NUMBER SENSE, PLACE VALUE AND "ODOMETER" PRINCIPLE IN DECIMAL NUMERATION: ADDING 1 TENTH AND 1 HUNDREDTH

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Having a facility with numbers (whole numbers and decimals) is essential for life skills. It requires an understanding and coordination of several powerful mathematical concepts (e.g., place value, additive and multiplicative fields, the role of zero, seriation) and principles (e.g., "odometer", associative, commutative distribute) (Baturo, 2000). This paper reports on 329 Year 6 students' proficiency with seriation of tenths and hundredths when instruction in these two places was complete. They were asked to add 1 tenth and 1 hundredth to a variety of given numbers with the overall results of 53.6% and 54.6% respectively. These results were evaluated in terms of the students' responses to basic place value tasks. This paper analyses the mathematical structure of the tasks, discusses some common error patterns discerned, and draws inferences for teaching.

The decimal number system is superficially simple but is structurally very complex. It is derived from the notion of a unit (Steffe, 1986), the importance (and complexity) of which is often not considered because of its deceptively simple association with one object. The decimal number system consists of simple yet powerful patterns within the domain of whole numbers and decimal numbers, patterns which should encourage the development of succinct yet global mental models within and between each of these domains. Within the domain of whole numbers, students are exposed to processes such as counting, comparing, grouping, regrouping, and approximating, most of which can be transferred without adaptation to decimal numbers (see Baturo, 2000, for a mathematical analysis of these processes). However, comparing is a process that must be adapted for decimal and common fractions (see, e.g., Resnick et al., 1989; Stacey et al., 2001). Furthermore, a study of whole numbers should develop an awareness of the commutative, associative, and distributive laws which can then be transferred without adaptation to decimal numbers (and to common fractions, percents etc.).

Another global principle that occurs in the decimal number system is referred to as the "odometer" principle in this paper. It underlies the counting process and requires an understanding that a place is "full" when it has 9 units (which could be ones, tens, tenths, etc) and that recording the next number requires a new position to the left of the place under consideration. Embedded in this principle is the notion of place value which requires an understanding of the role of the base, the order of places, and that numbers increase in value as they "move" to the left (and, conversely, decrease in value as they "move" to the right).

What makes the decimal number system powerful is the notion of place value and the use of 10 symbols only (base) to represent any number no matter how large or small

(Baturo, 1998). This capability is possible because of the multiplicative relationships (×, ÷) that relate adjacent (and nonadjacent places) places. However, Baturo (2000) found that top-performing students only have an understanding of the bi-directional multiplicative relations.

Research (Baturo, 1997, 2000; Bednar & Janvier, 1988; Fuson, 1990; Hiebert & Wearne, 1992; Jones et al., 1996; Ross, 1990) has produced a plethora of evidence that students (particularly from Years 1 to 4) have great difficulties in acquiring an understanding of place value. The general consensus seems to be that "children find place value difficult to learn and teachers find it difficult to teach" (Ross, 1990, p. 13). Baturo's numeration model (2000) gives some indication of the complexity of place value in that she indicates that there are three hierarchical levels which take account of Halford's (1993) complexity model in which unary relations (e.g., memory objects such as position, base, order) are less cognitively difficult than binary relations (e.g., equivalence such as 1 ten = 10 ones) which, in turn, are less cognitively difficult than ternary relations (e.g., transitivity where 10 ones = 1 ten, 10 tens = 1 hundred so 100 ones = 1 hundred).

In Baturo's model, Level 1 place value knowledge is baseline knowledge associated with position, base, and order, cognitions which are required for all numeration processes. Therefore, without these cognitions, students have no chance of processing decimal numbers with understanding. On the other hand, having Level 1 knowledge alone is not sufficient for a full understanding of all decimal-number numeration processes. Level 2 comprises unitising and equivalence, one or both of which are required for Level 3 knowledge which is associated with reunitising, additive structure and multiplicative structure. These cognitions appear to provide the superstructure for integrating the other levels and, for this reason, were defined as *structural knowledge*

Several researchers have pointed to the difficulty students have with grouping/unitising (Level 2) which involves quantifying sets of objects by grouping by 10 (in a base-10 system) and treating the groups as units (Fuson, 1988) and using the structure of the notation to capture the information about the groupings (Hiebert & Wearne, 1992; Ross, 1990). For example, Bednarz and Janvier (1988) viewed grouping as the basis for recognising and constructing multidigit numbers. Inherent in the process of grouping to form larger units or partitioning to form smaller units (called "superunitising" and "subunitising" respectively by Baturo, 2000) is the notion of base which gives rise to the positional aspect of place value.), From their study, Bednarz and Janvier noted that few children see the relevance of grouping tasks and the validity of doing and undoing groupings (partitioning) to solve multidigit number problems. Moreover, they found that some children form groupings, but only to count collections of objects (by tens in the decimal number system). That is, children do not appear to abstract the role of the base in grouping to form larger places.

