SECONDARY SCHOOL STUDENTS' ILLUSION OF LINEARITY: EXPANDING THE EVIDENCE TOWARDS PROBABILISTIC REASONING

Wim Van Dooren[1 2], Dirk De Bock[2 3], Fien Depaepe[2], Dirk Janssens[2] and Lieven Verschaffel[2]
[1] Research Assistant of the Fund for Scientific Research – Flanders (F.W.O.)
[2] University of Leuven and [3]EHSAL Brussels; Belgium

Many secondary school students have a strong tendency towards improper linear reasoning in the domain of geometry, e.g. by believing that if the sides of a figure are doubled, the area is also doubled. In this paper, the evidence for this "illusion of linearity" is expanded to a new application domain: probabilistic reasoning. The paper reports an empirical investigation on the ability of 10th and 12th grade students to compare the probabilities of different situations. It is shown that most students have a good capability of comparing two events qualitatively, but at the same time incorrectly quantify this qualitative understanding into linear relationships between the varying quantities. It is shown how the research findings can shed a new light on some well-known probabilistic misconceptions.

THEORETICAL AND EMPIRICAL BACKGROUND

Because of its wide applicability for understanding problems in mathematics and sciences, the linear (or proportional) relationship is a key concept in primary and secondary education. However, together with its intrinsic simplicity and self-evidence (see, e.g., Rouche, 1989) the reinforcement of the linear model may lead students to "the seduction to deal with each numerical relation as though it were linear" (Freudenthal, 1983, p. 267), a tendency which is sometimes referred to as the "illusion of linearity". This phenomenon can appear at different levels and in many domains of mathematics and science education, such as elementary arithmetic, geometry, algebra, probability and physics (see, e.g., De Bock, Verschaffel, & Janssens, 1999). The best-known case of the overreliance on the linear model is situated in the domain of geometry: many students of different educational levels believe, for example, that when the sides of a figure are doubled, the area and volume will be doubled too (National Council of Teachers of Mathematics, 1989). In the past years, we performed a series of empirical studies to evidence this irresistible tendency in secondary school students, to identify influencing task variables (see, e.g., De Bock et al., 1999) and to unravel the underlying problem-solving processes (De Bock, Van Dooren, Verschaffel, & Janssens, 2001).

Besides continuing our studies in the domain of geometry, we set up a new line of research which aims at searching for the illusion of linearity in other mathematical domains. The first new domain that we chose for exploration of the overgeneralization of proportionality, is probabilistic reasoning. As explained in Van Dooren, De Bock and Verschaffel (in press), this domain is particularly interesting since the learning of probability is often hindered by students' primitive conceptions, wrong intuitions, fallacies, etc. (see, e.g., Shaughnessy, 1992). Moreover, the notion of chance itself shows some very strong similarities to the notion of proportion (Fischbein, 1975; Truran, 1994), suggesting that the overreliance on proportions might very well occur when students approach probabilistic situations. Recently, we have performed a review of the literature on probabilistic misconceptions, and made an inventory of those specific misconceptions which are conceptually related to the (unwarranted) application of proportions. This inventory contains a wide variety of erroneous reasonings (both famous and intensively studied misconceptions and anecdotal phenomena) for which the illusion of linearity yields a proper explanation (see Van Dooren et al., in press; Van Dooren, De Bock, Verschaffel, & Janssens, 2001). The current paper reports the next phase in this new research line: after the conceptual analysis of linearity-related probabilistic misconceptions, we empirically tested whether the overreliance on the linear model is actually present in students' probabilistic thinking.

FOCUS OF THE CURRENT PAPER

The empirical study reported in the current paper focuses on one particular class of misconceptions that was distinguished in the theoretical inventory (Van Dooren et al., 2001, in press): the scope is on those misconceptions that could possibly be explained by erroneously assuming a linear relationship between the variables (n, k and p) of a binomial chance situation, on the one hand, and the final chance for success (P), on the other hand, and that can be illustrated by the following example:

> The participants in a television game can roll 12 times with a fair die. If they obtain at least 4 times a six, they win a car. At Christmas day, the game leader is in a generous mood and tells the participants that they get 24 instead of 12 trials, so that their chance for winning the car is doubled.

The probability P of winning the car in the regular game is about 12.5 %, and it is determined by three variables: n is the number of allowed trials (12), k is the required number of successes (4) and p is the probability for success in a single trial (the chance to obtain a six with a fair die is 1/6). The game leader is mistaken, however, when he claims that the chance P for winning the Christmas game is doubled at the moment when the number of trials is doubled ($n = 24$). He wrongly assumes a linear relationship between n and P, and he would be surprised that in fact, the probability of winning the Christmas game is not $2 \times 12.5 = 25.0$ % but 58.4 %!

