RESEARCH FORUM 4

Theme
From number patterns to number theory: issues in research and pedagogy

Coordinators
Rina Zazkis and Stephen Campbell

Contributors
Rina Zazkis and Peter Liljedahl, Nathalie Sinclair, John Mason, Tim Rowland

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Issues in Research and Pedagogy

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Session 1

- Introductions and opening remarks
- Short presentations (all interspersed with questions/answers, problems & discussion)
  "Making a case for number theory" by Rina Zazkis and Stephen Campbell
  "What is number theory? (Part 1)" by Stephen Campbell
  "Repeating patterns as a gateway?" by Rina Zazkis and Peter Liljedahl
  "For the beauty of number theory" by Nathalie Sinclair
- Plenary discussion
- Homework assignment (optional)

Session 2

- Short presentations (all interspersed with questions/answers, problems & discussion)
  "What makes an example exemplary? Pedagogical & research issues in transitions from numbers to number theory" by John Mason
  "Proofs in number theory: History and heresy" by Tim Rowland
  "What is number theory? (Part 2)" by Stephen Campbell
- Checking the homework
- Plenary discussion
- Concluding remarks

MAKING A CASE FOR NUMBER THEORY

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This research forum reflects on what number theory is, why it is relevant and important, how it connects with and builds upon number patterns, and with other research pertinent to this area (e.g., Campbell & Zazkis, 2002). Number theory is usually considered as a generalisation of whole number arithmetic or as a whole number specialised adjunct of algebra (Campbell, 2001). As such, it is not considered as a topic of study in K-12 mathematics education, in and of itself. In North America, for instance, the limited profile that number theory enjoyed in the NCTM *Standards* (1989) has been diminished in the NCTM *Principles and Standards* (2000). Nevertheless, introductory problems from elementary number theory typically abide and can be found sprinkled about various K-12 curricula around the world. On the bright side, there has been an increasing emphasis on using problems from number theory with respect to mathematical reasoning and proof in the middle grades, despite usually being presented under the guise of arithmetic or algebra.

The development and continual refinement of the number system is one of the great, and arguably the most important accomplishment in the history of mathematics. Yet, considering the whole numbers as a proper subset of the real numbers is a relatively recent innovation, and one that has evidently become widely incorporated in mathematics education around the world. There can be no denying the importance of rational and real numbers, practically and theoretically, in teaching and understanding arithmetic, algebra, and especially the calculus. Making a sound case for number theory, however, presupposes that one carefully discerns the integers and whole numbers from rational and real numbers, along with their respective properties, relations, and operations. Only in this way can number theory be properly and justifiably distinguished from and related to other topics in the mathematics curriculum.

A case for number theory can be built, and critiqued, from several different perspectives: its formal mathematical nature, particularly with respect to the additive and multiplicative structure of the integers and whole numbers, its profound beauty and mystique, its utility (or perceived lack thereof) and accessibility, not to mention its contribution to the history and philosophy of mathematics. We hope to touch on all these perspectives, and perhaps others, through the course of this forum. After a brief welcome and introduction to the goals and participants, we first consider the question of what number theory actually is. Campbell treats this issue by identifying some of the main concepts involved, illustrated through a classification schema for problems that comprise number theory in the K-16+ curriculum.

A natural access to number theory is provided through the exploration of patterns. Such exploration, which is typically practiced in elementary years, should not stop there. Zazkis and Liljedahl suggest that a more systematic approach to number patterns may serve as a gateway for introducing concepts and relations pertaining to number theory. They discuss trends and difficulties that surfaced in preservice elementary teachers' work with repeating patterns, connecting them with themes identified as troublesome in previous research. In rounding out the first session of this research forum, Sinclair makes a case for beauty, or aesthetic experience in mathematics education. She presents a model for describing the evaluative, generative, and motivational character of aesthetic activity, and illustrates various ways in which it applies to number theory.

It has become customary in mathematics education to argue for teaching the practical utility of mathematics through "real life" applications. We recognise the "practicality" of number theory in more advanced areas of application, such as cryptology, and that elementary number theory does not readily fit in the realm of applicability of "daily life." Knowledge of number theory will not help in calculating taxes or balancing checkbooks. We consider the utility of number theory from a different perspective – its utility for teaching and learning *mathematics*.

Topics from number theory, such as factors and multiples, provide natural avenues for developing mathematical thinking, for developing enriched appreciation and understanding of numerical structure, especially with respect to identifying and formulating conjectures, and establishing their truth. Applying his method of noticing to the question as to how certain task-exercises can lead to theoretical insight, Mason opens the second session by exploring how learners can make transitions from numbers to number theory. Continuing on this theme into more advanced levels, drawing on famous examples such as Gauss's method and Wilson's theorem, Rowland illustrates ways in which generic examples in number theory provide exemplary means for teaching and learning proof. Campbell brings the forum full circle by revisiting the question as to what number theory is by analysing why it is distinct from arithmetic and algebra, and how it can readily be treated as such in the early and middle grades.

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AUDIENCE PARTICIPATION ENCOURAGED AND FACILITATED THROUGHOUT BOTH SESSIONS

WHAT IS NUMBER THEORY?

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In making a case for number theory, it is helpful to consider various ways in which number theory can be defined and, what is related but not necessarily the same thing, to various ways in which it can be understood. In this paper both are taken to be involved in addressing the question as to what number theory *is*. One way to begin is to consider the kinds of problems involved. A problem classification scheme is introduced to provide further granularity in this regard. Number theory is then considered with respect to the field axioms implicitly underlying the K-12 curriculum in arithmetic and algebra. Whole number division with remainder, as a result of this analysis, is identified as *the* key factor in understanding number theory. This result supports evidence suggesting that understanding number theory is enhanced by quotitive dispositions toward division with remainder, and otherwise diminished by partitive dispositions (Campbell, 2002).

Defining Number Theory: Problem Classification

Number theory in K-12 and teacher education programs is typically restricted to introductory topics from elementary number theory. Such topics include the study of multiples, factors, and divisors, divisibility, divisibility rules, least common multiples (lcm), greatest common divisors (gcd), prime factorisation, prime and composite numbers, prime powers, relatively prime numbers, linear sequences, and so on. More substantive topics such as congruence relations, continued fractions, quadratic residues, Diophantine equations, and so on, are typically found in undergraduate number theory courses offered by departments of mathematics.

Most of these topics, along with the many problems in number theory that they readily give rise to, can roughly be grouped into four main problem classes: multiplicative, additive, linear, and non-linear. Although many problems fall clearly into one class or another, these classes are not disjoint. No claim is made or presumed regarding the comprehensiveness, uniqueness, or finality of this classification.

Multiplicative problems typically derive from the *fundamental theorem of arithmetic*, which asserts that every integer greater than one can be uniquely expressed, independently of order, as a product of prime numbers. Multiplicative problems presuppose divisibility properties of integers, and are often posed in those terms. Problems that require the use, identification, or derivation of various factors, multiples, and divisors are multiplicative. The following are examples of introductory multiplicative problems from elementary number theory:

Consider $M = 3^3 \times 5^2 \times 7$. Is M divisible by 2? 3? 5? 63? (Zazkis & Campbell, 1996a)

Find a number with exactly 13 factors (Mason, this volume).

Determining the number of factors (or divisors) of the integers provides a paradigm case for multiplicative problems in elementary number theory. For an illustration of how this problem can be implemented in the classroom, see Teppo (2002).

Additive problems are typically concerned with determining sums and/or sequences of summands of integers. A classic example here is Goldbach's conjecture that all even numbers greater than four can be expressed as the sum of two primes. Mason (this volume), for instance, invites us to attend to the patterns in the following series of equations involving sequential sums:

$1 + 2 = 3$

$4 + 5 + 6 = 7 + 8$

$9 + 10 + 11 + 12 = 13 + 14 + 15$

Other well known additive problems in elementary number theory include determining which numbers can be represented as sums of consecutive numbers, or consecutive odd numbers, or sums of squares, and so forth.

