# ON HEROES AND THE COLLAPSE OF NARRATIVES: A CONTRIBUTION TO THE STUDY OF SYMBOLIC THINKING*

Luis Radford

Université Laurentienne, Ontario, Canada

This paper is dedicated to Raymond Duval on the occasion of his retirement

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## INTRODUCTION AND THEORETICAL FRAMEWORK

The use of symbols in mathematics raises two different problems. The first one is linked to the mode of designation of the objects of discourse. The second one corresponds to the operations that are carried out on the symbols designating the objects. Although these problems are related, they are underpinned by different semiotic and cognitive demands. As far as algebra is concerned, the designation of objects of discourse requires a substantial reduction of vocabulary (Duval, in press). Indeed, while natural language accounts for a large set of words allowing one to *describe* objects (e.g. the next figure, the small rectangle), algebraic symbolism requires that these objects be designated using combinations of a *few characters* (*viz.* 0, 1, 2, ..., x, y, $\sqrt{\ }$ and the like). In previous papers (Radford 2001a, in press-a), I reported the tremendous difficulties that Grade 8 students had in finding a symbolic expression for the rank of the figure that follows the figure of rank 'n' in a pattern. Although the students could refer to the objects of discourse using more or less accurate descriptions in natural language, it took a long time before they could, with the teacher's help, figure out the expression 'n+1'. Algebraic language does not include adjectives, adverbs and other linguistic terms that prove to be crucial in natural language-based communication.

In addition to this, even if the students reported in my previous research could start designating simple objects with a few characters, they were not able to operate with symbols. They could not recognize that (n+1) +n, (n+n)+1 and 2n+1 referred to a same state of affairs. The problem is not merely the students' impossibility to operate with the unknown. As a matter of fact, at the same time that they were struggling with the generalization of patterns they were able to easily solve equations like 14+2e = 2+4e (see Radford, in press-b). The problem is related to the students' mode of designation of objects through algebraic symbolism.

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Indeed, in the designation of objects, the way signs stand for something else is related to the individuals' intentions as they hermeneutically unfold against the background of the contextual activity. In the designative act, intentions come to occupy the space between the intended object and the signs 'representing' it. In doing so, intentions lend life to the marks constituting the corporeal dimension of the signs (e.g. alphanumeric marks) and the marks then become signs that *express* something, and what they express is their *meaning*. The possibility to operate with the unknown thereby appears linked to the type of meaning that symbols carry.

Intentions occur in contextual experiences that Husserl called *noesis*. The conceptual content of such experiences he termed *noema*. Thus, noema corresponds to the way objects are grasped and become known by the individuals while noesis relates to the modes of cultural categorial experiences accounting for the way objects become attended and disclosed (Husserl, 1931).

Pursuing my investigation on students' semiotic processes of meaning construction and symbol use, in this paper I want to address the question of how the students' symbolic expressions are intended to convey meaning when the students proceed with the designation of objects and operate with the designating signs in a typical short story-problem. Within the sketched theoretical framework, the research question will be addressed in terms of the *manner* (the noesis) in which students use signs to *express* particular features (the noema) of the objects of discourse. After briefly commenting on the methodology, I will suggest a distinction between story-problems and symbolic narratives. This distinction will allow us to provide an interpretation of some 'nonsensical' symbolic expressions elaborated by novice students. I will then discuss the concept of nominalization whose theoretical interest is not simply to account for the introduction of unknowns in a problem. I intend it as a theoretical tool to examine how symbolic expressions become endowed with meaning in this limbo where we have neither fully left the original story (told in natural language), nor have fully entered into the symbolic narrative (told in symbols). The last section presents a short discussion concerning the problem of abstract or formal use of signs in obtaining the equation associated with the story-problem.

## METHODOLOGY

The data presented in this paper comes from my longitudinal classroom-based research program involving 4 classes of Grade 9 students. The classroom activities were designed to be carried out co-operatively by the students according to a small-group (2 or 3 students) working format and were usually followed by general discussions conducted by the teacher. Due to space constraints, I shall mention excerpts of the video-taped word-problem solving activity of 3 small groups of only one of the 4 classes. Transcriptions from the video-tapes were analyzed using the *qsr N5* software for interpretative, qualitative research (details in Radford 2000).

The mathematical activity was based on the following short story: "Kelly has 2 more candies than Manuel. Josée has 5 more candies than Manuel. All together they have 37 candies. »[1]. The same story was used to generate *three* problems involving transformations in the algebraic expression of the data. In problem 1, the students were asked to designate Manuel's number of candies by x, to elaborate a symbolic expression for Kelly and Josée, and, then, to write and solve an equation corresponding to the short story. Problems 2 and 3 included similar questions. The difference was that, in Problem 2, the students were asked to designate Kelly's number of candies by x while in Problem 3 the students were asked to designate Josée's number of candies by x.

