ON THE PATH TO PROPORTIONAL REASONING

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Four seventh grade students participated in a constructivist teaching experiment in which manipulatives within a computer microworld were used to solve fractional reasoning tasks followed by tasks that involve concepts of rate, ratio and proportionality. Through a retrospective analysis of video tapes, their thinking processes were analyzed from the perspective of the types of cognitive schemes of operation used as they engaged in the given problem situations. One result of the study indicates that the modifications of the students' available schemes of operation when solving the fractional reasoning tasks formed a basis for the cognitive schemes of operation used in their solutions of tasks involving rate, ratio and proportionality.

This paper focuses on Michael, one of four seventh grade students who participated in a study involving a constructivist teaching experiment as proposed by Steffe and Thompson (2000). In this study manipulatives within a computer microworld were used to solve fractional reasoning tasks and tasks that involve concepts of rate, ratio and proportionality.

BACKGROUND AND RATIONALE

It is commonly accepted among mathematics teachers and educators that reasoning involving the concepts of rate, ratio and proportionality proves to be difficult for most students. Because of the critical role that proportional reasoning plays in a student's mathematical development, Lesh, Post and Behr (1988) describe it as a watershed concept. That is, they refer to it as a cornerstone of higher mathematics and the capstone of elementary concepts. Thus, it is logical that there has been considerable research centered around these concepts that are embedded within multiplicative reasoning. The results of a study by Lamon (1993) indicate that before students proceed in using the formal representation of proportions and in using the cross-multiply-and-divide algorithm, they should be given extensive experience in exploring tasks involving multiplicative situations. One of the findings of Tourniaire and Pulos' (1985) review of the literature was that further research was needed in exploring the elementary proportional strategies and the problem context with its relationship to proportional reasoning acquisition. Studies by Thompson (1994) and Behr, Harel, Post & Lesh (1992) can be seen to be directed towards answering this call. Kaput and West's (1994) research can also be viewed as an answer to this call. The results of Kaput and West's (ibid.) study is used extensively in the study on which this paper is based.

The work in these former studies has included attempts to interpret the nature of concepts involved in proportional reasoning from the *adult's* perspective. The research in the present study (Nabors, 2000) is an attempt to clarify the nature of the *students'* concepts while engaged in such reasoning through producing cognitive models of students' mathematical reasoning from the students' perspective, thereby enriching former research. The cognitive schemes used to describe the mental operations and constructs of the students are based primarily on work by Steffe (1992, 1994), Sáenz-Ludlow (1990) and Olive (1996).

METHODOLOGY AND THEORETICAL PERSPECTIVES

A constructivist teaching experiment as proposed by Steffe andThompson (2000) was considered to be an appropriate methodology for the study because this type of teaching experiment was originally designed to formulate explanations and models of students' mental constructions based on intensive interaction between the researcher and the students. The theoretical underpinning of such a methodology is that of constructivism as perceived by Ernst von Glasersfeld (1995). In this epistemological perspective, knowledge is viewed as not being passively received but actively built up by the cognizing subject.

Four seventh grade students were selected from two general math classes through the use of a written pretest and a 20 minute interview. (Nabors, 2000) The students' responses were categorized according to Kaput and West's (1994) levels of proportional reasoning without instruction: (1) Coordinated build-up/build-down processes, (2) Abbreviated build-up/build-down processes using multiplication and division, and (3) Unit factor approaches.

In this study, these levels were interpreted through the use of the scheme theory mentioned previously. During the study the students used computer technology available in the computer microworld referred to as "TIMA: Bars"(Olive, 1996). In this microworld, the manipulable objects are rectangular regions that the student can make by clicking and dragging the mouse. The computer microworld provides a setting for the task situations that helps to discourage the students' use of school algorithms in their problem solutions. The physical hardware used to record the data collection consisted of the following equipment: a camcorder, a computer, a converter, a digital video mixer, a VCR and two television monitors. This technology enabled the researcher to work unassisted while collecting video and audio data.