Linear problems are exemplified by the "Division Algorithm" and the "Euclidean Algorithm." These two algorithms are not just methods for dividing integers and determining greatest common divisors respectively, they are two of the most fundamental theorems of number theory. Both of these theorems are characterised by a linear structure combining multiplication and addition. Linear sequences of integers also fall under this category. Zazkis and Liljedahl (this volume) focus on repeating number patterns and their classic relation to linear sequences.

Non-linear problems typically involve non-linear equations in one or more variables such as the classic problem of determining Pythagorean triples, namely, integer solutions to the non-linear equation: $x^2 + y^2 = z^2$. The generalisation of this problem, and the Pell equation, $x^2 - dy^2 = 1$, provide other good examples of classic non-linear problems of elementary number theory, even though in many cases more advanced non-integral methods are required to solve them. Determining integral solutions to non-linear equations are usually referred to as Diophantine problems, in honour of the Hellenic mathematician, Diophantos of Alexandria (fl. ~250 AD).

Although there are many important problems in elementary number theory that can readily be identified with one or another of these four classes, there are also problems that blur the boundaries. Consider Gauss's method for determining the sum of the first 100 non-zero integers (Rowland, this volume). The formulation of this problem is ostensibly additive, but the solution is a non-linear integer function (i.e., $[N(N+1)]/2$).

Aside from linear and non-linear problems, there are many others with combined emphases placed on addition and multiplication. Determining perfect numbers, in which divisors must be determined and also summed, provides a case in point. Other problems involving base representation, divisibility rules, distributivity, and congruence also place combined and deeply interrelated emphases on these two operations.

Defining Number Theory: Fields and Rings

With a sense of the kinds of problems to be found in number theory, let us turn now to some axiomatic considerations that distinguish number theory from the underlying assumptions upon which K-12 arithmetic and algebra are based. The set of integers constitute an algebraic structure called a *ring*, whereas sets such as the rational and real numbers constitute algebraic structures called a *field*. It is helpful in defining and understanding number theory to consider differences between these formal structures.

A field is a (not necessarily numerical) set F with two operations, called *addition* and *multiplication*, which satisfy a collection of axioms, known as *field axioms*. Field axioms can be separated into axioms for addition, axioms for multiplication, and the distributive axiom (i.e., distributive law) relating these two operations (Rudin, 1976).

The field axioms for addition and multiplication define the properties of closure, commutativity, and associativity for these operations. They also include axioms establishing the existence of operational identities and inverses. Uniqueness properties of operational identities (i.e., 0 and 1) and inverses (i.e., $-x$ and $1/x$, such that $x + (-x) = 0$, and $x(1/x) = 1$) can readily be deduced from these field axioms. The existence of unique operational inverses for all elements in the field (with, of course, the notorious exception of zero for multiplication) provides the logical basis for defining the operations of subtraction and division in terms of addition and multiplication respectively (i.e., $y - x \equiv y + (-x)$, and $y \div x \equiv y(1/x)$).

A (commutative) ring is a (not necessarily numerical) set R that conforms to all of the field axioms with the crucial exception of the axiom establishing the existence of operational inverses for multiplication. In a ring such as the integers, without multiplicative inverses, division simply cannot be defined or understood in the same sense as that operation is in a field.

This may not come as a surprise as there is another, more familiar way of reaching basically the same conclusion. It is well known that division as defined by the multiplicative field axioms suffers a lack of operational closure over the integers, when integers are considered independently as a subset of the rational or real numbers (e.g., there is no integral solution for $7 \div 5$). What *may* be surprising is that the lack of closure here is *not* due to the fact that some integers do not have multiplicative inverses. Rather, it is because no multiplicative inverses exist in the integers whatsoever. The apparent "exceptions" are a result of the divisibility property of the integers (e.g., $8 \div 4 = 2$ because $8 = 2(4)$), they are *not* (partitive intuitions aside) due to multiplication by a multiplicative inverse (e.g., $8(1/4) = 2$)).

The above considerations have not been raised in some covert "neo-new math" attempt to resurrect formalisms back into the K-12 curriculum. Rather they have been raised to emphasise, contrary to popular opinion, that division with integers and whole numbers is *fundamentally different* than rational number or real number division. There is evidence to suggest that when the two are conflated, trouble follows (Campbell, 2002; 2001).

Understanding Number Theory

This brief reflection on axioms defining rings and fields clearly and strongly indicates that understanding number theory in relation to and in contrast with arithmetic and algebra primarily involves questions pertaining to division. More specifically, this means, in teaching and learning number theory, that close consideration be given to understanding terms, procedures, and concepts pertaining to division with remainder (Campbell, 2002; Zazkis, 1998), divisibility (Brown, Thomas, & Tolias, 2002; Zazkis & Campbell, 1996a), and prime decomposition (Zazkis & Campbell, 1996b).

There are both reasons and evidence to suggest that when students have trouble understanding number theory, it is often because they are thinking of division with remainder (i.e., integer and whole number division) in terms of rational or real number division. Campbell (2002) noted that whereas 10 out of 10 students with a partitive disposition toward division were unsuccessful in conducting division with remainder using a calculator, 7 out of 8 students with a quotitive disposition toward division *were* successful. Campbell suggested those observations could be accounted for by structural similarities between the division algorithm (i.e., $A = QD + R$, where $0 \leq R < D$), which serves to define whole number and integer division, and the quotitive model of division.

These structural similarities alone indicate the quotitive model would be a much more appropriate model for teaching and learning division with remainder than the partitive model. These considerations, of course, take nothing away from, and can only complement the effectiveness of using the quotitive model for teaching and learning division with fractions. Partitive dispositions towards division, on the other hand, insofar as they allow for the possibility of non-integral quotients, seem at the very least to interfere with, if not completely undermine, students' understanding of number theory

Even in the limited case of divisibility, where one might be tempted to think that it applies, the partitive model seems quite at odds with the multiplicative structures upon which the concept of divisibility is based (i.e., for any non-zero integer D, D divides A if and only if there exists an integer Q such that $A = QD$). Partitive dispositions, at least insofar as they entail a whole number divisor, are more appropriately applied to and should be reserved for teaching and learning rational and real number division, not number theory.

Concluding remarks

Number theory (qua *arithmos*) dates back to the emergence of mathematics as a formal conceptual discipline (Campbell, 1999), and there are grounds to suggest that history can inform the psychology of mathematics education (Campbell, 2001). Logical and empirical grounds provided here further suggest that number theory can, and should be treated as a distinct conceptual field in the teaching and learning of mathematics. There are important ways in which number theory is formally and conceptually distinct from arithmetic and algebra, especially with respect to division.

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REPEATING PATTERNS AS A GATEWAY

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Consider the following problem:

> *Imagine a toy train, in which the first car is red, the second is blue, the third is yellow, the forth is red, the fifth is blue, the sixth is yellow and the same pattern repeats for all the cars. What is the color of the 100th car? If the train has 200 cars, what is the number of the last yellow car?*

Even very young children can engage in activity of continuing with this pattern. They can make a kids-train and declare or pick a label identifying the color. At an early age child's ability to continue the sequence can rely on recursive observations, that is an ability to relate items to adjacent items (such as blue after red, red after yellow, etc.), as well as on a "rhythmic" approach in memorizing the unit of repeat.

Work with patterns is justified in helping acquire mathematical reasoning that is important to learning – as a context for generalization, as a conceptual stepping stone to algebra, as a context for recognition, conjecturing and communication of rules (Threlfall, 1999). However, in order to achieve this relevance it is essential to develop a perception of the unit of repeat in a repeating pattern. Only then can one attend to the question *"What is the color of the 100th car"*. We further suggest that repeating patterns provide a vehicle for directing student's attention to the multiplicative structure of natural numbers, and in such provide a gateway to introducing the concepts of number theory.

