THE BRIDGE BETWEEN PRACTICAL AND DEDUCTIVE GEOMETRY: DEVELOPING THE 'GEOMETRICAL EYE'

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The dual nature of geometry, as a theoretical domain and an area of practical experience, presents mathematics teachers with the opportunity to link theory with the everyday knowledge of their pupils. Very often, however, learners find the dual nature of geometry a chasm that is very difficult to bridge. With research continuing to focus on understanding the nature of this problem, with a view to developing better pedagogical techniques, this paper reports an analysis of innovative geometry teaching methods that were developed in the early part of the 20th Century, a time when significant efforts were being made to improve the teaching and learning of geometry. The analysis suggests that the notion of the geometrical eye, *the ability to see geometrical properties detach themselves from a figure, might be a potent tool for building effectively on geometrical intuition.*

INTRODUCTION

The teaching and learning of geometry remains a major problem for mathematics education. As Villani (1998, p321-2) observes, in the conclusion to the ICMI study, "to build a [geometry] curriculum is a very difficult and demanding task" yet "teaching methods [in geometry] are even more important than content. And it is also more difficult to improve them". In a similar vein, the recent UK study of geometry teaching (Royal Society, 2001) concludes that "the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources" (p19).

One of the major characteristics of geometry, as each the aforementioned reports acknowledges, is its dual nature. Geometry is both a theoretical domain and perhaps the most concrete, reality-linked part of mathematics. This dual nature has dual consequences for the teaching and learning of geometry. While, hypothetically, the dual nature of geometry should help teachers to link mathematical theory to pupils' lived experience, in practice for many pupils the dual nature is experienced as a gap that they find very difficult to bridge. Thus, research continues to focus on the difficulties that pupils have in developing an understanding of geometrical theory and making the transition to formal proofs in geometry (see, for example, Arzarello *et al*, 1998; Malara and Iadorosa, 1997; Miyazaki, 2000).

While the use of software tools, such as dynamic geometry, is proving to be helpful (for recent research evidence, see the special issue of *Educational Studies in Mathematics* edited by Jones *et al*, 2000), there is an urgent need to develop a more effective pedagogical theory for geometry so that such tools can be integrated more successfully in mathematics classrooms. With a view to informing the development of better pedagogical models, this paper reports some of the findings from a study of forms of innovative geometry teaching that were developed in the early part of the 20th Century, a time when significant efforts were being made to improve the teaching and learning of geometry. The analysis of curriculum materials and associated teaching methods undertaken as part of this study focuses, in part, on ways of bridging the gap between practical and deductive geometry. The analysis suggests that much promise lies in the notion of the *geometrical eye*, a term coined by one of the major movers behind the reform of the geometry teaching in the early 20th Century, Charles Godfrey[1] (1910). Godfrey defined the *geometrical eye* as "the power of seeing geometrical properties detach themselves from a figure" (*ibid*, p197). This paper argues that this notion might be a potent tool for building effectively on geometrical intuition.

THEORETICAL CONSIDERATIONS

Of the range of theoretical work concerned with the learning of geometrical ideas, that of Piaget (and colleagues) and of the van Hiele's is probably the most well-known. In the Piagetian work (see, Piaget, Inhelder and Szeminska, 1960), one of the major themes is that a learner's mental representation of space is not a perceptual 'reading off' of what is around them. Rather, learners build up mental representation of the world through progressively reorganising their prior active manipulation of that environment. The van Hiele model also suggests that learners advance through levels of thought in geometry, characterised as visual, descriptive, abstract/relational, and formal deduction (see, van Hiele 1986). Both the Piagetian approach and the van Hiele model have been subject to critical review that is beyond the scope of this paper. Suffice to say that much additional research is needed on the relations between intuitive, inductive and deductive approaches to geometrical objects, the role and impact of practical experiments, and the age at which geometrical concepts should be introduced.

Geometry is an area of mathematics in which intuition is frequently mentioned. Views vary, however, about the role and nature of geometrical intuition, and how it might help or hinder the learning of geometry (and other areas of mathematics). Piaget, for instance, appears to suggest a hierarchy when he equates intuition to what he calls non-formalised operational thought:

> Although effective at all stages and remaining fundamental from the point of view of invention, the cognitive role of intuition diminishes (in a relative sense) during development. .... there then results an internal tendency towards formalisation which, without ever being able to cut itself off entirely from its intuitive roots, progressively limits the field of intuition (in the sense of non-formalised operational thought).
>
> Piaget 1966, p225

Van Hiele similarly gives intuition a relatively minor role in the latter stages of learning. In contrast, Fischbein suggests either a plurality or a dialectic when he writes that:

> The interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood.
>
> Fischbein 1994, p244

Geometers, nevertheless, tend to recognise the importance of geometrical intuition. In his classic text on geometry and the imagination, Hilbert wrote:

> In mathematics ... we find two tendencies present. On the one hand, the tendency towards abstraction seeks to crystallise the logical relations inherent in the maze of materials ... being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency towards intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations.
>
> Hibert 1932, piii

More recently, Atiyah writes:

> spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool...
>
> Atiyah, 2001

Yet not all mathematicians share this view. For many, intuition, even geometrical intuition, is not to be relied upon. As Tall (2000, p20) reminds use, the influential *Bourbaki* approach rejected any notion of geometrical intuition as being untrustworthy. Such a view remains fairly prevalent amongst many mathematicians.

