SUPPORTING STUDENTS' REASONING WITH DECIMAL NUMBERS: A STUDY OF A CLASSROOM'S MATHEMATICAL DEVELOPMENT

Ada Boufi Frosso Skaftourou
University of Athens

The focus of this paper is on students' developing conceptions about decimal numbers in a fifth grade classroom. To collect our data we organized a teaching experiment that lasted four weeks. Our students' progress was significant. Their development was supported by activities in the context of metric system measurement and with the model of the number line. In the analysis of our data we delineate the mathematical practices institutionalized by the classroom community. The shifts in this classroom's practices confirm our anticipated learning path.

Introduction

Research shows that students have many misconceptions about decimal numbers. In a recent comprehensive review of these studies (Stacey, 1998), we can notice that these misconceptions are mostly accounted for in psychological terms and they are not analyzed in relation to students' instructional experiences from their classrooms. In this way, teaching decimal numbers can not be adequately informed by available research data. How could teachers by just knowing the prospect of these misconceptions appearing avoid them? The only alternative we can imagine is for them to give their students detailed instructions at the risk of making mathematics more algorithmic than it is. On the other hand, if teachers try to follow the suggestions which are usually given for remedying these misconceptions, they will end up with a top-down structuralist teaching approach. Notwithstanding that, this approach has repeatedly been criticized as insufficient to assure quality in students' learning experiences (Gravemeijer, 1994).

A relational understanding of decimal numbers might be built if teaching had a different orientation. Such new teaching approaches are developed by researchers to promote a meaningful understanding in several mathematical areas (Carpenter, 1997; Cobb, 1999; Lampert, 1989; etc.). In these attempts, teaching is based on students' informal understandings and strategies, it supports the gradual mathematization of their contributions, and encourages communication in the classroom. We tried to develop a similar orientation in teaching fractions and decimals in a fifth grade classroom. At the end of our classroom teaching experiment our students did not show any misconceptions about decimals and their understanding was meaningful. Accounting for students' learning in this classroom could be useful for instructional design research. Also, the results of such an analysis might help teachers to reflect on their current instructional practices. In fact, it was on the basis of preliminary analyses of our students' learning that we were trying to support their developing understandings of decimals. Therefore, this paper presents a first attempt to account for our students' learning in the social context of their classroom by documenting their increasing understandings of decimals.

Theoretical framework

Our focus in presenting our results will be on the ways that taken-as-shared meanings of decimal numbers emerged in our classroom. This is particularly important if we consider these meanings, which originate in students' contributions, as constraining or enabling their individual activity. This consideration stems from the emergent perspective developed by Cobb and his colleagues (Cobb & Yackel, 1996). In this perspective mathematical learning is treated as both a social and an individual activity. More specifically, these two aspects of learning are taken as reflexively related. Individual students reorganize their activity as they participate and contribute to their classroom's mathematical activities. On the other hand, the students' collective activity does not exist separately from individual students' diverse ways of reasoning.

Methodology

The data we will analyze come from a fifth grade classroom in a school of Athens. These data include: 1. the videorecordings of all 17 lessons on decimal numbers, 2. the videorecorded interviews taken from most of the students upon the completion of our teaching experiment, and 3. the 21 students' notebooks. The presenting author taught most of the lessons in this unit, immediately after a fraction instructional sequence.

The activities used in our teaching experiment were a crucial factor for our students' developing understandings of decimals. But our students' activity as well as their further mathematical development was also influencing the selection and development of appropriate activities. Thus the set of instructional activities used in this teaching experiment (see the results section below) was not fixed in advance. However, our overall purpose guided the selection of an initial set of activities. This purpose was to support students' flexibility in reasoning with decimal numbers. For example, we wanted them to compare decimals like 0,06 and 0,6 by reasoning that 0,6 is bigger because it is ten times larger than 0,06, or by reasoning that 0,06 is very close to 0,5, while 0,06 is far from it as it is closer to 0,1. Gravemeijer's (1998) idea that students can reinvent decimals through the activity of repeated decimating along with his heuristic of emergent models were instrumental in helping us to foresee a path through which we could guide them to develop their understanding of decimals.