So, what is the color of the 100th car? This may be too challenging for a very young child. Let's think, in Polya's tradition, of a similar but simpler problem. What is the color of the 15th car? While young children will play out the sequence explicitly, older ones may start paying attention to the "unit of repeat". Another way to simplify the problem is to consider a two colour train. This may present a wonderful, and for some learners the first, opportunity to consider even and odd numbers. Extending the unit of repeat, that is, the number of colours in a repeating pattern, could introduce or foster a concept of a multiple in elementary school years.

Now let's imagine a 1000 cars toy train in a 7-colours repeating pattern (red, orange, yellow, green, blue, purple, white) and consider the color of the 800th car. A number theoretical analysis of the problem provides a systematic means for predicting the color of any car in the sequence. Strategies for determining such number patterns rely on introductory concepts of number theory, such as factors, multiples, and divisibility. A systematic solution may rely on division with remainder: 800 leaves a remainder of 2 in division by 7, therefore the 800th car is orange. An alternative strategy is to "count up from a multiple": every 7th car is white, therefore the 798th car is white, the 799th is red and subsequently the 800th car is orange. In what follows we describe the themes that emerged in analyzing the solutions to this problem in a group of preservice elementary school teachers and connect them to the findings of prior research on the understanding of concepts and relations underlying elementary number theory.

Multiplication and division, multiplication and addition

The strategy of "counting up from a multiple" was preferred. Even students who recognize and confidently implement both strategies are often unable to describe the connection, that is, to consider "remainder" as a distance from a multiple. Remainder is perceived as one of the numbers you get in performing division with remainder (Liljedahl & Zazkis, 2001). Participants' lack of connection between remainder and "distance from a multiple" is a particular explication of a lack of a more general connection—connection between division with remainder and multiplication. This issue is discussed in detail by Campbell (2002). A very illustrative example is presented in a request to determine quotient and reminder in division by 6 of the number A, where $A=147\times6+1$. Fifteen out of 21 participants in Campbell's study calculated the dividend A and used a long division algorithm in order to answer this question. Generalizing further, a fragile connection between multiplication and division with remainder could be seen as a manifestation of a "weak link" between multiplication and division in general. Consider for example the question of divisibility of M by 7, where $M=3^3\times5^2\times7$. As reported by Zazkis and Campbell (1996), it was not uncommon to calculate the value of M and then divide it by 7 in order to conclude divisibility.

In applying "counting up from a multiple" strategy, it was common not to focus on the closest multiple, but rather on a "convenient" multiple. For example, a student could start "counting up" from 700. However, rather than attending to 770, for example, as the next possible benchmark for a "easy" multiple, a students would engage in a "long" sequence of 707, 714, 721... untill 798 was eventually reached. We see in this strategy a strong preference towards addition, rather than multiplication. This can also be explained as incomplete understanding of the fact that repeated adding of 7 is equivalent to adding a multiple of 7. The theme of additive dispositions has repeatedly appeared in prior research (e.g. Brown, Thomas and Tolias, 2001).

Divisibility and division with remainder

The remainder strategy may appear as advantageous for a "mature" mathematical thinker. However, we observed instances of correct algorithmic applications of the strategy, without an understanding of "why it works". "Counting up from a multiple" strategy, though it may appear as less sophisticated, entails in it an important underlying idea of the number structure: the idea that "every seventh number is divisible by 7", or, in general, that "every *n*th number is divisible by *n*". Prior research shows that this property is not among the properties that students take for granted. For example, participants in a study of Zazkis and Campbell (1996) had difficulty in deciding whether there is a number divisible by 7 in a given interval of ten (they were asked to consider numbers between 12358 and 12368) without explicitly finding such a number in this interval. Furthermore, after correct calculation of remainder in division of 12358 by 7, not everyone could determine, without calculation, what would be the remainder in division by 7 of 12359.

In mathematics the idea of "every *n*th number is divisible by *n*" is naturally extended to the partition of natural numbers into disjoint sets by the relation of congruency modulo n. (That is, elements in each set of such partition are congruent to each other modulo n). This idea is expressed in terms of our pattern as "every seventh car is white". However, depending on where the count begins, it is equally true that every seventh car is blue and every seventh car is orange. While this may appear as "trivial" when considering cars, the mathematical manifestation of this property – that, for example, numbers leaving a reminder of 5 in division by 7 are "7 apart" on the number line – was not applied naturally by some participants. For example, having identified the color of a specific car as red, a participant had to count up in order to decide what would be the number of the next red car.

The ideas of partitioning of natural numbers have been explored further in the study that focused on arithmetic sequences (Zazkis & Liljedahl, 2002). In one of the interview questions students were asked to consider whether a given (large) number was an element of a given arithmetic sequence. Our analysis made a distinction between sequences of multiples (e.g. 3,6,9,12, ...) and sequences of so-called "non-multiples" (e.g. 2,5,8,11, ...). While the divisibility of a number by *d* (*d* is the common difference) gave a clear indication of belonging to a sequence of multiples, the lack of divisibility by *d* left students uncertain about its membership in a sequence of "non-multiples". This is another indication that extending the property of "every *n*th" from division to division with remainder should not be taken for granted.

Notable misgeneralizations

In previous sections we described students' approaches that may be seen as inelegant or mathematically unsophisticated, but at least they weren't wrong. In this section we describe several reoccurring mistakes related to recognition of numerical patterns.

- A notable misapplication of the above mentioned property "every 7th number is divisible by 7" appeared when students assigned to a car, which number was divisible by 7, the color "red", the color of the first car, rather than "white" - the color of the seventh car. These students noted the property of "every seventh", but applied it starting with the first, rather than the seventh, element (Liljedahl & Zazkis, 2001).
- Another improper generalization repeatedly appeared when participants generated the property of the *k*th attribute (blue) to "every *k*th" car, without attention to the unit of repeat. A claim such as "Since the blue car is in the 5th position, all the multiples of 5 will be blue" illustrates this approach.
- For a number of students that clearly and correctly identified that multiples of 7 are the key to solving the pattern, the ways to determine such multiples were improperly generalized. We witnessed claims that "the 137th car is white

because 137 is divisible by 7 as it ends in 7" and also "151 is divisible by 7 because of the sum of the digits" - which are based on overgeneralization of familiar divisibility rules for 5 and 3. Though these claims were infrequent, they confirmed appearance of similar improper generalization reported in prior research (Zazkis & Campbell, 1996).

- Consideration of a multiples of d (d is the common difference) is the key in situations of determining membership and generating elements in arithmetic sequences of multiples (described above). However, this strategy has been improperly extended in consideration of sequences of non-multiples as well. A claim such as "700 is an element in the sequence 8,15, 22, ... because 7 is a factor of 700" illustrates this extension. Similar phenomenon, referred to as "difference product" or "direct proportion", was observed by researchers investigating middle school students generalization of repeating patterns (Orton and Orton, 1999, Stacey, 1989).

Conclusion

There is a recent trend in mathematics education to make mathematics "relevant" to the students by presenting it in "context". "Relevance" and "context" are often interpreted as activities related to "everyday" or "real" life. In contrast, it is argued that numbers themselves offer a "context" for investigation and for rich background of ideas and experiences. We agree with Nemirovsky (1996) that "real contexts are to be found in the experience of the problem solvers" (p. 313), rather than in formulation of the problem.

Focusing on patterns is advocated in the research literature as a stepping-stone in the generalization approach to algebra (Lee, 1996). In particular, Threlfall (1999) argues that linear, or one-dimensional, repeating patterns represent the first step towards number patterns in algebra. Extending these claims, we suggest that attending to number patterns is also a stepping-stone to number theory. We believe that consideration of repeating patterns can either introduce or enhance concepts and relationships underlying elementary number theory, and especially the multiplicative structure of natural numbers.

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FOR THE BEAUTY OF NUMBER THEORY

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Imagine if—instead of following the established formal mathematical structures that dictate the basic guidelines of curriculum—we based our choices on more aesthetic aspects: which mathematics will students find appealing, wonderful, surprising and intrinsically satisfying? Why might we do this? Which topics would we choose? In this short article I will outline some of the potential aesthetic aspects of elementary number theory and thus provide additional support to several researchers' recommendations for its increased curricular emphasis (Campbell and Zazkis, 2002).

