FUNCTION MACHINES & FLEXIBLE ALGEBRAIC THOUGHT

Gary E. Davis & Mercedes A. McGowen

Rutgers, The State University of New Jersey, USA

William Rainey Harper College, Illinois, USA

We explore how college students understand ideas of functions, and which representations are productive for them in promoting their ability to work flexibly across representations. We use pre- and post-test scores, and triangulations via student self evaluations, to generate a hypothesis related to flexible thinking and success in algebra. We use confidence intervals to provide evidence for a highly significant change in student flexibility in algebraic thinking, and to assist in generating a plausible model of how the use of function machines in a developmental algebra course is instrumental in stimulating that flexibility.

INTRODUCTION

Students who present in developmental algebra, in universities and two-year colleges, commonly see mathematics as consisting solely of procedures and formulas. Unfortunately this vision of mathematics does not generally assist them in carrying out the procedures accurately, efficiently or appropriately, or in getting the formulas right in context. The unifying idea of a function is generally not something with which developmental algebra students are familiar or comfortable. Further their mathematical thinking, focussed principally on procedures and "correct" formulas, is largely inflexible and highly context specific. We relate the ideas of functions as a unifying concept in algebra, and flexibility of algebraic thought, through the use of function machines as a generic representation of functions with power to assist students in becoming more flexible and competent in their algebraic thinking.

Functions and function machines

Students often rely on intuitive and unreflective ideas of functional relationships. The subtlety of the function concept with its process-object duality and various representations proves to be highly complex. It is a concept, with wide-ranging powers and with widespread misunderstandings, that has been studied extensively in recent years (see, for example, Markovits, Eylon & Bruckheimer, 1986; Vinner & Dreyfus, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Harel & Dubinsky, 1992; Cooney & Wilson, 1993; Even, 1993; Cuoco, 1994; Thompson, 1994; Wilson & Krapfl, 1994; DeMarois & Tall, 1999; Lloyd & Wilson, 1998). Both teachers' and students' conceptions of functions are of enduring interest in mathematics education, because of the fundamental organizing and analytic role in mathematics played by modern reflective notions of functions. Various representations of function (table, graph, algebraic, for example) may be seen as ways of representing calculation of an input-output relationship. Functions viewed as input-output machines were studied in mathematics education as far back as 1965 by Peter Braunfeld. Tall, McGowen & DeMarois (2000) proposed the function machine box as a generic image that can act as a cognitive root, embodying the salient features of the idea of function, including process (input-output) and object, with various representations seen as methods of controlling input-output. The function machine embodies both an object-like status and the process aspect from input to output. Recent studies (McGowen, DeMarois & Tall, 2000; McGowen & Tall, 1999) indicate that the introduction of the function machine as an input-output box enables students to have a mental image of a box that can be used to describe and name various processes. This can take place often without the necessity of having an explicit process defined. Other forms of representation may be seen as mechanisms that allow an assignment to be made - by a table, by reading a graph, by using a formula, or by some other assignment procedure.

Flexibility

Krutetskii characterized flexible thinking as reversibility: the establishment of two-way relationships indicated by an ability to "make the transition from a 'direct' association to its corresponding 'reverse' association" (Krutetskii, 1969, p. 50). Gray and Tall (1994) characterize flexible thinking in terms of an ability to move between interpreting notation as a process to do something (procedural) and as an object to think with and about (conceptual), depending upon the context. In this article, flexibility of thought encompasses both Krutetskii's and Gray & Tall's ideas as facets of a broader notion of flexibility. In addition to reversible associations, and proceptual thinking, we are interested in connections between various representations of functions, including tables, graphs and algebraic syntax, which we refer to as conceptual. The proceptual divide (Gray & Tall, 1994) is, in a broader sense, part of a conceptual divide in which flexibility is compounded by student difficulties in using and translating among various representational forms.

The theoretical background on flexibility suggested we pair questions so that a student scoring correct on both of a pair was evidence of flexible thinking for that student. For example, for a simple function f we might ask: (a) what is f(5) ? and also ask (b) for what value or values of x is f(x) = 0? Essentially this is the difference between evaluating a function and solving for a value. 12 of a total of 31 questions were organized in pairs - 6 pairs in all. Question pairs addressed differing representations of functions: syntactic formulas, tables, graphs, and function machines. Success in both questions of a given pair was regarded as an indication of flexibility of algebraic thinking with respect to the representation for that question pair. Question pair 8 & 9 (Q8. Given $f(x) = x^2 - 5x + 3$, find f(-3); Q9. Given $f(x) = x^2 - 5x + 3$, find f(t-2)) relates to flexibility in that a student who can answer both questions correctly can substitute a number for a variable in an algebraic expression, and can also generalize the substitution from numbers to the situation of another algebraic expression in a different variable. Question pairs 10 & 11, 12 & 13, 14 & 15, and 16 & 17 are all concerned with evaluation versus solving. Success on both questions for any one of these question pairs indicates flexible algebraic thinking in the sense of Krutetskii. Details of question pairs 14 & 15 and 16 & 17 are given below in the results.

