COGNITIVE TENDENCIES AND THE INTERACTION BETWEEN SEMANTICS AND ALGEBRAIC SYNTAX IN THE PRODUCTION OF SYNTACTIC ERRORS

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In Filloy and Rojano (1989) we introduced the use of concrete models for the teaching of solving linear equations and studied the abstraction processes that take place when such models are put in work by 12-13 year olds. In this paper we discuss M and V cases where, on the face of elements provided by the same "concrete" model for the operation of the unknown, the evolution paths of their use for the resolution of more and more complex equation modes are dissimilar (in fact, antagonic). However, in spite of this antagonism there is a common tendency to abbreviate the processes involved and this generates (in both cases) a number of well known algebra learning obstacles and syntactic errors.

The literature about algebraic errors in the learning of algebra is mainly focused in its syntactic component: Matz (1982), Kirshner (1987), Drohuard (1992). Few works like Booth (1984) and Bell (1996) situate this problematic component in a more general context, for instance, problem solving. In this paper we analyse the interaction between semantics and algebraic syntax as a source of syntactic errors, when this ineraction takes place in teaching processes that involve concrete modeling. We argue that such analysis provides a perspective that allows us to give different explanations of the presence of some typical algebraic syntax errors.

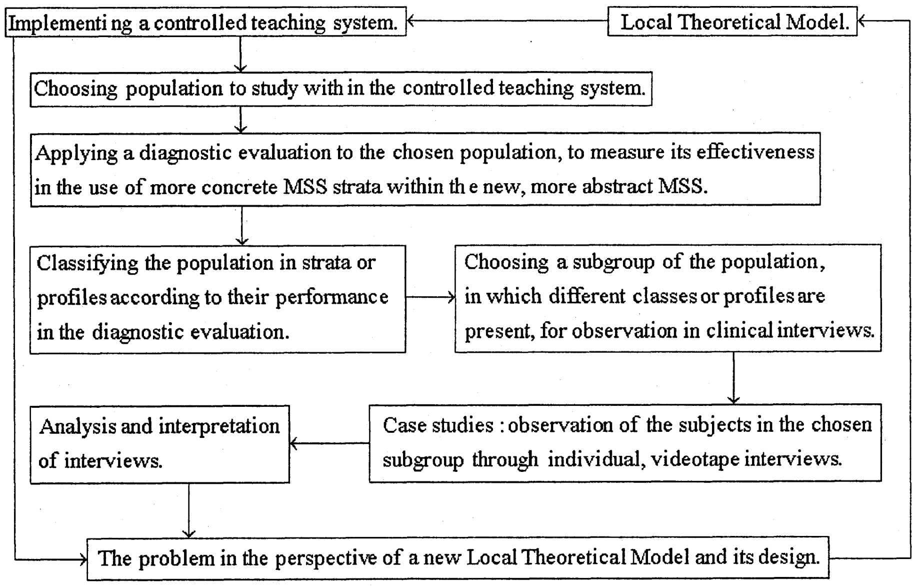
Undertaking a semantic introduction to new algebraic concepts, objects, and operations implies selecting a concrete situation (i.e., a situation which in some context is familiar to the learner) in which such objects and operations can be modeled. With this approach it is possible to resort to previous knowledge, in order to accomplish the attainment of new knowledge. This is one of the driving principles of modeling, the strengths and weaknesses of which become manifest at the time a specific model is put into operation (see Filloy/Rojano, 2001, for a more detailed description). In the cases we report here (V and M cases) the concrete situation in question is a geometric model that was used as a semantic introduction to the operation of the unknown for the resolution of the first non-arithmetical equations. In this model, the translation of the proposed equation into equalities between quantities or magnitudes in a more 'concrete' situation permits to find out the numnerical value of the unknown in the context of area comparison. The use of such a geometric model, then, presupposes a good handling of operations with areas. This handling, as can be verified in V and M's interviews, is a requirement that was covered in both cases. When this study was carried out, M and V showed to be highly proficient at school maths. They found no difficulties in handling the model during the instruction phase aimed at modeling the first non-arithmetic equations (equations of the form $Ax+B=Cx$, where A, B, and C are given positive integers, and $C>A$). It is in the transition to more complex modes of equations that modeling and actions in the model, in turn became more and more complex. In contrast to previous explanatios given with regards pupils' syntax errors, within the modeling realm it is possible to formulate explanations grounded on the nature of the model and the sort of cognitive tendencies displayed by the subjects.

