CONCEPT DEFINITION, CONCEPT IMAGE, AND THE NOTION OF INFINITE SUM IN OLD AND NEW ENVIRONMENTS

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The literature is rich in studies on the conceptual difficulties with the notion of limit of a sequence of numbers and infinite sum of numbers. In this study we analyzed the students' perceptions of the infinite sum of functions. Two different approaches were used: analytical and algebraic. In the first, the infinite sum is represented as a process and in the second, as an object. Mathematica software permits multiple representations of the same concept. The students used animation to focus on the "process itself" and not on the divergent process of adding terms to the sequence. Symbolic manipulations were utilized to create the illusion of an infinite number of calculations performed at once. The infinite sum as an object appeared clearer. However, not all the conceptual difficulties were resolved.

Students' conceptual difficulties in learning the notion of infinite sum are well documented. A number of double strands are observed when students try to understand the concept of infinite sum: the infinite sum as a *process* or as an *object*, the intuition of the infinite process as a *potentially infinite* process or as an *actual infinite* sum, and the reading of the equality $S = a_0 + a_1 + ... + a_n + ...$ from left to right or from right to left, which is cognitively different.

Herein, we describe a study that examined the effect of introducing the concept of infinite sum of functions using two different approaches in which those double strands are present. The research was conducted in the context of an experimental course on Approximation theory that was given to eleventh grade high school students. The concept of infinite sum was introduced by developing a function in infinite series. In this way an infinite decimal is obtained by substituting a particular value of x. Mathematica software was used. In particular, we utilized the software's ability to support multiple representations of the same mathematical concept.

Multiple representations were applied in two interpretations:

In the first, two different mathematical approaches were used, analytical and algebraic, the first representing the infinite sum as a *process,* a *potentially infinite* process, and the second as an *object,* an *actual infinite* sum. In the second interpretation, each of these two approaches to the infinite sum was investigated by multiple attributes of the Computer Algebra System: symbolic computation, graphics, and animation.

The use of animation in teaching the limit concept is discussed in Kidron et al. (2001). Animation gives the illusion of completing an *ongoing* process.

This article describes some efforts to partially overcome the difficulties that were pointed out by Núñez (1994). Núñez explained some difficulties concerning the concept of infinite sum by the fact that two types of iterations, of perhaps a different nature, are involved simultaneously: *the process itself* and *the divergent process of adding terms to the sequence* (an increasing number of steps).

We utilized Mathematica's capabilities in a way that reduced the two-dimensional analysis, which stems from the two types of iterations, to a one-dimensional analysis.

In the analytical approach, in which the infinite sum is represented as a *potentially infinite* process, the use of animation might help in focusing on *the process itself* and in differentiating it from *the divergent process of adding terms to the sequence.* In the algebraic approach, in which the actual infinite sum is represented as an object, the software creates the illusion of symbolic manipulation taking place on the actual infinity of terms. The illusion that the infinite calculations are performed at once might facilitate the transition from symbolic manipulation to a symbolic object.

Our study examined to what extent the interrelationship between the two approaches and the different attributes of the Computer Algebra System helped the students to perceive the infinite sum as a limit of the infinite process. We also investigated the readiness of the students to grasp the formal definition of infinite sum.

CONCEPTUAL DIFFICULTIES WITH INFINITE SUMS

The "finitist" character of our intellectual schemes might cause difficulties when we deal with the notion of Infinite Sum. Fischbein et al. (1979) observed that the natural concept of Infinity is in fact the concept of "Potential Infinity", which is simply a process that goes on without end, like counting without stopping. Lakoff & Núñez (2000) suggested that metaphorical thought might be necessary to conceptualize another infinity concept: the "Actual Infinity".

The intuition of infinity might become an obstacle to learning the formal definition of the concepts related to infinity (Cornu, 1991). Vinner & Hershkowitz (1980) introduced the terms "concept image" and "concept definition". The term "concept image" describes the *total cognitive structure that is associated with the concept which includes all the mental pictures and associated properties and processes.* The term " concept definition" is defined *as a form of words used to specify that concept* (Tall & Vinner, 1981). The concept definition of Infinite Sum is not necessarily linked to the concept image. There may be a gap between the mathematical definition of the concept and the way one perceives it.

