DESIGNING TO EXPLOIT DYNAMIC-GEOMETRIC INTUITIONS TO MAKE SENSE OF FUNCTIONS AND GRAPHS

Shinwha Cha and Richard Noss

Institute of Education, University of London, U.K.

ABSTRACT. We report on the results of two exploratory experiments which are part of a broader study seeking to find ways for students to express their knowledge of functions and graphs within dynamic geometry-based situations that do not explicitly involve algebraic representations. In the first experiment, the students displayed an intuitive appreciation of coordinates and graphs, understood geometrically without any algebraic representation. We attempted to capitalise on this intuition in a second experiment, designed to assist them in finding ways of expressing themselves algebraically. The findings suggest that this approach might be effective in activating new ways of viewing graphic representations and, from a research point of view, for helping to appreciate the expressivity of dynamic-geometrical media.

INTRODUCTION

This paper is based on an exploratory study in the area of using dynamic geometry (DG) to look at simple locus problems and conic sections. The overall aim is to link Euclidean geometry and analytic geometry, which tend to be regarded by students as separated subjects, often poorly understood. The aim of this phase of the work is to elaborate principles for design, and to study in some detail, the epistemological and cognitive opportunities and constraints of our approach.

It seems that students rarely see the geometrical sense underlying analytic geometry because tends to be dominated by algebraic formalism: as Mason (1997) points out, algebra is usually developed in the curriculum as a form of generalised arithmetic, with a "rush to symbolism" that is divorced from geometry. We are interested in developing an approach which preserves and extends geometric intuition and it turns out that there is an historical precedent for this: ideas for linking Euclidean and analytic geometry can be found in the history of mathematics, in particular in the ways that the ancient Greeks (Menaechmus, Apollonius, Pappus) developed an understanding of conic sections as loci and their 'equations' without any algebraic symbolism, using geometric constructions to express 'algebraic' relationships (Coolidge, 1940). This was a fundamental inspiration for the later inventors of analytic geometry and algebra (Vieta, Descartes, Fermat). What is notable about the Greek work is how the lack of an algebraic notation severely hampered their ability to appreciate the generality of what they had discovered. As Boyer notes:

> there appear to be no cases in ancient geometry in which a coordinate frame of reference was laid down a priori for purposes of graphical representation of an equation or relationship, whether symbolically or rhetorically expressed. Of Greek geometry we may say that equations are determined by curves, but not that curves were defined by

equations. Coordinates, variables, and equations were subsidiary notions from a specific geometrical situation (Boyer 1968, p.173).

BACKGROUND

The examples of locus first encountered by students are typically the circle, perpendicular bisector, and angle bisector. UK students are often simply taught to solve these simple locus problems in terms of a geometric construction, neglecting the 'pointwise' structure — i.e. they are seldom encouraged to express the properties common to points on the locus. Unsurprisingly, students often develop a superficial impression of the locus idea which can be summed up as: 'To solve this locus problem, construct this geometrical figure'. For example, to find the locus of points equidistant from two given points, you construct the perpendicular bisector. Or, as one student put it, "You just find two points and then join them up." This is a 'global' meaning for locus, seeing it as a whole shape, in contrast to a 'local' understanding, which means seeing the properties of individual points on the locus. The pointwise idea is fundamental in analytic geometry, beginning with an arbitrary point which satisfies given conditions and then generalising the point into an algebraic form so that the locus can be plotted in a Cartesian system.

METHODOLOGY

We carried out some experiments with several 14 year-old students, designed to compare students' understandings of the 'local' and 'global' structure of loci under the different cognitive and cultural influences of working with conventional tools (compass and straight-edge) or DG. In Experiment 1, the students were provided with worksheets that consisted of two tasks to be attempted, first using compass/straight edge and then with Cabri. For Experiment 2, working with Cabri only, we selected the two best students from those who participated in Experiment 1. The data shown here are based on audio transcripts of the students' conversations working with Cabri, their written responses to the worksheets, and their Cabri files.

