FLEXIBLE MATHEMATICAL THOUGHT

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We consider different aspects of flexible thinking in mathematical contexts, and illustrate these by examples from recent mathematics classrooms. We argue that aspects of flexible mathematical thought can be encapsulated as an ability to use long-term declarative knowledge in novel situations. Both Karmiloff-Smith's representational re-description and recent work in the psychology of memory are related to flexible thinking.

INTRODUCTION

Cognitive flexibility is a well studied, but not especially well-defined, notion in the psychology of mathematics education. Flexibility of thought is often treated in the psychological literature as a given which is approached operationally through measurement (Berg, 1948; Edwards, 1966; Busse, 1968; Jausovec, 1994), or via specific problems that are thought to require flexibility for their solution (Dover & Shore, 1991; Kaizer & Shore, 1995). With a few significant exceptions, studies in mathematics education treat flexibility generally as the capacity to exhibit a variety of novel strategies for solving problems (Dunn, 1975; Carey, 1991; Klein & Beishuizen, 1994; Vakali, 1994; Beishuizen et al, 1997; Heirdsfield, 1998; Imai, 2000; see also Shore, Pelletier & Kaizer, 1990). The significant exceptions are Krutetskii (1969b), Shapiro (1992) and Gray and Tall (1994). Shapiro was a PhD student of Krutetskii in Russia in the late 1960's. Krutetskii and Shapiro characterize flexible thinking as reversibility: the establishment of two-way relationships indicated by an ability to "make the transition from a 'direct' association to its corresponding 'reverse' association" (Krutetskii, 1969b, p. 50). Gray and Tall (1994) characterize flexible thinking in terms of an ability to move between interpreting notation as a process to do something (procedural) and as an object to think with and about (conceptual), depending upon the context:

> "We characterize proceptual thinking as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object. It is proceptual thinking that gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation." (Gray & Tall, 1994, p.7)

In this paper we consider in detail what aspects of "flexibility" might be considered reasonable and desirable in a mathematical context. Although this is a theoretical analysis, aimed at clarifying the notion of flexibility, we base our discussion largely upon empirical work carried out in a grade 6/7 problem solving class.

THEORETICAL BACKGROUND

Flexibility of thought we will argue, is basically an ability to use long-term declarative knowledge in novel situations (Squire & Kandell, 2000). As simple as this statement seems it encapsulates a lot of background discussion and thought from mathematics education and psychology.

Part of the reason that the psychological literature has had so much difficulty with flexibility is that it may not be, and probably isn't, a trait. Like bodily flexibility, some people may naturally incline to a greater degree of mental flexibility, but all people can improve their flexibility. This is an important issue to bear in mind in tests of flexibility: if flexibility is not a trait, as we suspect it is not, then tests of flexible thinking will generally have low reliability (Edwards, 1966), especially if practice on the tests substantially enhances one's flexibility of thought.

Flexibility of thought as conceived by psychologists working on the neuropsychology of memory, is essentially long-term declarative memory – memory that has been processed in its formation by the hippocampal system (Squire & Kandell, 2000). The significance of the hippocampus is that, according to Eichenbaum (1994, 1887), it establishes a "relational space" in which prospective long-term knowledge becomes embedded relationally with other long-term knowledge. The well-known properties of the hippocampus in relation to spatial orientation are just one aspect of a more general capacity of the hippocampus to represent knowledge and memory in ways that allow flexible use in novel settings. The basic idea is that declarative memory - long-term memory that, in students, is expressed through words or diagrams - is inherently flexible in nature. Neuropsychologists, particularly those engaged in memory research, tend to speak of "flexible memory representations" rather than "flexibility of thought". This aspect of flexible thinking, involving a transition from implicit to declarative memories, is also a prominent feature of Karmiloff-Smith's theory of representational re-description (Karmiloff-Smith,1992). One interpretation of her theory of representational re-description is that it involves a transition, for an individual student, from short-term working memory to long-term flexible declarative memory, and thereby achieves a greater capacity for storage and retrieval of relevant memories, and also a wider applicability of those memories.

