PROVING OR REFUTING ARITHMETIC CLAIMS: THE CASE OF ELEMENTARY SCHOOL TEACHERS

Ruth Barkai, Pessia Tsamir, Dina Tirosh and Tommy Dreyfus
Kibbutzim Teacher College Tel Aviv University

Abstract: We examined elementary school teachers' justifications to number-theoretical "for all" propositions and existence propositions, some of which are true while others are false. We also assessed whether teachers regarded their justifications as mathematical proofs. About half of the teachers produced formal algebraic proofs. A smaller number of teachers produced non-formal proofs appropriate for presentation in elementary school classes. However, a substantial number of teachers applied inadequate methods to validate or refute the propositions. Finally, many teachers were uncertain about the status of the justifications they gave.

The processes of examining the validity of conjectures, proving correct ones as well as refuting wrong ones are at the core of any student's mathematical development. It is therefore essential that teachers are intimately familiar with and confident in producing and reacting to arguments that purport to prove or refute mathematical statements that are discussed in their classrooms. Prior research (Jones, 1997; Martin and Harel, 1989; Movshovitz-Hadar and Hadas, 1988; Simon and Blume, 1996) has shown that not all is well in this respect.

This paper reports on a study aimed at examining elementary school teachers' justifications to number-theoretical propositions, some of which are true while others are false. We also examined teachers' views of the status of their justifications, that is, whether they regarded them as mathematical proofs. In this paper we shall focus on describing the various justifications that the teachers provided to "for all" propositions and to existence propositions.

METHOD

Participants

A class of 27 in-service elementary school teachers (25 women, 2 men) participated in the study. All were in their first year of a three-year professional development program at an Israeli university, a course that mainly focused on introducing the mathematics topics that are part of the elementary school mathematics curriculum from an advanced mathematical viewpoint. The participating teachers were engaged in teaching mathematics, as well as other subjects, in the upper grades of elementary school (grades 3-6). Their teaching experience varied considerably: Eight were in their first five years of teaching, eight taught between six and fifteen years and twelve taught more than 15 years.

Tools and Procedure

A questionnaire, including the following six, number-theoretical propositions, was administered to the teachers during a 90 minute session:

"For all" propositions

1. The sum of any five consecutive integers is divisible by five.
2. The sum of any four consecutive integers is divisible by four.
3 The sum of any three consecutive integers is divisible by six.

Existence propositions

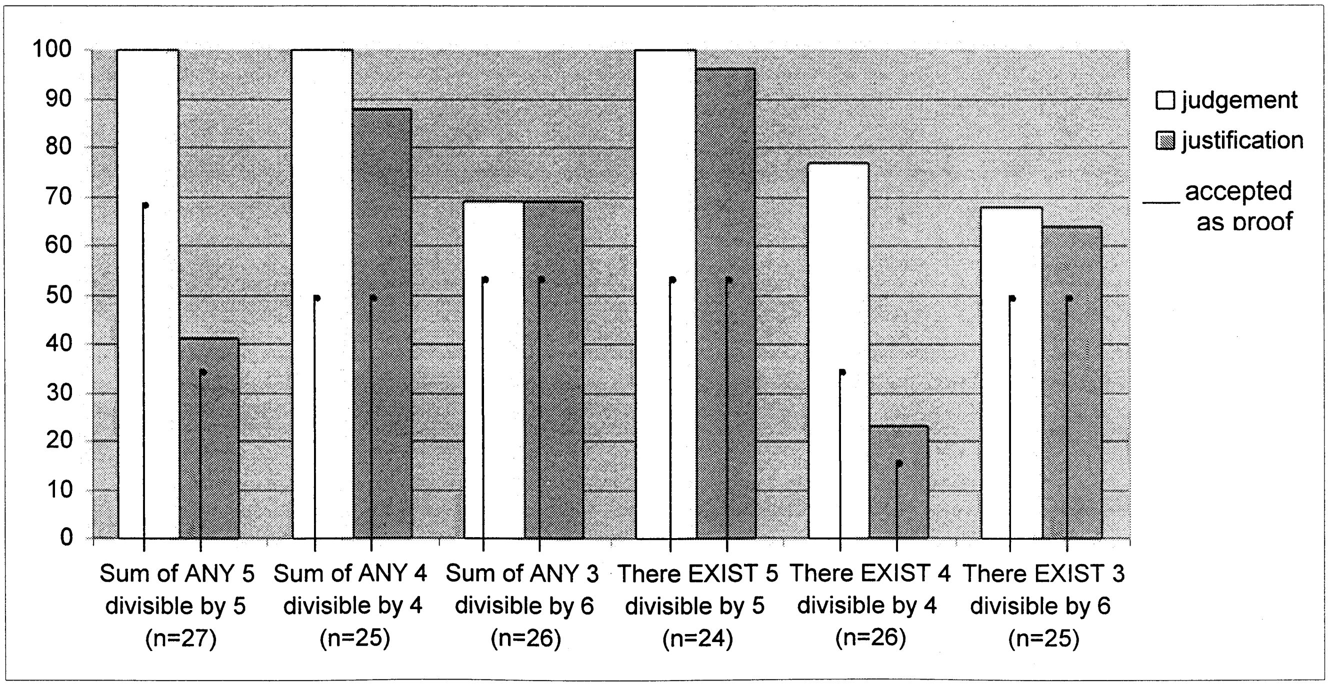
4. There exist five consecutive integers whose sum is divisible by five.
5. There exist four consecutive integers whose sum is divisible by four.
6. There exist three consecutive integers whose sum is divisible by six.

