RESEARCH FORUM 2

Theme
The nature of mathematics as viewed from mathematics education research

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THE NATURE OF MATHEMATICS AS VIEWED FROM MATHEMATICS EDUCATION RESEARCH

Introduction

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How we conceive of mathematics has a major bearing on our educational efforts. The nature of the mathematical ideas we consider essential for success in the new century, and the ways in which these ideas are conveyed during the school years, can facilitate or impede students' lifelong mathematical understanding, learning, and communicating. Though we have made significant advances in mathematics education research, fundamental issues of intellectual importance having political, social, and economic ramifications, continue to be debated. These issues include: (a) What counts as mathematics? (b) What is the nature of mathematical ideas? (c) What is the relative importance of these ideas to society? (d) What is the nature of the various representations these mathematical ideas may take (both internal and external to the student)? (e) What are the processes by which these ideas are understood by students? (f) How might we maximize students' understanding of these mathematical ideas?

Among the viewpoints are those that consider mathematics and the direction of its growth to be shaped by a complex system of cultural, social, and political forces (e.g., D'Ambrosio, 1999; Skovsmose & Valero, 2002). Lerman (2000) refers to the "social turn" in mathematics education, which came into being towards the end of the 1980s. The social turn saw the emergence of theories that view mathematics and mathematical meaning and reasoning as products of social activity. Social constructivists, for example, emphasize the processes through which consensus develops in determining the nature of mathematical knowledge and how it is constructed.

Complementing the sociocultural perspectives are those that draw on advances in cognitive science to explain aspects of the structure of mathematics and its development. Included here are analyses of external mathematical representational systems (e.g., mathematical symbolism, computer microworlds, structured analogues, diagrammatic structures) and internal systems (e.g., verbal/syntactic systems, imagistic systems, conceptual metaphors, mental models). The nature of the interactions within and between these representational systems is considered to play a powerful role in learners' mathematical growth.

Theories that lie within the broad sociocultural framework, along with the more cognitively oriented theories, are contributing to current debates about what mathematics students should learn, how they should learn it, and the extent to which school mathematics curricula should capture the essence of workplace mathematics (e.g., see Stevens, 2000).

A comparatively recent and controversial cognitive perspective on the nature of mathematics is that of "mind-based mathematics" (Lakoff & Nunez, 2000). Here mathematics is not inherent in the universe, nor is it merely a cultural artifact; rather it is shaped essentially by the nature and structure of human brains and minds via "conceptual metaphors."

There are varying perspectives among both mathematicians and mathematics educators on the aforementioned issues. It is important for educators to consider mathematicians' points of view, especially in light of the current curriculum debates highlighted by the media in several nations. Will mathematics educators and mathematicians find intellectually sound ways of connecting differing perspectives, or will existing gaps widen further?

The papers in this Forum provide critical debate on the issues we have addressed. Brian Greer argues that mathematics, mathematics education, and mathematics education research are situated in "sociohistory, culture, and politics." On a somewhat different note, Laurie Edwards presents a "personal journey" on the nature of mathematics, where she illustrates the perspective that mathematical ideas are shaped in fundamental ways by our embodied experience in the world. The importance of communication is emphasised in Anna Sfard's paper, where mathematics is seen as a discursive activity, that is, "a special way of communicating." Within her communicational framework, thinking is regarded as a special case of communicative activity. Another interesting perspective is presented by Shlomo Vinner who addresses boundaries, identities, and mathematical objects in his discussion of mathematics and mathematics education. Hartwig Meissner, on the other hand, explores the distinctions among "Einstellung" (attitude), "Vorstellung" (internal image) and "Darstellung" (external representation) in addressing mathematics and the processes of mathematics learning. Gerald Goldin extends Meissner's ideas in his discussion of representational systems, and provides critical thought on the sources of the "widening chasm" between mathematics and mathematics education.

Complexity of Mathematics in the Real World

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Perception of the relationships among mathematics, mathematics education, and mathematics education research used to be simple.

Mathematics was seen as a relatively well-defined, hierarchically structured, body of knowledge. Mathematics education meant transmitting this body of knowledge to each student up to an appropriate level in the hierarchy. Psychological research was expected to provide general theories of cognitive development and learning, with the assumption that these theories could be applied to the learning of mathematics as a domain and the improvement of (mathematics) education through generating hypotheses testable via standard experimental designs. Many mathematicians and psychologists taking a more or less informed interest in mathematics education feel comfortable with this simplicity (for example, in the context of the Californian Math Wars, see the analysis of the unholy alliance between psychologists and mathematicians by Jacob and Akers (1999)).

However, the situation has become more complicated.

