IMPLICATION AND MATHEMATICAL REASONING

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Implication is Omnipresent as a tool in mathematics. However this concept is neither clear nor easy. In this paper, we present a didactic analysis of implication under three points of view : sets, formal logic, deduction reasoning. For this study, our hypothesis is that most of the difficulties and mistakes, as well in the use of implication as in its understanding, are due to the lack of links in education between those three points of view. Then, we will show, thanks to the analysis of a problem from our experimentations, how the sets point of view can be implied in geometry, even with few knowledge.

INTRODUCTION

The implication seems to be a transverse mathematical object. Although it is in the heart of any mathematical activity, since it is essential for the formulation of proof, it does not have a definite place in French teaching, and is hardly ever taught.

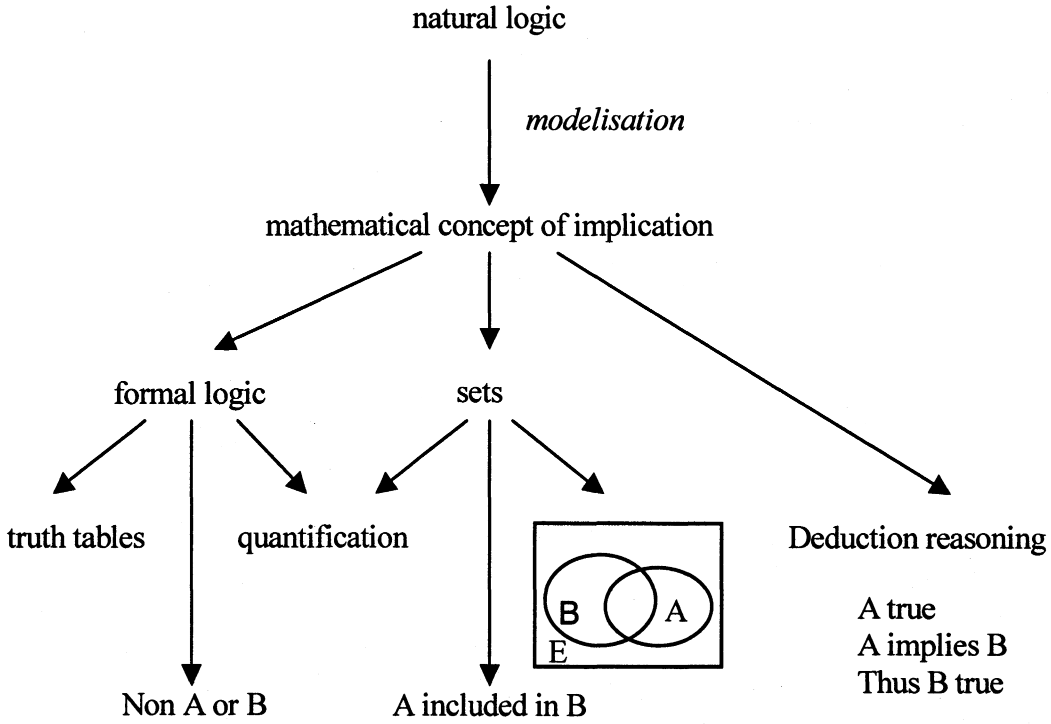
Moreover, the existence of the implication as an object of natural logic, leads to confuse it with the mathematical object. As a result, the implication seems to be a clear object. Yet, students have difficulties related to this concept until the end of university, especially with regard to necessary conditions and sufficient conditions.

Our theorical framework is placed in the theory of french didactics, in particular, we use the tools of Vergnaud's conceptuals fields theory and those of Brousseau's didactical situations theory. Our study is based on the work of V. Durand-Guerrier [Durand-Guerrier, 1999] on the one hand and of J. Rolland [Rolland, 1998] on the other hand. V. Durand-Guerrier shows, in particular, the importance of the contingent statements for the comprehension of the implication. J. Rolland, as for him, was interested in the distinction between sufficient condition and necessary condition.