In the above example, the mistake was a wrongly assumed proportional relationship between n and P. Analogously, we can think of situations where a variation of k or p is expected to have a proportional effect on P: the game leader might think that the chance for winning the car (P) is doubled when k is halved (e.g. only 2 instead of 4 sixes are needed to win the car), or that P is tripled when p is tripled (e.g. the goal is to obtain even numbers instead of sixes). Moreover, also the combination of two variables can lead to erroneous reasonings. For example, one could reason that the regular game is equally favourable as a game in which you get 24 trials, but have to obtain at least 8 sixes (n and k are doubled), or a game in which you have to obtain 4 even numbers, but at the same time get only 4 trials (p is tripled, n is divided by three).

RESEARCH QUESTIONS AND HYPOTHESIS

The goal of the current study is to test to what extent the above-mentioned linear misconceptions in binomial chance settings are actually present among secondary school students with and without formal instruction in probability in general and in the binomial probability model in particular. More specifically, the study aims at answering the following research questions:

- Do secondary school students have a good *qualitative* insight in the effect of a variation of the different variables (n, k and p) that determine a binomial chance setting?
- To what extent do these students have a tendency to *quantify* these qualitative insights *as proportional relationships* between n, k and/or p, on the one hand, and P, on the other hand?

Our hypothesis is the following. Since several authors (e.g., Fischbein, 1975) have claimed that even very young children have an elementary understanding of probability, we expect secondary school students to have a good qualitative insight in probabilistic situations. But because of the intrinsic simplicity and self-evidence of the linear model and students' well-established tendency to overrely on the proportional model in other mathematical subdomains, we expect that most of them will erroneously translate these correct qualitative insights into linear relationships between the available variables.

More concretely, we make the following predictions: First, we expect that the students will be able to make appropriate judgements when they have to qualitatively compare the probability of two events that differ with respect to one of the variables n, k or p. We expect that this capacity is present in students without formal probability instruction, and that the formal learning of probability will have an additional positive effect on it. Second, we predict that the large majority of the students will quantify these correct qualitative insights in terms of a linear function between n, k and/or p on the one hand and P on the other hand. We expect this tendency to be persistent: it will be present in students with and without formal instruction in the binomial probability model.

METHOD

A paper-and-pencil test was taken from 225 secondary school students divided in two age groups: 107 10th graders and 118 12th graders[1]. Participants had one hour to solve a test consisting of 7 experimental items and 3 buffer items, which were offered in

	n	k	p	$n \times k$
Qualitative items	1	1	1	
Quantitative items	1	1	1	1

Table 1 : Design of the experimental items

randomised order. The 7 experimental items were multiple-choice problems in which the students had to compare the probability of two specific events. All 10 problems were situated in the context of rolling fair dice. The design of the test is shown in Table 1. It can be seen that for each of the variables n, k and p there were two sorts of items: students had to make either a qualitative or a quantitative comparison between two situations. Additionally, there was one item in which n and k were varied simultaneously[2]. Table 2 gives an example of a qualitative item, a quantitative item (in which the variable n is varied), and the item where n and k are varied simultaneously.

Variation of *n*		**Variation of *n* × *k***
Qualitative	**Quantitative**	**Quantitative**
I roll a fair die several times. The chance to have at least two times a three if I can roll four times is ❑ *larger than* ❑ *smaller than* ❑ *equal to* the chance to have at least two times a three if I can roll five times.	I roll a fair die several times. The chance to have at least two times a six if I can roll twelve times is *three times as large as* the chance to have at least two times a six if I can roll four times. ❑ *This is true* ❑ *This is not true*	I roll a fair die several times. The chance to have at least two times a five if I can roll six times is *equal to* the chance to have at least once a five if I can roll three times. ❑ *This is true* ❑ *This is not true*

Table 2 : Examples of experimental items

For the *qualitative* items, the student had to indicate whether the first event had a higher, lower or equal probability as the second event. The correct answer always was either "larger than" or "smaller than". The *quantitative* items were necessarily formulated differently: they contained an explicit quantified comparison of the probabilities of the two events, and the students had to judge the correctness of this statement. The quantification was always done proportionally (e.g., in the example in Table 2: n is tripled, thus P is tripled). As a consequence, the correct answer always was "This is not true" [3]. For each item, the students were asked to indicate the correct alternative and, moreover, to write down an explanation for their answer.

