MAPPING OVERLAPPING WAVES OF STRATEGIES USED WITH ARITHMETICAL PROBLEMS

Robert (Bob) Wright
Southern Cross University
NSW, Australia

Peter Gould
New South Wales Department of
Education and Training

This study is based on a systemic, school-focused initiative and uses data from a very large number of children (23 121) who ages range from 4.5 to 9.9 years. Each child was assessed by their class teacher using an interview-based approach and a schedule of tasks. The study examines the use of Siegler's overlapping wave theory to map the range of levels of strategies used by children to solve addition and subtraction tasks. The levels of strategies are derived from psychological models developed by Steffe, and links are made to Gray's notion of preferential hierarchies. The results indicate that the overlapping wave theory is useful to demonstrate progression with age, of levels of strategy use.

A child provided with a pile of counters and asked to determine 6 + 3 could count out 6, then count out 3, and then count all nine items from one. This strategy which we call "counting from one three times" could be applied to a range of problems. Alternatively, the child might count-on from 7 to 9 and keep track of 3 counts. Strategies differ in the amounts of time their execution requires, in their processing demands, and in the range of problems to which they apply. On a problem-by-problem basis, children of the same age often use a wide variety of strategies (Geary & Burlingham-Dubree, 1989; Goldman et al., 1988; Gould, 2000; Siegler, 1988; Steffe et al., 1983). In a similar vein Carpenter et al. (1999) suggest that there is "a great deal of variability in the ages at which children use different strategies" (p. 26).

BACKGROUND

The Cognitively Guided Instruction (CGI) professional development program (eg Carpenter, Fennema & Franke, 1996) draws on word-problem research (eg Carpenter, 1985), and focuses on the ways teachers use knowledge of student's thinking in making instructional decisions. In New South Wales, a systemic professional development program, Count Me In Too (CMIT), has been developed to improve students' learning outcomes through a school-focused method of teacher learning (Bobis, 1997; Bobis & Gould, 1998), drawing on a research-based learning framework (see below). The learning framework used in the CMIT program is a synthesis of multiple research studies (Wright & Gould, 2000). CMIT emphasises the advancement of children's arithmetical solution strategies — a recognised need in teacher development:

> The research on addition and subtraction has identified a progression of concepts and skills that is generally not reflected in instruction. Most instruction jumps directly from the characterization of addition and subtraction using simple physical models to the

memorisation of number facts, not acknowledging that there is an extended period during which children count on and count back to solve addition and subtraction problems (Romberg & Carpenter 1986, p. 856).

Beyond recognising that children can use multiple strategies to solve arithmetic problems, the question arises: How do they construct such strategies in the first place? This question has been investigated from a range of theoretical viewpoints (Carpenter & Moser, 1984; Steffe & Cobb, 1988; Steffe, 1992). The research methods employed include longitudinal studies, constructivist teaching experiments and microgenetic studies — small-scale studies of the development of a concept (Siegler & Crowley, 1991; Kuhn, 1995). This paper reports a study involving a large population of children and uses the theory of overlapping waves of strategy use arising from the microgenetic approach (Siegler, 1996; Siegler & Jenkins, 1989). The overlapping wave theory is based on three assumptions: (a) children typically use a variety of strategies and ways of thinking to solve a given problem; (b) the diverse strategies and ways of thinking coexist over prolonged periods of time; and (c) experience brings changes in relative reliance on existing strategies and ways of thinking, as well as introduction of more advanced approaches.

Stages in number development. According to Fuson, "children in the United States display a progression of successively more complex, abstract, efficient, and general conceptual structures for addition and subtraction. Each successive level demonstrates cognitive advances and requires new conceptual understandings" (1992, p. 250). The CMIT program uses a research-based Learning Framework in Number (LFIN) (Wright, 1998; Wright & Gould, 2000; Wright, Martland & Stafford, 2000) which has as one of its key components, a progression of conceptual structures which we refer to as the Stages of Early Arithmetical Strategies, and which is based on psychological models developed by Steffe et al. (eg Steffe, 1992):

Emergent: A child who is an emergent counter may have some number knowledge but it is generally made up of discrete pieces of information. For example, a child may know some of the sequence of number words and be able to identify some numerals while still being an emergent counter.

Perceptual: A child at the perceptual stage can count perceived items, matching the number word sequence to the items.

Figurative: A child at the figurative stage can determine the total in two concealed collections of items but typically counts from one to do so.

Counting-on-and-back: A child at the counting on stage uses advanced count-by-one strategies to solve a range of addition and subtraction tasks. A number takes the place of a completed count and a child can count on or back to solve problems.

Facile: A child at the facile number sequence stage can use a range of strategies other than counting by one. This includes a part-whole knowledge of numbers that enables children to draw on doubles or known combinations to five or ten.

