CAN POOR STUDENTS IDENTIFY THE GOOD FEATURES OF A DEMONSTRATED PROBLEM SOLVING METHOD AND USE IT TO SOLVE A GENERALIZATION PROBLEM?

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This study investigated the effect of presentation of a good solution method on poor students' solving generalization problems and the difference between good and poor students' evaluation of a good method. Subjects were fourth, fifth and sixth graders. Three hints were presented :the suggestion of using a "solve a simpler problem" strategy, demonstrating how it is executed in solving a small term problem(n=4,5) and the application of the strategy to the far generalization problem(n=30). The hint was effective for poor students of every grade. Good students evaluated more often the aspect of structure for a good method.

1. Introduction

In mathematics lessons, it is important to learn mathematical thinking, including generalization. Generalization problems are problems that are solvable by finding a pattern of quantitative relationship in a given problem situation. There are three examples in Figure 1. Stacey (1989) pointed out that problems of this type are hard for school students. She contrasted success on and methods used on near generalization problems (where the number of a term is small e.g. finding the number of marbles with 10 marbles on a side in Problem C, Figure 1) and far generalization problems (where the number of a term is large e.g. side of 100 in Problem C). Students even find that near generalization problems are difficult to solve.

Ishida has investigated the processes used by good and poor problem solvers on generalization problems. Good students and poor students showed different approaches. Good students tried to find a mathematical expression when they solve a near generalization problem, and they applied it to solve a far generalization problem. However, poor students tended to use a "draw a figure" strategy on the near generalization and then they tried but failed to find a mathematical expression on the far generalization problem. This observation was consistent across grades 4, 5 and 6 (Ishida,1992; Ishida&Sato,1996).

Sato (1999) showed that it is helpful for some students to be asked to solve a near generalization problem before a far generalization problem. (For example, to be asked to find the number of stars for the 8^{th} figure as a sub question would make Problem A of Figure 1 easier.) Also Sato found that it is helpful for some students to be first asked to draw a small term figure and write a mathematical expression based on it before a far generalization problem. Those hints seem to be useful to remove some of the factors that affect the difficulty of solving a generalization problem. For example, students do not need to realise by themselves that they should use the "solve a simpler problem" strategy or set up a mathematical expression.

Kimura (2001) gave a further hint. In Kimura's study, two hints were presented: the suggestion of using a "solve a simpler problem" strategy (as Sato) and also demonstrating how it is executed in solving a small term problem. These hints were more useful than those of Sato for Grade 4 and equally helpful at Grades 5 and 6. But Kimura did not demonstrate the application of the strategy to a far generalization problem. If such a demonstration is given to poor problem solvers, can they use such a hint to solve a far generalization problem? This is one of the research questions for the present study.

Many studies have demonstrated the need to develop meta-cognition to work in problem solving situations. Evaluation of the mathematical value of methods relates to this meta-cognitive knowledge. Ishida (1998) has studied the choices that students in Grades 4, 5 and 6 make between four methods of solving far generalization problems. One method involved only drawing a figure and counting; the second provided a mathematical expression which did not reflect a relevant pattern; the third provided an expression linked to a simple generalizable structure; the fourth provided an expression in which the way it could be generalized was difficult to see. Most students (about 85 % including both good and poor problem solvers) selected the best method (third in list above). A few students (about 10%) selected the strategy of drawing a figure, which is a poor strategy for a far generalization problem. Their performance on the far generalization problem was lower than that of the others. The study also asked students to write the reason of their choice of method. The reasons were very varied, including references to the generality of the method, its effectiveness in producing an answer, the fact that it displays a simple structure, and the fact that the method is easy to understand. Effectiveness was often written by students in all grades. In the above study (Ishida, 1998), students selected preferred methods from a list of both good and poor problem solving methods. In the present study, only a good method is offered but students' evaluation of the good points of this method are examined. This is of greater relevance to teaching, where good methods are usually presented. Do good and poor students form different evaluations of them?

In summary, the present study asks two questions:

1 Is presentation of a good solution method useful to help poor students solve generalization problems?

2 Are there any differences between good and poor students' evaluation of a good method?

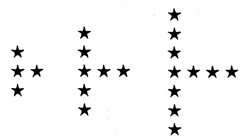
2. Procedure

The subjects of this study are 112 fourth graders, 102 fifth graders and 113 sixth graders, who are students of two elementary schools in Yokohama city, Japan. The three problems in Figure 1 were given to the subjects (each allocated 10 minutes) by a classroom teacher. Problems A and B were used to divide the students into good and poor problem solvers and Problem C was used to investigate the purpose of this study. The 94 students who succeeded on problems A and B were regarded as good problem solvers. The 166 students who failed to solve both problems were regarded as poor problem solvers. The numbers by grade are shown in Table 1. Sixty seven students who succeeded on only one of problems A and B are excluded from further consideration in this study.