EXPERIMENT 1: AN INTERESTING STRATEGY

For reasons of relevance and space, we report only the result of one part of this experiment. In this, we asked students to find the locus of points which have twice the distance from one given point as the distance from a second given point (a construction known as the Apollonius Circle). This is a difficult task for students who are not versed in geometry, and was certainly more complex than any simple standard loci question they had encountered.

The students needed to find points P where AP:BP is 2:1. The point P can be dragged by the mouse, and the two bars on the left represent the lengths of the segments AP and BP, dependent on the point P. Note that it was made easier for the students because they had to look for *equal* bars when the lengths are 2:1.

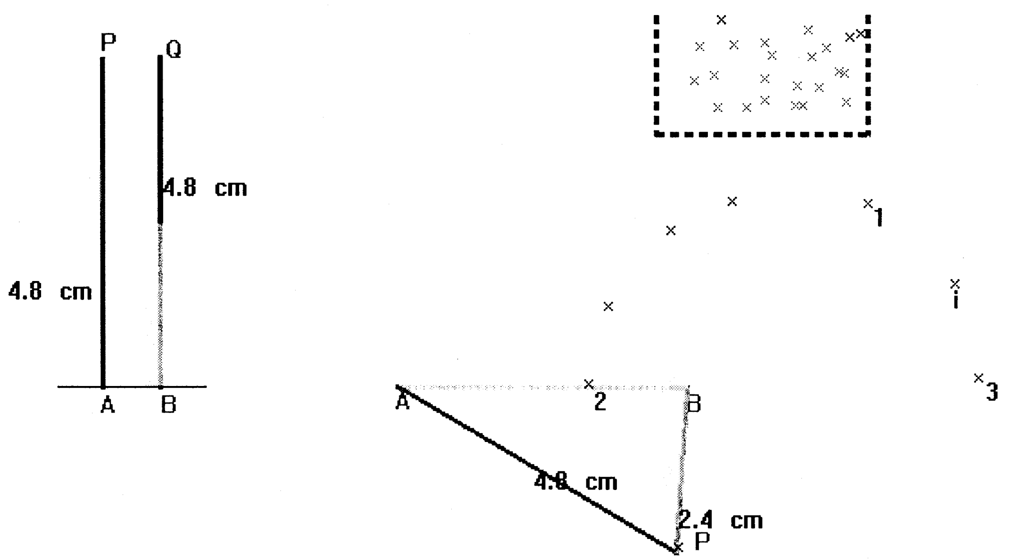

Figure 1: Students drag the point P so that the dynamic segments AP and BQ stays roughly the same length. Points can be dropped onto the screen by dragging them from the box at top right.

When the student finds a point P that has AP = 2BP she marks it with a cross (×), dragged from the box at top right (Arzarello et al, 1998, describe this kind of systematic dragging as "dummy locus", in which "dragging can act as a mediator between figures and concepts", ibid, p. 37). The points marked in Figure 1 indicate the order in which the students found them. Notice that they did not first find the 'internal' and 'external' points on AB, which one would normally start with in an analytic approach.

The locus is a (Apollonius) circle, and given that a circle is the most familiar curve, the students all identified it as such without further justification. However, a different Cabri situation provoked an interestingly different response.

