STUDENTS' CONCEPTIONS OF AN ACCEPTABLE GEOMETRIC DEFINITION

Karni Shir and Orit Zaslavsky

Technion – Israel Institute of Technology, Haifa

The current study focuses on students' conceptions of a mathematical definition. The research instrument consisted of written questionnaires and group activities. These activities aimed at eliciting considerations and argumentation surrounding students' decision-making process related to the acceptance of a statement as a definition of a well-known mathematical concept. The focal concept for this study was a square, because of students' familiarity with it. Data was collected from students' responses to written questionnaires and videotaped observations of their group activities. The findings suggest three main perspectives underlying students' conceptions of an acceptable mathematical definition.

PROLOGUE

We invite the reader to consider the following statements:

1. A square is a rectangle with four equal sides;
2. A square is a parallelogram with diagonals that are equal and perpendicular;
3. A square is a polygon with four sides, in which all sides are equal, and all angles are equal.

Which of the statements would you accept as a definition of a square? Which of the statements that you accept, do you prefer the most? Why?

The following three excerpts of 4 outstanding 12^{th}-grade students debating over these questions convey the spirit of our study, which we report hereafter.

Excerpt 1:

Erez:	[refers to statement 3] It's correct, but it **is not** a definition.
Yoav:	It's correct, and **it is** a definition.
Erez:	It has too many details.
Yoav:	Too many details, but it is still a definition.
Omer:	What does "too many details" have to do with that?

Excerpt 2:

Erez:	I don't accept statements (a) and (b) [refers to statement 1 above, in which a square is described as a special rectangle, and to another statement where a square is described as a special rhombus].
Yoav:	I don't either.
Omer:	Why?
Erez:	Because you need to know what a rhombus is, and you need to know what a rectangle is.

Omer:	So what?
Erez:	It is not acceptable to base a definition on other concepts.
Yoav:	In fact a square is a special rectangle or a special rhombus, so you can define it using these concepts.
Erez:	There is no doubt that it is correct, it is correct. But a definition, according to its nature, should be based on the lowest base.

Excerpt 3:

Omer:	Yoav, draw a square.
Yoav:	[sketches a square] o.k.
Omer:	Erez, draw a square.
Erez:	[looking at Yoav's square] You mean he didn't draw it with the diagonals?
Omer:	Exactly.
Erez:	What's the connection?
...	
Mike:	When you draw a square you see this [draws a square with his finger on the table], like he [Omer] told you, you see sides, you don't see diagonals that are perpendicular. A square is first of all 4 sides. You don't refer to the diagonals. The diagonals are a property of the square.
Erez:	So are equal sides and right angles, they are also properties.
Omer:	No, they are a definition.
Mike:	Right, sides are sides. They build the square. Sides and angles build the square. Diagonals don't build the square.

The current study is part of a larger one. A different portion of it, sharing some aspects of the theoretical background, appears in Shir & Zaslavsky (2001).

CONCEPTUAL FRAMEWORK

Definitions play a central role in mathematics and mathematics education. Recent studies on the notion of definition (e.g., Winicki-Landman & Leikin, 2000; Furinghetti & Paola, 2000; de Villiers, 1998; Borasi, 1992, 1987; Linchevsky, Vinner & Karsenty, 1992;) differ mainly with respect to the population under investigation (students vs. in-service or pre-service teachers) and the features of a definition that are focal to the study. The vast majority of these studies investigate how the participants view the minimal aspect of a definition. Others look into the arbitrariness aspect that is associated with the freedom to choose a definition among equivalent statements as well of the advantages of certain choices over others. In addition, Borasi (1992) studied gradual refinement processes directed towards reaching a valid definition. Our study looks into students' ways of thinking about mathematical definitions and their processes of reaching an agreement on the necessity of a broad range of features of mathematical definitions.

Features of mathematical definitions

There are various features of a definition that charaterize a mathematical definition (Vinner, 1991; Borasi, 1992; van Dormolen & Zaslavsky, 1999). For some of these features there is a consensus regarding whether they are strictly necessary or just preferable, while for others there is no such consensus. The following three features of a definition are commonly accepted as crutial, thus, a mathematical definition must be: *non-contradicting* (i.e., all the conditions of a definition should co-exist), *unambiguous* (i.e., having only one interpretation), and *logically equivalent* to any other definition of the same concept. In addition, there are some features of mathematical definitions that are necessary only when applicable: A mathematical definition must be *invariant* under change of representation. In addition, when possible and appropriate, definitions should be *hierarchical* (i.e., based on basic or previously defined concepts, in a non-circular manner).

