# MECHANICAL LINKAGES AS BRIDGES TO DEDUCTIVE REASONING: A COMPARISON OF TWO ENVIRONMENTS

Jill Vincent, Helen Chick and Barry McCrae

The University of Melbourne

*This paper reports part of a study investigating the use of mechanical linkages as contexts for establishing a classroom culture of conjecturing and proof in geometry at Year 8 level. The focus is on a comparison of students' use of a physical model and a Cabri Geometry model of a linkage, examining in particular how the students exploited the features of Cabri to assist them in producing conjectures and proofs.*

The lack of success with traditional methods of teaching geometric proof has prompted researchers to seek alternative approaches, involving new and old technology. Recent research on proof has focused on the role of dynamic geometry software, for example Cabri Geometry II™, as an environment for geometry learning, with debate as to whether these environments assist or, rather, hinder the development of deductive reasoning. Hoyles (1998), for example, expresses concern that unless we can develop a 'need for proof', dynamic geometry software may merely contribute to a 'data' gathering, empirical approach to geometry.

Bartolini Bussi (1998) reports on the use of historic drawing instruments to create an environment where students can 'relive' the work of mathematicians in theorem production. She describes the conjecturing and proof construction processes of a group of five year 11 students who investigated the geometry of Sylvester's pantograph, working with a physical model.

Bartolini Bussi notes that "producing the conjecture was difficult and slow" (p. 742), and that the students' written proof was incomplete and not in a logical sequence, but when they refined it with the teacher's help, it remained meaningful for them:

> The order of the steps recalls the sequence of production of statements, as observed during the small group work, rather than the logical chain that could have been used by an expert. Nevertheless it was easily transformed later with the teacher's help into the accepted format ... yet, what is important, the time given to laboriously produce their own proof ensured that the final product in the mathematician's style, where the genesis of the proof was eventually hidden, retained meaning for the students. (p. 743)

What is it, then, about an environment that assists students' conjecturing and proof?

## THE STUDY

The research presented here forms part of a study on mechanical linkages, using both physical and computer models, as an environment for Year 8 students to conjecture, argue, and construct proofs (see Vincent & McCrae, 2001). The linkages were carefully selected on the basis of appropriate geometry for Year 8. This report focuses on a comparison of students' use of a physical model and a Cabri Geometry focuses on a comparison of students' use of a physical model and a Cabri Geometry model of a linkage, examining in particular how the students exploited the features of Cabri to assist them in producing relevant conjectures.

The participants, who had no previous formal exposure to deductive reasoning, were from an extension Year 8 Mathematics class at a private girls' school in Melbourne. Prior to the research the following geometry was taught/revised: angles in parallel lines, triangles and quadrilaterals; Pythagoras' theorem; and similar and congruent triangles. Tchebycheff's linkage for approximate linear motion (Vincent & McCrae, 2001) was used to introduce the students to the need for proof by showing them that, although the midpoint of one of the links appeared to move on a linear path, closer investigation showed that the path was in fact not exactly linear. For each of the linkages studied, the students worked in pairs, constructing their linkages from plastic geostrips and paper fasteners, as well as having access to a Cabri model of the linkage which had been previously prepared by the teacher-researcher. The students were video-taped during their conjecturing, argumentation, and proof construction. At the conclusion of each task, the students were asked which model they considered most useful in helping them in their conjecturing and proof construction, and which one they most enjoyed working with.

**Sylvester's pantograph: Case studies of two pairs of students**

Before investigating Sylvester's pantograph, the students had completed pencil and paper proofs for the angle sum of triangles and parallelogram properties, and had worked with at least three other linkages. The students constructed the rhombus version of the pantograph, using the diagram shown in Figure 1, where OA = OC = AB = BC = AP = CP' and $\angle$PAB = $\angle$P'CB. They explored the linkage (Figure 2) for approximately 10 minutes before being given access to the Cabri model. They were then free to choose the model with which they preferred to work. Each pair of students worked for two 50-minute lessons, conjecturing, arguing, and constructing their proofs, with the teacher-researcher (TR) occasionally intervening.

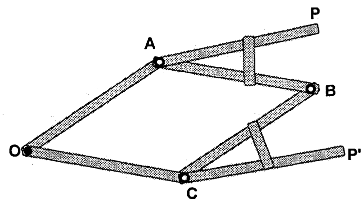



**Figure 1.**

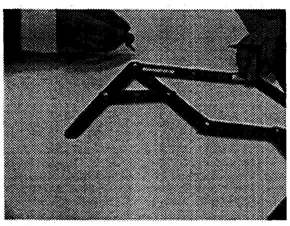


**Figure 2.**

**Students Ce and Ch**

Ce and Ch drew a shape on paper and traced over the shape with point P of their geostrip pantograph so that P' traced an image of their shape (Figure 3).