We will present three points of view on the implication and their place in French teaching. Then we will show, on some examples, the effects of a causal conception of the implication. Lastly, we will study a problem of geometry taken from an activity tested on beginner teachers of mathematics during year 2001.

THREE POINTS OF VIEW ON THE IMPLICATION

The notion of implication does exist in natural logic as it is necessary to our everyday life. The mathematical implication then seems a model of the natural logic implication. Like any model, this mathematical concept is faithful from certain angles to that of natural logic but not from others. This distance between the mathematical concept and the object of our everyday life leads to obstacles in the use of the mathematical concept. An epistemological analysis [Deloustal, 2000] enabled us to distinguish three points of view on the implication : formal logic point of view, deductive reasoning point of view, sets point of view.

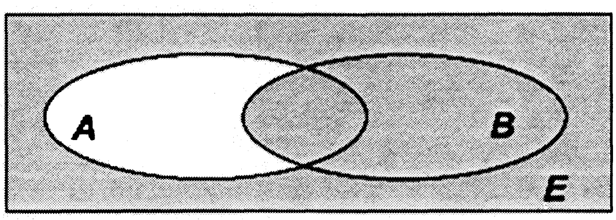


Of course, these three points of view are linked and their intersections are not empty. We will not develop here the formal logic point of view (for example truth tables or formal writing of the implication).

We call "deductive reasoning" the structure of an inference step : "A is true ; A implies B is true ; Thus B is true". Its ternary structure includes a premise "A is true", the reference to an established knowledge "$A \Rightarrow B$" and a conclusion "B is true" [Duval, 1993, p 44]. The reference statement may be a theorem, a property, a definition, etc. One thus builds a chain of inference steps : the proposition obtained as the conclusion of a given step is "recycled" as the entrance proposition of the following step. Therefore, in the deductive reasoning, the implication object is used only as a tool. However, in French secondary education, where this point of view is the only one, it often acts as a definition for the implication.

Generally speaking, having a sets point of view, means to consider that properties define sets of objects : to each property corresponds a set, the set of the objects which satisfy this property.

The sets point of view on the implication can then be expressed as follows : in the set ***E***, if ***A*** and ***B*** are respectively the set of objects satisfying the property A and the set of objects satisfying the property B. Then, the implication of B by A (i.e. $A \Rightarrow B$) is satisfied by all the objects of the set ***E*** excluded those which are in ***A*** without being in ***B***, i.e. by all the objects located in the area shaded below.



OBSERVATIONS IN FRENCH TEACHING

The definitions of the implication or of the associated terms[1] are hardly ever found in in school textbooks. They appear in some first years of university textbooks and in some new highschool textbooks (syllabus 2000).

There is a compartmentalization of these points of view in French school textbooks, no link is established. Whereas the ensemblist point of view is completely missing (only the sentence "A included in B if for every X, $X \in A \Rightarrow X \in B$" can be found in some university textbooks), and whereas the logical point of view appears only in some university textbooks, the deductive reasoning is dominating particularly in the secondary school where it acts as a definition.

Secondary school textbooks do not assume the definition of the implication, which is identified with the natural logic object.

> If... then: "standard" expression which tends to explain that if a property is satisfied, one can deduce from it that a second one is also. [in Mathématiques seconde, collection Pyramide, éd. Hachette éducation, 2000]

> An implication is a mathematical sentence indicating that a data (1) involves or implies a conclusion (2) [in Mathématiques seconde, Déclic, éd. Hachette, 2000]

Many definitions of the implication, within the register of the deductive reasoning, connote an idea of causality and even temporality[2]: "One has Q as soon as one has P" ; "If P is true then Q is true". This causal aspect is stengthened by the definition of the demonstration in school textbooks. Indeed, this one is presented like a succession of sentences, connected by theorems, properties or definitions, leading from the hypothesis to the conclusion.