## RESULTS AND DISCUSSION

### From Heroes to *Objectivities* [2]

One of the difficulties in dealing with problems involving comparative phrases like "Kelly has 2 more candies than Manuel" is being able to derive non-comparative, assertive phrases of the type: "A (or B) *has* C". If, say, Manuel has 4 candies, the assertive phrase would take the form «Kelly (Subject) has (Verb) 6 (Adjective) candies (Noun)». In the case of algebra, the adjective is not known (one does not know how many candies A has). As a result, the adjective has to be referred to *in some way*. In using a letter like 'x' (or another device) a new semiotic space is opened. In this space, the story problem has to be re-told, leading to what has been usually termed (although in a rather simplistic way) the 'translation' of the problem into an equation. I prefer here to use the term *symbolic narrative* since what is 'translated' still tells us a story but in mathematical symbols. Although there are similarities in the story problem and the symbolic narrative, the personages change. This change is best characterized as a noematic shift that brings forward certain parts of the story while putting others in the background. The 'heroes' –so to speak– of the re-told story are no longer Kelly, Manuel or Josée, but numerical relationships between the amount of candies that constitute the objectivities expressed in the new semiotic space (i.e. the symbolic-algebraic one).

Difficulties in accomplishing this noematic change or shift of attention may become an obstacle in the learning of algebra. Let us show an example in which we can see the students of Group 1 trying to produce symbolic expressions without achieving the aforementioned noematic change.

### Signs as marks in narrative acts

In this group, a (wrong) calculation with comparative phrases led the students to conclude as follows:

> Stacey: Kelly has 2 more candies than Manuel. Josée has 5 more candies than Manuel. Together, they [Kelly and Josée] have 7 more candies than Manuel.

Instead of transforming comparative phrases into assertive ones, the students changed the comparative phrase into an adverbial form ('more'), something that allowed them to rank the heroes in the story problem according to the number of candies that each one had:

> Stacey: Josée has 5 [*more*]. Josée has more, Kelly is second, Manuel third. Okay, so you put x that represents… no x that represents 7, okay? 12 [*as the result of 7+5*], 9 [*seen as the result of 7+2*] but I don't know how to find… […]. He [*Manuel*] has 7 less than these two put together (*she writes*) x–7 […] This [37] has to equal x-7 (*suggesting 37=x-7 or x-7=37*).

The transformation of comparative phrases into assertive ones is related to the possibility of explicitly taking into account the unknown amount of candies. However, the clear introduction of a letter for the designation of such an unknown amount of candies does not fix the problem. This is shown in the excerpt below (Line 2). When the teacher came to see the students' work, he realized that the students had not taken into account x as the amount of Manuel's candies. Trying to help, he said:

1. Teacher: Manuel is x.
2. Stacey: Yeah. Josée has 5 more candies than Manuel and the 3 together have 37 candies.
3. Teacher: Here, they are asking you to write the algebraic expression for the number of candies represented by Kelly. So, if he is x, she is what? That's what you have to figure out.
4. Stacey: (*while looking at the teacher, she says*) x-2.

Although the teacher's utterance took an elliptical form (Manuel *is* x), it was an attempt to cause the students to focus on Manuel's amount of candies. His attempt to shift the noematic content, nonetheless, was countered by a phrase (Line 2) that amounts to a monotone answer "Yeah, yeah, we know that".

Constructing a symbolic narrative for the story-problem requires a new approach: while the story-problem unfolds according to a left-to-right lineal reading (with eventual flashbacks) the starting point in the symbolic narrative does not have a permanent location. In the symbolic narrative, the *order* of discourse (to borrow Foucault's term) is different and the thematic character is about other things.

What, then, is the role of symbols in the students' previous symbolic expressions? We shall now see that the students' signs constitute short scripts recounting salient parts of the *original* story. Let us take a closer look at Stacey's algebraic expressions ("x-7", "x-2"). Each one of them is made up of three signs: the signs in the second one are: 'x', '-' and '2'. Their meaning, of course, is not the one required in the practice of algebra. We cannot say, however, that the expression is meaningless. The expression "x-2", which is polyphonic in tone in that it merges the teacher's voice (Line 1) and Stacey's understanding of it (Line 4), might be read as telling us that Manuel has a certain amount of candies ('x') and that he has two ('2') less ('-') candies than Kelly. Thus the sign '-' is *not* performing a subtraction on the unknown x but is an orienting mark of a short script about the story-problem. In a similar vein, the sign '7' in the expression "x-7" does not translate merely as "x minus 7". As indicated by Stacey's utterances, the number 7 comes to form part of the symbolic expression with an imported meaning so that each symbol in the equation tells us a part of the original story.

Later, the teacher came to inspect the group's work. He said:

Teacher: x is Manuel, right?