As mentioned above, a possible feature of a definition is heirarchy. We find it reasonable and useful, when applicable, to consider the level of heirarchy of a definition for concepts that are hierarchical in nature. For example, the 3 statements in the prologue are all hierarchical, yet they differ with respect to what we call *level of hierarchy*. Accordingly, statement 1 is based on a *rectangle* (thus we consider it the 1^{st} and highest level of heirarchy), statement 2 is based on a *parallelogram* (thus we consider it the 2^{nd} level), statement 3 is based on a *polygon* (which we consider the 4^{th} level). Note that a definition of a square based on a *quadrangle* would constitute the 3^{rd} level. We may continue in this way to even higher levels of hierarchy. The hierarchy of polygons to which we relate is rather common and concurs with the hierarchical classification of quadrangles opposed to the partition classification that de Villiers (1994, 1998) discusses.

Unlike the features described above, there are features of definitions upon which there is no consensus regarding their ultimate need. For example, it is not unanimously agreed upon whether a mathematical definition must be *minimal* (i.e., economical, with no superfluous conditions or information). While Hershkowitz (1990), Winicki-Landman and Leikin (2000), Vinner (1991) and Borasi (1987) claim that minimality is an ultimate requirement of a definition; others (e.g., de Villiers, 1998; Pimm, 1993) recognize the role of context with respect to the minimality requirement.

Another kind of distinction can be made between different types of definitions. A definition can be either *procedural* – by genesis, or *structural* - by a common property (Leron, 1988; Pimm, 1993)*. When geometric concepts are involved, we distinguish between a structural definition that relies on a property of certain parts of the object, and a property that is common to all, and only to, the points that constitute the object (i.e., definitions that are stated in terms of loci).

In our study we focus on definitions of a square that differ along the following dimensions: minimality, type (structural vs. procedural), and levels of hierarchy.

* Definitions can also be recursive (e.g., $n! = n(n-1)!$), or axiomatical (e.g., the definition of the Natural Numbers), however, our study does not address these types of definitions.

THE STUDY

The aim of the current study was to investigate students' conceptions of a mathematical definition. Thus, we decided to focus on alternative definitions of a familiar and well-known concept – a square.

Four 12th-grade top-level students participated in the study. The participants took part in three consecutive meetings dealing with alternative ways for defining a square. At the first meeting each student received a written questionnaire that contained eight equivalent statements (see Table 1), and was asked to reply to it individually. In the second meeting the four students were grouped together and were requested to relate to the same task and to try to reach an agreement. In the third and last meeting, the students were asked to reply again to the original written questionnaire individually. Table 1 presents the eight statements in the questionnaire, and their characterizing features.

A SQUARE IS:	Minimal	Type	Level of Hierarchy
(a) A rectangle with four equal sides	No	Structural	1
(b) A rhombus with a right angle.	Yes	Structural	1
(c) A parallelogram with diagonals that are equal, and perpendicular.	Yes	Structural	2
(d) A quadrangle in which all sides are equal and all angles are 90°.	No	Structural	3
(e) A quadrangle with diagonals that are equal, perpendicular, and bisect each other.	Yes	Structural	3
(f) A polygon with four sides, in which all sides are equal, and all angles are equal.	No	Structural	4
(g) The locus of points for which the sum of their distances from two given perpendicular lines is constant.	Yes	Structural (points)	Not Applicable
(h) An object that can be constructed (in the Euclidean Plane) as follows: Draw a segment; from each edge erect a perpendicular to the segment, in the same length as the segment (both in the same direction). Connect the other 2 edges of the perpendiculars by a segment. The 4 segments form a quadrangle that is a square.	Yes	Procedural	Not Applicable

Table 1: The Statements in the Questionnaire and their Characterizing Features

In constructing the different statements for the research instrument, we attention was given to several features. As mentioned earlier, the statements differ from each other with respect to three main constructs: minimality, type (procedural vs. structural), and hierarchy. The hierarchical statements differed from one another with respect to the level of hierarchy of the defining concept.

FINDINGS

Students' written responses to the questionnaire included 152 arguments: 108 arguments justifying the acceptance and 44 arguments justifying the rejection of a statement as a possible definition. The written arguments, as well as the arguments that were raised during the group discussion, were classified according to what seemed to be their underlying perspective: *mathematics*, *receptiveness*, or *figurative*.

Table 2 presents the distribution of types of arguments that the students used to support their decisions.

