MENTAL REPRESENTATIONS IN ELEMENTARY ARITHMETIC

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This paper discusses emerging themes about the nature of embodied objects and symbolic procepts in the context of elementary arithmetic. Drawing upon the mental representations and strategies children use whilst executing mental addition and subtraction the paper also aims to illustrate how the nature of base objects may be determined and used within elementary arithmetic. The paper indicates how the nature of base object may change in the development of arithmetical concepts and illustrates how for some it may be a stepping stone to more sophisticated thinking whilst for others it acts as epistemological obstacle preventing the attainment of such thinking.

INTRODUCTION

The notion of numerical concepts being formed from actions with physical objects forms the background for the conceived cognitive development of simple arithmetic (see for example Piaget, 1985; Gray & Tall, 1994). Encapsulation (or reification) theories suggest that a cognitive shift takes place between carrying out actions and the formation of numerical concept. How this process takes place, however, remains the subject of theory and open to debate. More recently, Gray and Tall (2001) suggested that a simple switch of viewpoint where theorised encapsulation (or reification) of a process as a mental object may be linked to a corresponding embodied configuration of the objects acted upon (the base objects) may reveal some powerful insight into the different ways which individuals construct mathematical concepts. Our effort to gain an insight into how this is happening has taken a route to consider mental representations.

This study considers the association between different kinds of mental representations projected by 8-11 year olds and the construction of arithmetical concepts. Different kinds of mental representations are identified based on what objects and actions are mentally represented when children are presented with an addition or subtraction. Our purpose is to compare the meaning embodied in the objects and their configurations as denoted by the children's mental representations with process-object abstraction.

Earlier research by Gray, Pitta, Pinto and Tall (1999) indicated that students who consider the descriptive qualities of the embodied object remain at a more primitive level whereas, those that rely on the more intrinsic qualities of the object such as the mathematical symbolism and its relationship to other objects move to a more sophisticated level.

THEORETICAL BACKGROUND

Encapsulation (or Reification) of a procedure

Piaget believed that learning as well as performing mathematics was a matter of active thinking and operating on the environment. This activity was strongly linked to 'physical experience' which "consists of acting on objects to discover the properties of the objects themselves... not from the physical properties of particular objects, but from the actual actions (or more precisely their co-ordinations) carried out by the child on the objects" (Piaget, 1973, p.80). His focus was on the way in which actions and operations became thematized objects of thought or assimilation (Piaget, 1985, p.49). Interiorising processes into mental objects is seen as a fundamental way of constructing mathematical objects; dynamic actions conceived of as conceptualised entities is now frequently associated with notions such as encapsulation (Dubinsky, 1991) and reification (Sfard, 1991). In the context of number what is clear is that there is a growing sophistication in the nature of the entities operated on, from physical objects to mental operations with the number symbols themselves. This development is manifest in an increasing detachment from immediate experience, the evolution of different aspects of counting and a change in the form of unit counted (Steffe, Richards, von Glasersfeld & Cobb, 1983).

In the context of elementary arithmetic the nature of the unit counted may be seen to be analogous with the nature of the object operated on. Tall, Thomas, Davis, Gray and Simpson (2000) draw the distinction between "perceived objects" and "conceived objects". The first is based upon perceptual information where the focus is upon specific physical manifestations of the object. The second occurs when the focus shifts from the physical manifestations to the actions/process performed on them. Such a distinction has been used by Gray and Tall (2001) to distinguish between the notions of embodied object and symbolic procept. The former "begins with the mental conception of a physical object in the world as perceived through the senses" (p.66) and can "only be constructed mentally by building on the human acts of perception and reflection (p.67)". Though they see an increasing sophistication in the notion of embodied objects, Gray and Tall see a significant distinction between embodied objects such as a triangle and a graph on the one hand and the symbols of arithmetic on the other. The latter

> "act as pivots between processes and concepts in the notion of procepts and provide a conscious link between the conscious focus on imagery (including symbols) for thinking and unconscious interiorised operations for carrying out mathematical processes"
>
> (Gray & Tall, 2001, p.67)

Within the field of elementary arithmetic the theorised encapsulation or reification of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon which we have indicated to be the base object. Counting processes operate on physical objects. Thus the seemingly abstract concept of number already has a primitive existence in the physical configurations of base objects. In the context of addition, for example, the base objects are initially physical objects, then they become figural objects but later these become redundant as they are subsumed within a counting process which itself can be compressed into the concept of sum.

