FROM THEORY TO PRACTICE: EXPLAINING SUCCESSFUL AND UNSUCCESSFUL TEACHING ACTIVITIES (CASE OF FRACTIONS)

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Abstract: *In a teaching experiment I examined a theoretical model of mathematics teaching and learning in practice. In this paper I focus on how the model can guide the teacher's thinking about students' understandings and the generation of activities that foster intended transformations in those understandings. As a researcher-teacher I taught, twice a week for four months, basic ideas of fractions to 28 third graders in a public school in Israel. The analysis of both classroom data and researcher's documented reflections indicates how the model can empower the generation and explanation of successful teaching activities, as well as thinking about and adjusting unsuccessful ones. It also highlights the importance of research-based, content-specific models of students' understandings for successfully implementing the model in practice.*

Recent reform documents in the USA (NCTM, 1991) stressed that a primary goal of mathematics teaching is to promote students' conceptual learning of key mathematical ideas and procedures. In line with a constructivist principle of active learning, teachers are encouraged to use valuable methods such as problem solving, manipulatives, and whole class discussions. Sometimes the techniques bring about conceptual understandings of the intended ideas but often times they don't. In both cases, what seems to be missing is a systematic, developmentally appropriate guidance for teachers how to generate and adjust activities that advance students to intended understandings. To provide such guidance, my colleagues and I (Simon et al., 2000; Tzur & Simon, 1999) have recently developed a model that relates learning with teaching of new mathematical conceptions. The purpose of the present study was to examine in practice how this model can guide the teacher in (a) generating and/or selecting activities that successfully bring about the intended learning and (b) thinking about and adjusting unsuccessful teaching activities.

CONCEPTUAL FRAMEWORK

The study was part of a four-month teaching experiment that drew on a social-constructivist approach (Cobb & Bauersfeld, 1995). This approach coordinates social and psychological perspectives of teaching-learning processes. The model at issue is an elaboration of the cognitive perspective and, itself, served as a framework for the study. Since the study focused on relationship between teaching activities and students' learning, I briefly present only components of the model that served in the data analysis of successful and unsuccessful teaching activities (for details see Simon et al., 2000; Tzur & Simon, 1999). I also present key constructs of a content-specific model of children's fraction learning that served me, as the researcher-teacher, in making sense of students' thinking.

Knowing, Learning, and Teaching. The model regards knowing and understanding as using a conception--a dynamic mental compound that relates between an activity and its effect(s) (abbreviated as A-E relationship). The compound is not mainly the activity nor the effect(s) but the dynamic relationship consisting of both. In this paper I will use the term A-E relationship and conception interchangeably. Learning is regarded as the process of transforming current conceptions into new ones and/or coordinating anew established conceptions (Piaget, 1985). The mental mechanism that enables learning is reflection on A-E relationship (abbreviated as Ref*A-E relationship). The term reflection refers to the continual mental comparison between one's goals for and effects of his or her activities. Note that from the learner's point of view, Ref*A-E relationship does not need to be directed toward making specific conceptual advances (awareness is not implied). Teaching is regarded as a cycle of four principal activities. It begins with inferring learners' current conceptions. It proceeds through hypothesizing a learning trajectory (Simon, 1995) from those current conceptions to the intended ones. It moves to designing and engaging learners in activities they can assimilate and use as a means to form the intended conceptions. Finally, it implies orienting learners' reflections on A-E relationships via probing questions and tasks which also serve to infer learners' conceptions, and so on.

Conceptualizing Addition of Fractions. The content-specific model that I used in the study does not regard unit fractions (1/2, 1/3, etc.) merely as parts of equally partitioned wholes. Instead, it regards unit fractions as particular relational quantities, that is, entities that stand in a multiplicative relation to a given whole (cf. Behr et al, 1992). For example, a given quantity called "a third" is not just one-out-of-three equal parts; rather it is a third because another quantity, which was designated as "a whole," is **three times as much** as the given quantity. (Children often say, "The given piece fits-in-the-whole 3 times.") Thus, learning to add unit fractions requires that learners establish those units as relational and symbolically meaningful entities in and of themselves. Only then addition, which learners have established for whole numbers, can be coordinated with and applied to those entities.

