QUADRATIC FUNCTION AND IMAGERY: ALICE'S CASE

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Abstract: The present paper is part of a more extensive study whose essential objective is to identify and analyse mental images that are constructed by students during mathematics learning process. In particular, it is one's intention to analyse the role that these images play in the realization of a mathematical task and to understand the origin of their development.

In this work, two mental images are analysed; these images are evoked by a 10th grade student while solving a task on the roots of a quadratic function. The origin of the identified images will be discussed while considering the use of a graphic calculator.

INTRODUCTION

The concept of function, considered by many investigators as one of the most important mathematics concepts, has acquired a central role in mathematics school curriculum (e.g. Ponte, 1984; Dubinsky & Harel, 1992). In spite the different didactics approaches suggested to introduce this concept - some years ago, this was done using a more algebraic perspective and, now, it is taking place using a perspective that values global understanding of the concept, considering all of its representations, with an emphasis on the graphic one, and making the most of the new technological potential (e.g., Eisenberg, 1992; Sfard, 1992) - it still seems to maintain its complexity in what respects the teaching-learning process (e.g., Sierpinska, 1992).

In fact, there is evidence that highlights the proliferation of mental images developed by students (e.g., Bakar and Tall, 1991; Resnick, Schwarz & Hershowitz, 1994; Ruiz Higueras et al., 1994) during the concept's learning process. These images and/or conceptions - rich probably due to the multiplicity of representations that the concept involves and the level of abstraction that those representations require for their understanding - might be idiosyncratic or be directly related to the approach context used (e.g., Eisenberg, 1992; Monk, 1992).

However, students' imagery does not appear exclusively connected to this concept and many investigators have dedicated their work to its study because they believe that it plays an important role in the learning process of mathematics (e.g., Clements, 1981, 1982), in the activity of "making sense of mathematics" (e.g., Dörfler, 1991, 1995; Presmeg, 1992; Reynolds and Wheatley, 1992; Solano and Presmeg, 1995) or in the development of mathematical reasoning at all levels (e.g., Thompson, 1996).

In this article, a broad perspective of imagery will be used - in which an image is "much more than a mental figure" (Thompson, 1996) - associated to the theory[1] of mathematics meaning construction proposed by Dörfler (1989, 1991, 1995).

Dörfler elaborates his theoretical perspective on the construction of mathematical meaning based on the notion of the image schemata. According to this author, an image schema is a set of cognitive interactions with, and cognitive manipulations of, a concrete carrier (an object as model, a material model or just an imagined one, a drawing, a graph or other representation of a mathematical concept...) that allows an individual to obtain a personal meaning of the concept in question. The concrete carrier does not represent the concept; it only serves as a referent for the individual - it is the cognitive activity developed with and on the referent that allows for the concept to be "present cognitively and mentally" (1991, pp.21) to the individual. In this way, the same carrier may serve different image schemata corresponding to different concepts and, conversely, the same concept may allow for various image schemata based on different carriers - it all depends on what constitutes the main focus of the individual's attention and on the properties, relationships or transformations that this person constructs.

As a cognitive process, the image schemata is idiosyncratic and can not be shared by the individual and his/her speakers; only the carrier can be used in the communicative process and it is here - the communicative process - that subjective deviations are corrected (negotiation of meanings) and socially normalised carriers are supplied (Dörfler, 1991).

The mathematical meaning, according to Dörfler (1991, 1995), has a predominant holistic aspect that corresponds to the pertinent image schemata for the respective concept[2]. Each mathematical concept may allow for, as previously stated, various image schemata, based on different concrete carriers, and each adequate image schemata constitutes, for the individual, a mathematical meaning of the respective concept. This meaning, which derives from the set of image schemata and the respective concrete carrier, serves as a basis for the individual's memory and other cognitive functions such as, for example, argumentation and logical inference (Dörfler, 1991).

In what concerns the specific context of mathematical learning, this author defends that "the cognitive manipulation of mathematical concepts is highly facilitated by the mental construction and availability of adequate image schemata" (Dörfler, 1991, pp.20), in view of which the teaching of mathematics should privilege this

[1] A theory essentially based on the notion of Johnson's (1987) and Lakoff's (1987) image schemata.

[2] The author refutes the idea according to which the mathematical meaning of a concept is constructed accumulatively from basic particles whose meaning is predefined or previously understood (the Bourbakist school is an example of this construction).

construction and availability. As it is not possible to construct pertinent image schemata without adequate concrete carriers and, also considering that these are acquired through appropriate fields of experience, Dörfler suggests that mathematics teaching should constitute, in its entirety, a good field for experience. Not only will this situation offer an accessible set of normalised concrete carriers - as potential referents for mathematical concepts - but, also, it will basically orient individual construction and appropriate development of image schemata, without which mathematics learning only restricts itself to "the formal, rule-guided manipulation of symbols without referents" (1991, pp.22). He also defends, so that an adjustment of individual schemata can take place to socially normalised patterns, the continued use of image schemata that explicitly have, in a conscious form for the individual's understanding, the way in which they were constructed. For the conscious construction of image schemas in this context, he proposes protocols of actions and processes (1989, 1991).

