ON LIMITED VIEWS OF THE MEAN AS POINT OF BALANCE

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It is clear to every high school student that the mean of two values is half their sum. Is it also clear that the mean is their point of balance? Not quite. In the course of studying pattern exploration disciplines of high school computer science majors, we noticed that a non-negligible number of students lack a clear view of the mean as the point of balance. The students were asked to design a computer program that inputs N, a positive integer, and outputs all the positive integer pairs <x,y> which average N/2. The majority of the students demonstrated limited orientation with patterns of the mean. In particular, a considerable number of them designed programs that "search for each x the y's which average N/2 with x". The student solutions, together with representative interviews, reflect diverse levels of mathematical insight and pattern recognition.

INTRODUCTION

One of the fundamental concepts in basic statistics, as well as everyday life, is that of the mean (average). The mean is introduced in primary school and invoked in numerous occasions thereafter, in mathematics, science, and other domains. The computation procedure of the mean is almost as fluent as the fundamental arithmetic operations of addition and division. However, the conceptual understanding of the mean is far less assimilated. In this paper we display a study on the limited conceptual understanding of the mean with respect to its notion as the center point of balance.

The lack of conceptual understanding of the mean was studied during the last two decades in several respects and for different group ages. Pollatsek, Lima, and Well (1981) reported on difficulties in conceptual understanding of the weighted mean among college students. Mevarech (1983) expanded their study and showed that non-mathematically oriented students "misconceive a set of given means under simple mean computation as ... satisfying the four properties of closure, associativity, identity, and inverse" (1983, p. 425).

Strauss and Bichler (1988) studied development aspects of understanding seven mean properties among young students of the ages 8-14. They found that the property of 'average as representative' was difficult for students to see. Mokros and Russell (1995) focused in mathematical representativness of the mean, and identified five basic approaches among 4^{th}, 6^{th}, and 8^{th} graders, and named them from the less insightful to the more insightful - average as 'mode', 'algorithm', 'reasonable', 'midpoint', and 'mathematical point of balance'. Cai (1998) noticed that a significant number of 6th graders were unable to correctly solve a contextualized average problem. Those who used algebraic representations performed better than others.

These studies demonstrated the gap between procedural and conceptual knowledge of the mean. While students are easily able to calculate the mean by straightforward sum-and-divide, many lack understanding of its meaning and have difficulties in performing insightful mean-related tasks. The gap between procedural and conceptual knowledge is apparent in a variety of mathematical domains, as discussed by Hiebert and Lefevre (1986), Bin-Ali and Tall (1996), and others. The lack of conceptual knowledge is strongly related to indications of the need for elaborating pattern exploration in mathematics (e.g., Schoenfeld 1992).

This paper extends the view on conceptual understanding of the mean, together with the formulation of arithmetic expressions. In addition, it illustrates the important role of pattern exploration and exploitation. Different from previous studies, in which students where questioned on particular values of elements, our study focuses on general formulation of tuples (pairs) of elements with the same mean. Its core emphasis is on conceptual understanding and arithmetic expression of the mean as the center point of balance. The participants were high school students, rather than college or elementary school students in the earlier studies.

The study was part of a broader inquiry into mathematical pattern exploration of high school computer science majors in solving algorithmic tasks. The broader study involved a series of short algorithmic tasks with emphasis on efficient algorithmic solutions. Although we expected some inefficient student solutions, we were surprised to see the extent of inefficiency and the lack of pattern recognition regarding the concept of the mean. As a result, we decided to carefully examine, analyze, and display the solutions.

In the next section we present the methodology used, involving both qualitative and quantitative evaluations. In the section that follows we display the results and focus on several solution categories that reflect gradual levels of mathematical insight. In the last section we discuss the results, relate them to previous work, and note on tying statistical concepts to pattern exploration and exploitation.

METHODOLOGY

Participants

The study involved 82 computer science majors (36 10th graders and 46 11th graders), with sound mathematical background. The students were selected from four different schools in different geographic locations in the country. All students had basic programming background, and were particularly aware of the fundamental concepts of correctness and efficiency of algorithms.

Task

In the broad study, the students were asked to design algorithmic solutions to four algorithmic tasks aimed at examining their mathematical pattern exploration. The students were explicitly told that they should write efficient solutions based on mathematical patterns they identify. We let each student write the solutions in his/her most convenient language – either algorithmic pseudo-code, or one of the programming languages Pascal or C. The mean-related algorithmic task was:

> **Same-Average-Pairs**:
>
> *Design* an algorithm/computer-program which inputs a positive integer N and outputs all the pairs <x,y> of positive integers which average N/2.
> *Write* a short explanation of your underlying idea.
>
> Note: As our focus was on mathematical pattern exploration, we told students that each of them can choose, according to his/her programming convenience, whether to consider pairs of the form <i,j> and <j,i> distinct. They could choose to output either both pairs or only one of them.

Our purpose in posing this task was to examine whether students view the mean as the center of balance of all pairs with the same average, and whether they can express it arithmetically.