THEORETICAL AND METHODOLOGICAL FRAMEWORK

In Filloy (1990) we introduced the methodological framework of **local theoretical models** in which the object of study is brought into focus through four inter-related components:

(1) **Teaching models** together with (2) **models of cognitive processes**, both related to (3) **models of formal competence** that simulate competent performance of the ideal user of a Mathematical Sign System (MSS) and (4) **communication models** to describe rules of communicative competence, production of texts, texts decoding, and contextual clarification.

The following scheme describes the rationale of the case study:



Results from the diagnostic test located V and M in the category of students with high proficiency in a) solving arithmetic linear equations; b) solving arithmetic word problems; and c) numeric skills. Before the interview, V and M had not been introduced to the learning of algebra. Once V and M got to solve the first non-arithmetic equations by means of the geometric model, they were faced with more complex modes of such type of equations as it is shown in the lists of the interview items below. It can be noticed that the list of items differ from one case to another, due to the specific characteristics shown either by the "semantic" case or by the "syntactic" case during the interview.

THE INTERVIEW ITEMS

We will write **IM.n** and **IV.n** for the **nth** item of Sequence I Series in the interview.

IV.1	$x+2=2x$	IV.17	$5x=3+2x$	IM.1	$x+2=2x$	IM.17	$3+2x=5x$
IV.2	$2x-14=4x$	IV.18	$6x+15=9x$	IM.2	$x+5=2x$	IM.18	$2x+3=5x$
IV.3	$2x+3=5x$	IV.19	$4x-3x=7$	IM.3	$2x+4=4x$	IM.19	$5x=3+2x$
IV.4	$x+5=2x$	IV.20	$4x+25=9x$	IM.4	$3x+8=7x$	IM.20	$5x=3+2x$
IV.5	$2x+3=8x$	IV.21	$7x+2=3x+6$	IM.5	$3x+8=6x$	IM.21	$6x+15=9x$
IV.6	$7x+6=8x$	IV.22	$13x+20=x+647$	IM.6	$2x+4=4x$	IM.22	$7x+2=3x+6$
IV.7	$8x+56=15x$	IV.23	$8x+30=5x+9$	IM.7	$x+2=2x$	IM.23	$13x+20=x+164$
IV.8	$6x+144=18x$	IV.24	$8x+986=12x+647$	IM.8	$7x+15=2x$	IM.24	$13x+12=x+144$
IV.9	$2x+3=5x$	IV.25	$7x+10=4x+4$	IM.9	$7x+15=8x$	IM.25	$10x-18=4x$
IV.10	$3x=4+2x$	IV.26	$9x+33=5x+17$	IM.10	$5x+12=9x$	IM.26	$10x-18=4x+6$
IV.11	$8x=5x+36$	IV.27	$9x+33=5x-17$	IM.11	$15x+13=16x$	IM.27	$7x-20=5x+30$
IV.12	$9x+90=19x$	IV.28	$5x-25=11x+3$	IM.12	$38x+72=56x$	IM.28	$10x-20=5x+30$
IV.13	$x+25=6x$	IV.29	$8x-10=6x-4$	IM.13	$129x+51=231x$		
IV.14	$x+5=2x$	IV.30	$23x-7=14x+2$	IM.14	$37x+852=250x$		
IV.15	$3+2x=5x$	IV.31	$18x-41=9x-5$	IM.15	$x+5=2x$		
IV.16	$5x=2x+3$	IV.32	$19x-3=4x$	IM.16	$2x+3=5x$		

ABBREVIATION PROCESSES

The development of the use of the concrete model is not uniform, it depends on the individual student's tendency to choose a particular approach (Filloy, 1991; Filloy/Sutherland, 1996).

