AN ACTIVITY FOR CONSTRUCTING A DEFINITION

Cécile Ouvrier-Buffet

Laboratoire Leibniz – IMAG – Grenoble (France)

The notion of definition is central in mathematics. We notice differences and analogies between axiomatic definitions in education (which often come at the beginning of a lesson) and definitions in mathematical research (which come generally at the end of a research process). The core of this paper concerns the activity for constructing a definition (called definition-construction). We aim to study a situation of definition-construction and to bring up conclusions about the nature and functions of definitions (constructed by students).

THEORETICAL FRAMEWORK AND EXISTING RESEARCH

In the usual mathematical activity of a researcher, a dialectic exists between the concept in construction as well as its definition which is constructed, too. According to Kahane (1999, p11):

> la transposition didactique est de règle en sciences: (…) l'exposé d'un sujet prend pour point de départ un aboutissement historique et réécrit l'histoire à l'envers. En mathématique, ce point de départ est une définition. [1]

Besides, Lakatos recommends a heuristic approach, in opposition to the usual deductivist one and underlines the dialectic between the construction of definition and the construction of concept within the framework of problems resolution. He develops two concepts in his thesis (1961): *zero-definitions* and *proof-generated definitions*. Thereby two functions of definition appear in a problem-situation: on one hand, *zero-definitions*, so called alluding to their place at the start of the investigation, are initial and tentative definitions and "the different choices of zero-definitions do not affect the domain of the proof" (p.71,ibid). They must be a little vague, and their heuristic rules correspond to Popper's remark [2]. This notion of *zero-definition* does not thwart Vygotsky's idea about language and verbal definitions; indeed Vygotsky (1962) studies the capacity to use language as a problem-solving tool and accounts for the importance of the naming process: the word guides and determines the course of action. Furthermore, Vergnaud (1991) describes three functions of language: communication, representation and contribution to the conceptualisation. On the other hand, Lakatos underlines the importance of the relation between proof and definition, and presents a new concept: the proof-generated concepts (i.e. generated by proof: it consists in establishing the domain of validity of a primitive conjecture) and their definitions, the *proof-generated definitions*. According to Pimm (1993), this notion

> ... seems particularly problematic in terms of teaching mathematics, because of needing to perceive the definition as a tool custom-made to do a particular job that cannot be known by those trying to learn it, certainly not with an order of presentation that seems to require definitions to come first (p.272).

We keep this idea of concept generated by proof as a possible situation for constructing a definition. We know that a lot of difficulties exist in teaching about understanding a concept with a formal definition. Annie and John Selden [3] focus their attention on the role of examining examples and non-examples in order to help students to understand definitions and ask how to help them to understand newly-defined concepts. Furthermore, we retain Vinner's hypothesis (1991,p79), who notes that "the ability to construct a formal definition is for us a possible indication of deep understanding" and explains the "scaffolding metaphor" which presents the role of a definition as a moment of concept formation. Vinner assumes that "to acquire a concept means to form a *concept image* for it (...) but the moment the image is formed, the definition becomes dispensable" (p.69,ibid) and proposes some interplay between definition and image. We suppose that *concept image* and *concept definition* are necessary to analyze an activity of *definition-construction*. Moreover, we share Vergnaud's idea (1991,p.135): "un concept ne peut être réduit à sa définition; c'est à travers des situations et des problèmes à résoudre qu'un concept acquiert du sens" [4]

An activity for constructing a definition

For the purpose of this article, to construct a definition is an activity which could concern three types of problems. P1 = the request of a definition (starting from given examples and counterexamples); P2 = a problem-situation whose resolution passes by the construction of an object (or a concept) and its definition (alluded to by Lakatos); P3= a situation of modelling. We will retain several aspects concerning the definition of a mathematical object : characterization of this object, naming process, relations between definition and proof.

About the naming process, two aspects emerge : the importance of the denomination (when the mathematical concept is of interest in that it can be used usefully) and, on the other hand, a denomination allows two mathematicians to speak about the same thing. And the expression "good definition" is frequently used, it means "precise definition" according to the mathematical accuracy and the arbitrary character of definitions, as they are presented in axiomatic form.

