UNPACKING STUDENT MEANING OF CROSS-SECTIONS: A FRAME FOR CURRICULUM DEVELOPMENT

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Recent literature indicates that there has been some change since Freudenthal (1973) expressed concern for the future of geometry instruction. This has mainly been due to research based around the van Hiele Theory. However, in the field of 3-dimensional geometry there is still room for great concern. In analysing the understanding expressed by students in their descriptions of a cross-section of a solid, the authors present a frame that has the potential to be used for curriculum development in 3-dimensional geometry.

While researchers lament the lack of importance attached to geometry in school curricula (Freudenthal 1973; Meissner 2001; van der Sandt 2001), they are unanimous in their espousal of its importance. Geometry offers a way to interpret and reflect on our physical environment, while spatial thinking has been suggested to be essential to creative thought. Recent literature (Meissner 2001; van der Sandt 2001) suggests that little has changed since Freudenthal (1973, p.402) expressed fears that "the days of traditional geometry are counted, ... there is cause to be concerned about the future of geometry instruction. The lack of attention to solid geometry is matched by the lack of published research on the topic. The only research reports on 3-dimensional geometry published in recent years appear to be on nets of solids (Despina, Leikin & Silver 1999; Lawrie, Pegg & Gutiérrez 2000; Mariotti 1989). Mariotti (p.260), in attempting to identify specific didactic variables related to the utilisation of mental images, hypothesised that there are two levels of complexity; the first level is when the image is global, and the second occurs when an operative organisation of images is required for coordination. Lawrie, Pegg & Gutiérrez (2000), in their analysis of students' responses expressing understanding of a net, demonstrated that there is a hierarchy of difficulties within Mariotti's second level.

If one considers geometry to be the poor relation of mathematics, 3-dimensional geometry must surely be the Cinderella. If the above situation is to be addressed, there is need first to address development of school curricula. A framework demonstrating the didactic hierarchy in 3-dimensional geometry needs to be determined.

RESEARCH DESIGN

The authors developed a test to investigate secondary-school students' understanding of various aspects of 3-dimensional figures, i.e., the solid form and its cross-sections, and nets of 3-dimensional figures. The analysis of students' perceptions of nets of solids has already been presented by Lawrie, Pegg & Gutiérrez ((2000). This report considers the coding of the students' responses to a second aspect, one in which students express their understanding of cross-sections of 3-dimensional figures. A cross-section is defined as:

> A section is a plane geometric configuration formed by cutting a given figure with a plane. A *cross-section* is a section in which the plane is at right angles to an axis of the figure. Daintith & Nelson 1989, p.290

Two frameworks, the SOLO Taxonomy (Biggs & Collis 1982) and the van Hiele levels of understanding (van Hiele 1986) are used in coding the nature of thinking displayed by the students in their responses.

BACKGROUND

The SOLO taxonomy

The SOLO Taxonomy (Biggs & Collis 1982) has been identified by several writers (e.g., Pegg & Davey 1998) as having strong similarities with the van Hiele Theory, despite some philosophical differences. It is concerned with evaluating the quality of students' responses to various items. A SOLO classification involves two aspects. The first of these is a mode of functioning, and, the second, a level of quality of response within the targeted mode. Of relevance to this study are the three modes, ikonic, concrete symbolic, and formal. Within each of these modes, students may demonstrate a unistructural, multistructural or relational level of response.

In the *ikonic mode* students internalise outcomes in the form of images and can be said to have intuitive knowledge. By contrast, in the *concrete symbolic mode,* students are able to use, or learn to use, a symbol system such as a written language and number notation. This is the most common mode addressed in learning in the upper primary and secondary schools. When operating in the *formal mode*, the student is able to consider more abstract concepts, and to work in terms of general principles.

A description of the levels that occur within the modes is given:

The *unistructural level* of response draws on only one relevant concept or aspect from all those available.

The *multistructural level* of response is one that contains several relevant but independent concepts or aspects.

The *relational level* of response is one that relates concepts or aspects. These relevant concepts are woven together to form a coherent structure.

The targeting of the concrete symbolic mode for instruction in primary and secondary schools, and the implication that most students are capable of operating within the concrete symbolic mode (Collis 1988) has resulted in the exploration of the nature of student responses within that mode. This has led to the identification of at least two unistructural (U) - multistructural (M) - relational (R) cycles within the concrete symbolic mode (Pegg & Davey 1998). One noticeable characteristic of the cyclic form of the levels is that the relational response (R_1) in the first cycle becomes a single entity as the unistructural element (U_2) in the second cycle. This cyclical nomenclature has been used in the coding of concrete symbolic responses for this study.