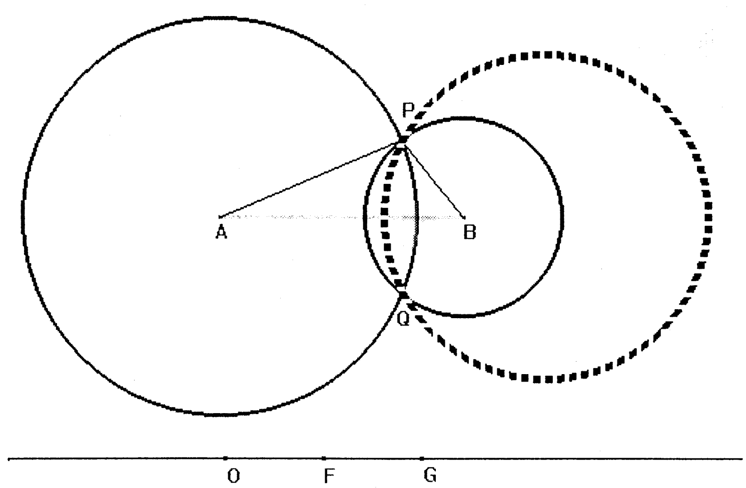

Figure 2: The Apollonius Circle using Cabri's Trace function

Given the set-up as shown in Figure 2, the students were firstly asked to make a guess about the shape of the locus as the control point F is dragged along OG (which determines the radii AP and BP). Before turning on *Trace* (used in preference to

Locus for reasons of simplicity), their first conjecture was that the locus might be a circle or an arc. But, suddenly, they shifted to say it is a line:

Alice: It's like an arc isn't it?

Jackie: It would be a circle. Can you move it [F] further?

Alice: (*repeatedly*) Always arc.

Jackie: They get, close it together. It's just part of a circle. It always is a circle.

Alice: It's a straight line. It's a straight line isn't it? It is between P and Q. *(keeps saying 'straight line')*.

Jackie: Agreed.

They turned on *Trace* and then they saw the circle. They said, "Uh… no its a circle". When they were asked why it's a circle, they replied as follows:

> "The distance between P and Q is a straight line. But they [the circles] move, they are together and then they are separate [and then] together again"

Clearly, they are using some mental image of a line. Of course, they have a limited range of experience to explain and make sense of geometric images. For example, in their experience having two points is often a cue to join them up, and this becomes a prototype strategy for what they are supposed to do with two points. Also, they had recently worked on a task involving a perpendicular bisector where they had joined two intersection points P and Q to make a line. However, notice that they say "they are together and then they are separate [and then] together". They are, it seems, *imagining* the line joining up points P and Q as they drag the point F, and have described the locus in these terms (see Figure 3).

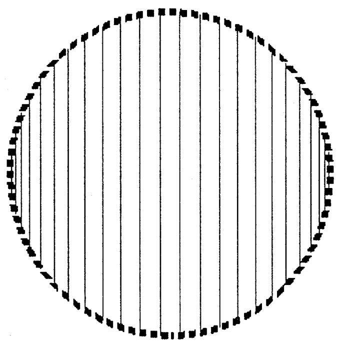

Figure 3: Visualising the locus as the end points of a changing line segment

We think that the students have hit upon a rather interesting way of making sense of the Apollonius circle. Seeing the dynamic image of the points P and Q (Figure 2) moving seems to have stimulated their geometric intuition in a novel way. Although it is not 'pointwise', it is a 'local' understanding. This kind of intuition, something like 'slicing a disc', can be found in the history of mathematics. It is very like the techniques of Apollonius and Dürer for constructing conic sections, which use a pre-Cartesian form of coordinates. And in the 17th century, Kepler used the idea of 'slicing' to find a formula for the area of an ellipse, which was a significant step in the development of the calculus (Boyer, 1968).

In the early 16th century, Dürer described a method to construct the conic sections of a right circular cone (Pedoe, 1976), based on Apollonius' study of the conic sections (Boyer, 1968, pp.164-5). Dürer's approach seems to be related to the strategy we observed with our students, in that it is a rather intuitive technique to draw a 'graph' without using algebraic representation. Figure 4 illustrates the method using Cabri. Suppose a cone is sliced by a plane. The triangle DEF shows the situation at the centre of the cone, where the plane cuts the cone at points A and A'. We make an object point M on the segment AA'. The distance A'M is the first quantity that we want to plot. For each value of A'M we need to know the width of the conic section at that point. Dürer found this by drawing the circle corresponding to the height of the cone at that point, and finds the width PQ by locating the intersection of a horizontal line through M with the circle. Finally, on the "axis" A'M we add the segment PQ perpendicular to A'M. And finally if we do *Locus*, we can see the locus of the conic section.