The question that these considerations highlight is how to resolve these apparently opposing positions, if this is possible. To address this question, and attempt to illuminate the relationship between practical and deductive geometry, we examine a time in mathematics education when these issues were being seriously tackled.

METHOD

In the UK at the beginning of the 20th Century, the teaching of geometry was in the spotlight. The suitability of the formal teaching of deductive geometry in the form of Euclid's Elements was being seriously questioned. The form of theorems and the order in which they should be introduced was the subject of great debate (see, Howson, 1982).

As part of a wider study of the process of change in geometry teaching in the early 20th Century, the historic sources examined in this paper are the geometry textbooks and educational writings of two influential scholars of the time, Charles Godfrey and Arthur Siddons. The pivotal role played by Godfrey and Siddons and the lasting importance of their textbooks in the history of the teaching of geometry is confirmed by Howson (1982) and Quadling (1996). The methodological approach is documentary analysis (Jupp and Norris, 1993) enhanced by the methodology for textbook analysis proposed by Schubring (1987).

The documents analysed include the two main textbooks by Godfrey and Siddons, *Elementary Geometry* (Godfrey and Siddons, 1903) and *A Shorter Geometry* (Godfrey and Siddons, 1912), together with many of their articles published in the journal *Mathematical Gazette*, a collection of educational writing published in the 1930s (Godfrey and Siddons, 1931) and a memoir published in the 1950s (Siddons, 1952). The analysis conducted for the study focused on the design and various roles of practical tasks in the textbooks and on their pedagogical purpose gleaned from documents written by Godfrey and Siddons. Through this analysis we consider Godfrey and Siddons' approach to the relationship between practical and deductive geometry, and propose some insights for future research.

PRACTICAL TASKS IN THE TEXTBOOKS BY GODFREY AND SIDDONS

In both *Elementary Geometry* (*op cit*) and *A Shorter Geometry* (*op cit*), and unlike that recorded in Euclid's *Elements*, the design was such that the early stages of each book comprised practical activities while the latter stages were devoted mainly deductive proof of various geometrical theorems. Throughout both of the Godfrey and Siddons texts, experimental tasks can be seen.

For example, through the exercise below, students would learn how to measure angles (Godfrey and Siddons, 1903, p. 12);

> Exercise 37. Measure the angles of your set square (i) directly, (ii) by making a copy on paper and measuring the copy.

An example of an exercise (Godfrey and Siddons; 1903, p. 28) that would lead students to discover a geometrical truth that is proved deductively in a later section of the book is as follows;

> Exercise 123. Cut out a paper triangle; mark its angles; tear off the corners and fit them together with their vertices at one point. What relation between the angles of a triangle is suggested by this experiment?

In some of the practical exercises, students apply chosen theorems in a practical way. For example, the following exercise would be undertaken after the students have learn that 'there is one circle, and one only, which passes through three given points not in a straight line'.

> Exercise 1162. (using graph paper.) Draw a circle to pass through the points (0, 3), (2, 0), (-1, 0), and measure its radius. Does this circle pass through (i) (0, -3), (ii) (1, 3), (iii) (0, -2/3)?

In summary (for more details of the analysis see, Fujita, 2001a and 2001b), the roles of these tasks can be categorised as follows:

a) Making students familiar with geometrical instruments and figures;

b) Leading students to discover geometrical facts; and

c) Applying the theorems to practical problems.

In addition, some of the exercises are included to justify geometrical facts through work on experimental tasks. Godfrey discusses the importance of this from a general educational point of view in one of his articles (see, Godfrey and Siddons; 1931).

THE RELATIONSHIP BETWEEN PRACTICAL AND DEDUCTIVE GEOMETRY

It is clear from the writings of Godfrey and Siddons that they considered that a disconnection between experiment and deductive geometry would be inappropriate in the teaching of geometry (Godfrey and Siddons; 1931, p21, Siddons; 1952, p9). Godfrey, for example, was explicit that mathematics cannot be undertaken by logic alone (Godfrey; 1910, p. 197). He wrote that another important 'power' is necessary for solving mathematical problems. This he called *geometrical power*, defined as "the power we exercise when we solve a rider [a difficult geometrical problem requiring the use of several pieces of theoretical knowledge]" (Godfrey; 1910, p. 197). To develop this *geometrical power*, Godfrey suggested that it was essential to train what he called the *geometrical eye*. This, Godfrey defines as 'the power of seeing geometrical properties detach themselves from a figure' (Godfrey; 1910, p. 197).