METHODOLOGY

For four months I conducted a classroom teaching experiment (Cobb, 2000), while teaching 28 third-graders in a public school in Israel twice a week. Prior to the period reported herein the students formed their conception of unit fractions (1/n) via the Repeat Strategy that they used in tasks of equally sharing a given whole among a number people. To partition the whole (24 cm long paper strips), they estimated the size (share) of one person, repeated it the number of times implied by the number of people, and compared the iterated whole with the given whole. Then came the critical step of the repeat strategy: adjusting the size of the initial piece and reusing the previous steps (if the iterated whole was longer than the one to be shared--they shorten the initial piece; If the iterated whole was shorter--they made the initial piece longer). When the present study began, 26 out of 28 students had established unit fractions as unique quantities. This was indicated by their ability to order unit fractions while explaining the inverse relationship between size and number of parts (e.g., 1/5 is larger than1/7 because to fit-in-the-whole 7 pieces one must make each piece smaller).

Data for the present study consist of:

1. Videotapes of 7 lessons conducted between December 10, 2000, and January 7, 2001. I taught the lessons on Friday and Sunday to create a sense of continuation while allotting the time needed between lessons for ongoing analysis and planning.
2. Tapes and transcripts of my reflections, recorded (audio) after every lesson.
3. Tapes and transcripts of audio-recorded conversations with the homeroom teacher (who observed and assisted my teaching) that took place 3 hours after every Sunday lesson.
4. Students' written works from classroom.

A retrospective analysis of the data focused on relating teaching activities and students' learning. The term 'successful teaching activity' is used to imply that it brought about students' learning of the intended idea ('unsuccessful' if it did not). The 7 lessons chosen represent both unsuccessful and successful activities.

ANALYSIS

This 3-part section tells the story of successfully teaching the students addition of like-denominator fractions. In the first part I portray my inference of the students' powerful understandings which prompted my decision to begin teaching them addition of unit fractions. In the second part I analyze my first, unsuccessful attempt to teach that topic. In the third part I analyze my second and very successful attempt to teach it.

<u>Part 1: Students' Thinking</u>. Before teaching addition of unit fractions, I evaluated the students' thinking to make sure that for them - these units were meaningful entities. On December 12, I engaged the students in a simple task. I gave each small group one long paper strip (the whole) and an envelope with nine, exact-size paper strips that I cut in advance (1/2 through 1/10). They were to randomly pick a piece from the envelope and find what fraction it was. After each group had picked at least three pieces we shifted to a whole class discussion. ('R' stands for researcher-teacher.)

R: (Holds an envelope like theirs) I pick a piece and I want to know what fraction is it of the whole on the board? Who can tell me how to figure it out? (Only a few students raise hands.)
R: Please go ahead and talk about this question again in your groups. (Group work, one minute.)
R: Okay, so what would I have to do?
Sally: Measure how many times the piece fits-in-the-whole.
R: How should I do that? Nora?
Nora: Put the piece exactly under the whole; mark its right end; then move it to the right and...
R: (Stops Nora before she explains the entire process) Okay, before I do this, what am I trying to find? When I'm done, what would I find? (A little later) Why am I doing what I'm doing?
Ruth: To find what fraction is it.
R: How would I know what fraction is it by doing what Nora said?
Ruth: You'll count how many such pieces fit-in-the-whole.

Reflecting on their activity of picking pieces from the envelope and stating what one <u>would have</u> to do was not a trivial task for the students. Thus, I asked them to re-discuss it over in the small groups so that their reflection would be based on what they found key to their activity. Then, by stopping Nora's articulate way of explaining the process, I attempted to orient students' reflections on the multiplicative relationship between their goal and activity. Key to my intervention was to let them carry out the activity in their mind, not in actuality. As a result of both interventions, Ruth was able to clearly state the intended relationship.

Following Ruth's reply we executed Nora's suggestion and all said that the piece was one-tenth because it fit-in-the-whole 10 times. So I asked, "What if I drew a piece that fits 7 times?" and Bob replied, "It would be one-seventh." To the same question only with '3 times' and '6 times' Yoel and Ellen replied 'one-third' and 'one-sixth,' respectively. The lesson on December 15 began with short individual work on a worksheet. Then, I presented a new situation. I put a long, red paper strip and said it was a whole, then put another red piece under it and the students easily detected that it was half of the longer strip. Then I put a green strip, shorter than the red whole but longer than its half, and under it a green strip that the students easily detected as one half of the green whole. Before I could say anything, Aaron brought up my intended issue.

