ABOUT THE FLEXIBILITY OF THE MINUS SIGN IN SOLVING EQUATIONS

Joëlle Vlassis
University of Liège

ABSTRACT

Among the difficulties the students have in solving equations with negative numbers, the 'detachment from the minus sign' (Herscovics & Linchevski, 1991) and solving an equations such as -x = 7 are often pointed out in the research literature. However, the study of their origin was not yet debated a lot. This article intends to propose an analysis of these difficulties. It reports results from a clinical study carried out with 8th grade students. The analyses stress that the presence of negative numbers in equations needs some level of flexibility in conceiving the minus sign that most students have not yet reached. They show that, beyond the equations and negative numbers, this lack of flexibility leads to an erroneous or superficial understanding of algebraic operations and symbolism.

INTRODUCTION

Many researchers showed that the presence of negative numbers in first degree equations with one unknown leads to several errors (Cortès, 1993, Gallardo & Rojano, 1994 ; Gallardo & Rojano, 1990; Herscovics & Linchevski, 1991; Vlassis 2001). Three obstacles are regularly pointed out by these authors and they thus seem to indicate major difficulties for students: (1) the avoidance of a negative solution (Gallardo & Rojano, 1990 ; Gallardo & Rojano, 1994 ; Glaeser, 1981; Vergnaud, 1989), (2) the 'detachment from the minus sign' (Herscovics & Linchevski, 1991 ; Linchevski & Herscovics, 1996; Vlassis 2001) and (3) solving an equation such as -x = a (Cortès,1993 ; Gallardo & Rojano, 1994 ; Vergnaud, 1989 ; Vlassis, 2001). The difficulties raised by the negative solution of an equation have already been analysed (Gallardo & Rojano,1990 ; Gallardo & Rojano, 1993 ; Vergnaud, 1989). However, whereas errors linked to 'the detachment from the minus sign' and those linked to solving '-x = a' have been often mentioned, their origin has not yet been clearly established. Historical epistemological arguments are usually pointed out when negative numbers are involved. We think that this perspective has to be enlarged to other analyses in order to define more completely the origin of these obstacles. In this article, we intend to propose an original approach of the difficulties stemmed from the 'detachment from the minus sign' and the solving process of -x = a. Problems associated with a negative solution to an equation will not be developed in this proposal. Our analyses are based on interviews of 18 students of the 8th grade. They show that beyond equations and negative numbers, those errors are due to a lack of flexibility in the conception of the operation signs, leading to a wrong or superficial understanding of the concepts involved in algebraic operations and symbolism.

METHODOLOGY

We interviewed 18 students of the 8th grade level at the end of the school year. These students learnt elementary algebra (reduction of expressions, distributive law, ...), the solving of first degree equations with one unknown and the negative numbers during their 7th and 8th grades. These students came from two different schools, one of them recruiting in a favoured population (6 students), the other one in a non-favoured population (12 students). The student selection was based on test results. Here is the 18 students distribution: 4 students of good level (results in the 80% range); 6 students of middle level (in the 70% and 60% ranges); 4 students of low level (in the 50% and 40% ranges); 4 students of very low level (less than 40%). The students that obtained 90% and more were not interviewed.

The aim of the interview was to list and analyse students' reasoning when they are confronted with negative numbers in solving 'arithmetical' and 'non-arithmetical' equations (Filloy and Rojano, 1989) and in the reduction of polynomial expressions. The interview also aimed at considering the students' conceptions of mathematical objects and operation signs in an expression such as '-18a - 2y + 5a – y'. The students were invited to tick the negative numbers of the expression. They also had to say what the letter represented and to give some examples of numbers that could replace 'y' in -2y and -y, and 'a' in -18a and + 5a.

