„ ... BECAUSE A SQUARE IS NOT A RECTANGLE“ – STUDENTS’ KNOWLEDGE OF SIMPLE GEOMETRICAL CONCEPTS WHEN STARTING TO LEARN PROOF

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The knowledge of concepts is essential for students when they start to learn proof. Empirical findings of a study with 106 grade 8 students show that there are deficits in students' concept understanding scheme for quadrangles. These deficits are particularly based on a different idea of classification of quadrangles and difficulties in the understanding of the mathematical language and thinking - problems which will cause difficulties regarding learning and teaching proof.

INTRODUCTION

The mathematical definition is one of the basic terms which together with axioms, theorems, proofs, corollaries, lemmas and propositions constitutes the skeleton of the mathematical theory. Though it is clear that a definition has to satisfy certain necessary properties like noncircular, non-contradictory, there is little agreement which properties are sufficient to constitute a good and elegant definition (cf. Shir & Zalavsky, 2001). However, elegant or not, mathematical definitions are prerequisites for the formulation of theorems and proofs and, therefore, they are essential for the development of mathematics as a deductive theory.

Teaching definitions in mathematics classroom is a task which depends on more than only mathematical requirements. In particular, in geometry classroom on the primary level students do not learn geometry as a deductive theory. Based on pedagogical and psychological reasons, geometry is introduced to young students as a theory of the visual space. It is obvious that in this stage the concept formation and the recognition of simple mathematical "theorems" is not undertaken by teaching formal definitions but mainly by examples and visual representations. The transition from everyday life thinking in the theory of the visual space to a scientific thinking in deductive geometry in higher grades is afflicted with many problems. These problems cause students' difficulties in advanced mathematical thinking as it is required in problem solving, reasoning and proof. One of the basic deficits can be found in the students' personal knowledge of concepts and its usage in a mathematical context.

THEORETICAL FRAMEWORK

Scientific thinking skills and prerequisites for learning reasoning and proof

The last decades many researchers from cognition psychology and mathematics education contributed to the description of the development of students' thinking skills. Research in this area shows that there are several restrictions in the preadolescent scientific thinking (e.g., Kuhn, 1989). For example, an empirical-inductive reasoning is typical for students on the concrete-operational stage, whereas a more hypothetical-deductive reasoning is typical for the formal-operational stage (Flavell, 1977). This hypotheses is supported by empirical findings of studies with grade 8 and grade 10 students' ability in mathematical reasoning and proof (Healy & Hoyles, 1998; Reiss & Thomas, 2001).

In a former study with students in upper secondary level we identified different prerequisites for the understanding of proofs in the geometry classroom. As described in Reiss, Klieme & Heinze (2001) geometrical competence is specifically influenced by methodological knowledge, declarative knowledge, metacognition and spatial reasoning. For the evaluation of students' declarative knowledge we chose the concept "congruence", a central concept of school geometry. The students were asked to give a definition, an example, a visual portrayal of the word "congruence" and to name a mathematical theorem in which the concept features. Our analyses have revealed considerable deficits in declarative knowledge. It emerged that even students at the end of secondary level often have only a vague intuitive understanding of concepts such as "congruence", that this understanding is restricted to examples, and that they have no exact mathematical knowledge of the respective definitions and theorems.

The last point corresponds to findings of several other studies: students' ideas about a geometrical concept and the definition of this concept are frequently inconsistent (e.g., Hershkowitz & Vinner, 1982; Wilson, 1990). The use of examples as an interpretation of definitions is often restricted to certain prototypes which, in addition, contain irrelevant characteristics. Distinguishing between these irrelevant characteristic properties and the relevant definitional properties is difficult for students (e.g., Burger & Shaughnessy, 1986; Wilson, 1990). Other problems related to definitions are the students' understanding of necessary and sufficient conditions in definitions and their imprecise use of words if they give definitions (Burger & Shaughnessy, 1985, 1986; Wilson, 1990). These deficits in students' ideas about geometrical concepts are strongly affected by the representation of these concepts in mathematics classroom and textbooks (e.g., Burger & Shaughnessy, 1985).

Concept definition, concept image and concept usage

As mentioned above we have to distinguish between the mathematical definition of a concept and the personal image of this concept in student's mind. To describe this fact Vinner and others introduced the theoretical model of the personal *concept image* (Vinner, 1991) which was used to identify students' ideas of different mathematical concepts like function, limit etc. (e.g., Tall & Vinner, 1981). The concept image is described as

> "... something non-verbal associated with the concept name. It can be a visual representation of the concept in case the concept has visual representations; it can be a collection of impressions or experiences" (p. 68, Vinner, 1991).

The concept image is evoked in the memory by the concept name. It is specific for an individual. The existence of a concept image is a necessary condition for the understanding of a concept. "To acquire a concept means to form a concept image" (p. 69, Vinner, 1991). The knowledge of a concept definition may be independent from the formation of a concept image: to know a concept definition does not imply to understand the concept.

