HOW STUDENTS STRUCTURE THEIR OWN INVESTIGATIONS AND EDUCATE US: WHAT WE'VE LEARNED FROM A FOURTEEN YEAR STUDY

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This talk reports on a portion of a fourteen-year study of the mathematical thinking of a cohort group of students that is based on an identifiable perspective on how mathematical ideas are built. Three videotape episodes are presented so that we can view together groups of students working together at three points in time: as fourth graders (ages 9-10), and in grades 10 and 11 (ages 15-16). Excerpts from interviews are offered to provide student perspectives on their own learning over the years.

INTRODUCTION

Choosing a theme for this paper has been difficult; fourteen years of research have provided many possibilities[1]. After much deliberation, I decided to focus on how students, working in small groups, structure their investigations and what we have learned so far by studying how their ideas develop. This talk will introduce some of the students who participated in elementary and secondary school, and later as college students. During the last few months, several of these students have talked with us about their participation in the long-term study.[2] These data along with interview data from their upper-high school years are included here.[3]

Videotape and written data come from students' early investigations of counting problems in elementary school, through their investigations of combinatorics and probability in middle and high school, and their investigations of ideas related to calculus in high school time (Kiczek, 2000; Kiczek, Maher & Speiser, 2001; Maher & Martino, 1996a, 1996b, 2001; Maher & Kiczek, 2000; Maher & Speiser, 1997; Martino, 1992; Muter, 1999; Speiser, 1997; Speiser, Walter, Maher, 2001). Observations and analyses of the students' early explorations provide foundations for later thinking about particular ideas in mathematics. Videotape data make it possible

[1] We would like to thank the Kenilworth students for their continued, invaluable contributions to our work. Thanks, also, to the research team for their dedicated work on the project.

[2] Jeff, Romina, Michael and Brian, now second year college students, joined graduate seminars in which video segments of problem solving were viewed together. The seminar sessions were audio or videotaped. In these sessions, the students talked about how they learned by working together.

[3] Fred Rica, Principal of the Harding Elementary School, Kenilworth invited me, in 1984, to visit his school and to observe mathematics lessons. These earlier classes, for the most part, emphasized drill and memorization. Inspired by the belief that the children's mathematics learning could be significantly improved, a three-year teacher development project was launched in 1984 between the Kenilworth Public Schools and our group at Rutgers (See O'Brien (1995) for a ten-year analysis of the teacher-development project). The longitudinal study was an outgrowth of the partnership.

to view these sessions together. The three episodes -The Gang of 4 (ages 9-10); Romina's Proof (ages 16-17), and the Night Session (ages 17-18) - taken together, provide illustrations over time of how the students worked together in small groups, and of how the teacher(s)/researcher(s) interacted with them. Finally, video clips of student interviews (ages 17, 18, and 19) provide further commentary by participants about their own learning. Jeff (March 2002), reflecting on his participation in the research since grade one, focuses on the depth of their investigations and the impression this left on him.

You didn't come in and say, "this is what we were learning today and this is how you're going to figure out the problem." We were figuring out how we were going to figure out the problem. We weren't attaching names to that but we could see the commonness between what we were working on there and maybe what we had done in school at some point in time and been able to put those things together and come up with stuff and to do these problems to come up with, what would be our own formulas because we didn't know that other people had done them before. We were just kind of doing our own thing trying to come up with an answer that was legitimate and that no matter how you tried to attack it, we could still answer it. It was a solid formula that works no matter how you tried to do it.

BACKGROUND, PURPOSES, RATIONALE

The study was initiated in 1989 with a class of 18 first-grade children at a public school in a working-class community (Martino, 1992; Maher and Martino, 1996b)[4]. The work reported here is a component of the longitudinal study of the development of students' mathematical ideas. Attention has been given to studying how learners build mathematical ideas, create models, invent notation, and justify, reorganize, extend, and generalize their ideas. Data come from a cohort group of students whose mathematical activity has been followed by the research team for over 14 years[5].

The main objective of our research has been to gain a deeper understanding of mathematical learning when particular conditions are in place. We have been interested in creating conditions whereby we can give children an opportunity to show us how they think about mathematics. These conditions are essential to the context of the study and may be helpful in understanding issues of commitment, motivation, and value to participating students. In the early years of the project,

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[4] The class was one of three first grades in the elementary school. The students in each class remained together for their first three years of elementary school. In grade four, new classes were formed. The study continued with a smaller subset of the original class and several other students who joined. The group that was followed for fourteen years consists of seven students; others (seven) participated for approximately eight years.