THE DETACHMENT FROM THE MINUS SIGN

Results

In 1991, Herscovics & Linchevski already stressed the following result: 'The detachment from the minus sign was somewhat of a surprise to us. The high incidence of this mistake indicates that the problem is not idiosyncratic but may well reflect unsuspected cognitive obstacles.' (p. 179). These authors defined that issue as 'a tendency to ignore the minus sign preceding a number'. In their experiment, that error concerned the 7th grade pupils that had not yet received any algebra teaching As for our 18 interviewed students, they had been learning algebra for two years. However, 'the detachment from the minus sign' still remains often observed and raised the following errors :

1) In solving an equation

error (a) : $2 - 3x + 6 = 2x + 18$ *becomes* $3x + 8 = 2x + 18$

error (b) : $2 - 3x + 6 = 2x + 18$

$$\begin{array}{c} 2 - 3x + 6 = 2x + 18 \\ {\scriptstyle -2x\downarrow} \qquad\qquad {\scriptstyle \downarrow -2x} \\ 2 - 1x + 6 = 18 \end{array}$$

Eight students on the 18 are concerned by one or the other of these errors.

- In case (a), 4 students simplify the first member and copy it, forgetting the minus sign before 3x. Cortès (1993) also stressed the high incidence of that error. This one is mainly made by good-level students (3 of good level and 1 of middle level). Here is how Coralie explains her error :

1 Interviewer: Which error did you made here?

2 Coralie : h'm, ... it's -3x.

3 Interviewer : Why did you forget your minus sign before 3x?

4 Coralie : Because there was a number before it.

5 Interviewer : Yes, and what did you think so?

6 Coralie : I deleted the 2 and the 6. Then I looked at the 3x and I forgot the minus sign before.

- In case (b), 4 students want to cancel -2x in each member and get 2 - 1x + 6 in the first member. This error is more typical of lower level students (2 of middle level, 2 of low level). The lowest students are not concerned since they did not begin any solving process. During the interviews, all the students who made that error explained that they removed 2x from 3x and that they obtained 1x. Then, three of them reintroduced the minus sign before 1x.

In both cases, the minus sign before a literal term is either completely forgotten (error a), or forgotten and then reintroduced (error b).

2) In the reduction of a polynomial expression

2.1. The following expression was presented to the students : -18a - 2y + 5a - y =

In order to simplify this expression, they had to group like terms. Four of them made a mistake when simplifying 'y terms'. They made - (2y - y) = -y instead of -2y - y = -3y. The reasoning is related to error (b) in the equation solving : again, the students took in consideration 2y and not -2y. In this case, the operation sign was forgotten and then reintroduced since they obtained -y. The error consisting in forgetting the minus sign before the 18a did not appear. No student operated -18a + 5a = 23a nor -23a. One of my hypotheses about the error on the 'y term' was that moving terms, in order to group them, 'made easier' the omission of the minus sign before 2y. Obviously, in order to solve the equation as well as to reduce the expression, it was needed to 'build' a new expression with the like terms (-3x - 2x in the equation and -2y - y in the expression).

2.2 In order to check that hypothesis, the following expression was proposed to the students : '-24b + 4b - 6a - 2a ='

I thought that, since it was not needed to change the place of the like terms, the error consisting in 'forgetting' the minus sign would appear less regularly. In such a case, in my opinion, the error consisting in operating 6a - 2a rather than -6a -2a should have appeared less regularly. I was disappointed. Five of the students simplified the terms -6a and -2a in -4a. The explanations were the same for the 5 students: 'I made 6a - 2a, that is 4a, and then, as there was a minus sign, I put it back before'. Again, the students concerned by this error are not the lowest ones. Here is their distribution: 1 is good level, 3 are middle-level and only 1 is low level. The error consisting in forgetting the minus sign before b was made by only 1 student, who obtained -28b for -24b + 4b.

Comments : from arithmetic operations to a combination of signed entities.

I was quite surprised by those results because, on the one hand, the 'detachment from the minus sign' concerns students of various competence levels, and not only low-level students. On the other hand, it is important to note that the students do not understand easily their error. Some of them, even among those of good level, did not see why I underlined an error in the first member of their equation when they wrote : 2 - 1x + 6. I had thus to re-formulate the operation like this : '-3x - 2x is -1x ? ', and to insist on the minus signs to make them finally become aware of their error. And even in this case, it was really very difficult for some of them to admit that -3x -2x produced - 5x and not -1x. Those results made emerge three main analyses :