Problem C asks students to solve a near generalization (n= 3, Question 2) and a far generalization (n = 77, Question 3), emphasizing the use of the mathematical expression generalized from Akira's solution. It gives the hints used by Kimura (2001) and also demonstrates the application of the strategy to the far generalization problem (n=30). The number 77 was selected to avoid false proportional solutions, which are common when student see simple multiplicative relations between the questions (Stacey, 1989). Question 1 of Problem C asks students to evaluate the benefits of Akira's method. The five choices that are presented are based on reasons offered by

Problem A

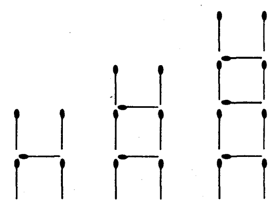
Figures are made as follows. How many stars are needed in the 100th place?



1st 2nd 3rd

Problem B

Using matchsticks ladders are made as following. How many matchsticks are needed to make ladder of 100 rungs?



1 rung 2 rungs 3 rungs

Problem C

Marbles are arranged like the following figures. Find the number of marbles needed to make a figure of 30 marbles to a side 30.

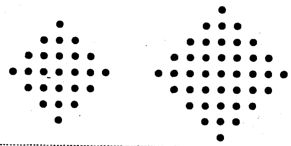
Akira is thinking of this problem.

He began by thinking it in the case of 4 marbles on a side and 5 marbles on a side.

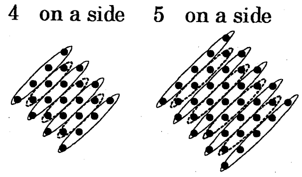
Then, he thought how to find a mathematical expression to get the answer to the problem.

30x30+29x29=1741 Ans. 1741 marbles

4 on a side 5 on a side



4 on a side 5 on a side



$4\times4+3\times3=25$ $5\times5+4\times4=41$

(1) How do you evaluate Akira's method? Select two viewpoints from the list.

- A I can find an answer correctly.
- B I can get an answer fast.
- C I can use this method when the number of marbles of a side increases.
- D He represents a mathematical expression.
- E He finds a group of marbles and uses it to make an expression.

(2) How many marbles are needed to make a figure of 3 marbles on a side? Using Akira's method, write a mathematical expression to find an answer.

(3) How many marbles are needed to make a figure of 77marbles on a side? Using Akira's method, write a mathematical expression to find an answer

Figure 1. Three problems used in this study.

students in Ishida's 1998 study, especially relating to use of mathematical expressions. Items A and B are aspects of effectiveness and item C is generality. Both items D and E refer to mathematical expressions but item E links the expression to the structure of the problem. Item D refers only to Akira's writing an expression.

3. Results

(1) The effectiveness of the hint in problem C.

The effectiveness of the hint given in Problem C is investigated first. In the analysis, we focused on a performance of setting up expression, so the following data was based on a correct answer for setting up a mathematical expression.

Table 1 shows the percentage of good and poor solvers who were correct on question (2) and (3) in problem C. Table 1 shows that the performance of poor students (those who had failed on problems A and B) was good on problem C. In both questions, the percentage correct increases from grade 4 to 6. Therefore the hint used in problem C effectively assists poor students to solve a generalization problem.

How did the hints offered by Problem C assist students? In Problems A and B, in every grade, students frequently gave wrong mathematical expressions by using proportion strategy (for example, 3 times 100 in problem A, 5 times 100 in problem B) and other wrong mathematical expressions which simply combined given numerals. These error types show that they seemed not to try to find a pattern or structure from figures given in problems. The hints in Problem C encouraged them to look at the structure.

Table 1 Percent correct on Problem C for good and poor problem solvers.

	Grade 4	Grade 5	Grade6
Good students	(N =17)	(N=35)	(N=42)
Question (2)	88.2%	100%	95.2%
Question (3)	76.5%	94.3%	95.2%
Poor students	(N=82)	(N= 41)	(N=43)
Question (2)	67.1%	80.5%	90.7%
Question (3)	47.6%	61.0%	81.4%

(2) How common errors reveal understanding of structure

Error analysis for poor students who failed to solve problem C is important to clarify their difficulty. The main classifiable errors were to give no answer, using the form "a x a + b x b" with wrong numbers, or using the incomplete form "a x a +". In total there were 26 students in Grade 4 who failed to solve both question (2) and (3) of problem C. For these, most gave no answer or an unclassified error. The "a x a + b x b" type of error did not appear very often. This trend was the same for Grades 5 and 6 although there were fewer students. On the other hand, poor students who failed only on question (3) in problem C made an error like "a x a + b x b" and "a x a + " . While they set up a correct mathematical expression on question (2), that is 3 x 3 + 2 x 2, they could not generalize it correctly. A high percentage wrote "77 x 77 + " and could not complete this with 76 x 76.