In their study with Year 1 students, Jones et al. (1996) developed a framework for nurturing and assessing multidigit number sense. The framework incorporated four components (*counting*, *partitioning*, *grouping and estimating*, *and number relationships and ordering*) which they saw as representing a sequence of understandings related to place value. They predicted and confirmed that counting was the pivotal component of their framework with grouping the key concept. Their study found that solution at the higher end of their framework was dependent on facility with (or understanding of) the lower components. Their framework is attuned to Steffe's (1986) framework with respect to units, namely, that there are four different ways of thinking about a unit, namely, *counting (*or *singleton) units*, *composite units* (grouping), *unit-of-units* (regrouping*)* and *measure unit,* with each type apparently representing an increasing level of abstraction.

Counting is the repeated addition of an iterated unit (counting forwards) or the repeated subtraction of an iterated unit (counting backwards). These notions hold true for counting whole numbers, decimal numbers, common fractions, etc. Counting from a place value perspective requires understanding of the odometer principle (i.e., grouping to form a new place after 9 ones, tenths, hundredths, etc is reached). Therefore, to be able to count/seriate successfully, students require an understanding of place value, order across and within places, the ability to group, and an awareness of when to group and of the effects of grouping.

THE STUDY

The study reported in this paper is part of a doctoral dissertation in which Year 6 students' understanding of the numeration processes (identifying numbers from a variety of representations, counting, comparing, ordering, approximating) relating to tenths and hundredths were assessed (Baturo, 1998). To this end, a diagnostic test was constructed to determine the robustness of Year 6 students' understanding of the numeration processes. The test was trialed with a convenience sample of 8 students, modified and retrialed with 156 students from two large suburban schools (referred to as Trial 1 in this paper), modified and administered to 173 students a year later (Trial 3) at the Trial 2 schools. All item directions were read to the students (but not the actual subitems) and special instructions issued. For example, for Item 1 (see Figure 1), a sample number (2.3) was written on the board and the students were told that answers such as "two point three", "two decimal three", or "two dot three" would not be accepted. They were told that they had to write the name of the decimal part. At the time of the test (early Year 6 in both trials), the students had received two years of instruction in tenths and one year of instruction in hundredths.

Item analysis

This paper reports on the students' performances on five of the diagnostic test items (see Figure 1). The first three were focused on assessing the baseline knowledge of position and order (i.e., Level 1 in Baturo's model) whilst the last two items focused on seriating, with most subitems at Level 1 but some at Level 2 (requiring grouping and thus more difficult). All subitems ranged from prototypic to nonprototypic cases (see, e.g., Baturo & Cooper, 1997) in order to assess the robustness of the students' knowledge. With respect to the items in this study, prototypic tasks are classified as those that are given so frequently in class that students can acquire syntactic rules for their solution (e.g., adding 1 to the rightmost digit when finding the number that comes next after a given number, a rule that can develop from an overuse of counting on with whole numbers). Thus, in this study, nonprototypic tasks were seen as those that would not normally be part of the students' experiences and could not be solved by syntactic rules. Figure 1 describes the tasks and the difficulty level of each subitem in terms of Baturo's (2000) numeration levels and expected prototypicality.

Item 1: Write these numbers in words: (a) 4.7 [1P]; (b) 6.39 [1P]; (c) 0.8 (1NP]; (d) 5.02 [1NP].
Item 2: Write these numbers: (a) nine, and 5 tenths [1P]; (b) 6 tenths [1P]; (c) four, and thirteen hundredths [1P]; (d) sixty hundredths [1NP]; (f) seven and one hundredth [1NP]; (g) three hundredths [1NP].
Item 3: Write the number that has: 2 tenths, 5 hundredths, 4 ones [1NP].
Item 4: Write the number that is 1 tenth more than: (a) 9.4 [1P]; (b) 2.9 [2P]; (c) 4.06 [1NP]; (d) 1.94 [2NP]; (e) 5 [1NP]
Item 5: Write the number that is 1 hundredth more than: (a) 2.76 [1P]; (b) 3.9 [1NP]; (c) 0.09 [2P]; (d) 6 [1P]; (e) 4.91 [1NP]

Note. 1 and 2 indicate Baturo's (2000) numeration levels; P = prototypic; NP = nonprototypic.