1) The students do not always forget the minus sign

The 'detachment from the minus sign' could be assimilated to 'an avoidance syntome'. Gallardo and Rojano (1990) already stressed on that difficulty in solving equations. A link could be established between that students' error and the historical development of the negative number where, at a given moment, scientists used some processes to avoid the obstacle of negative numbers (Glaeser, 1981). The 'avoidance syntome' actually exists by students (in the interviews, I could observe that equations such as 4 - x = 5 were modified into x = 5 - 4) and could explain errors such as the omission of the minus sign when copying a simplified equation (error (a) in equation solving). The results nevertheless show that not all the students forget the minus sign. On the contrary, most of them put it aside to make their operations, and then reintroduce it. Some students even tell us they put the expression between brackets and copy the minus sign before the answer they found. I would thus slightly qualify the Linchevski and Herscovics's definition (1991), adding that this error consists in ignoring or *temporarily omitting* the minus sign preceding a number or a literal term.

2) Moreover, the omitted minus sign is not any minus sign.

The omitted minus sign is located *in* the expression, before a number or a literal term. Coralie explains (line 4 of the interview here above), that what disturbs her is the number before the sign. The minus appearing in the beginning of the expression, such as in the expression -18a -2y + 5a - y is not or almost not omitted by the students. Another example of that difficulty can be found in the question about the mathematical objects of the expression -18a - 2y + 5a - y. When the students have to tick the negative numbers of the expression, almost all the students tick -18a, but -2y and -y are ticked more randomly. For the students I interviewed, the term -18a is the 'prototype' of a negative number and cannot be considered the same as -2y and -y. The minus signs in the middle of the expressions have an double status : they have to be considered like operation signs ('subtraction' level of Gallardo, 1994) and like signs attached to the number ('signed number' level of Gallardo, 1994). The difficulty does not depend on reaching one or the other level of conceptualisation, but rather on being able to go to one or to the other, according to the situation. The students that omit the minus sign do not consider that duality. On the contrary, they seem to keep an arithmetical and inflexible conception and do only consider the minus sign inside the expressions as the symbol of subtraction. This can explain why some of them put it aside in order to operate (it is the symbol of subtraction and it is thus not attached to the literal term), and then re-introduce it, once the operations are made.

3) The meaning of the operations changed

The introduction of negative numbers in algebraic expressions undoubtedly leads to another conception of the operations. In primary school, the students are used to operate on concrete entities, which means without signs, they have to subtract or add up. In expressions involving positive numbers, the students can keep most of that operation conception. For instance, in the expression 5b + 3c + 2b + 4c =, the plus signs can be seen only as operation signs. The students can then add up the 'b terms' and the 'c terms' which are seen as objects without sign. With negative numbers, that perspective isn't possible anymore. The students cannot consider anymore the operations as a set of numbers without signs (about numbers with and without signs, see Vergnaud, 1989) that have to be subtracted or added up. The duality of the operation signs implies to consider the expressions as a combination of signed entities, *i.e.* as '-18a followed by -2y, by +5a and by -y'. That notion requires a deep understanding of these mathematical objects. Signs need to be considered as attached to every term while acting also as operator. But, when it is needed to make the operations, the meaning of addition and subtraction is modified and becomes abstract. The first concrete arithmetical representations becomes then an obstacle to that evolution because, when it is about simplifying -3x - 2x, for example, the result (-5x) seems to be that of an addition (that of 3x and that of 2x). This perhaps explains one of the difficulties faced by the students. During the interviews, Julie (a middle-level student) helped me to think in that sense. When she understood her error (-3x - 2x is not -1x) and gave the correct answer (-5x), I was trying to make her say she had forgotten the minus sign before 3x.

Interviewer: What did you forget while making 3x - 2x? (expected answer : the "minus" before 3x)

Julie : to add them up.

I did not understand immediately what she meant. As to me, I did not see where it was about adding up. In fact, each of us referred to two distinct models. Julie was reasoning in very concrete terms, linked to her first arithmetic learning (if -3x - 2x = -5x, it is just like an addition); as to me, I referred to the model of the number line. It is possible that, for some students, the answer –5x is counter-intuitive, even if preceded by the minus sign. The meaning of the subtraction is still related to elementary arithmetic (the operation must produce a smaller result). The reference to the number line does not seem to be a conceptual instrument for these students, even though they have experimented that model during their learning of algebra.