First, mathematics continues to grow fast and computers have changed both its content and its methods. Consequently, questions of selection and arrangement arise – what parts of mathematics should be chosen and how should they be reorganized for education? Typically, curricula are largely the result of tradition and inertia and, insofar as growth occurs, it is mainly through accretion without radical restructuring. There is very little by way of principled design – consider the limited adaptation to the new representational systems afforded by computers, for example.

Second, the first wave of the cognitive revolution generated disequilibrium when it became clear that there was "de-emphasis on affect, context, culture and history" (Gardner, 1985, p. 41). The outcome was the "second wave" (De Corte, Greer, & Verschaffel, 1996, p. 497) which mathematics education research both contributed to, and was influenced by, in major ways. Methodologies became interpretative rather than scientific, with results that are liberating or anarchical, depending on your point of view. The work of some researchers now exemplifies Engestrom's proposed methodology for activity theory that puts it to "the acid test of practical validity and relevance in interventions that aim at the construction of new models of activity jointly with the local participants" (Engestrom, 1999, p. 35).

Inevitably, mathematics education researchers' views of mathematics have been complicated by their immersion in activity systems, including exposure to the culture of the classroom, the nature of schooling, and the politics of mathematics education. Mathematics, mathematics education, mathematics education research are all situated in sociohistory, culture, and politics.

To illustrate the foregoing comments, I offer sketchy outlines of three key characteristics of mathematics (revealing my own biases, naturally) and how they play out in mathematics education and mathematics education research.

The Two Faces of Mathematics

> On the one hand, mathematics is rooted in the perception and description of the ordering of events in time and the arrangement of objects in space, and so on ("common sense -- only better organized", as Freudenthal (1991, p. 9) put it), and in the solution of practical problems. On the other hand, out of this activity emerge symbolically represented structures that can become objects of reflection and elaboration, independently of their real-world roots. In the process, common sense is soon transcended, yet, time and again, the results of such elaborations have proved (often after a considerable lag in time) useful in theoretical descriptions of real-world phenomena and solution of real-world problems. (De Corte, Greer, & Verschaffel, 1996, p. 500).

The link between the two faces of mathematics is the activity of modeling. Typically, the modeling of a real-world situation leads to a range of solutions that need to be judged in terms of human criteria such as utility, purpose, and complexity. Introducing pupils early to this perspective may be considered part of the process of enculturation into the practices of mathematicians, yet until relatively recently, it has not received much attention (Niss, 2001; Verschaffel, 2002).

The Developmental Nature of Mathematics

"Mathematics grows ... by its self-organizing momentum" (Freudenthal, 1991, p. 15). In the course of the sociohistorical construction of mathematics, several developmental mechanisms may be identified:

(a) The disequilibrium that comes from lack of closure. The obvious example is the extension of the concept of number from its origins in natural numbers. (It seems to me that there is a clear parallel with Piagetian theory but I am not aware of anyone who has explored this idea in depth).

(b) Metaphorical extension, which has been elaborated in the recent book by Lakoff and Nunez(2000) (and see Edwards). Why are all those different things all called "numbers"? (Poincare defined mathematics as the art of giving the same name to different things).

(c) Variations on the theme of reification (e.g. Sfard, 1991, and see Vinner).

(d) Mediation by cognitive tools, as illuminated by the Vygotskian tradition – language (see Sfard), symbols, representational systems (see Goldin, Meissner).

(e) Systematization, including the development of axiom systems. The history of attempts to teach mathematics on this basis is well known.

It has been pointed out that a major reason for the difficulty of mathematics education is that children are expected to master in a few years concepts that took humankind millennia to develop. All of the above developmental processes have ramifications at the ontological level. In particular, analyses of developmental obstacles represent one broad focus for the continuing relevance and usefulness of cognitive analyses (Greer, 1996).

Mathematics as Cultural Construction

"Mathematics as a human activity" has become a principle cutting across developments in mathematics education, new directions in the philosophy of mathematics education (e.g. Hersh), and influences on mathematics education from critical pedagogy, ethnomathematics, feminist critiques, historical perspectives, and so on.

For balance, it should be remembered that the proof of Fermat's last theorem, and the pages of complex formulae that Ramanujan sent to Hardy also represent human activity and require an account of the coherence and continuity of cognitive processes within an individual brain over an extended period of time however mediated by social environments (Greer, 1996).

Mathematics as a Form of Communication

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Many different answers have been offered to the question *What is mathematics?* throughout history, but the definition given by Henri Poincare is the one which I find particularly useful. According to the French mathematician, *mathematics is the science of calling different things the same name*. This deceptively simple statement, if interpreted in a way not necessarily intended by Poincare himself, can be seen as a forerunner of the *communicational* vision of mathematics. In what follows, I outline this special approach in general terms. The presentation is organized as a series of questions and answers.[i]

Q_1. What is mathematics?

