A GROUP AS A 'SPECIAL SET'? IMPLICATIONS OF IGNORING THE ROLE OF THE BINARY OPERATION IN THE DEFINITION OF A GROUP

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The study from which this paper originates aims to contribute to the understanding of the difficulties that mathematics undergraduates encounter in a Year 2 course in Abstract Algebra. Analysis of their written responses to a question regarding a generic group on a set of four elements seems to suggest that such a group is not seen as a pair (set plus binary operation) but as a 'special set' where the axioms describe properties of the elements and not of the operation. Here we focus on certain implications of seeing a group as 'a special set': the students' occasional disregard for checking associativity (especially in a case where the group was presented in a table) and their neglect of the inner structure of a group (especially in their claim that two groups can be isomorphic but have different subgroup lattices).

The study of group theory gathered momentum in the mid-nineties with the work of authors like Ed Dubinsky, Uri Leron and Rina Zazkis (see references). The action-process-object-schema (APOS) theoretical framework was used to infer a general pattern for learning concepts like group, subgroup, cosets and normal groups. Also the learning of the concept of isomorphism (see Leron, Hazzan and Zazkis, 1995) was explored, and so was the use of Lagrange's Theorem (see Hazzan and Leron, 1994). The importance of studying this particular subject was suggested in Dubinsky et al. (1994):

> In many colleges, abstract algebra is the first course for students in which they must go beyond learning 'imitative behaviour patterns' for mimicking the solution of a large number of variations on a small number of themes (problems). ...An individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples, indeed including groups consisting of undefined elements and a binary operation satisfying the axioms. (p268)

This is still the case in most Universities in the UK, including the university where the present work was based. In this paper we want to show that this first encounter with this 'new' level of abstraction (see Hazzan, 1999) appears to be problematic as the students find it difficult to co-ordinate all the parts needed in a conceptualisation of the notion of group.

Research Issue. The main research issue we want to address in this paper regards the learning of the concept of group. More specifically, our conjecture, in the light of the evidence drawn from our study, is that the most difficult obstacle that the students face while progressing in forming a group schema is understanding that a group is formed by a pair: a set and a binary operation. In a study of the students' written work (see Methodology below) we observed this difficulty in the context of an exercise in which the students had to demonstrate an understanding of the axioms in the definition of a group (closure, associativity, neutral element, inverse element) and appeared to treat these properties, especially that of associativity, as properties of the elements of the group rather than properties of the binary operation. We also note that, while closure, neutral element, inverse element and commutativity are properties that can be checked by applying the binary operation on *two* elements, associativity involves operating on *three* elements and in a particular order – which, in the context of the particular exercise we examine here, was significant.

Methodology. This study is funded by the Nuffield Foundation and will last six months (October 2001 - March 2002). It is a small, exploratory data-grounded theory study (Glaser and Strauss 1967) of the mathematical writing of the students (55 in total) in the Year 2 Group Theory course of the 3-year degree in Mathematics at the University of East Anglia in the UK. It is the third phase in a series of small Nuffield-funded projects run by the two authors during the last two years as an on-going collaboration between the School of Mathematics and the School of Education at UEA (see for example Nardi and Iannone 2000). The aims of the study are: identifying the major problematic aspects of the students' mathematical writing in their drafts submitted to tutors on a fortnightly basis; increasing awareness of the students' difficulties for the tutors at this University's School of Mathematics; providing a set of foci of caution, action and possibly immediate reform of practice; and, setting foundations for a further larger-scale research project. The first two phases focused on Year 1 Calculus, Linear Algebra and Probability courses. The current phase focuses on the Year 2 Abstract Algebra course.

The course ran for 5 weeks at the beginning of the Autumn Semester 2001, with 4 hours of lectures per week. There were 3 seminar sessions for each of three groups of 15-20 students. The lectures were traditional front-teaching sessions. The seminars were run by a seminar leader (the first author) and a seminar assistant. The lectures were observed by the first author in order to become familiar with the examples and the notation the students were exposed to as well as with their general reactions to the new content. At the end of each seminar session the students were asked to submit a selection of the exercises in an exercise sheet. Their responses were then marked by the seminar leader and the seminar assistant. These written responses to the problems sheets administered at each of the three seminar sessions form the bulk of data gathered for this project.

The first level of data analysis, Data Analysis Version 1, involved the production of a student–by–question table that focused on the written responses of 15 out of 55 students registered for the course. This table summarised observations and comments of the first author on the students' responses to the set homework. After this table was completed, informal conversations were held with the seminar assistant to record his observations and comments after having marked part of the homework of the remaining student cohort. Following a detailed discussion of Data Analysis Version 1, the second author produced Data Analysis Version 2, a question by question table where the major issues were summarised, characteristic examples of the students' work were referred to and links with current literature were made. In these analyses it became clear that probing further into the students' thinking would be greatly helped if they could be asked to provide justification for parts of their writing, for example, via interviewing them. An initial step in this direction has been taken with an extensive interview of one of the students (parts of the transcript we use here to support the evidence from the students' written work).

