LEARNING FROM LEARNERS: ROBUST COUNTERARGUMENTS IN FIFTH GRADERS' TALK ABOUT REASONING AND PROVING

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I, a teacher-researcher, presented my fifth graders with an interesting but incorrect student-constructed proposal first seen in 1996 (Zack, 1997). The students used the patterns they had detected while solving the chessboard task to formulate counterarguments. The five types of counterarguments which emerged offer insights into the children's understanding of the mathematics. The children's perspectives, in turn, changed my understanding in substantive ways.

For the past ten years, as a teacher and researcher in a fifth grade classroom, I have been studying the problem-solving work and talk of ten to eleven year-old learners, with a particular interest in recent years in the children's notions of proving. This study is a follow-up to a study reported previously at PME (Zack, 1997). In that paper I showed how, in solving a variant of the chessboard problem, one team is convinced that their patterns work, and they use what they know of their patterns to refute an interesting but incorrect proposal put forth by the other two group members, Ted and Ross. I subsequently decided to offer the Ted-Ross idea to all of my fifth grade students for consideration (1999-2000, 2000-2001). My aim was to see whether, and if yes, when and how the fifth graders would refute this idea. It has been noted that "'wrong' ideas can be opportunities for important mathematical discussions and discoveries" (NCTM, 2000, p. 191). The authors of the recent NCTM (2000) *Principles and standards for school mathematics* envision children constructing valid arguments and evaluating arguments of others, detecting fallacies and critiquing others' thinking, and reasoning about mathematical relationships, such as the structure of a pattern (p. 188). I will describe the diverse counterarguments which emerged, and dwell on one in particular, the "array strategy" counterargument, which surprised and intrigued me. I have shown elsewhere (Zack, 1996, 1997, in press) and will also point out here in regard to the children's "array strategy," that the children's ways of seeing have changed my understanding in fundamental ways.

THE SCHOOL COMMUNITY AND CLASSROOM SETTING, AND ASSIGNED TASKS

St. George's is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds; the population is predominantly English-speaking. The homeroom class size in the 1999-2000 year was 27, in the 2000-2001 year was 26; the work, however, is always done in half-groups (13/14 children in each group) of heterogeneous ability. Problem-solving is at the core of the mathematics curriculum in this classroom. The school and classroom is one in which the children are expected to publicly express their thinking, and engage in conjecture, argument, and justification. The groundwork laid during the year by the teacher included an expectation that the children would be looking for patterns, and that they could be nudged to think about the mathematical structure underlying the pattern (Zack, 1997).

Mathematics class periods are 45 minutes, and twice a week are extended to 90 minutes. In addition to the in-class problem-solving sessions, each week the children also work on one challenging problem for which they are expected to record their work and reflect on their strategies. They write in a Math Log which serves as the initial basis of their group discussions in class. In class much of the session is conducted by the children as they discuss the problem first with a partner, then in a group of four or five, and finally with the entire group of thirteen students.

The children are videotaped throughout the school year as they work in their groups. In addition to the videotape records, data sources include focused observations, student artifacts (Math Logs), teacher-composed questions eliciting opinions (written responses), and retrospective interviews.

The mathematical context of the problem/discussion

The task is a variant of the 'Chessboard' problem (see Mason, Burton & Stacey, 1982). The work was assigned as follows:

> **Task 1:** Find all the squares [in a four by four grid given as a figure]. Can you prove you have found them all?
>
> **Task 2:** What if . . . this was a 5 by 5 square? How many squares would you have? Extensions were subsequently posed.
>
> **Task 3:** What if this was a 10 by 10 square? What if this was a 60 by 60 square? How many squares would there be?

Interesting talk about proving arose with the 'What if it was a 60 by 60 square?' task. The proposal that 2310 would be the answer for the sum of squares for a 60 by 60, what I have called the Ted-Ross strategy, was first seen in 1996 (Zack, 1997), and then came up again in subsequent years. In the 1999-2000 and 2000-2001 school years, I distributed to all the students in the class the proposal which Ted and Ross made in 1996, and asked the students to respond. It was presented to each pair in late April after they had completed their discussion about their work done independently in their Math Log on the task, 'How many squares would there be if it were a 60 by 60 square?' The 'Ted-Ross question' was posed as follows:

> Imagine that two of your classmates, Ted and Ross, came up with the following solution for the 60by60: The answer for the 10by10 grid is 385 squares. So take the answer for the 10by10 square (385) and 10 x 6 = 60 so multiply 385 x 6 = 2310 and you have the answer for the 60by60. What would you say?