> This emphasises a proceptual structure that consists of a theory of related procepts, including the base objects on which the processes act, the symbols as process and concept, and the concept image on which the processes act. (Gray & Tall, 2001, p.68)

This papers attempts to consider the way in which children's explanation of the way in which they solve elementary number tasks may give us insight into the nature of the object that is being used. We feel that the distinction is essentially one that is to be made in terms of the use of external or internal representations. The latter suggests the notion of mental representation and in the sense that is discussed within this paper these mental representations may be embodied objects or symbolic procepts. Clearly a mental conception of a finger is an embodied object and of course the actual physical object 'finger' is not. Both may be referred to as base objects. However, we suggest that the 'threeness' associated with the display of three fingers is embodied. Clearly then, the mental representations of numerical objects reported by Seron, Pesenti, Noel, Deloch, and Cornet (1992) may also be seen as embodied objects. Their subjects reported seeing simple digits or numbers, numbers transformed into patterns as found on a dice, numbers with colour and numbers as on a number line. They suggest that quantity directly represented by "patterns of dots, or other things such as the alignment of apples or a bar of chocolate (p.168) may be deemed to be analogical. In our sense these objects are embodied — they arise initially in perception but they can carry mental ideas. Thus the dots on the dice may carry the idea of five.

In this study one portion of the complex structure of mathematics is isolated, that of mental additions and subtractions to 20. By considering children's mental representations and more specifically the object of the representation whether embodied object or symbolic procept, it tries to make sense of the influence that they have on children's successful understanding.

METHOD

The research was conducted in a "typical" primary school in the English Midlands. Sixteen children aged 8-11 years old, representing the extremes of numerical achievement in each of the four years - thus two at each extreme of each year - were presented verbally with mental additions and subtractions to 20. The children's numerical achievement was measured by criterion based test results available in the school and a numerical component which formed part of a larger study of which this paper is part.

For these combinations children were reminded that the object of the interview was for the interviewer to get a sense of the approach the child used to solve particular items and to gain some indication of what was happening in the child's head while doing these mental additions and subtractions.

Classification of the Results

To classify children's representations we have adopted some of Steffe's *et al* (1983) classifications for two reasons, since it is a finely-grained analysis of counting units and provides valuable insight into numerical development.

Automatic Responses: These were identified whenever a child indicated that "I just knew it" and no overt or covert actions appeared to be associated with the response. Here the base objects will be the inputs, for example 3 and 5, although it is difficult to discern whether an individual who knows facts projects the use of a mentally embodied object or the use of a symbolic procept (see abstract representations below).

Perceptual Representation: Strategies are applied with the support of physical items, for example fingers. Physical items are the base objects. The embodiment is the mental association made between the configuration of the fingers, as sets to count, and the quantity counted.

Verbal Representations: Here the number word is taken as a substitute for countable items. Typical examples include:

> Five, counted in my head three, four, five. (Y3-, 3+2)
>
> 9, 8, 7,... just said that to myself. (Y6-, 9-7)

Here the base objects are mental embodied objects. The number words, are embodied as mental conceptions of the number counted.

Figural Representations: Here the counting process is taking place in the absence of actual items but is associated with visual or verbal analogues of the items:

> I saw a line of numbers. It was one, two, three, ...13, 14, 15. After 10 the numbers got bigger. I counted from 1 until I got to 8. (Y3-, 3+5)

The mental conceptions of fingers are embodied objects.