METHOD

Data were collected from a cohort of students taking a common 16 week course in developmental algebra at a two-year college. Data were of two sorts: pre-test and post-test scores (weeks 1 and 16 respectively; identical questions), and student written self-evaluations, turned in by week 16. Of the 135 students initially enrolled in the course, 87 students completed the course and were taught in 6 sections by 4 different instructors. The text for the course (DeMarois *et al*, 2001) emphasized the use of function machines, as well as syntactic algebraic formulas, tables, graphs and finite differences. However, function machines are not mentioned explicitly after chapter 3. Instructions given to students for their self-evaluations included: "What mathematics have you learned during this time? Write a summary of what you have learned." Students were asked to cite specific examples of mathematical growth, and to place their mathematical understanding, knowledge and skills competencies, in the context of Clarke, Helme & Kessel's (1996) criteria for meaningful learning in mathematics. They were not explicitly or specifically asked to address function machine or other representations for functions.

RESULTS AND ANALYSIS

Students' pre- and post-test responses were significantly different. We present results for the set of linked pairs of questions: 6 pairs, comprising 12 questions of the complete set of 31. The inter-item reliability (Cronbach's alpha) of the linked pairs of questions on the pre-test was 0.43 and on the post-test 0.88 (identical questions on the pre- and post-test). Since the questions did not change from pre-test to post-test, we conclude that this cohort of students saw the questions as having greater internal consistency at the end of the course than at the beginning. The correlation between pre-test and post-test scores for the linked pairs of questions was small and statistically significant ($r^2 = 0.04$, $p < 0.05$). This indicates no linear relationship between pre-test and post-test scores on the paired questions: students did not increase uniformly and proportionately from pre-test to post-test.

A confidence interval analysis for the paired two-sample difference in proportions, between the pre- and post-tests, of students who correctly answered both of a linked pair of questions yielded the results shown in table 1, below. Note that for the first linked pair (questions 8 & 9) one section comprising 12 students was not available for analysis, so the data there comes from 75, rather than 87, students.

The differences in proportions are all significant at the 99% level: no 99% confidence interval for the difference in means contains 0. In the context of the other question pairs, the least significant difference in proportions correct on both of a linked question pair is that for questions 14 & 15:

Questions 14 & 15: A function table copied from a TI-82 graphics calculator is shown:

Question 14: What are the output(s) if the input is -2?

Question 15: What are the input(s) if the output is -3?

The most significant difference in proportions correct on both of a linked question pair is that for the linked pair of questions 16 and 17, which are as follows:

Questions 16 & 17: Consider the equation y = 3x – 7.

Question 16. What are the output(s) if the input is 5?

Question 17. What are the input(s) if the output is 0?

Question pair	95% CI	99% CI
8 & 9	[0.26, 0.49]	[0.22, 0.52]
10 & 11	[0.21, 0.48]	[0.16, 0.52]
12 & 13	[0.20, 0.44]	[0.16, 0.47]
14 & 15	[0.05, 0.27]	[0.01, 0.31]
16 & 17	[0.57, 0.77]	[0.52, 0.71]
18 & 19	[0.09, 0.25]	[0.06, 0.29]

Table 1: 95% & 99% confidence intervals for differences in proportions pre- to post-test, of students getting both question pairs correct. Note that none of the confidence intervals contains the value 0 and that the confidence intervals show effect size.

The confidence intervals, at varying confidence levels, and the relative effect sizes can be seen in a visually striking way via confidence bands. These consist of a plot of the confidence interval, for the difference in proportions, versus the specified confidence level. In Figure 1, below, we illustrate these confidence bands, for confidence levels between 0.95 and 0.99, for the question pairs 14 & 15 and 16 & 17.

