DESIGN PRINCIPLES FOR TASKS THAT SUPPORT ALGEBRAIC THINKING IN ELEMENTARY SCHOOL CLASSROOMS[1]

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The purpose of this work is to describe design principles for mathematical tasks in elementary school that have been transformed from arithmetic tasks to those that exploit algebraic thinking. In particular, we discuss design elements of these 'algebrafied' tasks that can elicit students' activity of generalizing, where our particular mathematical focus is on generalizing from numerical patterns to describe functional relationships. We also consider ways in which arithmetic, particularly number and operation, are embedded in the creation and analysis of patterns. Classroom data in which third-grade students solve the Handshake Problem is used as a context for elaborating these ideas.

BACKGROUND FOR THE STUDY

Algebraic Reasoning as a Way of Thinking Mathematically

Traditionally, the focus of elementary mathematics has been deeply oriented to arithmetic and computation, with little attention given to the relationships and structure underlying simple arithmetic tasks. However, there is a growing recognition that algebraic reasoning can emerge from and simultaneously enhance elementary school mathematics (NCTM, 2000) and can consequently build habits of mind that will prepare students for the more complex mathematics of the new century (Romberg & Kaput, 1998). By algebraic reasoning, we refer to students' activity of generalizing about data and mathematical relationships, establishing those generalizations through public conjecture and argumentation, and expressing them in increasingly formal ways (Kaput 1999). Indeed, algebraic reasoning extends far beyond a traditional view of algebra as an act of syntactically-guided symbolic manipulations and can occur in various interrelated forms, including (a) the use of arithmetic as well as mathematical and non-mathematical situations as domains for expressing and formalizing generalizations; (b) generalizing numerical patterns to describe functional relationships (including covariation); and (c) generalizing about mathematical systems abstracted from computations (Kaput, 1998). Our interest in fostering the development of algebraic reasoning in elementary school mathematics prompted us to consider the kinds of tasks that might be used to exploit mathematical ideas that the discipline recognizes as "algebraic" and to examine characteristics of these tasks that signal this. We discuss our partial results here and report on them more widely in Blanton and Kaput (2001).

The GEAAR Professional Development Project

The work reported here is drawn from our multi-year, district-wide professional development program, Generalizing to Extend Arithmetic to Algebraic Reasoning (GEAAR), designed to develop elementary teachers' ability to identify and strategically build on classroom opportunities for algebraic thinking. In particular, we engage in year-long professional development with teachers in a strategy directed toward classroom-grounded change along three dimensions: (1) building opportunities for algebraic thinking from teachers' available instructional resources; (2) building teachers' capacity to identify spontaneous opportunities for algebraic thinking in the classroom through a focus on student thinking; and (3) building teachers' capacity to create a classroom culture that can support active student generalization and formalization within the context of purposeful conjecture and argumentation. As part of the project, teachers are asked to contribute "algebrafied" tasks that they have developed or selected for use in their own classrooms. During one of our sessions with teachers (Kaput and Blanton, 1999), we and the teachers came to value the feature that generalizing a functional relationship could be achieved through an analysis of the *form* of non-executed sums describing a total for some prescribed quantity (e.g., the number of 'right-side-up' triangles in a large triangle subdivided into smaller triangles). As a result, we began to focus in our sessions with teachers on a genre of tasks with this characteristic and to explore how these tasks could further support algebraic thinking and simultaneously leverage students' experiences with arithmetic, particularly number and operation.

We base the findings reported here on a one-year case study of a third-grade classroom, the teacher of whom has been one of our central participants in the GEAAR Project. The data from which we draw our results were taken from a 45-minute classroom episode that is part of a larger data corpus including student work, the teacher's (Jan – pseudonym) reflections, and classroom field notes, all collected during one academic-year period. In the classroom episode, students solved an 'algebrafied' version of the Handshake Problem, that is, they solved a task traditionally formulated as a single-numerical answer problem that had been transformed to the following:

> *If 5 people in a group shake hands with each other once, how many handshakes will there be? What if there are 6 people in the group? Seven people? Eight people? Twenty People? Write a number sentence that shows your result. Show how you got your solution.*

We selected this problem because it reflects a genre of problems used in our seminars with teachers, and subsequently in their own classrooms, that illustrate the potential for arithmetic tasks to support algebraic thinking, particularly as it relates to generalizing numerical patterns in order to describe functional relationships. In what follows, we draw on students' experiences with the Handshake Problem to describe several of the principles of these tasks that we see as supporting algebraic thinking in conjunction with students' experiences with arithmetic, particularly number and operation.