[5] Earlier work for this study was supported, in part, by National Science Foundation grants MDR9053597 (directed by R.B.Davis and C.A.Maher) and REC-9814846 (directed by C.A. Maher) The opinions expressed are not necessarily of the sponsoring agency and no endorsement should be inferred.

when the research was conducted in classrooms, these conditions, negotiated with the participating school district administrators and classroom teachers, guided the establishment of the context for our research. What came to be called "Rutgers mathematics" occurred in the early elementary years four to six times a year, for three days duration. For two of the days, their math period was extended approximately an hour to an hour and a half. The third day was the regularly scheduled math time of about 45 minutes. Students continued conversations through follow-up individual or small group interviews the same or the following week. When feasible, the classroom teacher observed the interview.

Interview design was motivated by our observations of the children doing mathematics in the classroom and of our study of videotapes and researcher notes. For example, we might notice that there would be an interesting idea that was being pursued by a child or a group, and we would ask the children to tell us about what they were doing. In the interview setting, we invited the reconstruction and extension of ideas put forward. To a large extent, the direction we took in the presentation of investigations was inspired by what the children showed us in their talking, drawings inscriptions, and building of physical models.

Students were invited to think about mathematical situations, often over long periods of time. They were asked to present their ideas with suitable justifications that were convincing to them (and to us) and to consider generalizations and extensions. They were not graded for their work; they revisited problems over months and years; they offered arguments for the validity of their solutions. Ideas were listened to and treated with respect by the teacher/researcher(s). In the early years, we called the students' attention to the variety of ways they represented their ideas, with the intent of making public both similarities and differences in their thinking (Maher, 1998a). We invited reflection and discussion among students about such differences (Maher, 1998b; Maher, Martino & Pantozzi, 1995; Maher, Davis & Alston, 1992). The children, in justifying their ideas, provided arguments that exemplified several important types of mathematical proof, for example, proof by contradiction, proof by cases, and reasoning by induction. In the later years, we observe students using all these forms of reasoning as natural parts of their discourse.

During the first eight years, the study was classroom based. Since high school, cohorts of students participated in small group after school sessions. This came about, at least in part, because the county school district formed a regionalization of the high schools. This resulted in the closing of the Kenilworth high school for several years. Most of the children, upon completing 8 years of elementary school, continued to a regional high school in another town. Others attended parochial schools in the area. Public pressure from Kenilworth citizens (and from some neighboring communities) to regain their local high school led to a public referendum that resulted in a vote to de-regionalize the schools, an historical event for the state. Some local high schools, including Kenilworth, re-opened in 1997 and most of the original students in the study were reunited. In that interim year, we continued our research with several students after school hours in private homes. The interest of students to continue once their high school was re-opened sparked the continuation of what we refer to as "after-school" mathematics. In 1996, classroom research was replaced largely by small group research.

THEORETICAL PERSPECTIVE ON LEARNING

We investigate the development of mathematical ideas by examining, from moment to moment, the development of students' thinking as it is indicated by their conversation and their inscriptions as they work on well-designed investigations.[6] The guiding framework for this level of analysis comes from research on representations (Davis, 1984; Davis & Maher, 1990; Davis & Maher, 1997; Goldin 2002, 2000; Kiczek & Maher, 1998; Kiczek, Maher & Speiser, 2001; Maher & Martino, 1997; Maher & Davis, 1995). In doing mathematics, mental images can be formed by individuals, to be used in building representations of mathematical ideas. These representations can be carried forth and used, and revisited and modified, in the light of new experiences. Although the internal, cognitive representations are not available to us and perhaps the individual, certain features of them can be made public and open to discussion. This can occur as ideas are explained, justified and shared with others. To represent an idea, an individual may create a structure or present a notation[7]. In this way, the ideas are made public in their discourse in the form of explanations, actions, writings, and notations. Records make possible later re-examination of the relationships between ideas. In this way, the ideas can be discussed and reflected upon (Dörfler, 2000).

RESEARCH METHODS

We regard events as connected sequences of utterances and actions by the learners. An event is called "critical" when it demonstrates a significant advance from previous understanding, or a conceptual leap in earlier understanding, or the identification of a cognitive obstacle (Maher & Martino, 1996a; Kiczek, 2000; Steencken, 2001). These episodes are obvious and striking in that they can be connected to prior events. Identification of critical events makes it possible to examine their influence on later understanding and to trace the development of ideas.