I will speak first about the different kinds of counterarguments, and what the ten to eleven year-old students draw upon to shape their counterarguments.

FIVE COUNTERARGUMENTS

In the first study (Zack, 1997) Will, Lew, and Gord, in 1996, constructed three counterarguments (CA), namely CA #2A, #2B, and #1, in that order, in their quest to refute their partners', Ted's and Ross's, idea. A number of other counterarguments emerged in subsequent years, and all are described below using categories outlined in late 2001.

CA #1: 60 by 60 gives you 3600 squares of the smallest size, the 1by1 squares (not counting the others) and 3600 is already larger than 2310.

CA #2: The growth pattern is one in which the pattern of differences between the numbers grows as the numbers get bigger. There were a number of counterarguments which dealt in some way with the idea that the numbers increase in size, and are numbered 2A through 2D below.

(2A) The answer for the [number of squares in the] 10by10 (which is 385) is not double the answer for the [number of squares in the] 5by5 (which is 55). If you could just double the answer for the 5by5 to get the answer for the 10by10, the answer for the 10by10 would be 110, and it is not. (There were variants on this as the children chose different pairs to present their arguments, e.g. the 4by4 and the 8by8, the 3by3 and 6by6, but the reasoning was the same.) This counterargument proved hard to express. A few of the students (5, and 2 in the respective years) could not understand Ted and Ross's method. The majority however did understand Ted and Ross's method, and understood as well the connection between Ted and Ross multiplying by 6, and the presenters of the counterargument choosing to show that when one multiplied by two it did not work.

(2B) The pattern does not stop at 385 and then 'restart' itself. It keeps growing.

(2C) Look at the string of numbers (1x1, + 2x2, + 3x3 . . .). By the time one reaches a certain point, for example, the 20by20 square, the answer for the total number of squares in the 20by20 was a number already greater than 2310.

(2D) Refer back to a task previously done in class, one which had a similar growth pattern. For example, in the task 'What is the number of diagonals in an n-sided polygon?' one could not take the answer for the number of diagonals in a 10-sided polygon, multiply that number by 2 and get the answer for the number of diagonals in a 20-sided polygon.

CA #3: If one takes a 10by10 grid and lays six of them side by side, representing what Ted and Ross are suggesting, one omits counting all the different-size squares within each grid, and the squares which overlap the (10by10) grids. (This counterargument was offered only twice, by Jake in 2000, and by Dora, in 2001)

CA #4: There are 51 times 51 10by10's in a 60by60. This statement was offered once only as a counterargument, by Jake in May 2000, immediately after he presented CA #3. This counter argument will be elaborated below.

CA #5 If theirs (Ted and Ross) works, you should be able to take the answer for the 6by6 (91) and multiply it by 10 and it should give you 2310, but it does not. One student, Theo, presented this argument, in 2001.

The distribution of the different kinds of arguments is presented in Table 1. At times a student constructed more than one counterargument. In most cases it was 2 counterarguments, and in one case 4 (May, 2000), in the years 1999-2001. There is therefore a discrepancy between the total of counterarguments, that is, 20 and 23, and the total number of students who offered them, that is, 13 and 18, in 2000, and 2001. The total class size each year was 27 and 26 children respectively.

	CA #1		CA	#2		CA #3	CA #4	CA #5	Total
		#2A	#2B	#2C	#2D				
1999-2000	6	4	3	5		1	1		20
2000-2001	5	8	4	2	2	1		1	23

Table 1: Distribution of the five counterarguments

There was potential seen for a sixth counterargument, but the presentation of that counterargument did not materialise. In this case, the child, Kate, used inductive reasoning, appeal to empirical evidence, and reference to authority to support her way of arriving at her answer for the 60by60. She stated that "Leo's way", that of adding 1x1 plus 2x2 plus 3x3 and so on, as used by her classmate Leo, had been tried and tested for many examples by many of the students and it had proven true. She rejected the Ted-Ross strategy, giving the following reasons: "They didn't prove it and they didn't check it. They can't prove it's right." However, she did not then proceed to offer a counterargument to disprove their method. Had she made the move, she might have pointed out that using Leo's way and with the help of the calculator, she had arrived at a sum far larger than 2310. (In fact, Kate had arrived at the correct answer, 73 810, one of only three students to have the correct sum.) Although she could not be certain that her answer was correct, she might have insisted that the discrepancy in size deserved attention and deliberation. The point she might have made was both that the Ted-Ross method does not work, as the answer must be a far larger number, and that the Ted-Ross method is not needed because they are convinced that Leo's way works and will result in the correct answer (provided that the calculator work is done without error).