PRINCIPLES OF PATTERN-ELICITING "ALGEBRAFIED" TASKS

Principle 1: The tasks promoted the use of number and number sentences as objects for reasoning algebraically.

The tradition in elementary school mathematics of finding "the answer" (usually a single numerical value) can disguise the algebraic potential of tasks such as the Handshake Problem by diverting students to a process of finding patterns in a sequence of numbers rather than looking for a functional relationship between varying quantities. In the Handshake Problem, when students rushed to compute the total number of handshakes for groups of varying size they were left with a sequence of numbers, not number sentences, which limited their capacity to develop a predictive model. While finding patterns in a sequence of numbers is a useful task, we argue that it does not engage the full algebraic character of the task because it limits the scope of predictability to consecutive states. The power of functions rests ultimately in their capacity as mathematized objects to allow predictions beyond the scope of known data. To this end, we were interested in suspending computation so that students could reason with the *forms* of the number sentences and thereby deduce a functional relationship. With the Handshake Problem, we found that by using sequences of number sentences and deliberately not computing them, students were able to attend to the sums for their shapes as inscriptions, rather than as instructions to perform procedures. By analyzing these forms, they were able to determine a relationship between the size of any group and the number of handshakes. In essence, treating these non-executed sums as algebraic objects allowed for an analysis of co-variation in the data so that students were able to generalize that the number of handshakes in a group of any size would be "the sum of the numbers from one (or zero) up to one less than the number of people in the group".

Another important feature of these tasks is that they allowed for the *algebraic use of number.* A mature understanding of the concept of function ultimately requires students to attend to the notion of arbitrariness. Numbers can be used algebraically when the type or size of number chosen requires the student to think about structure and relationship between quantities, not simply arithmetic operations on quantities. With the Handshake Problem, by asking students to determine the number of handshakes for a group of 20 people without knowing the previous cases, we maintain that '20' was treated algebraically because it called on students to analyze the structure embedded in the *forms* of a sequence of arithmetic models, or number sentences.

Principle 2: The tasks involved sequences of computations that could be exploited to engage students arithmetically.

The dominant role of arithmetic in elementary schools requires that we think about how to make its role in the more complex processes of algebraic thinking explicit. Tasks such as the Handshake Problem can deeply support arithmetic while simultaneously engaging students in algebraic processes. For instance, mathematizing the act of shaking hands required students to understand the correspondence between a collection of counted handshakes and the number representing it, as well as how to operate additively to find a total amount. These particular aspects of the task were more significant for those teachers in earlier grades where counting and numeracy were in earlier stages of development. Students' facility with number and operation were also strengthened by looking for efficient counting strategies, as the following excerpt illustrates. In it, students were calculating the number of handshakes in a group of size 12, that is, they were computing the sum '11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1'.

1 Teacher:	Did anybody change the order of those numbers in any way when you added them?
2 Student:	Yes. You can change them to tens.
3 Teacher:	You made tens out of these?
4 Student:	We put 11, 10, 9, 8...(voice trails off). You don't need to [add 1 through 7] over again.
5 Student:	All you have to do is put 11, 10, 9, 8 on top of all the numbers we had.
6 Student:	You don't say 12.
7 Teacher:	You don't say 12? Why don't you say 12?
8 Student:	Cause you can't shake your own hand.
9 Teacher:	Now, wait a minute. You just said...
10 Student:	...that we didn't have to count 7, 6, 5, 4, 3, 2, 1, 0...because we knew they already equaled 28.
11 Teacher:	Okay, that's pretty good!...Zolan did something that I really liked.... What did you do?
12 Zolan:	I added the 11 and 10 together. (From the result of 21, he then subtracted 1.) Then I added the extra 1 of the 11 to the 9 and that made a 10.

Zolan had commuted the numbers in such a way that he could add groups of 10. Later in the episode, another conversation occurred about a strategy for computing the sum of the number of handshakes for a group of size 20. At students' suggestion, Jan had written '0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19' on the board. She asked, "Is there a way that we can change the order of these numbers and make them easier to add up?"