Ce: It's the same but it's not in the same direction. It's been turned.

Ch: [Spreads her thumb and fingers to compare the size] It's the same size. It's turned 45 degrees.

TR: How do you know it's 45 degrees?

Ch: Just guessing, 'cause it was about half 90.

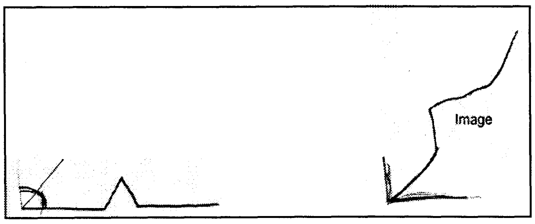



**Figure 3.**

Ce and Ch were confident that the image was the same size as the original, but they were less confident of their conjecture about the angle of rotation, particularly as there seemed no obvious reason why it should be 45 degrees. When given the Cabri model, their initial reaction was to use the same approach as for the geostrip model, but they were not sure how to proceed. With *Trace* on P', Ch carefully dragged P in the shape of a square so that P' traced the same shape. She then switched on *Trace* for P and tried to drag P so that P' moved around its previous path. However, she had trouble controlling the mouse to follow the small shape and began to move P haphazardly, before dragging it horizontally across the screen (Figure 4), when she suddenly noticed that the paths of P and P' were converging:

Ch: [Excitedly] Goody goody goody … we've got 'em to meet! And then look …

Ce: Oh, we've got the angle there …

Ch: We'll make a segment. And from there down to about there. [Draws segments over traces of P and P' — see Figure 5] Measure the angle … 30.1. [Measures angles PAB and P'CB] 30! 30 and 30. So it rotates the angle of that [PAB or P'CB]!

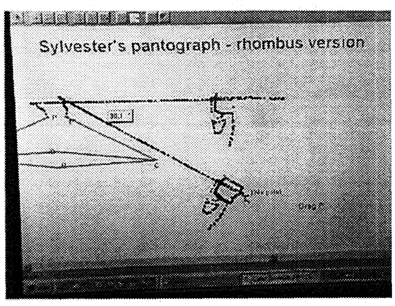
Ce: Yes!

Ch: They're the same distance [indicates from P to O and from P' to O] … maybe if we draw segments. [Ch draws segments OP and OP' — see Figure 6]

Ch: Let's measure that angle … [ POP'] … it looks like the same angle again.

Ce: 30! So that [OP'] is always turned around 30 degrees from that [OP].

Ch and Ce recognised that their method of drawing segments over the converging traces (see Figure 5) involved some error, so they seemed unconcerned by the discrepancy between the measured angles, 30° and 30.1°; the size of angle POP' seemed to carry greater significance. Ch and Ce were now confident about the rotation of the image: "The image rotates [through] the angle of BAP".



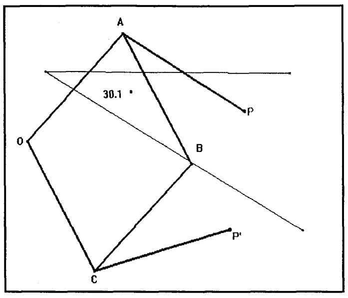




**Figure 4.** **Figure 5.**

Ce and Ch were now able to explain why their drawing and its image were the same size, and were able to write the geometric proof (Figure 7):

Ch: They're congruent triangles [OAP and OCP'].

Ce: Yeah, 'cause the sides are the same [OA, OC; AP, CP'] and so are those angles [OAP and OAP'].

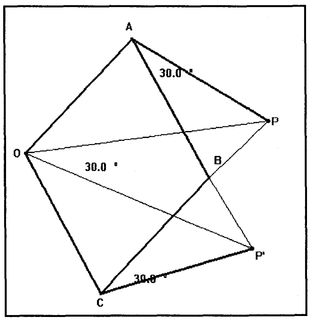
TR: Why are those angles equal?

Ch: The [opposite] angles in the rhombus are equal and then they both have the same fixed angle bit added on.

Ce: So therefore OP and OP' are equal.

TR: And what does that tell you?