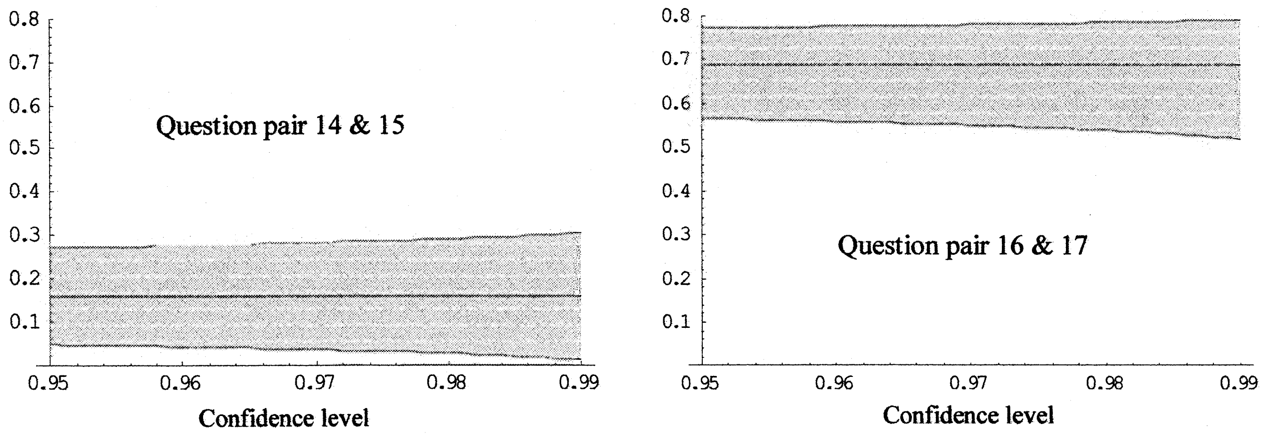

Figure 1. Confidence bands: confidence intervals for a difference in proportions, pre-test to post-test (vertical axes) versus confidence level (horizontal axes). The straight horizontal lines show the difference in proportions for the sample of 87 students

Student written self-evaluations

Student written self-evaluations indicated that many feel a function machine model assists them to make sense of notation, to organize their thinking, and to produce equations to describe data. References to input and output occurred in the work and interviews throughout the semester of students who were successful. They used the function machine notion to organize their thinking as they worked problems and interpreted notation. In contrast to the more successful students, the least successful students made very few references to function machines in their work or in the vocabulary they used. The least successful students demonstrated little or no improvement in their ability to think flexibly. Students were deemed "least successful" by a combination of post-test, final examination, and final portfolio grades - all of which form the components of a student's final grade for the course. Usually the top 15% and the bottom 15% of the class, after final course grades are determined, are examined to identify the most and least successful students, given the 15% criteria. Given class sizes, this is generally the top 3 to 4 and bottom 3 to 4 students in any given class.

Excerpts from self-evaluations for some of the more successful students are provided below.

> Student A: I know that function machines are good models for mathematical relationships when one wants to clearly identify and separate the input, output and process. I also can look at function notation, f(x), and understand what it is stating: That f is a function of x. I see the input (x), the output f(x), and the process, when in a relationship form like f(x) = 3x + 10.... I know now that 'solving' can refer to many different things including: 1) solving for an unknown variable; 2) solving a system of equations (where the given functions share the same input and output values); and 3) solving an equation (finding the input value(s) that produce an output of 0).

> Student B: Function machines are a great way to visualize the process of inputting, finding # processing, and finding the output.... Before I never knew there was a difference between evaluating, simplifying, and solving an equation. We EVALUATE the function to find the output of a function when the input is given. After we evaluate, we simplify using various mathematical properties such as associative, distributive, and identify. We use order of operations to obtain the output or the answer. We SOLVE to find the input, given the output.... Numerically, a linear relationship is the change between two outputs divided by the change between corresponding inputs that always produce a constant slope.

> Student C: I now have the skills to interpret and use mathematical notations appropriately, reflected in my work and ability to interpret a function machine; convert from one type of mathematical notation to another, convert a function machine to an equation/equation to function machine.... I have identified inputs and outputs, which was the key to answering the questions, appropriately and accurately in my tests on questions involving: recognizing the given slope to make a table, showing input/output;

identification of a given notation and breaking it down by input/output; naming input/output from a given function machine or equation; finding output if given input/finding input if given output.... I am knowledgeable in understanding the difference between evaluating and solving: solving for X; evaluating for Y.... I have learned more about equations and functions during the past 4 months than I have ever learned in my lifetime.

Student D: I had no idea what function is and what a relation is, but now I do. Relation is mathematical notation with at least two variables. Independent one is input. Dependent is output. I know that relation, which has only one output for each given input is a function. If I see f(8) = x + 7, I know I am given an input. f(x) = 8 gives me output. I can work with ordered pairs. In case (8, 45), I see 8 as input and 45 as output.

CONCLUSION

The major lesson to be learned from our analysis of the data collected on the pre- and post-tests and student self-evaluations is that some factor (or factors) across 6 classes, with 87 students and 4 instructors, is associated with a dramatic change in flexibility of algebraic thinking at the whole class level. An emphasis on functions as machines is the novel element that is likely to assist students to form mental images to assist them in the interpretation and use of syntactic algebra. How far and wide, and how strong across ability ranges, this effect might be we do not yet know. The evidence we have presented suggests that the representation of functions as function machines assists students' understanding of or development in algebra, and provides them with models that will require no significant re-shaping and modification in further mathematical studies.