A_1: It is a kind of discourse (a way of communicating)

The first thing to notice in Poincare's definition is that by putting the issue of *naming* in the center of our attention, it implies that mathematics is, in principle, a discursive activity. In other words, mathematics is a special way of communicating. One can oppose saying that it is *thinking* rather than *communicating* that should be given prominence in the definition. My answer to this is that thinking is already included in the term *communication*. Indeed, according to the basic tenet of the communicational framework, *thinking can be regarded as a special case of communicative activity*.

Q_2. What renders mathematical discourses their unique identity?

A_2. Their use words, their visual mediators, and their special routines.

After bringing the discursive activity to the foreground, Poincare gives a hint as to what makes mathematical communication distinct: It is the mathematicians' special propensity for unifying many different things under the same name which is the hallmark of the mathematical discourse. True, using the same word as a signifier for many different signifieds is not unique to mathematics – this activity is the very essence of *conceptualization*, and as such it is a vital ingredient of any communication. Mathematics, however, exceeds all the other types of discourse in the range of things included under each of its terms. This special tendency of mathematicians to speak of sameness even when what reveals itself to their eyes (and ears) appears different, is known as their propensity for *abstracting*.

Please note that within the communicational approach, the adjective 'abstract' refers to the way words are being used in the discourse, and not, as is often the case within other conceptual frameworks, to a special property of objects that are being talked about. More generally, *the use of words* is the first of several properties that one has to consider while trying to decide whether the given discourse can be called mathematical. While becoming a participant of the mathematical discourse, the learner often modifies her uses of known words and then introduces new words which from now on will serve as common names for sets of things that until now were never considered as "the same".

Two additional dimensions along which mathematical discourse can be distinguished from other types of communication are their special *mediating* *tools* (or simply *mediators*), that is, visual means with which people help themselves while communicating; and their distinct *discursive routines* with which the participants implement well defined types of tasks. Let us say a few words about each of these special discursive features.

Mediators. Unlike in the less abstract, more concrete discourses which can be visually supported with objects existing independently of the discourse itself, mathematical communication is mediated also, and sometimes exclusively, by symbolic artifacts specially designed for the sake of communication. Contrary to what is implied by a common understanding of a tool in general and of symbolic tools in particular, within the communicational framework one does not conceive of the communication mediators as mere auxiliary means that come to provide expression to pre-existing, pre-formed thought. Rather, one thinks about them as a part and parcel of the act of communication and thus of cognition.

Discursive routines are patterned discursive sequences that the participants use to produce in response to certain familiar types of utterance expressing a well-defined type of request, question, task or problem[ii]. In the case of mathematical discourses, the routines in question are those that can be observed whenever a person performs such typically mathematical tasks as calculation, estimation, explanation (defining), justification (proving), exemplification, etc. The routines with which interlocutors react to the given type of request (e.g. "estimate" or "justify") may vary considerably from those employed in response to a similar question asked in everyday setting. One of the special characteristics of full-fledged mathematical discourse is that its routines are particularly strict and rigorous.

Finally, let me explain why the question I am answering now speaks of "mathematical discourse*s*", with the plural form implying that there is more than one type of communication that can count as mathematical. Although the same words can be used on many occasions, the rules that regulate this use may vary from one setting to another. Similarly, although seemingly speaking of the same things (quantities, geometric shapes) discourses may differ in their mediators and in their routine interpretation of what appears as the same tasks. Thus, we have a good reason to speak of different types of mathematical discourse, distinguish between everyday mathematical discourses, school mathematical discourse, and the discourse of professional mathematicians (cf. Rittenhouse, 1998).

Q_3. Why do we need mathematical discourse?

A_3. For the sake of economy of communication, for its maximal effectiveness, and to solve problems that could not be solved before.

The brief answer A_3 above points to three reasons because of which mathematical discourses came into being and developed the way they did. The last of these reasons seems quite obvious, so I will elaborate here only on the other two.

The economy of communication is attained by the very property Poincare was talking about: By calling different things the same name, mathematical discourse *subsumes* several former, independently existing discourses, turning them into discourses "about the same thing" and making it possible to express in the new language everything that can be said in any of them with their own special signifiers. For instance, while saying that "three and two equals five" we simultaneously express a truth about fingers, dollars, kilograms, and infinity of other countable objects. The successive discursive "squeezing" exists also within the mathematical discourse itself. For example, the discourse about *functions* subsumes discourses about *graphs* and the discourse about *algebraic expressions*.