Data Source. Our main sources of data are: (1) behavior of students as they work on mathematical investigations recorded on videotape; (2) written work of students; (3) follow-up interviews of individual or small groups of students, often taking the form

[6] The mathematical investigations posed are central to the research. To view the tasks, visit the Robert B. Davis Institute for Learning website, http://www.rbdil.gse.rutgers.edu

[7] Willi Dörfler (2000) makes an important distinction between the visible representations and the internal, cognitive (re)presentations that are not visible to us.

of teaching experiments; (4) individual student interviews; and (5) researcher notes. Groups of us observed the children, took notes, noticed behaviors, and developed interests in what they produced individually, together, and through sharing with others. At various times, students revisited tasks and talked about their ideas. It was not uncommon for ideas to re-emerge for discussion over long periods.

Videotaping. Videotapes are made of nearly all sessions or task-based interviews. These have three main forms: (1) videotapes of task-based (or "clinical") interviews, where there would usually be one interviewer, one student, and two camera operators (one to record work; the other to record the conversation.) Also, two or more observers take notes, but not in view of the student; (2) videotapes (made outside the classroom in an office or quiet setting) similar to those just described, with 2-5 students working together on a task. During student investigations, there is usually no interviewer present. There are two cameras and a sound technician; (3) videotapes made in actual classroom settings, but otherwise similar to the second category with small groups of students working together. There are three cameras and two sound technicians. Our research results emerge through systematic study of extensive, archived videotape data, often from tape segments, which now, because of recent data, have been re-analysed from new directions, with newly developed tools and a more detailed framework.[8]

FRAMEWORK FOR ANALYSIS

A framework is offered that takes into account how ideas develop and travel within the group and how the teacher/researcher interacts in the process. The analysis begins with the identification of critical events. The mathematical content of each critical event is identified and described, taking into account the context in which the event appears, the identifiable student strategies and/or heuristics employed, earlier evidence for the origin of the idea, and subsequent mathematical developments that follow its emergence. Together, these components provide a "trace", the data tracking the development of the idea(s) (Maher & Martino, 1996a; Maher, Pantozzi, Martino, Steencken & Deming, 1996; Kiczek, 2000; Steencken, 2001).[9] We identify and code their traces in the form of diagrams. Concurrently, transcripts are verified, and explicitly co-ordinated to diagrammed events. Our interpretations evolve from all of these.

Event diagrams and codes.[10] Each critical event defines a timeline, consisting of a past, a present and a future.

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[8] See Davis, Maher & Martino (1992) for a discussion of using videotapes to study the construction of knowledge.

[9] The set of connected critical events with past, present and future defines a "pivotal strand". See Kiczek (2000) and Steencken (2001) for a discussion of pivotal strand.

[10] The framework was developed for the National Science Foundation grant MDR9053597 and further elaborated and extended in the project work. See Speiser (1998) for a prototype-coding

past————————present————————future——

(critical event)

The critical event itself defines the present. Prior images to which the critical event folds back[11] define the past (both recent and more distant), while later events which help us understand (or fold back to) the present critical event define its future. The timeline is followed in strands of analysis, all of which are coded.

Constructing a storyline. Coded nodes denote events along the timeline, and descriptive codes are used to mark strands of events which we call the "flow of ideas". The construction of a storyline begins with the flow of ideas. We examine and identify codes and their respective critical events in an attempt to trace an emerging and evolving story about the data. A storyline is constructed from a coherent organization of the critical events, and often involves complex flowcharting. Hence, the process of producing a trace involves identifying a collection of events, coding those events, and then interpreting them, to provide insight into a student's cognitive development. The trace contributes to the narrative of a student's personal intellectual history as well as to the collective history of a group of students who collaborate.

Constructing narrative. In our model, a narrative phase enables researchers to view the recorded material from the data set holistically. Although they appear last, interpretative actions actually begin from the inception of research; they are originally formulated through theoretical perspectives and research questions of interest (Powell, Francisco & Maher, 2001).

TASKS AND VIDEO SEGMENTS[12]

Three tasks and video episodes are considered here. The tasks are selected from the counting/combinatorics strand and the episodes span an eight year period.

Grade 4 Task: Building Towers. Convince each other and the researchers that you found the number of towers that could be made 3-cubes tall, selecting from two colors.

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scheme for critical events and Powell, Francisco & Maher (2001) for further discussion of the framework and methodology.

[11] See Pirie & Kieren (1994).

[12] See the Private Universe Project in Mathematics (PUP-Math), 2001 for video episodes and accompanying workshop materials at http://www.learner.org/channel/workshops/math and Maher, Alston, Dann & Steencken (2000).

