MATHEMATICAL REASONING IN CONTEXT: WHAT IS THE ROLE OF SCHOOL MATHEMATICS?

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This report considers the appropriateness of modelling and of a radical situated paradigm for working out advanced (non-intuitively obvious) mathematical tasks in work contexts. A case study of vocational school students and expert practitioners solving geometry problems in a computer-drafting context indicates that neither of these paradigms is appropriate. For the students school mathematics and drafting context are, above all, important as frames of reference. The practitioners do employ school mathematics but their way of mathematising undergoes a complex transformation that cannot be simply explained by either paradigm and which requires further conceptualisation and research.

INTRODUCTION

The relation between school and out-of-school mathematics is an important issue which has been the object of many research studies. The apparently self-evident idea that mathematical knowledge, learnt at school, is simply applied in out-of-school practices via some sort of mathematical modelling is more of a myth than a matter of empirical observation (Bishop, 1988, p. 8). Situated cognition and related theories have pointed to a marked discontinuity between school and out-of-school practices (Lave, 1988). In short, mathematical practices in specific activities, e.g. in work or buying context, were found to have 'little' in common with school-learnt mathematics. Many studies (e.g. Masingila, 1993; Millroy, 1992; Scribner, 1984) have confirmed that contextual activities impose many conventions, social and activity-related constraints, indeed, different conceptualisations of apparently purely mathematical problems, and that practitioners, as they solve such problems, do not base their mathematical reasoning on school-learnt knowledge.

In this report I present a self-contained part of a recently completed broader study about geometric thinking in an out-of-school context. The research was motivated by the observation that studies about the discontinuity between school and out-of-school mathematics considered mathematical practices where it would hardly make sense to use school-learnt procedures, for they are either ineffective or the problems are too complicated to be solved by analytical means or they are so simple that the solutions can be easily learnt from peers or can be even self-invented. The research considers the situated vs. modelling dichotomy in activities and situations where advanced (i.e. non-intuitively obvious) school-like mathematical knowledge could be profitably applied. Several aspects of the situated vs. modelling paradigm of mathematisation in out-of-school contexts are examined: the mathematical actions in activities (Magajna & Monaghan, 1998), learning mathematical concepts in activities (Magajna, 1999), and mathematical reasoning in activities. In this report I focus on the last issue: how do people reason when they solve non-trivial geometric problems which arise in the case of designing geometric shapes on computers using professional software for computer aided design (CAD). In particular I consider the following questions:

1. Do practitioners in working out geometric problems (where school mathematical knowledge can be profitably used) switch to school mathematics by using some sort of modelling or do they stick to activity practices?
2. Do practitioners, as they solve geometric problems in context, take into consideration, in any way, the context (e.g. the available tools, allowed ways of working, the required precision)?
3. How does personal mathematical knowledge and work expertise reflect in the way practitioners work out geometric problems in activity?

METHODOLOGY

The research questions were explored in two activities closely related to computer technical drafting: designing moulds for glass containers (a work activity) and learning computer-drafting in a school for machine technicians (a school-learning activity). Since the aim of the study was to find cases of qualitatively different ways of mathematical reasoning in context the participants were purposefully selected from those that took part in the broader research. In this part of the research eight participants were studied as they solved problems given in a computer-drafting context. Two of the participants were machine technicians with six and 15 years of experience, selected from a group of six designers and technologists in a small mould-making factory in Slovenia. As part of the broader research the group was studied using several techniques (interviews, ethnographic observations, scheduled observations, document analysis). The other six participants were selected from a class of 22 students, aged 18, from a vocational school of machine technicians in Slovenia. The students attended a computer-drafting course, which lasted several months (two hours per week) and were observed as part of the broader research. The six participants were selected on the basis of a special filtering procedure aimed at selecting students that avoid and students that are inclined to use non-intuitively obvious ideas in computer-drafting.

The participants' reasoning in solving geometric problems was studied by analysing verbal protocols of participants' speech as they solved geometric tasks in a drafting context under think-aloud instructions. In simple terms, the method consists in recording and analysing the vocalisations of subjects who work out a task and simultaneously talk, i.e. they are verbalising what is 'going through their mind' as they work on a task. More details about the method are in (Ericsson & Simon, 1985).

Each of the eight participants was asked to work out, under think-aloud instruction, 3-6 drafting tasks which contained non-trivial geometric problems. Figure 1 shows two examples of such tasks. The participants worked in front of a computer using the drafting program they commonly used. Their utterances (while thinking aloud) were tape-recorded, and their hand-sketched and computer drafts were also recorded. The protocols were literarily transcribed and segmented using easily applicable criteria of pauses between thoughts and the timing of each segment was measured.

