CHANGE IN PUPILS' APPROACH TO PROPORTION PROBLEMS THROUGH CLASSROOM TEACHING: FROM THE VIEWPOINT OF MEDIATION BY SYMBOLS

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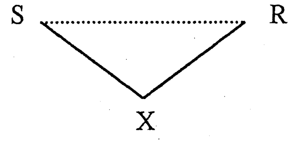
This article is part of a larger study that examines the development of pupils' proportional reasoning through classroom teaching. In a fifth-grade classroom, five pupils with varying proportional reasoning ability were studied by using qualitative research method. In this article, the pupils' approaches to the same proportion problems before and after a series of 11 lessons on "quantity-per-unit" are compared. Results show that the major change after the lessons was their more active use of notations in dealing with the aspect of ratio and proportion. Especially there are notations that lead one to make aware of his/her own thinking, i.e., make it articulated, systematic, conscious and intentional. These significant notations were observed clearly in four pupils. Still, specificity of the notations varied depending on the pupils' conception of invariant relation between two quantities.

BACKGROUND OF THE STUDY

The purpose of this study is to examine how the development of pupils' proportional reasoning takes places in the mathematics classroom. A great deal of studies on proportional reasoning has shown different strategies children use and relationships of them with task characteristics (e.g., Vergnaud, 1993; Kaput & West, 1994). Along with the result of investigation in everyday life and in workplaces, researchers began to discuss the inextricable links of people's thinking strategies with contexts (e.g., Nunes et al., 1993; Hoyles et al., 2001). In this study, classroom is also recognized as a unique place where students meet with different culture of mathematical ideas, feelings and discourse, which they will not know in their daily livings (Nagano, 1997). Indeed, in the classroom pupils' meet different ideas of ratio and proportion, novel notations and specific ways of attending and using the symbols. Here, it is essential not only identify different strategies but also to look closely at pupils' use of notations and associated meanings they give to them.

Hino (e.g., 1996, 1997) illustrated that pupils' use of notations, which is introduced in class, is changing in nature. To discern the change, Hino (1999) proposed three-phase model of evolution of pupils' use of introduced notations: "label use," "positive use" and "effective use." Case studies were done in different classrooms where notations such as "a÷b" ("a" and "b" belong to different kinds of quantities) or "y=mx" were introduced. They showed that many pupils stayed at the incipient "label use" in spite of the fact that the teacher was hurry in proceeding to the "effective use." Then what notations are productive and contribute to their evolution of use and eventually their development of proportional reasoning?

To delve into the question, the perspective of mediated action by symbols is taken (Vygotsky, 1978). As shown in the figure, elementary form of behavior is the direct reaction to the tasks in front of one's eyes (S-R). On the other hand, structure of sign operations requires another form of behavior, in which indirect link between stimulus and response is made and function as a secondary stimulus (S-X-R). This is called



"mediated action by symbols." The intermediate link not only improves previous thought operation, but also makes such operation qualitatively new and enables one to control his/her own action with the help of extrinsic stimuli. Vygotsky says that base of higher psychological processes is one's ability to create and use these artificial stimuli intentionally.

Studies on mediated action in the context of teaching and learning of ratio and proportion are not many. Lo & Watanabe (1997) traced one pupil's change in ratio and proportion schemes in the laboratory setting. The pupil initially found certain correspondence between two quantities by trial and error approach. As the sessions unfolded, he began to express his strategy by way of table. The table gradually contributed to shaping his direction of problem solving process; and later he came to curtail the table process by using multiplication. This process indicates that his thinking became to be mediated by tables ("X" in the figure above). The study shows the potentiality of analyzing pupils' change and role of notations in approaching proportion problems from the perspective of mediation by symbols. Still, there are unanswered questions, e.g., "what and how 'X' is chosen," "whether pupils choose introduced notations in class as 'X'," or "there are some relationships between the chosen 'X' and his/her proportional reasoning ability." In this article, five fifth-graders' approaches to the same proportion problems before and after a series of 11 lessons are compared. Specifically, the following questions are pursued:

- In what way can pupils' changes in approaching proportion problems be viewed by paying attention to the mediation by symbols?
- What kinds of symbols are chosen as the mediator and in what way do they function in pupils' thinking?