Propositions 1 and 4 are both true, because the sum of any sequence of five consecutive integers is divisible by five. Propositions 2 and 5 are both false, because no sum of four consecutive numbers is divisible by four. Proposition 3 ("for all") is false and Proposition 6 is true, because the sum of some sequences of three consecutive numbers is divisible by six while the sum of others is not. Half of the teachers responded first to the "for all" tasks while the other half responded first to the existence tasks.

The teachers were asked to consider each proposition and (1) decide whether it is true or false and justify their claim; (2) determine whether, in their opinion, the lecturers would consider their justifications as mathematical proofs. It was assumed that the lecturers were perceived by the participants as representatives of the mathematical community. Therefore, this was a measure of the extent to which the teachers viewed their own justifications as mathematical proofs.

RESULTS

The results of both the "for all" and the "existence" tasks are summarized in the following Figure. The figure presents, for each proposition, the number of teachers who responded to it, the percentage of correct judgments (left, white bar) and the percentage of correct justifications (right, shaded bar).



Furthermore, the figure presents the percentage of teachers who expected their judgments to be acceptable by the lectures (line inside left bar) and the percentage of teachers who had given a correct justification and expected it to be acceptable by the lecturers (line inside right bar). It should be noted that justifications were classified as incorrect only if they contained mathematical errors or if they were inadequate in terms of the methodology that was used to prove or refute a proposition (e.g., using supportive examples to prove a "for all" proposition or using a counterexample for refuting an existence proposition).

"For all" Propositions

Proposition 1: <u>The sum of any five consecutive integers is divisible by five.</u>

All participants correctly stated that this proposition is true. However, only 41% accompanied their correct claim by correct justifications (see Figure). Almost all these justifications were algebraic proofs (33%) and they were commonly regarded as proofs by those who provided them (30%). A typical proof was:

> Shelly: x+x+1+x+2+x+3+x+4=5x+10. Any number substituted for x is multiplied by 5, 5x is divisible by 5, 10 is a multiple of 5, so 5x+10 is always divisible by 5.

