MISUNDERSTANDING OF VARIABLES BY STUDENTS IN AN ADVANCED COURSE

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The paper proposes a theoretical framework to analyse the understanding of the roles of literal symbols in algebraic tasks basing it on the distinction between free and bound variables. This approach is applied to study the difficulties of advanced high school students with the notion of the parametric representation of a plane. It is shown that algebra courses tend to form a limited understanding of the notion of variable which creates an obstacle for a learner in advanced courses.

INTRODUCTION

The students' difficulties with the mathematical language has been the traditional concern in the introductory algebra courses. Some time ago the issue has also got an attention in the field of advanced mathematical thinking: for example in relation to the teaching of linear algebra. It was shown, for instance, that students do not interpret rightly the formulas of linear algebra and do not have for the latter appropriate set-theoretical meanings (Dorier, Robert, Robinet & Rogalski, 2000). Other researchers had also suggested that studying linear algebra requires from a student a good understanding of the mathematical syntax and use of variables (Sierpinska & Nnadozie, 2001).

There are enough signs, however, that many of the difficulties with understanding the mathematical language originate as early as at school and are not overcome by the students in their passage from school to college (Ursini & Trigueros, 1997). It was repeatedly shown, for example, that an average high school student doesn't realise the difference between unknown, parameter and functional variable. Some studies had related this obstacle to the historical development of algebra and deficiencies of the dominating structural approach in teaching of elementary algebra (Sfard & Linchevski, 1994), whereas others, to inherently complex propositional nature of algebraic tasks, especially when they contain parameters (Bloedy-Vinner, 1994). At this moment there is a growing agreement that the 'algebraic sense' developed by a learner may be to a large extent equated with his or her mastery of the concept of variable. Ursini and Trigueros suggested that a learner must understand the three main uses of variable: as unknown, as a general number, and in functional relationships (Ursini & Trigueros, 1997). Bills showed in a recent study that there are many problem solving situations which require from a student not only to recognise different uses of literal symbols, but to shift attention from one to another meaning of the same symbol in order to succeed with a task (Bills, 2001). The latter, like other studies, used its own, though more extensive than others, list of the roles of literal symbols in the algebraic tasks. This leads to the following methodologically important questions: To what extent do the lists proposed by different studies correspond to each other? What is their relation to the mathematical status of literal symbols as defined in mathematical logic? And can the research on algebraic thinking draw on it to describe the roles of literal symbols a more unified and simple way?

The author believes in a positive answer to the last question and suggests as a guiding principle the distinction between free and bound variables used in mathematical logic. This theoretical framework is then used to describe the difficulty of advanced high school students with such notion of linear algebra as the parametric representation of a plane. It is shown that their difficulty follows from the limited understanding of the notion of variable. The paper ends with the discussion of the historical and didactical aspects related to the teaching the notion of variable in introductory and advanced algebra courses.

THEORETICAL FRAMEWORK

Whereas there are numerous terms used by researchers for the roles of letters in algebraic expressions, in mathematical logic a literal symbol which doesn't denote a mathematical constant is either free or bound variable. This useful distinction was introduced in mathematics by Peano and has become the standard approach in any formal analysis of the mathematical language (Taylor, 1999). According to this distinction, the free variable in an expression stands for any one unspecified element of a set, while the bound variable denotes collectively all the elements of a set. The bound variables are used in such mathematical expressions as sums, integrals, limits, equations of curves, etc. An assignment of a certain value to the bound variable is irrelevant procedure, as far as the corresponding mathematical entity is defined by all the possible values of that variable. When variable in an expression is treated as free, it may, on the opposite, be assigned any value, and the resulting expression is treated as a particular case of the initial generic form. The main experiences with free variables in algebra courses are related to their use as generalised numbers and to equations. Finding among the values of a free variable those which satisfy an equation, corresponds to solving the equation in relation to this variable, which is then called unknown. If the equation contains besides the unknown an additional free variable, then the latter is called the parameter for this equation. The parameters may also appear in the formula of a function in addition to the independent variable, which has the status of the bound variable. The status of variable in a certain expression may depend on the goal of a task and may be changed in the course of the task. For example, a parameters of an equation or function may be 'binded' when one is interested to consider instead of one equation or function corresponding to an arbitrary value of a parameter, the properties of the entire family of equations or functions corresponding to all the possible values of the parameters. Another important example is the introductory course of linear algebra: binding of variables in an arbitrary linear combination produces a subspace. Many additional examples may be found in the theory of groups and functional analysis. On the school level more often happens 'unbinding' of variables: for example, in evaluating functions for certain values of independent variable or in finding certain points on a curve.