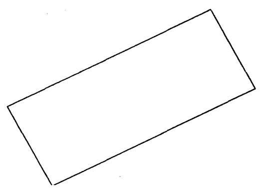
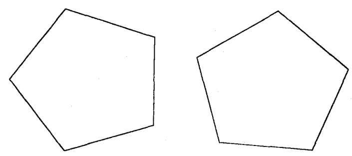
Task 3. From a CAD you have imported a shape representing a 5 mm thick metal plate to be milled.  Your task is to find out whether the plate was designed as a perfect rectangle.	*Task 6.* On your CAD you have got a drawing, like shown below, representing two separate pieces to be machined. Your task is to find out whether you should machine two copies of the same piece or should you machine two different pieces. 

Figure 1. Two examples of tasks used for verbal protocols.

ANALYSING THE PROTOCOLS AND RESULTS

Of the eight selected participants one was not able to think aloud. The other seven participants solved altogether 32 geometric problems. The recorded protocols gave rise to 2843 segments, 773 of which were excluded from further consideration because were not directly related to the solved problems (e.g. comments on saving files to the computer disk). The analysis proceeded along two directions: a qualitative analysis of the solutions of the tasks, based on clustering techniques, and a quantitative analysis of the protocols, which is described below.

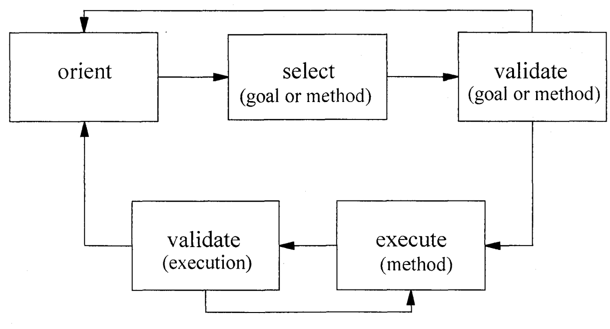


Figure 2. The assumed processes in reasoning.

The quantitative analysis of the protocols has to be considered to be exploratory because of the small number of observed tasks and because of the fact that the participants were purposefully selected. In the analysis two elements were considered: 1. the cognitive processes during the problem solving, and 2. the context (school mathematics or computer-drafting) to which the participants were referring while solving the tasks. The cognitive processes were analysed using a cognitive model, based on Saxe's (1991, p. 16-23) description of cognition. Figure 2 shows a simplified schema, derived from the model. The related categories used for coding the protocol segments are listed in Table 1.

Category	Description of segments
Context	Referring to the problem at a meta-level, e.g. any reference to drafting or school-learning activity, tools, artefacts, conventions, or social relations.
Ignored	Making statements related to the problem but not to the solving process, e.g. students' utterances solicited by experimenter for clarification.
Orient	Looking for a goal or method; stating given, observed or inferred facts; quoting facts or methods without relating them to the worked task.
Select	Selecting, declaring, naming a goal or method with the expressed intention to work on it or to consider it as a possible direction of work.
Execute	Executing a method either mentally, on paper or on computer.
Validate	Checking whether the selected goal or method would lead to the desired result, e.g. expressing arguments, questions or confirmations about the correctness or appropriateness of a goal or method.

Table 1. Categories related to the assumed cognitive model.

The assumed cognitive model and the categorisation were satisfactory in the sense that the segments could be reliably coded and reasonably followed the flow from Figure 2. For example, the segments from the Orient category were followed by another Orient segment in 61% of cases, by a Select segment in 15% of cases, by an Execute segment in only 6% of cases and by a Validate segment in 3% of cases.

Another categorisation of the segments of the verbal protocols, shown in Table 2, was related to the important issue of whether participants, in solving a geometric problem in a drafting context, were 'reasoning' in terms of school mathematics or in terms of contextual (drafting) activity. It was initially envisaged that it would be extremely difficult, if not impossible, to determine whether someone is referring to one or another context but this was not the case because there are concepts and operations in computer-drafting that are very unusual or not even meaningful in school-geometry context (e.g. setting the snap interval of the computer mouse – an important issue in defining geometric entities in computer-drafting) and vice versa (e.g. quoting a theorem or writing down an equation). However, in most cases the segments are meaningful in both contexts (e.g. measuring a distance or drawing a line segment). It was fortunate that the participants were thinking aloud in Slovenian while the software they were using was in English, so that it was easy to identify segments that certainly referred to drafting commands on computer.