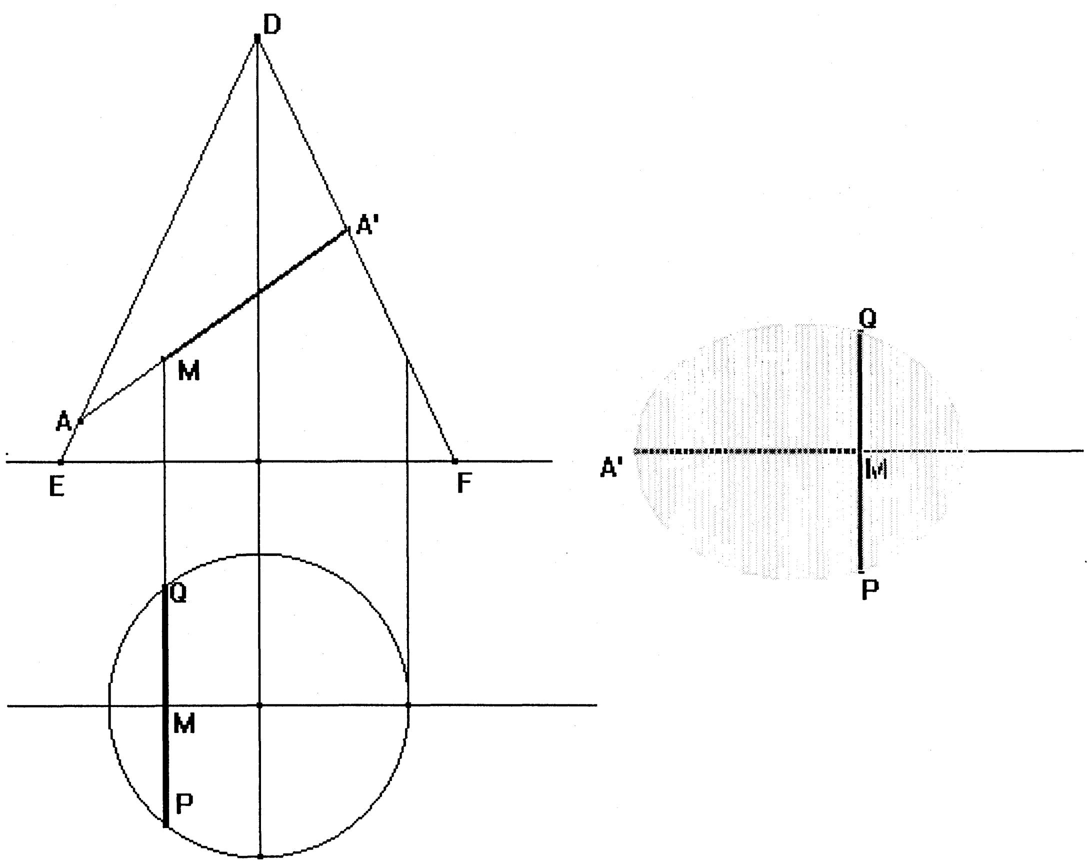

Figure 4: A Cabri version of Dürer's construction of the conic sections.

It seemed to us that 'Apollonian coordinates' in Cabri might provide some similar cue for our students, allowing us to build on their existing intuitions to extend, geometrically, their knowledge of functional relationships without necessary recourse to algebra.

EXPERIMENT 2: GRAPHING WITH 'GEOMETRICAL COORDINATES' WITHOUT ALGEBRA

We designed an exploratory sequence in which two students were invited to make geometric constructions and to investigate the relationships between different distances and areas in the construction, plotting the relationships on Cartesian axes using the *Measurement Transfer* function of Cabri. In these tasks we were interested to see how students interacted with and interpreted the Cabri situations, rather than (at this stage anyway) looking for learning outcomes.