***Figure 1.* Items [and difficulty level] designed to assess Year 6 students' understanding of place value and seriation.**

Items 1, 2 and *3* assessed the students' baseline knowledge of *place value* (Level 1). Items 1 and 2 focused on position and naming whilst Item 3 focused on the order of the places. In Item 1, the first two numbers were considered to be more straightforward than the last two which incorporated a zero. Of these latter two, the last number was expected to produce lowest performance because of the internal zero. It was also expected that naming hundredths would produce a lower performance than naming tenths. *Item 3* was included to determine whether students used the semantic understanding of the places or whether they relied on the syntactic "left to right" order in which the digits of a number are usually written.

With respect to the seriation tasks, *Item 4* required the students to record the number that was 1 tenth more than the given number and, as such, could be considered to be a place value item because the students had to identify the place to change first. The subitems included prototypic and nonprototypic cases. Prototypic cases included 9.4 where the digit to change first is in the rightmost place and does not involve grouping, and 2.9 where the digit to change first is in the rightmost place and does involve grouping. Nonprototypic cases included numbers where the digit to change first is not in the rightmost place either because it is an internal place (4.07, which requires no grouping; 1.94, which requires grouping) or because the given number was a whole number (5).

Item 5 was similar to Item 4 except that students were required to add 1 hundredth to the given numbers. With respect to the prototypic examples (2.76, 0.09, 4.91), although they are structurally similar in that hundredths are provided and therefore 1 can be added to the rightmost place, they were deliberately selected to identify syntactic learners. For example, does a student see a "9" and automatically elicit the regrouping process (probably because of having been exposed to prototypic examples)? The nonprototypic examples (3.9, 6) were expected to produce 4/4.0/4.00 and 7 or 6.1 respectively.

The seriation items reported on in this paper were based on Jones et al.'s (1996) lowest component (counting) in their place value framework whilst the subitems required an application of Steffe's (1986) simplest notions of unit, namely, counting/singleton units and composite units (grouping).

Results

Table 1 shows that the means per item were quite poor across both trials. With respect to Items 1,2, and 3, the results indicate that the students did not have the prerequisite baseline place value knowledge (i.e., the names and order of the decimal-fraction places) that will enable them to process decimal numbers with understanding. Thus, not having baseline knowledge of the place names, it is then not surprising that the students were unable to add 1 tenth and 1 hundredth to the given numbers (Items 4 and 5 respectively). Table 1 also shows that, for the place value items, students generally performed much better on Item 2 than on Item 1, both of which were assessing knowledge of place names/position. (See Discussion.)

Table 1: ***Means (%) for All Items for Individual Trials and Overall Study***

	Items 1-5 Means				
Study	Item 1	Item 2	Item 3	Item 4	Item 5
Trial 2 (n = 173)	66.8	79.4	61.3	56.4	56.0
Trial 1 (n = 156)	55.6	71.9	61.5	50.4	53.1
All (N = 329)	**61.5**	**75.8**	**61.4**	**53.6**	**54.6**

Table 2 shows that, within Item 1, the subitems relating to tenths generated markedly better performances in Trial 3 than those related to hundredths (80.3, 79.8 as opposed to 56.1, 50.9 respectively).

Table 2: ***Means (%) for the Subitems in Items 1 and 2 for the Trials and Overall Study***

	Items 1 and 2 responses											
	Item 1 (write in words)					Item 2 (write in digits)						
Study	4.7	6.39	0.8	5.02	**M**	9.5	0.6	4.13	0.60	7.01	0.03	**M**
Trial 2 (n = 173)	80.3	56.1	79.8	50.9	**66.8**	85.0	85.0	83.2	83.2	71.7	68.2	**79.4**
Trial 1 (n = 156)	68.6	50.0	50.6	53.2	**55.6**	85.3	77.6	80.8	71.8	57.1	59.0	**71.9**
All (N = 329)	**74.8**	**53.2**	**66.0**	**52.0**	**61.5**	**85.1**	**81.5**	**82.1**	**77.8**	**64.7**	**63.8**	**75.8**

In Trials 1 and 2, the tenth item with zero, 0.8, generated much lower performances than the tenth item with no zero (4.7). With respect to Item 2, Table 2 shows that, generally, performance decreased markedly over the last three subitems, each of which was classified as nonprototypic because they included a zero, particularly the last two which required an internal zero.

With respect to Items 4 and 5, Table 3 shows that the students performed best on the prototypic examples, namely, adding 1 tenth to 9.4 and 1 hundredth to 2.76 (where 1 could be added to the rightmost digit). Although Item 4b (2.9) and Item 5d were also prototypic in this regard, they had the extra distraction of "9" which either didn't evoke the odometer principle in Item 4b or erroneously evoked the principle in Item 5d.