Introduction

There are no shortage of quotes, such as Bertrand Russell's assertion that mathematics possesses a "supreme beauty... capable of a stern perfection such as only the greatest art can show" (1917, p. 57), in which mathematicians profess the beauty and elegance of their discipline. Many mathematicians have also claimed that the aesthetic plays an important if not necessary role in mathematical activity, a claim that has had curiously little impact on mathematics education research—partly, perhaps, because we know so little about the roles of the aesthetic in the production and appreciation of mathematical knowledge.

Given its low profile—and even, at times, its outright dismissal—in the mathematics *education* research community (as opposed to the professional mathematics community), why should the aesthetic play any part in determining curricular and educational goals? There are three different though related reasons that I have formulated: a) the aesthetic should play a major role in mathematics education because it plays a fundamental role in research mathematics; b) the aesthetic should play a major role in mathematics education because aesthetic ways of thinking and knowing are central to learning and meaning-making; and c) the aesthetic should play a major role in mathematics education because providing students with access to aesthetic experiences—that is, intrinsically satisfying experiences of mathematics inquiry—is a basic goal of education. [1]

Since I adopt a broad interpretation of the aesthetic that extends beyond restrictions to responses to objects of 'art,' and that encompasses other domains of inquiry through which humans attempt to make their experiences meaningful and 'fitting,' I begin by elaborating a model of the roles of the aesthetic in mathematical activity.

[1] In the following section, I provide warrants for the first reason. I have provided arguments for the second reason in Sinclair (2002b). The third argument rests on particular goals for education that some may disagree with; it loosely corresponds to the Deweyian (1938) goal of "assisting the personal growth and development of individuals."

Model of the roles of the aesthetic in mathematical activity

Through an empirical/analytical interdependency methodology (Toulmin, 1971) I have shown that the aesthetic dimension of mathematical activity is not merely a fanciful, romanticised, after-the-fact judgement of mathematical beauty; but rather, it permeates mathematical activity and purposively animates mathematical knowledge (Sinclair, 2002b). I have identified three distinguishable aesthetic characteristics that play varying roles in mathematical activity: the evaluative, generative and motivational.

The evaluative characteristic concerns the aesthetic nature of mathematical products and the role of the aesthetic in making value judgements on these products (including theorems, proofs, definitions, diagrams, questions and theories). The evaluative characteristic of aesthetics is not just about objectively deciding whether a proof is elegant. Rather, it is involved in a mathematician's decision-making about truth, understanding and value. Without aesthetic judgement, it would be very difficult to distinguish the select results—of the thousands and thousands produced each year—that are worthy of further attention, of recognition, of passage into the 'canon' of textbooks, and of funding.

The generative characteristic focuses on the role of the aesthetic in inventing or discovering mathematics; it may be the most difficult of the three characteristics to discuss explicitly, operating as it most often does—for mathematicians at least—at a subconscious or tacit level (Hadamard, 1945; Poincaré, 1908). The aesthetic choices guiding mathematicians' processes of inquiry manifest a form of understanding that is qualitative, that is, neither formal nor propositional. These choices are made based on sensitivities to aesthetic qualities such as symmetry, analogy, simplicity and order. Mathematicians as well as philosophers (Dewey, 1938; Polanyi, 1962) have argued that the process of (mathematical) inquiry relies on the extra-logical form of thinking of the generative characteristic of aesthetics as much as on the more frequently cited logical and sequential forms of thinking.

The motivational characteristic relates to the role of the aesthetic in prompting or inspiring mathematical activity. This character is necessary to the process of selection and initiation in the mathematician's work. But more importantly it is directly involved in motivating inquiry as it directs the attention of the mathematician—what will be noticed—and frames the types of questions a mathematician will ask about a certain situation. Thus the motivational characteristic not only determines the field(s) and problems on which a mathematician works, but actually shapes the mathematician's inquiry (Dewey, 1938).

My model of the roles of the aesthetic in mathematical activity also accounts for a more encompassing theme in mathematical inquiry not explicitly mentioned in the literature: the aesthetic qualities of experiences which mathematicians describe having that unify and make memorable some of their encounters with problematic situations. This experiential dimension—pervading feelings and responses that arise out of mathematicians' experiences—contrasts with the more distinctive, cognitive roles of aesthetics identified above (see Sinclair (2002a) for an analysis of the aesthetic as a theme in mathematical experience).

The aesthetic in number theory

The following 1989 NCTM Standards quote nicely draws attention to the aesthetic potential of elementary number theory: "Number theory offers many rich opportunities for explorations that are interesting, enjoyable and useful. These explorations have payoffs in problem solving, in understanding a developing mathematical concepts, in illustrating the beauty of mathematics, and in understanding the human aspects of the historical development of number" (p. 91). What is meant mean by "interesting" and "enjoyable"? What is meant by the phrase "illustrating the beauty of mathematics"? Are these aesthetic opportunities actually valued as fundamental to mathematics learning by the education community or are they seen as epiphenomenal? By analysing five different dimensions of elementary number theory in light of my aesthetic model, I would like to inquire further into these aesthetic qualities of mathematics, as well as better articulate why they should be valued.

<u>Children have number</u>

Non-negative integers—the primary objects of elementary number theory—are pervasive in children's lives, as are the operations of multiplication and division. These are objects and operations that children see, know and use "... students from middle school through college feel comfortable dealing with whole numbers" (Selden and Selden, 2002, p. 214). This level of comfort and acquaintance can help in conjecturing and problem solving, and I claim it can also stimulate both the generative and motivational characteristics of aesthetics. In terms of the generative, students working with number theory situations or activities have the cognitive liberty of concentrating on exploring, guessing, and 'playing with' without being distracted by having to simultaneously extend their conceptions of mathematical operations and objects. Silver and Metzger (1989) have shown how this cognitive liberty is crucial to enabling aesthetic thinking—even in 'expert' mathematical activity.

A higher level of comfort and acquaintance can also stimulate the motivational characteristic of aesthetics, particularly with respect to the potential for surprise and paradox: the more you know about something, the more you will expect certain regularities based on your experiences, and the more likely you will be surprised by contradictions or interruptions. Zazkis (1999) illustrates a simple example of potential surprise with respect to the number for factors of an integer F(n): students expect that the greater A is, the greater F(A) should be (or if $A_1 > A_2$ then $F(A_1) > F(A_2)$), using the reasoning that works well in everyday contexts that 'the more of A, the more of B.' Students are surprised by mathematics when their expectations, even those based on their intuitions, are contradicted. For instance, students can be surprised that a small change can make a big difference, that a random collection of objects can share a common property, or that analogies can prove to be non-analogous. Moreover, as Movshovits-Hadar (1988) argues, surprise can not only provide exciting experiences for students in the mathematics classroom, it is also deeply connected to learning. The feeling of surprise stimulates students' curiosity which can, in small steps, lead towards intelligibility; it makes the students struggle with their expectations—and with the limitations of their knowledge—and thus their intuitive, informal, and formal understandings; and lastly, it provokes students to develop an appreciation of the significance of the concept in question.

Humans have had number for a long, long time

Mathematicians are often aesthetically motivated by a feeling that mathematics is somehow transcendental—providing them with a sense of order and certainty in our otherwise chaotic world. Similarly, there is an appeal to the seemingly transcendental types of problems and concepts that have been with us—or could have been with us—since the time of the Ancient Greeks: mathematicians continue in the tradition of Pythagoras to look for proofs of the irrationality of 2; many questions about prime numbers, factoring, and partitioning are as vibrant today as they were 2000 years ago. The realisation that she is posing a problem, or working on one, that Zeno or Euler could have posed might stimulate in a student an aesthetic motivation—one that is much more profound than the purported motivation factor associated with the 'real-life' problems pervading so many textbooks and curricula.

More than meets the eye!