The lack of a decimal monetary system in our country at that time was a serious difficulty. We thought that our students could be offered the opportunity to see decimals as describing the results of repeated decimating in the context of activities related to the standard metric system measures, which are commonly used in our country. Before involving students in activities in the context of measuring lengths, we decided to start from a fictitious scenario, similar to one found in the "Mathematics in Context" curriculum unit "Measure for Measure" (Britannica, 1997). Our students heard a story about a visit to an amusement park where a competition was organized on a special wheel of power. The winner would be the person who managed to turn this wheel more than the other participants would. In the context of this narrative the problem of finding the winner brought to the fore the issue of precisely measuring the turn of the wheel. Students made a number of suggestions including: halving the wheel and then again halving the half of the wheel, or making divisions like those on a clock face. Next, the teacher announced that this wheel was repeatedly divided by ten and students accepted this as an efficient way of measuring precisely the players' strength. The relationships among decimal fractions were then explicitly discussed. This was meant to prepare the exploration of the subunits' relationships on the meter stick, which was familiar from previous grades. So, when the meter stick was introduced students compared it to the decimally subdivided wheel and easily identified their similarities. These discussions concerning the wheel of power and the meter stick together with the students' learning history from fractions constituted the background for their engagement with the decimal instructional sequence. Also, we took into account that students had had a few lessons on decimals in the fourth grade. We hoped that students' reasoning with the meter stick, would come to function as a model of their activity on the wheel, i.e. of measuring in "ones", "tenths", "hundredths", etc. Eventually, we anticipated that reasoning with the meter stick would serve as a model for reasoning with decimal numbers. Through this shift, by embodying the results of repeated decimating, the symbol of decimal number would evolve into an experientially real mathematical object independent of their activity.

The above mentioned hypothesized path will be documented by utilizing the construct of practice (cf. Cobb&Yackel, 1996). Thus, the criterion we will use in describing the mathematical practices that emerged in our classroom community will be the legitimacy / acceptability of the students' explanations in the classroom as well as their content.

Results

After the introductory discussions on the wheel of power and the meter stick, the teacher asked students how they would find 75 cm on a double number line that she had drawn on the board as a simplified form of their meter sticks. Students offered several partitionings (see Figure 1). The variety of students' partitionings was not totally unexpected. Students had already connected the idea of refined measurement with the use of common fractions. From the outset, it seems that students' prior experiences from fractions were going to influence the course of their decimal understanding, just like in history where the origin of decimals can not be separated from fractions.

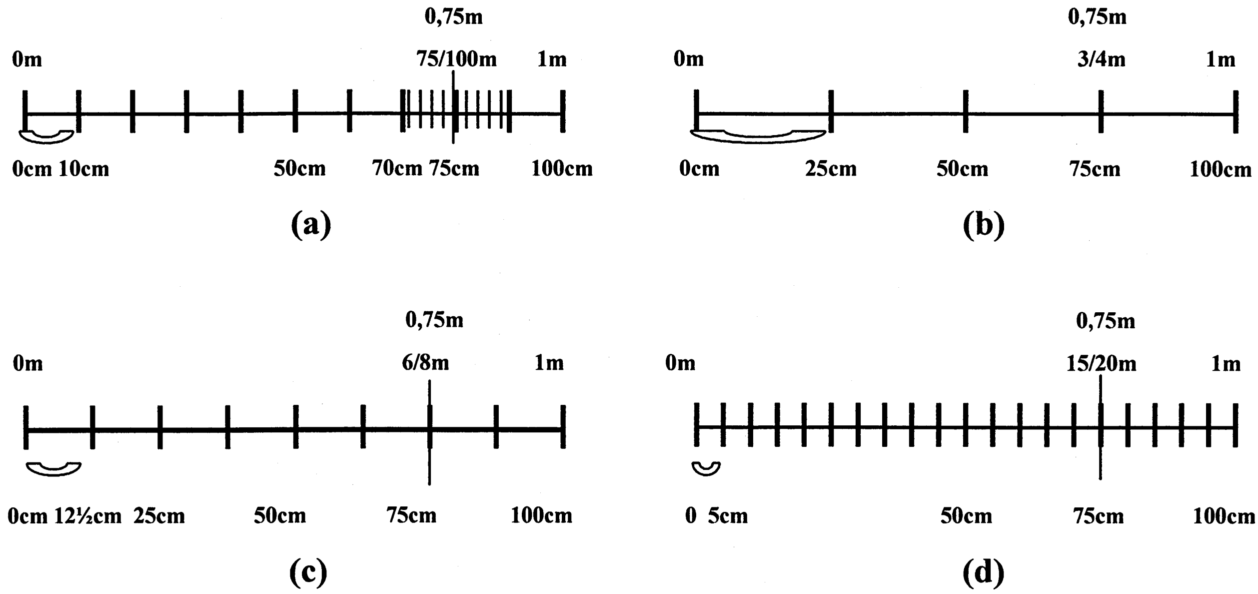

Figure 1: Place 75cm on the number line

The number line seems to have played a significant role in this episode. As our instructional intent is for students to reason with decimals independently of the imagery of the number line, inevitably the meaning of the number line will change: from students' using it with a context-specific meaning (measuring strength on the wheel or length with a meter stick) to using it in order to think about relations among units of a different rank. Therefore, analysis of subsequent episodes will be focused on students' interpretations of their activity as they reason with the number line.

