RESEARCH FORUM 1

Theme
Abstractions: Theories about the emergence of knowledge structures

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ABSTRACTION: THEORIES ABOUT THE EMERGENCE OF KNOWLEDGE STRUCTURES

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INTRODUCTION

Tommy Dreyfus and Eddie Gray

Since there is no universally accepted research paradigm in mathematics education, theories and terminology tend to multiply. It is therefore one of the tasks of the research community to critically compare theories that deal with closely related issues and have similar aims. The setting of a research forum at PME conferences is one of the few opportunities where attempts at such comparison can be undertaken in public by a large group of researchers.

Several theories for processes describing the emergence of mathematical knowledge structures (abstraction) have been put forward recently. While these theories differ in many respects, they have a common goal: they aim to provide a means for the description of processes during which new mathematical knowledge structures emerge. Thus, they have the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice.

Three theories of abstraction have been selected for discussion in the Research Forum on the basis of their similar aims but different approaches. The selection was made so as to achieve, within the limited space and time available, a wide variety with respect to the theoretical underpinnings of the approaches, the formal or informal nature of the emergent knowledge, the role of context in the process of abstraction, the importance the theories attribute to contextual factors, and the degree to which they are anchored in instructional design. While a main aim of each of the three theories is to describe processes (rather than outcomes) of the emergence of knowledge during learning activities and a secondary aim is to contribute to the design of learning, the assumptions of the three theories differ considerably. For example, the theory by Tall & Gray is predominantly cognitive (with links to neuro-physiology), while context is accorded a limited role; on the other hand, the theory by Gravemeijer is predominantly contextual. Similarly, instructional design interacts with the theories in very different ways: For Tall & Gray, instructional design is a result of the research undertaking, for Schwarz, Hershkowitz & Dreyfus, it is the location of the research undertaking (in the sense that the research takes place in an environment designed for instruction), and for Gravemeijer instructional design is at the origin of the research undertaking.

A theory cannot possibly be presented appropriately in the limited space allotted to each group of contributors within this written presentation of the Research Forum. Therefore, the following contributions by the three presenting groups only give a summary overview over each of the theories, and provide the reader with ample references for further reading. Many of these references are accessible from the conference website. They are also available from the authors. The papers included hereafter have been written following a set of fairly detailed guidelines. The aim of these guidelines was to define a number of dimensions for comparison of the theories. Specifically, each paper

(i) Gives a description of what their theory is about,
(ii) Identifies the assumptions being made by the theory,
(iii) States the theses of the theory (what does the theory say?), including a detailed discussion of the meaning of the terms that are used,
(iv) Discusses the aim(s) and applications of the theory,
(v) Gives evidence concerning the validation of the theory (for example empirical research).

Authors have been asked to be specific about the meaning of their terminology to enable the identification of cases where either the same term is used with different meanings or where different terms are used to describe closely related phenomena. First and foremost, the term 'abstraction' is likely to mean different things to different people; similarly, the term 'context' may be given a rather narrow or a very wide interpretation. In addition, each theory uses its own idiosyncratic terms.

The two papers by the reactors stress commonalties and differences between the three theories, for example with respect to the underlying definitions of abstraction, the domains of applicability of the theories, the empirical evidence validating the theory, and the role which context plays in the process of abstraction.

Whilst our theme examines from a contemporary perspective theoretical issues that have been of interest to PME members over the past quarter of a century, two broader issues are also relevant to our discussions.

First, there is an issue associated with "having abstracted". Though we are examining theoretical perspectives of the role of abstraction, its contribution and influence on different modes of thinking displayed by mathematics students leads us to ask to three general questions, which can and should be expected to arise for discussion out of the reaction papers are:

- When do we know students have abstracted and what student behavior attests to this?
- What happens if students do not abstract?
- How do we encourage abstraction?

Secondly, there is an issue associated with coherence and unification. In the opinion of the presenters and the reactors, the domain of theoretical physics has no exclusive right to yearn for a unified theory. Even though mathematics education as a scientific discipline is a few centuries younger than physics, we believe that this is the time to start the work of combining, merging and fusing our theories, and thus to make them more widely known, applicable and applied. One milestone on this road will be the use of different theories to analyze the same data set and thus to directly confront the theories. We hope that the discussions of the research forum will give rise to such undertakings, as well as to some speculation on a possible unification of the theories into a larger framework. It is this hope for progress in the direction of fewer, more widely known, more widely agreed, and more widely applied theories that has motivated this research forum.

ABSTRACTION AS A NATURAL PROCESS OF MENTAL COMPRESSION

Eddie Gray & David Tall

Introduction

The term 'abstract' has its origins in the Latin *ab* (from) *trahere* (to drag) as:

- a verb: to ab*stract*, (a process),
- an adjective: to be *ab*stract, (a property),
- and a noun: an *ab*stract, for instance, an image in painting (a concept).

The corresponding word 'ab*stracti*on' is dually a process of 'drawing from' a situation and also the concept (the abstraction) output by that process. It has a multi-modal meaning as process, property or concept. Piaget distinguished between construction of meaning through *empirical abstraction* (focusing on *objects* and their properties) and *pseudo-empirical abstraction* (focusing on *actions* on objects and the properties of the actions). Later *reflective abstraction* occurs through mental actions on mental concepts in which the mental operations themselves become new objects of thought (Piaget, 1972, p. 70). In Tall et al, 2000, we reviewed ideas in the literature and concluded that elementary mathematical thinking uses reflective abstraction both by focusing on *objects* (for instance, in geometry) and on *operations* on objects represented as *symbols* (in arithmetic, algebra, etc). In the latter case we see symbols used dually as process and concept and have formulated this in terms of the notion of *procept* (Gray & Tall, 1994, see also below). At a later stage, in advanced mathematical thinking, the focus changes to *properties* (of objects and operations) formulated as fundamental axioms for mathematical theories.

Our hypothesis is that different forms of abstraction lead to different type of cognitive development and in turn, to differing cognitive problems. Empirical and reflective abstraction in shape and space lead to a van Hiele type development that we see as the growing dominance of verbal description over visual perception, as language refines our imagery and leads to increasingly sophisticated forms of mathematical structure and proof. Pseudo-empirical and reflective abstraction in arithmetic, algebra and calculus naturally focus on our notion of *procept*. Increasing focus on properties and deduction lead to a property-based axiomatic theory where the process of proof leads to the concept of theorem which may then be used as steps in building up a systematic formal theory.