As Selden and Selden (2002) write, "it is a well-known 'folk theorem' among mathematicians that number theory questions and conjectures can be easy to state but phenomenally hard to prove" (p. 214). For mathematicians this is a frequent source of aesthetic motivation: questions that are easy to state can elicit in them not only with the same sense of wonder that Aristotle claimed grips philosophers, but can provide them with a way of connecting with non-mathematicians and mathematicians alike—everyone can understand Goldbach's conjecture! In a related way, elementary number theory features many of the aesthetic wonders of mathematics—ideas such as infinity and scale, which Egan (1990) has shown have intrinsic appeal to students. Such ideas can thus initiate Whitehead's (1929) romantic stage of mathematics learning during which the "subject-matter has the vividness of novelty; it holds within itself unexplored connexions with possibilities half-disclosed by glimpses and half-concealed by the wealth of material" (p. 29). Whitehead argued that this stage was crucial and necessarily antecedent to the stages of precision and generality which students are often pre-maturely pushed into.

A study of patterns

The types of problems that crop up in elementary number theory are, as Ferrari (2002) notes, particularly well-suited to those actions of pattern-discernment and exploration which are eminently available to students. This can stimulate the motivational, generative, and evaluative characteristics of aesthetics. Humans are good at patterning—it is how they learn about and appreciate their world. When students recognise that they can activate a behaviour they know they are good at, they can derive the same kind of aesthetic appeal that motivates mathematicians to seek out and take on the types of problems that activate the ways of thinking with which they have had success—whether its symbolic juggling or visual representation. Also, many of the problems and objects of number theory can be patterned in concrete, accessible forms (for example: actual numbers as opposed to, say, abstract algebra objects; the patterned numbers in the colour calculator (Sinclair, 2001); figurate numbers; Pascal's triangle[2]) on which the qualitative, aesthetic ways of reasoning can more easily operate. Finally, with such pattern-rich, accessible situations, students can undertake that most crucial of mathematical activities: problem posing (Brown and Walter, 1983). It is with the act of posing problems—deciding whether they are worth pursuing and convincing others that they are—that the evaluative characteristic of aesthetics is first operative for mathematicians, and can be first stimulated in students.

A mathematical showpiece

Of course, the evaluative characteristic is also operative in the judgements that mathematicians make about their results, be they theorems, definitions or proofs. It is this aspect of the aesthetic on which mathematics educators have traditionally focused—concluding that although it is important in mathematics, it cannot be introduced to, or appreciated by students until the 'basics' have been learned (e.g. Dreyfus and Eisenberg, 1986). Their conclusion is based on the fact that students do not have the same aesthetic responses as mathematicians. However, Brown (1973) has shown that students do have aesthetic responses—based on different criteria perhaps than those of mathematicians—which contribute to their sense of the value of mathematics and thus serve a similar animating purpose. While Brown's students responded to the degree with which a solution to a number theory problem revealed their own path through the problem, students can also appreciate, for instance, the efficiency of a solution (as evidenced in their responses to Gauss' purported method for summing the integers from 1 to 100), the cleverness of a proof, and the perspicuity of a diagram; these are all criteria that mathematicians such as Hardy (1940) have proposed for judging the worth of mathematical entities. Geometry has traditionally been the topic area through which students are exposed to that ultimate mathematical entity of proof (and its related arts of conjecturing and verifying), but apart from an exposure to the various proofs of the Pythagorean theorem—maybe—they have little opportunities to respond to the various elegant, brutish, insightful, practical ways that mathematicians have of explaining and convincing. However, through elementary number theory, even students with little mathematical background can encounter many of these methods such as proofs by contradiction, induction, generic examples (see Rowland, 2002), visual proofs and counterexamples.

[2] In fact, Hawkins (2000) argues that by working with a combination of number and form—of the spatial and pictorial with the analytic means of understanding—that students' aesthetic engagement can best be stimulated and supported.

Concluding remarks

This article is but an initial, and very general, attempt at analysing the aesthetic potential of elementary number theory in mathematics education. Although my model of the roles of the aesthetic in mathematical activity is both theoretically and empirically grounded, more inquiry into the particular responses of students would greatly help in designing activities that can best capitalise on this aesthetic potential.

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WHAT MAKES AN EXAMPLE EXEMPLARY?: PEDAGOGICAL & RESEARCH ISSUES IN TRANSITIONS FROM NUMBERS TO NUMBER THEORY

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Worked examples have been used in teaching mathematics since the earliest of historical records, at least. In order for such examples to be useful, learners must see them as exemplifying something. If that 'something' is, for example (sic) 'the mystery of mathematics', 'the impossibility of doing something similar myself', or 'the ridiculousness of all this stuff', then learners are unlikely to make mathematical progress. If on the other hand the 'something' is a class of problems and a collection of techniques and ways of thinking, then the worked examples have served at least some of their purpose. Looking through a particular and seeing generality is a form of, and a building block for, generating a theory. When a learner has constructed their own 'theory', their own seeing of how 'to do' a whole class of problems, then real learning is taking place. In the language of Sinclair (this volume), learners participate in an aesthetic of efficiency and compactness afforded by their awareness of generality, or what the examples are exemplifying. Rowland (this volume) provides further examples of techniques for supporting learners in seeing the general through the generic particular (see also, Mason & Pimm, 1984).

The fact that text authors have, throughout the ages, inserted worked examples and only in some periods tried to start with general rules before illustrating their use in particular, suggests that in most generations teachers have been aware of the difficulty of proceeding from the general to the particular (at least unless very carefully guided: see Davydov 1990).

Method of working

My method of enquiry is to identify phenomena I wish to study, and to seek examples within my own experience. I then construct task-exercises to offer to others to see if they recognise, or can be directed to recognise, what I find myself noticing. Through refinement and adjustment of task-exercises in the light of experience and of reading relevant literature, I both extend my own awarenesses, and offer others experiences which may highlight or even awaken sensitivities and awarenesses for them. These sensitivities and awarenesses may inform their future practice. As task-exercises are developed and shared, actions which exploit what is noticed for the benefit of learners are incorporated. My method does not attempt to capture or cover the experience of readers. Rather it aims to make contact with that experience, perhaps challenging interpretations, perhaps pointing to other features not previously noticed. The data of this method are the experiences generated, the sensitivities to notice which are enhanced. If you recognise at least something of what I am talking about as a result of having worked on the task-exercises, then you may be stimulated to look out for similar experiences in the future, and over time, begin to act upon what you notice. Validity in this method lies in you finding your actions being informed in the future, not in what I say (Mason, 2002).

Examples

Consider the following task:

> *Find the greatest common divisor* (gcd) *of 84 and 90. Find their least common multiple* (lcm) *as well.*

You can imagine a whole page of these sorts of questions, and if you do, then you have a sense of a class of tasks (for more on classifying problems in number theory, see Campbell, this volume). But do learners also have a sense of this class, even when they have worked through the page? Could they make up for themselves versions which 'showed possible difficulties that can arise in some cases' or which 'illustrate what things can happen'? Are they aware of two approaches (factoring and the Euclidean algorithm), and do they have criteria for choosing one over another? If such tasks are augmented a little:

> *Multiply your two answers together and compare the product with* 84×90

something more is suggested, which can be pushed further into the realm of generality with something like

> *Might this always happen? or Will this always happen? or even, Can you find two numbers for which this does not happen?*

The indefinite pronoun 'this' may invite learners to clarify what the 'this' is, or may leave them helpless, depending on their past experiences with these kinds of question. To prompt learners to think of factoring, the task

> *Find a number with exactly 13 factors*

makes use of a largish number (13) which invites simplification (try 3 factors, try 1 factor!) and then re-generalisation beyond the 13, with extensions concerning characterising or describing the class of all numbers with a given odd number of factors, and finding the smallest such number. There are also extensions to even numbers for the bold. This is typical of the approach developed in the 60s and 70s in the U.K. (Banwell *et al.,* 1972) in that learners are invited to undertake a task in which they make choices in order to simplify on the way to re-complexifying for themselves.