The average increase in syntactic skills over the 16 week course was very significant. The linked question pair showing the highest gain consisted of syntactic algebra questions 16 & 17, asking about input and output for the function f(x) = 3x – 7. The mean gain, across the cohort of 87 students, for the average number of correct answers to both these questions, from pre-test to post-test, was a very high 0.82 (ref. Hake, 1998). This is very strong evidence that students were flexible, in the sense of Krutetskii, in interpreting this type of syntactic algebra question by the end of the course, compared with a general inflexibility at the beginning.

A function machine representation was used throughout the text (DeMarois *et al*, 2001) and the course as a means of helping students make sense of notation and to understand processes of "solving an equation" versus "evaluating a function". The pre-post tests show a statistically highly significant increase in students' ability to use algebraic notation. This, taken together with student self-evaluations, suggests strongly, but does not prove, that a function machine representation provides students with a way of organizing their thinking and with a more process-oriented view of mathematics.

This approach may not be appropriate for all students at all levels. However, for students who have had algebra previously and place into a developmental algebra courses, starting with a process notion of function and a function machine representation seems to help them make sense of notation. Such students believe they need to be able to manipulate symbols successfully and, as a consequence of this belief, focus on being able to "do algebra": that is, manipulate syntactic expressions. A function machine representation provides them with a potential means to organize their thinking so they can be more successful in dealing with notation than in their past experiences with algebra.

Student work and writings are replete with references to "input" and "output" in their concept maps, their explanations, and in their self-evaluations - indications that introduction and use of function machines impacted their thinking and provided an organizing tool or principle throughout the semester.

We hypothesize that use of a function machine representation of functions correlates with a student's ability to work flexibly across syntactic, graphical, and tabular representations. More specifically, we hypothesize that:

(1) a student's ability to form mental images of functions as machines;

(2) use of a graphing calculator as a concrete manifestation of a function machine; and

(3) use of finite differences and finite ratios to make sense of parameters and to help students have some understanding of where equations come from, given data;

are jointly highly and significantly correlated with their ability to utilize syntactic algebraic representations of functions and, to some extent, correlated with an ability to utilize graphical and tabular representations of functions. The pre- and post-test data we have presented, together with student self-evaluations, provides a strong foundation for this hypothesis. The class of students we studied dramatically increased syntactic algebraic skills over a period of 16 weeks. Many of the students stated explicitly this was due to understanding functions as processes, through a representation as function machines, particularly the ability to form images of function machines, and through the use of finite differences in conjunction with function machine representations.

REFERENCES

Clarke, D., Helme, S. and Kessel, C. (1996). Studying mathematics learning in classroom settings: Moments in the process of coming to know. NCTM Research Presession. San Diego, California.

Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about functions: Historical and research perspectives. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating Research on the Graphical Representation of Functions*, pp. 131-158. Hillsdale, NJ: Erlbaum.

Cuoco, A. A. (1994). Multiple representations of functions. In J.J. Kaput & E. Dubinsky, (Eds.), Research Issues in Undergraduate Mathematics Learning, pp. 121-141. Washington, D.C.: MAA.

DeMarois, P. & Tall, D.O. (1999) Function: Organizing principle or cognitive root? In O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of PME, vol. 2, pp. 257– 264. Haifa: Israel.

DeMarois, P., McGowen, M. & Whitkanack, D. (2001) *Applying Algebraic Thinking to Data*. 2nd Edition. Reading, MA: Addison Wesley Longman.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, 24, 94-116.

Gray, E.M. & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116-140.

Harel, G. & Dubinsky, E (Eds.) (1992) *The Concept of Function: Aspects of Epistemology and Pedagogy*. Washington, D.C.: MAA.

Krutetskii, V.A. (1969). An investigation of mathematical abilities in schoolchildren. In Kilpatrick, J. & Wirszup, I. (Eds.) *Soviet Studies in the Psychology of Learning and Teaching Mathematics*, vol I, pp. 5-57. Chicago: University of Chicago Press.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60, 1-64.

Lloyd, G. M. & Wilson, M. (1998) Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, 29 (3), 248-274.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics*, 6(2), 18-24.

McGowen, M., DeMarois, P. and Tall, D. (2000). Using the function machines as a cognitive root. *Proceedings PME-NA 22*, 247-254. Tuscon, Arizona.

Tall, D., McGowen, M., & DeMarois, P. (2000). The function machine as a cognitive root for the function concept. *Proceedings, PME-NA 22*, pp. 255-261. Tuscon, Arizona.

Thompson, P. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, E., Schoenfeld, A. & Kaput, J.J. (Eds.). Research in Collegiate Mathematics Education, I. CBMS Issues in Mathematics Education, Vol. 4, pp. 21-44.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20, 356-366.

Wilson, M. R., & Krapfl, K. M. (1994). The impact of graphics calculators on students' understanding of function. *Journal of Computers in Mathematics and Science Teaching*, 13, 252-264.