We have a great empathy for the notion of different *modes* of operation as proposed by Bruner (1966) and, more particularly, in the SOLO taxonomy of Biggs and Collis. (1982). For instance, it is possible to build a holistic embodied mode that relates to the enactive/iconic modes of Bruner or the sensori-motor/ikonic modes of Biggs and Collis, before gaining an insight in proceptual (concrete-symbolic) terms; or, at a later stage in advanced mathematical thinking, in formal-deductive terms. Tall (1999) considers the distinct forms of proof available in these various modes as the child develops cognitively into a mathematical expert. Tall (2002) reviews calculus in terms of an enactive-iconic approach manipulating graphs, symbolic-proceptual representations (manipulating formulae) and formal proof (in analysis).

In this short paper we do not have space to attend to our full theoretical perspective. We focus only on the abstractive processes occurring in constructing *procepts* in arithmetic, algebra and symbolic calculus and how differing types of symbol (whole numbers, fractions, algebraic expressions, (infinite) decimals, limits) give rise to distinct problems of concept construction and re-construction.

Five Aspects

The research forum is designed to focus on five aspects, given in (a)-(e) below.

a. What is the theory about?

Our theory grows as a result of our quest to understand not only *what* students do in constructing symbolic mathematics, but *how* they do it. We believe that abstraction is a *natural* consequence of human brain function. At any given time human thinking occurs dynamically as a process, whereby items evoked in the focus of attention are manipulated mentally as concepts. It is the duality of symbols in arithmetic, algebra, etc as both process and concept that is the basis of our theory.

b. What assumptions are being made?

We assume that abstraction is a natural product of human mental activity, in which a complex parallel-processing organ solves the problem of complexity by focusing on essential structures that enable decisions to be made. Sometimes this process of abstraction is a conscious reflective act, but much of it does, and must, occur unconsciously to enable the brain to focus only on essential elements. There is physical evidence that over time routinising tasks uses less brain capacity:

> As a task to be learned is practiced, its performance becomes more and more automatic; as this occurs, it fades from consciousness, the number of brain regions involved in the task becomes smaller. (Edelman & Tononi, 2000, p.51)

There is also a compression in the nature of the symbolism being used:

> I should also mention one other property of a symbolic system – its compactibility – a property that permits condensations of the order $F = MA$ or $S = ½gt^2$, …in each case the grammar being quite ordinary, though the semantic squeeze is quite enormous.
>
> (Bruner, 1966, p. 12.)

We do not have the data to link mathematical activity in a one-one mapping to neurophysical phenomena, steps in this direction (eg Dehaene, 1997) are still in their early stages. However, the underlying biological basis of mathematical thinking in a brain ill-built for numerical computation and formal logic, is a vital underpinning for our own reflections on how mathematical thinking develops.

c. What does the theory claim? What terms are used and what do they mean?

The notion of procept (as given in Gray & Tall, 1994) is seminal in what follows.

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.
> … A *procept* consists of a collection of elementary procepts which have the same object. (Gray &

We follow Davis (1983, p. 257) in defining a *procedure* as an explicit step-by-step algorithm for implementing a process and see a spectrum of increasing power through the usage of procedure, process and procept. We do not agree with Sfard or Dubinsky that the development invariably proceeds in a sequence we describe as procedure-process-procept. In particular, as students become more sophisticated, they may sense an intuitive holistic grasp of the overall ideas in, say, an embodied mode before concerning themselves with the specific procedures that may be seen to occupy a particular role within a symbolic or formal mode of operation.

We do not have a theory that tells us how *all* individuals can be helped to move through all of these modes. (Indeed, *no-one* has such a theory at this moment in time.) Instead, in the growth of symbols, we find a bifurcation between those who concentrate more on the procedures associated with symbols, who have a greater cognitive strain to overcome, and those who develop a proceptual system switching flexibly between process and concept to construct a more powerful generative mental structure. This does *not* mean that students necessarily remain in a fixed part of the spectrum. However, we do have considerable evidence that there is a bifurcation in performance between those who remain entrenched in procedures and those who develop more flexible proceptual thinking, so that progress to greater sophistication is more difficult for some and easier for others.

d. What are the aims of the theory and what are its applications?

The initial aim of our theory of the proceptual growth of symbols is to try to explain why some students are so highly successful with symbols, whilst others are procedural at best and could, at worst, be overwhelmed by the complexity of mathematics. To move towards this overall goal we focus on the different ways that procepts arise in cognitive development. These include

(1) ***arithmetic procepts***, 5+4, 3×4, $\frac{1}{2}+\frac{2}{3}$, 1·54÷2·3, all have built-in algorithms to obtain an answer. They are *computational*, both as processes and even as concepts. *Fractional procepts* behave differently because the focus moves from sharing procedures (eg divide into 4 equal parts and take 2) to equivalent fractions, which from our viewpoint are seen as processes that have the same effect (divide into 4 equal parts and take 2, has the same effect as divide into 6 equal parts and take 3).

(2) ***algebraic procepts***, e.g. $2+3x$, can only be evaluated if the value of x is known and so involves only a *potential process* (of numerical substitution) and yet the algebraic expressions themselves represent manipulable concepts (manipulated using algebraic rules of equivalence).

(3) ***implicit procepts***, such as the powers $x^{\frac{1}{2}}$, x^0 or x^{-1}, for which the original meaning of x^n no longer applies but the properties need to be deduced using the power law $x^m \times x^n = x^{m+n}$ (which also no longer has its original meaning!)

(4) ***limit procepts***, $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x-a}$ or $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ etc, have ***potentially infinite processes*** 'getting close' to a limit value that may not be computable in a finite number of steps.

(5) ***calculus procepts***, such as $\dfrac{d(x^2e^x)}{dx}$ or $\int_0^{\pi}\sin mx\cos nx\,dx$ focus again on finite operational algorithms of computation (the rules for differentiation and integration).

This reveals that each of new form of procept has its own peculiar difficulties that makes abstraction qualitatively different in each case. We believe that knowledge of these specific difficulties is essential to help a wider spectrum of students to succeed in the longer-term process of successive abstractions.

e. How has the theory been validated?