SOLVING $-x = a$ OR A MINUS SIGN CAN HIDE ANOTHER ONE.

Another example of the students' lack of flexibility about the minus sign was pointed out when solving the equation $-x = 7$: many students made a confusion between the minus sign before the x and that of the solution -7. The difficulties in solving that kind of equation have already been underlined by Cortès (1993), Gallardo and Rojano (1994), Vergnaud (1989) and Vlassis (2001). Vergnaud (1989) thinks that difficulty comes from the negative solution. I do not think the explanation is mostly based on the negative value of the solution. Without denying the obstacle related to a negative solution, it is shown that other equations produced the same type of solution but didn't present so much problems to the students. According to Gallardo and Rojano (1994), the problem is that the students don't decode $-x = 7$ as $-1x = 7$. Our analyses also show that students do not interpret the equation in that way. We did not find out any explanation of that kind in our interviews. It seems that this formulation is difficult to be understood by students at that school level (Vlassis, 2001). It does not represent, however, the only possible way of solving.

Results

A major part of the issue seems to come from the ambiguity of the minus sign before the x. Among the 15 students who solved the equation, five of them gave 7 as the answer of the equation and 10 gave the correct answer. The students who gave 7 as the solution were not able to consider –7 as the solution since the minus sign was already noted. But the most interesting analyses come from the students who gave the correct answer. I first asked them to explain how they had found their solution. Here are the types of answers I obtained : *(a)* the value of x is 7 and, since there is a minus sign before, I added it up (2 students). *(b)* I transferred the minus sign to the other member (5 students). *(c)* The value of x is -7 since, when I put -7 instead of, I get $-(-7) = 7$ (2 students). *(d)* I took the opposite (1 student). Seven of the 10 students justify their answer only on basis of algebraic rules - true or invented- (answers *a* and *b*). When I asked to these students to explain me more about the method they follow, they told me it was just like that, that it was a trick in algebra. Thus I invited the 10 students to check their solution. All the students, but the two who had done it spontaneously, were perturbed by my question. Most of them did not understand what I wanted. When I reminded them that the solution of an equation is checked by replacing the letter by its numerical value, some of them thought their answer was wrong : 'If I replace x by -7, I get $-7 = 7$, which is not correct, so my answer is wrong'. It turned out to be quite difficult to make them understand that the minus sign before the x was not the same as that which was attached to the 7 of the solution and to make them copy the equation again, replacing the x by -7.

Obstacles to solving $-x = 7$

The difficulty in solving this equation is due to the multiplicity of concepts it covers :

- First, the students must be able to detach the minus sign from the letter x. They must consider the sign attached to the numerical value of the letter, *independently* from the sign attached to the letter. The students' answers show it is an obstacle quite difficult to overcome. It seems that -x is considered as a 'prototype' of a negative number (as well as -18a) and that they see -x in an inflexible way, as a 'signed number'. Another example of that difficulty appeared during the discussion about the expression -18a - 2y + 5a - y. When the students had to decide whether the letter y of -2y could replace a positive number, a negative number, or both, most of them answered without hesitation a negative number. Only one student told me it was not possible to answer the question because he did not know whether the letters replaced negative or positive numbers. Finally, when the students were asked to give examples of numbers that could be replaced by y, they gave examples of positive numbers, without looking disturbed by the apparent contradiction of their reasoning. I then asked whether 'y' could replace, for instance, -6. The answer given by a large half of them was 'No, since the minus sign is already wrote'.

- The students must be able to enlarge their notion of 'signed number', but also to juggle with high levels of conceptualisation of negative numbers (Gallardo, 1994) : that of 'isolated number' (the solution is a negative number), that of the formal conception of a 'negative number' (a letter replaces a number: the negative numbers are part of the numbers that could be replaced by a letter). The 'relative number' level (the opposite) is not compulsory but it rather represents a way to give sense to the equation.
- Decoding -x = 7 in -1. x = 7 seems to be a problem because it needs not only to detach the minus sign from x, but also to attach it to a "hidden" number (number 1) and to consider the presence of an operation (that of a multiplication), whereas a lot of students still just see a number in -x. It seems that such decoding of the implicit imbedded in the algebraic symbolism is out of sight for numerous students of that school level.