To consider an activity of *definition-construction* requires a change of point of view (relatively to handbooks) which consists in accepting the provisional status of a definition, the multiplicity of the definitions of the same object (thus equivalent definitions), the dialectic between definition and properties and the operational aspect of a definition in a proof. We will explain our point of view about *definition-construction* and possible analysis with the presentation of an activity (type P1).

PRESENTATION OF THE ACTIVITY

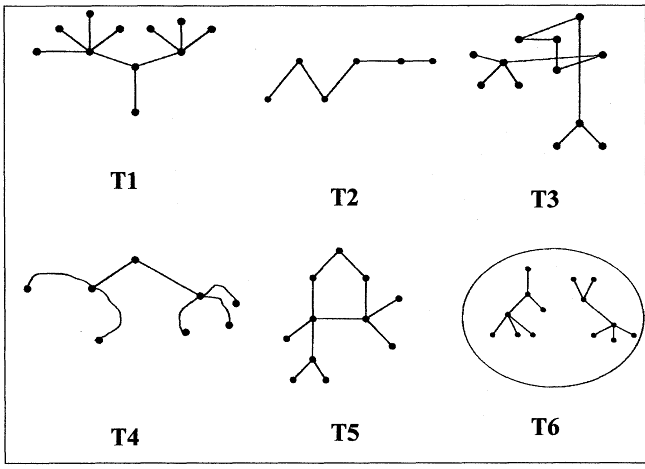
For the activity of *definition-construction* the mathematical object that was selected was the tree, for several reasons: in France, it is a familiar object in teaching, used in the handbooks as a tool of representation (it is recommended by the official secondary syllabus), however it is absent as a mathematical object, hence there is no institutionalised definition (before University). An experimentation by Balmand (2001) proved that definitions and properties of this mathematical object are unknown to French teachers (discrete mathematics are not learnt in France before University). However, the tree is a "natural tool" of representation and resolution, which can be used in restricted fields (combinatory, probabilities).

From a mathematical point of view, it is an accessible object (by its representations) but hard to theorize and to define owing to the difficult concept of connectivity. Let us notice that the students did know neither the word 'connected' nor the concept 'connectivity' before our activity. Moreover, the tree has equivalent definitions which are different in nature. We don't claim to construct all the aspects of the mathematical concept 'tree' but some of them, and we assume that the construction of a mathematical concept is required for knowing and mastering this concept.

We chose to call it "thingummy" (a neutral name, whose semantic meaning is attached to nothing) to avoid students connecting too quickly with the meaning of tree as it is used in probability (otherwise we assume that it would stall the situation). Etymologically, to define means to delimit (one defines an object compared to another in order to find out a criteria of recognition) and we believe the construction of a definition is possible starting from examples and counterexamples.

Presentation

First, 4 examples and 2 counterexamples were proposed to students, with the question



1- How could you define the mathematical object 'thingummy', knowing that :

T1,T2,T3 and T4

are representations of 'thingummy' and T5,T6 are not representations of it ?

Figure 1 : examples and counterexamples of "thingummy" (first question)

We think that a definition is not a finished product in itself, so we proposed a second question (when the students think they have done with the definition): "**2-** <u>Exercise</u> : Let G be a graph (i.e. a collection of dots and lines between two dots) connected (i.e. in only one piece). Prove that G admits a spanning tree (i.e. a tree with same vertex set than G)" [5]

Analysis

There are a lot of mathematical definitions of "tree": let G be a graph on n vertices. Then G is a tree if and only if one of the following equivalent assertion holds:

Definitions	Nature of the definition
Def1- G is connected without cycle.	Perceptive
Def2- Between any two vertices of G, there exists a unique path.	Perceptive
Def3- G has no cycle and n-1 edges	combinative (counting)
Def4- G is a connected graph with n-1 edges	combinative (counting)
Def5- G has no cycle and if we add a new edge then we create a unique cycle (which means that G is a maximal acyclic graph).	Dynamical definition: requires action on the object
Def6- G is connected and if we remove any vertex v then G-v is disconnected (which means that G is a minimal connected graph).	Dynamical definition
Def7- A tree is a vertex (basis) or a tree T for which we add a new vertex adjacent to only one vertex of T (induction step).	Constructive, inductive and dynamical definition

Table 1 : mathematical definitions of 'tree'

The exercise has two aims. First, it allows students to return to the definition, and second, to use (and perhaps reconstruct) the definition. To achieve the proof means to use a definition to plan an overall structure of the proof [6]. Two definitions of a tree are particularly "effective" to write the proof. We propose the main ideas of these proofs:

Resolution with <u>inductive definition</u> : we search the tree directly and we avoid cycles ; we choose a vertex and we connect it (by edges) to its neighbours which are not yet connected (it is possible because the graph is connected) and so one.