Emphasised in the LFIN is that children frequently use strategies that are less sophisticated than those of which they are capable. This may happen for one or more reasons. A child may use a basic strategy because it is easier and although it may take more time this may not be of concern to the child. Alternatively, some feature of the child's thinking immediately prior to solving the current task may focus the child's attention on a less sophisticated strategy than the child is capable of. The emphasis in LFIN just described accords with the first two listed assumptions of Seigler's overlapping wave theory (see above).

Preferential hierarchies. Gray (1991) investigated the strategies used by 72 children to solve addition and subtraction tasks. The children were equally spread across the six age ranges of 7+, 8+, ... 12+ at each of three teacher-defined ability levels - below average, average and above average. Addition strategies were classified as known fact, derived fact, count-on, or count-all; and subtraction strategies were classified as known fact, derived fact, count-up or count-back, or take-away. Gray describes these four-level classifications of addition and subtraction strategies as preferential hierarchies. If unable to solve a task by immediate recall (ie known fact) the child reverted to what might be regarded as a preferred level. Gray identifies "two distinct approaches to the regression" (p. 569-70):

> The first makes use of other known knowledge, the deductive approach [used mostly by above average and average children]. The second is dominated by the use of counting, the procedural approach [used mostly by below average children]'....What has become fairly clear ... is that the below average ability child is neither successful at learning the number bonds nor in making use of the ones that they do know .... [For younger below average children] memory is abandoned for a procedure that involves the use of physical or quasi-physical objects. The bits they do know do not appear to be held together, with the result that this change in strategy may involve the child in long sequences of counting....In contrast, condensing the long sequences appears to be almost intuitive to the above average child. This eventually becomes the cornerstone to their higher level of attainment; they can take short cuts and operate with increasing levels of abstraction.

Gray's preferential hierarchies suggest that students might use strategies that are less sophisticated than those of which they are capable.

Mapping strategy use. Carpenter and Moser (1984) identified inconsistency of strategy use as an issue. "When children have several strategies available, they often use them interchangeably rather than exclusively using the most efficient one. Even when a more efficient strategy like counting-on from larger has been acquired, children often revert to a less efficient strategy like counting all" (p. 189). In the CMIT program, strategy use is documented using an interview-based assessment which we call the Schedule for Early Number Assessment (SENA) (NSW Department of Education and Training, 2000) and the subsequent analysis of children's responses. Each child's performance is recorded as the highest level of strategy use demonstrated during the interview. Thus the child's highest level of strategy is taken to indicate the child's level of conceptual development in number in terms of the model of stages of early arithmetical strategies.

Research question. This report addresses the following research question: Is the overlapping wave theory useful to demonstrate progression with age, of levels of strategy use?

METHOD

The data reported here is derived from analyses of individual interviews (using SENA) of 23 121 children, across 327 schools in New South Wales. The children were in classes from Kindergarten to Year 4 and ranged in age from 4 years 6 months to 9 years 11 months. Each child was interviewed by their classroom teacher at the start and end of the CMIT classsroom project. In each school the team of participating teachers was assisted by a district-based mathematics consultant trained in CMIT. In each class, the assessment interviews of at least three students were videotaped. These tapes were used in consultant-led, professional development meetings focusing on children's solution strategies. The meetings constituted a forum for collaborative planning of instruction aimed at advancing children's strategies. As well, the consultants assisted the classroom teachers with assessing, analysing and planning for teaching. Analyses of the assessment interviews enabled determination of each child's stage of early arithmetical strategy, and levels of facility with number words and numerals (eg Wright, Martland, Stafford, 2000). This report focuses only on the children's stages of early arithmetical strategy, as determined in the first assessment interview, that is, the initial interview. Any child whose data set was incomplete was excluded from the analysis.

RESULTS AND DISCUSSION

Table 1 shows the numbers of students in each age group, determined to be at each stage of early arithmetical strategy use, at the time of their initial interview. These numbers are consistent with the notion that children's highest level of strategy use tends to increase with age.

Table 1.

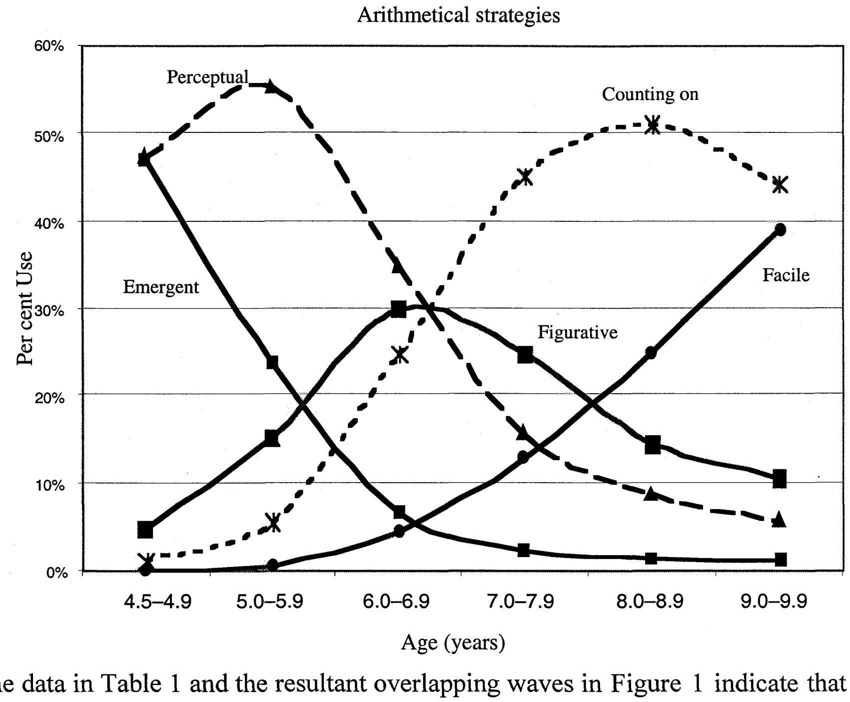
Number of children at a each stage, for each age group (n = 23 121)