The issue of effectiveness must be considered when one asks why the meta-rules of mathematical discourse developed the way they did. It seems that it has always been an undeclared hope of the mathematicians to create a discourse that would leave no room for personal idiosyncrasies and would therefore lead to unquestionable consensus. Such consensus would imply certainty of mathematical knowledge. The exacting rules of the modern mathematical discourse are the result of unprecedented efforts of 19th- and 20th-century mathematicians to attain this unlikely goal.

Q4. What is mathematics learning?

A4. To learn mathematics means to change one's discourse

Learning mathematics may now be defined as an initiation to mathematical discourse. It is important to note that the introduction to a new form of communication never starts from zero. Whether the discourse to be learned is on fractions, triangles, functions or complex numbers, it will be developed out of the discourses in which the children are already fluent. If so, to investigate learning means getting to know the ways in which children modify and extend their discursive ways in the following three respects: in vocabulary they use, in the mediators they employ, and in the discursive patterns (routines) they follow.

Q5. How does the learning occur and what can we say about teaching?

This is the very central question math ed researchers are asking. The issue is extremely complex and it would be imprudent to try to summarize it in a few sentences. I thus leave this last question without an answer. Here, let me just say a few words about the expected impact of the communicational conceptualization on the vision of learning and teaching mathematics.

Perhaps the most dramatic difference between the more traditional, cognitivist vision of mathematical thinking and the one discussed in this paper lies in their conception of the origins of mathematical learning: The traditional approaches assume that learning results from the learner's attempts to adjust her understanding to the externally given, mind independent truth about the world, and thus imply that, at least in theory, the learning could occur without the mediation of other people. In contrast, the idea of mathematics as a form of discourse stresses that individual learning originates in communication with others and is driven by the need to adjust one's discursive ways to those of other people.

What is the added value of this conceptual shift? First, if we agree that the site of mathematical learning is *between* people rather than beyond them, we also realize that social and cultural factors are those that enable the process of learning in the first place. Second, the communicational conceptualization helps us to see an inherent complexity of learning: The idea of thinking as a form of communication and of mathematics as a kind of discourse, if taken seriously, makes us realize that in the process of learning mathematics, the students' awareness of the proper use of words and symbols must precede their ability to account for this use. This vision of learning is bound to entail a revision of some popular interpretations of the idea of learning-with-understanding. Finally, the communicational approach brings second thoughts about many other pedagogical believes as well. As has been argued in many places, some of these beliefs must be modified, while some others would better be abandoned altogether. Much work is yet needed to examine the practical value of this theoretical change.

The Nature of Mathematics: A Personal Journey

Laurie D. Edwards, St. Mary's College of California

What counts as mathematics? What is the nature of mathematical ideas?

The questions that frame this Research Forum are clearly foundational to the practice of mathematics education. I would like to address these questions not by proposing definitive answers, but by reflecting on my own experience as a researcher over the past 18 years. During this time, my own thinking about the nature of mathematics has evolved, in parallel with the emergence of the theoretical frameworks discussed in this Forum. I hope that the examination of a particular "case" of changing theoretical perspectives in a single body of research may be instructive.

My first major research project involved the creation of a computer-based learning environment for a specific mathematical domain, transformation geometry. As with many studies of students' mathematical thinking, the research revealed "errors" in the children's thinking, interpretations that differed from accepted mathematical truth. An example of such an "error" is described in this passage:

> The rotate bug...is an error in conceptualizing a transformation...Instead of imagining the entire plane rotating around the center point...these students thought that the shape would first slide over to the specified point, and then turn around it in place." (Edwards, 1989, p. 107-8).

The characterization of the students' interpretation of rotation as an error, as well as the entire framing of the research, reflected an objectivist view of mathematics (Edwards & Núñez, 1995); indeed, it exemplified what Lakoff and Núñez call "the Romance of Mathematics" (Lakoff & Núñez, 2000). According to this view, "Mathematics is an objective feature of the universe; mathematical objects are real; mathematical truth is universal, absolute, and certain" (ibid., p. 339). In other words, mathematics has a transcendent existence, apart from any human knowledge of it. The implication of this view is that our role as educators and researchers is to design more effective instruction about, and representations of, this mathematical reality.