Our data (summarized in Tall, Gray, *et al*, 2001) reveals both general themes and specific information on cases (1)-(5) above. The general themes illustrate diverging approaches from procedural to proceptual in a spectrum of students from elementary arithmetic (Gray & Tall, 1994), through algebra (DeMarois, 1998; McGowen, 1998; Crowley, 1999), symbolic calculus (Ali, 1996), and on to formal mathematical theory (Pinto, 1998). In addition, qualitative differences in imagery emerge from different forms of abstraction (Pitta, 1998; Gray & Pitta 1999), leading to differing levels of success in the longer term, depending on whether children continue to focus on real-world situations and imagery, or move on to a more flexible proceptual hierarchy (Gray *et al*, 1999). The data from the above-mentioned studies reveal how differing contexts pose significantly different kinds of cognitive problems in both the nature of the procepts concerned and the procedure-process-procept spectrum of student activity. We believe that these difficulties are best handled by the learner supported by a mentor who is aware not only of the mathematics but of the underlying cognitive structures.

This aspect of learning is complementary to the desire of Schwarz *et al* (this forum) to theorize about a general strategy for encouraging abstraction in context. We suggest that it is a laudable aim to have a general theory of construction, but we observe that specifics often overwhelm the broad sweep of such a theory. From the learner's point of view, different obstacles occur in different contexts. The acquisition of mathematical knowledge from early years to undergraduate level involves a variety of reconstructions. Each new reconstruction refines that which was established earlier so that effective reconstructions contribute to the organic nature of growth in the embodied and proceptual modes of operation and on to a close harmony between wider aspects of concept image and concept definition in advanced mathematical thinking. Our central concern is not just how we can encourage students to make abstractions, but also to find why some students succeed so effortlessly and others can fail so badly at making the necessary reconstructions. Our empirical evidence provides an insight into a possible answer—inappropriate abstraction from mathematical activity.

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ABSTRACTION IN CONTEXT: CONSTRUCTION AND CONSOLIDATION OF KNOWLEDGE STRUCTURES

Baruch Schwarz, Rina Hershkowitz and Tommy Dreyfus

The construction of abstract knowledge structures is central in human learning, including mathematics education. As practitioners who are informed about recent theoretical research, we have been deeply involved in curriculum design, development, and implementation. Our approach to abstraction is thus a product of our interest both in theory concerning abstraction and in experimental observations of activities in schools in which we judged that a process of abstraction has been evidenced.

Many researchers have taken a predominantly theoretical stance and have described abstraction as some type of decontextualization. For example, Piaget has proposed that abstraction consist in focusing on some distinguished properties and relationships of a set of objects rather than on the objects themselves. Abstraction is thus a process of decontextualisation. According to Davydov (1972/1990), on the other hand, abstraction starts from an initial, undeveloped form of knowledge and ends with a consistent and elaborate knowledge structure.

A Definition for Abstraction, the Nested RBC Model, and Consolidation

Leaning on ideas of Davydov and other researchers, and in view of our experience in classrooms and our need for an operational definition, we translated our theoretical principles into the following more applicable definition:

> *Abstraction is an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure.*

The term *activity* in our definition is directly borrowed from Activity Theory (Leont'ev, 1981) and emphasizes that abstraction is an activity, a chain of actions undertaken by an individual or a group, and driven by a motive, which occurs in a specific context. Context is a personal and social construct, which includes the student's social and personal history, conceptions, artifacts, and social interaction. The term *previously constructed mathematics* refers to two points: One, that outcomes of previous processes of abstraction may be used during the present abstraction activity; and two that a process of abstraction leads from initial, unrefined abstract entities to a novel structure, as posited by Davydov. These two points show the recursive nature of abstraction. The phrase *reorganizing into a new structure implies* the establishment of connections, such as inventing a mathematical generalization, proof, or a new strategy of solving a problem. The novel structure comes about through reorganization and the establishment of new internal and external links within and between the initial entities. We very intentionally used the word *new* to express that, as a result of abstraction, participants in the activity perceive something that was previously inaccessible to them. Finally, we borrowed the term *vertical* from the Dutch culture of Realistic Mathematics Education, in which researchers relate to vertical mathematization as to an activity in which mathematical elements are put together, structured, organized, developed etc. into other elements, often in more abstract or formal form than the originals. It is mainly this integration that comes about by the establishment of new connections during processes of abstraction, which we wanted to express by means of the term vertical.

According to this definition, abstraction is not an objective, universal process but depends strongly on context, on the history of the participants in the activity of abstraction and on artefacts available to the participants. In this sense structure is internal, "personalized".

The study of abstraction raises a methodological challenge. Whichever its definition is, abstraction implies mental activity, which is not observable. Since we want to empirically investigate processes of abstraction, we need to devise a way to make them observable. Put otherwise, we need to use (theoretical) spectacles, which let us see processes of abstraction, as they occur during students' activities. And it is precisely this view of abstraction as activity, which provides us with the desired spectacles: Activities are composed of actions – and actions are frequently observable. The question, which actions are relevant for abstraction, we answer with reference to Pontecorvo & Girardet (1993): *Epistemic actions* are mental actions by means of which knowledge is used or constructed. Epistemic actions are often revealed in suitable settings. Therefore, settings with rich social interactions are good frameworks for observing epistemic actions. Coming back to our experimental research we were able to identify three particular epistemic actions, which are constituent of abstraction, and provide a strong indication that a process of abstraction is happening: *Recognizing, Building-With and Constructing*, or RBC. In summary, we consider these epistemic actions because they characterize abstraction and because they are observable. In other words, they provide us with an operational description of processes of abstraction.

Constructing is the central action of abstraction. It consists of assembling knowledge artefacts to produce a new knowledge structure to which the participants become acquainted. *Recognizing* a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation. *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. The same task may thus lead to building-with by one student but to constructing by another, depending on the student's personal history, and more specifically on whether or not the required artefacts are at the student's disposal.

The three epistemic actions are the elements of a model, called the dynamically nested RBC model of abstraction. According to this model, constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognizing actions are nested in building-with actions and in constructing actions. Moreover, constructing actions may themselves be nested in further constructing actions.

On the basis of observations reported below and elsewhere, we postulated that the genesis of an abstraction passes through (a) a need for a new structure; (b) the construction of a new abstract entity; (c) the consolidation of the abstract entity through repeated recognition of the new structure, building-with it in further activities with increasing propensity, and using it in further constructions.

Stage (c), the *consolidation* of the newly knowledge structure, seems *a priori* to be linked to the following behaviors: (i) the reconstruction of the new structure or its actualization by recognizing it in different contexts, (ii) its use with increasing facility for building-with in different contexts, (iii) its use in the construction of further structures for which it is a necessary prerequisite, (iv) its verbal articulation, possibly during or after an activity of reflection such as reporting or summary discussion in class. Thus the term *consolidation* denotes a progressive familiarization and further use observable through recognizing and building-with actions in these four types of situations.