We need to distinguish two types of students who failed on problem C: those who failed on both question (2) and (3) and those who failed only on question (3). In the former case, they did not understand about Akira's method at all. They did not understand that Akira tries to find a pattern in a small term situation to set up a mathematical expression that can be generalized easily. In the latter case, they understand the pattern shown in Akira's method and can easily apply it to the very near generalization problem but cannot apply it to a far generalization problem. This shows that they do not understand the mathematical expression on a small term situation as revealing a mathematical structure.

(3) Comparing good and poor students evaluation of the good method demonstrated in problem C

In question (1) of problem C, students were asked to select two viewpoints to indicate what makes Akira's solution good. The reason why we let them select two items is that this solution method has several good points. Because the hint was not equally helpful for solving problem C for all grades, we analyze the data by grade. Table 2 shows the percentage of students who selected each item. Since the students could select two viewpoints each, the theoretical totals across the rows are 200%. However, not all students selected two, so the totals are less than 200%.

Table 2 Percentage of students selecting items A to E by grade and problem solving group.

		A correctness	B effectiveness	C generality	D expression	E structure
Grade 4	Good	5.9%	11.8%	35.3%	70.6%	70.6 %
	Poor	23.2%	12.2%	42.7%	64.6%	43.9%
Grade 5	Good	17.1%	0%	48.6%	62.9%	68.6%
	Poor	26.8%	9.8%	51.2%	61.0%	43.9%
Grade 6	Good	16.7%	4.8%	52.4%	52.4%	66.7%
	Poor	23.2%	14.0%	58.1%	55.8%	37.2%

The significant difference between good and poor students' selection appears on item E in every grade. This item refers that whether they identify as most important the aspect of structure or pattern to help for generalization. Poor students select this aspect less than good students.

This table also shows that most of good students in each grade commonly identified item D (expression) and item E (structure) as an advantage for Akira's method, while poor students selected commonly item D (expression) more than E (structure). They only paid attention to the surface solution, identifying only that Akira has written a mathematical expression. They did not identify that the expression was linked to the structure.

4. Discussion

(1) The effect of presenting other's method on poor solvers' solving a generalization problem

The hint used in this study assisted poor students to solve a generalization problem. Specially, most of Grade 5 and 6 students succeeded. Many of the poor students who could not solve a far generalization problem without a hint, were able to solve the far generalization problem in Problem C. As they can learn how to solve Akira's problem (n=30) from Akira's method, they understand the way a pattern and its mathematical expression can be used more generally, but they need to learn to find the pattern and expression for themselves. Kimura (2001) pointed out that offering the hint of a figure that presented the structure and a mathematical expression based on this structure in a small term situation assisted poor Grade 5 and 6 students to solve a generalization problem. The result of this study supported Kimura (2001) and extended it.

This result suggested that one difficulty of poor students in solving a far generalization problem was the lack of a way to approach a generalization problem. If they learn Akira's method and have a skill to solve it, their performance will improve. However, the hint was not helpful for some poor students. More research is needed to identify which factor prevented them from solving the far generalization problem.

(2) Poor students could not understand the usefulness of structure in solving generalization problems. Error analysis showed that poor students could not understand the relationships of quantity in small term problems. The result of evaluation of Akira's method also demonstrated that there was a significant difference between good and poor students in selecting a goodness of "structure" from Grade 4 to 6.This may suggest that students who have difficulty for solving generalization problems have difficulty in recognizing the mathematical structure in figures.

It is very interesting that the difference of viewpoint for evaluation between good students and poor students appears to be in noticing a simple structure. Good students learn the importance of finding a pattern through their regular mathematics lesson, but this is not true for all students. Teachers could use this information to be more explicit about developing pattern finding ability through mathematical lessons. Even students from poor group 1 need to develop their ability to notice the structure of a quantitative relationship.

As noted above, the mathematical structure of Problem C (a quadratic relationship) is different from the linear structure of problems A and B. The structure of problem C is more complex than other two problems, so a further study could investigate the effect of the hint on problems similar to problems A and B.

Most of good students from Grades 4 to 6 selected item E (structure). However, the percentage of good students selecting item C (generality) by good students increased from Grades 4 to 6. This result suggests that understanding of structure is different through the grades and that as they grow older, students come to understand the role of a mathematical expression from both aspects of structure and of generality.

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