Grade 10 Task: Ankur's[13] Challenge. Find as many towers as possible that are 4-cubes tall if you can select from three colors and there must be at least one of each color in each tower. Show that you have found all the possibilities.
Grade 11 Task: Pascal's Triangle. For Pascal's triangle, how does the addition rule work?

In a May 1999 interview, seventeen-year old Jeff describes the way the students work. He reports that members of the group would put forth their ideas, review them, and select the most salient. He points out that the ideas of others are to be valued.

Well, we break up into groups...like five groups of three, say, and everyone in their own groups would have their own ideas, and you'd argue within your own group, about what you knew, what I thought the answer was, what you thought the answer was and then from there, we'd all get together and present our ideas, and then this group would argue with this group about who was right with this...

Grade 4 Task: Building Towers. In grade 3, the students worked on building towers, 4-tall and 5-tall, selecting from two colors. Sixteen months later, in the fourth grade, they investigated towers 5-tall. About one month later, a group interview with fourth graders: Jeff, Michelle, Milin, and Stephanie was conducted. In this session, we were especially interested in what made the students' reasoning convincing. For about half an hour, the students shared their different approaches.

Grade 4 Video Episode: Gang of Four.[14] In grade four, Jeff, Michelle, Milin and Stephanie discovered the idea of mathematical proof. For at least a year before this, the children had been building arguments in their block tower investigations in which they controlled for variables, argued by cases, and used inductive reasoning and argument by contradiction. For example, in the case of towers, students noticed that, as they built from towers of height n to height n+1, thy could choose one of two colors, thereby doubling the number of towers. So they investigated a doubling idea and come up with a doubling rule and posited this as a generalization. Other students, in looking for patterns, recognized certain organizations that accounted for all possibilities of a given height, and suggested an argument by "cases".

In an interview, Mike (May 1999) talks about exploring and gaining understanding:

In our class, all we did was just explore. We took days at a time, and I have a good understanding of it...like, if you were going to, I guess, a normal class, you'd have to be, like, only selected kids might understand it. But in a class where everybody's working

[13] For a full discussion, see Muter (1999).
[14] See Maher & Martino (1996a, b) for a transcript of the videotape session and an analysis of the data.

together, everybody's a part of the teaching, and everybody, or at least the majority of kids will understand it.

Grade 10 Task: Ankur's Challenge. In the 10th grade, five students (Ankur, Brian, Jeff, Romina, and Michael) met again and considered variations of the tower problems. First, they were asked to find all 5-tall towers, choosing from colors, red and yellow, such that each tower contains 2 red cubes, and to justify their solution. Mike and Ankur quickly solved the problem. While they were waiting for the rest of the group, Ankur poses another problem:

How many combinations can you make with towers four tall, selecting from a choice of 3 colors, and using at least one of each color in every tower?

They worked on Ankur's new problem for approximately 15 minutes. Mike and Ankur, after calculating that there were 81 total towers when selecting from 3 colors, returned to the conditions of the problem and, by subtraction, came up with 39 towers. Romina, working with Jeff and Brian, said that there were 36 towers.
Michael continued to work on the problem by himself. Unaware of the work of Romina and her group, he asked to hear their solution. Romina went to the chalkboard and presented her justification.[15] She indicated that the set of all possible towers could be partitioned into six groups. Since every tower would have two of one color, Romina focused on the placement of the duplicate color, using x and o. She indicated that for each placement of the first or duplicate color, there would be two possible combinations for the second and third colors. She also indicated that these combinations would have two opposite arrangements for the second and third colors. She then tripled the 12 possibilities to represent every color, concluding that there should be a total of 36. Romina was asked to write her solutions. Figure 1 shows the refinements in her written work presented at the next after-school session.

Grade 10 Video Episode: Romina's Proof. In an earlier session, Michael introduced the idea of using binary notation to count towers. Other students soon integrated binary numbers in their coding. When Michael indicated that he was "ready to listen", Romina shared her solution with the group.

Romina (March 2002) comments on the way they worked:

If I didn't understand the problem, or if I didn't work enough to it, by myself to understand, and I guess if Michael didn't know where I was heading with what I was doing, and if I didn't understand where the other person was heading I would like to work on it before I came up with a couple of options myself to see which one we take.