13 Student:	You can take the zero and put it, you can change it around by putting the 19 where the zero was and that would be easier. (We infer that the student meant to add '19 + 0'.)
14 Teacher:	Can I do this? If I do this, 19 + 0....
15 Student:	That equals 19.
16 Teacher:	What else could I do?
17 Student:	You could do 18 + 1.
18 Student:	What about 17 + 2?
19 Teacher:	What about 17 + 2? Wait a minute. Let me just try this. So I've used those two (19, 18) and these two numbers (0, 1). (Jan drew lines on the board connecting the numbers in each pair.) Let me see 17 + 2. Okay. These are pairs aren't they? How many pairs of numbers do you think we're going to be able to make out of these 20 numbers? ...Anthony, how many do you think we are going to make?
20 Anthony:	Ten.
21 Teacher:	Why?
22 Anthony:	Because you need to get 1 from each side.
23 Teacher:	Let's see if we do make 10 pairs of numbers.
24 Student:	Each one (pair) is equal to 19.

After they had established that there would be 10 pairs of numbers, the sum of each being 19, Jan said, "Now I've got to add up all these 19's. What is this?

25 Student:	Repeated addition. You could do times.
26 Teacher	I could do times?
27 Student:	Nineteen times 10....190.
28 Teacher:	How did you figure that out so quickly?
29 Student:	I just changed that to 9 and added a zero.
30 Teacher:	Why?
31 Student:	Because the one is a 100 and 9 is 90.

We found that when asked to find efficient strategies for computation, students looked for relationships in the arithmetic models in order to organize numbers in ways that exploited productive pairings. For example, recognizing that they could generate a sequence of sums of 19 by adding 0 to 19, 1 to 18, 2 to 17, and so forth, students used this to more effectively 'add' by multiplication. Moreover, students' proficiency with these maneuvers required an understanding of operation, particularly the generalized commutativity of addition and a sophisticated concept of 'counting on'. Indeed, while data from observations earlier in the year showed that 'counting on' was an emerging part of these students' cognitive schemes, the

Handshake Problem seemed to more fully form this concept in their thinking. In particular, students were able to count on from a previously determined sum and interact with that sum as an independent object. All of these processes, such as counting on, commuting numbers, and mathematizing handshakes, occurred in meaningful contexts in which students explored issues of numeracy and increased their facility with number and operation. Moreover, they were simultaneously engaged in the more complex processes of algebraic thinking about relationships and properties of whole numbers and, especially, attention to forms of the number sentences from which they could deduce general relationships. As a result, the arithmetic was not neglected but became an integral part of students' algebraic reasoning.

Principal 3: The tasks allowed for the enactment of actions and situations familiar to students.

By this we mean the tasks allowed students to construct a sequence of number sentences, or arithmetic models, from a context that accessed their everyday experiences (e.g., shaking hands) and in such a way that sequential cases could be explicitly examined to see how perturbations in one case affected other cases. We conjecture that to have these explicit cases, or number sentences, available as permanent artifacts of the task enabled students to move more easily between the abstraction of the models and the source for interpreting the models, namely the physical context of the problem. In contrast, reducing a model to a computed sum would have concealed the physical actions embodied by the representation. In essence, we maintain that students' analysis of the relationship between the number of handshakes and the amount of people in the group was facilitated by symbolizing a familiar action and by keeping that action explicit. To the extent that the sequential nature of these tasks and the explicit number sentences they generated made functional relationships more recognizable, we maintain that this principle supported students' opportunity for algebraic thinking.

CONCLUSION

Tasks such as the Handshake Problem represent one (of many) genre of 'algebrafiable' tasks that can enrich elementary school mathematics. We found that tasks which leveraged students' arithmetic knowledge and included some act of mathematizing a phenomenon so that a mathematical abstraction had its representation in a familiar context and where that abstraction was left in an explicit form (e.g. '1+2+3+4+5+6') could help access students' capacity for algebraic thinking. One significant feature of such tasks is that they embed the arithmetic in a context that requires more complex mathematical thinking, such as (a) knowing to and knowing how to represent data; (b) using arithmetic (number and operations) to model a phenomenon that involves variation; (c) examining how perturbations in a phenomenon affect a model; (d) reasoning algebraically about the forms of sequences of number sentences; (e) deepening arithmetic reasoning to support the appropriate use and choice of operations and to understanding relationships between numbers in order to facilitate computation; and (f) the algebraic use of numbers and number sentences. Thus, it is our claim that these types of tasks offer students a mathematical experience where arithmetic and algebraic processes interact symbiotically.

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NOTES

[1] The research reported here was supported in part by a grant from the U.S. Department of Education, Office of Educational Research and Improvement to the National Center for Improving Student Learning and Achievement in Mathematics and Science (R305A600007-98). The opinions

expressed herein do not necessarily reflect the position, policy, or endorsement of supporting agencies.