THE CASE OF MICHAEL

The objective of the pretest and the subsequent interview is to explore the level of proportional reasoning each student seems to possess at the beginning of the experiment. The pretest is composed of seven problem situations covering the topics of money exchange, recipe formulations, mixtures, magnification and work which are consistent with those found in former research in which proportionality is being investigated among middle school students.

Pretest Results.

Michael's written results of the pretest indicates that he possesses Level I (Kaput and West, 1994) proportional reasoning operations. That is, he used additive reasoning in his attempts to solve the tasks. Therefore, in an effort to further investigate his potential proportional reasoning, he is given a problem in which a hook is placed on one pole and he is asked to indicate where a hook should be placed on a second pole so that there would be a proportional relationship between the hooks and poles. Michael understands that there is a need to maintain a constant ratio between the hooks and poles. Thus, in spite of the additive nature of his responses to most of the tasks, there are two important insights that his responses on the pretest illustrate. They involve his awareness for comparing units for the calculations in the tasks involving recipes and in his recognizing the need for maintaining a constant ratio in the pole problem during the interview. Due to the importance former research gives to *units* in the area of rate, ratio and proportion, (Lamon, 1993; Kaput and West, 1994), these insights lead to the categorization of Michael as being one of the two stronger reasoners in the study. That is, he had available schemes of operation that characterize Kaput and West's (1994) Level II proportional reasoning before instruction.

Part I: Fractional Reasoning.

In Part I of the teaching sessions, Michael was asked to solve tasks involving fractional reasoning by using manipulatives within the computer microworld: Tima: Bars. (A bias going into the study was that the fractional reasoning involved in the tasks in Part I would be sufficient for the reasoning involved in the tasks in Part II of the study.)

Michael quickly learned to use the actions of the microworld. The first task involved the construction of a bar that would be 3/4ths of a given bar. Michael partitioned the given bar into 4 equal parts and used the computer to dis-embed 3 of the four parts to get the new bar.

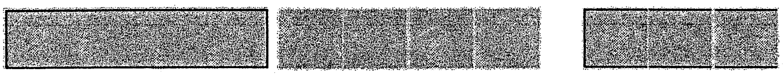

Figure 1

This was accomplished with no hesitation and illustrated, at the very least, the use of a partitive fraction scheme. "The partitive fraction scheme is regarded as an initial fraction scheme since it is here that the child begins using standard fraction words while working with partitive units." (Tzur, 1995, p. 135) Michael knew he could iterate 1/4 three times to get 3/4 or iterate it four times to get the whole. This task was also given to show that the whole is not destroyed through this operation. Thus, a whole-to-part concept (Sáenz-Ludlow, 1990) could be shown or promoted through this task. The next task presented Michael with strong perturbations:
The given bar is 5/7 of another bar. Make the other bar.

This task was given to test for and/or promote reversible operations. That is, in order to make the required bar, one needs to be able to 'see' or think of the other bar as being made of 7 sevenths. Then one needs to see the given bar as being made up of 5 of those sevenths.

M: The other bar is 5/7 of this one? (A typical response.)
T: No, the given bar is 5/7 of another bar.

Michael first partitioned the given bar into 5 vertical pieces and then partitioned the same bar into 7 vertical pieces. It is possible that he was trying to see the other bar as if it were superimposed on this bar; trying to see both bars at the same time. The resulting partition did not consist of equal parts, so he erased the bar. Then he partitioned a new bar into 7 pieces, copied it and pulled out 2 pieces of the new bar and joined it to the copied bar. Realizing that it did not give him what he wanted, he erased everything and made a new bar and pondered it for several minutes.

T: Can you picture both bars in your mind... at the same time?
M: Yeah.
T: Which one is bigger?
M: The second one.