Moore (1994) extended the ideas of Vinner to the *concept-understanding scheme*. The concept-understanding scheme contains three aspects of concept understanding: the concept definition, the concept image and, as a third aspect, the *concept usage*. The concept usage "refers to the ways one operates with the concept in generating or using examples or doing proofs" (p. 252, Moore, 1994).

Both, Moore (1994) and Vinner (1991), described the main problem for students using concepts in their mathematical activities. If a student is confronted with a mathematical problem in which a certain concept appears, then in his or her mind the associated concept image is evoked. To get a correct solution of the problem the student has to check, if the operation done with the evoked personal concept image is compatible with the concept definition. Here a difference between every day thinking and scientific thinking appears: in every day thinking, in general, it is not necessary to check the concept definition (in many cases such a concept definition even does not exist, e.g., for the concept "tree"). In a mathematical context, in general, it is indispensable to compare the personal concept image with the concept definition. The ability to "switch" between the personal concept image and the concept definition is essential for the solving processes in a mathematical context. It can also be observed in the research process of mathematicians: mathematicians do not retrieve definitions and theorems from their memory to construct logical deductions. On the contrary, first they do not pay attention to each detail in the process but consider the line of argumentation in broad terms and recognise important properties and connections. Finally, if they know how to argue they will construct a mathematical proof using formal definitions and theorems (cf. Koedinger & Anderson, 1990).

Partitional and hierarchical classification

When discussing the concept understanding scheme of geometrical concepts like triangles or quadrangles, we must also consider the classification of these concepts. As described in de Villiers (1994) there are two main classification types: the hierarchical and the partitional classification. *Hierarchical classification* means the classification of a class of concepts in such a manner that the more particular concepts form subclasses of the more general concepts (class inclusions). In contrast, in a *partitional classification* the various subclasses of concepts are considered to be disjoint from one another. For example, in the first case we can define squares as special rectangles and rectangles as special parallelograms. In the second case a square is not a rectangle and a rectangle is not a parallelogram. Since in mathematics classifications and the associated definitions are arbitrary in a certain sense, the choice for a hierarchical or partitional classification is a question of convenience and economical and personal reasons. In general, mathematicians prefer a hierarchical classification for triangles and quadrangles.

There are several studies which show that students in lower secondary level still tend to a partitional classification in the case of quadrangles (e.g., Burger & Shaughnessy, 1985; de Villiers, 1994, 1998). Moreover, de Villiers (1994) showed that even students who exhibit excellent competence in logical reasoningstill prefer to define quadrangles in partitions, if given the opportunity. He suggests to treat the classification of concepts like triangles and quadrangles in such a way that a meaningful discussion is possible. As described in deVilliers (1998) by comparison of advantages and disadvantages the students will then realise that hierarchical classifications are more economical than the partitional ones.

RESEARCH QUESTION AND DESIGN OF THE STUDY

According to the theoretical framework described above we investigated aspects of the concept understanding scheme of students who started to learn proof. As concept we chose quadrangles, in particular, squares and rectangles which are well known to these students. The research question to be addressed in this paper is the following:

- Are the students' concept understanding schemes of (special) quadrangles sufficient for solving problems in different situations like recognising equivalent descriptions, finding counterexamples and distinguishing between sufficient and necessary conditions?

For this research question we considered three items of a study which was carried out for the investigation of informal prerequisites for informal proofs. In this study five mathematical principles (definitions, equivalent descriptions, arguments and proof, logical implication and counterexamples) were represented by ten items in a paper and pencil test. This test was administered by a teacher in four classes of grade 8 in a German Realschule (Realschule means lower secondary school for students with an average proficiency level). Altogether the sample comprises of 106 students (50 female, 53 male, no data 3). The students were asked to answer the test in 45 minutes. A detailed description of this study is given in Heinze (to appear).

The three items related to the research question are the following:

1. The recognition of equivalent descriptions of a square: Six descriptions of quadrangles were given and the students had to mark which of them describe squares (multiple choice).
2. The finding of a counterexample: We presented the following problem: "Klaus considers squares and rectangles. He says: 'In each quadrangle each angle is 90°.' Karin says: 'This is not true'. How can you show, that Karin is right?".

3. The distinction between necessary and sufficient conditions: Here we asked: "If a quadrangle is a rectangle, then the opposite sides are parallel. Consequently: If the opposite sides of a quadrangle are parallel, then it is a rectangle. Is this true?"

RESULTS

Table 1 shows the results of the first item (Which quadrangles are squares?):

	Quadrangles with ...	**Frequency**	**Percentage**
(a)	four sides of same length and all angles of 90°	88	83,0 %
(b)	four angles of 90 °	48	45,3 %
(c)	three angles of 90° and two neighbouring sides of same length	6	5,7 %
(d)	all sides of same length	34	32,1 %
(e)	opposite sides are parallel	63	59,4 %
(f)	four sides of same length and one angle of 90°	19	17,1 %

Table 1: Correct answers for the recognition of a square as a special quadrangle

If we consider the alternative descriptions of a square (answers (a), (c) and (f)), we see that 83 % of the students know that quadrangles with four sides of same length and four angles of 90° are squares. Answer (f) (four sides of same length and one right angle) is recognised by 17,1 % and statement (c) (three angles of 90° and two neighbouring sides of same length) is accepted by only 5,7%. Conversely, more than half of the students think that each quadrangle with four right angles is a square (b) and more than two thirds believe that each equilateral quadrangle is a square (d). The fact that, in general, a parallelogram is not a square (e) is known by nearly 60 % of the students. It is interesting to see that about two thirds of the students think that an equilateral quadrangle is a square (d) but only 17,1 % gave the answer that an equilateral quadrangle with one angle of 90° is a square (f).

