DESCRIBING YOUNG CHILDREN'S DEDUCTIVE REASONING

David A Reid

Acadia University

This paper reports results related to the development of a consistent descriptive language for research on mathematical reasoning. Ways of reasoning deductively are highlighted, using examples drawn from observations of young students. One-step deductions versus multi-step deductions, known versus hypothetical premises, and single versus multiple premises, are used to distinguish different ways of reasoning.

This paper reports results related to the development of a consistent descriptive language for research on mathematical reasoning. These results arose out of a long term research project (the PRISM project [1]) aimed at elaborating and clarifying previous models and terminology for describing reasoning. The model now in use for this research project describes reasoning across five dimensions: need, target, kind of reasoning, formulation and formality; and has been used to describe reasoning of students of all ages (Reid 1995a,b, 1997, 1998, in press). In this paper one dimension, ways of reasoning, will be highlighted, using results drawn from observations of students aged about seven years.

THE MODEL

The PRISM project took as its beginning point a model for reasoning outlined by Reid (1995a, 1996b). It includes four dimensions for describing reasoning. *Need* includes the needs to explain and to verify mathematical statements and to explore to discover new statements. This dimension of the model was inspired by the work of Bell (1976) and de Villiers (1990). *Kind of reasoning* includes reasoning deductively, inductively and by analogy, and was inspired by the work of Polya (1954/1990). *Formulation* refers to the degree of awareness the reasoner has of their own reasoning. *Formality* refers to the degree to which the expression of the reasoning conforms to the requirements of mathematical style. The work of Lakatos (1978) and Blum & Kirsch (1991) inspired this dimension. One of the refinements of this model that has resulted from the PRISM project is the addition of a fifth dimension, *target*, that describe who the reasoning is for: a teacher, a peer or oneself.

This model of reasoning is compatible with those being developed by others. For example, Sowder and Harel (1998) have outlined a model describing what they call "proof schemes". In terms of the PRISM model it offers additional detail concerning kinds of reasoning but is limited to those kinds of reasoning related specifically to the need to verify.

THE CONTEXT

The episodes of mathematical activity that will be described here were recorded as grade two students (aged seven or eight years) worked in small groups at their classroom mathematics centre, one of five learning centres that the students moved through on a weekly rotation. The regular classroom teacher supervised the other centres while another teacher, working as a research assistant in the class, supervised and interacted with the students at the mathematics centre. Mathematics centre activities included playing games, reading and discussing stories, and engaging in geometric activities with pattern blocks and geoboards.

The research assistant recorded the activities at the mathematics centre on both video and audio tape, and produced summaries of the students' mathematical activity based on these recordings and her own observations. The author re-viewed the tapes, discussed the students' mathematical activity with the research assistant and occasionally visited the classroom as a participant/observer.

The results below evolved through an enactivist research methodology (Reid, 1996a). In keeping with the enactivist position that all learning is structure determined, the learning of individual researchers concerning the data is acknowledged as being determined by the structure of the researchers. At the same time, the researchers' structures co-emerge with their environments, which include physical artefacts (e.g., video tapes) that constrain the researchers' learnings, and other human beings whose structures impose additional constraints, if the researchers wish to maintain communication with them. Enactivist research methodology seeks to build on this co-emergence, through the mechanism of multiple perspectives.

Perspectives can be multiple in several ways. Several observers will inevitably observe from their own perspectives, each of which has something to offer. One researcher re-viewing a video tape on several occasions does so with different perspectives each time. Data represented on video tape, audio tape or in transcript affords further perspectives. All of these are ways of obtaining additional perspectives can be involved in enactivist research. In the case of the results reported here, multiple perspectives arose from the individual structures of the research assistant and the author, from multiple viewing of tapes, and from re-presentations of the data as video tape, audio tape and written summaries.

RESULTS

One of the aims of the PRISM project is to explore how the model must be adjusted when used to describe the reasoning of students of different ages. In the case of students in the first years of schooling, the dimensions of need and target require no adjustment. The dimensions of formulation and formality are not of much interest for this age as all the reasoning observed was unformulated and hence informal. It was found to be necessary to refine the description of deductive reasoning, in order to capture any hint of such reasoning present in the students' mathematical activity. For this reason the results described here focus on describing this dimension.

To describe deductive reasoning precisely it was found to be necessary to distinguish between one-step deductions and multi-step deductions. In addition the number and nature of the premises of the deduction were found to be useful for distinguishing kinds of deductive reasoning. The interaction of these two criteria and the descriptive categories arising from them is summarised in Table 1.