The validation of the theory through empirical research

We characterized abstraction as a process taking place in a complex context that incorporates tasks, tools and other artifacts, historical background of the participants, as well as the social and physical setting. Abstraction processes are then context dependent. However, we claim that the ways in which these processes are taking place and become operational have a universal structure. This structure was elaborated and partially confirmed in Hershkowitz, Schwarz and Dreyfus (HSD, 2001). Further studies were partially designed for confirmation of the model and partially designed for extending it.

In HSD we showed that the dynamically nested RBC model fits the genesis of abstract scientific concepts acquired in activities designed for the purpose of learning. A first validation of stage (a) and (b) of the genesis of abstraction according to the model was obtained in a case study with a single ninth grade student who was interviewed while solving a problem, a suitable computer program being at her disposal.

We showed that the model describes the mechanism of processes of abstraction. As such it contains the main invariant features of abstracting as a thinking process. Moreover, the model is apt to take context into account.

HSD also revealed a methodological problem: the occurrence of processes of abstraction cannot be ensured; rather, students can only be presented with opportunities for abstraction. The creation of such opportunities presents a challenging design problem since it depends on the contextual factors mentioned above. We tried to elicit strong motives such as the need to justify a just discovered claim, the need for solving a problem as well as conflict situations in order to augment the opportunities for abstraction. This corresponds to stage (a) of the genesis of abstraction according to the model.

Dreyfus, Hershkowitz and Schwarz (2001a; 2001b), tested stages (a) and (b) of the model in a richer context, in which two peers interacted to construct new knowledge. The study focused on the social dimension of the process of abstraction. Two parallel analyses were carried out on the same protocols: the analysis of epistemic actions according to the model of abstraction as well as an analysis of the interaction. The study showed far-reaching parallels between the two analyses. In other words, we enhanced the RBC model of abstraction so as to describe processes of abstraction by interacting pairs of students and patterns of distribution of abstraction between collaborating peers. The parallel analyses led to the identification of types of social interaction that support processes of abstraction.

While research concerning stages (a) and (b) can be done within one activity, investigation of consolidation processes requires at least a medium term research, where one can analyze processes occur among successive activities. Such an analysis demands the elaboration of powerful methodologies with the help of which individual history of individuals evolving in changing learning environments should be traced. A small number of studies in this direction have already been undertaken (Dreyfus & Tsamir, 2001; Tabach, Hershkowitz & Schwarz, 2001; Tabach & Hershkowitz, 2002). A first attempt at an empirically based theory for consolidation emerged from a sequence of interviews about the comparison of infinite sets with a single talented student. It showed that consolidation may occur both as a result of problem solving activities and as a result of reflective activities, and that it can be identified by means of the psychological and cognitive characteristics of immediacy, self-evidence, confidence, flexibility and awareness.

The significance of our theory of abstraction concerns theoretical, psychological and educational issues: Since our research is empirical, it has the potential to yield insights on processes of abstraction and consolidation as they develop, and to confront these empirical insights with the theoretical constructs of the model. Abstraction and consolidation as the central components of construction of knowledge are investigated in relation to the history of the participants, and to series of activities in a social context. On the basis of the theory, we also expect to articulate educational design principles for sequences of activities that are intended to lead to abstractions and their consolidation.

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BUILDING NEW MATHEMATICAL REALITY, OR HOW EMERGENT MODELING MAY FOSTER ABSTRACTION

Koeno Gravemeijer

Abstract

In general, mathematics is thought of as abstract, formal knowledge. Within this view, the key problem for mathematics educators is to shape mathematics instruction that helps students in bridging the gap between informal, situated knowledge at one hand, and abstract, formal mathematical knowledge at the other hand. A rather common view is that students have to abstract from their informal knowledge; they have to decontextualize, or to cut the bonds with reality. In this paper, an alternative view is presented that does not take its point of departure in the metaphor of a gap between abstract, formal knowledge and informal knowledge, but in an emergent approach, within which formal mathematics grows out of the mathematical activity of the students. The latter view is part of work in the area of instructional design.

Abstraction as the Creation of New Mathematical Reality

The deliberations on the issue of abstraction that will be presented here grew out of an effort to further explicate and elaborate the domain-specific instruction theory for realistic mathematics education (RME) (Treffers, 1987). As part of this effort, this domain-specific instruction theory has been recast in terms of instructional design heuristics (Gravemeijer, 1994). The elaboration of one of those design heuristics—concerning emergent models, or emergent modeling—created the need to further investigate the underlying, implicit, notions of abstraction (Gravemeijer, 1999). The emergent-modeling heuristic assigns a role to models that differs from the classical role of models in mathematics education: instead of trying to concretize abstract mathematical knowledge, we try to help students model their own informal mathematical activity. In doing so, we attempt to foster a process, within which a *model of* their own informal mathematical activity gradually develops into a *model for* more formal mathematical reasoning for them. In contrast with the gap metaphor, formal mathematics is not seen as something separate, existing independent of a knowing agent. Instead, formal mathematics is seen as emerging alongside with the model-of /model-for transition.

When speaking of formal mathematics, we hasten to say that in RME, formal mathematics is not seen as something "out there". Instead, formal mathematics is seen as something that grows out of the students' activity. For us, the notion of "abstraction" is tied to a progression from informal to more formal mathematical reasoning, which in turn is tied to the creation of new mathematical reality. So instead of "cutting bonds with (everyday-life) reality", we want to stress "construction". Informal, situated knowledge is the basis upon which more formal, abstract mathematical knowledge is build.

Our claim is that the emergent-modeling design heuristic helps instructional designers in developing topic-specific instruction theories and corresponding instructional activities that support learning processes in which students construe new mathematical reality. In order to clarify the emergent modeling heuristic, we will briefly describe an exemplary instructional sequence.

This exemplary sequence, which concerns linear measurement and flexible arithmetic, was developed in connection with a teaching experiment carried out at Vanderbilt University (Cobb, Stephan, McClain, and Gravemeijer, in press; Stephan, 1998). The underlying idea is that measuring by iterating measurement units can give rise to the construal of a ruler and that the ruler can subsequently support arithmetical reasoning about problems concerning incrementing, decrementing and comparing measures.