CONCLUSION AND FINAL DISCUSSION

The analyses presented in this article show that the presence of negative numbers in the equations causes some errors with various origins : they are related, not only to negative numbers and equations, but also to essential algebraic concepts. These difficulties appear indeed as an expression of a lack of understanding of the concepts involved in the algebraic operations and symbolism. The 'detachment from the minus sign' is performed by students that are not aware of the duality of the minus sign in algebraic expressions: they always consider these expressions as operations performed on numbers without signs. To give some meaning to the reduction of these expressions, they must be able to consider them as a combination of signed entities. This ability needs not only some flexibility in the meaning given to operation signs that are both attached to every term and operation sign, but also an abstract vision of subtraction and addition. That flexibility of the minus sign is also required to solve significantly an equation such as -x = 7. The minus sign in the initial equation is the sign attached to the letter x, but it is not attached to the value of x, which can be itself positive or negative. Students meet some significant difficulties at that level. Even those who find the correct answer explain it with algebraic rules and are not able to re-write the equation, replacing x by its value -7. The minus sign before the solution -7 is very often mistaken with the minus sign attached to x. We can conclude that the students' expertise doesn't consist only in reaching such or such level of conceptualisation of the negative numbers, but also in being able to adapt this level to the situation. It also seems that a part of those difficulties is linked to a pseudo-structural notion of algebraic expressions, as think Sfard and Linchevski (1994, p. 222). « Students acted as if they were handling some kind of objects, but their thinking was completely inflexible and the appropriate kind of structural interpretation was unavailable... The majority of pupils could not provide any sensible justification for the permissible operations and it was obvious that for them these were no more than arbitrary « rules of the game »

References

Cortès, A. (1993). Analysis of errors and a cognitive model in the resolving of equations. *Proceedings of the XVII Annual Meeting of* PME. Tsukuba, Japan : Ed. University of Tsukuba, vol 1, 146-153.

Gallardo, A. (1994). Negative numbers in algebra, the use of a teaching model. *Proceedings of the XVIII Annual Meeting of PME*, vol II, 376-383.

Gallardo, A. & Rojano, T. (1990). Avoidance and acknowledgement of negative numbers in the context of linear equations. *Proceedings of the XII Meeting of PME-NA*. Mexico, vol II, 43-49.

Gallardo, A. & Rojano, T. (1993). Negative solutions in the context of algebraic word problems. In J.R. Becker & B.J. Pence (Eds), *Proceedings of the XV Meeting of PME-NA*. Pacific Grove, CA, USA : Asimolar Conference Center, vol 1, 121-127.

Gallardo, A. & Rojano, T. (1994). School algebra, syntactic difficulties in the operativity with negative numbers. In D. Kirshner (Ed.), *Proceedings of the XVI Meeting of PME-NA*. Baton Rouge LA, USA : Louisiana State University, vol 1, 159-165.

Glaeser, G. (1981). Epistémologie des nombres relatifs. *Recherche en didactique des mathématiques*, 2, *3*, 303-346.

Filloy, E. & Rojano, T. (1989). Solving equations : the transition from arithmetic to algebra. *For the learning of the mathematics*, 9, *2*, 19-25.

Herscovics, N. & Linchevski, L. (1991). Pre-algebraic thinking : range of equations and informal solution processes used by seventh graders prior to any instruction. F. Furinghetti (Ed.), *Proceedings of the XV Annual Meeting of PME*. Assissi, Italy, vol.3, 173-180.

Linchevski, L & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra : operation on the unknown in the context of equations. *Educational Studies in Mathematics*, *30*, 39-65.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification : the case of algebra. *Educational Studies in Mathematics*, *26*, 2-3, 191-228.

Vergnaud, G. (1989). L'obstacle des nombres négatifs et l'introduction à l'algèbre. Construction des savoirs. *Colloque International Obstacle Epistémologique et Conflit socio-cognitif.* CIRADE, Montréal, 76-83.

Vlassis, J. (2001). Solving equations with negatives or crossing the formalizing gap. *Proceeding of the XXV International Group for the Psychology of Mathematics Education*. Utrecht, The Netherlands : Freudenthal Institute, vol. 3, 375-382.