MATHEMATICAL SYBOLISM : A FEATURE RESPONSIBLE FOR SUPERFICIAL APPROACHES ?

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Abstract

In arithmetical problem solving process, young pupils' competencies were stressed on through analyses of their informal solving strategies. Other research works have shown pupils have significant difficulties to use the mathematical symbolism. Symbols seem to be meaningless for some of them and that feature could be responsible for the starting of some superficial approaches. Through individual interviews, we analysed the approaches developed by first graders who were not able to solve some problems. These solving errors are mainly due to conceptual difficulties, whereas the number sentences-writing task seems to develop superficial approaches.

BACKGROUND

Today's research concerns mainly focus on studies about the links between mathematics and everyday life situations. Those links can be considered according to a double research aspect, depending on whether we are interested in the use of school knowledge in situations outside of the school, or whether the issue is considered as the integration of real life knowledge into the school (Verschaffel et al., 2000). Research pointed out a gap between both types of knowledge. One of the reasons of that gap could be found within mathematics themselves: *at school, children learn rules and symbols but tend to lose the relationship to what these symbols represent* (Asman and Markovits, 2001, p. 66).

In early school years, the link between mathematics and real life situations should appear through arithmetical problem solving. More specifically, the word problems should help to establish some relationship between, on the one hand, the mathematical symbolism used to represent additions and subtractions, and on the other hand, the actions on the objects or the relationships mentioned through described situations.

Some reflections about the first mathematical learnings could bring some light on the general issue mentioned here above. Several studies (see Fuson, 1992, for a review of the literature) focused on the analysis of informal solving strategies developed by young pupils. The categories elaborated by the authors helped to enlighten a large variety of the strategies aiming at modelling the actions or the relationships described in the problem. Hiebert and Lefevre (1986) think that at that level, pupils generally do not develop any superficial strategies and that only the problems that are understood can be solved.

Other researches (Carey, 1991 ; Carpenter et al., 1983, 1988 ; De Corte and Verschaffel, 1985 ; Fagnant, 2002) focused on the analysis of the mathematical symbolisation abilities shown by young pupils. Most of these studies are based on individual interviews and help to analyse the relationships between the informal solving strategies and the number sentences writing task. These studies pointed out specific difficulties at the stage of symbolisation: pupils are not always able to produce a correct number sentence when they are confronted with a situation, even if they solved it correctly.

Furthermore, various studies have shown that young pupils' informal strategies, based on the word problem analysis as well as on a modelling of the involved actions or relationships, are often replaced by superficial strategies in which the pupils only wonder which operation they are going to perform (Carpenter et al., 1983 ; Verschaffel and De Corte, 1997).

Why do pupils develop superficial strategies in which mathematics consist in applying operations without trying to build some sense related to the suggested situations? Given the difficulties enlightened by the studies focused on mathematical symbolisation, it is possible that *part of the answer may lie in the transition from informal knowledge of addition and subtraction to the formal symbolic system of arithmetic* (Carpenter et al., 1983, p. 55). Generally, nothing is done to connect number sentences to the informal strategies of young pupils, they are not linked to concrete referents and we can therefore wonder which meaning they acquire. *Many students seem to learn symbols as meaningless marks on paper... After instruction on writing number sentences (equations) many first graders still not see the connection between the story problems and the number sentences that represent them* (Hiebert and Lefevre, 1986, pp. 21-22). That lack of meaning attributed to symbolism at its very introduction could be a feature responsible for superficial strategies.

This study aims at clarifying this issue through a quite particular perspective. Through the data collected during the individual interviews, we will focus particularly on the results of pupils who were unable to solve and to symbolise some problem situations. Most current errors made by young pupils consist in answering with on of the given numbers. On the contrary, wrong-operation errors are the most current among pupils with several months of experience in formal arithmetic in general and in problem solving in particular (Carpenter et al., 1983). The first types of error are generally due to conceptual difficulties (misunderstanding of some terms, lack at the semantic schemata,... - Lewis and Mayer, 1987 ; Stern, 1993 ; Riley et al., 1983). Wrong-operation errors can be explained in various manners; they can notably result from superficial strategies. It is generally considered (Verschaffel and De Corte, 1997) that the reason of these approaches lie in the stereotyped nature of the problems and in some focalisation of teaching on the final product, to the detriment of the process (not enough attention is paid to the building of the representation). The introduction of the mathematical symbolism could also be an explaining feature; that's what we are going to investigate now.