After a series of preparatory activities, the students start measuring with stacks of ten unifix cubes. They first iterate units of ten, then adjust by adding or subtracting ones. In this manner, measuring with tens and ones helps the students in structuring the number sequence up to 100. Next, the students create their own paper strip that is ten unifix cubes long. With that, a basis is being laid for the construction of a measurement strip that comprises ten units of ten; each subdivided into ten units of one cube. The idea is that, thanks to the history, measuring with the measurement strip is grounded in the imagery of measuring with units of ten and one. Thus, for the students, measuring with the strip signifies iterating a unit of ten cubes and a unit of one cube. Next, a shift is made from actually measuring items to reasoning about lengths when solving tasks around incrementing, decrementing and comparing lengths of objects that are not physically present (i.e. comparing the measures of the heights of sunflowers in the context of a sunflower contest). These tasks offer opportunities for developing solution methods based on curtailed counting—using the decimal structure as a framework of reference. Numbers close to a decuple, for instance, can be identified by using that decuple as a referent, e.g. $64 = 60 + 4$; $40 = 35+5$. These relations can be exploited when analyzing patterns that correspond with jumps of 10. An empty number line is introduced as a means for symbolizing measurement strip-specific, arithmetical solution methods that are grounded in reasoning with "tens & ones" (see fig. 1). A jump on the number line describes a move on the measurement strip that in turn can be seen as corresponding with iterating unifix cubes or smurf bars.

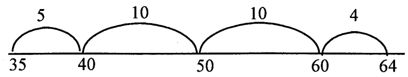

Figure 1. 35-29 on the empty number line.

Finally a generalization is made from magnitudes to numerical quantities in general; the students are asked to solve various addition and subtraction context problems, while using the empty number line as a means to record and support their thinking.

Emergent Models

We will use this example to explicate the emergent modeling heuristic. We may start by noting that the label "model" is used in a metaphorical sense. There is an overarching model that takes on various manifestations. We may characterize the series of symbolizations within which the model manifests itself as a chain-of-signification (Stephan, 1998). In the exemplary sequence the ruler is conceived as the overarching model. The idea is that the ruler emerges as a *model of* iterating a measurement unit (or measurement units). In this sense, the ruler is grounded in the activity of measuring. Gradually, however, the ruler changes character, as the attention shifts from measuring to reasoning about the results of measuring. Finally a schematized ruler becomes a *model for* reasoning about arithmetical relations between numbers up to one hundred.

Key for us is that the shift towards more formal mathematical reasoning is connected with the creation of a new mathematical reality. In the example sequence, we may conceive this new reality as constituted by numbers up to 100 as entities in a framework of number relations. What is expected is, that in the course of the sequence, a shift is taking place in which the student's view of numbers transitions from referents of distances to numbers as mathematical entities. This shift involves a transition from viewing numbers as tied to identifiable objects or units (i. e. numbers as constituents of magnitudes; "37 feet") to viewing numbers as entities on their own ("37"). For the student, a number viewed as a mathematical entity still has quantitative meaning, but this meaning is no longer dependent upon its connection with identifiable distances, or with specified countable objects. In the student's experienced world, numbers viewed as mathematical entities derive their meaning from their place in a network of number relations (see also Van Hiele, 1973). Such a network may include relations such as 37=30+7, 37=3x10+7, 37=20+17, 37=40-3. The critical aspect of this network is that the students' understanding of these relations transcends individual cases. That is, when students form notions of mathematical entities, or mathematical objects, they come to view relations like the above as holding for *any* quantity of 37 objects (including a magnitude of 37 units). We would denote this conception of numbers as mathematical objects that derive their meaning from a framework of number relations as new mathematical reality.

As an aside, we want to remark that we prefer to limit the use of the model-of/model-for terminology to those more encompassing shifts where one can speak of the creation of new mathematical reality. We may further note that this creation of new reality is reflexively related to the model *of* to model *for* transition. On the one hand, the students' actions with "the model" foster the constitution of new mathematical reality (in our example, a framework of number relations). On the other hand, through the students' development of this new mathematical reality, "the model" *can* take its role as a model for mathematical reasoning.

Aims and Applications

The emergent modeling heuristic may guide instructional designers by asking them to think through the endpoints of a given instructional sequence in terms of new mathematical reality; to describe what mathematical objects the students are expected to construe, and how these relate to some framework of mathematical relations. They are further advised to think through the model-of/model-for transition, which for instance means, to indicate what informal situated activity is being modeled, and what a potential chain-of-signification might look like. In connection with the above, the heuristic suggests points of attention for the enactment of the instructional sequence. It highlights that formalizing is not equal to, and cannot be forced by, the use of formal notations. Instead formalizing grows out of a shift of attention towards mathematical relations. The aforementioned considerations will indicate what those relations are, what the mathematical issues are that are to become topics of discussion, and what role the various tools/symbolizations may play.

The emergent modeling heuristic implicitly or explicitly plays a role in various RME designs (e.g. Streefland, 1990). The role of emergent models is older that the explicit characterization presented here. However, more recently this heuristic has explicitly guided design and analysis in a number of developmental research (or design research) projects. In this respect, we can claim that this heuristic is validated in a number of teaching experiments. Next to experiments at the primary-school level, like the aforementioned numberline experiment, we want to mention research in *data analysis* (Cobb, in press), and research on *differential equations* (Rasmussen, 1999).

Acknowledgement

The analysis reported in this article was supported by National Science Foundation (NSF) Grant REC 935387 and by Office of Educational Research and Improvement (OERI) Grant R305A60007. The opinions expressed in this article do not necessarily reflect the view of either the NSF or the OERI.

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REACTION

Anna Sierpinska

This "reaction" will be about my rather unsuccessful attempts at understanding the three proposed theories. Words were familiar, all right. But they were put together in strange juxtapositions and concatenations. I could *recognize* the words or parts of them - activity, perception, abstraction, procedure, process, concept, proCEPT, PROcept, object, structure...- but I couldn't *build-with* them, never mind *construct* anything remotely resembling a *structure* with them.

I was taught in school that "meaning belongs first of all to the world of objective-historical phenomena" (Leont'ev, 1959, p. 223), and that the content of an individual mind is a result of "an assimilation of the experience of the previous generations of people" (ibid.), but here I was, confronted with the meanders of individual consciousnesses of several "Robinson[s], making [their] own independent discoveries on a desert island" (Leont'ev, 1959).