Procedure

The students were given up to 25 minutes to solve the above task (as part of 90 minutes provided for all four tasks). One source of our data was the 82 solution sheets we obtained from the students, with algorithms and descriptions of their underlying ideas. In addition, nine of the students were asked to elaborate on their solutions in individual interviews.

RESULTS

The solutions to the task varied considerably. Only one third of the students demonstrated a clear view of the mean as the center point of balance. The rest of the students showed limited degrees of insight. We refer to the different degrees of insight through five patterns that characterize the task output:

- *element range* – the range of the output elements is 0..N.
- *unique pairing* – no range element has more than one pairing element.
- *pair symmetry* – the two elements of an output pair are located on two different 'sides' of the mean, at the same distance from it.
- *pair arithmetic* – one way of arithmetically expressing pair values in the task is: <N/2-d,N/2+d>. Another way is: <d,N-d>. Since we are interested in integer values, the second expression is more convenient to utilize.

- *pair adjacency* – adjacency in the sequence of 'same-mean pairs' corresponds to adjacency on the line of natural numbers.

We divided the student solutions to four different categories based on their insight of the above patterns.

- **Clear-View**: Solutions that reflect clear view of the five patterns.
- **Incomplete-View**: Solutions that reflect incomplete view of the *pair arithmetic* and the *pair adjacency* patterns.
- **Search-View**: Solutions that reflect no clear view of the *unique pairing*, *pair symmetry*, *pair arithmetic*, and *pair adjacency* patterns. These solutions are based on inefficient search "for the y's that match each x".
- **Dim-View**: Solutions that do not reflect any clear pattern. Some of these solutions only include a search for pairs of the form <x,x> or <x,x+1>.

In what follows we describe each of the above solution categories.

Clear-View Solutions

34% of the students (28 of the 82) provided algorithmic solutions that reflect clear view of the mean as the center point of balance of the output pairs. The code typical for these solutions was:

```
for i from 0 to N do
   write (i, N-i)
```

Those students who chose not to output both <x,y> and <y,x> bounded the "for" loop by $\lfloor N/2 \rfloor$ (rather than N). This loop outputs the pairs starting from the range ends towards the mean. A couple of students output the pairs in the opposite order – from the mean towards the ends. Below are two student explanations.

Omer: The mean of two numbers is exactly their middle, thus if we increase the smaller by 1 and decrease the bigger by 1 we will preserve the same middle point, which is the mean.

Nimrod: Every two numbers have the same mean as the mean of the lowest number (of the range) plus a constant and the highest number (of the range) minus that constant; and that constant may be from one of the (range) ends until half the difference between them.

Both students explicitly expressed the patterns of *pair symmetry* and *pair adjacency*. Their algorithmic solutions expressed the additional patterns.

Incomplete-View Solutions

18% of the students (15 of the 82) provided algorithmic solutions that reflect partial view of the mean as the center point of balance. They all noticed the *element range* and the *unique pairing* patterns. Many also had 'a picture' of *pair symmetry*. But their notions of *pair arithmetic* and *pair adjacency* were incomplete. They noticed that each element within the range 0..N may be in at most one output pair, but they were not clearly convinced that each element will indeed be in the output. Their programs generated the relevant symmetric pairs, but checked for each pair whether its mean indeed equals N/2. One representative solution was:

```
for i from 0 to N do
   if (i+(N-i))/2=N/2 then write (i, N-i)
```

We wondered how is it that the students who provided this solution did not see that the condition in the **if** statement is completely unnecessary. The interview below sheds some light on that:

Interviewer: Amit, what is the idea underlying your solution?

Amit: When you take a number from the beginning and a number from the end, 0 and N, and then add and reduce 1, you get the potential pairs.

Interviewer: What do you mean by 'potential pairs'?

Amit: 0 and N have the mean N/2, also 1 and N-1, ... as **i** grows you get more pairs ...[pause]... I am not sure about some of the numbers further on ...

...<some additional interaction>...

Interviewer: Can you explain the condition (i+(N-i))/2=N/2 in your code?

Amit: For example, for i=0 it will be true ... then, for 1 also ... then ...???

Amit did not immediately notice the algebraic equality in the condition that he himself wrote. While he did have some notion of symmetry (<0,N>, <1,N-1>), he had a difficulty in expressing it with an arithmetic expression. He also had a vague view of the complete sequence of output pairs.

Search-View Solutions

32% of the students (26 of the 82) provided solutions which reflect a view that does not adhere to any of the patterns: *unique pairing*, *pair symmetry*, *pair arithmetic*, or *pair adjacency*. Many simply generated all the <x,y> pairs in the range 0..N (a total of $(N+1)^2$ pairs). Some let x be in the range 0..N/2 and y in the range N/2..N. The only pattern to which these student solutions clearly referred was *element range*. A common solution in this category was:

```
for i from 0 to N do
   for j from 0 to N do
      if (i+j)/2=N/2 then write (i,j)
```

From interviews with some of the students we learned that they did not have a clear picture of the output pairs, and quite a few were not sure of the *unique pairing* pattern. The following interviews with Kfir and Anat illustrate that.