METHOD

In this study, data were collected in a fifth-grade classroom in a public elementary school, rural area of west Japan. There were 9 boys and 11 girls in the classroom. The author and a graduate student observed the lesson everyday while the teacher taught the chapter on "quantity-per-unit." We recorded each lesson by two video cameras and audio tape recorders, and by our field notes. We also conducted written tests for the entire pupils before and after the lessons. Moreover, as for the focused pupils, we interviewed them before, during and after the lessons. In the interviews, we asked the pupils to explain their approaches to some of the problems in the written test, and also to solve additional proportion problems to look closely at their approaches.

In the present article, the five focused pupils' approaches toward proportion problems before and after the lessons (abbreviations "B-L" "A-L" are used below) are compared on the individual basis. Such comparison is possible since many of the problems used in B-L and A-L are the same. The problems are composed of different situations and different ratio complexities. Their ratio complexities are classified into 4 levels according to Hart (1981):

Level 1: No rate needed or rate given.
(A car goes 200 km in 5 hours. If it keeps the same speed, how much time will it take to go 800 km?)

Level 2: Rate easy to find or answer can be obtained by taking an amount then half as much again.
(15 candies cost 240 yen, then how much will we pay for 20 of the same candies?)

Level 3: Rate must be found and is harder to find than above.

(A car goes 200 km in 5 hours. If it keeps the same speed, how much time will it take to go 75 km?)

Level 4: Must recognize that ratio is needed, the questions are complex in either numbers needed or setting.

(A rectangle (length : width = 6 : 8) is enlarged by copy machine. When the width of new rectangle is 12, how will the length be?)

Note: Examples are taken from the problems used in this study.

In the written test, four Level 2 problems and three Level 4 problems were prepared. In the interview, problems were given from the lower level to higher levels by fixing the situations as "speed" and "fertilizer." For example, in the speed situation, under the condition of driving a car at the speed of 200 kilometers in 5 hours, distance (time) was varied and corresponding time (distance) was asked. For each situation, two Level 1 problems, one Level 2 problem and one Level 3 problem were prepared. In the actual interviewing process, according to the response by each pupil, the interviewer judged whether to give problems with higher levels. Therefore, numbers of problems given to the pupils were not equal.

The focused pupils were chosen based on their performances, learning styles, seat locations and cooperativeness. Three boys (Yoshi, Himu and Honda) and two girls (Kawa and Ueda) were finally chosen (all names are pseudonym).In B-L, Yoshi, Kawa and Ueda had difficulty for approaching both the Level 1 and 2 problems. Himu was okay for the Level 1 but had difficulty for the Level 2 problems. Honda had no problem for the Level 2, did almost correct for the Level 3, but failed for the Level 4 problems.

During the 11 lessons, the teacher introduced quantity-per-unit (intensive quantity) to the pupils. By following the textbook, she dealt mainly with population density and velocity. The series of lessons began with discussing how to decide which of the two places is more crowded. Among different opinions, some pupils proposed to coordinate one of two quantities into the same amount and compare the amounts for the other quantity. The teacher pointed out the important idea of coordination and introduced the quantity-per-unit (number of people per 1 m^2). In the sixth and seventh lessons, the pupils were engaged in the activity of measuring speed. For the purpose of deciding the fastest in class, they again used the quantity-per-unit and developed a formula of speed ($v=d\div t$). In the subsequent three lessons, the teacher introduced the double number lines to represent the relationship among four amounts in two quantities.

RESULTS

Pupils' Performances across Problems with Different Levels of Ratio Complexity

Table on the next page shows percentages of correct answers for the five pupils in B-L and A-L for all problems given in the written tests and in interview sessions. The percentages were calculated according to levels of ratio complexity. The table shows that their performance for Level 2 problems increased in A-L. Kawa's percentage for Level 1 problems also increased. On the other hand, Honda's percentages did not show any change even though he showed highest performance among the five pupils in B-L.

Changes in Pupils' Approaches to the Same Problems

Then, how did the pupils' approaches to the same problems differ between B-L and A-L? For this purpose, their approaches to the same problems were put in order. In total, there were 51 such cases. Among them, it was possible to identify some differences in 26 cases. Almost all cases for problems in Level 4 were discarded at this point because no differences were seen in the pupils' approaches; they kept using 'incorrect addition strategy.' Honda's cases were also excluded due to the same reason. As for the 26 cases, the differences in the pupils' approaches in B-L and A-L were compared and contrasted, which led to four categories. In this section, they are described briefly with some illustrations.