Premise(s)	One-step	Multi-step
One	Specialisation	[Not observed]
Two or more	Simple one-step deductive reasoning	Simple multi-step deductive reasoning
Two or more, at least one Hypothetical	Hypothetical one-step deductive reasoning	Hypothetical multi-step deductive reasoning

Table 1: Kinds of deductive reasoning observed

Specialisation

The most common type of deductive reasoning observed in the grade two classroom was *specialisation.* Specialisation is determining something about a specific situation by applying a general rule that applies to it. Specialisations can involve simple or complex general rules. Several specialisations from a general rule involving multiple attributes occurred when the children were playing Set. The game of Set involves determining if three cards satisfy the conditions that define a "Set." In the standard game those conditions are that the three cards be identical, or all different, in each of four attributes: colour, number, shading, and shape. In this case the children played a variation in which the striped shading was removed from the deck, and the open shading was treated as a fourth colour: white. The teacher defined a Set as three cards that were different colours, shapes, and numbers.

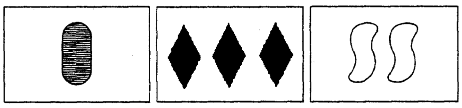

Figure 1: 1 red oval, 3 green diamonds and 2 white squiggles make a Set. (In this game variant a set is defined as three cards of different colour, shape and number.)

After playing for a while, the teacher asked the children to explain why 1 red oval, 3 green diamonds and 2 white squiggles are a Set (See Figure 1). Cynthia replied, "They're all...different. They're all different shapes. They're all different colours." Alison added, "They're all different numbers." And Jared summed up, "Different colour, number and shape." Jared states the general rule that they are using to identify Sets, which he is applying to this specific case.

Simple deductive reasoning

Simple deductive reasoning is deducing a conclusion from two or more established premises. It can be one-step or multi-step, of which one-step is more common.

When a grade two student makes a simple one-step deduction it is not likely to be clearly stated. Following the reasoning can be difficult. For example, consider this statement made by Maurice when playing the game Mastermind with the teacher (See Figure 2).

"It's blue. Cause if there's three there. I changed the blue and I only got two."

					Score
Guess 1	blue	orange	yellow	green	3 white pegs
Guess 2	green	brown	yellow	orange	2 white pegs
Target (Hidden)	green	red	blue	yellow	

Figure 2: The Mastermind board as it appeared when Maurice made his simple one-step deduction. (The object is to guess the colours and order in a four colour pattern picked by one's opponent. A white scoring peg indicates that one of the pegs in the guess is the right colour but in the wrong place.)

The teacher had asked Maurice if he knew anything new after receiving the two white pegs for his second guess. Maurice's response can be re-expressed as:

It's blue. Cause…	Blue is correct because:
if there's three there	1. Three of the colours in my first guess are correct
I changed the blue	2. And the only relevant change I made in the colours from my first guess to my second guess was leaving out blue
and I only got two	3. And only two colours in my second guess are correct

He has taken three statements about the situation and concluded a fourth statement that follows logically from them.

While simple one-step deductions are the building blocks of proving, they need to be assembled into chains to make a proof. Reasoning with chains of deductions is called simple multi-step deductive reasoning. It is difficult to observe in early elementary classrooms, both because it is relatively rare and because young students rarely articulate their reasoning. It is sometimes possible, however, to conclude that multi-step deductive reasoning has taken place by observing the conclusions students come to and the information they had to work with.

For example, when Maurice was playing Tic Tac Drop, a computer game, he made the comment, "If he don't put it next to me, I won," immediately after placing his first marker (Marked as O1 in Figure 3).

1	2	3	4	5	6	7
	X4	**O3**	**O1**	**O5**	**X2**	

Figure 3: Tic Tac Drop. The board after Maurice's game with the computer. The object of the game is to get three of one's markers in a row. Markers can only be added directly above the other markers in a column. O represents Maurice's markers. The numbers indicate the order in which they were placed.

On a previous day Maurice had described a general rule he used when playing a related game, Connect Four, at home. That rule was: *If you have three markers in a row with both ends free, then you can win.* It depends on the winning condition of Connect Four, which is getting four in a row. Because the winning condition for Tic Tac Drop is different, Maurice's general rule for Connect Four is not directly applicable, so he cannot be specialising from it. While we cannot know for sure how Maurice came to his conclusion that he would win if the computer did not place a marker next to his, it seems plausible that he deduced this new general rule either from the strategies he already knew from playing Connect Four or from analysing the new situation in terms of general features (e.g., two in a row with free ends). In both cases he would have had to make a chain of deductions to get from what he knew to his conclusion. As it happened, the computer played its marker in column six, and Maurice won.