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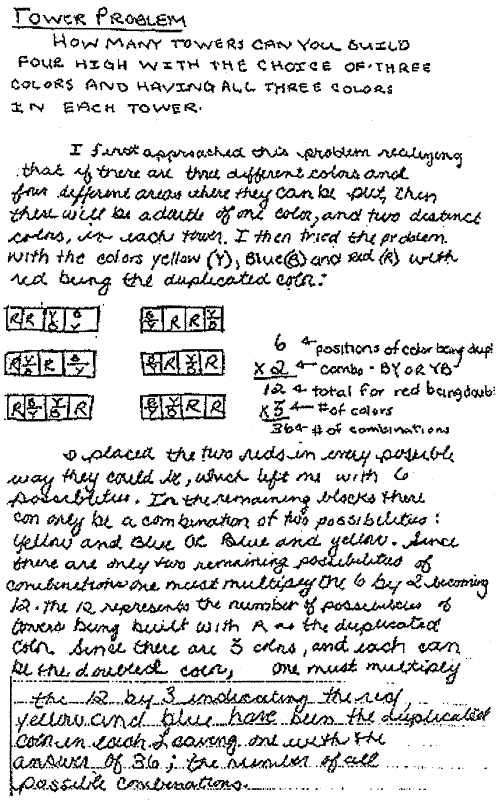
[15] See Muter (1999) and Muter & Maher (1998) for a more complete discussion



TOWER PROBLEM

HOW MANY TOWERS CAN YOU BUILD FOUR HIGH WITH THE CHOICE OF THREE COLORS AND HAVING ALL THREE COLORS IN EACH TOWER.

I first approached this problem realizing that if there are three different colors and four different areas where they can be put, then there will be a double of one color, and two distinct colors, in each tower. I then tried the problem with the colors yellow (Y), Blue (B) and Red (R) with red being the duplicated color:

6 ← positions of color being dupl
× 2 ← combo - BY or YB
12 ← total for red being doubl
× 3 ← # of colors
36 ← # of combinations

I placed the two reds in every possible way they could be, which left me with 6 possibilities. In the remaining blocks there can only be a combination of two possibilities: yellow and blue or blue and yellow. Since there are only two remaining possibilities of combinations one must multiply the 6 by 2 becoming 12. The 12 represents the number of possibilities of towers being built with R as the duplicated color. Since there are 3 colors, and each can be the doubled color, one must multiply the 12 by 3 indicating the red, yellow and blue have been the duplicated color in each. Leaving one with the answer of 36; the number of all possible combinations.

Figure 1. Romina's written work of her solution to Ankur's Challenge.

Grade 11 Task: Pascal's Triangle. In May of their junior year, the high school students returned to school one evening around 7:30 pm for a research session. The researcher began by asking the students to review what they had discussed in their pre-calculus class earlier that day. They reported that the class had learned to use a calculator to find the coefficient of any term in the binomial expansion without having to write out rows in Pascal's Triangle. The students were also asked why the addition rule for Pascal's Triangle worked. In response, they showed a 1-1 correspondence between terms in Pascal's Triangle and choices in particular pizza and tower problems. During their discussion of these problems, they gave meaning to the addition rule. [16]

<u>Video Episode: The Night Session.</u> The students were asked to write the general n^{th} row in Pascal's triangle, using the bracket notation for nCx. They responded to the researcher's request to formulate the addition rule with this notation. They explained the correctness of the notation by referring first to particular cases of the pizza problem,[17] and then to the meaning and structure of the addition rule as additional toppings are added.[18] Jeff, assisted by Michael, wrote the following equation on the chalkboard:

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Challenged by the researcher to express their result in factorial notation, the students worked together to produce the following equation:

$$\left(\frac{n!}{(n-x)!x!}\right) + \left(\frac{n!}{(n-x+1)!(x+1)!}\right) = \left(\frac{(n+1)!}{(n-x)!(x+1)!}\right)$$

After succeeding to write the addition rule for nCx, Jeff remarks:

Do you know, like, how intimidating this equation must be like if you just picked up a book and looked at that.

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[16] For a full discussion, see Kiczek (2000); Kiczek & Maher (1998); Kiczek, Maher, & Speiser (2001).

[17] The metaphoric reference is to the general Pizza Problem. Since grade 5, the students worked on variations of Pizza investigations. The reference here is to a general problem of finding how many different pizzas that could be made using any number out of, say n, different topping choices and of considering how they could account for organizations of pizzas as additional toppings are made available.

[18] A detailed analysis of the session is described in Dörfler & Maher (in progress).

Mike (April 2002), in an interview 3 years later, was again asked how he might explain the addition rule. Mike recalled that the group had given a general rule in the 1999 after-school session; he immediately began to reconstruct it. This time, Mike called a row, r [to denote the number of toppings], and a "spot" in the row, n. He used the notation to show that "r choose n" plus "r choose n+1" equals "r+1 choose n+1", referring again to adding pizza toppings to explain the rule.