Level	Yoshi		Kawa		Ueda	
	B-L(%)	A-L(%)	B-L(%)	A-L(%)	B-L(%)	A-L(%)
1	2/4(50)	1/2(50)	1/3(33)	2/2(100)	2/3(66)	1/2(50)
2	3/5(60)	5/6(83)	4/5(80)	6/6(100)	2/6(33)	6/6(100)
3	0/1(0)	1/1(100)	-	0/1(0)	-	-
4	0/3(0)	0/3(0)	0/3(0)	0/3(0)	0/3(0)	0/3(0)

Level	Himu		Honda	
	B-L(%)	A-L(%)	B-L(%)	A-L(%)
1	4/4(100)	-	-	-
2	2/6(33)	5/6(83)	4/4(100)	4/4(100)
3	0/2(0)	0/2(0)	1.5/2(75)	1.5/2(75)
4	0/3(0)	1/3(33)	0/3(0)	0/3(0)

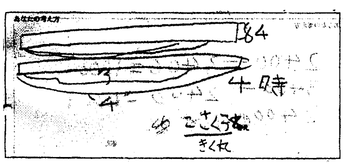
Note 1: "a/b" designates that the pupil got correct answers for "a" problems out of "b" problems given to the pupil.
Note 2: The shadowed cell designates that the percentage in A-L is higher than that in B-L.

Making one's thinking process explicit by numerical expressions. Compared with B-L, in A-L some pupils came to denote their process of thinking on the sheet of paper by using numerical expressions. Although the number of such cases was small (2 cases in Yoshi and Himu), the difference was remarkable in the eyes of observers.

The case by Himu is described here. He made his process of thinking explicit by verbalizing and denoting it on paper by both sentence and numerical expression. As the figure below illustrates, in B-L he tried but failed to express the idea of coordinating quantities to make two speeds comparable by using tape diagram. When asked how he did it in the subsequent interview, he again failed to make the idea explicit and showed insecurity in his answer.

Problem: (A table is shown. It shows that Sakura-maru goes 84 km in 4 hours and Kiku-maru goes 66 km in 3 hours.) Which ship goes faster, Sakura-maru or Kiku-maru?

B-L:



A-L:

(writing on paper)
"Let me make both [are going in] four hours,
$66 \div 3 = 22$ $22 \times 4 = 88$. Answer Kiku-Maru"

In A-L, in order to make the two speeds comparable, he transformed the information "66 km in 3 hours" to "88 km in 4 hours" by keeping the rate constant. The idea of coordination was shown clearly in his writing. It is remarkable to know the difference of his notations (diagram vs. numerical expressions).

Becoming conscious of the quantity-per-unit. All the five pupils were observed to use the unitary strategy in B-L. In using the strategy, pupils need to find the quantity-per-unit and to make use of it to solve the problem. In B-L, even though the pupils used the strategy, they tended to be less conscious of what they try to find and how they make use of it. It was noticed that the consciousness of the quantity-per-unit was enhanced in A-L. Eleven of such cases were discerned, especially in the approaches by Yoshi, Himu and Ueda.

The pupils' growing consciousness of the quantity-per-unit accompanied the expression of rate orally or in written form. Yoshi came to describe the quantity-per-unit in words, to interpret the quantity and use the information in making decisions, and to apply the quantity to the relevant problems. For example, in B-L he calculated "66÷3=22" and "84÷4=21" for the ship problem above. However, he was not able to describe what "22" and "21" refer to. In A-L, he said clearly that both numbers refer to "the distance that the ship would go in 1 hour." Becoming conscious of the quantity-per-unit and being able to use it better was his most notable change.

The growing consciousness was seen in various places in their thinking processes. Himu was unclear in finding the quantity-per-unit in B-L. He seemed to exclusively rely on his guesswork in head. In A-L, he came to find the quantity-per-unit at some time by using division, and at other time by using multiplication (e.g., "はてな掛ける4は20" (meaning "?×4=20")). Here, his guesswork was replaced by more systematic and reliable means. Compared with the others, Ueda showed scalar operator strategy more in B-L, though she was clever in using numerical patterns rather than making sense of the strategy. In A-L, her use of the unitary strategy increased. Thinking of quantity-per-unit enabled her to overcome difficulty where the scalar operator strategy induced remainder:

<u>Problem</u>: This wire weighs 390 grams in 6 meters. How much does it weigh in 10 meters?
<u>B-L</u>: Ueda tried the division 10÷6 but got confused because it would not divide evenly. So, she did "10-6 = 4 390×4 = 1560 (g)."
<u>A-L</u>: Her reasoning is seen from her work on paper, "390÷6 = 65 65 = weight in 1 meter." It continued such as "6 meters = 390. Still 4 meter less. 65×4 = 260 390+260 = 650 (g)."