Because of the emphasis on arithmetic in early elementary mathematics, the context in which students are most likely to evidence simple multi-step deductive reasoning is in the course of solving problems involving arithmetic. The following example occurred when the teacher posed the following problem to Maurice and Saul:

> First there were 8 cookies. The children got four each. Then some more children came in. Then they got 2 each. How many people came in?

Both Saul and Maurice reread the question several times, which was written on a piece of paper. They tried to solve it with paper and pencil only. Saul circled the number eight in the question. Another boy, Ira, interrupted and stated, "Very easy!" but Saul disagreed, "No! Eight take away two is six. — First there were eight cookies. Then the children got four each. Oh! Let me see. First there were 8 cookies... — Four people!"

The teacher asked Saul, "Did four people come in? How many people were there at the beginning?"

"There were two."

"You think there were two people in the beginning and then four people came in?"

"No, then two more people came in. That made four. Because eight, — let me think about that again. — Um, this is hard. — I think there were four or six people."

The teacher asked Saul how he was going to figure out if there were four or six people, "Are there any parts that you can figure out?"

Saul drew a diagram on the bottom of the page showing eight cookies, in two rows of four. He circled the two groups of four. Then he drew a line through each group of four, creating four groups of 2. "There were four people in there." He said, and then he reread the question again silently.

Maurice agreed that there were four people. The teacher asked once again how many people came in. Maurice replied, "Two."

To solve this problem Saul and Maurice had to first determine the number of children originally present, then the number of children present after the new ones arrived, and then the difference between the two numbers. Each one of these requires a one-step deduction. It is the linking of them together that makes this an example of multi-step deductive reasoning.

Hypothetical deductive reasoning

Thus far the deductions we have seen involve reasoning from something that is known. In mathematics proofs however it is often necessary to reason from a hypothesis, something that is not known to be the case, either to show that it cannot be the case (as in a proof by contradiction) or to show that if it were the case for one number it would also be true for the next number (as in a proof by mathematical induction). Such reasoning, because it involves a hypothesis, is called hypothetical deductive reasoning. Although hypothetical deductive reasoning is often thought to be more difficult than simple deduction from known statements, it can be observed in the reasoning of early elementary school students, in both one-step and multi-step forms.

An example of a hypothetical multi-step deduction occurred during a game of Mastermind (See Figure 4). After giving Kyla two white pegs for her third guess, the teacher asked her which one she thought might have been in the right place. Kyla pointed to the blue in the first row and then changed her mind. "I never got a black one right there [pointing to the blue in the second turn]." She then indicated that green could not be correct either in the first try. "Cause on this one [turn three] I didn't get a black." After that Kyla stated that orange on turn one must be in the correct spot but then realised it can not be. "Cause I got a black one right here — no! Oh my! It's yellow." Kyla's reasoning includes three hypotheses: That blue is in position three, that green is in position four, and that orange is in position two. Having arrived at contractions from each of these hypotheses in turn she concludes that the one remaining case (yellow in position one) must be correct.

					Score
Guess 1	yellow	orange	blue	green	1 white peg, 1 black peg
Guess 2	brown	green	blue	orange	2 white pegs
Guess 3	blue	orange	yellow	green	2 white pegs
Target (Hidden)	yellow	red	orange	brown	

Figure 4: The Mastermind board after Kyla's third guess. (The object is to guess the colours and order in a four colour pattern picked by one's opponent. A white scoring peg indicates that one of the colours in the guess is the right colour but in the wrong place. A black peg indicates one of the colours is in the right place.)

SUMMARY

Describing the reasoning of young children is difficult. Quite often they do not articulate their thinking clearly, and they tend to make use of many implicit assumptions (Anderson, Chinn, Chang, Waggoner, & Yi, 1997). This makes it all the more important for researchers investigating children's reasoning to be precise in their descriptions of that reasoning. For example, while of the reasoning described here is deductive, but there is a significant difference in sophistication between a specialisation and a multi-step hypothetical deduction. Omitting to note that difference would be a serious weakness in a description of a child's mathematical activity. The distinctions I have made in this paper, along with the terminology I and others have outlined elsewhere, should help provide researchers with the tools they need to observe and describe children's reasoning more precisely.

NOTES

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