Lakoff and Núñez acknowledge that there is no way to determine, empirically, whether mathematics indeed has such a transcendent existence. However, it is clear that the teaching and learning of mathematics always takes place within specific social contexts, and that simply characterizing students' understandings as "correct" or "incorrect" does not go very far in helping to improve learning. Thus what Lerman has called "the social turn" in mathematics education has come to the fore (Lerman, 2000). This change in focus from evaluating the adequacy or inadequacy of individual cognition to investigating the irreducibly social nature of learning and teaching emerged in my own research as well, One specific area in which this framework became important was in the investigation of mathematical explanation and informal proof. I first used the transformation geometry microworld with 11-year-olds. In addition to the occasional "bug" in the students' understanding of the transformations, I also found that few students were able, independently, to generate explanations or informal proofs for the patterns they were guided to discover in the microworld. At the time, I attributed this to the students' age and level of intellectual development. I expected that when I used the microworld with older students, they would be able to, fairly spontaneously, notice and explain these informal theorems that seemed so obvious in the microworld. This turned out not to be the case: the older students behaved very much like the younger children with regard to their mathematical explanations – neither group was able to produce such explanations without some degree of scaffolding and interaction with the researcher. This led me to reconsider the nature of mathematical explanation and proof. Rather than expecting that, given a dynamic and accurate representation of a domain, students would be able to discover and explain pre-existing mathematical truths, I came to think of proving as a social process, one which needs to be explicitly modeled and scaffolded (Edwards, 1997).

Thus, in my own personal journey in thinking about the nature of mathematics, I moved from assuming that mathematical ideas were "out there," waiting to be discovered, to thinking of mathematics as a product of social interaction, a kind of language, a human practice with norms that must be learned over time. Yet the fact that mathematics is learned and practiced within social contexts begs an important question: within a given social context, why is it that mathematical ideas take the form that they do? And how is it that humans, as cognizing creatures, are able to co-construct systems of mathematical knowledge that are mutually intelligible? One answer, of course, might be that mathematical ideas take the form they do simply because of their objective, transcendent reality, that human beings are simply "perceiving" the way things are, mathematically. I found this answer unsatisfying, in part, because it seemed to set mathematics apart from all other products of human history and cognition. Instead, I found work on conceptual metaphor (Lakoff, 1987; Lakoff & Johnson, 1980) and embodiment (Varela, Thompson & Rosch, 1991) to be evocative, in pointing to a deeper level of cognitive structure upon which much of human thought and language is construc-ted. The reason that mathematical ideas take the form that they do, and the reason they are mutually intelligible, is because they are, at a foundational level, built upon the common experience of being humans, with the same kinds of minds and bodies, living and growing in the same physical world (Lakoff & Núñez, 2000; Núñez, 2000, Núñez, Edwards & Matos, 1999).

A concise statement of the implications of embodiment for understanding mathematics can be found in the work of Lakoff and Núñez:

- Mathematics, as we know it or can know it, exists by virtue of the embodied mind.

• All mathematical content resides in embodied mathematical ideas.

• A large number of the most basic, as well as the most sophisticated, mathematical ideas are metaphorical in nature.

(Lakoff & Núñez, 2000, p. 364).

I would like to offer an example of the application of this perspective by returning to the "rotate bug", described above. This interpretation arose after the students were introduced to what was, for them, a new mathematical idea – they had never been taught about geometric transformations before. Yet the "idea" of turning was not new to them – indeed, the embodied experience of moving through the world, from a very early age, includes innumerable instances of turning one's own body. However, this experience of turning is different in an important way from the mathematical version of rotation instantiated in the microworld. This general transformation, or mapping of the plane, could take place around any arbitrary center point, whether this point was part of, or distant from, the block letter L used to show the transformations.

The conceptual construction that the students made of the new mathematical idea of rotation of the plane was shaped by their embodied experience of turning in the physical world: the rotate "bug," in which rotations always take place around a point *on* the L-shape, can be seen as a metaphorical mapping from the experience of turning one's own body in place. It is worth pointing out that this metaphorical mapping was unconscious: there was no socially-communicated introduction of the metaphor; instead, the physically-grounded source domain existed prior to the introduction of the mathematical idea, and shaped its assimilation in the children's minds.

In fact, the researcher did introduce an explicit metaphor or image to help the students extend and generalize their understanding of rotation. I asked the students to think about an object at the end of a string, which could be turned, with the other end of the string being fixed in place. This explicit, socially communicated metaphor helped, I believe, to bridge students' initial "local" interpretation of rotation to the more general or global mathematical one.

These remarks are intended to communicate aspects of a personal intellectual journey, yet this journey is not one in which prior theoretical commitments are left completely behind. Putting aside the question of the objective existence of mathematics (which seems to be something of a religious question), I still believe that much of mathematics is socially constructed, and that in understanding teaching and learning, we must attend to particular social and cultural contexts. However, *what* is constructed, within these contexts is not arbitrary: mathematical ideas, as they exist within, and are shared between, actual human minds, are shaped, in fundamental ways which we are still in the process of understanding, by our embodied experience in the world.

Boundaries, Identities and Mathematical Objects – Should we bother?