The benefit of having diverse counter-arguments

Some might say that one counter-example is all one needs. Others might suggest that one should consider in the main the most elegant one. However, having a diverse assortment of counterarguments is beneficial. The counter-arguments tell us different things about the mathematics in the task. A look at the counterargument helps me as a teacher to come to a better understanding both of the mathematics, and of the children's understanding of the mathematics. CA #1, that 3600 is already bigger than 2310, is the 'simplest' and some of the children voice their appreciation of it as the most powerful. Mark, who had used the Ted-Ross strategy in his independent work, states that he was quickly convinced by his partner Chris's refutation: "I did what the person did on the sheet and he [Chris] proved me wrong. . . . [Chris] ended up showing me in like one rather small sentence that it was wrong, that sixty times sixty is three thousand six hundred." CA #2 and the talk about growth rates reveals an understanding of the mathematics pattern which is at work here. The children at times themselves refer to the fact that CA #2A is difficult to articulate. However, those who propose and those can understand CA #2A -- and try to show for example that the answer for the 10by10 is not 'double of 'the answer for the 5by5 -- reveal an understanding of the method proposed by Ted and Ross, and its underlying structure.

The students "learn to describe relationships that hold across many instances and to develop and defend arguments about why those relationships can be generalised and to what cases they apply" (NCTM, 2000, p. 190). Lew, for example, builds upon what he knows of the number of little squares in the 5by5 square, and applies it to the 60by 60. This constitutes the foundation of CA #1. Lew explains the connection and justifies how he and his partners arrived at the number 3600 (of the littlest size square); while gesturing to both Gord's and Will's Math Logs, he points out: "Because here you would do five times five to see how many little squares there are, so we did sixty times sixty . . ." The students also push for reasons. When Adele keeps saying that the Ted-Ross strategy sounds good, stating: "I think it's right but I don't know how it works," Maggie (who has just finished presenting CA #1 to her group of five) insists: "Then you have no reason to think it's right." Soon after she repeats again: "But Adele, really, I think you should have a reason to think it's right."

The students who have generalised a rule after testing it, use what they know and trust it. Reid (in press) has shown for example how Will (from the Zack 1997 study) formulates a conjecture about one of the patterns he has seen, tests it, and once he feels sure that the pattern works, he is prepared to generalise it as a rule. Tom Kieren spoke recently about a "truth box" in relation to cases in which the children are secure with the knowledge, and do not feel they have to continue to test (Gordon Calvert, Zack, & Mura, in press). I hear them predict with confidence, and state that 'it will always work like this'. There are a number of patterns which are pivotal, all of which the children introduced to me. One, the 1, 4, 9, 16 'criss-cross' pattern (see Zack, 1997); two, adding 1 + 4 + 9 + 16 without having to worry about the number of squares (see Reid, in press); three, the pattern of differences between the above numbers a difference which increases by 2, i.e., 3, 5, 7 (Zack, 1997); and the "array strategy," a means whereby the children see that they can determine the number of squares there are of each particular size, which will be described below,.

The "array" strategy

The counterarguments are robust in that they are deeply rooted in the children's grasp of various aspects of the mathematics of the task. I have seen how the students formulate generalisations about observed regularities in regard to diverse patterns which they have detected (NCTM, 2000, p. 262). I will use as an example a counterargument which startled me, namely CA #4, when Jake said that there are 51 times 51 10by10s in a 60by60.