Finding a number with exactly 13 factors is not a single task but an entry point into a whole domain of tasks which include finding numbers with a given number of factors, how many smaller numbers less than itself are relatively prime, among others (see Banwell *et al.,* 1972 p. 37, pp. 102-3). Such a task domain includes various variants in presentation as well as in content. It is of pedagogic use only if it becomes a vehicle for learners to use their natural powers to imagine and to express, to generalise and to particularise, to conjecture and to convince, and if their attention is drawn to the fact of these powers and to ways of refining and honing them. For example, although an observer might say that they detected the emergence of a theory about the structure of numbers as products of primes, and about the parity of the number of divisors, participants might be wholly unaware that that is what they were doing. Their attention is likely to be confined to trying to sort out their ideas and to justify their conjectures. Yet what they are doing is what mathematicians do. Becoming aware of the emergence and articulation of a 'theory' can both inspire and support the development of a learner's mathematical self esteem.

Another collection of tasks which promote number factoring are obtained by 'undoing' the first task:

> *Find a pair of numbers whose* gcd *is 6; find another pair; and another ... leading to the learner deciding spontaneously to classify or characterise all such pairs. Similarly, find a pair of numbers whose* lcm *is 1260, leading to, how many different pairs can have a specified* lcm*?*

Participants are not only developing a theory about the structure of pairs of numbers with a given *lcm*. They are also experiencing the use of their power to organise and characterise in a mathematical context, much as mathematician are wont to do. In other words, by being immersed in such tasks learners are likely to develop a 'theory' of what doing mathematics is like, a theory which would be very different from a theory developed as a result of only attending lectures and doing routine exercises (Watson & Mason, 1998).

A further advantage of refining and honing their powers to think mathematically is that learners are less likely to be caught by, or to persist in, spontaneous theories such as that 'all functions are monotonic': if $x > y$ then it is likely that $f(x) > f(y)$ (Zazkis, 1999), or that all functions are linear: e.g., $\sin(A + B) = \sin(A) + \sin(B)$ and $\ln(A + B) = \ln A + \ln B$. These are all too common manifestations, even by learners who when questioned directly know that they are false generalisations. The problem is perhaps that learners are entirely unaware that they have these theories, and they are not in the habit of testing theories and looking for counter-examples. In the midst of a complex problem, they simply do not have sufficient free attention to monitor what they are doing.

The temptation of authors and teachers is to lay out examples, perhaps as tasks, and to expect learners to build on the experience of a succession of tasks in order to become aware of, or to experience that succession. But as Kant has quite rightly pointed out in his *Critique of Pure Reason*: a succession of experiences does not add up to an experience of that succession.

Evidence of this phenomenon can be found in many different contexts. For example, in a professional development session I offered the following sequence of statements, with lots of pausing so that participants would see that they were supposed to check each equation in turn, and to attend to the patterns between equations:

$1 + 2 = 3 \qquad 4 + 5 + 6 = 7 + 8 \qquad 9 + 10 + 11 + 12 = 13 + 14 + 15$

'We' (meaning I wrote what some of the participants said) had written down two more rows. I asked for an expression of generality. One person suggested

$n + (n + 1) = \ldots$ but was then a bit stuck.

It transpired that his attention was entirely on the first equation. The others were not seen as part of 'the experience to be generalised'. Rowland (2000) met the same thing with pre-service primary teachers. Asked to check that:

$$3+4+5=3\times 4,\quad 8+9+10=3\times 9 \quad \text{and} \quad 29+30+31=3\times 30$$

and then to write down a statement in words generalising these three examples, many wrote nothing or nothing that could be construed, and some wrote a false generalisation such as:

three consecutive numbers added together equals the product of the first two

achieved by attending solely to the first 'example'.

It is quite likely that unfamiliarity with being asked to express a generality produced a tunnel vision effect, so that attention became focused on but one instance. The fact that the two threes play different roles could be overlooked, which would make the conjectured generality at least understandable. In the case of the sequence above, progress was made by asking people to chant the equations out loud but with emphasis first on the first number, then in a second pass, on the last.

Watson (2000) has pointed to the phenomenon of 'reading with the grain' and the necessity of 'reading across the grain' in order to experience structure. Thus in the Tunja sequences (Mason, 1999; 2001), which I developed in order to promote simultaneous work on factoring quadratics and on multiplication of negative numbers, I have found as expected, that non-mathematical audiences are perfectly capable of working with the grain—that is of following a pattern which is closely related to counting numbers and perhaps square numbers. For example:

$$3\times 5=4^2-1 \qquad 4\times 6=5^2-1 \qquad 5\times 7=6^2-1$$

can be extended 'downwards' to more equations by observing the counting-numbers in sequence. Being directed to read across the grain—that is to relate both sides of each 'equation'—leads to the realisation that the symbolic expressions must be the same, somehow. This is one of the necessary awarenesses that make algebraic manipulations meaningful. Different looking expressions can nevertheless express the same thing, so there ought to be a way to get from one expression to the other simply by manipulating symbols. The sequence can also be pushed backwards, against the grain, to reveal necessary facts about multiplication of and by negatives, given that we want the 'pattern' to continue.

As is well known, specific kinds of questions can lead learners to unexpected meta-generalisations. For example, being asked to express a pattern as a general formula can lead them to a pattern of behaviour which avoids the intended inner-task (Tahta, 1980) and exercises only the outer-task, namely, to find a formula which fits. As mathematicians well know, though intelligence testers seem yet to discover, no sequence, even the Tunja or the 'Consecutive Sums', uniquely defines its next term. There must be some source for a pattern which is agreed. Thus the sequence:

$$1, 2, 4, 8, \ldots$$

can have many different fifth terms (indeed, any fifth term, but see Sloane (1973) for examples of sequences which count things, and which begin this way). If the sequence is counting some aspect of a sequence of pictures, such as the number of regions of a circle formed by 0, 1, 2, 3, … chords, or the number of regions of space formed by 0, 1, 2, 3, …

planes in general position (Polya, 1965), then it is essential to have a statement of that generality before embarking on trying to find a general formula.

But even where learners are frequently engaged in formula-finding, the whole exercise can turn into 'train-spotting' (Hewitt, 1992) rather than productive mathematical thinking which exercises and refines the power to generalise.

Consider then some further options for extension.

The act of finding the *gcd* and *lcm* of two numbers can be seen as functions, but this requires the learner to step out of immediate action-experience, and to contemplate the whole. Having achieved some measure of competence with these calculations, the calculations themselves can be seen as objects. This is the domain of reification, when a process is also experienced as an object. In Mason *et al.* (1985) this was used to characterise one of the major steps in appreciating algebra, when expressions like $3x + 4$ come to be seen both as a specification of a calculation process, and as the object resulting from that process, and this dual nature then becomes the essence of algebraic expressions which replace numbers as the objects to be manipulated. Sfard (1991; 1992; 1994) developed the notion of reification while (Gray & Tall, 1994) used the term *procept* to indicate the evolution of a concept from carrying out a process as a theorem-in-action (Vergnaud, 1981) to seeing the process as an object in itself. Notation for the process helps enormously, for once something is named, it comes into psychological existence.

One way to stimulate learners to experience calculations as processes is some variant of:

> *Tell an absent friend how to calculate the* gcd *and* lcm *of a pair of numbers.*
>
> *Program a machine to find the* gcd *and* lcm *of a pair of numbers.*
>
> *I am going to be given a pair of numbers, but I can't tell you what they are at the moment (or I have a friend who has a pair ...). Please tell me how to calculate their* gcd *and* lcm.

Suddenly what seemed almost frighteningly open becomes bounded. A theory might just be possible. Notice that I do not provide my 'answers' nor even my theories in the sense of ways of seeing. For once theories are published, pedagogic value leaks away.