> To prove that the statement "P implies Q" is true, is to prove that, on the basis of the hypothesis P is true, one reaches, by observing rules of calculation, theorems, definitions, the conclusion Q is true. [in Mathématiques seconde, IREM de Poitiers, éd. Bréal, 2000]

CAUSAL CONCEPTION OF THE IMPLICATION

A conception is "a set of rules, practices, knowledge which make it possible to solve a class of situations and problems in a satisfactory way, whereas there is another class of situations where this conception fails, either that it suggests false answers, or that the results are obtained with difficulty and under adverse conditions." [according to Brousseau, 1997]

We understand by "causal conception of the implication" all the rules, practices and knowledges related to the interpretation of the sentence "A implies B" by "A is the cause of B". This conception of the implication is obviously very close to natural logic and its validity field[3] is wide, it includes, in particular, all usual problems requiring a deductive reasoning. As we showed in the preceding paragraph, this conception is strengthened by teaching practices, but leads to inconsistencies in the use of the mathematical concept. Indeed, from this interpretation, one can easily derive the interpretation "A is the cause of B and only A" then the interpretation "A is the cause of B, A thus precedes B" which we will call "temporal conception of the implication". This last interpretation is reasonable within natural logic since in the physical universe, the cause precedes the effect ! However, it leads to a paradox in the use of the mathematical implication : if $A \Rightarrow B$ is translated by "A is the cause of B and thus A is before B", how to accept that B is a necessary condition for A ?

Here are some examples from our experimentations[4] which illustrate errors that one can explain by the causal conception. They are Sarah's answers, Sarah who studies for the competitive examination to be a teacher in secondary education.

Give the negation of $P \Rightarrow Q$[5]

Sarah: Q can exist without P existing

Are there implications between these expressions "M is a necessary condition for T" and "T is a sufficient condition for M"[6] ?

Sarah: There are no implications between those expressions because T requires M and having T is sufficient for having M. [...] For me "M necessary condition for T", that means that necessarily one must have M to have T. Thus M implies T. In the second one, having T is sufficient for having M, therefore T implies M.

The "causal conception of the implication" model may explain many errors, in particular those due to the implication "which is not in the right way". Experimentations at university showed that, contrary to a widespread idea, a logic lecture is not enough to get rid of this conception and of the errors which result from it.

RESEARCH HYPOTHESIS

The experiments carried out for three years, within the framework of our research, have shown that the implication was not a clear object even for beginner teachers and that the difficulties were primarily due to a causal conception of the implication. Yet this causal conception is not only present in natural logic, but it is also strengthened by French teaching practices. Lastly, this conception may "live" in spite of a logic lecture. Following these comments, we formulate the research hypothesis :

it is necessary to know and establish links between these three points of view on the implication for a good apprehension and a correct use of it.

In the following paragraph we show that a problem of geometry, using only easy properties, may question the reasoning in a non obvious way and allow a work on the implication under the sets point of view.

STUDY OF A QUESTION OF THE GEOMETRY-PROBLEM

Let ABCD be a quadrilateral with two opposite sides with the same length. What conditions must diagonals satisfy to have : two other parallel sides (P1)[7] ?

In an usual French secondary education problem, the implication is generally in the way "A $\Rightarrow$ B" where A and B are known. To solve the problem, one makes then use of a deductive reasoning in which A is considered as true. This implication usually takes place in a specific class of objects, for example quadrilaterals, triangles or parallelograms... One considers, in fact, the implication : "in ***H***, A $\Rightarrow$ B", but ***H*** is implicit when the corresponding class is "institutionalized", that is to say very well known and used, like in teaching. Indeed, if the implication is in the parallelograms class, to solve the problem one uses properties of the parallelograms implicitly (for example, convexity), without expliciting this restriction.

The search for sufficient conditions follows the model "in ***H***, (A?) $\Rightarrow$ B", i.e. what is the property A that is enough for the objects of the class ***H*** to satisfy, if they are to satisfy the property B as well. This is hardly ever practised in French teaching.

