UNDERGRADUATE STUDENTS' VERIFICATION STRATEGIES OF SOLUTIONS TO COMBINATORIAL PROBLEMS

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This study is part of a larger study focusing on one of the difficulties in solving combinatorial problems, namely, on the verification of a solution. Our study aimed at identifying verification strategies students employ when solving combinatorial problems, and at evaluating their level of efficiency in terms of contribution to reaching a correct solution. Forteen undergraduate students participated in this part of the study, of which 8 were interviewed individually and 6 in pairs. During the interviews the participants were asked to solve 10 combinatorial problems, thinking aloud, and to try to verify their solutions. Five verification strategies were identified, two of which were rather frequent and more helpful than others. The most frequently used strategy was the least helpful.

BACKGROUND

Combinatorics is one of the important areas of discrete mathematics, which is "an active branch of contemporary mathematics that is widely used in business and industry" (NCTM, 2000, p. 31). It is significant to people's everyday experience as well as their professional practice, and is connected to various strands of mathematics and other disciplines, (e.g., computer science, communication, genetics, and statistics).

Combinatorics is a topic that can, and furthermore should, be an integral part of the mathematics curriculum, from early elementary grades through senior high school (English, 1993; NCTM, 2000). First, because the fundamental principles of Combinatorics are rather easy to comprehend, and rest on very little factual knowledge and memorization. Secondly, many combinatorial problems can be solved intuitively, yet yield themselves to the development of systematic exploration, argumentation and proof (Maher and Martino, 1996). Third, combinatorics provides a rich source for challenging mathematics. Most combinatorial problems echo the spirit of the NCTM (2000) and qualify as "good problems" (ibid, p. 52), as they have the potential of integrating multiple topics, involving significant mathematics, making connections to the learner's personal experiences, fostering the use of different solution strategies, linking various representations, and forming different levels of difficulty.

In spite of the above, Combinartorics is considered one of the more difficult mathematical topics to teach and to learn. Most problems do not have readily solution strategies, and create much uncertainty regarding how to approach them and what strategy to employ. Moreover, verifying an answer to a combinatorial problem is a particularly difficult task, because there are no guaranteed ways to ensure the detection of an error. There are numerous examples in which different solutions to the same problem, resulting in different answers may seem equally convincing. In addition, the detection of an error does not necessarily yield a correct the solution. Several studies support the assertion that students encounter many difficulties in solving combinatorial problems and shed light on some factors contributing to these difficulties (Batanero et al, 1997a, 1997b; English, 1991; Fischbein & Gazit, 1988; Kahneman & Tversky, 1973). Although verification plays a critical role in problem solving (Polya, 1957; Schoenfeld, 1984; Silver, 1987), we have not found any studies that have investigated the ways in which students and teachers cope with the difficulties associated with the verification of combinatorial problems. Our study is a step towards understanding students' approaches to verifying solutions to combinatorial problems.

THE STUDY

Goals: The main goal of the study reported hereof was to identify and characterize the ways in which students deal with the need to verify their solutions to combinatorial problems and the verification strategies they employ.

Participants: The participants in the study consisted of 14 undergraduate students all of which had completed a basic course in combinatorics prior to the study.

Method: In order to be able to follow, describe, and understand the problem solving processes of the participants a qualitative method was employed. Data collection was done by audio taped interviews and field noted observations.

Borrowing from Schoenfeld (1985), the participants were encouraged to work in pairs when engaged in the problem solving tasks. Those who felt comfortable to work with a peer were grouped in pairs. Thus, there were 3 pairs of students, each of which worked collaboratively on the combinatorial problems they received. The remaining 8 students worked individually and were prompted by the researcher to think aloud. Altogether, 11 interviews were conducted with the students. All interviews were transcribed and coded.

Every problem, for each individual or pair, at each stage of the interview, was coded according to a number of parameters, including its degree of correctness (correct, partially correct, or incorrect), the number of attempts to verify it, and when applicable, the verification strategies employed. For most of the analysis we combined the partially correct and the incorrect solutions in one category named "Incorrect Solutions".

Research Instruments: The interviews were semi structured and consisted of two parts: In the first part of the interview participants were asked questions related to their personal background, as well as their experience with and attitudes towards combinatorics. The second and main part of the interview focused on problem solving and verification of solutions to ten combinatorial problems.

The main interview included a set of 10 combinatorial problems specially designed for the study. The problems varied with respect to their underlying model, according to Dubois (1984): a selection model, a distribution model, or a partition model. All problems required only basic combinatorial tools for solution. However, in order to create some uncertainty that would foster the need to verify the solutions, each problem required the use of some blend of principles and operations (combinations, arrangements, and permutations) for its solution. In addition, 2 of the 10 problems were very similar problems, and were given to the participants consecutively. This was done in order to see whether the participants, first, identified the connections between the two problems and secondly, built their solution and verification of the second problem on the first one.