In the same interview, Mike talked about how he looks for relationships while he works on problems:

The process while I am doing the problem...I just start understanding more that this is related to that...how this is related to just a triangle that's made up of numbers. At first when they showed us a triangle, we didn't know that has anything to do with...once you start understanding things have a relation to each other you just start convincing yourself...and then you come to a point where you know it's right, or you think it's right.

STUDENTS REFLECT ON THEIR LEARNING

Through a series of individual and small group interviews, we present student views on how they structured their learning, thereby gaining insight into their views of the process.

Just giving the answer was never enough. Jeff (May 1999) indicates that the students themselves took on the expectation for presenting a careful argument. Consequently, they reviewed their own argument, focused on meaning, and anticipated questions and "holes". They questioned each other and put ideas together before offering their solution to the researchers. Listening to and asking questions of each other were essential components of the process of working together.

Just giving the answer was never enough, in order to do it. You'd have to have a good, like, structural record. It's almost like doing, like a proof...like you need to show every step from point A to point B...you couldn't just, like, skip some things and jump around. You had to go straight, and everything had to be written out and good, and ... understanding, and if you had a problem with somebody, to ask another question about it, so you ended up doing whole types of things, just to get from the beginning to the end, and through it, that's how you really understand what you were doing, that's why we'd learn, like what we were doing without actually calling anything a certain thing...

We would come to it ourselves. Jeff (May 1999) talks about not being told how to do things.

And then, like, later now, we would be doing things, like, "Oh, that's what we were learning." because Rutgers never really told us what the answers were, or what we were actually doing...like, 'This is what we're going to do today; it's called the... theorem,' or anything like that. We would come to it ourselves, and then later, we would realize that that's what we were doing this whole time.

Jeff (March 2002) indicates a building process.

If we tried to just present a final thing and really didn't know it from the beginning we couldn't explain it in a way that that you would accept from us. So in order to explain it in a way that you would accept we'd really have to start from bare bones, from the beginning.... We didn't start talking about what we were doing with you until very late in what we were doing. There was not a lot of communication back from them to us about the work we were doing.

We got so in-depth. Jeff (May 1999) reviews what they accomplished.

Well, even though we didn't spend much time together, and they [researchers] only came a few times a year, we did so much, we covered so much, we got so in-depth on topics, that it leaves an impression. I mean, we could talk about doing the blocks in first grade, and we can almost go through problems: We did shirts and pants in second grade. I mean, how many other people can tell you the math that they were doing in second grade...like a word problem, you know? Because you go in deep, you work on it so much, and you go so far into it, that it just sticks with you...That's why it leaves such an impression, because of the depth you get into it...

We just sat and thought for hours a day. Romina (July 1999) talks about the confidence she gained. She indicates that they spent days thinking about problems and that presenting them to others was a valuable undertaking.

We did a lot of problem solving. We did a lot of thinking. We just sat and thought for hours a day, and we came up with a lot of interesting things. We were able to go in front of a large audience and talk about our ideas and argue our points, and prove our points. I think it was a very good experience.

I think a lot of what we were doing was working together. Jeff (February and March 2002, respectively) talks about the benefits of collaboration and the frustration of working alone.

Well that's...how we got to wherever we were going...we were like four different people with four ideas and we all thought we knew something on how to do a problem but...you

just cover so much more when everyone is discussing what you're doing, I mean that's what it was really all about…that's really how we got anywhere was kinda work, doing our thing together, you know, and using what we each knew, to work something out.

I think it would have been very different if it was all of us producing our own solutions….I think a lot of what we were doing was working together. I think when you are working alone, when you reach a part where you don't know anymore it is very easy to just be frustrated and say I don't know anymore. I'm not going to do this. I can't think about this. Like forget it. I think that by working with everybody when you got to that point, you can kind of peak over a little bit and it was all right...it was encouraged. That allowed everybody to really we could all move forward.

Everything has to make sense. Romina (March 2002) talks about understanding.

Everything has to make sense in my terms. Someone else may have done it already in a book, but I just don't understand it unless I try it myself and put it in my own term.