Table 3: ***Means (%) for Items 4 and 5 for the Trials and Overall Study***

	Items 4 and 5 responses											
	Item 4 (add 1 tenth)						Item 5 (add 1 hundredth)					
Study	9.4	2.9	4.07	1.94	5	**M**	2.76	3.9	0.09	4.91	6	**M**
Trial 2 (*n* = 173)	81.5	64.2	50.9	38.2	47.4	**56.4**	61.8	47.4	60.7	59.0	50.9	**56.0**
Trial 1 (*n* = 156)	74.4	58.3	42.9	43.6	32.7	**50.4**	64.1	60.9	68.6	31.4	40.4	**53.1**
All (*N* = 329)	**78.1**	**61.4**	**47.1**	**40.7**	**40.4**	**53.6**	**62.9**	**53.8**	**64.4**	**45.9**	**45.9**	**54.6**

The students' responses were analysed for error patterns and very few consistent errors were made by the same students across the range of subitems within an item, indicating that these students' responses were contingent on the "look" of the number. Generally, individual error patterns were discerned in Item 1. For example, if a student wrote "four point (or decimal/dot) seven" for Item1a, s/he invariably wrote the remaining three numbers using the same syntax. However, across the 337 students, some errors occurred frequently. For example in Item 1, the prototypic examples (4.7, 6.39) produced an error rate of 22.8% and 46.8% respectively. The errors could be classified as "syntactic reading" (e.g., four point seven) or misidentification of places (e.g., 4 tenths or 4 tenths and 7 hundredths). Syntactic reading accounted for 14.9% of the errors whilst misidentification accounted for 11.2% of the errors. For Items 4 and 5, students' errors on all subitems could be categorised as syntactic responses to overpractised tasks. That is, students either added 1 to the rightmost place or added 1 to the ones place irrespective of the number given. For example, when adding 1 tenth to 4.06 (4c), the most common responses were 4.07 (30.4%) and 5.06 (6.1%).

Furthermore, each subitem across the five tasks generated at least 20 different types of errors. The disparity between the number of error types within each item and the percentage of most common errors revealed the idiosyncratic nature of the responses as the students strived to make sense of the tasks. For example, in Item 2c, students gave 23 different erroneous responses but the most common error (4.013) occurred in only 2.7% of the 17.1% of these responses.

CONCLUSIONS

The results on all of these elementary place value and seriation tasks are of concern considering the length of school time devoted to developing students' understanding of decimal numbers. Almost 40% of these students could not read or write decimal numbers to hundredths, and approximately 50% could not add 1 tenth or 1 hundredth to a variety of decimal numbers. Without adequate foundational knowledge, students cannot develop the facility for numbers (i.e., number sense) that is the focus of current mathematics syllabi (e.g., NCTM, 2000).

These students' lack of ability to read and write decimal numbers can be explained by the "short-cut" method of reading fractions (e.g., "four point seven" rather than "four and seven tenths") that predominates in Queensland schools. Hence, students are not practised in reading decimal numbers so that the name of the fractional part is heard. This practice is also prevalent in reading common fractions, for example, "two over three" rather than "two thirds". Baturo and Cooper (1999) suggested that this form of syntactic reading in the early years of learning is detrimental to students' ability to process fractions with understanding. Moreover, the poor performances on such a low-level task as Item 3 (61.4%) suggest that the students have come to rely too heavily on the "left to right" syntactic rule when recording numbers (as indicated by their responses to Item 1, namely, 61.5%).

The seriation items evoked low performances, even on the level 1 prototypic tasks (4a–78.1%; 5a–62.9%) suggesting that adding 1 to a decimal place was, in fact, a nonprototypic task for these students. The most common error responses, namely, to add 1 to the ones place indicate that, for these students, seriation tasks had been limited to whole numbers. Table 3 revealed that when the difficulty level was increased, the results decreased. For example, the generally poor results for the Level 2 grouping items (4b, 4d, 5c) indicate that grouping is not a simple process for young students. The results of both of these seriation tasks indicate the fragile nature of students' understanding of place value and seriation. Students cannot seriate without good place value knowledge and are limited if they cannot employ the odometer principle because of lack of understanding of grouping.

With respect to teaching numeration processes, Ross (1990) claimed that children learn to represent numbers with concrete manipulatives (as practised in Queensland schools) through following the teacher's directions rather than from thinking about what they have constructed. The results of this study suggest that teachers need to be more creative in the types and levels of examples they provide to ensure that students have the robustness and flexibility of knowledge that is required for number sense.

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