Symbolic notation is another source of conceptual difficulty. The symbol $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ represents both the process of n tending to infinity and the concept of limit. It is a *procept* (Gray & Tall, 1994). Another source of cognitive complexity is embedded in the symbolic notation "=" in $S = a_0 + a_1 + ... + a_n + ...$. An understanding of what "=" means requires a cognitive analysis of the mathematical ideas involved (Tall, 2000).

STUDENTS' PERCEPTIONS OF INFINITE DECIMALS IN PREVIOUS STUDIES

The following perceptions of infinite decimals were observed:

Infinite sums exist only in theory (Kidron, 1984). *The infinite decimal is perceived as one of its finite approximations* (Kidron & Vinner, 1983). *The infinite decimal is perceived as a process not as a product* (Kidron & Vinner, 1983; Monaghan, 1986). The students also expressed a generic concept of *measuring infinity:* 1+1/2+1/4+1/8+...is stated to be $s = 2 - 1/\infty$ because "there is no end to the sum of segments" (Fischbein et al.,1979). In Kidron (1984), the students in comparing $0.\dot{9}$ and 1, claimed that $0.\dot{0}1$ is a legitimate number that is equal to $1\text{-}0.\dot{9}$. The existence of a conflict in comparing $0.\dot{9}$ and 1 and the path dependence of the decisions are mentioned by Tall (1976).

The following perceptions of infinite decimals were observed in some computer environments: Monaghan et al. (1994) pointed out difficulties that result from attempts to use the computer for learning the concept of limits. Nevertheless, the limit sum as an object appeared clearer. Sacristan (2001) presented some results demonstrating that in exploring different types of representations, including visual ones, the students gradually understand how a process can continue infinitely and not grow to be infinite. Tall (2000) indicated that reading the equality $a_0 + a_1 + ... + a_n + ... = S$ from left to right or from right to left can relate to ideas that are cognitively different. For example, most of the students regarded 0.1+0.01+0.001+...= 1/9 to be false but 1/9 = 0.1+0.01+0.001+... to be true. The students were interviewed and as a possible explanation, Tall suggested that when reading from left to right, the first statement seems to represent a potentially infinite process that can never be completed, whereas the second shows how 1/9 can be divided out to get as many terms as are required. Tall also noted that the students were reluctant to accept a formal definition of infinite sum that does not agree with their personal experience.

THE TEACHING EXPERIMENT

High school students learning at the highest level (grade 11, N = 63) participated in the research. The laboratory consisted of 20 PCs, each equipped with Mathematica software. The author taught the students six hours a week; two of the six hours in the PC lab were devoted to topics in Approximation theory.

The concept of infinite sum was introduced from two different points of view. The algebraic approach represents the polynomials with "an infinite number of terms" as an *object*. The analytical approach describes the *process* of the different polynomials approaching a given function. We pointed out the importance of interacting with each of the different representations. Moreover, we emphasized the importance of establishing links between the different types of representations. The **algebraic approach** is Euler's intuitive idea of expressing non polynomial functions as "infinite polynomials"; it relates to the concept of Infinity as Actual

Infinity. Euler's approach is from left to right: you start from a given function f(x) = P(x)/Q(x) and you seek a "polynomial with an infinite number of terms" such that $f(x) = P(x)/Q(x) = a_0 + a_1x + a_2x^2 + ...$ The students used Mathematica to *follow the original text of Euler* (translated into English in Euler, 1988). Following Euler's instructions, the students carried out the "continued division procedure": they assumed that there is an infinite series such that P(x)/Q(x) = $a_0 + a_1x + a_2x^2 + ...$ and then calculated the coefficients $a_0, a_1, a_2, ...$that satisfy the equality by multiplying both sides of the equality by Q(x) and comparing the coefficients as Euler did.