Abstract Representations: The use of the term is based on the notion of Steffe *et al.* (1983) but in the current context not only did a child not require to construct countable units but the symbol was identified as the object of thought. The classification was most frequently associated with derived facts. A typical response could be:

> [Said to myself] the difference between six and seven is one, so two sixes are 12 and 12 plus 1 is 13. (Y4+,7+6)

This category suggests the use of symbolic procepts.

All responses were videotaped and supported by field notes. The results are established from analysis of the video transcriptions.

RESULTS

Analysis of Results

Children's combinations to twenty

As illustrated in Table 1 (which illustrates the percentage use of each representation) perceptual and verbal counting responses were given more often by low achievers whereas automatic and abstract responses were given more often by high achievers. Very few figural responses featured in either of the two groups.

	Automatic	Abstract	Perceptual	Verbal Counting	Figural
High achievers	64	29	4	2	1
Low achievers	10	8	61	17	4

Table1: Representations of Mental Arithmetic: Number combinations to twenty

Low achievers' combinations to twenty

Perceptual representations featured strongly amongst 'low achievers' and this may have been associated with vocal or sub-vocal counting. Most frequently fingers, these perceptual units, became objects of thought which were sequentially tagged in the counting procedure. Tagging was mostly overt in that children looked directly at their fingers, tagging those on one hand with those on the other, or, if the number was relatively large, tagging through touching the desk or even the nose. On some occasion motor acts were used as substitute for tagging. At times the counting was not associated with any obvious tagging. Children would "feel" movements in their fingers without any obvious sign that they were doing so. When this happened, it seemed that the younger the child or the more difficult the sum the more exaggerated this movement was. Therefore, while a Year 4 'low achiever' said:

> 7, 8, 9, 10, 11 and I was counting on my fingers. (Y4-, 4+7)

a Y6 child said

> I was counting in my head 6, 7, 8, 9 but had it as a finger feeling. (Y6-, 6+3)

We conjecture that the need to use perceptual items amongst the 'low achievers' reinforces the evidence that symbols need to be concretised.

There were few instances where low achievers made reference to figural representations. These may be seen in the context of analogical representations. In the instances associate with number combinations to 20 these figural representations resembled the number forms of Seron *et al* (1992) in that they were based on the number line:

> I saw the number line in my head and saw a lot of numbers in my head on a number line. There are lots of numbers but it depends on how high the numbers are. High means how big the numbers are. This time it was three plus four. (Y3-, 3+4)

The smaller combinations were often associated with verbal counting and the numbers themselves served as countable objects.

> I was thinking of it. Add. It is 11. I said 10, 11 to myself. (Y6-, 9-2)

> 5. I counted in my head 3, 4, 5. (Y3-, 3+2)

Often the verbal tone or double counting accompanied the counting act, although this sometimes led to errors:

> Got to 13 to add up to 17…13, 14, 15, 16, 17. 13. One 14, two 15, three 16, four 17, five …said all in the mind. (Y6-, 17-13)

In all the above examples there is an intrinsic similarity, 'low achievers' use a lengthy counting procedure as if it is a generalisable process. Irrespective of whether the object is a concrete finger or a mental number line or a number word they are essentially doing the same thing. Low achievers seem to be trapped in the only process that one is able to do with base units that are physical — counting. The overemphasis on embodied objects and on surface characteristic of objects and processes does not allow children to see the power of symbolic procepts. Therefore, things are becoming increasingly difficult as numbers are getting bigger.

High achievers' combinations to twenty

High achievers' responses were mainly 'automatic' and 'abstract', the first category indicating an almost immediate response whereas in the second the symbols dominated the children's thinking.

The symbols were frequently associated with the input combination or with the final solution.

> I saw the eight then the two and then I see them altogether. (Y5-, 8-2)

The tendency to see the input symbols and/or the final output was relatively common amongst those 'high achievers' who reported seeing symbols though other characteristics were also apparent.