Caroline: Yes.

Stacey: (*interrupting*) So, x minus…

Teacher: (*continuing his utterance*) Kelly has 2 more candies than Manuel. Let's suppose that Manuel has 20 candies, how many candies would Kelly have?

Stacey: 22?

Teacher: 22. *(He looks at Caroline).* If Manuel had 30 candies, how many …

Stacey: (*interrupting*) 32.

Teacher: *(He looks at Jessica).* Therefore, um, what did they do to find Kelly?

Stacey: You put the 2.

Teacher: (*correcting*) You add 2.

Stacey: (*having understood how to algebraically express the relationships, says, referring to Josée*) There you add 5. […] So, it's x+5. (*The students write 'x+2' and 'x+5'.*)

Teacher: Then, this (*indicating the question about the equation for the problem on the page*) would be equal to what? This is an equation so it has to equal something. (*The teacher is called by another group and feeling that the students are on the right track he leaves.*)

Caroline: (*adding the 3 algebraic expressions*) So, if I have 3x+7, *(she looks at Stacey)* 3x+7? […] That means 3, no, 3x+7. This equals 37?

Stacey: (*recognizing the number 7, says*) I don't believe that! 3x+7 is equal to 37! … oh!

We see how using the elliptic formula 'x *is* Manuel' and through a calculation on numbers (which functions here as the ground of *noesis*, i.e. the meaning-conferring act), the teacher shifts the students' attention to the relationships between amounts of candies. What is important, though, is not that the students could write the sought symbolic expressions. The important point is the emergence of a kind of awareness that, in the symbolic expressions, the heroes, without being thrown away, are put in the background and predications in the symbolic narrative are done about other things, about *objectivities*. Perhaps elliptical formulas based on the verb 'to be' of the kind "x is Manuel" are not the best way to forge the distance between the story-problem and the symbolic narrative. And, perhaps, the use of the verb 'to have' would have been more suitable in terms of the goal of the activity (of course, we became aware of this only after the activity was analyzed). Nevertheless, in the classroom context, the choice of the elliptical formulas allowed the students to start moving into the realm of algebraic symbols and to begin learning the incredible amount of meaning that these phrases encompass despite the dramatically limited number of signs they use.

**Nominalization**

Groups 2 and 3 did not face the same difficulties as Group 1. For instance in Group 2, we find Anik saying:

Anik : Okay. … Manuel is going to be the variable x. *(she points to the paper)* like… like if they want… find the equation there… the equation for Kelly is… because … uh, she has 2 more than Manuel. Manuel has… has the amount x. So x+2 because we don't know, x is how many Manuel has. Right? So, she [Kelly] has … *(she points to the paper)* has like whatever Manuel has +2.

We see how the comparative phrase was transformed into an assertive one ("she has like whatever Manuel has +2"). By introducing the letter x (in "Manuel is going to be the variable x" and "Manuel has … has the amount x"), Anik (first using the verb 'to be' and then the verb 'to have') opens the door that leads to the symbolic narrative. We can see, despite the final reformulation at the end of her utterance, how the heroes start fading away. The insertion of x as a designation of Manuel's number of candies, allows room for a *nominalization*, that is, a process in which something becomes enabled to function as the subject or the object of a verb. In saying "whatever Manuel has", the expression can now become the noun in the assertive phrase "Kelly has (noun) +2". It is indeed interesting to notice that, without help, Group 1 could not offer nominalizations. Groups 2 and 3, in contrast, did offer clear instances of nominalizations. Here is an example, taken from Group 3, concerning Problem 3 (where x designated Kelly's number of candies).

1. Michelle: Kelly... *(inaudible)* ... There the x is all moved around. They're trying to trick us. So if Kelly has 2 more candies than Manuel, then Manuel has 2 candies less than Kelly, right? […] But now that Kelly is x, minus 2 ...
2. Jessy: *(interrupting)* Yeah, yeah.
3. Michelle: I'm thinking… Josée has 5 more candies than Manuel. So Manuel has x-2. Then Josée has 5 more than that, right? So x-2 in brackets ... +5."