One task was the familiar optimisation problem of finding the maximum area for a rectangle whose perimeter is fixed. As the construction was explained to the students, we discussed for some time what would be the shape of the graph of the area as the side lengths are varied by dragging a vertex. They easily sketched a parabola-like shape.

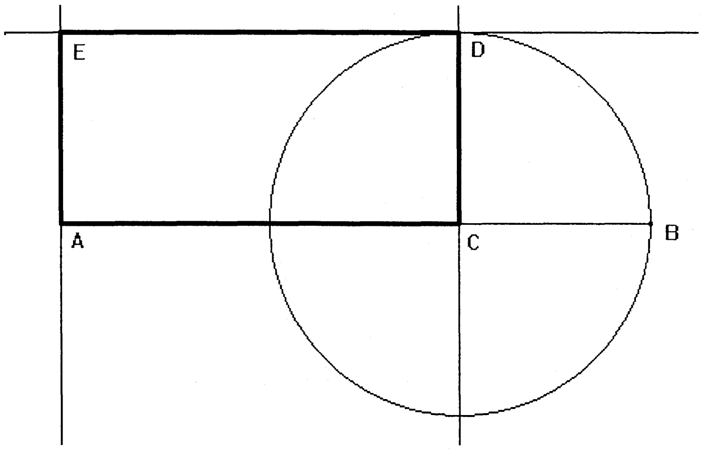

Figure 5: The rectangle area task

They constructed, by following instructions from a worksheet, the rectangle (Figure 5) and informally (without measurement) observed how dragging the point C affects the area and perimeter of the rectangle, and in particular, where the greatest area occurs. Afterwards we turned on the measurements for them to view, and by looking at the change of the numerical value for the area they noticed it had a maximum when the rectangle become a square. These students had had some experience with this kind of problem, and when we asked them about the relationships involved, they found it natural to talk about a formula although they could not precisely express it in symbols. They knew that a parabola-like shape was reasonable for the function, and they noticed that the area of the rectangle can have the same value at two different values of the side length.

After explaining to them how to plot a graph (using *Show Axes* and *Measurement Transfer*) they were able to construct a 'plot', based on *Locus*, for the graph of the rectangle area against side length (Figure 6).