ASPECTS OF FLEXIBLE THINKING IN MATHEMATICS

Apart from a capacity for using novel strategies to solve problems, which is the most commonly used definition of flexibility in the mathematics education literature, we illustrate, below, a number of aspects of flexible thinking that are consistent with the neuropsychology of declarative memory. These all involve students in a "re-orientaton": of place, situation, of person, direction, or interpretation of notation. This re-orientation is consistent with the nature of long-term declarative memory as essentially relational (Eichenbaum, 1994; Squire & Kandell, 2000). From this perspective cognitive neuropsychology may be catching up with the ideas of Skemp (1976).

Capacity to interpret someone else's thinking

A student might explain another student's thinking and use it, build off it, try to prove that it is invalid, or ask questions about it. Consider the following example. A grade 6 student, Amanda, presented to her class a connection between three problems: (1) building all possible towers 4 high, using black or purple cubes; (2) finding all ways 2 teams can win the World Series playoff in 4 games; and (3) finding all possibilities for walking stairs 4 high, taking double or single steps. The following day a student, Jessica, gave a presentation to the class trying to explain what Amanda was saying, and convince the class that Amanda's connection was not valid. Jessica showed a key that equated a black cube, *b*, with *Y* (representing the Yankees) and a *2* (representing a double step). She did the same for the other possibilities: *purple* ↔ *M* ↔ 1. She showed a four tall tower (*bpbp*) that should, under Amanda's connection, be equivalent to an example of a way to play the World Series in four games (*YMYM*), and equivalent to a possibility for walking stairs four high (*2121*). She stated the connection would not work "because it would go overboard": 2121 would make six stairs high, instead of four.

This is an example, on Jessica's part, of interpreting another student's thinking (in this case to show a connection was invalid). The situation that is novel for Jessica is Amanda's schema of connections. Jessica brings to this novel situation her recollections of working on all three problems and is able to argue that Amanda's connections do not do what Amanda intended.

Using an idea or strategy across different contexts or changing an existing strategy to fit a new context

Consider a grade 6 student, David, who mapped out an equivalence between the number of handshakes at a 4 person party (where each person shakes hands once with every other person at the party) to the "two purple case" of building towers four-tall, choosing from two colors, black and purple. He labeled the position of each block beginning at the top of a tower, working down (1, 2, 3, 4), focusing on the position of the two purple blocks. He also labeled the people at the party 1, 2, 3 and 4 and color-coded them. He began with the numerical pair "1, 1" and claimed that two purple blocks cannot both be placed in position 1, and, corresponding to his labelling, person 1 cannot shake hands with themselves. He then listed "1, 2" and said the two purple blocks would be in the top two positions and, correspondingly, person 1 would shake hands with person 2. He listed "1, 3" and "1, 4" showing how the towers would look, and represented this with two people shaking hands. He did not count "2, 1" because, according to him, person 1 already shook hands with person 2 and the purple block was already in position 1 and 2. He continued to show each possibility with towers and people and why he would or would not list each. Here he used the same strategy for two different problems, creating an exact mapping of possibilities from one problem to the other. In essence David extracted the scheme of working systematically through the number pairs "a, b" with a < b, and interpreted both the tower building problem and the handshake problem from this point of view. This involved a double re-orientation: first interpreting towers as handshakes and vice versa, and then seeing both problems as instances of ordered pairs "a, b" with a < b. David utilized his knowledge of building towers, and of calculating handshakes, in a novel setting in which number pairs were his focus of attention.

~~1, 1~~	1, 2	1, 3	1, 4
~~2, 1~~	~~2, 2~~	2, 3	2, 4
~~3, 1~~	~~3, 2~~	~~3, 3~~	3, 4
~~4, 1~~	~~4, 2~~	~~4, 3~~	~~4, 4~~

Table 1. David's explanation that C(4,2) = 6, relating towers of height 4, with 2 black and 2 purple blocks, to handshakes between 4 people.

Using multiple representations for the same mathematical problem or using multiple representations to express the same idea

Use of multiple representations is an essential aspect of mathematical flexibility. One can see it operating superbly at an advanced level in Thurston (1997), where, for example, different representations of manifolds and knot groups are utilized to enhance understanding.

We give an example of multiple representations by considering Jessica as she built towers two blocks tall from those one block tall by placing a black cube and purple cube under each possibility. She showed this by drawing an arrow from the original tower to the two new towers that she created and numbered each pair with the tower it came from. She also did this to create all towers four blocks tall from those towers three blocks tall. She later changed her recording to a tree diagram to show how to recursively build all towers of any height. Jessica re-oriented herself through encapsulation of the recursive process of building towers via a tree diagram. The tower building problem as conceived as a branching process (from our perspective, not Jessica's) and the tree diagram was used as a representation of this branching process.