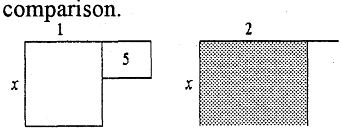
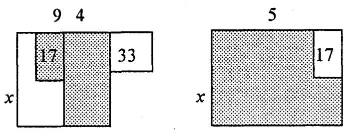
Two extreme cases were detected in the interview: In one case (V case) with an operative tendency, the development anchored to the use of the model context even when the equation types required very complicated modelling procedures. This is the *semantic cognitive tendency* case. In the other case (M case) there was a constant search for the syntactic elements present in the actions on the model as they were repeated in equation after equation and in type after type. The subject broke away from the semantics of the model with a more abstract languaje through the creation of personal codes, belonging neither to the model nor to algebra. This is the *syntactic cognitive tendency* case.

Notwithstanding the bias that the subject's own tendency introduces in his or her use of the concrete model for the resolution of the new equations, there exists, as can be observed from these extreme cases, a common tendency. This tendency consists in abbreviating both the translation processes from the equation to the model and the actions performed in the model (or in the equation itself). In the following subsections, we discuss the abbrevation processes related to both cognitive tendencies, the semantic and the syntactic tendencies.

A semantic tendency

V, shows distinctive evolution lines with regards the abbrevation processes: on the one hand, 1) there are those of stability, progress and generalization of the graphic abbreviation, with the intervention of anticipatory mechanisms regarding the actions, and on the other 2) the transference and discrimination of strategies for area comparison, for each mode of equation.

V's resolution examples:

IV.14 *x+5=2x* Areas comparison	The comparison of areas and the writing (and verbalization) of the simplified equation is done at the same time - without difficulty - without help.	*Obs.: In the areas comparison V draws some more lines in the incomplete figure. This stage and the written form of the simplified equation are almost mixed up, due to the speed with which V completes them. From Vs' manner of writing the simplified equation, making very clear the coefficient 1 of 'x', it could be said that this equation is very close to the context of the model(1x represents an area). Very probably V solves this equation through a specific fact.*
Construction of the simplified equation	V: "We take away this piece of land (referring to the rectangle of 1 x 'x') and we are left with this which should be one(referring to the remaining front)": V only draws some additional lines for the comparison.  V: "One by 'x' should be equal to five". At the same time writes, making clear the coefficient 1 of 'x' : $1x = 5$	
IV.27 $9x+33=5x-17$ Construction of the simplified equation	 V: "Four plus seventeen". She writes 4×17. V: "...Times 'x'?".	*Obs.: Mistaken operation between terms with different geometric dimensions, at the moment of constructing the simplified equation.*

Also, in M's performance with the model, abbreviation processes are observed, but in this case such processes tend to not merely automate the actions in the 'concrete' model (as is observed in V's case), but to extract the syntactic elements present in these actions, so they can, in turn, be modeled in a more abstract language and thus make it possible to detach herself from the semantics of the more concrete model.

A syntactic tendency

In M's case, her very concern about finding syntactic rules for the resolution of the new equations leads to a reflection on the actions that are always performed, one case after the other, and this presupposes (as in V's case) an abbreviation process of such actions. The search for syntactic rules is carried out, in this case, by describing the actions in a more abstract language: such a description requires a synthetic version of the actions.

This synthesis of the actions is achieved through an abbreviation of the resolution process, both in the concrete model and in her translation into the more abstract language. During this translation process, graphs are created (arrows as personal codes) that do not belong to either algebra or to the context of the 'concrete' model. These personal codes allow M to understand, orient, and **represent** the operations outside of the model, although it should be pointed out that such extra-model representations have the defect of not being adequate ways of representing the result of the operations. Such a representation appears in terms of the operations that led to a simplified equation, without making it clear that it (the simplified equation itself) is a final state of those operations. This situation (having in mind just the operations), very frequently leads to an aberrant operation between terms of different degrees.