METHOD

The reported here study is a part of a larger project evaluating the difficulties of the Israeli 12th grade students with their regular vector geometry course. The sample consisted of 10 classes (N=214) of most advanced students (top 10% of the student population which pass the mathematics matriculation exam on the highest, '5 units', level) chosen from 5 academically high standing schools. The teachers, who volunteered to participate in the study, all had at least the second academic degree and included also the Heads of the mathematics departments of these schools. The purpose of the described below questionnaire was to test the students' general understanding of the parametric representation of a plane which had been one of the core notions of the course. The teachers were acquainted with the questionnaire in advance and recommended to deliver it in their classes in the most appropriate moment after the notion of the parametric representation would be well mastered by the students. Some of the students were interviewed after they had completed the questionnaires, and it allowed to clarify their written answers and the meanings they hold about the parametric representation. Another way to get a better insight into the students understanding of this notion was to suggest the same questionnaire to the academics from the mathematics and theoretical physics departments and analyse the differences between their and the students' approaches. Due to the space limitations, the results for only two of the items of the following questionnaire will be presented.

Let $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$ be the parametric representation of a plane. Find out for each of the following two representations whether they may represent the plane identical to that of the given representation. Explain your answers!

1. $\underline{x} = \underline{a} + t\underline{u} + t\underline{v}$ 2. $\underline{x} = \underline{a} - t\underline{u} + s\underline{v}$

RESULTS

Though the first two items were not expected to present any difficulty, they revealed a serious misunderstanding by the students of the role of the parameters in the parametric representation of a plane. Only 32% of the students in the sample had answered correctly that the first item didn't represent the same plane, whereas the rest wrongly claimed that the item could be a representation of the plane identical to that of the given representation. These students treated the expression $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$ as a symbolic name of a plane which is completely specified, if each of the lettered symbols were assigned a certain value. It included the parameters t and s which were understood by the students as two certain numbers. The students' explanations could be be divided into three described below categories.

<u>Category 'Formal'</u>. The students in this category had considered the parametric representation as a formal expression. In order to find out whether two representations correspond to one and the same plane, the students equated them and

solved corresponding equations, treating the letters as unknowns. Thus the common conclusion in this category was that the item may represent the same plane as the given representation, if t equals to s :

> 'The planes do not coincide, since one of the direction vectors is not identical to that in other representation The planes will coincide only if t=s'.
>
> 'Yes, t and s are given numbers. It may happen to be the case that these s and t will be equal'.
>
> 'No, it may represent only the planes for which t=s, but all other planes are not possible to represent by this equation'.

Category 'Geometric'. These students assumed that the parametric representation is a way to express the geometric fact that two intersecting lines, or correspondingly, two vectors applied to a certain point, determine one and only one passing through them plane. For these students the main elements of the parametric representation are vectors $\underline{u}$, $\underline{v}$ and $\underline{a}$, since geometrically they are enough to determine a plane. The values of the parameters t and s are not important, because a pair of vectors $t\underline{u}$ and $s\underline{v}$ still determine the same plane as vectors $\underline{u}$ and $\underline{v}$. Therefore, the parameter was considered to be a free variable which may be substituted by any letter or number without changing the corresponding plane. The following are the examples of the explanations in this category:

> 'Yes. It doesn't matter whether s=t or $s \neq t$, what matters is that the vectors are the same'.
>
> 'Yes, since s and t in the given representation are variables, so that even if we substitute s=t, like it is in this item, the representation will represent the original plane'.
>
> 'Yes, it may be, because s is a variable, so it is possible to write instead of s anything, including t'