Take, for example, ABSTRACTION. I am used to thinking of abstraction as a *dual* mental activity whereby some aspects of the object of thought are ignored while other are highlighted. For example, if the object of my thought are integers and I decide to ignore multiples of 2, then all that is highlighted are the remainders, 0 or 1, and I end up with the even/odd distinction (or the *concept* of even and odd numbers, if you will). If I now highlight the mental process which led me to the construction of the even/odd numbers construction, and disregard the fact that I was ignoring multiples of 2, and decide to now remove from the field of my attention multiples of 3, 4, or any number n, for that matter, then I end up with the concepts of Euclidean division, and congruence modulo n. I can further ignore the specific nature of particular integers, look at the whole arithmetic of integers from afar, highlight only its ring structure and ask myself if I could not do something similar with other rings as well. I may fancy taking R[x] and decide to ignore multiples of $x^2 + 1$. Then what is highlighted forms a structure strikingly similar to the field of complex numbers. This chain of ignoring and highlighting is usually called generalization: a process of abstractions which starts from some object of thought O_1 and arrives at an object of thought O_2 such that O_1 is a special case of O_2. If abstraction is understood this way, as an act of ignoring/highlighting, then it appears as an "elementary particle" in the process of mathematical thinking. One would hardly want to call the whole process of theory building in mathematics "abstraction". Even the processes of single concept construction involve more than a few acts of abstraction. This concept of abstraction is too elementary to capture what happens in processes of mathematical thinking. It is also not specific to mathematics nor any scientific knowledge for that matter. Abstraction is an elementary operation in any kind of thinking. For example, we engage in abstraction when we move from saying that our neighbors seem to be a happy couple to thinking about happiness in general. Therefore, in speaking about mathematical thinking, we need more specific concepts such as generalization and concretization, formalization and de-formalization, algebraization and geometrization, axiomatization and modeling, etc.

The above socio-cultural notion of abstraction appeared not to satisfy the authors of the RBC theory presented in this forum. SHD (here and in the sequel, G, GT, and SHD will indicate the three groups of authors, according to the initials of the family names of their authors) were inspired by Davydov's definition of abstraction, which they interpreted as, "abstraction starts from an initial, undeveloped form of knowledge and ends with a consistent and elaborate form. It proceeds from the idealization of the basic aspect of practical activity involving objects to cognitive experimentation characterized by the fact that one (a) mentally transforms objects during the activity and (b) forms a system of connections between these objects". I didn't quite see how this description would exclude mental activities such as fantasizing. Imagine a poor man sitting there in his boat with a fishing rod and idealizing the basic aspect of his activity as dreaming about a big catch. Cognitively experimenting with his vision of a big catch, and dreaming about how this would impress his wife upon his return home, his vision was suddenly transformed into a half-fish/half woman. This way, the notion of mermaid has been proved to be a socio-cultural consequence of the division of labor in the poor man's household, and fantasizing – a special case of abstraction.

SHD ' own definition of abstraction was as follows: "Abstraction is an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure". Now, this was very confusing because the statement was at the same time very restrictive (restricted to mathematical abstraction) and very general. It was general because it seemed to identify all construction of mathematical knowledge with mathematical abstraction. It was confusing also because it was circular. The circularity was, in particular, in the definition of the genesis of abstraction as "passing through (a) a need for new structure; (b) the construction of a new *abstract* entity; (c) the consolidation of the *abstract* entity…". Thus the product of abstraction should be an "abstract entity", but I was not informed what the authors understood by "abstract entity".

While SHD stressed that, for them, abstraction was a mental *activity*, GT assumed that it is was a *product* of mental activity, "in which a complex parallel processing organ solves the problem of complexity by focusing on essential structures that enable decisions to be made". GT's notion of abstraction had the properties of the notion I was used to, namely those of ignoring some aspects while focusing on some other aspects, and I felt quite comfortable with it. I was less at ease with the rest of the theory and especially with the interpretations of the students' mathematical behavior that this theory afforded. The focus was entirely on the biology of cognition, in abstraction from the social and institutional situation, in which the learning of mathematics normally takes place. For GT, "essential structures" of the problems they presented to the students were always certain mathematical structures, those structures they were themselves most familiar with. In the experiment with children being shown five red cubes and asked what would be worth remembering about them, children who chose to remember that there were 5 cubes happened to belong to the group of high achievers; those who chose to remember the color, the pattern or configuration were from the low achievers group. But who decided that remembering the number of the cubes was the right thing to do? Isn't this a matter of didactic contract? In a mathematics class, numbers are important. In communication or arts class, color and arrangement could have tremendous importance. Aren't some children failing because they have not figured out what is the didactic contract in each particular class? Because they have not figured out how school works and what one is rewarded for? It is extremely dangerous to explain success and failure in mathematics at school by cognitive factors alone. We must take the didactic system as a whole and the student as a "perfinking" person (perceiving, feeling and thinking, David Krech cited in Bruner, 1987) in it as a whole. "Success in mathematics" is an institutional measure, not a measure of cognitive progress or capacity.

The examples of the contrasting behavior between high and low achievers in the area of algebra again suggest that low achievers are those who are bad at noticing what are the rules of the game; in this case – what are the formal conventions of writing algebraic expressions, and when two functions are to be considered the same. They are strangers in the school mathematics culture. They would rather use a different syntax to express things, and how you get a result is important for them. How you get a result is important in programming computers; a(u + v) and au + av, where u and v are vectors and a is a scalar are different functions in a CAS. The first involves n additions and n multiplications; the second - 2n multiplications and n additions. How do you know that in this particular algebra class this does not matter? How do you know what is important and what is not in a particular culture? This is not mathematics thinking; this is socio-cultural thinking and some people are better than others in assimilating into a foreign culture. Some keep their *terrrribel forrreyn akzent* for ever.

G, representing RME (realistic mathematics education) defines abstraction as an activity, which is comprised in the processes of mathematizing and seen as a two-way process: from less formal to more formal and the other way round. As in GT, abstraction is not an important concept in RME. More important is the assumption that, for the purposes of mathematics education, mathematics should be seen as a human activity and not as a library of accomplished and polished theories. RME is a project of curriculum development, not a theory, although the developers have started formulating their epistemological assumptions and principles in view of building techniques (heuristics), technologies and theories of mathematical instruction. SHD claim that their theory is also an outcome of curriculum development activities but their focus in the papers is on a theory of learning. RME researchers focus on the design of tasks embedded in long term curriculum activities. They describe the activities and try to justify the design. Little is said about the classroom experimentations and the notion of "success" of an experiment is not defined.