Line 1 indicates a change of meaning. Although the sentences "Kelly has 2 more candies than Manuel" and "Manuel has 2 candies less than Kelly" refer to the same state of affairs, the meaning is not the same (as in Frege's famous example also discussed by Husserl: *the victor at Jena* versus *the vanquished at Waterloo*). The meaning changes because of differences in the way of attending the object –the noematic content is not the same. In the last part of Line 1 and the first part of Line 3, Michelle establishes Manuel's amount of candies. The insertion of the sign 'x' allows for a *first nominalization* which makes possible the phrase 'Manuel has x-2' (an important hybrid phrase where meaning is lent from the story problem to the symbolic narrative). In the second part of Line 3, the attention is focused on 'x-2' only. Instead of seeing this expression as expressing any of its various possible meanings (e.g. 'the amount of Kelly's candies minus two' or 'Manuel's amount of candies'), Michelle proceeds to a subtle and fundamental suspension of these by using the deictic 'that'. In doing so, a *second nominalization* is produced: the referent is formally nominalized and can thereby become the noun of the verb 'to have' in "Josée *has* 5 more than *that*". As mentioned in the Introduction, the theoretical interest of nominalizations is to inform us how symbolic expressions become endowed with meaning in this limbo where we have neither fully left the original story, nor have fully entered into the symbolic narrative. In particular, nominalizations make it possible to see how high-order meanings are made available for further predication. Let us now discuss the didactic problem of the operations with signs necessary to obtain the equation.

**The collapse of narratives**

This is an excerpt from Group 2 during their discussion about Problem 1:

1. Anik: Yeah … Well guys … *(She takes the papers)* what we're trying to do is to put [*the symbolic expressions*] with the people, okay? Kelly has 2 more than Manuel. Manuel has x. Plus 2 is what Kelly has. [She] has whatever he has +2. Okay. That's going to be x+2, that's in brackets, plus x+5. That'll be what Josée has, plus x, that'll be what Manuel has [*she intends the expression '(x+2)+(x+5)+x'*].
2. Luc: Equal to what? 30? 37? (*Chantal writes 2x+5x+x.*)
3. Anik: (*looking at Chantal's symbolic expression*) 2x, I don't think so.
4. Chantal: Why not?
5. Anik: *(She points to the paper)* because there you are about to do 2 times x.
6. Chantal: No.
7. Anik: Here we're doing 2+x. (*Anik writes (x+2)+(x+5)+x*).
8. Luc: (*looking at Anik's expression*) You group them together, you group all the x's *(Chantal erases what she had written.)*
9. Anik: (*Talking to Luc*) No, no!
10. Luc: Yeah! You group all the x's.
11. Anik: No! Wait guys! Wait! *(She points to the paper).*
12. Luc: Oh my God!
13. Anik: I just want to explain it to you. Guys, here! He has … she has x+2, right?

In line 2, Chantal uses a syntax based on the criterion of juxtaposition of signs. The sentence is structured in the manner of a narrative where signs become encoded as *key terms* – much as pictographic signs used by Mesopotamian scribes did in the proto-literate periods ca. 3300-2900 BC where, e.g., a set of pictograms representing "sheep" "two" and "temple" may mean "two sheep delivered to (or received from) the temple" (see Radford 2001b, 28-33). The expression 2x does not mean twice x or two times x. For Chantal, 2x does convey the idea that Kelly has 2 more candies than Manuel, and this is why she is surprised (line 6) that Anik could have interpreted it in a different way. But the previous dialogue shows another feature of the students' struggle with algebraic symbolic language. In line 8, Luc proceeds to collect similar terms. This action is radically opposed by Anik. Why? The reason is that the collection of similar terms means a rupture with their original meaning. All the efforts that were made at the level of the designation of objects to build the symbolic narrative have to be put into brackets. The whole symbolic narrative now has to collapse. There is no corresponding segment in the story-problem that could be correlated with the result of the collection of similar terms, i.e. with 3x+7. Anik's desperate effort not to lose track of the narrative meaning is clear in line 13.

## CONCLUDING REMARKS

Focusing on a story problem, in this article, I dealt with two main points: (1) the designation of the objects of discourse in the construction of symbolic narratives and the meaning of symbolic sentences, and (2) some of the problems arising in the operations that are carried out with signs that recount the symbolic narrative. As for the first point, the analysis of some key lines in the students' dialogue suggests that the students' success in constructing the symbolic narrative depends on their ability to move across different layers of noematic content. We have indeed seen the interplay between the various meanings and the dynamics required to enrich, shift, and abandon these meanings as well as the role played therein by nominalizations. As for the second point, the classroom observations intimate how difficult it may be to tackle what I termed the *collapse of narratives*. The constitution of meaning after such a collapse deserves more research. While Russell (1976, p. 218) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention, a noematic change: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl 1961, p. 79), which led him to talk about signs having a *game signification*. I think that the richness of Husserl's metaphor resides in its stressing the cultural, conventional role of rules. Since convention and arbitrariness are two different things, the weakness of the metaphor is that it does not help us to see the rationale behind its conventional nature.

### Notes

1. Story-problems of this kind have been investigated in depth by Bednarz and Janvier (1994) in terms of the effect that different kinds of comparative relationships (e.g. additive vs multiplicative comparisons) have on the students' strategies.

2. Husserl (1961, 44) coined the term *objectivity (Gegenständlichkeit, obectité, objectidad)* to refer not necessarily to an individual thing but also to complex things, categories and states of affairs as they become the referent in sentences.

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