Interviews with the students had confirmed that for many of them any two given values of the parameters were considered to be enough to determine a plane. The following is an excerpt from an interview with one of the students Ron after he completed the questionnaire:

I:	So you say, that it doesn't matter what the parameters are. Can you describe me the case, of say, t=s=2?.
Ron:	OK, we have a direction vector which is in the same direction but twice as long as vector u and another one which is also twice longer than v. But still it is the representation of the same plane.
I:	...How?
Ron:	Well, imagine two vectors $\underline{u}$ and $\underline{v}$ in space. I can put a plate on them in only one way, so that its position will be quite determined. Now you have vectors $2\underline{u}$ and $2\underline{v}$. They still define that same plate: it doesn't matter infinite or not. Do you see?

I: Yes, I see...

To provide more evidence that the students saw nothing wrong in the substitution of a parameter by a specific number, an additional item $\underline{x} = \underline{a} + 100\underline{u} + s\underline{v}$ was included in the questionnaire delivered in four classes of the sample (N=74). The success with the first item in this sub-sample was 32%: the same as in the whole sample. The success rate with the additional item turned out to be only slightly higher: 42%. Most of the students, who erroneously claimed that the first item could represent the plane identical to that of given representation, answered the same to the additional item: 35 students out of 41. In the category 'Formal' this proportion was the highest: 26 out of 28; that is, 93% of the students in this category gave the same answer to the both items.

Category 'Analgebraic'. The term 'analgebraic' was first introduced by Bloedy-Vinner (Bloedy-Vinner, 1994) to describe the students' behaviours which go against the basic algebraic conventions. The students in this category not only assumed that the parameters t and s denote two certain numbers, but that these numbers must be necessarily different, since they are denoted by different letters:

'No *t* is not equal to *s*'.

'No, the direction vectors are not the same'.

'No, the lengths of the direction vectors have become equal'.

.The distribution of the students' answers to this item according to the described categories is presented in Table 1:

Table 1: Distribution of the students' wrong answers according to the categories.

'Formal'	'Geometric'	'Analgebraic'	no explanation
55	40	18	28

Considering the item $\underline{x} = \underline{a} - t\underline{u} + s\underline{v}$, most of the students answered, like they did for the previous item, that it represents the same plane as the given representation $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$, which in this case was the right answer. The close analysis of the students' explanations had shown, however, that many of them were based on the same erroneous assumption that parameters in the parametric representation of a plane are certain fixed numbers. For example, one of the very common answers was: *'the same plane, because the direction vector -t$\underline{u}$ is just opposite to t$\underline{u}$'.* This utterance means that for a student *-t$\underline{u}$* denotes only one, rather than a set of vectors corresponding to all real values of *t* and that the letter *t* is assumed to denote in different representations one and the same number. Another type of erroneous explanations which accompanied the correct answer to this item was: *'yes, it is a linear combination of the same direction vectors'.* This answer indicates that a student doesn't understand the difference between a generic linear combination and the parametric representation of a plane which is the set of all linear combinations described by this generic form. .

Among those who didn't answer correctly to this item, most had formally equated the item and the given parametric representation of a plane and, after subtracting the terms $s\underline{v}$ in the resulting equation, concluded that *'the item represents the same plane only if t=0'*. Only few among these students had realised the contradictory nature of this conclusion: *'Not the same plane. If t= -t, then t=0, rather than being any real number'*. In fact, by equating the representations in order to get the conditions of their equivalence, the students were employing a sound mathematical procedure. However, the performance even of those who knew that the parameter should be any real number, was lacking the understanding that the parameters in different representations vary independently one from another and for that reason can't be denoted in the resulting equation by the same letters.

DISCUSSION

The study has revealed the students' serious misunderstanding of the role of the parameters in the parametric representation of a plane. That this misunderstanding is rooted in the students' one-sided experience with variables, becomes evident if one compares the answers of the students and the experts to the same questionnaire. Most of the students saw the item $\underline{x} = \underline{a} + t\underline{u} + t\underline{v}$ as a result of the assignment of a particular value to one of the variables of the representation $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$. For all the academics, on the other hand, it was obvious that the number of the parameters in the two representations was different. Unlike the students, the academics knew that the parameter in the parametric representation of a plane is a variable which is not characterised by its particular values. The students' unawareness that there are different types of variables and that the parametric representation requires the bound ones, made them readily accept that $\underline{x} = \underline{a} + 100\underline{u} + s\underline{v}$ is also the parametric representation of a plane.