> I saw a picture of 9-2, black. (Y6+, 9-2)
>
> I saw 9 and 7 flash, not as a sum but just 9 and 7. I then saw 2 much stronger. (Y6+, 9-7)

'High achievers' also reported symbolism associated with the formation of derived facts:

> I said [to myself] seven and seven and take away one. Told you thirteen. (Y3+, 7+6)

> I *saw* 13-2 and *thought* I would split it up into different parts to make it easier. This stood out. Saw 13-3 then 10. Then I saw 10-2 this stood out. Then I saw 8. 13-3 and 10-2 stayed at the same time. I saw 8 on its own. (Y6+, 13-5)

In the second example the child seems to be oscillating between visual representations "saw" and verbal representations "thought". This relationship between visual and verbal mental representations was one that clearly gave the child the power to complete the combinations. The visual image of the high achiever generates the idea and then it is put aside so that she can focus on the relational characteristics. The symbols are thought generators and not the mental embodied objects used to carry out an action. Of course the fact that their mental representations are dominated by symbols, which can either be seen or talked about, gives them this flexibility.

'High achievers' are not only able to filter out the surface characteristics of embodied objects and the lengthy procedures but they are also able to avoid "difficult" numerical combinations by transforming the question to a more manageable one, or one that is derived from an easier known fact:

> I said to myself the difference between 6 and 7 is 1 so 2X6=12, 12+1=13. (Y4+, 7+6)

'High achievers' seem to be able to filter out surface characteristics from the embodied objects, condense the counting procedure or most often omit it altogether and carry out their mental processes in a truly abstract fashion. Their emphasis lies on the more intrinsic qualities of the mathematical symbolism, process to do and concepts to know. Another strength of the mathematical symbolism is that it can be seen and also talked about. It is very rare that an embodied object can be interpreted as a process and a concept or carried in the mind as a visual sign or a verbal word.

DISCUSSION

Hearing the combinations triggered the 'low achievers' to carry out a procedure; in this instance an overwhelming desire to count. It could be hypothesised that in their failure to recall combinations the representation that the 'low achievers' used, evoked either physical objects or mental conceptions of these objects. In the first instance, their mental representations were embodied as general representations of the number sequence. This general mental representation was retrieved either as physical objects or as an embodied object, such as the number line. Qualitative differences in the nature of the base objects were determined through this real or imaginary difference. Thus the 'specificity' of the number sequence is not only identified through the inclusion of the numbers that need to be counted but also through the different objects used. Therefore, it seems that the choices the children make (consciously or subconsciously) focus on the nature of the counting procedure and representation (mental or physical) they need to support this procedure. Increasing procedural efficiency may determine both the counting strategy, for example, count up as opposed to count back, and the nature of the base unit to be used. However, within the three categories of representations identified, physical, figural and verbal we may see the gradual shift in the nature of the base unit from perceptual to a mentally embodied object.

In contrast categories identified for 'high achievers' illustrated extensive use of the retrieval of known facts, either to give an automatic response or in order to proceed to a derived fact. Clearly the nature of the entities operated upon have changed. Now they signify conceptual entities that appear to exist "independently of the child's actual or represented motor activity" (Cobb, p.168). The 'high achievers' are performing the operations of addition and subtraction on symbolic procepts. It is conjectured that underpinning this approach is the power that emerges from their representational flexibility. They seem to be carrying out a 'search' for the most appropriate number fact that can be used. They retrieve it and either present it as the answer or manipulate it in order to reach an answer. The former is not easy to qualify, the latter is the essence of proceptual thinking. The symbolic procept acts seamlessly to switch between a mental concept to manipulate to a process to carry out.

Essentially the role of base objects may be seen as a stepping stone to higher order concepts. However, they may have specific meanings for some individuals that act as epistemological obstacles that prevent a hierarchical development that is essential to progress. These differences would seem to become apparent very early within the child's mathematical development.

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