains or justifies the error. (Some errors may be symptomatic of a misconception, while others may not). If teachers can predict their pupils' errors, does this mean they can diagnose them? This diagnosis is an essential link between 'case knowledge' (in the sense of knowledge about typical errors based on classroom practical experience) and 'propositional knowledge' (in this case knowledge of students conceptual development and misconceptions). We suggest that the link between 'case knowledge' and 'propositional knowledge' is generally best conceptualised not just as a cognitive one, but one which is socially structured, and in particular it can be mediated by tools-in-practice. We will develop this theoretical analysis of cognitive and situated perspectives on Shulman's categorisation further in the presentation.

The purpose of the paper was not to generalise empirically from these teachers' pedagogical content knowledge but to suggest a methodology for evaluating and maybe developing this knowledge. There seems to be a gap between pupils' difficulties and teachers' perception of these difficulties. Our concern is to provide research findings and propose a methodology that will help to bridge this gap.

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