After the first student's double decimating in order to place 75 cm on the number line, the teacher asked students to describe the result of his repeated decimating in meters. Apart from students' response 75/100m, the request for a decimal number in meters resulted in several answers: 75,0cm, 7,5m, 7,5cm, 0,75m. These answers became a topic of discussion and invalid answers were rejected. There were students however, who were unwilling to accept 0,75m as a legitimate way of notating. A possible explanation might be that in their prior experiences from mixed numbers "0" was not used to describe the absence of whole units. As students negotiated their different ways of symbolizing the results of repeated decimating, the first practice of connecting decimals to the records of decimally partitioning the number line seems to emerge. Subsequent episodes gave students additional opportunities to negotiate their ways of symbolizing the results of repeated decimating. For example, a few days later, the difference between 0,10 and 0,100 emerged as a topic of discussion. As students had not yet created the decimal number as a true symbol of decimating, the equivalence of these two decimals was not self-evident.

The "75cm episode" showed us that a possible source of students' difficulties was their familiarity with arbitrary partitioning. By including tasks that involved students in enlarging the unit of measurement for lengths not convenient for the use of different than decimal partitionings, we tried to support their attempts to create an operational meaning for decimals. As an example, 9cm were easily converted to a decimal after decimating the number line (see Figure 2).

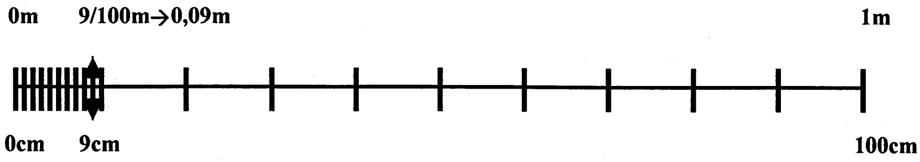

Figure 2: Place 9cm on the number line

Christos' activity on this number line was still dependent on the imagery of the meter stick. How else could he place 9cm on the number line? On the other hand, in contrast to other students, he consistently used decimal fractions before attempting to write down the equivalent decimal number, in this as well as in other similar tasks. Moreover, his behavior is representative of what some few other students were also doing in this type of activity. In accounting for these behaviors we conjectured that these students were applying the well-known trick of writing the numerator of the decimal fraction and then holding so many digits after the decimal point as the number of zeros in the denominator of the decimal fraction. Instead of coordinating the records of decimating with the digits of the decimal number, these students seemed to rely on numerical patterns. At the same time, their participation in the classroom mathematical practice of connecting decimals to the records of their decimating activity was unhindered. Their solutions based on their reasoning with decimal fractions were not challenged. In fact, it can be argued that these students were not in a position to see decimals in relation to repeated decimating, as they were not reasoning with the number line. In the Greek language the prefixes used for the subunits of the meter are identical to the fractional names used to describe their relationships to it. "Tenth-meter" is the word used for decimeter, "hundredth-meter" for centimeter, and so on. So they could well write the decimal fraction without paying any attention to what they had been doing on the number line. In such a case, the only purpose for carrying out the decimating process could be to specify the distance of a given length on the number line. Therefore, for these students understanding of place value would remain instrumental if further opportunities to reflect on the results of their decimating activity in connection to decimal numbers were not given. To this end, we involved students in the following activity.

Students were asked to place different decimal numbers with up to two decimal digits on schematized rulers. These rulers were already divided in centimeters or decimeters. Our main purpose in introducing this activity was for students to realize how repeated decimating was signified by the decimal digits. Students were encouraged to describe the decimal digits of the given decimals in terms of their actions of placing them on these rulers. The topics of conversation that emerged in the context of this activity provided opportunities for students who relied on tricks to reorganize their activity. Now, it seemed that the classroom's activity with the number line implied the numerical values of the decimal digits. In other words, it became a true model of measuring with units of different decimal rank.

With the next activities we tried to support students' reasoning with decimal numbers. We thought that if the values of the decimal digits would become independent of students' activity on the number line, this would allow students to focus on relationships among decimal numbers. Students' activity on the number line would evolve into a model for reasoning with decimal numbers. Gravemeijer (1997) notes that: "This transition from 'model-of' to 'model-for' implies a process of 'reification'. [...] what is reified is the process of acting with the model, not the means of symbolization itself". In our case, the activity of repeated decimating had to be reified. Signs for this reification emerged, in the context of several activities. The following episodes will clarify how this process took place in our classroom.