What are the criteria of success? What have been the proofs of success? Results on TIMSS? G claims that in RME the developmental research is "*evolutionary* in the sense that theory development is gradual, iterative and cumulative. There is no theory with which to start. The initial, global theory is elaborated, refined and explicated during the process of designing and testing" (G, 1998, p. 282). This suggests the hope that the law of the "survival of the fittest" will guarantee progress in the long run. However, the law of the survival of the fittest does not imply that the best curricula and best conditions for learning mathematics are eventually going to be achieved. What is "the fittest" is often governed by the law of least resistance or the tendency of the educational system to short-circuit all scholarly activities that are costly in managerial effort and time and are not directly related to the preparation of students for passing the final examinations.

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ABSTRACTION: WHAT THEORY DO WE NEED IN MATHEMATICS EDUCATION?

Paolo Boero

I will divide my reaction into two parts. In the first part I will follow the grid *("what is the theory about",* etc.). In the second part I will discuss the need for a theory of abstraction in mathematics education, and the requirements that, in my opinion, such a theory should meet, then I will reconsider the three theories from this personal point of view. In the sequel, G, GT, SHD will indicate the three theories (according to the initials of the family names of their authors).

According to the Grid...

Most of suggested reading criteria (*What is the theory about? What assumptions are being made? What does the theory claim? What terms are used and what do they mean? How has the theory been validated?*) are related to minimal, necessary requirements that each theory (as a "scientific theory") must meet in human sciences (psychology, anthropology, sociology, etc.). The second part of the last reading criterion (*What are the aims of the theory and what are its applications?*) refers to a specific challenge for theorists in mathematics education.

More or less explicitly each presented theory satisfies the first and the third requirement. Here it seems to me that theories G and SHD deal with subjects that are rather close to each other (a common title might be "abstraction in context", yet with a different meaning of the word "context" – see later), while in the case of GT the theory deals with mathematical content and related individual learning processes.

Different ways of satisfying the second requirement are followed within the above presentations of the three theories. In the case of GT, "assumptions" are intended as general assumptions derived from other theories in order to create an environment where the theory can develop and be better understood; in the case of G and SHD, general assumptions are internal to the theory and constitutive of the core of the theory itself. We can observe how G aims at self-sufficiency in the presentation of the theory, while SHD refers to existing general theories (especially "activity theory"). The second criterion poses some problems: in the case of G, what meanings may be attached to crucial terms in a self-sufficiency perspective? (See later). In the case of GT and SHD, what relationships to establish with related theories?

Concerning the latter problem, we can identify different attitudes in mathematics education research as well as in the GT and SHD presentations. In the GT case, an autonomous elaboration (the procept theory) is linked to existing general theories in the field of psychology (Piaget's *"Pseudo-empirical and reflective abstraction in arithmetic, algebra and calculus naturally focus on our notion of procept."*) and used to reinterpret some other theories in the field of mathematics education: *"We do not agree with Sfard or Dubinsky that the development invariably proceeds in a sequence we describe as procedure-process-procept"*. In my opinion the legitimacy of these links and re-interpretations should be carefully discussed. As concerns the specific section "*What assumptions are being made?*" in the GT presentation, in my opinion the need for this kind of discussion becomes ever stronger: for instance, beyond heuristic hints, what are the precise relationships between the "*semantic squeeze*" in Bruner's quotation, the *"reduction in brain area involved"*, considered in neurophysiology studies, and the construction and functioning of procepts? In the SHD case, activity theory is taken as a fundamental reference. In my opinion, the adoption of an activity theory reference paradigm needs to consider the teachers' role as constitutive of the "learning" process (in our case, of the "abstraction" process). Indeed in Vygotsky's seminal work it is well known that a crucial, recurring term is "*obucenie"*, that means *"teaching and learning"*. Yet I see (in the article as well as in the SHD forum presentation) that this "joint activity" aspect is not sufficiently developed.

Let us come now to the most critical criterion: *What terms are used and what do they mean?* It is clear that the danger for a person like me, who was educated as a mathematician, did research in mathematics for some years and still teaches mathematics at the University level, is to apply such criterion in the strict way he generally uses when dealing with his students' mathematical performance. It is true that in the human sciences domain it is very difficult to give "definitions" in the same, strict sense. In most cases, definitions are reduced to some evocative words that suggest a meaning, and then the context provides the full meaning. But I think that within the same theoretical construction (a theory), or the presentation of a theory, a crucial term must have a rather precise meaning (in order to establish whether or not an object or a situation falls within its semantic domain) and keep it. From this point of view, I find that in G and SHD the meaning of some crucial terms is not sufficiently clear, while the meaning of other terms seems to change during the presentation of the theory. In particular, I refer to the following terms:

"Formal mathematics" (in G): "formal" according to a high level of formalisation? And/or according to a social (or academic) consensus about ways of presenting relevant concepts, validating statements, etc.?

"Mathematical reality" (in G): what is its psychological and epistemological status? A subjective construction (or re-construction)? A historically shared and inheritable production, rooted in mankind's needs and experiences? A set of shared conventions?

"Structure" (in SHD): one part of the axiomatic organisation of mathematical knowledge (e.g. "the structure of group")? And/or the overall organisation of mathematical knowledge? And/or the organisation of mathematical thought?

"Context" (in SHD): in mathematics education, like in psycholinguistics, the word "context" takes different meanings:

- that of "situation context": those factors affecting the mathematical performance that are related to the situatedness of the students' activity (including social relationships in the classroom, environmental factors, etc.);
- that of "task context": the task evokes specific "realities" and constraints; as a consequence, behaviours, schemes, etc. related to those "realities" are activated;
- that of "inner context": in this case the attention is focused on the (internal) representation of the subject's past and present experience.

These different meanings of the word "context" suggest different perspectives under which teaching and learning mathematics in the classroom can be considered. For instance, in the case of abstraction the second perspective suggests to choose peculiar tasks suitable for it, while the first perspective suggests to take into account the social interactions that the teacher must "orchestrate" in the classroom.

Concerning the *aims* and the *applications* of the three theories, they are very different. Here again there are strong analogies between G and SHD (the theories are intended to provide useful tools to plan and/or improve teaching projects, and better interpret what happens in the classrooms where planned teaching is implemented). In the case of GT, the focus is on interpretative aims and in particular on explaining *"why some students are so highly successful with symbols, whilst others are procedural at best"* (etc.). In my opinion, in mathematics education we need both types of theories, bearing in mind that a theory of the second type can develop in (or support) a theory of the first type, and that a theory of the first type can provide interesting research questions for theories of the second type.