For each problem there were 1-2 occasions when the interviewer prompted the student (or pair of students) to verify the solution. Accordingly, for each problem the interview consisted of 2-3 stages: the first stage V_1, in which the student(s) solved the problem with no prompting at all, the second stage V_2, in which they were prompted to verify their solution, independently of its correctness, and the third stage V_3, in which only those that held an incorrect solution were prompted to verify their solution once again, disclosing the fact that their solution was incorrect.

FINDINGS

Through the 11 interviews that were conducted with the students (3 with pairs of students and 8 with individual students) we received altogether the total of 108 solutions.

The findings include a description of the different types of verification strategies employed by the participants and an analysis of their usefulness.

The Verification Strategies

Altogether, 219 attempts were made by students to verify their solutions at Stages V_1, V_2, and/or V_3. All attempts to verify a solution were analyzed and classified into 5 main categories of verification strategies:

1. Verification by *reexamining* the solution;
2. Verification by *adding justifications* to the solution;
3. Verification by *evaluating the reasonability* of the answer;
4. Verification by *modifying some components* of the solution;
5. Verification by *using a different solution method.*

We turn to a description of the verification strategies.

<u>Strategy 1</u>: Verification by *reexamining* the solution

The participants who used this strategy reexamined their solution by going over and checking all or parts of it a second time, without adding any substantial justifications to their solution. This kind of checking focused on various aspects of the solution, such as rechecking their calculations or the extent to which their original plan for solution was carried out. In several cases, this strategy served as a springboard for a more profound strategy of verification.

<u>Strategy 2</u>: Verification by *adding justifications* to the solution

The participants who used this strategy added justifications to their solution to support it. The justifications referred to either a particular step in the solution or to a more global aspect of the solution. Generally, the justifications were of three mutually related types:

One type was directed to clarify and support some (or all) specific parts of the solution. A second type aimed at justifying in a more global way the model (or formula) used to solve the problem, by showing how the conditions and nature of the problem match the model. A third type of justification was based on an analogy to another, more familiar or previously solved problem, the solution of which was known to the participant. This strategy was particularly helpful for improving partially correct solutions, in which the general solution strategy was appropriate, however, there were some steps in which an error occurred in applying it.

Strategy 3: Verification by *evaluating the reasonability* of the answer

The participants who used this strategy looked at the final result they had obtained and tried to examine its reasonability either by an intuitive estimate, or more commonly, by calculating the size of the outcome-space. In a number of cases, the participant noticed that the result s/he had reached was larger than the outcome-space, which did not make sense. In some cases this strategy led to the identification of wrong answers, however, it was not helpful in locating the specific erroneous considerations and steps in an incorrect solution.

Strategy 4: Verification by *modifying some components* of the solution

The participants who used this strategy made one of the following modifications to their original solution: They either altered the representation they had used in their solution or tried to apply the same solution method by using smaller numbers. Those who used the former approach tried to represent the situation of the problem in a different way, mostly by using some visual symbols (e.g., circles, squares, blocks etc.) to represent the different components of the problem. Unfortunately, this kind of attempt did not prove helpful for them, because they repeated the same considerations and arguments as in their original solution, failing to identify any faulty step. One student used the latter approach, that is, used smaller numbers. This enabled him to detect an error and identify where he went wrong in his solution.

Strategy 5: Verification by *using a different solution method*

The participants who used this strategy employed a completely different solution method for the problem. In these cases, the new solution method led to either the same result that they had reached or to a different one. Table 1 presents the different cases according to the correctness of the first and second solution methods, the difference between the results obtained in each way, and consequently, the decisions the participants made within this strategy of verification.

As seen in Table 1, Strategy 5 was used altogether in 57 cases, of which, 43 cases were with an incorrect solution, and 14 cases with a correct solution. Of the 43 cases with an incorrect solution, in 12 cases this strategy proved helpful in correcting their solutions. In one case a student managed to reach a correct solution in the second time, however, made a wrong choice, thus remained with his first incorrect solution. Interestingly, there were two pairs of students and one individual student who reached the same incorrect result in two different solution methods, thus their answers remained incorrect leaving them more confident of their solutions.

First Method	**Second Method**	**Comparison of Results**	**Choice of Solution**	**Individual Students** (8)	**Pairs of Students** (3 pairs)	**Total**
Incorrect (N=43)	Correct (N=13)	Different	Second	11	1	12
			First	1	-	1
	Incorrect (N=30)	Same	Same	1	2	3
		Different	Second	15	2	17
			First	9	-	9
			None	-	1	1
Correct (N=14)	Correct (N=12)	Same	Same	3	9	12
	Incorrect (N=2)	Different	Second	2	-	2

Table 1: Decisions made by participants employing verification Strategy 5

Of the 14 cases with a correct solution, 12 cases remained correct in both solution methods. However, this strategy led to two unfortunate decisions (in stage V_2), where a wrong result was obtained the second time, causing a switch from a correct to an incorrect solution. Luckily, in the following stage (V_3) they both corrected their solution.

Some Comparisons between the Verification Strategies

Altogether, there were 219 verification attempts made, in various stages, to incorrect as well as correct solutions. One hundred and sixty six attempts were made by students working individually, and 53 by students working in pairs. Table 2 presents the distribution of the number of verification attempts by the different verification strategies. It should be noted that the distribution of the verification attempts of only the incorrect solutions was similar to the one in Table 2.