Our problem suggests yet another approach. The hypothesis "to have two same-lenghted opposite sides" does not, here, represent a class of "institutionalized" objects. This hypothesis must thus remain explicitly present during the resolution. Since the class ***H*** does not exist as such in teaching, it is necessary to come back to the associated property. There is thus, on the one hand, H which one knows and A that one does not know and, on the other hand, B that one knows. Between them, there is an implication whose direction is not given since we did not specify if the requested conditions were necessary or sufficient.

Let us present, now, three approaches which may induce different solving strategies.

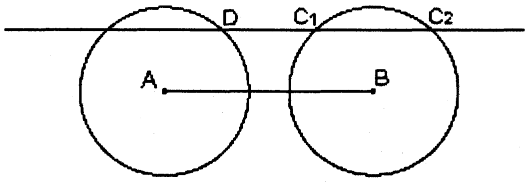
The first approach raises the question of sufficient conditions. One may list conditions on diagonals (same length, perpendicular) and then check if these conditions, added with the hypothesis H, imply the conclusion B. This approach puts back the problem within the deductive point of view. Then, the found conditions are known as sufficient, but this strategy is "expensive".

The second approach refers to known objects. Some quadrilaterals which satisfy both H and B are well known, for example squares, rectangles, parallelograms. Besides, the properties of their diagonals are also well known, and then one can work directly with equivalences. However, if some conditions may be cheaply found, this strategy does not give the exhaustiveness of the results, all the configurations are not a priori reached.

Lastly, the third approach raises the question of necessary conditions. Which objects satisfy both H and B ? Then, what properties have their diagonals ? This approach is basically related to sets point of view. Since the set ***A***, such as the set ***B*** contains the intersection of ***A*** with ***H*** is sought (in terms of properties : A such as (H and A) $\Rightarrow$ B), seeking fisrt the intersection of ***H*** with ***B*** (i.e. the objects satisfying both the properties H and B) seems natural. Then, there are two ways to study those objects which satisfy H and B, either to be in ***H*** and add the property B, or to be in ***B*** and add the property H. The first strategy is closer to the text of the problem but the second one looks easier. It seems, indeed, easier to draw two parallel sides than two same-lengthed opposite sides.

First sets point of view strategy : H then B (H : two equal opposite sides)

Once the points A and B placed in the plane, the hypothesis (H), AD=BC, means that the points C and D belong to two same-rayed circles respectively, one centred on B, the other centred on A. Once D placed, the property (B) "two other sides parallel", means that C is the intersection of the straight line parallel with (AB) containing D with the centred on B circle. There are two intersection points C1 and C2.



Two configurations are thus obtained : isosceles trapezium (ABC1D) and parallelogram (ABC2D) [fig.1]. But one must not forget that, once A fixed, one may still change the distance AB, the ray of the circles and the position of D (linked to that of C) on its circle. So, when the two circles intersect, there is a new configuration : a cross quadrilateral called CQ1 (ABC1D) [fig.2]

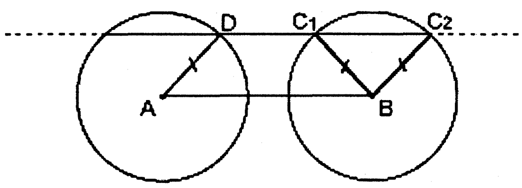


Fig.1

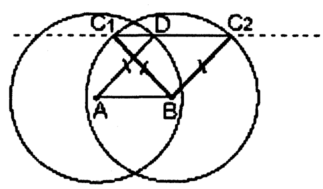


Fig.2

So there is the implication : (H and B) $\Rightarrow$ (parallelogram or isosceles trapezium or cross quadrilateral CQ1). We thus know the configurations which satisfy both H and B, it remains then to find the conditions on the diagonals.

However, for a quadrilateral, being a parallelogram is equivalent, to having diagonals which cross in their middle. This property is well-known by French pupils. Isosceles trapeziums and cross quadrilaterals CQ1 have same-lengthed diagonals. Now remains to see whether "to have same-lengthed diagonals" (A1) is a sufficient condition, i.e. if the implication, within the quadrilaterals, (H) and (A1) $\Rightarrow$ (isosceles trapezium or cross quadrilateral (CQ1)) is true.