RESULTS

Tables 3 and 4 give an overview of the answers of the 10th and 12th grade students respectively on the seven experimental items.

As predicted, the large majority of the students in both age groups performed very well on the qualitative items. In about 90 % of the cases, the correct alternative was chosen, indicating that even before formal instruction in probability, students have a good qualitative understanding of how the probability in a situation evolves when an aspect (n, k or p) of this situation changes. Contrary to our prediction, the 12th

Variable	Qualitative items			Quantitative items		
	Correct	Incorrect	No answer	Correct	Incorrect	No answer
n	87.9	10.3	1.9	15.9	84.1	0.0
k	83.2	15.9	0.9	17.8	81.3	0.9
p	93.5	6.5	0.0	16.8	82.2	0.9
n × k				22.4	77.6	0.0
Total	88.2	10.9	0.9	18.2	81.3	0.5

Table 3 : Frequency (in %) of correct and incorrect answers of the 10th graders on the experimental items

Variable	Qualitative items			Quantitative items		
	Correct	Incorrect	No answer	Correct	Incorrect	No answer
n	92.4	6.8	0.8	18.7	76.3	5.1
k	78.8	19.5	1.7	38.1	61.9	0.0
p	97.5	2.5	0.0	16.1	83.1	0.8
n × k				30.5	67.8	1.7
Total	89.5	9.6	0.8	25.8	72.4	1.9

Table 4 : Frequency (in %) of correct and incorrect answers of the 12th graders on the experimental items

graders, who had already met the binomial probability distribution in their curriculum, performed only slightly better on these qualitative problems than the 10th graders (89.5% versus 88.2% correct answers), but considering the already high performance of 10th grade students, there might have been a ceiling effect. This good qualitative understanding of the chance situation was present for all of the three items. For the *k*-problem, there was a higher error rate which is probably due to the inversed effect of *k* on *P*: if *more* successes are needed, there is *less* chance of succeeding.

As expected, the high performance on the qualitative items is in sharp contrast with the low score on the quantitative problems. For this last category, the students most frequently chose for the incorrect alternative, which expressed a proportional relationship between *n*, *k* or *p* and *P*. Apparently, the vast majority of the students agreed with a proportional quantification of their correct quantitative insights. Also for the problem in which *n* and *k* were varied simultaneously (see Table 2), most of the students believed in the linear effect. The 12th graders performed slightly better on the quantitative problems (on the average, 25.8 % correct answers) than 10th graders (18.2 % correct answers) but as expected, the tendency towards linear modelling is still strongly present in these students. In particular, 12th grade students had a better score on the *k*-problem than 10th graders. Apparently, after instruction in probability, more students were aware that when the number of required successes is doubled, the probability is not necessarily halved.

We will also perform a qualitative analysis of the written notes and explanations of the students to investigate what specific strategy led the students towards the correct or incorrect answering alternative. A first round of qualitative analysis of the written notes and explanations accompanying the incorrect answers on the quantitative problems already revealed that more than 80 % of the incorrect answers on the quantitative problems can be clearly identified as resulting from students' overreliance on the linear model. Examples of such statements – referring to a linear relationship between the variables in the *n*-problem in Table 2 – are:

> In the first case, you can try three times more to obtain the same result (two sixes), so it is evident that you have three times more chance of winning.

> The chance of getting 2 sixes in 12 trials is a lot bigger than getting 2 sixes in 4 trials. And 12 is three times larger than 4, so the statement is true.

A more fine-grained analysis of students' written notes and explanations will provide richer data on the mechanisms and origins of students' improper proportional reasoning in binomial probability situations.

CONCLUSIONS AND DISCUSSION

The results of this study have confirmed our global hypothesis. Secondary school students have a good qualitative understanding of probabilistic situations, and are able to compare two such situations that differ in one variable. The understanding is even present in students without formal instruction in probability. At the same time, however, most students have a strong tendency to incorrectly quantify their correct qualitative insights as linear relationships between the variables in a binomial chance situation. In our multiple-choice items, the large majority of students chose for the alternative that stated a linear increase (or decrease) of the probability of the described event if one or two variables in the situation were increased (or decreased). This tendency towards linear reasoning is strongly present in all students in the research group, even those who met the binomial probability distribution in their mathematics curriculum.

Further qualitative analyses of the answers, as well as in-depth interviews with selected students will have to clarify how different components in the students' mathematical knowledge base lead them towards unwarranted linear reasoning.

Another remaining question, which we will address in our future research, is to what extent our findings are significantly affected by the way in which the test items were administered to the students. It could be argued that so many students fell into the linearity trap because they were seduced to do so, by confronting them with proportional statements with which they either had to agree or disagree. The tendency to reason linearly might considerably decrease when an open-answer format is used.