CONCLUSIONS AND IMPLICATIONS

We have engaged in a research program, extended over 14 years, within which sense making has become a cultural necessity. An aspect of this culture has been the emergence, beginning in the elementary grades, of argumentation, justification, proof making, and generalization. Such processes have developed in the context of coherent strands of mathematics. The reflections by students about their learning over the project years gives further insight into the process of how they worked together and structured their learning. They reported that giving the answer was never enough. They understood that they would be expected to provide a written account to support their reasoning and that details in arguments were important. They accepted that they were not to be told the answers or how to solve a problem and took on early the expectation that they would produce the result and offer appropriate support for it. The support came from convincing first themselves and then each other. Student expectations guided how far they were willing to go in solving problems. These expectations came from interactions with researchers, who challenged them to be attentive to details, provide evidence for their results, and consider extensions and generalizations to investigations posed. The students reported that they were aware of what would be asked of them and took upon themselves on the responsibility of developing satisfactory solutions beforehand. Their work and conversation, backed by extensive interviews, indicate that they maintained high expectations for themselves and for each other, and these expectations help explain the way they worked together, over months and years. As evidenced by their comments, they took on, progressively, responsibility for their own learning and for the maintenance of communication and collaboration in their working groups. They reported increased confidence over the long run, which may explain, at least in part, their evident commitment and responsibility for helping the project to continue and evolve through time.

REFERENCES

Alston, A.S., Davis, R.B., Maher, C.A., & Martino, A. (1994) Children's use of alternative structures. Proceedings of the Eighteenth Annual Conference of the International Group for the Psychology of Mathematics Education. (Vol. 2, pp.248-255). Lisboa, Portugal, University of Lisboa.

Davis, R. B. (1984). Learning Mathematics: The cognitive science approach. Ablex.

Davis, R. B. & Maher, C. A. (1997). How students think: The role of representations. In L. English (Ed.), Mathematical reasoning: Analogies, metaphors, and images (pp.93-115). Hillsdale, NJ: Lawrence E. Erlbaum Associates.

Davis, R. B., & Maher, C. A. (1993). Schools, mathematics and the world of reality. In R. B. Davis & C. A. Maher (Eds.), Schools, mathematics, and the world of reality. (pp. ix-xii). MA: Allyn & Bacon.

Davis, R.B. & Maher, C. A. (1990). What do we do when we do mathematics? In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist Views on the Teaching and Learning of Mathematics Journal for Research in Mathematics Education Monograph No. 4, (pp. 65-78). Reston, VA: National Council of Teachers of Mathematics.

Davis, R. B., Maher, C. A. & Martino A. (1992). Using videotapes to study the construction of mathematical knowledge of individual children working in groups. Journal of Science, Education and Technology, 1(3), 177-189.

Dörfler, W. (2000). Means for meaning. In P. Cobb & E. Yackel & K. McClain (Eds.), Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design (pp. 99-131). Mahwah, NJ: Lawrence Erlbaum,

Dörfler, W. & Maher, C. A. (in progress). Pizzas and towers as thinking tools. [Manuscript under preparation].

Goldin, G. A. (2002). Representation in mathematical learning and problem solving. In L.D. English (Ed.), Handbook of international research in mathematics education (pp.218). Mahwah, NJ: Lawrence Erlbaun Associates Publishers.

Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. Mathematical Thinking and Learning, 2(3), 209-219.

Kiczek, R. D. (2000). Tracing the development of probabilistic thinking: Profiles from a longitudinal study. Unpublished doctoral dissertation, Rutgers University, NJ.

Kiczek, R. and Maher, C. A. (1998). Tracing the origins and extensions of mathematical ideas. Proceedings of the Twentieth Annual Conference of the North American Group for the Psychology of Mathematics Education. Raleigh, North Carolina, 377-382.

Kiczek, R. D., Maher, C. A., & Speiser, R. (2001). Tracing the origins and extensions of Michael's representation. In A. A. Cuoco & F. R. Curcio (Eds.), The role of representation in school mathematics (pp. 201-214). NCTM Yearbook 2001. Reston, VA:

Maher, C.A. (1998a). Kommunikation och konstruktivistisk undervisning (Communication and constructivist teaching).In Arne Engrstom (Red.), Matematik och reflektion (pp.124-143) Lund, Sweden: Studenlitteratur.

Maher, C.A. (1998b). Constructivism and Constructivist Teaching - Can They Co-exist? In Ole Bjorkqvist (Ed.), Mathematics Teaching from a Constructivist Point of View (pp.29-42). Finland: Abo Akademi, Pedagogiska fakulteten.

Maher, C. A., Alston, A., Dann, E., Steencken (2000). Private Universe Project in Mathematics series content guide: A professional development workshop series for K-12 teachers of mathematics. Cambridge, MA: Harvard Smithsonian Center for Astrophysics and the Robert B. Davis Institute for Learning.