In this present piece of work, part of an interview will be analysed. It seems to illustrate the existence of idiosyncratic images that interfere, in a decisive way, with the solution of a mathematical task. In particular, the way that two mental images are used is analysed; these images were evoked by a 10th grade student while solving a task on the roots of a quadratic function. The idiosyncrasy of one of the identified images is discussed, thus trying to establish its origin. The influence of the use of graphic calculators in the development of that image will be speculated.

DATA PRESENTATION AND ANALYSIS

The excerpt that is presented is part of a more extensive interview elaborated for a presently on-going investigation project.

The student, Alice (15 years old), is a 10th grade urban secondary school student. She likes maths and she does not easily give up when she finds tasks difficult.

The interview, which is semi-structured and task-based, consists of essentially two moments: (a) a moment[3] during which the student tries to explain and illustrate how many roots a polynomial function of the third degree has; (b) a second moment in which the student argues that a quadratic function has always two roots. In the present article, the arguments presented by the student during the second moment will be analysed (the first moment was reported elsewhere).

The analysis of this moment in the interview reveals the existence, in the student's way of thinking, of two images of "roots of a function": a visual image associated to the

[3] This first moment consists of two phases inserted by the analysis of the roots of the quadratic function; this is, the analysis of the quadratic function roots appears during the explanation of the cubic function roots: the student tries to explain that a cubic function has one, two or three roots (first moment). This explanation is interrupted so that the root of a quadratic function may be analysed (second moment); the cubic function is approached again to systemize the roots (conclusion of the first moment).

graphic representation of a function - *'the roots are the points of intersection of the graph with the xx axis'*; a non-visual image where she tries to establish connection between specific mathematical vocabulary - *'if it is of the third degree, it has three roots, if the second, it has two'*. These two images coexist and are evoked throughout the interview.

Roost of a quadratic function[4]: Alice's arguments

(...)

Interviewer: *So, the second degree has always two roots?*

Alice: *Hum. Yes.*

Interviewer: *Can you give me an example?*

The student draws the example in figure 1 - she begins by representing the axis, she draws without hesitation, from left to right, and highlights the points of intersection of the graph with the *xx* axis.

At this moment the investigator, trying to explore the stability of the idea 'that a quadratic function always has two roots', suggests the application, to this parabola, of a translation associated to a vertical vector. The student accepts the suggestion and draws the new situation (figure 2).

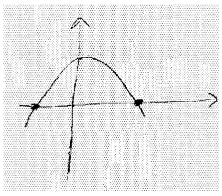

Figure 1. Second degree function with two roots.

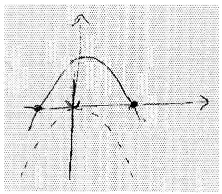

Figure 2. Translation applied.

Interviewer: *How many roots are there?*

Alice: *Also two... there are two; it's zero here* [she indicates the origin of the coordinates] *b'cause it's tangent...two.*

Interviewer: *And what if we apply another translation associated to a vector of greater length...*

Alice: *Down more?*

Interviewer: *Yes. How many roots will it have?*

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[4] During the first part of the interview, when explaining the roots of a cubic function, the student arrives at a conflicting point where, on the one hand, she affirms that a cubic function has one, two or three roots while, on the other, she states that 'if is of the third degree, it has three, if of the second, it has two'.

Alice: *Ahh, it'll probably have the same* [she pauses a few seconds - she looks at the paper and, with the pencil close but above the paper, she makes some gestures above and below the *yy* axis; she lightly sketches a point on the *yy* axis and draws a part of what would be the new parabola, she does this so lightly that you almost can not see it (figure 3)]. *But I don't know if it's still a second degree. Is it?*

Interviewer: *What do you think?*

Alice: *The form is, ... of a quadratic function.* [She pauses] *Hum. I don't think so. It's zero only here* [she indicates the vertex of the third parabola that she did not actually draw][5] *I don't think so* [pause] *but these* [referring to the second and third parabolas] *come from this one* [she indicates the first parabola], *they're written from this one and this is a quadratic function* [she points to the first*], this one... I think... is too* [indicating the second] *and this one...logically should be* [she pauses for a few seconds during which she looks at the paper. She shakes her head negatively and state]...*Maybe it is.* [She giggles].