Ch: That's why the copy is the same size.




**Prove:** OP = OP'

**Proof:** $\angle OAB = \angle OCB$ (rhombus)

$\angle PAB = \angle P'CB$ (Given)

$\therefore \angle PAO = \angle P'CO$

$\therefore \triangle P'OC \cong \triangle POA$ (SAS)

$\therefore OP' = OP$

**Figure 6.** **Figure 7.**

Their next task was to prove their conjecture that the angle of rotation POP' was equal to the fixed angles PAB and P'CB. Ch drew segments from P to B, B to P' and P to P' on the worksheet diagram (see Figure 8).

Ce: Well … could we make a triangle there? [points to O and from P to P']

Ch: But B's not in line … see …why don't we use the angles in the rhombus?

Ce: OK, let's put some letters in. We'll call this [PAB] *a* and these [BCP' and POP'] *a*.

TR: We don't know yet …that's what you're trying to prove.

Ce: Oh, yeah, we'll call it [POP'] *b*, and we'll call this [OAB] *c* and this [OCB] is *c* too.

Ch: And call those *d* [points to AOP and COP'].

Ce: And those [APO and CP'O] are *d* as well.

Ch: Oh, yeah … isosceles triangles.

Ce: And this [ABC] is *e*. OK … *e* equals 2*d* plus *b*.

Ch: *d* plus *d* plus *a* plus *c* equals 180 'cause that's a triangle

Ce: And 2*d* plus *b* plus *c* equals 180 'cause angles in the rhombus.

Ch: So *b* must equal *a*!

Ce: Yeah, that's it!

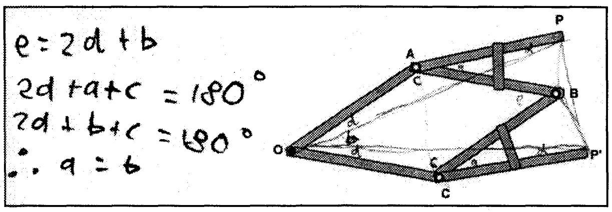



**Figure 8.**

Both students then wrote correct proofs; Figure 9 shows the proof written by Ch.

Prove: ∠POP' = ∠BCP' = ∠PAB

Proof: ∠COP' = ∠CP'O (isosceles triangle)

= ∠AOP = ∠OPA (congruent triangles) = d

let angle OCB = c

d+d+c+a = 180° (triangle)

∠AOC + ∠OCB = 180° (rhombus), let angle POP' = b

∴ d+d+c+b = 180°

∴ b = a

∴ ∠POP' = ∠BCP' = ∠PAB

**Figure 9.**

## Students Lu and Li

Like students Ch and Ce, Lu and Li were reasonably confident that the geostrip pantograph was producing a congruent image but they too were less certain about the angle of rotation and unsure how to determine the angle with the geostrip model:

Lu: Well, it creates a mirror image … well, not exactly a mirror image but …

Li: … if it was more accurate.

Lu: A congruent shape of the shape and it goes in a circle. If you kept on following the trace it would go round and round [points to rotation of the linkage about O]

Li: Maybe that angle [points to the fixed angle, PAB] … I'm not sure … maybe not …

Lu: When you move it [points to P] up at the beginning it [points to P'] moves around and then it just makes the rest of the shape.

Li: I was thinking that many degrees [points to angle PAB] …but I don't know why.

At this point the girls were given the Cabri pantograph. They used the same approach in Cabri as they had done with the geostrip model, drawing a triangle and moving P around it so that P' traced the image (Figure 10). Because of the transitory nature of Cabri traces, Li then drew a triangle over the image trace.

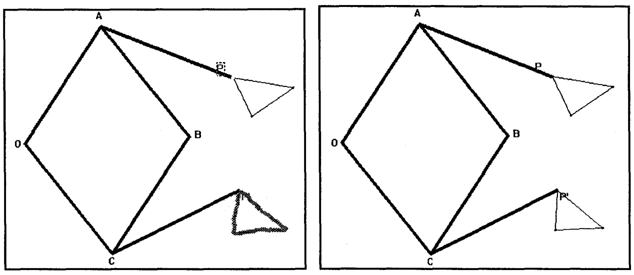



**Figure 10.**

Lu measured angles PAB and P'CB as 30 degrees (Figure 11), then carefully dragged the original triangle, placing it over the image in order to measure the angle between the two triangles. Seeing the linkage as an accurate geometric diagram seemed to encourage the girls to add construction lines and to notice congruent angles.