R: Aaron raised a point that I would like you to discuss as a question in your groups. (Holds up the green and the red halves) Both the red half and green half are halves, but the red half is longer than the green half. How do you explain this? (Group work, 1.5 minutes.)
R: (To the whole class) Okay, what if I would put a red third and a green third on the overhead. Would the red third be longer or shorter than the green third?
Jack: The red third will be longer.
R: Why?
Jack: Because the red whole is longer than the green whole. Therefore, [all] parts of the red should be longer than parts of the green whole. So the red third must also be longer.
R: (A little later) Why is the red third longer than the green third?
Noel: Because to fit-in-the-red [longer] whole three times the third has to be longer.
Judah: And to fit-in-the-green [shorter] whole three times the third has to be shorter.
(A little later, without any prompt, Lee says he has a different idea and comes to show it.)
Lee: If you take this (puts the green third below the red half) it also comes as a half.

The students' work on different-size halves and thirds indicated a significant understanding of unit fractions: they saw them as multiplicative relations with respect to a given whole. They could anticipate the effects of the activity to determine a fraction and hence could shift their attention from one whole to another as a means to compare parts of those wholes. Moreover, Lee's initiative indicated a powerful understanding—he was able to re-designate a part as a whole and compare it to a part of a different whole. I thought they were ready for addition.

Part 2: Unsuccessful Attempt. On December 15, toward the lesson's end, I distributed a worksheet that introduced a coloring activity with which I planned to initiate adding of unit fractions. They worksheet showed a whole partitioned into 6 equal parts. I asked the students six questions that involved coloring certain parts and they completed the first two questions. The first asked them to color a part and say what fraction it was (answer - '1/6'); the second asked to color another part on that whole and say what fraction was that part only (answer - '1/6'). To my genuine surprise, some replied '1/5' to the second question whereas others simply did not know. In my written notes on the matter I wrote:

Journal entry, Saturday, December 16: I thought again on tomorrow's lesson, Dec. 17 [my initial plan was to go on with coloring]. Already by the end of class I thought, 'What might have been the reason that the [coloring] activity caused some students difficulties to disembed [dissociate] the pieces of the partitioned whole?' In class I thought that this is only a difficulty of unclear convention, hence I'll have to tell them to count the colored parts. Then I realized - I should focus on the A-E relationships involved in the task, which re-focused my attention on 'What activity could be focusing the students' attention when coloring?' The coloring activity creates a background-foreground problem and masks the counting activity to which I wanted to orient them. Coloring is an activity of pointing at the specific unit [to count] but it creates problems that I must also think about in terms of conceptualizing the situation, not merely in terms of conventions. The specific conception is the understanding that each part is 1/6 no matter what its color.... Because the students completed only the first two tasks I had the time to reflect on their thinking and on my initial tendency (in class) to dismiss the problem [convention]. My focused reflection about the activity as a plausible source for the students' difficulties led to a change in my plan for the next class. First, I'll ask every child to point to each part, colored or not, and ask what fraction was it of the whole [anticipated – '1/6' for each]. Only when I'll be clear how students 'see' the colored parts [separately] shall we move to adding fractions.

The entry above indicates both the power and limitation of the model. It provided powerful guidance to focus my reflection on plausible conceptual sources for students' difficulties. In particular, it guided me to infer the particular activity that the students might have been using to assimilate the task. My knowledge of content-specific models of students' fraction learning, brought forth the 'disembedding activity' notion. I used this notion to bridge (so I thought) the gap between my expectation that coloring parts will simply serve to point at individual parts to be counted and the students' difficulty to see the parts individually. The limitation is that a teacher's inference can be inappropriate. Here, I did not infer that a problem to isolate parts could result from the coloring activity itself. In that case, just asking students to identify each part as 1/6 could not help because it only dealt with effects of the coloring activity.

On December 22 we returned to the coloring activity via the 6-question worksheet. This was a daunting, teacher-led lesson. I read out loud the first question. The students replied that the first colored part was 1/6 of the whole. I read the second question. A few students replied that the second colored part was ... 1/5 of the whole. On the spot I thought that we must establish some convention, so I told the students that we also have to count the colored parts.

I then asked the third question, 'What fraction were both colored parts combined?' which fostered a very interesting learning experience. During a 3-minute small-group work, four (out of six) groups had figured out that the two parts combined would fit-in-the-whole three times, hence it is one-third. I devoted 8 more minutes for a whole class discussion about this exciting, student-generated idea. Then, 10 minutes to the end of the lesson, frustrated for not achieving my goal, I decided to present the conception of adding non-unit fractions as if it was a matter of learning a new convention. I told them that just like when one adds apple plus apple to get two apples or a flower plus a flower to get two flowers, one adds one-sixth plus one-sixth and gets two-sixths. Never before or after this repetitive teaching activity had I used such a method and, not surprisingly, it was unsuccessful. For example, when asked to color and add the third part students responded, 'I don't know,' '1/3,' '1/2,' and only a few wrote '3/6.' Students' difficulties on December 22 perturbed me greatly and led me to back to the model as a guide for resolution.