Forming An Image of the Concept of Group. The data we use to raise the issue of the students' understanding of the concept of group originate in the first cycle of data collection and in the students' responses to the following question:

Q1.5: Write down all group tables for a group of four elements. Hence show that there are two essentially different such groups, both commutative. (Consider group tables obtained by merely renaming elements as essentially the same). How are they best distinguished? For each make a list of all the subgroups.

An understanding of group, subgroup and commutativity as well as a "naive" concept of isomorphism (see Leron, Hazzan and Zazkis, 1995: "Two groups are isomorphic if they are the same except for notation") are necessary here. Moreover let us observe that, at this point in the course, students had only just been introduced to the concept of isomorphism, and this is why the question setter suggested a criterion for judging when two groups are "essentially the same".

Fifty-two pieces of homework were handed in that week and all the students but one attempted this question. The most common response consisted of:

- a list of the four tables that can be obtained from a set of four elements;
- a declaration of some of the obtained groups to be isomorphic, hence the existence of "two essentially different groups";
- a list of the subgroups of each of these groups.

Commutativity was dealt with by observing symmetry around one of the diagonals of the tables. In the following we offer certain observations on and exemplify the students' responses.

A major common characteristic of the responses is that none of the students checked whether a group that satisfies these tables actually exists. The students constructed the tables possibly bearing in mind the direction of the lecturer (as recorded in the first author's observation of the lectures) to check that each element of the group appears only once in every column/row. This way of proceeding takes account of the properties of inverse, neutral element and closure but leaves out the checking of associativity. The fact that the table has to be shown to be associative, or else that the students have to produce an example of a group that satisfied the group table, went missed. One interpretation of this fact (see also interview extract later) is that, at this stage, the students deal with the concept of group as a "special set". The schema the students are referring to is the "set" schema and the properties are checked as to be properties of the elements and not of the binary operation defined on the set to form the group. If we agree that, by placing an emphasis on the *order* in which elements are operated on, associativity is the property of a group that refers more to the operation and not so much to the elements, this is not surprising. In short, instead of having a concept of group consisting of three interrelated schemas: set, binary operation and axioms (see Brown et al. 1997), the set schema dominates leaving the binary operation schema, and the checking of certain properties, neglected.

One implication of this domination of the set schema is evident in the response to Question 1.5, offered by Jo:

5. ①

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

②

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

③

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

④

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

① is distinguishable because it is cyclic.
The others can be seen as the same group
as they are isomorphic
All tables are symmetrical about the line from top left
to bottom right and are therefore commutative.

The subgroups are:

① {e, b} {e} {e, a, b, c}
② {e, a} {e} {e, a, b, c} {e, b} {e, c}
③ {e, a} {e} {e, a, b, c}
④ {e, c} {e} {e, a, b, c}

In Jo's script three groups are said to be isomorphic. Yet a few lines later they are shown to have different subgroup lattices. The concept of 'naive isomorphism' seems to be affected by a failure to recognise a group as a pair. The notion that the binary operation defined on the set induces naturally an inner structure of the group (its subgroup lattice) is missed and this inner structure is not regarded as something that characterises a group. Therefore, two groups are seen to be 'essentially the same' even if their inner structure is different. Another student, Hazel, also does not seem to see any contradiction in the claim that two groups are isomorphic but have different subgroups:

5. All gp tables for 4 elements:

	e	a	b	c
e	e	a	b	c
a	a	○		
b	b		□	
c	c			

First row and column are fixed
○ can be e, c, b
□ can be e or a after e in ○
~~c or e~~ after c in ○
~~e or a~~ after b in ○

So 4 tables:

1)

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

2)

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

3)

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

4)

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

There can divided into two different groups; these that are isomorphic ie 1), 2), 3) so that elements can be renamed to produce another table
eg to turn 1) into 2) swap the e's and a's round.
4) is not isomorphic but is distinguished by having a cyclic propertie ie the rows and columns follow the order of e,a,b,c but starting on different letters

All the tables are commutative, that is, They are symmetrical about the diagonal axis from the top left to the bottom right.