The interview continues to explore the possibilities in determining the roots of a function, when it is defined by an equation and not by its graph, but the student is unable to do so. During the dialogue, another graph appears (figure 4) and the student is once again asked to analyse this function's roots.

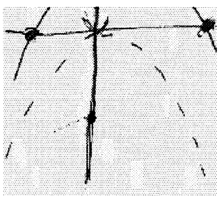

Figure 3. Enlargement of part of the third parabola.

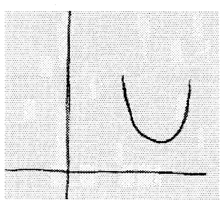

Figure 4. Another example of parabola.

Alice: (...) *there's always root, but...* [she pauses for a few seconds] *we can adjust here the graph* [she moves her open hand along the *yy* axes] *we make a translation...if we put the vertex here...*[she indicates the *xx* axes]...*it'll have a double root.*

The student's hand movement seems to imply that it is possible 'to adjust the axis' in such a way as to make the quadratic function always have roots, as if it were possible

[5] It is notorious to see how much the student wants the quadratic function to have roots to the point where the parabola's vertex is considered a root.

to physically hold the parabola and place it in a new more adequate position, in relation to the axis, so that the graphic could intersect the *xx* axis.

The student continues to think about the parabola and tries to find an argument that will allow her to accept the fact that a quadratic function may not have roots without contradicting her non-visual image. She tries to explain how, even after having stated that a quadratic has always two roots (and she continues to believe it is thus), it is possible to obtain a graph that does not intersect the *xx* axes.

Alice: *It could if...it can be written through this one here, ...if it is the result of a translation of a function...that has, that has...roots. This one is written through this one here above...*[she points to figure 2, the second parabola]*...it will be written...if it's the result of a translation, I think so.*

What the student seems to be trying to explain is that a quadratic may not have roots if it is the result of a translation applied to a quadratic that does; in other words, if the expression that defines it can be written from the expression of the original quadratic, associating to it the corresponding translation calculations. For this student, a translation does not seem to be an application that transforms a parabola into another parabola but, rather, something that allows for the moving of the parabola to another place: the parabola is always the same one, and since it is possible to put it in a convenient place so that it can have two points of intersection with the *xx* axis (or one tangent point), then any quadratic has, always, two roots.

DISCUSSION

The following discussion is based on a preliminary analysis of the data.

This episode seems to reveal the student's preference for her visual reasoning component, in this specific task, given that every time she is asked to supply an example, she does so using the graphic representation. This preference, reported, for example, by Norman (1992) but contrary to what other investigators have given accounts of, may be related to the fact that the graphic component was initially the most explored representation within the study of functions in the classroom (even though the quadratic function had been systematically focussed as a whole, which allowed for the interaction of the various representations) or, still, the continued use of the graphic calculator and, some times, of the computer.

The two evoked images of the 'roots of a function' - a visual one associated to graphic representation of a function, *'the roots are the points of graph intersection with xx axis'*; and, the other, a non-visual one related to specific mathematical vocabulary terms, *'if it is of the third degree, it has three roots, if of the second, it has two'* - coexist throughout the task appearing to take on a similar level of credibility for the student.

The visual image, *'the roots are the points of graph intersection with xx axis'*, is a standard image introduced by the teacher (and, also, by school textbooks) to help with graphic interpretation of a function.

The non-visual image, *'if it is of the third degree, it has three roots, if of the second, two'*, seems to be idiosyncratic given that no other subject involved in the study has referred to it. It also appears to have assumed a role of 'uncontrollable image', in Presmeg's (1992) and Aspinwall's et al. (1997) sense, given that it is maintained even when confronted with evidence to the contrary, and it overlaps the visual image not letting the student irrefutably accept the fact that a quadratic function may not have roots.

The origin of this non-visual image seems to be associated to a third one that interferes, in a direct and determinant way, with the task's resolution. For the student, a translation is not an application but 'a vehicle of transportation' that allows for the moving of the parabola in such a way that it may preserve one of its characteristics: to have always two roots.

This third identified image - one of translation seen as a 'vehicle of transportation' - might be connected to the kind of, and writing used in, some exploratory tasks, in the study of function families, using graphic calculators; for example: "Use your graphic calculator to study a and b parameter variation effect in functions of type $f(x) = ax^2 + b$. What happens to the graph when parameter b is altered?" Answers to a proposal of this type usually appear as, 'the function graph has suffered a translation associated to the vector...' or 'the parabola suffered a displacement of ... units'. In this specific case, phrases such as 'the function graph suffered ...' or 'the parabola suffered...' may lead students to develop images such as the one evoked by Alice. (This point needs further investigation.)

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