Journal entry, Sunday, December 24: Immediately after Friday's class, I thought that the coloring activity failed to foster the intended learning because it might not be appropriate. The homeroom teacher said that the students would need to work on something concrete, to 'feel the units in their hands.' I thought, 'True, but coloring is a concrete activity.' Reflecting on that thought, I realized the need to have an activity that focuses students' attention on counting, where they separately point to each part while accumulating the parts. This need to figure out such a counting activity focused me back on the problematic issue of disembedding. I immediately asked myself why didn't I think about that issue earlier. This self-criticism pushed me to re-examine my reasoning about the failure and I realized that a wrong inference about the students' current understanding was also a possible. Thus, I decided to re-evaluate students' thinking before using a different activity for coordinating their established counting operations with the newly established unit fractions.

The entry above indicates the practicality of a model that combines general constructs of learning and teaching with articulated, content-specific constructs. The model implies to re-examine not only the activity but also students' thinking and enables the teacher to analyze specific component understandings needed for successfully learning to add unit fractions. Thus, consciously using the model to guide my reflection on students' work resulted in executing a specific plan that, here, rectified my previous inferences. On the basis of my re-examined inferences I generated a teaching activity that was more likely to foster the intended coordination between students' counting operations and unit fractions.

Part 3: Successful Attempt. On December 31 the students worked on a 2-page worksheet that contained 4 problems designed for the re-examination. I read every student's worksheet as soon as he or she finished it and this re-confirmed my inference. Thus, I began teaching them the game that I designed in line with my December 24 reflections. Each pair received a paper strip - the whole. Each student received 6 pieces marked 1/10. First, they checked that a part was, in fact, 1/10. Then, taking turns, they rolled a die and laid down as many 10ths as it indicated. Each student was supposed to say the names of the fractions as they accrued, write the fraction that each of them laid down on a chart, add their two fractions, and figure out game points (0, 1, 2, or 3, governed by chance) that the pair received in that turn. Before the lesson ended we had enough time to let two students demonstrate one turn of the game.

On January 5 the students quickly got engaged in the joyful game, which quickly brought forth their counting activity as a means to add fractions. They realized that what matters in adding and writing the results is the two non-unit fractions each player had and hence applied addition of whole numbers. The pairs quickly completed 10 turns of the game (10ths) and I gave them another worksheet for a game with 8ths. I moved about pairs to watch their work. Although successful, I was aware of two issues that may be hidden. First, several sums were improper fractions (e.g., 5/10+6/10, 6/8+4/8) and I needed to know what sense they made of them. Second, I wanted to make sure that as they were coordinating counting with unit fractions they were not losing the meaning of the units added.

R: (About 15 minutes to the end of lesson) Here's a question that we would talk about later. Some of you saw that sometimes we go beyond the whole. Please pay attention to the number of parts in such situations.

R: (A whole class discussion about 8 minutes later) There were times when you both rolled the die and got a number of pieces that exceeded the whole. Let's talk only about the game with 10ths. We have a whole and we have 10ths. When does it go beyond the whole? Please raise your hands if you can explain it. You may like to recall what you've just done in the game. (Several students raise hands.) Let's hear from Joe.

Joe: With twelve- and with eleven [pause], 10ths.

(R. takes a minute to write on the board Joe's examples, 6/10+5/10=11/10, 6/10+6/10=12/10.)

R: Joe says that in these cases we have more than the whole. If you think that Joe is right please just raise your hand quietly. (A majority of the students raise their hands.)

R: Who thinks that Joe is wrong? (Nobody raises a hand.) Aaron, would you explain Joe's idea?

Michael: (Bursts out) Because the whole is ten 10ths.

R: (Hisses them) Only Aaron.

Aaron: Because, because ten 10ths is one whole.
R: And so if we have 11/10, or 12/10 (points accordingly on the board), then it is more than the whole?
Many students: This happened to us, too.
Tom: This did not happen to us with 10ths but it did with 8ths.... We got ten 8ths!

The model-based question I asked in the first line of the protocol above meant to orient the students' reflection on relations between their activity and an effect of mathematical interest (to me). Consequently, students related between their activity—adding unit fractions beyond the whole—and the numerosity of the partitioned whole. Thus, I fostered an understanding beyond simply being able to add unit fractions. This was particularly indicated in Tom's voluntary contribution, he learned to conceive the partitioned whole as an entity and could then make the abstract application to his personal experience of 10/8.