List of all subgps: 4) has subgroups: {e}, {e, a, b, c} and {e, b}
1) has subgps {e}, {e, a, b, c}, {e, a}, {e, b} {e, c}
2) has subgps {e}, {e, a, b, c}, {e, a}
3) has subgps {e}, {e, a, b, c}, {e, c}

Hazel's actions, typical in the written responses we have examined, appear to be heavily table-based. Her understanding of the notion of isomorphism seems to involve processes she calls 'swapping' and 'turning into'. Could we suggest that this firm adherence to/dependence on table-based actions, while facilitating the construction of the groups with four elements (and the checking of properties such as commutativity), at the same time places an obstacle in the students' constructing an image of the group's inner structure (as well as perhaps distracting them from checking associativity which requires operating upon three, not two, elements)? That

Hazel's notion of isomorphism is problematic is suggested also by the grammar/syntax she uses when she talks about isomorphic groups: she doesn't talk explicitly about, e.g., 1 as isomorphic to 2 but she writes '4 is isomorphic' (to what?).

This problematic understanding of the 'naive' concept of isomorphism was widespread across the responses we analysed – as was the exclusion from the list of subgroups of the trivial ones ({e} and {e, a, b, c}) such as Steven:

Subgroups

2: {e, a}, {e, b} and {e, c}
3: {e, b}

and the inclusion of subsets that were not closed or did not include the identity element. For example Wayne – who also throughout his writing does not adopt a consistent and conventional bracketing system to denote a set:

Can divide into 2 ~~groups~~ different groups
i) Group has 2 self-inverting elements (tables 1, 2, 3)
ii) Group has all self-inverting elements (tables 4)
Both commutative as the tables are symmetric on the left diagonal.

Subgroups for i)
(e), (a), (b), (c) and (e, a*) [and (e, a, b, c)]
* a, b, or c depending on which element is self-inverse.

Subgroups for ii)
(e) (a), (b), (c), (e, a), (e, b), (e, c) [(e, a, b, c)].

This suppression of the role of the binary operation and the tendency to attribute properties of the binary operation to the elements of the group was also evident in one interview with Wayne held at the end of the Group Theory course (we conducted this interview to test whether interviewing could illuminate us any further about the reasons behind certain parts of students' writing). The student responded to our question about the difference between a set and a group as follows:

> **Wayne:** Hum ... A group has axioms ... certain axioms that have to hold. One is associativity. That is basically if you got elements, I don't know, a, b, and c in the group then ... then, a plus b, well, say that it is plus the operation [...] Yes, Then a plus b ... plus c is equal the same as a plus...b plus c. That's the first property. You have got some neutral element. Which basically says that, say call it e, ... then if you multiply by any of these elements in the group you still get ... you get the same thing out. And then from that ... And also you have got an inverse. You have got a and some d they come together to come the inverse. a plus d is the inverse. Basically.

In Wayne's words "a group has axioms" that "have to hold". These axioms include associativity, neutral and inverse element. In this extract the binary operation is mentioned – albeit alternately referred to as 'plus' and 'multiply' (which is an often problematic choice of words - see Nardi 2000). He also seems to confuse the use of the words "inverse" and "neutral" in "a and some d they come together to come the inverse. a plus d is the inverse" but still his description of the properties seems to indicate a clear understanding of how they characterise the relationship between the *elements*. Later on he links these properties to his perception of a group:

> **Wayne:** Oh ... I think that the group is a special kind of set, basically, where you got certain properties, basic properties, axioms, basic axioms that have been defined and for groups these hold, no matter what. And so ... that's why I see a group as a special kind of set. That's ... is not a set ... a group is like, ... a group is a set, but is a special kind of set that has properties that can be defined, can be shown to be true. For each group that you have. That's basically...

In Wayne's words a group is a set whose elements happen to have certain properties, it is a "special set". No reference is made to the group's inner structure and to the central role the binary operation plays in the formation of this structure.

Summary and Conclusion. In this paper we discussed Year 2 mathematics undergraduates' developing notions of a group in the context of their first encounter with a generic group on four elements.

Following strictly table-based actions the majority of the students constructed the four possible tables but, by leaving out the checking of associativity, they refrained from checking/showing whether a group that satisfies these tables actually exists. We conjectured that, by strictly remaining within a table-based action schema, the students demonstrated a dominant image of a group, not as a pair (set with a binary operation), but as a "special set". In this schema the properties of associativity, inverse and neutral element are attributed to the group (and in particular to its elements) and not to the binary operation.

There seem to be certain implications of the above conceptualisation of a group as a "special set" in the students' responses. One is that an image of a group that neglects the role of the binary operation – namely, the inner structure that the binary operation yields – makes the emergence of problematic images of isomorphism possible. Such images include: two groups can be isomorphic but have different subgroup lattices.

Another implication of the above involves an image of a subgroup as a subset of a group - evident in the students' responses where subsets that were not closed or did not include the identity element were presented in the list of subgroups of the group.

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