Michael then made and partitioned a bar into 5 parts. Next he made a bar that *appeared* to be two sections longer. I interpret these actions to indicate that he knew the second bar would be two parts longer, but couldn't make the connection of how to make it *exactly* the right amount longer. So I gave him another problem with numbers representing more familiar 'parts of wholes'; that is, the given bar is 3/4ths of the second bar. Again, he tried to follow the same procedures and finally said: "I don't know..". It appears that he was unable to make a modification of his existing schemes in order to solve the task. I went back to more familiar parts of wholes and instructed him to make a bar from the given bar knowing that the given bar is one half of the bar he is to make. At this point Michael copied the given bar twice and joined the two new bars to get a correct second bar. To get the second bar in this case does not require any kind of partitioning of the given bar. He just doubled the first bar by copying and joining. He had to realize that two halves make a whole. I then asked him to make a second bar given that the first bar is 2/3rds of the second bar. This time, without hesitation, he copied the given bar, partitioned the given bar into 2 equal pieces, pulled out one of the sections of the given bar and joined it to the copy to represent the requested second bar. (See next page, figure 2.)

M: Is that it?

Michael had constructed a correct solution, but because he still asked if it were correct, I was not sure that an accommodation, a modification of a conceptual structure in response to a perturbation, had been made. Next, he attempted the task dealing with 3/4ths. He made a bar, partitioned it into 4 parts, pulled out one part and joined it to a copy of the given bar and turned to me as if for approval. I read the problem again, and without realizing it, emphasized the 3 in 3/4. Michael said: "Oh." He then partitioned the given bar into 3 parts, copied the bar, pulled out one part and joined it to the copied bar. He commented: "Now, I get it." At this point an accommodation had taken place because he was able to assimilate the situation into his existing scheme of adding a part of the given parts to make a whole composite unit.

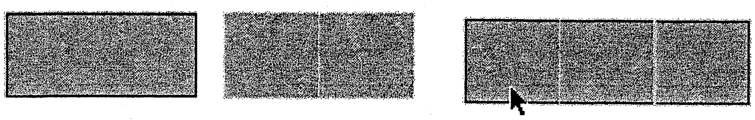

Figure 2

I hypothesize that Michael had modified his part-to-whole scheme to include viewing the whole given bar as a part of another bar. Up to this point he had viewed a whole bar as a whole composite unit from which he could pull out a part. In order to check to see if learning, "a permanent modification of a conceptual structure in response to perturbation" (Steffe and Wiegel,1996, p. 496), had taken place, I gave the following task:

A given bar is 5/3rds of another bar. Make the other bar.

With an extremely surprised, puzzled look, he asked:

M: Five thirds?!

T: Yes.

This requires a huge conceptual leap in that not only are reversible operations involved, but the given bar is greater than a whole. With little hesitation though, Michael made a bar and partitioned it into 5 parts, pulled out 3 parts and turned to me as if for verification. (See figure 3.)

Figure 3

It is my hypothesis that Michael's correct solution to this task indicates that he reorganized his iterative fraction scheme to include reversible operations. He reasoned that 5/3 could be obtained by iterating 1/3 five times and 1/3 iterated three times would be 3/3.

The previous exercises were given primarily to determine the availability of reversible operations with fractions. It appears that Michael did not have such operations available at first, but was led to develop them through reverting back to a situation in which assimilation and accommodation could take place. According to Glasersfeld (1995, pp. 62-63), "...cognitive assimilation comes about when a cognizing organism <u>fits</u> an experience into existing sensorimotor or conceptual structures it already has." In this case, Michael was familiar with doubling and halving and with parts of wholes. I interpret his responses to indicate that he possessed a *unit fraction scheme* (Sáenz-Ludlow, 1990) which allowed him to consider both whole-to-part and part-to-whole relations. I hypothesize that his reversible operations were basically brought forth through the use of his partitive fraction scheme. But more than this, he was dealing with fractions such as 5/3 which represents the meaning of fractions beyond the part-of-a-whole concept. Thus, his partitive fraction scheme was modified to include a unit fraction scheme and an iterative fraction scheme which allowed him to extend his fractional reasoning to include operating with fractions representing more than a whole.