M's resolution examples:

IM11 $15x + 13 = 16x$	$15x + 13 = 16x$ $15x + 13 = 16x - 15x$ M. "Sixteen x minus fifteen x is one; then, one plus thirteen is fourteen"	*Obs.: The fact of not registering 'x' as a part of the result of subtracting terms in 'x' (15x and 16x), leads M. to effect an aberrant operation between terms of different degrees: one (x) plus thirteen.*
IM13 $129x + 51 = 231x$	$129x + 51 = 231x$ $129x + 51 = 231x -$ $129x + 51 = 231x - 129x = 102x$	*Obs.: M develops her own code of signals in order to leave written traces of those actions that have already been done, as well as those to be performed and of intermediate states in the resolution of the equation:*

M: "Therefore 'x' is equal to two", the answer is wrong.	*a) Indicating, by means of arrows, the transfer of terms, through the corresponding inverse operation, from one side of the equation to the other.* *b) Indicating, by means of arrows, the equality between terms in this case, such equalization to tantamount to the so-called simplified equation.* *c) Indicating, by means of chains of equalities, both operations between terms and the results of such operations.*

Different Tendencies

The antagonism of these two tendencies (V's and M's) becomes clear by merely observing their respective interviews. However, from their comparative analysis a couple of remarks regarding the aspects that are common to both cases, deserve to be emphazised. On the one hand, 1) it can be seen that in spite of the aforesaid antagonism, there is a common tendency to abbreviate the processes (following, in each case, its' own execution path), and on the other hand, 2) during such abbreviation processes a number of obstacles and errors, also common, are generated, and can be considered as typical in the later syntactic handling of symbolic algebra. In one case (V's), the abbreviating tendency consists of trying to lighten the operations performed in the model, while staying within it. To this end, attention must be given to the actions (translation, comparison, etc.) that are performed again and again. This reflection, in turn, leads to an abbreviation process of such actions. It is through this abbreviation that some parts of the concrete model are lost. On the one hand, 1) the "bottom of the terrains", (the linear dimension which corresponds to the unknown), is a situation that leads to hiding the operation of the unknown. On the other hand, 2) the area condition of the constant term is also lost, as well as its' operative handling. This provokes a tendency of performing the addition of 'x' with the terms of degree zero, resulting in the aberrant operation that has been formerly pointed out between terms of different degrees.

SYNTACTIC ERRORS

The generation of the same type of syntactic errors in the two cases discussed here is not accidental and can be explained from a more general level of analysis. In teaching by means of models there is the danger that the main virtue of any concrete model (namely, that of seeking support in previous knowledge) becomes the main obstructor in the acquisition of such new knowledge. In the cases of the children interviewed, whom are left to develop by themselves the use of the geometric model, it happens that the model component which tends to abbreviate, and therefore, to hide the operation of the unknown, will persist in both cases. In cases such as Vs', subjects possessing a strong semantic tendency have this happen due to the fact that the automation of actions in the model weakens the presence of the unknown throughout the procedure. In cases such as Ms', this tendency is due to the effects of the creation of personal codes, to register intermediate states of the originally proposed equation. Corrections, in each case, are of a local nature and according to the subjects' tendency. Thus, when there exists a leaning towards staying in the model, the correction of the syntactic abberration mentioned above is performed in the model itself, for only the model semantics can make such an aberration evident. On the other hand, in the case of a syntactic tendency, the correction takes place currently with other events in the syntax, namely, an essential modification of the notions of equation and the unknown.

GENERAL DISCUSSION

By way of conclusion, the interaction between semantics and algebraic syntax which takes place along the abstraction processes of operations performed with algebraic objects (that have been given meanings and senses within the context of a concrete model) is modulated when learning the algebraic language by tendencies in the subject and by features of the specific model being used. There are, however, some aspects of such an interaction that remain constant when changing the subject's tendency factor, or the type of model. These essential aspects in the relationship between semantics and algebraic syntax reflect, in turn, essential aspects of another interaction, the one occurring between the two basic model components: the reduction to the concrete, and the detachment from the semantics of the concrete. The transference of the problem, semantics vs. algebraic syntax to a level of model actions allows one to close the existing teaching gaps between these two algebra domains. The analysis of this interaction between semantic and syntax at this new level, points to the necessity of intervening through the teaching model at key moments at the beginning of algebraic language use.

In a forthcoming paper we will indicate how this dialectic semantics/syntax; the theoretical description of the relationship between the deep and superficial forms; the generative and transformational aspects to describe the grammar (see Kirschner, D. 1987 and Drouhard, J. P. 1992) of algebraic mathematical sign system syntax, can be linked with the explanation of why errors are committed when following a rule is needed to utilize one or more previous and competently used rules.

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