Underlying Perspective	**Reasons for Acceptance:** The statement is ...	**N**	**Reasons for Rejection:** The statement is ...	**N**
Mathematics	Correct (constitutes a necessary and sufficient condition for the concept)	25 (23%)	Incorrect (doesn't describe the concept)	1 (2%)
	Equivalent to a known definition	9 (8%)	Not structural	8 (18%)
	Useful	14 (13%)	Not useful	2 (5%)
	Minimal	1 (1%)		
Receptiveness	Simple or clear	30 (28%)	Complicated	11 (25%)
	Based on basic concepts	14 (13%)	Based on concepts that are not basic	19 (43%)
	Short	3 (3%)	Long	1 (2%)
	Captures unique features	6 (5%)		
	Elegant	3 (3%)		
	Familiar	2 (2%)		
Figurative	Based on properties of integral parts of a square	1 (1%)	Based on properties of latent parts of a square	2 (5%)
Total		108 (100%)		44 (100%)

Table 2: Arguments for Accepting or Rejecting a Statement as Definition of a Square

By a *mathematics perspective* we refer mainly to arguments in which logical considerations were involved (e.g., evaluating a statement on the grounds of its correctness, namely, on whether it is both a necessary and sufficient condition for a square). Other considerations related to the extent to which the statement is useful in mathematics (e.g., for proving or for classifying examples and non-examples) or to the legitimacy of using definitions that are not in the form of a conditional statement (e.g., procedural). Excerpt 1 in the Prologue is an example of students' employing mathematical considerations in their discussion.

By a *receptiveness perspective* we refer to arguments that focus on the communicative nature of a definition. According to this perspective a statement was evaluated based mainly on its clarity and whether it is comprehensible, within reach to those who deal with it, captures the essence of the concept, and based on accessible concepts (see Excerpt 2).

The last category of arguments, which we call a *figurative perspective*, has to do with the way the participants perceive a geometric object and its different components. According to this perspective, there is a distinction between the parts that seem integral to a geometrical object (such as the sides and angles of a polygon) and those that are often hidden (such as the diagonals of a polygon). In accordance with the figurative perspective, the latent parts are not equally considered integral parts of the object. Arguments of this category focused on the issue of whether it is legitimate to define a figural concept by properties of its latent parts (e.g., congruence of its diagonals). Thus, this perspective is characterized by the reluctance to accept statements that are based on properties of latent parts of a square (see Excerpt 3).

Note that about half the arguments (45%) for accepting a statement were based on mathematical arguments, while there was less mathematical support (25%) for rejecting a statement. On the other hand, receptiveness considerations played a major role both in accepting (54%) as well as in rejecting a statement (70%).

The group discussion (in the second meeting) yielded more insight to students' conceptions regarding what a (good) mathematical definition is. During this discussion the students found out that they don't agree on what conditions a mathematical definition should satisfy. One of the questions that they raised dealt with the correctness of the statement, as in the following two excerpts:

Mike:	All the statements can serve as definitions.
Erez:	All the statements are correct, there is no doubt about that.
Omer:	No, (h) can't, because a definition should be more abstract and (h) is... a...
Erez:	(h) is an instruction how to construct a square.
Mike:	No way, (h) is too long.
Yoav:	(h) constitutes construction instructions, it's not...
Erez:	[interrupts] It's a description of how to build a square.
Omer:	So, our question is whether we agree to accept it as a definition, [writes while speaking: w-e do-n't ac-ce-pt...] we don't accept statement (h) as definition, because?
Erez:	because it's not a definition, it's an instruction regarding how to build.

The next excerpt occurred after reaching a consensus regarding Statement (h). In this part the students are discussing Statement (c):

Yoav:	[refers to Statement (c)] We don't accept it. It doesn't mean that it is not correct. We just don't accept it.
Erez:	It is obviously correct, all the statements are correct, so should we accept them all as a definition?!
Omer:	If it is correct, why can't it be a definition?
Erez:	[turns to Omer] Is statement (h) correct? Is it correct? Is statement (h) correct or isn't it?
Mike:	Statement (h) is correct.
Erez:	[turns to Omer] So why didn't you accept statement (h) as a definition?
Mike:	[turns to Omer] Yes, why didn't you?
Erez:	[turns to Omer] Give yourself an answer for that.
Omer:	Because it isn't abstract.
Erez:	This statement [refers to Statement (c) (see Excerpt 3 in the Prologue);] is also correct, but I am not willing to accept it, in the same way that statement (h) is correct and I was not willing to accept it.

Eventually, Erez succeeded to convince all the others that correctness of a statement is not a sufficient condition for a mathematical definition, and that there are many other considerations that should be involved in the decision process.