R: (On Jan. 7, after returning the all-correct worksheets to the students) Today, I want us to make a rule that will help you from now on. (Writes on the board and says) 'How do we add fractions?' We want to come up with a rule that always works. And just as usual, we want to see that everybody understands why this is the rule. (A little later) Okay, now I'll write the question the board. (Writes on the board while saying) 2/7+4/7= . We want to know how much is it, this time with no dice and no pieces. And I wrote these two only as an example. Because the question is not simply if you know the answer but if you can state a rule how will we always know. (About half of the class raises hands.) Hmm, okay, Jan?
Jan: Six-sevenths.
R: And how should I write it [on the board]?
Jan: (Directs R.) Six; fraction-bar; Seven.
R: Jan, how did you know this was 6/7 and not 9/10 or 85/12?
Jan: I did it in my head.
R: What did you do in your head?
Jan: I added 2 and 4 and got 6.
R: (Asks who agrees. About half agree, no one disagrees. R. calls on Tom who said he was not sure.)
Tom: Ah, 2+4 is, in fact, 6. But I was not sure that she has to write: 'fraction-bar, 7.'
R: Good observation. Tom says that Jan added the 2 & 4, but why do we have to write 7 here (points to the sum's denominator)? Why don't we add 7+7 and write 14? Many think it's obvious but Tom asks us why.
Aaron: I think she cannot explain if it's supposed to be 7 or 14, but her answer is fine.
R: Why can't we explain this?
Aaron: Because...
R: 'Because' is not an answer that helps us to understand. (A little later, Noel raises his hand) Noel?
Noel: Ah, this is what we learned, six **is** with sevenths (his hand movement indicates, 'this is obvious').
R: Why is it with 7ths and not 10ths, 14ths, 3rds?
Noel: (Voluntarily gets up and goes to the board) Look. These (the 2/7) are 7ths. And if you add 7ths to it (points to the 4/7), it must come out with 7, not with 14.
R: (A little later) Noel, why must it be 7 and nothing else?
Noel: Because if you have 2 and 4, and they **are 7** [he seems to mean 7ths], and you add them together, the '7' does not go away [i.e., the parts, 7ths, do not change].
R: (To the entire class) Look, we added the 2 and 4 but we did not change the 7. Why?
Nora: (A little later) We knew we had two-sevenths and four-sevenths, so we say it's six-**sevenths**.
(A little later R. engages them in small-group work to write their rules. Five out of six groups write rules that clearly indicate their understanding. A student from each group reads their rule.)
Judah: When adding fractions of the same size [denominator], the number on the bottom always remains the same and you have to add the numbers on the top.
R: (Two minutes before the end of the class.) We would now use your rule to solve some problems. (R. distributes a worksheet of 28 problems of adding like-denominator fractions, e.g., 3/9+4/9, 2/5+4/5. All 27 students in attendance completed the entire worksheet, all correct, in less than 5 minutes.)

The last two protocols indicate that the model-based game successfully fostered students' coordination between counting activities and unit fractions. It was a powerful activity that promoted students' thinking about the unit fractions as invariants. Thus, it succeeded where the coloring activity failed. Key to the creation of a successful activity was my model-based reflection on the previous failure: analyzing students' understandings and articulating ways in which their understandings support counting activities.

DISCUSSION

This study examined how a general model of teaching-learning a new mathematical conception, combined with content-specific models of students' thinking, can guide a teacher in practice. It demonstrated how useful models are in devising successful teaching activities and in analyzing and adjusting unsuccessful ones. It highlighted the critical role that well articulated, content-specific models play in making sense of students' evolving understandings and in creating tasks and activities that both 'meet' the students' current conceptual state and foster an intended advance. That is, it indicated that the model can help the teacher to understand the activity from the student's point of view and thus to consider the pedagogical potential of an activity. A particular, content-specific implication of the study is that coloring activities, which are frequently used in schools, may be very difficult and sometimes misleading even for students with robust understandings.

The study also supported the key role that the postulated mechanism of learning a new conception—reflection on A-E relationship—plays in advancing students' conceptions. This implies a critical role for teachers. Engaging students in thoughtful activities must be followed with thoughtful probing of reflective processes in directions that can foster the intended learning. In this sense, the study implies that the model of teaching and learning is a plausible goal for teacher education. That is, in order for teachers to be articulate about their teaching activities, especially unsuccessful ones, they need to develop understandings of both general constructs of learning and teaching and content-specific models regarding students' thinking so they can intentionally focus students' reflections.

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