METHODOLOGY

This study focuses on 25 first graders who were selected randomly from 6 distinct classes of 4 different schools. They did not receive any teaching related to problem solving. Mathematical symbolism was introduced very early in the year but the word problems are only used in grade 2, in order to illustrate the applicability of formal operations. They are not used in grade 1 *to promote a thorough understanding of the basic arithmetic operations* (Verschaffel and De Corte, 1997, p. 69). The methodology consisted in individual interviews about the 14 problems of the Riley et al.'s classification (1983). These interviews were performed in the middle and at the end of grade 1. It was an approach almost similar to that used by De Corte and Verschaffel (1985) : the interviewer reads loudly the problem, the child retell it and tries to solve it with the help of physical objects, he is invited to write down a number sentence which corresponds to the story and/or to his solving strategy, the interviewer repeats the question to the child and asks him to tick the answer within his number sentence, the interviewer finally asks the child whether he could propose another number sentence (flexibility – Carey, 1991). The task requiring to tick the answer within the number sentence is essential for the future analysis. One of the advantages of the interview technique is that it helps to check whether the child tick really the number he identifies as the solution to the problem (the answer to the asked question). When this requirement is just proposed as a paper/pencil test, children tend to tick rather systematically the number located just after the equal sign (Fagnant, 2002), because they consider it as signifying "becomes" or "results in" (Fuson, 1992). Another advantage of that procedure is that it helps to analyse the solving strategy developed by the child before he produces his number sentences and, therefore, the link (or the absence of link) between informal and formal solving strategy.

We considered that the problem solving process was correct when it helped to end up in the expected answer. For number sentences, several criteria were taken into account: the number sentences must present a correct structure and be built from the expected number triple. Moreover, the answer must be correctly identified in the number sentence. For the present analysis, we chose to focus on the pupils who were neither able to solve nor to symbolise the analysed problem(s). We thus try to analyse the solving errors themselves and those in the number sentences. That could help us to perceive how the first ones can be explained by conceptual difficulties, whereas the second ones would rather be due to superficial approaches. And that is how we will try to support the hypothesis according which the introduction of mathematical symbolism could be a feature responsible for superficial approaches.

RESULTS

The analyses are mainly focused on 3 problems of the classification of Riley et al. (1983) : compare 1 (*Peter has 5 candies. Ann has 11 candies. How many more candies does Peter than Ann have?*), compare 3 (*Peter has 3 apples. Ann has 9 more apples than Peter. How many apples does Ann have ?*) and compare 5 (*Ann has 11 books. Ann has 6 more books than Peter. How many books does Peter have ?*).

The errors at the level of the informal solving strategies

For the three problems, we can notice that most errors at the solving level are given-number errors (the number to which it is referred in the relational sentence: respectively 11, 9 and 6 for the three problems). To explain the error committed in solving compare 3 problem, we can consider that the child interprets the relational sentence (*Ann has 9 more apples than Peter*) as a belonging sentence (*Ann has 9 apples*) or as two distinct sentences (*Ann has 9 apples. She has more apples than Peter* – Riley et al., 1983). We list 27 errors of that type (19 in January and 8 at the end of the school year). For compare 1 problem, we can extend this explanation: "*How many more candies does Peter than Ann have?*" becomes "*How many candies does Ann have*?". We list 18 errors of this type (12 in January and 6 at the end of the school year). According to the *Consistency hypothesis* (Lewis and Mayer – 1987), compare 5 problem would be presented in a structure that does not correspond to the child's "preferences". It is the reason why the child would try to modify the relational sentence : "*Ann has 6 more books than Peter*" would *become "Peter has 6 less books than Ann*". An incomplete modification would then make him think that "*Peter has 6 more books than Ann*" and would lead him to propose "17" as the solution (6 more than 11). We only list 2 cases of this type at the end of the school year. Furthermore, considering the relative sentence as a belonging sentence could lead the child to answer "6" *(Peter has 6 books)*, whether he made the complete modification (*Peter has 6 less books than Ann*) or not (*Peter has 6 more books than Ann*). We list 11 errors of this type (6 in January and 5 at the end of the school year).

It is also interesting to observe that a superficial key-word strategy (based on "*more than*") would bring to produce the answer inverse to the expected solution for compare 1 and 5 (for compare 3, that brings to the expected solution). No case of this type has appeared for compare 1 (no pupil answered 16) and only two cases of this type emerged for compare 5 (they can also be explained thanks to the *Consistency Hypothesis*). The remaining errors are either omissions, or errors difficult to explain but which are not an indicator of superficial approaches.

In relation to our hypothesis, errors at the solving level could be explained by conceptual difficulties rather than by superficial strategies. Stern (1993) had end up to similar conclusions through several studies aimed at compare problems.

Errors at the level of number sentences production

Now, what does happen when pupils are requested to produce a number sentence whereas they solved the problem by proposing a data from the problem text (given-number errors)?

About half of them do not propose any number sentence, which is quite a "logical" behaviour in relation to their wrong solving process. Some pupils propose a wrong operation, quite difficult to interpret. These number sentences are detached from the informal solving and indicate some type of gap between both tasks. Finally, the remaining pupils propose number sentences corresponding to the sum of the data from the text (the brackets indicate the number identified by the child as the solution of the problem) : 4+(9)=13 for compare 3 (11 number sentences) ; 5+(11)=16 for compare 1 (3 number sentences) and 11+(6)=17 for compare 5 (3 number sentences).