The van Hiele Theory

Pierre van Hiele's (1986) work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels, the ones commonly displayed by secondary students and most relevant to this study, is given for 3-dimensional geometry (Pegg's (1997) differentiation between Levels 2A and 2B is used):

Level 1 Perception is visual only. A solid is seen as a total entity and as a specific shape. Students are able to recognise solids and to distinguish between different solids. Properties play no explicit part in the recognition of the shape, even when referring to faces, edges or vertices.

Level 2A A solid is identified now by a single geometric property rather than by its overall shape. For example, a cube may be recognised by either its twelve equal edges or its six square faces.

Level 2B A solid is identified in terms of its individual elements or properties. These are seen as independent of one another. They include side length, angle size, and parallelism of faces.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised. Symmetry follows as a consequence. Simple proofs and informal deductions are justified. Families of solids can be classified.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions in definitions are used with understanding, as are equivalent definitions for the same concept.

Van Hiele saw his levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s). Van Hiele also saw (i) the levels as discontinuous, i.e., students do not move through the levels smoothly; (ii) the need for a student to reach a 'crisis of thinking' before proceeding to a new level; and (iii) students at different levels speaking a 'different language' and having a different mental organisation.

ANALYSIS OF RESPONSES

The 3-dimensional geometry test designed by the authors was given to students from all except the final year (i.e., to Years 7 to 11, ages 12 to 17 years) of four secondary schools in a rural city in New South Wales, Australia. The students were of mixed ability, and were not drawn equally from across the years; rather, the researchers were dependent on the availability of classes and the goodwill of the schools and their teachers. This resulted in there being many more students from Years 7 and 11 than from Years 8, 9, and 10.

Because of page limitations, this paper considers in depth the responses to one of the questions on cross-section of 3-dimensional figures, namely:

"Describe in as much detail as possible what is a cross-section of a solid."

An indication of the difficulty students experienced with the notion of cross-sections is that only 181 students attempted the question. The response rate differed across the various years, with 18, 12, 31, 51 and 69 from Years 7, 8, 9, 10 and 11 respectively. It should be noted that in Australia cross-sections are considered in a global sense in primary schools and are not considered in any greater depth until late in Year 8, with further treatment in Years 9 and 10.

Analysis of responses (SOLO)

The responses were coded first with relation to the SOLO Taxonomy. Of the 181 responses, it was possible to code 177 (98%); only 4 responses were considered uncodable. No responses were identified as belonging to either the ikonic or formal modes, i.e., it was considered that all responses were concrete symbolic.

Initially, the responses were classified according to their cycle within the mode. This was done depending on whether a response expressed the notion of a cross-section being a single cut or halving of the solid (first cycle), or whether the response inferred multiple cuts or slices (second cycle). After considering several features in responses, e.g., the cut, direction, result of the cut, together with various combinations of these, it was felt that the significant factors for indicating levels in the responses were:

- single versus repeated cut(s) or slice(s);
- whether the result of the cut(s)/slice(s) was acknowledged;
- qualification of the result, i.e., the linkage between the cut(s)/slice(s) and the result(s).

The term 'slice' rather than 'cut' was used frequently by students giving second-cycle responses, yet did not appear in responses of the first-cycle. In the first-cycle responses, students frequently used the terms 'cutting in half", 'cutting down the middle', 'bisecting'. Almost all students applied their notion of a cross-section to figures with a constant cross-section (prisms and cylinders), not mentioning cross-sections of pyramids nor 3-dimensional figures in general.

This initial classification of responses led to the following description of each SOLO level, U, M, R, for the two cycles in the concrete symbolic mode (Table 1).