The fact that the introductory algebra courses still do not include the issue of different types of variables, has several historical and didactical reasons. First, it should be noted that the distinction between free and bound variables is of relatively recent origin. Almost up to the beginning of the 20th century, the mathematics were quite satisfied with the Euler's simple distinction between variable and constant quantities and meeting, for example, something of the type $ax+by+c=0$, one knew exactly which is which. It was not until Russell had challenged this tradition and proclaimed on the first pages of his 'Principles of mathematics' a new, 'only variables' age (Russell, 1903/1964):

> '...But unless we are dealing with one absolutely particular line, say the line from a particular point in London to a particular point in Cambridge, our *a, b, c* are not definite numbers, but stand for *any* numbers , and are thus also variables'.

Russell had, of course, good reasons for this claim. He developed the notion of variable as a set-building device which allowed to deal with a family of equations, rather than with its arbitrary representative; to form functions, functions of functions -

all this by just prescribing for a free variable in a mathematical expression to become a bound one. He had also suggested special symbols to mark bound variables, but this and other similar suggestions had never been universally accepted, so the type of variable is still inferred only from the context. On the other hand, the mathematicians find even some advantage in this notational ambiguity, since it allows to deal with different meanings of the same expression by just imagining that corresponding variable had changed its type. Furthermore, this ability for mental 'binding' of variable has become a prerequisite for learning advanced mathematics in which this feature has become widely used.

One of the conspicuous features of the school mathematics curriculum is that though it is permeated with the distinction between bound and free variables and is concerned that the students master it, there is no any attempt to teach it explicitly. Probably the first occasion when the students have to deal with the notion of a bound variable is related to functions. However, as this study shows, this doesn't necessarily led even the advanced students, who had a year-long introductory calculus course, to master this notion in a way that allowed to them to recognise and apply it in linear algebra. Avoiding the issue of variables also in this latter context, may only multiply the students' misunderstanding, because the dual use of variables there is quite common. For example, in the expression $t\underline{u} + s\underline{v}$ the letters t and s are bound variables, if it is meant as the parametric representation of a plane, but are free variables, if it is intended as a general linear combination of vectors $\underline{u}$ and $\underline{v}$. The difference is not well expressed in the ordinary language, as in the both cases one would likely to say that the lettered symbol stands for any number. Indeed, in their explanations that the item $\underline{x} = \underline{a} + 100\underline{u} + s\underline{v}$ is the parametric representation of the same plane as $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$, some students wrote that the parameter may be 'any' number, meaning by it **any some** number; what a teacher or a textbook meant by using the word 'any' in relation to the parameteric representation, was that the parameter should take on **all** the real numbers, that is to be the bound, rather than free variable.

The 'silencing' of this distinction is the worst possible choice in the case of the parametric representation of a plane, because the textbooks introduce it starting with an arbitrary point on a plane and then describe its possible positions on a plane as all the linear combinations of the basis vectors. This passage remains an impossible mental exercise for a learner who is neither prepared for set-theoretical arguments, nor is proficient in bound variables. This shows the dilemma of the current algebra courses: on the one hand, due to certain didactical tendencies of the last decades the textbooks try not to use the set-theoretical notations, in order not to embarrass the students; on the other hand, they by all means avoid any informal or kinematically coloured expressions, like 'parameter ranging over all numbers' or 'a point moving along a plane'. It is worth to note that the university textbooks seem to have much less constraints in this sense and do not hesitate to explain to a reader that 'the parameter t runs through all numbers' (Lang, 1966, p.13).

CONCLUSIONS

The study has described the difficulties experienced by advanced high school students with the notion of the parametric representation of a plane. It is shown that this difficulty followed from their lack of understanding of the difference between free and bound variables in mathematical expressions. This distinction, though basic in mathematics, has not received any attention in the current algebra courses. It is suggested to direct more research and didactical efforts to this issue which may significantly improve the situation.

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