The total number of correct answers for each student is presented in Table 2:

corr.	**Frequency**	**Percentage**
0	0	0 %
1	21	19,8 %
2	35	33,0 %
3	34	32,1 %
4	15	14,2 %
5	1	0,9%
6	0	0 %

Table 2: Total number of correct answers for each student

Furthermore, if we consider only the correct descriptions in the first item (answers (a), (c) and (f)), then 71,7% of the students recognised one of these, 13,2% two and only 2,8% (three students) recognised all three correct descriptions of a square among the six given answers.

A deeper analysis of the students' answers for this first item shows that there is indeed a certain ordering of the different descriptions by difficulty: the easiest cases are (a), (e) and

(b) (in this ordering), i.e., students with three or more correct answers mostly have these three cases correct.

Table 3 gives the results for the second item (In each quadrangle each angle is 90°):

	Frequency	Percentage
correct	68	64,2 %
false	22	20,7 %
no response	16	15,1 %

Table 3: Results for the second item

More than 60 % of the students gave a correct answer, nearly all of them gave a counterexample. About 20% gave a false answer, half of these students said that it is true that each angle in each quadrangle is 90°.

Table 4 presents the answers for the third item (necessary and sufficient condition):

	Frequency	Percentage
correct with reasons	2	1,9 %
correct without reasons	8	7,5 %
correct with false reasons	30	28,3 %
false	24	23,1 %
no response	42	39,6 %

Table 4: Results for the third item

About 37 % gave a right answer, whereby many students (28,3 %) gave false reasons for their response. More than half of these cases with false reasons (16%) was based on a partitional classification of quadrangles (without class inclusions). Typical answers were "The opposite sides of a square are also parallel, but a square is not a rectangle." In addition to this, we noticed that many students do not distinguish between the concepts of square resp. rhombus and quadrangle. Furthermore, it is remarkable that nearly 40 % of the students gave no (mathematical) response to this item. Often they wrote, that this item is too complicated or that there is no logic in this question.

DISCUSSION

For an interpretation of the described results it is necessary to analyse the different items and their requirements. In particular, the six problems in the first item require different kinds of thinking. For example, for the cases (c) and (f) it is not possible to find a correct answer without an analytical process of thinking. For these cases we get only a small number of correct answers which indicate that the students mainly did not use analytical approaches. The results for the two cases (d) and (f) (Is an equilateral quadrangle resp. equilateral quadrangle with one 90° angle a square?) supports this fact: 61 students (57,5%) said that an equilateral quadrangle *is* a square and an equilateral quadrangle with one 90° angle *is not* a square. The problems in this case may be based on the students' difficulties with sufficient and necessary conditions and the understanding of the "mathematical language" ("one 90° angle" means "only one 90° angle").

Restrictions in the understanding of sufficient and necessary conditions and the language resp. thinking in mathematics can be also identified in the third item (If the opposite sides of a quadrangle are parallel, then it is a rectangle). Though this item is similar to case (e) in the first item, it is more difficult for the students (37,7% to 59,4% correct answers). This may be caused by the explicit question if the *necessary* condition "opposite sides are parallel" of a rectangle is also *sufficient.* A fact that is also influenced by deficits in mathematical language is the students' preference of the partitional classification of quadrangles. Nearly half of the answers which were accepted as correct (16% of 37,7%) were based on this classification. The frequently given reason "a square is not a rectangle" may also be caused by the interpretation of the word "is" as "is equal to" (cf. de Villiers, 1994).

The best results the students obtained for the second item ("In each quadrangle each angle is 90°."). Here, about 64% gave a correct answer. Nevertheless, 15% of the students gave no response and 20% gave a false response. For 12% of the students with false response we identified a restricted concept image of quadrangles (In a quadrangle each angle is 90°). As one can expect, these students also achieved poor results for the other items.

The results show that many students have deficits in the concept understanding schemes for the discussed quadrangles. In particular, if they have to use the concepts for certain problems they remain on using their personal concept image and ignore the concept definition. In addition, the findings support the results of de Villiers (1994) that many students prefer a partitional classification for quadrangles. This seems to be related to a wrong understanding of the mathematical language and thinking.

The difficulties described above are problematic in a stage where the teaching and learning of reasoning and proof begins. In particular, the fact that in mathematics classroom teachers and a part of the students have a different understanding of the classification of concepts and of the mathematical language and thinking cannot be considered as a basis for first steps in advanced mathematics. It is essential for a successful instruction in advanced mathematics that teachers and students agree on a common view on the basic mathematics.

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