Table 1: Response descriptors for SOLO levels with examples of student responses

Level	Description
Single cut	
U_1	One cut e.g. *cut a solid in half; a solid is divided through the middle.*
M_1	One cut and look at the result e.g., *what you see when you cut a solid into two pieces; cut down the middle and the cross-section is the result.*
R_1	The effect of the cut; qualifying the 'look' e.g., *It is when you cut it in half and both sides match.*
Multiple cuts/slices	
U_2	Multiple cuts/slices e.g., *when the solid is cut straight anywhere; ... is cut into slices.*
M_2	Multiple cuts/slices and the resulting shape stays the same e.g., *A cross-section of a solid is an imaginary line that can be cut anywhere so that the same shape remains throughout.*
R_2	Direction of the cut is the integrating factor that links multiple cuts and same shape e.g., *The section where the solid is cut down, parallel to the front face. The cross-section is the same shape as this face.*

Of the four responses considered not able to be classified in accordance with the descriptors listed in Table 1, two very similar responses appeared to reflect the appearance of a pyramid if viewed from above the vertex;

> *The cross-section of a solid is the section where, when drawn, all the vertices intercept each other; straight down the middle.*

In the third response the student, while demonstrating understanding of the general concept of a cross-section, gave a response for topography;

The shape meaning the ascension and descensions of the shape – just like the cross-section from one point to another on a topographic map.

The fourth student's response seemed to reflect general geometric awareness, but no specific knowledge of a cross-section:

A particular part to the solid, looked at thoroughly unto detail. Things which might be taken into account are length of sides, edges.

The results of the analysis of the 177 responses are shown in Table 2. As the number of students giving a response to the question varied, results of the analysis will be given as percentages in parenthesis after the number of students.

Table 2: Results of analysis for SOLO levels of student responses (with percentages)

Level	Total	Year 7	Year 8	Year 9	Year 10	Year 11
U_1	43(24)	9(5)	6(3)	7(4)	11(6)	10(6)
M_1	63(36)	7(4)	2(1)	16(9)	18(10)	20(11)
R_1	37(21)	0	4(2)	7(4)	10(6)	16(9)
Subtotal	143(81)	16(9)	12(7)	30(17)	39(22)	46(26)
U_2	5(3)	1(1/2)	0	0	1(1/2)	3(2)
M2	18(10*)	1(1/2)	0	0	4(2)	13(7)
R_2	11(6)	0	0	0	4(2)	7(4)
Subtotal	34(19)	2(1)	0	0	9(5*)	23(13)
TOTAL	177(100)	18(10)	12(7)	30(17)	48(27)	69(39)

* percentages may not tally because of rounding

Analysis of responses (van Hiele)

The students' responses were also coded according to the van Hiele levels of understanding. It was considered that there were no students displaying understanding at either Level 1, the visual level, or at Level 4, the formal level in which logical reasoning can be demonstrated. All students were considered to be displaying reasoning at Level 2A, i.e., identifying a single geometric element, Level 2B, i.e., identifying individual elements, or at Level 3 in which students display a degree of analysis or logical ordering. Table 3 gives descriptors of the van Hiele levels and the correspondence with the SOLO levels, and Table 4 gives the results of the analysis of student understanding as van Hiele levels. Again, percentages are given in parenthesis.

Table 3: Response descriptors for van Hiele levels

van Hiele Level	Description	Corresponding SOLO level
2A	Perceiving cut(s)/slice(s)	U_1; M_1; R_1; U_2
2B	Acknowledging that there are multiple cross-sections which are the same shape	M_2
3	Recognising that the multiple cuts/slices are in a certain direction – the integrating factor	R_2

Table 4: Results of analysis for van Hiele levels of understanding (with percentages)

Van Hiele Level	Total	Year 7	Year 8	Year 9	Year 10	Year 11
2A	148(84*)	17(10)	12(7)	30(17)	40(23)	49(28)
2B	18(10*)	1(1/2)	0	0	4(2)	13(7)
3	11(6)	0	0	0	4(2)	7(4)
Total	177(100)	18(10)	12(7)	30(17)	48(27)	69(39)

*percentages may not tally because of rounding

CONCLUSION

The analysis given in this report demonstrates a hierarchy of difficulties in the understanding of cross-sections of solids. Not only does the hierarchy fit into Mariotti's (1989) second level of complexity in the manipulation of mental images in which an operative organisation of images is required for coordination, it also provides a frame for the development of a curriculum for instruction in cross-sections of 3-dimensional figures. This frame could be expanded to encompass the study of the whole of 3-dimensional geometry and spatial reasoning. However, care should be taken that within a curriculum, adequate time is allotted to the various hierarchical strata demonstrated above as levels in the two SOLO cycles. In particular, there should be adequate time allowed for students to develop through the first SOLO cycle, thus allowing students to come to an understanding of Level 2A in van Hiele. Instructors, in observing students from their higher level of understanding, do not always consider the importance of establishing sound understanding at the lower levels. To move students from one level to the next requires interaction between the teacher, students and subject matter and interaction requires time.

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