For that, the sets point of view is necessary again, we have to study the quadrilaterals which satisfy (H) and (A1).We will not detail the rest of the solving, but let us say that these two conditions bring obviously the isosceles trapezium and the cross quadrilateral CQ1 but also a cross quadrilateral CQ2 (cross quadrilateral linked to parallelogram) which does not satisfy the conclusion (B). The condition "having same-lengthed diagonals " is thus not sufficient and will have to be restricted to exclude CQ2. The final solving of this exercise is not the subject of this article, but we wanted to show how this problem, with easy objects, can question the implication.

Second sets point of view strategy : B then H (B: two parallel opposite sides)

To express (B), one draws two parallel straight lines, one including A and B, the other including C and D. A, B, D being fixed, there are two points C so that AD=BC (ie so that (H) is satisfied) : C1 and C2. Two configurations are then obtained : parallelogram (ABC1D) and isosceles trapezium (ABC2D) [fig.3]. But,again, A being fixed, B and D can move on their line. Thus, when [BC] crosses [AD], a new configuration is obtained : a cross quadrilateral CQ1 (ABC2D) [fig.4]. The rest of the solving is the same as in the previous strategy.

Fig.3

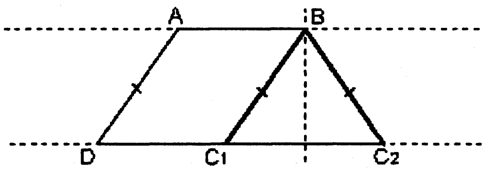
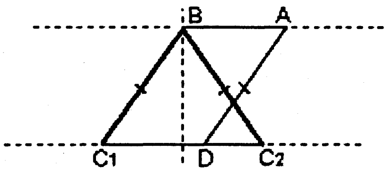


Fig.4



This second strategy, also based on the sets point of view, is not as far-reaching as the preceding one. Indeed, proving that the conditions are sufficient needs to come back to the hypothesis (H) and thus needs to use the previous strategy.

CONCLUSION

The analysis of the students' answers of the whole geometry problem is still in progress. However, we can already say that, although students first found this exercise very easy, its solving required a very long research in groups. The answers are incomplete and, in the end, the students declared this exercise complicated.

Robert: "It is an exercise which as a teacher, I would not give before university"

The exercise fulfiled its role, as for the work on the implication, since discussions about necessary and sufficient conditions took place in the groups in an explicit or implicit way. In addition, the exercise also fulfiled its role, at least partly, as for the work on the sets point of view. In particular, to have the exhaustiveness of the results, the groups, which had drawn apart the cross quadrilaterals, had to take them again into account.

These results are to be placed among others. Indeed, this problem of geometry forms part of a six hour experimentation including other stages of work, in particular, studies, in groups, of written proofs and of a problem of discrete mathematics. Moreover, this experimentation takes sense when one knows that it was preceded by two others, carried out in 1999 and 2000.This problem of geometry is, thus, to consider as part of a broader context. Now, remains to finish the analysis of these results and to connect them together, this is our goal for next year .

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1 We take into account the following expressions : P implies Q ; P brings to Q ; P thus Q ; Q is a consequence of P ; if P then Q ; P $\Rightarrow$ Q ; P is a sufficient condition for Q ; Q is a necessary condition for P.

2 This will be detailed in the following paragraph.

3 We call validity domain of a conception the goup of the situations wich may be correctly solved with the practicies associated to this conception.

4 These experimentations were carried out in june 1999 with four mathematics students, [Deloustal, 2000].

5 This expression is equivalent, in mathematics, to the expression " P and Non Q".

6 These two expressions are equivalent, in mathematics, to the expression : "T implies M"

7 There were two other questions : two 90 degrees angle (P2) ? ; two other same lengthed sides (P3) ?

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