It should be noted that there were no conspicuous differences between the students who worked individually and those who worked in pairs, in the distribution of use of the different verification strategies. The most frequently used strategies were Strategy 1 (38%), Strategy 2 (26%), and Strategy 5 (26%).

Types of Verification Strategies	Individual Students (8)	Pairs of Students (3 pairs)	Total
1.Reexamining the solution	66	17	83
2.Adding justification to the solution	42	15	57
3.Evaluating the reasonability of the answer	10	1	11
4.Modifying some components of the solution	6	5	11
5.Using a different solution method	42	15	57
Total	166	53	219

Table 2: Distribution of attempts to verify a solution by the type of verification strategy employed, in percents

A further analysis of the use of the different verification strategies focused on the extent to which these strategies were helpful in leading the participants to improving their solutions. In this analysis we considered an improvement as a change from an incorrect solution to a partially or completely correct solution, or from a partially correct to a completely correct solution. Altogether, there were 27 improved solutions as a result of employing a verification strategy (note that some improvements are attributed to more than one verification strategy).

Table 3 presents the distribution of the improved cases by the different verification strategy and by group. As shown in Table 3, the most helpful verification strategies were the 2nd (in 12 cases) and the 5th (in 13 cases). Note, that there were 5 cases in which a combination of verification strategies led to an improved solution. For example, one student used Strategy 3 to detect that an error had been made, then moved to Strategy 4 through which he identified the kind of error that he had made, and finally used Strategy 5 to correct his solution.

Types of Verification Strategies	Individual Students (8)	Pairs of Students (3 pairs)	Total
1. Reexamining the solution	2	0	2
2. Adding justification to the solution	8	4	12
3. Evaluating the reasonability of the answer	3	0	3
4. Modifying some components of the solution	1	0	1
5. Using a different solution method	11	2	13
Total	25	6	31

Table 3: Distribution of cases in which the use of a verification strategy was helpful in the process of improving a solution, by type of verification strategy and group

A close look at the number of cases, for which the use of a verification strategy was helpful, points to a considerable difference in favor of the individual students compared to the pairs. However, this is due to the fact that the students who worked in pairs were significantly better in reaching correct solutions to begin with.

Another measure of efficiency of the verification of a solution was obtained, for each stage separately, by the percentage of the number of cases in which the use of a verification strategy was helpful out of the total number of incorrect solutions that were verified at the respective stage. Thus, 13% of the incorrect solutions in stage V_1 were improved, 12% of the incorrect solutions in stage V_2 were improved and 30% of the incorrect solutions in stage V_3 were improved. Clearly, there was an increase in the efficiency of the verification attempts in stage V_3, when the students became aware that their solution was incorrect.

DISCUSSION

Our findings support the merit of encouraging students to verify their solutions to combinatorial problems. Apparently, when pushed to it, students are capable, to a certain extent, of finding efficient ways to verify their solutions. We suggest that through such experiences students are likely to become aware of the potential of verifying their solutions, and hopefully, will be motivated to verify their solutions out of their own initiative. The differences that were found with respect to the efficiency of the various strategies may serve as a springboard for teaching problem solving in combinatorics with a focus on metacognitive processes, in general, and on the use of more efficient verification strategies, in particular.

As shown above, Strategy 1 was the most frequently used, however, it turned out one of the least efficient strategies, in terms of helping students shift towards an improved solution. The low efficiency level of Strategy 1 is in accordance with the view that merely going over a solution of a mathematical problem is insufficient for verifying it (Polya, 1957; Schoenfeld, 1985). For those who applied Strategy 2, which turned out considerably helpful, what proved useful was mainly the need to add justifications to the various steps in a solution. This was helpful particularly in detecting minor errors. Unlike Strategy 2, Strategy 3 was not very frequent. This is mainly because estimating an expected outcome in a combinatorial problem is extremely hard to do. Fischbein and Grossman (1997) found that when asked to estimate such results, students usually gave lower estimates than the actual number. Thus, Strategy 3 was helpful in detecting that an error had occurred only in cases when the answer that was obtained was larger than the size of the outcome-space and when the student(s) applying this strategy compared these two numbers. Strategy 4 was also the least frequent. We speculate that most of the students were not familiar with this strategy and did not know how to apply it. In another part of our study, in which expert mathematicians were interviewed about useful verification strategies to combinatorial problems, they recommended Strategy 4 as one of the most helpful methods, provided the solver does it carefully and is aware of some underlying subtleties of using smaller numbers. Finally, we turn to Strategy 5, that was both frequent and rather helpful, and point to its limitation. There were three cases in which students reached the same incorrect answer in two different solutions strategies. This raised their degree of confidence in their incorrect solution.

To conclude, we bring a quote from a student, conveying the limitation of Strategy 5: A pair of students who reached stage V_3 was told that their solution was incorrect. In response, one of them suggested solving the problem in a different way. Before they started to carry out his plan, he turned to his partner and said: "if we reach the same result in a different way then the answer is right and the question is wrong".

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