Maher, C. A. & Davis, R. B. (1995). Children's explorations leading to proof. In C. Hoyles and L. Healy (eds.), Justifying and Proving in School Mathematics, (pp. 87-105). London: Mathematical Sciences Group, Institute of Education, University of London.

Maher, C. A. & Davis, R.B. (1990). Building representations of children's meanings. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist Views on the Teaching and Learning of Mathematics Journal for Research in Mathematics Education Monograph No. 4, (pp. 79-90). Reston, VA: National Council of Teachers of Mathematics.

Maher, C. A. & Davis, R.B.& Alston, A. (1992). Teachers paying attention to students' thinking. Arithmetic Teacher, 34-37.

Maher, C.A. & Kiczek, R.D. (2000). Long Term Building of Mathematical Ideas Related to Proof Making. Proof and proving in Mathematics Education [Online]. http://www.cabri.net][Preuve/ICME9TG12/ICME9TG12Contributions/MaherICME00.html

Maher, C. A. & Martino, A. M. (2001). From patterns to theories: Conditions for conceptual change. The Journal of Mathematical Behavior, 19 (2), pp. 247-271.

Maher, A.M. & Martino, C.A. (2000). Teacher questioning to promote justification and generalisation in mathematics. Journal of Mathematical Behavior, 18(1), 53-78.

Maher, C. A. & Martino, A. M. (1997). Conditions for conceptual change: From pattern recognition to theory posing. In H. Mansfield & N. H. Pateman (Eds.), Young Children and Mathematics: Concepts and Their Representation. Sydney, Australia: Australian Association of Mathematics Teachers.

Maher, C. A. & Martino, A. M. (1996a). Young children invent methods of proof: The "Gang of Four." In P. Nesher, L.P. Steffe, P. Cobb, B. Greer and J. Goldin (Eds.), Theories of mathematical learning (pp. 431-447). Mahwah, NJ: Lawrence E. Erlbaum.

Maher, C. A. & Martino, A. M. (1996b). The Development of the idea of mathematical proof: A 5-year case study. In F. Lester (Ed.), Journal for Research in Mathematics Education, 27 (2), 194-214.

Maher, C. A. & Martino, A.M. (1992). Teachers building on students' thinking. Arithmetic Teacher, 32-37.

Maher, C. A., Martino, A.M., & Pantozzi, R. (1995). Listening and questioning better: A case study. In L. Meira & D. Carraher (Eds.), Proceedings of the 19th International Conference for the Psychology of Mathematics Education (Vol. 3, pp. 82-89). Brazil.

Maher, C. A., Pantozzi, R., Martino, A. M., Steencken, E. P. & Deming, L. S. (1996). Analyzing students' personal histories: Foundations of mathematical ideas. Paper presented to the American Educational Research Association, New York.

Maher, C. A. & Speiser, R. (1997). How far can you go with block towers? Stephanie's Intellectual Development. Journal of Mathematical Behavior, 16(2), 125-132.

Martino, A. M. (1992). Elementary students' construction of mathematical knowledge: Unpublished doctoral dissertation, Rutgers, University, NJ.

Muter, E. M. (1999). The development of student ideas in combination and proof: Unpublished doctoral dissertation, Rutgers, University, NJ

Muter, E. M. &. Maher, C. A. (1998). Recognizing isomorphism and building proof: Revisiting earlier ideas. In S. Berenson et al. (Eds.), Proceedings of the twentieth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 461-467). Raleigh: ERIC Clearing House for Science, Math and Environmental Education.

O'Brien, M. J. (1995). Changing a school mathematics program: A ten year study. Unpublished doctoral dissertation, Rutgers, University, NJ.

Pirie, S. & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it. In W. Dörfler & P. Cobb (Eds.), Educational Studies in Mathematics (Vol. 26, 2-3, pp. 165-190). Dordrecht, The Netherlands: Kluwer.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2001). An analytical model for studying the development of mathematical ideas using videotape data. Paper presented at the 23rd Annual Meeting of the PME-NA, Utah.

Speiser, R. (1998). Unpublished research notes on prototype-coding schemes.

Speiser, R, Walter, C. N. & Maher C.A. (2001). Representing change. In M. van den Heuvel-Panhhizen (Ed.), Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education, (Vol. 4, pp.209-216). The Netherlands: Freudenthal Institute, Utrecht University.

Steencken, E. (2001). Tracing the growth of understanding of fraction ideas; A fourth grade case study. Published doctoral dissertation. Rutgers University, NJ. Dissertation Abstracts International 3009381.