Other interesting questions, which were raised and discussed by the group of four included: Is it legitimate to define a concept based on properties of its latent parts (e.g., equal diagonals)? (as in Excerpt 3 in the Prologue); On what kind of concepts can a definition be based? (as in Excerpt 2 in the Prologue); Must a definition be minimal? (as in Excerpt 1 in the Prologue); Should the term (i.e., the concept name) indicate the meaning of the concept?

There were only 2 statements on which there was a unanimous agreement between all four students in their pre and post written responses, as well as in their final group decision. For Statement (d) they all agreed that it is acceptable as a definition, while for Statement (h), they agreed that it was unacceptable. Interestingly, in the latter case they expressed no doubts at all, while in the former case they had a stage in the second session in which they reconsidered their views.

The group discussions proved particularly influential with respect to the views held by the participants regarding the remaining 6 statements. Thus, for each of these statements at least one student changed his standpoint as a result of the interaction with his peers. Interestingly, students continued to think about the issues they had discussed together even after the second session was completed. Consequently, there were 3 occurrences of a student changing the standpoint that he held at the end of the group discussion, and responding differently in the third session (i.e., in the second time the written questionnaire was administered).

CONCLUDING REMARKS

Our findings suggest that the kind of task that was designed and implemented within the study has the potential of creating a rich and stimulating learning environment, in which the learners are motivated to interact meaningfully on their own, taking an active part in genuine mathematical inquiry surrounding different features of a mathematical definition, and engaging in argumentation and justification, in the sense that Yackel (2000) discusses and in the spirit of NCTM (2000). This activity also proved valuable as a research tool aimed at identifying students' conceptions of a mathematical definition and its roles, and tracing the changes in their thoughts as a result of interactions with each other. Presumably, such activity may enable students to develop a view of mathematics as a humanistic discipline in which there is room for various opinions (e.g., Borasi, 1992). In fact, at the beginning of the study the students were convinced that a textbook definition is unquestionable, while at the end they became aware of the arbitrariness of the choice of definition and of their right to question a given definition and suggest alternative ones that are supported by valid considerations.

REFERENCE

Borasi, R. (1992). *Learning Mathematics Through Inquiry.* NH: Heinemann Educational Books, Inc.

Borasi, R. (1987). What is a Circle? *Mathematics Teaching, 118,* 39-40.

de Villiers, M. (1998). To Teach Definitions in Geometry or to Teach to Define? *Proceedings of PME 22, vol. 2,* 248-255. South Africa: University of Stellenbosch.

de Villiers, M. (1994). The Role and Function of a Hierarchical Classification of Quadrilaterals. *For the Learning of Mathematics, 14* (1), 11-18.

Hershkowitz, R. (1990). Psychological Aspects of Learning Geometry. In: P. Nesher and J. Kilpatrick (Eds.). *Mathematics and Cognition.* NY: Cambrige University Press.

Furinghetti, F. & Paola, D. (2000). Definition as a Teaching Object: a Preliminary Study. *Proceedings of PME 24, vol. 2,* 289-296. Japan: Hiroshima University.

Leron, U. (1988). On the Mathematical Nature of Turtle Programming. In D. Pimm (Ed.), *Mathematics, Teachers and Children* (pp. 185-189). London: Hodder and Stoughton.

Linchevsky, L., Vinner, S. & Karsenty, R. (1992). To Be or Not to Be Minimal? Student Teachers' Views about Definitions in Geometry. *Proceedings of PME 16, vol. 2,* 48-55. USA: New Hampshire University.

NCTM (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.

Pimm, D. (1993). Just a Matter of Definition. *ESM, 25,* 261-277.

Shir, K. & Zaslavsky, O. (2001). What Constitutes a (Good) Definition? The Case of a Square. *Proceedings of PME 25, vol. 4.* 161-168. Netherlands: Utrecht University.

van Dormolen, J. & Zaslavsky, O. (1999). The Many Facets of a Definition: The Case of Periodicity. *Aleh – The (Israeli) Senior High School Mathematics Journal,* No. 24, 5-16, in Hebrew.

Vinner, S. (1991). The Role of Definitions in Teaching and Learning of Mathematics. In: Tall, D. (Ed.) *Advanced Mathematical Thinking.* Dordrecht: Kluwer.

Winicki-Landman, G. & Leikin, R. (2000). On Equivalent and Non-Equivalent Definitions I. *For the Learning of Mathematics, 20*(1), 17-21.

Yackel, E (2000). Explanation, Justification and Argumentation in Mathematics Classrooms. *Proceedings PME 25, vol. 1,* 9-24. Netherlands: Utrecht University.