Shlomo Vinner, University of Israel

The proposal for this research forum raises the question whether mathematics educators and mathematicians will find intellectually sound ways of connecting their differing perspectives and reinforce each other's ideas, or whether the existing gaps will widen further. I assume that the mathematicians mentioned here are the university mathematicians who teach tertiary mathematics. Some of them are not interested in teaching mathematics since their main interest is mathematical research. Others, in case they care about teaching, have their own views on how to do it and do not believe that mathematics educators have useful advice for them. Usually, mathematicians have vague ideas about who we are and what we do (there might be some exceptions). So, who is going to listen to us? One answer is that we can listen to each other. This is quite common for academic circles. A parody about such circles appears in Davis and Hersh (1981) where a handful of devoted mathematicians who work on the decision problem for non-Riemannian hypersquares is described (pp. 34-39). If we do not want to stay like them in the isolated ivory academic towers the alternative is to look for communities who can use our research findings. Such a community, and perhaps the only one, is the community of mathematics teachers. However, if we want to approach them it should be done within their intellectual frameworks and in their language. The nature of mathematics is undoubtedly an issue with which they have to be involved. But to what extent? Thus, this forum, whose title is *The Nature of Mathematics as Viewed from Mathematics Education Research,* is a good opportunity to raise some questions about mathematics education research that deals with the nature of mathematics. To be more specific, my question is the following: What aspects of the nature of mathematics are relevant to the community of mathematics teachers, and what aspects should be kept for our closed circles where we can discuss any subject at any level of sophistication. Asking that, I am, in fact, raising two questions. One is about boundaries and the second one is about identity. The one about boundaries is: What are the boundaries of mathematical education research that are relevant and meaningful to mathematics teachers. The one about identity is: What is the purpose of mathematical education research? In fact, this is an identity question about our group and it has been raised in the past several times by several people. One of them was in Ballachef's letter from 1996. Questions about boundaries and identity have more than one answer. So, what I suggest here is only one possible answer out of many. I suggest trying to define a restricted domain of mathematical education research which I will call the core and which will have an immediate simple application to the practice of teaching. Some other issues, which imply a level of sophistication that teachers do not have, will be considered as peripheral.

The nature of mathematics has many aspects. One of them is the nature of mathematical objects. Some time ago, a student of mine (she is a junior high mathematics teacher) came to me complaining: "You sent us to take a course in the philosophy of mathematics," she said, "and the lecturer spent three weeks discussing the question: what are objects and what are mathematical objects? What is the point of it?" I was quite irritated by the question but as a teacher, I have trained myself to control my reactions and to try to tolerate and to understand my students' views. "Isn't this question relevant to our research forum?" I asked myself and decided to discuss it here. First of all, I would like to explain why the question of mathematical objects is such a crucial question in the philosophy of mathematics. According to mathematical logic and model theory, mathematics is a collection of theories about mathematical systems. A mathematical system is a set of abstract objects with relations and operations that fulfill certain primary conditions. The mathematician's task is to discover some interesting claims about the mathematical systems implied by these primary conditions. Whether you accept this or not, it can explain why the problem of mathematical objects is so crucial for the philosophy of mathematics. But is it also so crucial to mathematics education research? Even a short literature survey will show us that many mathematics educators are involved in investigations about this issue. In a paper by Tall et al (2000), there is an attempt to draw certain boundaries between some approaches to mathematical objects in mathematical education research. When one speaks about boundaries one has to speak also about territories, but Tall et al. (p.233) speak about scopes, not about territories. Hence, what I say here is my interpretation of their formulation. Their paper discusses three approaches to mathematical objects: In the first two, an attempt is made to explain how mathematical objects come into being in the human mind: it is either by encapsulation (Dubinsky, 1991) or by reification (Sfard, 1991). The third one (Gray and Tall, 1994) does not bother with the question of how mathematical objects come into being. It assumes that people think and speak about mathematical objects. However, it draws our attention to the fact that some mathematical terms and mathematical symbols are ambiguous. These terms and symbols denote both processes and objects or, if you wish, both processes and concepts. This led Tall and Gray to invent the notion of procept. If you look at it this way, the discussion in the above paper (Tall et al, 2000) is, in fact, about boundaries. If you agree to accept mathematical objects without trying to ask how they are formed then some controversies are moved from the *core* to the *periphery*. A serious objection to excluding the mathematical objects from the *core* might claim the following:

People fail in mathematics because they have not constructed in their mind the mathematical objects required in order to perform the mathematical tasks imposed on them. We should lead them through well-designed activities in order to construct in their mind the required mathematical objects. Therefore, these activities should be an essential part of the mathematical education research core. More specifically, we should make our students go through many processes that will eventually become (by encapsulation or reification) mathematical objects.