Final comments

Terms such as *investigative teaching* (and its variants such as *discovery learning*) provoke extreme reactions in many audiences, while *lecturing* and *starting from the abstract* provoke similar reactions in different audiences. Neither reactions are helpful as they are based on emotive associations with general labels, rather than precise details of pedagogic strategies. When teaching that is even marginally effective is examined closely, aspects of both pedagogic stances, of starting from the particular and the concrete and starting from the general and the complex will be found to have value. Strict adherence to one format is likely to foster the pedagogic theory that 'this is always how things are done', whereas variation in approach is more likely to broaden learners' views of what mathematics is about, what questions it addresses, and what methods it employs (Watson & Mason, 1998). Above all, the most important theory we want learners to construct is that they do actually possess the requisite powers to do mathematics and to think mathematically. Then they can make an informed choice as to whether to develop and make use of those powers within mathematics in the future.

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PROOFS IN NUMBER THEORY: HISTORY AND HERESY

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My purpose in writing this paper is to advocate the use of particular-but-generic proof strategies in undergraduate classrooms and in textbooks, in order to convince students of the truth of number-theoretic theorems and student-generated conjectures. The domain of number theory lends itself particularly well to generic argument, presented with the intention of conveying the force and the structure of a conventional generalised argument through the medium of a particular case. The potential of the generic example as a didactic tool is virtually unrecognised. Although the use of such examples has good historical provenance, the suggestion that they might be an alternative to formal proof tends to be viewed as a kind of heresy from the perspective of modern proof practice.

Procedures and proofs

The use of examples to point to abstract concepts and to general *procedures* is commonplace pedagogical practice (see e.g. Mason's paper for this research forum). In the field of number theory, a case in point might be explication of the Euclidean algorithm for the greatest common divisor of two natural numbers. Beginning with, say, 194 and 40 the demonstration proceeds:

$$\begin{array}{rcl} 194 & = & 4 \times 40 + 34 \\ 40 & = & 1 \times 34 + 6 \\ 34 & = & 5 \times 6 + 4 \\ 6 & = & 1 \times 4 + 2 \\ 4 & = & 2 \times 2 + 0 \end{array}$$

In order to apply the procedure to another pair of natural numbers, the student needs to become aware of the status of each number in each row of the procedure, and how each row relates to the next. That is, not only to agree that each line is a true statement, but to appreciate how it has been initiated and structured. As teacher, I might assist this by (say) underlining the quotients 4, 1, 5, 1, 2 in red. I might draw diagonal lines joining the divisor and remainder in each line to the dividend and divisor, respectively, in the next e.g. joining the two 40s, the two 34s, and so on. (It is relevant to pause to reflect on how you made sense of the previous sentence: perhaps the example was more illuminating than the somewhat archaic expression of the general procedure that preceded it.) The choice of example (194, 40) was made in recognition of its merits in its own right and relative to some alternatives. I judge it to be preferable to (194, 48), which is a poor paradigm because, for that pair, the algorithm terminates too soon. I would also avoid (144, 89) for a different reason: although it has good 'length', it conveys difference rather than division. Try it, if the intention of that remark is not self-evident. I would resist (97, 20) in recognition of my own liking for coprime pairs despite their particularity.

Much less common is the use of examples to explain why general relationships might hold: in short, to *prove*. One reason why this might be the case is clear enough—because one or more examples cannot prove a statement about an infinite category of cases. Yet there is a sense in which the *presentation* of a single example can speak for some general truth, and for some general argument above and beyond the particularities of the example itself. Such examples, suitably structured to be not just a confirming instance but a chain of reasoning, are known as generic examples. As Balacheff (1988) so clearly and elegantly puts it:

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class. (Balacheff, 1988, p. 219)

The generic example serves not only to present a confirming instance of a proposition - which it certainly is - but to provide insight as to *why* the proposition holds true for that single instance. The transparent presentation of the example is such that analogy with other instances is readily achieved, and their truth is thereby made manifest. Ultimately the audience can conceive of no possible instance in which the analogy could not be achieved.

Un peu d'histoire

The story (probably apocryphal, but see Polya, 1962, pp. 60-62 for one version) is told about the child C. F. Gauss, who astounded his village schoolmaster by his rapid calculation of the sum of the integers from 1 to 100. Whilst the other pupils performed laborious column addition, Gauss added 1 to 100, 2 to 99, 3 to 98, and so on, and finally computed fifty 101s with ease. The power of the story is that it offers the listener a means to add, say, the integers from 1 to 200. Gauss's method demonstrates, by generic example, that the sum of the first $2k$ positive integers is $k(2k+1)$. Nobody who could follow Gauss' method in the case $k=50$ could possibly doubt the general case. It is important to emphasise that it is not simply the *fact* that the proposition $1+2+3+ \ldots + 2k = k(2k+1)$ has been verified as true in the case $k=50$. It is the *manner* in which it is verified, the form of presentation of the confirmation.

Paul Hoffman recounts the story in his best-seller *The Man Who Loved Only Numbers* (Hoffman, 1998). His comment on it (quoting mathematician Ronald Graham) is a telling testimony to the genericity of Gauss' method.

> What makes Gauss' method so special .... Is that it doesn't just work for this specific problem but can be generalised to find the sum of the first 50 integers or the first 1,000 integers ... or whatever you want. (p. 208)

In introducing the notion 'generic example' to audiences of all kinds - undergraduate and graduate students, mathematics education conference-goers, 'general audiences' - I routinely choose Gauss' method as a paradigm of the genre. We should not be surprised that Gauss, of all people, should have provided it. Ironically, his *Disquisitiones Arithmeticae* established the 'modern' standard for generality in number theoretic proof arguments.

By contrast, Pierre de Fermat (1601-65) was notorious for stating number theoretic results in the absence of formal proof. In particular, it was Euler who gave a general proof of the 'Little' Theorem (if p is prime and a, an integer, p divides a^p-a) some decades after Fermat stated it. In a recent article, Burn (2002) offers some suggestions concerning the kinds of reasoning that Fermat himself might have used to establish the truth of some claims associated with his Little Theorem, made in a letter to Mersenne in 1640. These claims were developed in the course of Fermat's search for perfect numbers. A natural number (such as 6 or 28) is said to be *perfect* if it is equal to the sum of its divisors including 1, but not itself. Around 300BC, Euclid had established that the set of perfect numbers can be identified with integers of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime. In his letter to Mersenne (after whom such primes are named), Fermat claimed that if n is composite then $2^n - 1$ is not prime. The proof amounts to the observation that $2^a - 1$ divides $2^{ab} - 1$. The converse, however, is false: in 1536, Hudalrichus Regius had shown that although 11 is prime, $2^{11} - 1$ is composite, and so $2^{10}(2^{11} - 1)$ is not a perfect number.

Fermat made a claim which was to transform the previously Herculean task of determining whether or not $2^p - 1$ is prime for a given prime p. In effect, Fermat claimed that if an integer of this form has a prime factor, then that factor is of the form $2kp + 1$ (the factor 1 is covered by k=0, and it follows from Fermat's Little Theorem that $2^p - 1$ itself is of the form $2kp + 1$). Thus, to decide whether a proper factor of $2^{11} - 1$ exists, we only need to consider 23, since this is the *only* prime of the form $22k + 1$ with square less than $2^{11} - 1$. In fact, $2^{11} - 1 = 23$x89. (Note that 89 is also of the form $22k + 1$, as expected).

In his letter, Fermat exemplifies this statement about prime factors of $2^p - 1$ with reference to this case i.e. when $p = 11$. Burn (*ibid.*) reconstructs the argument that Fermat might have given with reference to this particular-but-generic case. Burn then continues: "Now we generalise the generic example of factorising $2^{11} - 1$ by expressing the argument algebraically". Of course, this accords with good modern practice, although Burn does not suggest that Fermat, having established the generic, then required the general formulation to be convinced of the general case.