Two teachers, Odette and Anna, provided non-algebraic proofs. Odette attempted (and succeeded) to "cover all possibilities":

> Odette: I first tried all the possible examples of 5 consecutive numbers within the first ten numbers: 1+2+3+4+5=15, 2+3+4+5+6=20, 3+4+5+6+7=25, 4+5+6+7+8=30, 5+6+7+8+9=35, 6+7+8+9+10=40, 7+8+9+10+11=45, 8+9+10+11+12=50, 9+10+11+12+13=55, 10+11+12+13+14=60.
> All other sums of five consecutive numbers are created by adding one of these sequences to certain multiples of 10 (for instance, 44+45+46+47+48 is composed of 5 times 40 and the sequence: 4+5+6+7+8=30 that we had before. The number 40 is composed of 10 times a number and since 10 is divisible by 5, 40 is also divisible by 5). The same holds for three digit numbers (since 100 is divisible by 5), for four digit numbers, etc. So each such sum is divisible by 5.

Odette regarded her justification as a mere example, insufficient for proving the proposition. She wrote: "I still need to find a rule, beyond my examples".

Anna examined the general structure of any sequence of five consecutive numbers and gave the following, generic proof (Balacheff, 1987):

> Anna: When we add five consecutive numbers, for instance, 1,2,3,4,5, we can look at these numbers in the following way: We have the number in the middle (3), then, the sum of the two numbers that stand next to it from both sides (2 on one side and 4 on the other) is 3-1 plus 3+1. The -1 cancels the +1 and we get 2*3. Similarly, 5 is 3+2 and 1 is 3-2 so we have, again, twice the number in the middle. All in all, the sum is 5 times the number in the middle, and 5 times any number is divisible by 5.

Unlike Odette, Anna was sure that her justification is a mathematical proof.

Most incorrect justifications consisted of providing one or more supporting examples (52%). These justifications were perceived as proofs by 33% of the teachers. Two teachers provided an improper algebraic justification. They reached the expression 5x+10, but then "solved" it in the following manner:

> Sofie: x+x+1+x+2+x+3+x+4=5x+10. 5x=-10, x=-2. The answer is (-2, -1, 0, 1, 2). I got an answer; therefore the sum is divisible by 5.

While one of these two teachers viewed her justification as a mathematical proof, the other expressed reservations, stating that she was unsure whether one example (-2, -1, 0, 1, 2) is sufficient to prove a proposition.

Proposition 2: <u>The sum of any four consecutive integers is divisible by four.</u>

The figure shows that in this case, all teachers correctly stated that the proposition is false and all those who justified this assertion provided correct justifications to their claim (three teachers stated that the proposition is false but provided no justification). Most justifications consisted of one or more counterexamples (72%). Some correctly commented that a single counterexample is sufficient for refuting a claim while others doubted the status of such examples. Indeed, half of those who provided counterexamples regarded them as proofs (36%).

Sixteen percent of the teachers provided algebraic proofs and regarded them as proofs. A typical proof was:

> Ramit: x+x+1+x+2+x+3+x+4=4x+6, 4x is divisible by 4 but 6 is not divisible by 4. Therefore the sum 4x+6 is not divisible by 4.

Proposition 3: <u>The sum of any three consecutive integers is divisible by six.</u>

Here, unlike the previous cases, not all the teachers correctly judged the proposition as false (see Figure). While most teachers (69%) correctly stated that the proposition is false, 23% incorrectly claimed that it is true and 8% were unsure about its status.

All those who correctly argued that the proposition is false accompanied this judgment with correct justifications. Most correct justifications (42%) consisted of counterexamples. Maria, for instance, explained:

> Maria: 2+3+4=9, 4+5=6=15, 7+8+9=24, 8+9+10=27, 10+11+12=33, 14+15+16=45, 16+17+18=51, 27+28+29=51. I found many examples for which the sum is not divisible by 6. Therefore, the statement is incorrect.

Unlike Maria, Lily provided only one counterexample and noted that one such example is sufficient for refuting a proposition of this kind:

> Lily: This is a general proposition. We need one counterexample to prove that it is false: 0+1+2= 3 and 3 is not divisible by 6.

Of the 50% who stated that the proposition is false and provided one or more counterexamples, most (38%) expected the lecturers to accept them as mathematical proofs. Those who did not (11%), were either unsure that examples are sufficient to refute a given proposition or stated that examples are not mathematical proofs and further explained that they do not know the proof and therefore all they could do was provide some examples.