Since I wish to avoid controversies I will not argue with this claim. I would only suggest, as an alternative working assumption, a different view. Mathematical objects are a special case of abstract objects. The problem of abstract objects is widely discussed in the philosophy of language. There is the classical distinction between concrete nouns and abstract nouns. Please, note that we are speaking here about nouns and not about objects. Some nouns or noun phrases denote well-defined concrete objects. For instance: *The dog of my mother in law.* On the other hand, many nouns do not denote any object. For instance, is there any object in our world that is denoted by the noun *milk*? The question becomes even more embarrassing when abstract nouns are discussed. Are there objects in the world denoted by *love, peace* or *compassion*? Surprisingly enough, in our thoughts we relate to these nouns *as if* they denote objects. This is *reification.* The Webster's Ninth New Collegiate Dictionary suggests that the word "reification" is in use at least since 1846 and *it is the process or result of reifying. To reify is to regard something abstract as a material or concrete thing.* Thus the working assumption which I suggest claims the following:

(1) There is a is a tendency in languages to introduce nouns even when no objects are involved (it probably facilitates talking about certain things.)

(2) Reification occurs spontaneously the moment a noun is introduced.

Because of time limitation I will bring only two short examples to support the above claim. 1. When teaching limits in Calculus, many of us use the term "infinity" instead of saying "increasing unboundedly." However, in calculus (contrary to set theory), there is no object the name of which is infinity. In spite of that, many calculus students think of infinity as an object. 2. Even languages of primitive cultures have abstract terms. Levi-Strauss, the famous French anthropologist, in his book The Savage Mind (1966) illustrates this by some examples. Two of them are the following: *In Chinook, a language widely spoken in the north–west of North–America, the proposition "The bad man killed the poor child" is rendered as: "The man's badness killed the child's poverty." And for "The woman used too small a basket" they say: She puts the potentilla-roots into the smallness of a clam basket"* (p.1). (The issue of reification is widely discussed in the major works of Quine (1960), 1981, 1995)). Finally, I would like to relate again to the claim that people fail in mathematics because they fail to construct the mathematical objects involved in their mathematical tasks. There are so many potential reasons for failure that it is impossible to isolate one factor and to claim that it is the cause for failure.

The way I suggest to understand the procept paradigm by Tall and Gray allows us to speak about processes, objects and concepts as primary notions. Namely, we do not have to explain what processes, concepts and abstract objects are. On the other hand, the procept theory points at a major obstacle in the learning of mathematics - ambiguity. Ambiguity is a serious obstacle in communication. On the other hand, it also enriches communication immensely. I suggest, however, that this issue will not be included in the core of mathematical education research.

Einstellung, Vorstellung, and Darstellung

Hartwig Meissner, Westf. Wilhelms-Univ. Muenster, Germany

Einstellung, ***Vorstellung***, and ***Darstellung*** are keywords to describe the process of learning and understanding mathematics. Analyzing this process we do not rely on philosophical theories (Kant's ontology). We base our theory of learning and understanding mathematics on the following assumptions:

(1) Mathematics is "something" which exists independent from human beings or from human brains like trees, birds, genetic codes, time, space, electricity, gravity, infinity, ...

(2) There are external representations of mathematical ideas, ***Darstellungen***, which we can read, or see, or hear, or feel, or manipulate, ... These *Darstellungen* can be objects, manipulatives, activities, pictures, graphs, figures, symbols, tags, words, written or spoken language, gestures, ... In a *Darstellung* the mathematical idea or example or concept or structure is

hidden or encoded. There is no one-to-one-correspondence between a mathematical idea, concept, etc. and a *Darstellung*.

(3) Human beings are able to "associate" with these objects, activities, pictures, graphs, or symbols a meaning. That means each *Darstellung* evokes a personal internal image, a ***Vorstellung*** (cf. concept image, Tall & Vinner, 1981). Thus *Vorstellung* is a personal internal representation, which can be modified. Or the learner develops a new *Vorstellung*. A *Vorstellung* in this sense is similar to a cognitive net, a frame, a script or a micro world. That means the same *Darstellung* may be associated with many individual different internal representations, images. Each learner has his/her own *Vorstellung*. And again here, there is no one-to-one-correspondence between a *Darstellung* and a *Vorstellung.*

(4) The process of building up a *Vorstellung* depends very much on the basic mentality of the learner, i.e. on his or her ***Einstellung***. The *Einstellung* includes affective components like attitudes, beliefs, emotions, values (Goldin). The *Einstellung* affects attitudes towards learning in general, towards mathematics in general, towards problem solving, or towards the specific learning "environment". The *Einstellung* is a product of social interactions (with parents, teachers, peers, etc.), of genetic factors, of cultural or historical impacts, etc. Positive *Einstellungen* in the class room are necessary for a successful teaching-learning process. A learner with a negative *Einstellung* probably will not be very successful. *Einstellungen* work like a filter or a catalytic converter in the transformation processes *Darstellungen* ↔ *Vorstellungen.*