Raising hypothetical problem situations based on existing problems

These are essentially "what if" questions, and involve imagined scenarios, regarded as similar to an existing problem. For example, a grade 6 student Michelina asked in relation to building towers: "What if we have three colors to choose from, instead of two?" This question indicates that Michelina was not stuck in the constraints of the problem but imagined, what was to her, a similar problem in which she could utilize three, rather than just two, colors. Again, this involved a re-orientation into scenarios that are similar to those already encountered, but not yet acted out.

Using notation ambiguously

This is Gray & Tall's (1994) definition of flexible mathematical thinking. They give an example of moving comfortably from using the division sign to a fraction bar representing division.

Consider the example of a grade 6 student, Jessica, who explained that the number of possibilities doubled when she moved from towers built from black and purple blocks, of a given height, to towers of height one greater. She described doubling by multiplying by two. She then to switched to doubling by adding the number plus itself. This way of describing doubling apparently allowed her to see things more clearly. After she began describing doubling this way, she saw that her possibilities were doubling recursively. (Initially she had doubled the *height* to get the number of possibilities). This allowed her later to make a connection with doubling of the number of possible towers and her strategy of placing a black and purple cube under each tower to get all possibilities for the next height. The ability to see the process and results of calculation in different lights is a form of mental re-orientation: one obtains alternative and complementary points of view.

Transition from a 'direct' association to a corresponding 'reverse' association

This is Krutetskii (1969b) and Shapiro's (1992) definition of flexibility of mathematical thought. A clear example of his aspect of flexibility occurred when sixth graders found a formula for the handshake problem for different numbers of people at a party. Michelina asked if they could find a formula when they know the number of handshakes and don't know the number of people. Two further examples, one from a college level remedial algebra course, the other from a pre-service elementary methods course, also illustrate this aspect of flexibility. For the first example, consider the remedial college algebra student who wrote, as part of his final self evaluation:

> I now have the skills to interpret and use mathematical notations appropriately, reflected in my work and ability to interpret a function machine; convert from one type of mathematical notation to another, convert a function machine to an equation/equation to function machine.... I have identified inputs and outputs, which was the key to answering the questions, appropriately and accurately in my tests on questions involving: recognizing the given slope to make a table, showing input/output; identification of a given notation and breaking it down by input/output; naming input/ output from a given function machine or equation; finding output if given input/finding input if given output.... I am knowledgeable in understanding the difference between evaluating and solving: solving for X; evaluating for Y.

This student's work indicated that he did indeed know the difference between evaluating an expression and solving an equation, a distinction he did not make at the beginning of his course.

Second, consider the example of an elementary teacher who, in a mathematics methods course, turned around a problem, utilizing a reverse association. The problem was to begin with a sheet of paper in one person's hands, to tear it into two and pass one piece to some one else. That procedure was repeated until everyone had a piece of paper. The problem was to figure how many pieces of paper there were after the last tear. The reverse problem, raised by a teacher on the course, was to count the number of people in the room and figure how many tears were needed so that everyone had a piece of paper.

CONCLUSIONS

Building on the work of Krutetskii (169b), Gray & Tall (1994) and other research in mathematics education we identify several major aspects of flexibility of mathematical thought:

1. Use of novel strategies to solve problems.
2. Ability to establish both direct and reverse associations.
3. Ambiguous use of notation.
4. Capacity to interpret someone else's thinking.
5. Using an idea or strategy across different contexts or changing an existing strategy to fit a new context.
6. Using multiple representations for the same mathematical problem or using multiple representations to express the same idea.
7. Raising hypothetical problem situations based on existing problems.

These aspects of flexibility are, from our point of view, all manifestations of a fundamental ability to utilize what is stored in long-term memory in novel settings: of place, time, direction, representation, or person. This ability we hypothesize is related to the capacity of the hippocampus to establish long-term relational memories. A model of Davis, Hill & Smith (2000) suggests that a teacher can act as an external agent in taking students' procedures, helping to make them explicit and emotionally colourful, and so assist in transferring those procedures into explicit long-term memory. The classroom practices of the first author are entirely in accord with that model. Whether, and to what extent, all students can increase their flexibility of mathematical thought is a question we are currently investigating.

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