To illustrate what Godfrey means by the *geometrical eye*, consider an example:

> If A, B are the mid-points of the equal sides XY, XZ of an isosceles triangle, prove that AZ=BY' (Godfrey and Siddons; 1903, p. 94).

When we consider this problem, we would not be able to prove this statement unless we could 'see', first of all, that, for example, triangle AYZ and triangle BZY are likely to be congruent. A study, carried out in Japan, showed that some students could not 'see' which triangles, in the problem above, were likely to be congruent. Nakanishi (1987), who tried to identify the difficulties that students experience when they solve geometrical problems, gave 87 Japanese students aged 14-15 the isosceles triangle question quoted above. Even though 65 students could prove this problem correctly, nine of them were not sure what to do because they could not see any congruent triangles, and four of them could not focus on an adequate pair of congruent triangles (*op cit* p71).

Godfrey stated that this kind of 'power' would be essential to solve geometrical problems, and it was experimental tasks that would make possible to train the 'geometrical eye' at any stages in geometry:

> There must be a good foundation of practical work, and recourse to practical and experimental illustration wherever this can be introduced naturally into the later theoretical course. Only in this way can the average boy [*sic*] develop what I will call the 'geometrical eye'.
>
> Godfrey; 1910, p197

When we analyse Godfrey and Siddons' textbooks from this point of view, we can find that practical exercises were placed in various places in the deductive stages in *Elementary Geometry* or *A Shorter Geometry*. For example, before the theorem 'a straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord', the following exercises were included in *A Shorter Geometry* (Godfrey and Siddons, 1912, pp. 151-2),

Exercise 877. Draw a circle of about 3 in. radius, draw freehand a set of parallel chords (about 6), bisect each chord by eye. What is the locus of the mid-points of the chord?

Exercise 878. Draw a circle and a diameter. This is an axis of symmetry. Mark four pairs of corresponding points. Is there any case in which a pair of corresponding points coincide? (*Freehand.*)

Exercise 879. What axes of symmetry has (i) a sector, (ii) a segment, (iii) an arc, of a circle?

These exercises would make students aware of the symmetry of the circle as well as leading them to discover the theorem (notice that a symbol in the textbook against each exercise denoted that each required classroom discussion between the teacher and students). Also, to prove the theorem, the fact that the triangles OAD and OBD (see the figure 1 below) are congruent needs to be shown. Our analysis suggests that the precursor exercises are designed to help the students to 'see' the congruency of the triangles. That is, that the exercises are purposefully included to develop the students' *geometrical eye*.

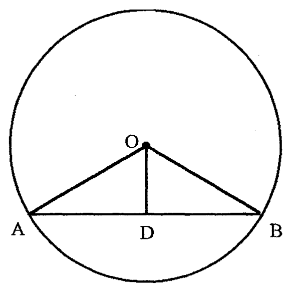

Fig. 1

CONCLUDING COMMENTS

In a number of countries, the early stages of geometry in schools comprise practical activities such as the drawing and measurement of geometrical figures. Later stages of schooling are then devoted to deductive geometry. The specification for geometry in the Japanese 'National Course of Study' (Japan Society of Mathematics Education, 2000), for example, takes such an approach. While this is somewhat in line with the van Hiele (1986) model of learning in geometry, the relationship between practical and deductive geometry remains unclear, and, in particular, the transition between them is one of the major concerns in the study of the teaching of geometry.

A major improvement in geometry pedagogy would be to improve on calls to develop geometrical intuition by linking more directly with geometrical theory. This would entail developing pedagogical methods that mean that a deductive and an intuitive approach are mutually reinforcing when solving geometrical problems (see, Jones 1998).

This paper argues that Godfrey's notion of the *geometrical eye* might be a potent tool for building effectively on geometrical intuition. As we have shown through our analysis, Godfrey and Siddons considered that practical and deductive geometry should be combined in the latter in the teaching of geometry. Godfrey considered that the *geometrical eye* would be essential for successfully solving geometrical problems, and that it should be trained by practical tasks at all stages of geometry. It is illuminating that innovative teachers 100 years ago mentioned the importance and roles of visual images in geometry, and it is worth considering the following issues in the future research for improvement of the teaching of geometry in primary and secondary schools. Future research could examine whether it would be possible to define more clearly the notion of the *geometrical eye*, what the relationships are between difficulties of proof in geometry and the *geometrical eye*, and how (or whether) it would be possible to train students' *geometrical eye* though practical tasks.

NOTES

1. Godfrey's use of the term *geometrical eye* predates that of Wittgenstein (see Wittgenstein, L.J.J.: 1958, *The Blue and Brown Books*, Oxford, Blackwell, pp. 73-74) and has a different sense.

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