Resolution with <u>connected minimal definition</u> : the idea is to remove some edges of the given graph in order to obtain a tree. We search if an edge exists of which suppression does not disconnected the graph: if such an edge does not exist, we have a tree (with def6). If such an edge exists, we remove it and we search a new one to remove. When we can not remove any edge, we have a tree whose same set of vertices than G.

RESULTS

Devolution of the problem

The students were not reluctant to do the first question. We note that to define is not an activity strictly reserved for mathematics and this can support the devolution of the problem. Moreover the examples and counterexamples allow a process of comparison. Lastly, the students feel they have enough elements. In order to explain the main results, we will study three representative groups (called group 1, 2 and 3).

Defining-methods: defining by genus and differentia

Aristotle's defining method by genus and differentia [7] is to indicate what specific object a word means ('thingummy' here), to take a bigger class (graph) within which that object falls, and then to try and see what distinguishes it from the rest of that class. For Aristotle, a definition is a discourse according to specific rules (about language, syntactic and semantic rules). The chosen activity of *definition-construction* is by genus and differentia, so called because the students try to distinguish the examples from the counterexamples. But their regulations are not specifically like Aristotle's rules. Let us describe some of the main students' regulations and conceptions.

The two expressions "sufficient definition" and "minimal definition" were often used. These search for a sufficient definition proves that they seek a sufficient condition allowing the recognition and/or the construction of the mathematical object i.e. they evaluate up to what point their definition does not relate to too large a class of objects. When the students talked about 'minimal definition', it was not a matter of minimal sufficient condition, but short sentence. Moreover, the two properties 'connected' and 'acyclic' were mixed-up (conceptually and at the linguistic level) as testified by this extract (the student does not distinguish 'connected' and 'acyclic') :

Yohan: a unique path: that leads to only one condition. Instead of having two conditions to check on each figure each time.

The wish to define the "vague" terms of their definition was expressed by the students rather quickly when they defined 'thingummy' as a "set of dots and lines without cycle and in only one piece". This last property was "fuzzy" for them and some of them worked on the definition of connectivity to redefine "in only one piece" by "a path exists between two unspecified dots of the figure". We assume that when the students have given a name to a property ("in only one piece" for connectivity and "closed polygon" or "circuit" for cyclic), that allows them to work on this property (cf. Vygotsky) and to overstep simply verbalising the representation.

Produced definitions

Vygotsky notes the deep discrepancy between the formation of a concept and its verbal definition. With the examples and counterexamples, the students described the representation of 'thingummy': "in only one piece, without cycle". We notice that this *zero-definition* was as narrow as to coincide with def1 (table1). But the students did not agree to accept this assertion as a mathematical definition (although it could be a "good" one for a researcher because it gives the structure of the mathematical object) in accordance with one of their conceptions: a mathematical definition should be specified. So they questioned the mathematical object, its properties and its representation. In the table 2, *unfinished proof-generated definition* concerns the *proof-generated definition* and means that it was unfinished in terms of its form and its use. We notice that def5 (table1) was not constructed by the students: this is an inappropriate exercise to make it worthwhile.

Students' statement	Group	*Reference definition*	*Production phase*	*The role in exercise*	*Our status stand point*
In only one piece without cycle	1,2,3	Def 1	1	Idea of the proof (students'understood' the proof but did not know how to say it.)	Zero-definition
Inductive def.	2,3	Def 7	1	Proof	Definition
A unique path	1,2,3	Def 2	1	Idea of the proof	Definition
n dots and (n-1) lines	1	Def 3 or 4	2	Proof	Property
	3	Def 4	They add 'connected' (end of 2)	Usable only for a small graph in order to obtain a tree.	Definition
Minimal connected graph	1,3	Def 6	2	Unused	Unfinished proof-generated definitions
Remove a line it disconnects	3	Def 6	2	Unused	

Table 2 : description of students' statements

Functions of produced definitions

The research of a definition was guided here by functions of definition : to communicate, to recognize, to build and to prove.