These two cases show that the pupils took the idea flexibly and made use of it in the specific problematic situations they were facing with.

Emergence of notations that assist thinking. One notable feature of pupils' behaviors in A-L is that they put some notations down on paper when they had difficulty in proceeding their thinking. There were 4 such cases. It occurred when the level of ratio complexity was raised to Level 3. In such situations, they were either degenerated into the incorrect addition or quit their thinking in B-L. The pupils' restorations by the help of notations were seen in Yoshi, Himu and Kawa.

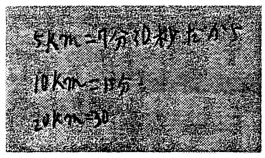
Both Yoshi and Himu wrote down the notations that designate correspondence between two quantities (e.g., "1 hour = 40 km"). Both pupils used the notations in the context of burden to proceed to their thinking. The case of Yoshi in A-L is illustrated below. He was approaching one of the speed problems (Level 3) that was totally out of his reach in B-L.

<u>Problem</u>: When a car goes at the speed of 200 km in 5 hours, how much time does it take to go 75 km?
<u>Yoshi</u>: 75 km? It sounds hard… (Hino reads the problem again.) Oh, 75… (he catches glimpses of the wall clock.) for 30 minutes it's 20 km (he is thinking with his eyes to the clock) Well, well… it is difficult (he looks bitter.) well… to go 75 km, how many tens of minutes? mm… what is the answer, answer, answer (he wrote

"Answer" and "1 hour") well… 7 minutes 30 seconds, oh, wait a second (he wrote "since '5 km = 7 minutes 30 seconds'.)

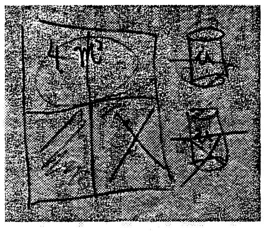
Hino: You don't need to get the answer. Would you tell me how you thought about it?

Yoshi: How about the remaining 30 km? well, let me put this, since 2 (meaning 20), km is for 30 minutes, so, as for 10 km, let's make both just a half, well for 15 minutes, 15 minutes, it's 10 km (he writes "10 km = 15 minutes".) and, for 20 km it's 30 (writing), since [the problem asks] 75, for 1 hour, it's 40 km, for 1 hour, it's 3, no, no, no… 35 remain, so let me add these two, 45, 45 (he adds 15 and 30), and add 7 minutes (he developed written calculation of 45+7=52.) it's 52, so, 1 hour 52 minutes 30 seconds (writing).



His inscriptions are shown in the figure. In his writing, the correspondence between the two was taken for granted and even stressed by the symbol "=." He was also aware of the distinction between the two quantities (which is on the left and on the right) and kept balance between the two (if you multiply one, you must multiply the other). This type of notations must have assisted him greatly to sort out the different amounts and quantities at hand.

In the case of Kawa, not numerical expression but drawing served as a clue in recognizing the correspondence. In B-L, she approached one of fertilizer problems by guesswork and eventually paid attention to the difference (minus 0.5 m^2 of land). In A-L, she was appealed by the difference again. However, this time she modified her thinking toward building up strategy. Here, Kawa relied on the picture she had just drawn (see the figure at right). She recognized that one small rectangle (1 m^2) corresponds to half package and therefore she needs to minus 1 m^2 to find the area for one and a half packages of fertilizer ("then this part is here, this part is here, and this part is here…, and this part is missing…."). She had written the same drawings in B-L, however, they were not functioned as a clue to modify her approach.



Attempting to use introduced notation. In A-L, uses of introduced notation, double number lines, were observed in 3 cases of Kawa and Ueda's thinking. However, neither used the notation in the way that the teacher taught. Even when they successfully represented the four amounts on the two lines, it was another hurdle for them to reason from the two lines. In addition, there were 4 cases that show the pupils' confusions for meaning of division. Using the number lines requests the pupils to identify and connect four amounts in two different quantities and to relate them with quotative division. This hurdle was high enough for them who had already possessed misconceptions on division in B-L.