Category	Description of segments
Context	Referring to the problem at a meta-level, e.g. any reference to drafting or school-learning activity, tools, artefacts, conventions, or social relations.
Ignored	Making statements related to the problem but not to the solving process, e.g. students' utterances solicited by experimenter for clarification.
School geometry	Referring to geometry related ideas which are clearly not meaningful or common in computer-drafting activity and which are clearly not related to any computer (drafting) command.
Unclear	Referring to geometry related ideas that are meaningful in the context of school geometry as well as in relation to computer-drafting.
Drafting	Referring to geometry related ideas that are clearly not meaningful in the context of school geometry.

Table 2. Categories related to the context of thinking.

Here is a brief summary of the results of the (exploratory) quantitative analysis of the 32 geometric tasks. Recall that the tasks were solved by eight participants (one of which was not able to think aloud). The quantitative and qualitative analysis of the protocols (as well as other performances not considered in this report) indicated that the participants clearly split into three groups: the two expert practitioners (the WORK group), three students that avoided using advanced mathematics (the CAD group), and two students that showed a preference for applying advanced mathematics in drafting context (the GEO group). The unit of analysis used in comparing the three groups was the solved task. For each solved task the fractions of time spent in each category related to cognitive processes was computed (and similarly for categories related to contexts). To determine significant differences in the distributions of codes between tasks worked out in the three groups a one way ANOVA was performed for each code. The results are presented in Table 3. From the table it can be read, for example, that validation took on average 16% of time for the students from the CAD group, on average 13% of time for the students of the GEO group, and on average 21% of time for the participants of the WORK group. The value p ($p=0.297>0.05$) indicates that in this respect the groups do not differ significantly. From Table 3 it is evident that the three groups do not significantly differ with respect to the assumed cognitive model. However, the participants from the three groups behaved quite differently regarding the context they referred to as they were solving the tasks.

	Common		Cognitive model				Context of thinking		
Group (Tasks)	Context	Ignore	Orient	Select	Exe cute	Vali date	Draft	Unc lear	Sch ool
All (32)	1%	13%	23%	10%	35%	17%	27%	50%	8%
CAD (13)	2%	16%	20%	9%	37%	16%	40%	40%	2%
GEO (8)	0%	15%	26%	12%	34%	13%	25%	50%	10%
WORK (11)	2%	9%	24%	10%	34%	21%	13%	63%	13%
Significance p	0.475	0.387	0.687	0.609	0.949	0.297	0.004	0.031	0.119

Table 3. Average fractions of time per task spent by participants of different groups in categories related to cognitive model and the categories related to context of thinking.

DISCUSSION OF THE RESULTS

School mathematics in drafting context. In spite of the fact that in drafting activity procedures tend to be routinised and, for the sake of productivity, reduced to push-button operations, some students and practitioners use methods and reasoning patterns similar to those commonly found in school mathematics. A practitioner, for example, solved Task 3 (see Figure 1) by measuring the lengths of both diagonals: since they were not congruent, he deduced that the quadrangle is not a rectangle. Practitioners and some students also wrote down equations containing trigonometry functions or ratios related to similar triangles and tried to solve them. In general, the students from the GEO group and the practitioners from the WORK group did not hesitate to draw attention away from the computer drawing and to study the geometric properties of the figures. In contrast, the students from the CAD group, in general, tried to solve the problems by looking for appropriate computer commands. Though the tasks were set in computer-drafting context, some participants did not automatically stick to the computer-drafting practices.

The role of the context. The practitioners very sporadically mentioned anything about context. Altogether they spent only 1% of time on segments anyhow related to the contextual activity, this occurred mostly when they were stuck. The fact that the participants did not reason about the context at the beginning of the task (and similarly not at the end of the tasks) by itself indicates that they their way of working out geometric tasks was not based on mathematical modelling.

Though the participants spent very little time considering anything related to the context, the context played an essential role in solving geometric problems. The students from the GEO group evidently placed the task immediately (without even thinking about it) in school-geometry context, and similarly the students form the CAD group placed the tasks in drafting context. The students from the GEO group spent significantly less time in drafting considerations (e.g. looking for an appropriate command, working on a command) and used more advanced mathematical ideas than the students from the CAD group. This indicates that, for students (novices), the context serves as a frame of reference in making sense of a task and in indicating the way of working out the task. Though the tasks were given in a drafting context, some students perceived them as 'purely' mathematical tasks to be solved with mathematical means, and some perceived them as drafting tasks to be solved with techniques specific to the used software or activity. Typically, the students from CAD group solved Task 3 by interactively rotating the figure to an almost horizontal position and reasoning about whether the line segments are parallel on the basis of the discontinuities of lines due to the resolution of computer display.