	4.5 – 4.9 yrs	5.0 – 5.9 yrs	6.0 – 6.9 yrs	7.0 – 7.9 yrs	8.0 – 8.9 yrs	9.0 – 9.9 yrs
Emergent	552	1406	388	113	39	18
Perceptual	555	3254	2032	825	262	97
Figurative	56	887	1749	1321	441	182
Counting on	13	322	1445	2413	1563	772
Facile	0	29	259	682	765	681
Totals	1176	5898	5873	5354	3070	1750

Figure 1 is obtained from the grouped frequency distribution data in Table 1 as follows. For each stage of early arithmetical strategy a line graph is plotted across the six age ranges using the cell numbers from Table 1 expressed as percentages. Each cell number is expressed as a percentage of the total number in the corresponding age range. Curve smoothing is then applied to the resulting line graphs. The overlapping wave theory suggests that children in a given age range have access to more than one level of strategy. Figure 1 gives an indication for each age range, of the proportion of children for whom a given strategy level is their most advanced.

Figure 1.

Strategy use by age



The data in Table 1 and the resultant overlapping waves in Figure 1 indicate that the overlapping waves theory is useful to demonstrate progression with age, of levels of strategy use. All of the levels of strategy use are present in the population at any given age range with the exception of the facile level at the lowest age range (4.5-4.9), but there is significant variation in the degree to which levels of strategy use occur. Perceptual counting appears to peak between 5 to 6 years of age and counting-on appears as the dominant strategy used between 8 to 9 years of age. A significant range and change of strategy use appears to occur between 6 to 7 years of age. A positive feature of this approach to mapping children's levels of early arithmetical strategy use against age is that it incorporates both a sense of directionality in children's learning and a sense of children's movement among the levels. At the same time, applying overlapping wave theory in this way does not capture the insidious nature of inefficient strategies. Inefficient arithmetical strategies continue to be used by some students long after they need to, simply because they work. Thus inefficient strategies can be very persistent. A child asked to find 8 + 3 could count out 8, then count out 3 and finally count all the objects to obtain an answer. If this strategy persists in later years, the amount of mental effort needed to obtain the answer can make it difficult to achieve a necessary cognitive reorganisation.

Overlapping waves theory specifies four dimensions along which learning occurs: (a) acquisition of novel ways of thinking; (b) more frequent use of the more effective ways of thinking from among the existing possibilities; (c) increasingly adaptive choices among alternative ways of thinking; and (d) increasingly efficient execution of the alternative approaches (Siegler, 2000). Clearly, Gray's (1991) notion of preferential hierarchy has implications for the second and third mentioned dimensions. In reviewing video-taped assessment interviews we have seen that occasionally, children become quicker at using inefficient strategies. As well, children do not necessarily spontaneously progress to higher levels of strategy use. Thus it seems that instruction should focus directly on the development of higher levels of strategy use.

CONCLUSION

One of the features of assessment in the CMIT project is explicit acknowledgement that children are not always consistent in their choice of strategy. Consequently, the assessment tasks have been designed to elicit the most advanced strategy that children can demonstrate. Although a child might be able to count on or use non-count-by-one strategies, it does not mean that he or she will always do this. Competent adults might revert to perceptual counting for comfort when placed under stress or if materials are present. In our view, any approach to mapping children's solution strategies needs to take account of changes of strategy use. This study shows that an overlapping wave model of strategy use is can be applied to a large cross-sectional sample of children. Mapping individual strategy use as a function of age is a useful graphic organiser for capturing students' arithmetical strategies. The overlapping wave model constitutes a broad response map for displaying children's solution processes, describes the flow of development as observed in the population and suggests movement between the various strategies is possible at a range of ages. Thus a model arising from a microgenetic approach can be applied in a macrogenetic cross-sectional study of arithmetical solution strategies. This is consistent with the view that "the most important aspect of a research study is the constructs and theories used to interpret the data" (Kilpatrick, 1981, p. 27). Changing the theoretical lens need not distort the view.

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