Using the **analytical approach** the students plotted two functions, f(x) and g(x), whose formulas were different but whose plots were similar, near x = 0. The notion of the order of contact was introduced. The two curves, y = f(x) and y = g(x), have an order of contact n at x = 0 if: f(0) = g(0) , f '(0) = g'(0), f''(0) = g''(0), $..., f^{(n)}(0) = g^{(n)}(0)$. As an application, the students were asked to find the polynomials of degree 2, 3, 4, ... that have the highest possible order of contact with a given function at x = 0. Mathematica helped the students solve the relevant system of equations. Using the analytical approach, the students calculated polynomials with a *finite number of terms,* the Taylor polynomials, which approach a given function. Acting with this representation, they observed that when the degree n of the approximating polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ increases, the approximation is better. The students made graphical representations of their results. By means of animation (Kidron, 2000, example 1) they were asked to "encapsulate" the *process* into an *object* (Dubinsky, 1991). The animation permitted them to see the dynamic process in one picture. Animating the approximating polynomials $a_0 + a_1x,\ a_0 + a_1x + a_2x^2, .., a_0 + a_1x + a_2x^2 + ... + a_nx^n, ..$ approaching the given function f(x), may help in identifying the equality $a_0 + a_1x + a_2x^2 + ... + a_nx^n + ... = f(x)$. In both representations the students performed algebraic manipulations of the two sides of the equality in order to find the unknowns $a_0, a_1, a_2, ...$which are the coefficients of the Taylor series.

DATA ANALYSIS AND RESEARCH METHODOLOGY

Each year, one class (grade 11) participated in the experimental course throughout the academic year. The course was given three times. All together, 3 grade 11 classes participated in the experiment. We noted the students' questions and remarks during the sessions, and collected the Mathematica files of their examples. We gave the students a questionnaire that was designed to elicit their conceptions of infinite sum. The same questionnaire was given each year. In the second and third years, a full session was dedicated to the formal definition of the infinite sum of functions as the limit of an infinite sequence of partial sums. We present here a class discussion and some findings from the questionnaire in different years.

The class discussion When we replace a given function f(x) by its Taylor polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, we just leave off the "infinite tail". Some students had problems with this "infinite tail".

Dina: How could we speak about a graph that describes the error $|f(x) - P_n(x)|$? This difference is the "infinite tail": $a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + ...$ How could this difference be well defined?

A similar reaction is shown in the following class discussion. The teacher proved Taylor's theorem at x = 0. In one step of the proof, in order to calculate the derivative of a certain term, she mentioned that the error term $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$ is a constant.

Julia: Why is d a constant?

Teacher: d is the difference between two constants when we approximate f(x) with a polynomial with a given exponent n at a given point.

Julia: But d is the "infinite tail"... how could it be a constant?

Tomer: We compute the error for a given n.

Julia: But $d = a_{n+1}x_0^{n+1} + a_{n+2}x_0^{n+2} + ...$ so how could it be that d is a constant?

Tomer: $d = a_{n+1}x_0^{n+1} + a_{n+2}x_0^{n+2} + ...$ is an infinite sum that is equal to a given number.

Ron: Yes! For example, you have 1/2+1/4+1/8+1/16+1/32+...which equals 1.

Julia: Will it not be bigger than 1 when we continue to add terms?

Dan: In the example 1/2+1/4+1/8+1/16+1/32+...the infinite sum is a defined number but there are other examples in which the infinite sum is not a given number. It tends to ∞!

Adi: So, how could we know if $d = a_{n+1}x_0^{n+1} + a_{n+2}x_0^{n+2} + ...$ is an infinite sum which is equal to a given number?

Yifat: In the last lab we have seen animations showing that when $n \to \infty$, the expression $f(x) - (a_0 + a_1x + a_2x^2 + ... + a_nx^n)$ tends to 0. Therefore the expression $d = a_{n+1}x_0^{n+1} + a_{n+2}x_0^{n+2} + ...$ is a given number and not an expression which tends to ∞.

The questionnaire We report on responses to three of the questions.

Question 1 deals with the meaning of the statement:

$1/(1-x) = 1 + x + x^2 + x^3 + ... + x^k + ...$ for $-1 < x < 1$.

We observed three categories of perceptions of the infinite sum of functions:

I The infinite number of terms is perceived as a very large but finite number of terms: **the infinite sum is perceived as a finite approximation**.

II **The infinite sum is perceived as a process and not as a product**; it is perceived as an ongoing process that passes through an infinite number of terms: $1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3, ...$

III **The infinite sum is perceived as a limit**, as the product of the infinite process.