The remaining teachers who correctly stated that the proposition is false provided algebraic justifications (19% of the teachers) and all but one of them regarded these justifications as proofs. The most prevalent algebraic justification was:

> Lima: x is the first number in the sequence x+x+1+x+2=3x+3. The sum of this sequence is divisible by 3. We have to show that the sum is also divisible by 2. If x is even then 3x is even but if we add 3 we get an odd number, so 3x+3 is not divisible by 6. If x is odd then 3x is odd and 3x+3 is even and then it is divisible by 6. The proposition does not hold for all sequences. Therefore, it is false.

Miki gave another argument that used algebraic notation:

> Miki: 1+2+3=6, 6 is divisible by 6; 0+1+2=3, 3 is NOT divisible by 6; 4+5+6=15, 15 is NOT divisible by 6; 7+8+9= 24, 24 is divisible by 6.
> The sum of all sequences of three consecutive numbers x-1, x, x+1 is divisible by 3, because, x-1 + x + x+1 is 3x (-1 and +1 cancel each other and we get 3 times x). But 3x is divisible by 6 only if the middle number, x, is even.

All teachers who incorrectly claimed that the proposition is true provided supportive examples. Only one of them expected his justification to be accepted as a mathematical proof. Those who were unsure about the status of the proposition provided both supportive and counterexamples. They did not expect that their justifications would be accepted as mathematical proofs.

Existence Propositions

Proposition 4: <u>There exist five consecutive integers whose sum is divisible by five.</u>

All participants correctly stated that this proposition is true and all but one participant accompanied their correct claim by correct justifications (see Figure). The most prevalent justification consisted of one or more adequate examples (50%), yet, only half of those who gave such examples regarded them as mathematical proofs.

Two teachers explicitly related to the type of this proposition, arguing that one supportive example is sufficient for proving it. Ora, for instance, wrote:

> Ora: If the claim starts with "there exist" it is sufficient to provide one example to prove existence. In this case, 1+2+3+4+5=15, and 15 is divisible by 5.

Algebraic proofs were given by about a third of the teachers (38%). Odette and Anna, who provided non-algebraic proofs to the matching, general proposition (Proposition 1), repeated the same justifications for the existence proposition. This time, however, both viewed their justifications as mathematical proofs. We note that none of the teachers referred to the fact that Proposition 4 trivially follows from Proposition 1, even though 50% of them had worked on proposition 1 shortly before working on Proposition 4. Vinner (1983) has observed similar behavior among senior high school students.

Proposition 5: <u>There exist four consecutive integers whose sum is divisible by four.</u>

Most teachers (77%) correctly stated that this proposition is false. Yet, 15% incorrectly claimed that it is true and 8% were unsure about its status. This was the only proposition to which less than a quarter of the teachers provided both correct answers and correct justifications. Most correct justifications (19%) were algebraic proofs that were presented by the same teachers to both this proposition and to the matching "for all" proposition (Proposition 2). While 12% of the teachers viewed their algebraic justifications as mathematical proofs, 7% were unsure that such justification refutes the proposition.

Sami provided an original, correct justification:

> Sami: There are no four consecutive numbers whose sum is divisible by 4. I'll prove it in the following way:
> The sum of the first sequence of numbers: 0+1+2+3=6, is not divisible by 4.
> The second sequence is formed by enlarging each number in the sequence by 1. Instead of 0, 1, 2, 3 we get 0+1=1, 1+1=2, 2+1=3, 3+1=4 so we get 1, 2, 3, 4. The sum this time is 10 because we added 1 to each of the four numbers in the previous sequence. Since 6 is not divisible by four, and we added four, the sum (6+4) is also not divisible by 4. The next sequence 2, 3, 4, 5, is created in a similar manner, and the sum is 14. Consequently, the sum of each sequence is larger by 4 than the previous sum. Since we start with a sum that is NOT divisible by 4, the jumps by four always bring us to new numbers that are not divisible by 4.In this way we can show that the sum of no such sequence is divisible by 4..