Teaching and learning Wilson's theorem

The obscure Cambridge mathematician John Wilson is remembered to this day on account of a theorem stated in 1770, a century after Fermat's demise:

p is prime if and only if $(p\text{-}1)! = \text{-}1 \pmod p$

To be precise, it was Edward Waring, Isaac Newton's successor (and Stephen Hawking's predecessor) as the Cambridge Lucasian Professor of Mathematics, who stated the result of his former student, Wilson. In the best traditions of the time, neither Wilson nor Waring managed to prove the theorem: its status seems to have been a conjecture, the outcome of inductive reasoning from examples. It fell to Lagrange to give the first proof of Wilson's theorem, in 1773. How then, might we approach the genesis of the theorem and the construction of its proof with the hindsight and didactic insights of the twenty-first century? What might a generic proof of that theorem look like?

As a preliminary, we would need to know that ± 1 are the only self-inverse elements under multiplication modulo p. Now consider the prime number 13 (17 or 19 would do equally well) and list the reduced set of residues modulo 13:

1 2 3 4 5 6 7 8 9 10 11 12

Pair each of the numbers from 2 to 11 with its (distinct) multiplicative inverse mod 13: (2, 7), (3, 9), (4, 10), (5, 8), (6, 11). 1 and 12, of course, are self-inverse. [I usually link the elements in the inverse-pairs with lines on a chalk board]. Clearly, the product of these integers from 2 to 11 must be congruent to 1^5, i.e. 1, modulo 13. Therefore $12! \equiv 1 \times 1 \times 12$ (=12) mod 13. The argument is generic, since 13 was in no way an untypical choice: the pairing would work equally well with any prime.

The scene now shifts to a session with class of about 20 first-year undergraduate joint honours mathematics-with-education students. I could have stated Wilson's theorem and proved it formally in five minutes. In fact, it took an hour to make some conjectures and to work on proof. This is what happened.

First, I asked them to evaluate 4! mod 5, 6! mod 7, 10! mod 11, and to write down a conjecture. The most common version of the conjecture was $n! \equiv n \bmod (n+1)$. The 'for all n' seemed to be implicit. I asked them to evaluate 5! mod 6. They did, and they were visibly surprised by the refutation. I asked whether they could modify the conjecture. At first they homed in on the even/odd distinction between moduli, but n=8 led to further refutation and eventual restriction to prime values of n+1. n=12 provided a further confirming instance. I proceeded to an interactive presentation of a generic proof, inviting Sonia to pick a prime between 11 and 19. She chose 19. I got them to list 1 to 18 and work on inverse pairs in table-groups, during which Simon spontaneously explained to his colleagues why 18! *had* to be 18 mod 19. I asked him to repeat his reasoning to the class, and wrote his explanation on the whiteboard. He picked out eight inverse pairs, and explained why the product of the integers from 2 to 17 inclusive would have to be 1 mod 19. They dutifully copied Simon's argument. Later, I enquired what would have happened if we had looked at 28! mod 29, and Abby explained why it would have to be 28, again referring to inverse pairings of the integers from 2 to 27, although without feeling the need to identify the pairs this time. "Does everyone agree?", I asked. They agreed. One shouldn't read too much into such consent, however pleasing; nevertheless, Abby, at least, had convinced me that she had appropriated the proof-scheme.

The next day, at a tutorial meeting, I asked five members of the class to write it out the proof (that, for primes p, $p! \equiv p-1 \bmod p$) in conventional generality. Their responses were unaided and individual. It should be borne in mind, as I indicated earlier in this paper, that these students will have had little experience of composing formal proofs. Nevertheless, they all indicated in their writing that the genericity of the case p=19 had been apparent to them. Moreover, their argumentation and use of notation would have satisfied any examiner. Hannah's response, which was typical, was as follows.

$(p-1)(p-2)(p-3)(p-4) \ldots 2 \times 1$

Every element of M_p[1] has an inverse, because M_p is a group.

We know (from work on primitive roots) that only p-1 has order 2. Therefore p-1 is self-inverse. All other members of M_p apart from 1 must have a distinct inverse.

Each inverse pair when multiplied gives 1 mod p.

This gives $(p\text{-}1)(1^{1/2(p-3)})1 \equiv (p\text{-}1)! \bmod p$

Therefore $(p\text{-}1)! \equiv (p\text{-}1) \bmod p$

Only Zoë gave evidence of some insecurity in this intangible world that lies beyond examples. Her proof was much the same as Hannah's, but began with identification of the inverse pairs in the case p=11 (transfer to other examples) and concluded the comment:

I tried to find a formula for the inverses, for example p-2 has inverse p-6 (but only for p=11). I have been unable to do this.

For Zoë, mere knowledge of the *existence* of distinct inverses in the range 2 to p-2 is not enough. What is not clear is whether it leaves *her* cognitively insecure, or whether she believes that I (in my role as assessor) will expect more.

Abby's proof was elegantly and lucidly expressed, but stated that there are $^1/_2(p\text{-}1)$ inverse-pairs rather than $^1/_2(p\text{-}3)$. A case an error of manipulation, but not one of conception.

Generic arguments and cognitive unity

The domain of elementary number theory lends itself remarkably well to generic argument, presented with the intention of conveying the force and the structure of a conventional generalised argument through the medium of a particular case. One reason for this might be that, in the choice of examples, one seems to be spoiled for choice: there are an awful lot of integers (or primes, or whatever subset is called for) compared, say, with groups, or topological spaces. This is not to say that the choice of a generic example is an arbitrary one: it can be (and in a sense, it *ought* to be) a conscious pedagogical act. Some examples work better than others do for particular purposes - they carry and convey the generalisation rather better because the salient operations on the variable(s) can easily be tracked through the argument. Some tentative principles for the selection of generic arguments and the construction of generic arguments in number theory are given in Rowland (2002). I conclude, however, with some cautionary remarks.

First, the proof of Wilson's theorem given above crucially depends on knowing and being certain that 1 and p-1 are the only self-inverse elements under multiplication mod p. How shall we establish that result? It emerges readily (as a conjecture, of course) from examples, especially when the contrast is made with non-prime moduli. The usual proof runs as follows: if $1 \le a \le p\text{-}1$, and $a^2 \equiv 1 \bmod p$, then $p \mid (a\text{-}1)(a+1)$ and so $p \mid (a\text{-}1)$ or $p \mid (a+1)$. Whence a=1 or a=p-1. The essence of this argument is the solution of a

[1] M_p denotes the group {1, 2, 3, …, p-1} under multiplication mod p.

quadratic equation in a multiplicative modular group, which seems to rule out a generic presentation entirely free of algebraic symbolism. It is true that I could take p to be 13, and argue that a must be 1 or 12, but I can't seem to side-step arguing from (and with) $13 \mid (a-1)(a+1)$. Perhaps someone will delight me by convincing me that I'm wrong on this point.

Secondly, the converse of Wilson's theorem [if n is composite then $(n-1)! \neq n-1 \bmod n$] *appears* to lend itself wonderfully well to generic exposition. Take the case $n=10$. Now $9!=362880$, so $9!\equiv 0 \bmod 10$. Yes, but why? Because 9! includes factors 2 and 5. Since 10 is composite it can be decomposed into the product of two factors, both strictly between 1 and 10, so both occur as terms in 9! It is thus apparent that if n is composite then:
$(n-1)! \equiv 0 \bmod n$

However, whilst this conclusion is certainly true, the argument does not, in fact, transfer to all composite numbers. In those special cases when n is the square of a prime p, the only possible decomposition of n into the product of two factors, both strictly between 1 and p-1, is $n = p \times p$. The factors are not distinct and it is not the case that both occur as terms in $(n-1)!$ It is not difficult to make a separate argument for these cases, but they can easily be overlooked, and caught in the shadow, as it were, of the earlier generic argument.

Notwithstanding these cautionary words, there seems to be a good prospect of developing and offering a systematic didactic technology of formal proof in number theory, building on skilfully-constructed generic examples. There is evidence that such an approach to proof is supportive of the *cognitive unity* of theorems, that is to say "the continuity ... between the process of statement production and the process of its proof, as well as providing meaningful examples". (Mariotti, Bartolini Bussi, Boero, Ferri and Garuti, 1997).

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