	First year N=22	Second year N=22	Third year N=19
Category I	14%	9%	10.5%
Category II	46%	45.5%	26.5%
Category III	36%	45.5%	47.5%

Table 1: Distribution of the categories of perceptions in question 1

Question 2 Complete: $1+1/2+1/4+1/8+.....+1/2^K+...=$

Our aim was to check the students' perception of the infinite sum of numbers as a limit.

	First year	Second year	Third year
$1+1/2+1/4+1/8+.....+1/2^K+...=2$	50%	57%	61%

Table 2: Distribution of answers to question 2

In the first year, no one linked question 2 to the expansion of 1/(1-x) in the power series in question 1. In the second year, 33% of the 57% linked question 2 to the expansion 1/(1-x) = $1+x+x^2+x^3+.....$ (in the third year, 36% of the 61%). These students indicated that they could look at the process from the opposite side, from right to left, $1+x+x^2+x^3+.....=$ 1/(1-x) and then substitute x= ½. This ability did not eliminate the students' difficulties in perceiving the infinite process as an object:

Daniel: As a consequence of the expression 1/(1-x) = $1+x+x^2+x^3+.....$, $1+1/2+1/4+1/8+.....+1/2^K+...$ is supposed to be equal to 2 if K = ∞, but since there is no such thing 'infinite' the expansion tends to 2.

Question 3 Find: $\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)$. Is $0.\dot{9}$ equal to 1 or less than 1? Justify your answer.

	First year	Second year	Third year
$\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)=1$ and $0.\dot{9}=1$	14%	27%	32%
$\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)=1$ and $0.\dot{9}<1$	57%	41%	21%
$\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)\to 1$ and $0.\dot{9}<1$	29%	32%	47%

Table 3: Distribution of answers to question 3

Some perceptions already seen in previous studies reappear:

Orit: The lim tends to 1, but will not reach it since infinite exists only in theory

Aviv: 0.999…is less than 1 because it always lacks 0.000….01 (an infinite number of 0) in order to be 1.

The legitimate $0.\dot{0}1$ re-emerges!

DISCUSSION AND CONCLUSIONS

The class discussion indicates that a conflict exists. The teacher presented the error term as the difference of two constants $d = f(x_0) - (a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n)$. Julia looked at the representation of d as an infinite sum $d = a_{n+1}\, x_0^{n+1} + ...$ She realized that the difference between two constants is a constant and yet she was not ready to accept the infinite tail as a constant. Julia looked at the infinite sum as an unending process. In the class discussion, the dynamic images produced by animation were present in the students' minds even when the computer was turned off. The animation enabled them to concentrate on *the process itself* (the error term tends to zero) and to differentiate it from *the divergent process of adding terms to the sequence*. It helped the students understand that an infinite sum is not necessarily an expression that tends to ∞, and that it could be equal to a given number. One of the purposes of the study was to investigate the students' ability to perceive the infinite sum as a limit, as an object. In our research we encountered some perceptions of the infinite sum that were observed in previous studies. However, there was also an indication that some students were beginning to see the infinite sum as an object (43% of all the students expressed their perception of the infinite sum of functions as a limit; 56% of all the students wrote $1 + 1/2 + 1/4 + 1/8 + ... + 1/2^K + ... = 2$). A possible explanation for this is that the students, looking at the same concept from different approaches, developed a more balanced view of the infinite sum as process and concept. We quote also Euler's reason for applying an algebraic approach to subjects that are usually discussed in analysis: *in order that the transition from finite analysis to analysis of the infinite might be rendered easier.* We also wanted to investigate the students' ability to grasp the formal definition of an infinite sum. The percentages of answers where students perceived the notion of the infinite sum as a limit, as an object, were slightly higher in the second and third years (these students were given the formal definition of an infinite sum). It seemed that the influence of providing the formal definition was minor. However, this impression changed examining the students' explanations to their answer $1 + 1/2 + 1/4 + ... + 1/2^K + ... = 2$. Some of the students who received the formal definition were able to link between this question and the first question, where $1/(1-x) = 1 + x + x^2 + x^3 + ... + x^k + ...$ for $-1 < x < 1$. In order to do so, they had to look at the equality from the other side (from right to left) and to replace ½. Yet, Daniel's answer to question 2 illustrates the fact that the flexibility to read the equality from both sides does not eliminate the conceptual difficulties in *the transition from finite analysis to analysis of the Infinite*.

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