Most teachers who incorrectly claimed that the proposition is true explained that they tried several examples and none of them fulfilled the condition. Yet, they felt that if they continued their search they would eventually find a supportive example. The teachers who were unsure about the status of the proposition went through a similar process (namely, trying, with no success, to find supportive examples) but they remained uncertain whether such an example exists.

Proposition 6: <u>There exist three consecutive numbers whose sum is divisible by six.</u>

Most teachers (68%) correctly stated that this proposition is true, and all but one of them accompanied their correct claim by correct justifications (One teacher provided no justification). The most prevalent justification consisted of one or more supportive examples (40%) most of which (28%) were regarded as mathematical proofs. The remaining justifications were algebraic and, much like in the case of propositions 1 and 4, they were identical to the algebraic justifications that were presented by the same teachers to the matching, "for all" proposition (Proposition 3).

The teachers who incorrectly stated that the proposition is false (24%) provided as justification one or more examples of three consecutive numbers whose sum is not divisible by 6. Most of them (20%) regarded these examples as counterexamples and expected the lecturers to accept their justifications as proofs. Amit, for instance, wrote "a counterexample is sufficient to refute any proposition, including this one". It seems that Amit, and others, overgeneralized a scheme that holds for refuting "for all" propositions and used it for refuting existence propositions as well. Two teachers (8%) could not decide whether the proposition is true or false, because they found both supportive and counterexamples.

FINAL COMMENTS

1. There is a wide consensus in the mathematics education community that teachers should encourage students to make mathematical conjectures and investigate them (e.g., NCTM, 2000). A growing number of studies identify conjectures, generalizations, refutations, and even proving among elementary school students (e.g., Ball and Bass, 2000; Lampert, 1990; Maher and Marino, 1996; Zack, 1997). Students present conjectures of different types (e.g., "for all", existence, true, false) and apply different ways to verify them. Teachers, the representatives of the mathematical community in class, have a crucial role in establishing the various sociomathematical norms in general and those related to justifications, argumentation and proofs in particular (Yackel and Cobb, 1996). To fulfill this role, it is crucial that teachers be knowledgeable about different types of propositions and ways of proving or refuting them. Our paper shows that a substantial number of elementary school teachers applied inadequate methods to validate and to refute various propositions. Two salient cases relate to the use of examples: Supportive examples were used by about half of the participants to prove "for all" theorems and about 20% of the participants refuted an existence proposition by counterexamples. These findings call for more attention to this aspect of elementary school teachers' knowledge in professional development programs and in research.

2. Teaching mathematical reasoning and proof in elementary school is a very demanding job. When an elementary school teacher is confronted with a student's conjecture, such as the ones given here, he/she should be able to both examine its validity and to find suitable, non-formal ways to communicate about this conjecture with his students. The propositions that were presented in this study could relatively easily be proved or refuted with algebraic tools. Thus, algebra is a powerful tool that could be used by teachers in elementary schools to determine the validity of number-theoretical propositions and other propositions that could be raised by students in class. However, our data show that only about half of the teachers used algebraic tools in their attempts either to prove or to refute at least one of the propositions. These teachers usually did so in an appropriate manner and most of them were confident that their justifications are mathematical proofs. Moreover, about 20% of the teachers were aware of a need for a more general proof for some propositions, but noted that they did not have the required knowledge. Thus, it seems essential that professional development programs attempt to enhance elementary

school teachers' algebraic reasoning so that they would be able to use this knowledge to determine the validity of their students' conjectures.

3. As stated previously, it is essential that elementary school teachers be able to present non-formal proofs in their classes. A number of such proofs were presented by some teachers in our study (e.g., Odette, Anna and Sami). It is important to find out how students and teachers conceived these justifications: Do they regard them as proofs? How do they explain their decisions? Do teachers who have better algebraic reasoning accept these justifications as proofs? These and other related issues are currently under investigation.

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