DISCUSSION AND CONCLUSION

The pupils' performances on different levels of ratio complexity did not show steady change between B-L and A-L. Since the quantity-per-unit was taught in class, problems with Levels 2 and 3 were more to be targeted. Accordingly, the pupils' correct responses to Level 2 problems increased. Still, their development was not clearly seen across levels. Nor were their performances stable. In A-L, they still had difficulty in lower level problems.

Nevertheless, the results show that major change was seen in their more active use of notations in dealing with the aspect of rate and proportion. Notations that appeared in A-L varied, including different numerical expressions, drawings, expressions of correspondence between two quantities and the double number lines. They mediated the pupils' thinking by modifying the stimulus situation as a part of the process of responding to it (Vygotsky, 1978, p14). Such notations were recalled or generated according to the pupils' needs. They may be classified into three types by the extent to which they contribute to the pupils' thinking:

- Notations borrowed (e.g., Kawa & Ueda's use of the double number lines).
- Notations making aware of one's thinking (e.g., Yoshi's use of expression of correspondence, Kawa's use of drawing)
- Notations inseparable to thinking (e.g., Honda (described later)).

An important observation of this article is that there are notations in the second type, or the notations that lead one to make aware of his/her own thinking. The categories outlined supports this argument. The pupils became more articulated in their reasoning both orally and in written form, which enabled them to reason beyond guess and ambiguous image. Their uses of the quantity-per-unit became more conscious. They came to find the quantity-per-unit systematically, to make decision based on information of the quantity, modify their thinking, and use the quantity-per-unit in order to complement their insufficient parts of reasoning. The pupils who previously depended only on memory also came to proceed by controlling their own thinking with the help of notations. Vygotsky & Luria (1994) regard that changing relationship between speech and action is characteristic to the process of emergence of higher-order mental functioning. According to them, child speech is at first accompanied by activity and simply reflects on it, but it gradually shifts to the starting point of the process of activity, precedes the action and comes to possess the function of directing the activity itself. The observations of using notations in more articulated, systematic, conscious and intentional ways made in this article do indicate that this crucial change is seen in their approaches between B-L and A-L.

Notations that came to play these significant roles in the pupils' approaches to proportion problems also varied. Importantly, the notations that express correspondence between two quantities are widely used. Not only in written form, they were also heard from the pupils.

These significant notations did not exist uniformly out there but depended delicately on the conceptions of ratio and proportion that the pupils had developed earlier. The pupils were building their use of notations on their own strategies. This is shown, for example, in the cases of Ueda and Himu. Both made sense of the quantity-per-unit in A-L. However they began to use it differently. For Ueda, it served as a clue in overcoming the difficulty of division failure. On the other hand, for Himu it gave a reliable way in place of guesswork to find the quantity.

Another observation is the pupils' difficulty in the use of double number lines. For those pupils, this introduced notation did not necessarily become significant notation. It again suggests its dependency on the pupils' conception of ratio and proportion, especially invariant relation between two quantities. In developing the expression of correspondence, it is speculated that they attend only statically to the two quantities (c.f., "ratio" by Thompson, 1994). On the other hand, the double number lines request them to conceive the quantities as varying each other while they maintain the rate invariant. This gap must be profound for the pupils. Honda seems to be the only pupil who had mastered the double number lines. He solved them in the following way.

Problem: This wire weighs 390 grams in 6 meters. How much does it weigh in 10 meters?

B-L:	$6\times5=30\div3=10$	A-L:	$6\div3=2$	
	$390\times5=1950\div3=650$		$390\div3=130$	
	650 g		$2\times5=10$	
			$130\times5=650$	650 g

Honda's approach can be said to be an interiorized version of the use of double number lines. His conception of invariant relation was stable and also flexible. For Honda, parallel expressions with multiplication/division by the same numbers were inseparable to his thinking, or functioned as more than significant notations.

TASKS FOR THE FUTURE

The analysis is still going on. It is now analyzed what had occurred in the course of learning of the five pupils during the series of lessons on "quantity-per-unit." Questions that lead the analysis include: what kind of notations did they bring to the instruction?; what kind of notations did they generate and modified during instruction?; what had influenced on such notational activities?; and when and where the pupils came to develop significant notations? Preliminary observations show that from the very beginning the pupils select specific notations based on their taste, and that instruction influences the emergence of consciousness of the pupils' uses of notations indirectly. Information on these matters will contribute to revealing how social aspects of learning impact individual's development of proportional reasoning and to designing lessons that take care of each pupil's positive resources to be used in refining their conceptions (Moschkovich, 1998).

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