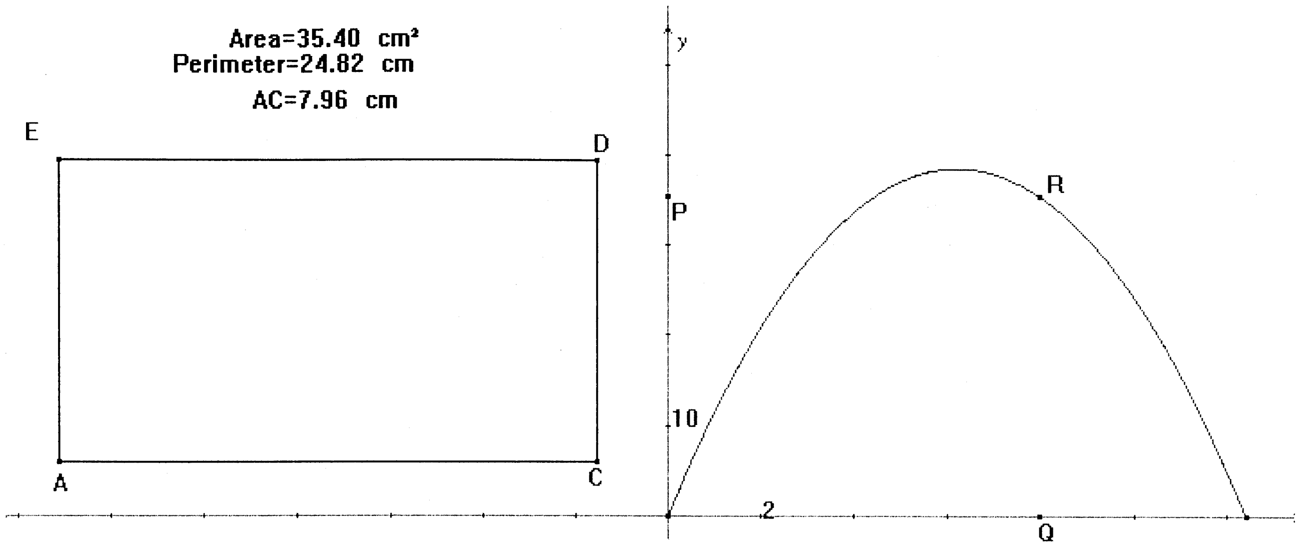

Figure 6: A graph of rectangle area against side length in Cabri

The final step was to introduce algebra into the problem: we asked them *If CD has length x and the perimeter has length P what is the area?*

In response to this question, they articulately expressed the structure of the algebraic expression of the area of the rectangle. We pushed them towards using the quantities x and P, and after some prompting to think about a specific numerical example, they eventually said:

A: … oh, I know it's P take away 2x and then divide by 2, then that number times x.

However, they had great difficulties in *writing* the algebra, and were unable successfully to construct the expression for the area. The problem, we think, came from the students' limited ability to think with 'abstract' symbols, compared with thinking about symbols as labels on a concrete geometrical figure. They held onto the idea that a quadratic function is associated with a parabolic shape, but arrived mistakenly at the expression for area $S = P - x^2$ (although wrong, we carried on the discussion with it in order to probe the students' thinking). Surprisingly, Jackie thought this equation did *not* represent the graph that they plotted in Cabri, because she sketched *S* as a ∪-shape, reasoning that by substituting numbers at a few positive values (x = 1, 2, …) the value of *S* is decreasing and must therefore reach a minimum at a later point — a reflection of how she was taught to plot graphs in the classroom by calculating a few points and then joining them up. Though there are no problems plotting a line from its equation using this approach, with higher degree equations it can be misleading. By contrast, the approach from geometry using loci gives students a sensible whole graph.

CONCLUDING REMARKS

In the experiments we have described, locus played a mediating role to afford an expression for generalising the relations between quantities. As one student put it:

"Oh, the Locus [is] where we can go! That's all the places they [the points or segments] can go"

This is a semi-abstract situation in advance of using algebraic expressions, in which quantities are 'numericalised' from the concrete geometric constructions. Given the tendency in the UK curriculum to substitute empirical experiment and measurement for the elaboration of structural relationships, we believe that we might have hit upon a way that students can be led naturally to algebraic representations and functional relationships. In our experiment, for example, from observing a geometric construction by dragging, the students elaborated their thinking about quadratic relationships, the symmetry of the situation and the existence of a maximum. More importantly they were able to express some of this knowledge geometrically. As another example, x=constant and y=constant graphs are among the most difficult equations to learn about algebraically (and these students struggled with pre-task questions we posed on this topic), but when the students 'accidentally' created an x=constant graph by plotting area against perimeter, they seemed to have no hesitation in reading off the vertical line graph in terms of its equation.

The exploratory studies we have undertaken so far have proved sufficiently interesting from cognitive and epistemological points of view, to provide grounds for cautious optimism. In future work, we intend to explore ways in which we can design didactical situations that afford students the opportunity to coordinate quantities by geometric construction in a DG environment, and further, to use this as a stepping stone towards the expression of functional relationships in algebraic terms.

REFERENCES

Arzarello, F. et al (1998). Dragging in Cabri and modalities of transition from conjectures to proofs in geometry. *PME 22 Proceedings*, vol. 2, 32–39.

Boyer, C. (1968). *History of Mathematics*. New York: John Wiley.

Coolidge, J. L. (1940). *A History of Geometrical Methods*. Oxford University Press.

Mason, J. (1997). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran and L. Lee (Eds.), *Approaches to Algebra*. Dordrecht: Kluwer.

Pedoe, D. (1976). *Geometry and the Visual Arts*. New York: Dover Publications.