Finally, we want to show in an exemplary way how our research can shed some new light on a number of other well-documented misconceptions in the domain of probability, by looking at them from the perspective of the improper assumption of linear relationships between quantities. A first example is a problem used by Tirosh and Stavy (1999, p. 190):

> The Carmel family has two children, and the Levin family has four children. Is the probability that the Carmels have one son and one daughter larger than/equal to/smaller than the probability that the Levins have two sons and two daughters?

More than half of the 7^{th} to 12^{th} grade students in their study answered that the probabilities are the same. According to Tirosh and Stavy (1999, p. 190), students here applied the intuitive rule Same *A*–Same *B*: "Because the target boys:girls ratio in the two families was the same (1:2), the probability would therefore be the same." A second example comes from Fischbein (1999), who found that the majority of 5^{th} to 11^{th} grade students answered erroneously on the following problem:

> The likelihood of getting heads at least twice when tossing three coins is smaller than/equal to/greater than the likelihood of getting heads at least 200 times out of 300 times.

For a similar problem, Fischbein and Schnarch (1997, p. 103) argue that "The principle of equivalence of ratio imposes itself as relevant to the problem and thus dictates the answer."

Taking into account our own research findings, we believe that these misconceptions can be explained in a related, but somewhat different way, with a stronger emphasis on linear functions instead of ratios. In the study reported above, we observed that many students believe that (1) P is doubled when n is doubled (2) P is halved when k is doubled and (3) these two effects can also play simultaneously. The improper assumptions of linearity might also be a valid explanation for Tirosh and Stavy's and Fischbein's findings. We are even able to provide a useful framework for understanding a famous historical problem: Chevalier de Méré knew that it was advantageous to bet on at least one six in 4 rolls of a single die. He deduced that it should be equally advantageous to bet on at least one double-six in 24 rolls of a pair of dice. This did not yield the financial gain he had hoped for… Probably, most of our students would also agree that multiplying n by 6 and dividing p by 6 is a neutral operation for P.

REFERENCES

De Bock, D., Van Dooren, W., Verschaffel, L., & Janssens, D. (2001). Secondary school pupils' improper proportional reasoning: An in-depth study of the nature and persistence of pupils' errors. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 313–320). Utrecht, The Netherlands.

De Bock, D., Verschaffel, L., & Janssens, D. (1999). Some reflections on the illusion of linearity. In P. Radelet (Ed.), *Proceedings of the Third European Summer University on History and Epistemology in Mathematical Education* (Vol. 1, pp. 153–167). Leuven / Louvain-la-Neuve, Belgium.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children.* Dordrecht: Reidel.

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics, 38*, 11–50.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education, 28*, 96–105.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Rouche, N. (1989). Prouver: Amener à l'évidence ou contrôler des implications? [Proving: Bringing back to the evidence or verifying the implications?] *Actes du 7ème Colloque inter-IREM Epistémologie et Histoire des Mathématiques* (pp. 8–38). Besançon, France.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics thinking and learning* (pp. 465–494). New York: Macmillan.

Tirosh, D., & Stavy, R. (1999). Intuitive rules and comparison tasks. *Mathematical Thinking and Learning, 1*, 179–194.

Truran, J. M. (1994). Examination of a relationship between children's estimation of probabilities and their understanding of proportion. In J. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 337–344). Lisbon, Portugal.

Van Dooren, W., De Bock, D., & Verschaffel, L. (in press). De illusie vanineariteit in kansrekening: Een terreinverkenning [The illusion of linearity in probability: An exploration of the domain]. *Nieuwe Wiskrant*.

Van Dooren, W., De Bock, D., Verschaffel, L., & Janssens, D. (2001). Searching for the illusion of linearity in probabilistic misconceptions: A conceptual analysis. *Poster presented at the 25th Conference of the International Group for the Psychology of Mathematics Education.* Utrecht, The Netherlands.

---

[1] According to the Flemish curriculum, probability (and the binomial probability model in particular) is a part of the mathematics curriculum in the 11th grade.

[2] As can be seen in the $n{\times}k$-problem, the simultaneous variation of two variables (in a proportional way) necessarily implies a quantified statement (i.e. the *equality* of the probabilities). Therefore it was impossible to formulate a qualitative item for this category.

[3] The buffer items were manipulated so that they were similar to the qualitative items (but 'equal to' was the *correct* alternative) or to the quantitative items (but this time, 'This is true' was the *correct* alternative).