In his solution, Kfir enumerated both **i** and **j** from 0 to N.

Interviewer: Kfir, why did you write the nested loops with **i** and **j**?

Kfir: I needed to find for each element the elements that can be paired with it.

Interviewer: Can you say something about the number of elements that may pair with a particular element?

Kfir: Well … 1 has only one match, but 2 may have more …

Interviewer: How many? For example let's say that N=100.

Kfir: Then, 1 has one match – 99; but 2 may have many …?

Interviewer: ???

Kfir: Eh, … 98 is one match …<trying further>… maybe only 98? … Yes …

He gradually started realizing *unique pairing*, *pair symmetry*, and *pair adjacency*.

In a different class we interviewed Anat. Her solution included the enumeration of: **i** for 0 up to N/2 and **j** from N down to N/2+1.

Interviewer: Anat, what is the rational underlying your solution?

Anat: I divided the numbers into two groups – those from 0 to N/2 and those from N/2 to N; for each number in the first group there is a search for a match in the second group.

Interviewer: Why did you do that?

Anat: When we deal with average you cannot immediately tell both numbers; I need to loop through the 'upper' group in the order to search for corresponding matching …

Later in the interview Anat conjectured the *unique pairing* and *pair symmetry* patterns, but she had a difficulty to formulate a proper arithmetic expression.

Dim-View Solutions

16% of the students (13 of the 82) provided solutions that reflect no clear recognition of any of the five patterns. From their solutions it was even unclear whether they noticed that the range of element pairs is 0..N. Some designed solutions that output a single pair: <N/2,N/2>, or <N/2-1,N/2+1>. The latter case shows some 'picture' of symmetry, but only with respect to one output pair. Some provided solutions in which all the pairs of the form <i,i+1> were generated and checked for the possibility of having the mean N/2.

Interviews with two students who provided solutions in this category revealed that they searched for some time to find pair examples. They mentioned rather quickly the output pair <N/2,N/2>, for the case that N is even. The case of odd N was more subtle. We posed the task of generating the pairs which to N. This assisted one of them, who noticed the similarity to the original task. Shortly after, she managed to write down relevant pairs, mentioning that "the sum question is much easier". We got this response from students of other categories as well (see next section).

DISCUSSION

The results of this study showed diverse levels of recognition and exploitation of patterns of the mean. The task posed to the students involved the formulation of a general scheme, based on the patterns common for all pairs that have the same mean. As in all the previous studies of conceptual understanding of the mean, here too, the task focused on a perspective more involved than the simple computation of sum-and-divide. The new contribution of this study stems from examining the formulation of a general expression rather than calculation of values for particular examples. In addition, the participants were high school students (rather than 4^{th} to 8^{th} graders or college students).

In order to properly answer the task, students had to identify and utilize the five patterns *element range*, *unique pairing*, *pair symmetry*, *pair arithmetic*, and *pair adjacency*. The pattern of *element range* is related to the property of extreme points examined by Strauss and Bichler (1988). The pattern of *pair symmetry* corresponds to Mokros and Russell's midpoint representation. The pattern of *pair arithmetic* is correlated with Cai's (1998) observation of algebraic representation.

Our division of the diverse solutions into different categories characterizes four different levels of mathematical insight with respect to conceiving the mean as the center of balance. The first, Clear-View category, includes the students who realized and efficiently exploited all the five patterns. Only one third of the students were in this category. The second, Incomplete-View category, includes about a fifth of the students, who realized the first three patterns but did not clearly see the latter two. Their solutions were still rather efficient. The third, Search-View category, includes almost a third of the students, who exploited the first pattern, but not any of the other four. Their solutions were correct, but very inefficient. The fourth, Dim-View category, includes the students with very vague insight.

We posed to the interviewed students the task of calculating all the pairs of elements which sum to N (rather than average N/2). The first, Clear-View category students instantly noticed the identity between the two tasks. Students from the other categories noticed it more gradually, and indicated that the sum task is easier (to some - "much easier") then the average task. One explanation of this phenomenon is the gap in familiarity between the notions of the sum and the mean. Apparently, conceptual understanding of the sum is more developed, even for high school students, than conceptual understanding of the mean.

Shaughenessy (1992) indicated, in his review on reflections in probability and statistics, the limited perspectives presented to students. One study that elaborates perspective diversity and conceptual understanding of the mean is that by Meyer, Browning, and Channell (1995). Their study introduces four activities for elementary and middle school students, involving concrete values. Our study suggests the need for further insight elaboration of the mean, for more mature students. General formulation, based on recognition and exploitation of patterns of the mean, as in the task presented here, should enhance conceptual understanding as well as competence in arithmetic and algebraic representation.

The results in this study add not only to the study of statistical conceptions but also to the research on limited pattern exploration. In particular, it illuminates the need for elaborating mathematical insight in algorithmic problem solving. It is our hope that this study will encourage further elaboration of students' pattern exploration both in statistics and algorithmics.

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