Expert practitioners regarded the context quite differently than the students. On one hand they had considered the broad context of the task – in solving Task 6 (see Figure 1) a practitioner, for example, asked whether the shapes have to be milled from inside or from outside and he mentioned what one should take care of when positioning the workpieces on the milling machine. On the other hand, though the practitioners claimed they had forgotten all mathematics they learnt in school, in solving the tasks the mathematics they used was much richer than even those of the students from the GEO group. They used trigonometry, wrote equations and made non-trivial deductions. This apparently paradoxical behaviour is perhaps due to two causes. First, the practitioners spent very little time on drafting considerations (see Table 3), for they used the software in a very fluent, almost automatised way. The fluency in managing the activity-related apparatus seems to be a necessary condition for 'advanced mathematical reasoning' in other activities. Second, compared to the students, the two participants used many mathematical ideas and procedures similar to those found in school classes even though they did not perceive these to be mathematical. I hypothesise that, as participants join an activity, their mathematical knowledge undergoes a transformation. Apparently, on one hand their visible mathematics reduces to some more or less routinised practices, but on the other they learn to relate school mathematics to the activity environment, how to 'reason mathematically' using the structural resources found in their activity and how to govern their mathematical thinking by the contextual activity.

Individual differences in solving mathematical tasks in context. One of the aims of the research was to point to different ways of using school mathematics and activity-related knowledge in solving mathematical tasks that occur in other activities. The most important differences in this respect have been considered above, but other differences between the participants from the three groups also emerged. Expert practitioners, for example, in solving harder tasks worked in parallel on two or more different directions, while the students commonly stick just to one solution. Another interesting trait of expert practitioners was to work on solutions for which they knew in advance to be only apparently correct. Usually they were able to use such solutions as a step towards the final solution. Such differences are probably due to the expertness of the practitioners.

FINAL REMARKS

This study indicates that when the use of advanced mathematical ideas in out-of-school practices is considered neither the radical situated view (according to which people work out tasks by relying on activity-related practices) nor the modelling paradigm (according to which tasks in activities are worked out by applying school-learnt knowledge) seem to be valid. Inexperienced students solve geometric problems given in an out-of-school activity by placing them in a context - which can be school mathematics as well as the contextual activity - and solve them according to conventions common to the considered context. For them, the context is, above all, a frame of reference for setting the meaning of the task and for setting the way of working out the task. As practitioners acquire expertise their school mathematical knowledge and mathematical reasoning appear to undergo a complex transformation which is not explained by either paradigm. Further research and different conceptualisations are required to describe mathematical thinking in (work) context (Noss, Hoyles & Pozzi, 1998).

REFERENCES

Bishop, A. J. (1988) *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*. Dordrecht: Kluwer Academic Publishers.

Ericsson, K. A. & Simon, H. A. (1985) *Protocol Analysis: Verbal Reports as Data*. Cambridge Ma: MIT.

Lave, J. (1988) *Cognition in Practice*. Cambridge: Cambridge University Press.

Magajna, Z. & Monaghan, J. (1998) Non-elementary Mathematics in a Work Setting. in A.Olivier & K.Newstead (eds.) *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*. Vol. 3. Stellenbosch. p. 231-238.

Magajna, Z. (1999) Making sense of informally learnt advanced mathematical concepts. in O. Zaslavsky (ed.) *Proceedings* of the *23rd Conference of the International Group for the Psychology of Mathematics Education*. Vol. 3. Haifa. p. 249-256.

Masingila, J. O. (1993) Learning from Mathematics Practice in Out-of-School Situations. *For the Learning of Mathematics*, 13(2), p. 18-22.

Millroy, W. L. (1992) *An Ethnographic Study of the Mathematical Ideas of a Group of Carpenters*. Reston VA: National Council of Teachers of Mathematics.

Noss, R., Hoyles, C. & Pozzi, S. (1998) *Towards a Mathematical Orientation through Computational Modelling*. ESCR End of Award Report (version II).

Saxe, G. B. (1991) *Culture and Cognitive Development: Studies in Mathematical Understanding*. Hillsdale NJ: Laurence Erlbaum Associates.

Scribner, S. (1984) Studying Working Intelligence. in B. Rogoff & J. Lave (eds.) *Everyday Cognition: Its Development in Social Context*. Cambridge MA: Harvard University Press. p. 9-40.