Concerning *validation of theories*, it seems to me that (in relationship with their specific aims) each theory meets this requirement. However I must say that it is met not so much in the above papers as in the articles included in the references: this is an unavoidable, necessary consequence of the space limitations of presentations.

Do we Need a General Theory of Abstraction in Mathematics Education? What Kind of Theory?

Let us consider the following examples:

- a right-angled triangle is drawn on the blackboard; students draw right-angled triangles on their copybooks; the teacher illustrates Euclid's theorem;
- the teacher writes on the blackboard: $(uv)'=u'v+uv'$, then $\int xsinxdx=$, then illustrates and justifies the well known method of integration of the $xsinx$ function based on the law of derivation of products of functions;

- the teacher draws a square on the blackboard, then one diagonal, then he proves (by the usual "reductio ad absurdum" proof) that the diagonal and the side of a square are incommensurable: .... $d^2=2s^2$, $(d/s)^2=2$, etc.
- the teacher establishes a 1-1 correspondence between the set of even numbers and the set of all natural numbers, then defines "infinite sets" as those sets which are equivalent to a proper subset.

In each of these cases mathematicians recognise some specific aspects of "abstraction". Mathematics educators have tried to deal with these aspects in different ways. For instance, C. Laborde and B. Capponi define a (geometric) figure as the set of couples (O,d_i), where O is the geometric object (e.g. the right-angled triangle) and d_i is one of the drawings that constitute the 'material representation' of the geometric object. Therefore the figure *"is the product of the abstraction process performed by the subject when, starting from a drawing (signifier) he or she thinks about the represented geometric object".* This definition of figure is useful to deal with some difficulties that students meet when they approach geometrical reasoning (for instance, the reference to the peculiarities of a drawn right-angled triangle, or the stereotyped representation of the height of a triangle). These considerations are specific to the kind of abstraction inherent in the first situation. For the second situation an entirely different theoretical approach to abstraction is needed. Indeed, the 'material representation' is related to the represented mathematical object in a completely different manner: on one side the 'material representation' is much more distant from the mathematical object, on the other it becomes the starting point for a chain of transformations performed on the written expressions according to general syntactic rules. The third example shows some partial similarities both to the first example and to the second. Let us consider the fourth example: the idea of "epistemological obstacle" was elaborated in order to cope with students' difficulties inherent in "accepting" some cultural, "abstract" constructions like the "equivalence" between a set and a proper subset, which contradict our usual experiences about the sets of objects that we can describe extensively (i.e. by listing all their elements).

I add, as explicitly quoted in GT and in SHD, that general theories of abstraction already do exist in psychology.

A recurrent question for me, when reading the contributions for this panel, was: do we need a **general** theory of **mathematical** abstraction in **mathematics education**, i.e. a general theory suitable for describing and interpreting many typical phenomena of "abstraction" that intervene in teaching and learning mathematics and, possibly, controlling them (i.e. planning teaching in order to get the best results) by selecting pertinent variables and coming to predict the effects of actions on them?

In my opinion, a general theory of mathematical abstraction that would be of interest for mathematics education purposes should:

- cover most forms of abstraction currently met in teaching and learning mathematics at different school levels;

- interpret difficulties met by students in tackling abstraction in their approach to mathematical knowledge;
- point out relevant variables accessible to educational intervention;
- take into account relevant research in the field of epistemology of mathematics (as concerns reflection on abstraction) and cognitive sciences (as concerns general theories about abstraction, or specific theories about mathematical abstraction).

According to the first three requirements that I propose, each of the three theories offers some relevant contributions but also shows important weaknesses. G shows (through the example sketched in the PME contribution and other evoked examples) how some processes of abstraction can work, helping both the designers of teaching projects and teachers to plan and manage suitable classroom situations. But it seems to me that G does not cover processes of abstraction that are needed when a "break" is unavoidable in the transition from mathematical experience in real contexts to "formal mathematics". Moreover, the role of mediation played by the teacher, which is particularly crucial in this case, is not explicitly dealt with. GT is suitable to cover many "abstract" mathematical objects, but it seems to me that in my first example it does not provide much help in dealing with students' difficulties in managing the "abstract" notion of right-angled triangle (e.g. in the case of stereotyped images). And (being mainly a theory about mathematical objects in their relationships with generating processes) I see it might have some difficulties in dealing with the "abstraction" inherent in the activities (e.g. students' mathematical argumentation). SHD is a very general theory and it surely covers abstraction in a wide sense, but its current generality implies that some peculiarities are lost in specific cases. Perhaps in the future this theory will become suitable to deal in a productive way with all the examples provided at the beginning of this Section, but then the subject's individual processes should be better investigated in relation with the "situation context", the "task context" and the teacher's mediational strategies.

As concerns the fourth requirement that I propose, it seems to me that the three theories do not take sufficiently into account important, new streams of research in epistemology of mathematics, psychology and neurophysiology that are developing in different research communities. Recent joint studies in the fields of epistemology and neurophysiology (for an example the project "Geometry and cognition" at the ENS, in Paris) show the possibility of a convergence on the idea that even at the highest level of "abstraction" productive reasoning relies upon very "concrete", body related intuitions. This approach puts again into question, but within a new perspective (neurophysiology investigation), the idea of a purely conventional character of axioms and axiomatic theories; the anti-logicist positions have opposed this idea during the whole XX century under different perspectives (mainly philosophical or ideological or based on introspection). This approach also draws an almost entirely new picture of how high level professional mathematical activities are performed. It seems very interesting and promising that this stream of research goes in the same directions of some other streams of research in different disciplines (in particular, "embodied cognition", in psycholinguistics and psychology).

If it is true that "thinking abstract objects as if they were concrete" is possible, but "thinking in an abstract way" is impossible (as far as productive thinking is considered), then a theory of abstraction suitable for educational purposes should take charge of the whole complexity of the relationships between mathematical objects, the thinking processes concerning these objects, their "situatedness" in the classroom environment (the mediational role of the teacher being a crucial issue), and the most suitable "task contexts" for meaningful mathematical abstraction (both as concerns the mathematical content involved and the body-rooted processes). From this wide and very demanding point of view I think that the presented theories offer some important contributions, but we are still far from a comprehensive theoretical answer to the challenge of mathematical abstraction in mathematics education.