Michael was able to solve the remaining tasks in Part I of the study tasks without significant perturbations. He (as the other students) did experience difficulty in naming the different fractional parts of consecutively constructed bars. For example, the task of making a bar, constructing a bar that is 1/4 of the given bar and then constructing another bar that is 2/3 of the one just made, caused perturbations in terms of naming the fractional value of the new parts. He viewed each newly constructed bar as an entity unto itself and not as a fractional part of the previous bar, etc. I hypothesize that Michael was working in his zone of potential construction (Steffe, 1992) because he could name the fractional parts when reminded after each construction, that each new bar was constructed from a previous bar.

Part II: Word Problems.

The word problems presented in Part II of the study are generally considered, from a school math perspective, to involve reasoning that involves concepts of ratio, rate and proportion. An analysis of Michael's attempts to solve one such problem in terms of the schemes he used in his solutions is given here.

Work Problem: Adam can mow the grass of a rectangular field in four days. Jim can mow the same field 12 days. How many days would it take Adam and Jim, working together, to cut the grass? (Task adapted from Abramovich,1996).

Michael immediately made a bar and then made two copies of it. (See figure 4.) Each bar represented the field of grass to be cut or the whole job. One bar was partitioned vertically into 4 parts with each part representing the amount of grass Adam could cut in one day. The second one was partitioned vertically into 12 parts with each part representing the amount of grass Jim could cut in one day. In both cases, he viewed the whole field (the whole bar) as a composite unit or whole. In the first solution Michael set up a three-to-one units coordination scheme, a 3-to-1 ratio, between the parts of the bars. This indicated to him that it would take Jim 3 days to cut the same amount of grass that Adam could cut in one day. Next he made a one-to-one coordination in which he matched one day's work of Adam with one day's work of Jim, three consecutive times. This 'used up' three of the vertical parts in the bar representing Adam's work. The last part of Adam's bar is recognized as the part of the field that Jim cut, thereby indicating that it takes three days to cut the field when working together.

Michael's solution to the work problem involves a 3-to-1 units correspondence scheme to indicate the amount of work performed by one person in each day. From the perspective of school math, it might be said that he illustrated the concept of rate when he talked about the amount of work per day. From the perspective of this study, rate was defined as: a reflected abstracted constant ratio (Thompson,1994). I hypothesize

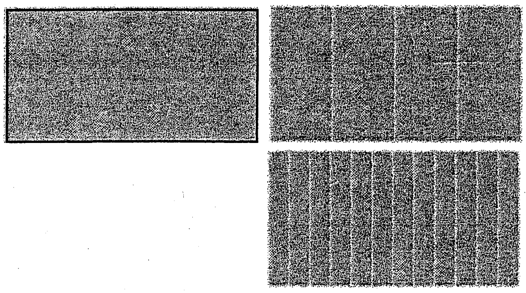

Figure 4

that Michael was viewing the 3-to-1 ratios as static ratios involving a comparison of invariant quantities. That is, he had yet to view the constancy of the ratio from the viewpoint that the ratio would remain constant regardless of variation of quantities being compared. His reasoning did not demonstrate this level of abstraction.

DISCUSSION.

Michael left the experiment possessing more sophisticated fractional reasoning operations available to him than when he entered the experiment. My conjecture is that one of the reasons he was able to conceptualize problem situations that had previously involved constraints for him on the pretest is because he was first met at his level of reasoning and understanding. Next he was placed within mathematical situations which promoted the reorganization of his present schemes of operation to form operations that allowed for solutions requiring more sophisticated schemes of operation. His solving of *all* of the written tasks on the post test, which were similar to the tasks on the pretest, appears to be a byproduct of the increased sophistication of his available operations. With this being said, it is my hypothesis that Michael had still not demonstrated proportional reasoning from a Piagetian perspective. For Piaget (Inhelder, B. and Piaget, J., 1958), proportional reasoning involves a relationship between two relationships, a second order relationship. This entails a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons. For Michael (as with the other participants) proportional reasoning was still at the stage of involving the comparison of static ratios.

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