In one of the first lessons, the teacher asked students to find a length that would be very close to 3 meters. She drew a number line on the board (see Figure 3) and the first offer of 2,99 meters was placed on it. She then asked for a decimal number even closer to 3 meters. The number 3,0 was offered by a second student but it was rejected by other students as being equal to 3 meters. Another student suggested 2,999 meters. To the teacher's question of how far is 2,99m from 3m this student answered 1cm and when he was asked to place his number on the number line, he said: "I would divide it in 10 pieces, and it will be at the ninth small line". This episode closed with the specification of the distance that 2,999m had from 3m.

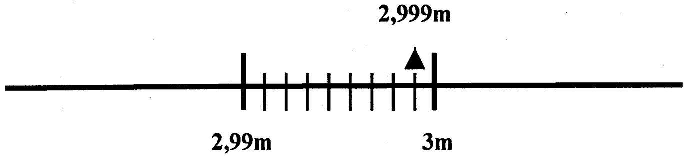

Figure 3: Find decimals close to 3m

The student's reasoning concerning the placement of 2,999m on this empty number line shows that for him the decimating process was already reified. The fact that other students did not ask any clarifying questions suggests that they could follow his argument. However, the focus of their comments indicates that they were not in a position to anticipate decimating activity. Many students objected that their classmate's drawing was not precise. As they said the small lines were not equidistant and the distance of 2,999 from 3 was longer than 1mm. The imagery of the meter stick was still too dominant. Therefore, reasoning in terms of repeated decimating was not yet a practice.

Later, students' work in the context of the activity with the schematized rulers might be considered as providing opportunities for many students to reflect on their decimating activity. So when for example, students had to place decimals with two decimal digits on the ruler with decimeters, they started anticipating their decimating activity. However, their anticipations did not become an explicit topic of discussion. Discussions were mostly focused on the results of their decimating activity and not on the students' intentions. An activity where students were asked to find decimal numbers in between other given numbers allowed students to make their reasoning in terms of repeated decimating explicit. When the teacher asked for a decimal at the midpoint of the distance between 8,6 and 8,7 students explained their solutions as follows: Stavroula said that the number will be 8,65 and then she explained her thinking by drawing a number line on which she had first placed 8,6 and 8,7. Then she divided the distance between the two numbers in ten parts and she showed 8,65 on the fifth line. Costas placed the two given numbers on a number line and then he explained that the wanted number will be at 865cm. In his mind, he had already transformed the given numbers in centimeters. Then he placed 8,65 at the midpoint of the distance separating the two given numbers. Helias tried to find 1/2 of 1dm. He said that it will be 5cm and then he added 0,05 to 8,6.

In all of the above solutions, students reasoned with the decimating process. Moreover, in the last two solutions the decimating process has already been concealed behind students' reasoning. The same finding concerns students' reasoning in converting fractions to decimals. Progressively they institutionalized a formalization of the decimating process in a long division scheme. For example, students jointly transformed 1/3m to 0,3333…m by making three successive divisions (100cm:3, 10mm:3, 10dmm:3). It was in this task that the notion of the infinite number of decimal places first struck them. However, the presence of the metric unit names in students' explanations might be considered as showing students' inability to escape from the imagery of measuring length. But the adaptability of their ways of acting as we posed tasks in different contexts, documents that their understanding of decimals was not context dependent. Without losing their ties to the phenomena that gave rise to them, decimal numbers have now become mathematical objects that students could reason with.

The practice of reasoning with decimal numbers seemed to be well established by the end of the decimal instructional sequence. Actually, students were able to reason not only with decimal numbers but at the same time to incorporate it into reasoning with fractions or whole numbers. Their sense of decimals is illustrated by students' reasoning on the task 4X0,75. The teacher had notated their solutions on the board as follows:

0,75	75cm × 4	0,75=0,25+0,25+0,25
+ 0,75	/\	1←0,75
150 cm→1,5m	(70+5) × 4	1←0,75
+150 cm→1,5m	(70 × 4)+(5 × 4)	+1←0,75
3 m	280cm+20cm	3
	2,8m+2dm	
	\ /	
	3m	

75cm	0,75m=1m-0,25m	0,75=3/4
+75cm		
150cm	4×1=4m	3/4+3/4=6/4
+150cm	-4×0,25m=1m	4×0,75=6/4+6/4=12/4=3
300cm=3m	3m	

Conclusion

In studying our classroom's mathematical development through an introductory course on decimal numbers, we identified three mathematical practices These practices represent a joint accomplishment of this particular classroom community. Individual students' participation in these practices was varied. For example, at the end of our teaching experiment, most of our students could reason with decimal numbers. However, there were students who were still reasoning in terms of repeated decimating. As the final episode shows, these students' participation in the classroom discourse was by no means insignificant.

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