Definition in order to communicate means to explain to another persons what a tree is i.e. the properties and the construction of the mathematical object.

Definition in order to build is alluded to by these students (group3):

Arnaud: here is the definition to create a thingummy, and there the definition to know if we have a thingummy or not.

Yohan: but to check, I prefer the idea of the path. If one wants to check whether the figure is a thingummy or not, one takes the first definition (a unique path), and if one wants to build a thingummy, one takes the inductive definition.

Group3 spoke about "good definition". For us it means algorithmic definition,

Yohan: It is a good mathematical definition when one can make a program [...] One can make a data-processing program which checks the thingummy, if it gives the good result each time, that means that the definition is good, inevitably since the computer does not think.

and/or it allows the *recognition* of the object :

Fabien: lets consider one must give a definition. If I, who don't know the thingummy, I have it explain to me by somebody, I inevitably see what it is.

The last function of a definition was *to prove*: the interest of equivalent definitions (multiplicity of the possibilities in a demonstration) is alluded to by group1 and we find the connection between a definition and the function which it will be able to have, in particular in a proof (operational definition) :

Arnaud: And then finally, especially the inductive definition, that facilitated the second exercise to us.

Yohan: Yes, one can choose a definition to prove. In fact a few moments ago, it was not useful. If you want to draw a tree, you don't need to know that.

DISCUSSION AND CONCLUSION

We raised some properties in the protocols, for example: "when one adds a dot to a tree, one always has a tree". Another property emerges at the time of the proof: "when a line is removed, that disconnects". Why did these "potential" definitions not emerge under the status of definition? Which leads us back to the question: which criteria allow a definition to be recognized (perceived) as such? Let us notice perhaps that these two characterization properties of the tree alluded to above contain a dynamic aspect in opposition to the static representation of the tree available to the students' experience. Moreover, we already mentioned the gap between definition and characterization properties which can be one of the causes of the non-emergence of these definitions as such. This gap means that a definition is enough, what can appear later only represents properties, as testified by this extract (group2):

Vincent: That will be included in the first definition (...) I have the impression that we have finished our work.

Angelique: It will be the same thing. Actually it all boils to giving properties. Finally they are not definitions, but properties for me.

We would like to stress the following : group1 has not reconstructed an appropriate definition in order to solve the exercise because the constructed definition (first question) has a form and a content that institutionalised it. Moreover, the definition represents a common knowledge for the resolution of a problem (Balacheff,1987).

It's possible to make students construct a definition of an object which is accessible by its representations. It was through examples, counterexamples and the produced definition that the students were able to build their *concept images*. We assume that a definition is not a finished product, so the necessity of the exercise. It appears that this activity of *definition-construction* could be a part of a process of concept acquisition. We would like to study the dialectic between the formation of a mathematical concept and its definition (with *definition-constructions*' activities) or more precisely the role of a *definition-construction* in learning. And also, the students' conceptions on mathematical definitions could be an obstacle to the concept formation. These conceptions concern the dynamic and static points of view, the functions of a definition and the gap (and the relations) between a definition and characterization properties.

NOTES

1. "the didactical transposition is the rule in sciences : (…) the presentation of a subject takes as a starting point a historical result and rewrites the history in reverse. In mathematics, this starting point is a definition".
2. "we are always conscious that our terms are a little vague… and we reach precision not by reducing their penumbra of vagueness, but rather by keeping well within it, by carefully phasing our sentences in such a way that the possible shades of meaning of our terms do not matter. That is how to avoid quarrelling about words." (p19. Popper (1945) *The Open Society*, Vol.II – London ; Henley : Routledge & Kegan Paul)
3. See columns of research sampler (www.maa.org).
4. "a concept cannot be restricted to its definition (…) it is through situations and problems to be solved that a concept acquires meaning".
5. A graph is made up of dots (vertex/vertices) connected by lines (edges).
6. For the possible ways of operating with definitions in doing proofs, see Moore.
7. *Topiques & Seconds Analytiques* (in Aristote, *Organon*, trad. J.Tricot, Paris, Vrin, Ed.1965).

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