About compare 3, the number sentences could had been considered as correct if the solution identification would not have been taken into account. For compare 1 and 5, number sentences lead to produce the operation inverse to the expected solution, but the answer identified by the child is a data from the problem text. For the three problems, it can be supposed that the number sentence is the result of a superficial strategy based on the key words (*more*) (or an « *extract-and-add* » strategy – Bebout, 1990). These number sentences are detached from the pupil's problem solving approach and the final answer is not considered as the problem solution. Some pupils interpret the number corresponding to the data sum as being what the children (of the problems) have together. Other pupils are not able to give a concrete meaning to that number. In any case, the children are able to explain their number sentence in relation to their understanding of the situation.

If we had tried to interpret the number sentences only, we would have been led to consider that the number sentence « 4+9=13 » was correct in relation to compare 3 problem and that the number sentences « 5+11=16 » and « 11+6=17 » resulted from superficial strategies in relation to compare 1 and 5 problems. The implementation of the interview procedure (solving before writing the number sentence and task requiring to tick the answer) helped us to observe the gap between the solving process and the number sentence. It is thus really the symbolisation task (and not the solving process) that seems to be responsible for the superficial strategies of some children.

Though that type of error does not occur very often, we have to admit that it appears systematically for most problems of the Riley et al.'s classification (1983). For instance, for combine 2 problem (*Peter has 5 candies. Ann has also some candies. Together Peter and Ann have 11 candies. How many candies does Ann have?*), a current error at the solving level is to answer by proposing the great number (the whole). This error can be interpreted as a conceptual one (Riley et al., 1983). We do not list any answer resulting from an inverse operation (which could be the mark of a superficial approach) but the number sentence production task leads some pupils to propose number sentences of that type (5+11=16). For combine 1 problem (*Peter has 4 apples. Ann has 9 apples. How many apples do Peter and Ann have together?*), some pupils answer proposing both numbers of the terms (4 and 9 – confusion in words "together" = "each one"). The number sentence production task leads them to propose the number sentence (4)+(9)=13 ; considered as incorrect, since "13" is not identified as the solution (pupils do generally not give it any concrete meaning).

CONCLUSIONS AND DISCUSSIONS

The results presented here above focus on a few cases only. However, it is important to note that this type of analysis turns out to be relevant for most proposed problems : if pupils do not seem to develop superficial approaches at the level of the problem solving process, the task requiring to produce a number sentence seems, on the contrary, to generate the development of superficial strategies, of the " *extract-an-add*" type (Bebout, 1990). Faced with addition problems, that strategy leads to produce a number sentence which would had been considered as correct if the solving identification criteria had not been taken into account. Faced with subtraction problems, that leads to produce the number sentence inverse to the expected one. The gap between the solution and the number sentence is important : the child generally does not consider the number resulting from the number sentence as the problem solution.

To strengthen those observations bringing out a tendency to develop a superficial approach at the very moment of the number sentence production, we can briefly evoke the results concerning children who have been enable to symbolise the problems they had yet correctly solved. The wrong number sentence analysis (that we have no opportunity to detail here) shows also a strong tendency to develop superficial approaches of the " *extract-and-add*" type. Furthermore, even the results of the pupils who had correctly solved and symbolised the problems reflect that tendency to detach completely the symbolisation stage from the solving one. The change 2 problem (*Peter had 12 cherries. Peter gave 7 cherries to Ann. How many cherries does he get now*?) helps us to illustrate both cases. The child correctly solves the problem by taking 12 blocks, by removing 7 blocks, then by counting it remains 5 blocks (5 is thus proposed as the solution of the problem). He may then produce either the incorrect number sentence "12+7=19" or the correct one "7+(5)=12" (this type of number sentence is produced by about half of the pupils who propose a correct number sentence faced to that problem).

The various results presented here show the detachment between the solution process and the number sentence. Those difficulties can partly be explained by the way the symbolism is introduced : *Because the operations are initially learned outside the context of verbal problems and children are simply told that addition and* *subtraction can be used to solve these problems, they have no basis for using their natural intuition to relate the problem structure to the operations they have learned* (Carpenter et al., 1981). That way to proceed places the pupils in a difficult position : not only they do not attribute easily any concrete meaning to the mathematical symbolism, but they are too "constrained" to develop superficial approaches in order to perform tasks which are not much connected to their intuitive understanding of the situations.

Faced with those observations, we are induced to think that the first mathematical learnings are already partly responsible for the behaviour of « *suspension of sense making* » (Verschaffel et al., 2000), noticed among older children. There is indeed a risk that the pupils consider mathematics as meaningless and consisting in applying the rules taught by the teacher and that they do finally not understand a lot (Schoenfeld, 1992). In other words, it can be said that at the early school years, children have some opportunities to make them think that mathematics are quite distinct from real life.

In the classrooms we worked in, the problem solving teaching is coming later on, in grade 2, with the objective of showing the applicability of mathematics in everyday life situations. The results we got make us think it would be very more productive to build learning of problem solving on basis of pupils' informal solving strategies, rather than to choose to be supported by a basic expertise of operation techniques (wrongly considered as a pre-requisite). If problem solving teaching could begin earlier, we would get more chances to be able to build on the pupils' experience (*i.e.*, on their informal approaches), to give more meaning to the mathematical symbolism, to "avoid" the development of superficial approaches and then, perhaps, to create from the very beginning a less important gap between the mathematics and the real world…

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