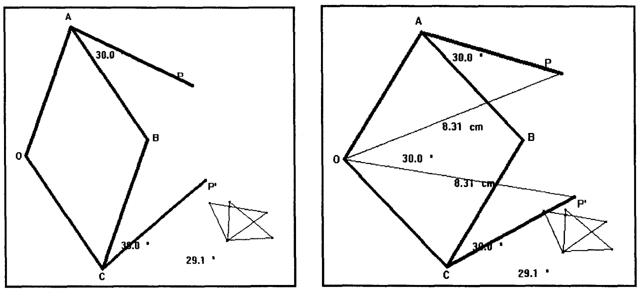



**Figure 11.**

Having drawn and measured segments OP and OP', and measured angle POP', the girls now understood how the pantograph was working:

Lu: 29.1 … it's point 9 off.

TR: So what is your conjecture?

Lu: The angle which the copy of the shape rotates is that angle of the pantograph [PAB].

Lu and Li then began writing their proofs, with Li demonstrating that she had a clear understanding of the side-angle-side condition for triangle congruency. The girls discussed and sorted their statements into a logical sequence as they wrote, for example:

Li: Let's do the sides first. OA equals AP equals OC equals CP' … then angle OCP' equals … OAP because they both have 30 degrees … they share 30 degrees … we shouldn't do that yet [erases] … angle OA … angle OAB equals …

Lu: Angle OCB.

Li: And angle BCP' equals BAP because given … OCB plus BCP' …

Lu: Those two added together, that whole angle … that means …

Li: Once we've proved that angle, then the whole thing's easy 'cause side angle side … see if you have two sides and how big it's going to be in between … when you join them up the triangles will be the same …

## DISCUSSION

In their responses to the linkage questionnaire which they completed after working with the pantograph, all the students agreed that operating models of the linkage made the geometric properties more obvious. Even though three of the four girls had enjoyed working with the geostrip model more than with the Cabri model, they all believed the Cabri model to be more useful than the geostrip model for finding out why the linkage worked. Significantly, perhaps, Ch — who had become quite excited by the converging traces of P and P' (see Figure 4) — indicated that she had enjoyed the Cabri model more than the geostrip model.

The students' use of Cabri was by no means restricted to dragging. The function of the pantograph led naturally to the use of the Cabri *Trace* facility, and transformation of the screen construction indicated where additional construction lines, for example, OP and OP', might be useful. In other linkages explored by the students, tabulation of angle measurements was also used. The students' prior familiarity with the features of Cabri was therefore an important aspect of their successful use of the Cabri linkage in helping them to produce their conjectures and construct proofs.

The ease with which they could trace the paths of points, add construction lines, and accurately measure angles and distances meant that the students made little attempt to return to the geostrip model after they had been given the Cabri model. Once the students were confident of their conjectures, they tended to work with paper and pencil diagrams during the proof construction. This was, however, not always the case; the students worked with a number of linkages during the classroom research and with some of these they would sometimes go back to the physical linkage to check an observation they had made with the Cabri model, or return to the Cabri model while they were constructing their proof.

The students in the study described by Bartolini Bussi (1998) were required to produce their own geometric drawing of the linkage from the model and to provide a description from which another person could construct it. This resulted in the need to determine the structure of the linkage, for example, by measuring the lengths of the linkage bars. This was, of course, an important aspect of Bartolini Bussi's project, which aimed to make these historic drawing instruments 'transparent' to students. The Cabri model, on the other hand, provided a ready-made geometric figure, which allowed the students to focus more immediately on the functional relationships within the linkage, rather than its basic structural properties. This may have contributed to the ease with which these Year 8 students were able to present their statements in a logical order, compared with the initial difficulty experienced by the Year 11 students in the study described by Bartolini Bussi (1998).

It would seem, then, that the imagery, both static and dynamic, of the Cabri environment made a substantial contribution to the conjecturing process, but also challenged these students to produce an explanation. The features of dynamic geometry software — constructions based on Euclidean geometry, accurate measurements, the tracing of loci and the drag facility — make the software highly suitable for exploring the geometry of linkages. Rather than eliminating the need for proof, the convincing evidence and the unique opportunities for exploration and discovery which the software provided gave the students the confidence and desire to go ahead and prove their conjectures. However, the tactile experience and satisfaction of working with actual physical linkages may also represent a significant motivational aspect, at least for some students, and should not be over-looked.

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