Jake's second counter-argument came swiftly upon the heels of his presentation of CA #3 without any overt deliberation or elaboration with his partner or small group members. This took me by surprise, and I had to sit down and figure out by working down from 60 times 60 1by1's, to 59 times 59 2by2's, etc., to 51 times 51 10by10's that it was indeed so. In the follow-up interview (May 23, 2000), when asked how he had come to that answer, Jake said at first that he could not recall, and two of his peers were asked what they thought, and how was it that they knew he was correct. Dexter said: "I visualised (the) others would be off the grid." Another clue to Jake's approach might be found in his partner Ari's explanation of his/their way. In answer to the question "Why does it go down by one?" (i. e., from 60 x 60, to 59 x 59 etc.), Ari said: "The size goes up by one row of squares so there's one fewer." Ari's reasoning introduced to me a new perspective: First there are 60 x 60 little 1by1 squares with none going off the grid. The next size square, the 2by2, is one square larger than the 1by1 going up and across, and therefore only 59 x 59 2by2's fit on the grid. Jake quickly dropped 9 squares down to get 51 x 51 10by10's, while I had to work it out one step at a time.

Jake may possibly have worked out yet another generalisation. He says the following during the interview as part of his response to how he arrived at the 51 x 51 10by10's:

> I remember Jennine showed there are four 2by2's [in a 5by5]. The one itself plus 3. There are 50 rows plus the one at the bottom that's 51.

Jake may have made the following connections, building upon Jennine's idea: In the 5by5, 5 minus 2 (the size of the square) gives you three more 2by2 squares which you can fit; three plus the first one gives you four 2by2's (along one dimension) in a 5by5. Hence there would be 4 times 4, or 16, 2by2's. In a 60by60, it would be 60 minus 10 (the size of the square) which gives you fifty more squares which you can fit on the grid; fifty plus the first one gives you fifty-one 10by10's in a 60by 60. Hence there would be 51 times 51 10by10's in a 60by60 (David Reid, personal communication, January 5, 2002). Jake found it difficult to articulate how he had arrived at his counter-argument of 51times51 10by10's. Doubtless, articulation of the idea is challenging because the procedure itself is complex

For the past three years a number of students have 'seen' and have attempted to explain in writing and in talk their strategy of how to figure out how many squares of different sizes are contained within a large square, the "array strategy." These students are seen to develop and test conjectures about mathematical relationships. They work with generalisations they have made while doing their first tasks; they apply to the larger square, the 10by10, the generalisations about how many squares one can fit along two dimensions, that is both vertically and horizontally. For example, Clare in discussing the 10by10 showed how she arrived at 64, or 8 times 8, squares of the 3by3 size, saying: "It would only be 8 [3by3's], cause you would go off the grid otherwise." I saw this "array strategy" for the first time in 1999 in one child, Walt's, work. In the past two years, seven and two students respectively have independently or with partners constructed and used this 'rule' about the array to arrive at their answer for how many there are of each size square. However, Jake was the only student seen to use this 'rule' to counter and refute.

As a result of the children's work I am constantly developing my own knowledge of children's thinking and my understanding of the mathematics in the task. My learning trajectory of the "array strategy" had its genesis in Lew's sliding of each size square across each row, from left to right (proof by cases) in 1996; was extended by students seeing the array as a way to arrive at the answer for the number of each-size square (Walt, 1999); and was further extended by Ari and Jake's knowing why it works that way (2000). This informs my work with future cohorts of children, in a continuing cycle of generative growth (Franke, Carpenter, Levi & Fennema, 2001).

The teacher's role

The children who invented the "array" procedure have a tacit but not conscious awareness, and work ahead of formal instruction; indeed, in this case they were teaching me. One of my enduring questions is in regard to the role the teacher should play in making more explicit what some of the children are doing naturally. Richard Pallascio suggested recently that the teacher is instrumental in making certain aspects visible (Gordon Calvert, et al., in press). The students use sophisticated reasoning, but may well not see the power in the reasoning they are doing. The teacher can bring to the surface some of the implicit items. She can point out the mathematical concepts the students have used perhaps without being conscious of them, and the types of arguments they have used. The teacher can come back to look at what the students have said, and to connect their talk with the ways in which a mathematician would express those ideas. My personal goal as teacher lies in encouraging the children to keep in touch with their personally meaningful ways of seeing, while valuing and learning the conventions of their culture. As I have elaborated elsewhere, the challenges inherent in that endeavor are substantial (Zack, 1999). Most important is to encourage the students to stay connected to the meaning they make personally.

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