In this paper I concentrate on *Vorstellungen* and *Darstellungen.* The process of building up a *Vorstellung* very much depends on the already existing internal representations ("assimilation, accommodation" according to Piaget, "coherence, connectedness" according to Greeno) and on the already existing "subjective domains of experiences" (Subjektive Erfahrungsbereiche, Bauersfeld). Learning mathematics now means that the learner has to build up a *Vorstellung* which "corresponds" (especially in the sense of Greeno) as much as possible to the mathematical idea / concept / structure. But the learner does not experience the mathematical idea / concept / structure directly, the learner only is confronted with (different types of) *Darstellungen*. Figure 1 presents a summary of these ideas.

We are interested in the cognitive processes[1]. What does "learning" mean? And when do we "understand"? Learning obviously is the process of building up an "adequate" *Vorstellung* of a given mathematical situation (by means of "appropriate" *Darstellungen*). But the *Vorstellung* is individual and cannot be inspected or evaluated directly. With other words, there is no direct way to evaluate "adequate" or, there is no direct way to evaluate the degree of "understanding".

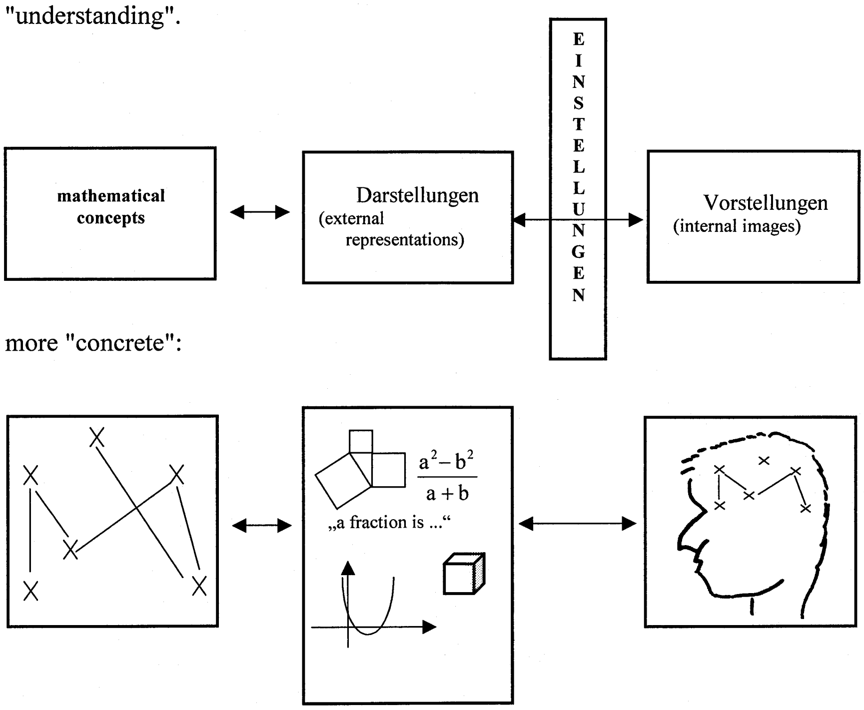

Figure 1: Examples of *Vorstellungen* and *Darstellungen*

We only can judge a *Vorstellung* by a corresponding *Darstellung*, that means "communication" is necessary. To prove understanding the learner must transform the individual *Vorstellung* into a *Darstellung*. And when the learner's *Darstellung* corresponds with one of the expected *Darstellungen* we may assume that the learner did understand. The problem is obvious. We do not judge a *Vorstellung* but we interpret a performance.

[1] Despite a suggestion from the IC to "facilitate greater insight and engagement" we will continue the paper with the German words which are more precise than English translations.

To distinguish the *Vorstellung* from the performance we will speak of a ***conceptual understanding***[2] when the learner has an adequate *Vorstellung*, i. e. his or her internal representation corresponds appropriately to the given situation. A conceptual understanding also may be intuitive or unconscious. And of course, a conceptual understanding only can be demonstrated indirectly up to a certain degree, consciously or unconsciously, see examples.

To detect the student's conceptual understanding we still need *Darstellungen* from the student. But we must allow flexibility in the use of *Darstellungen.* Not the *Darstellung* itself is important but the *Vorstellung* behind that *Darstellung.* When there are misunderstandings concerning a specific *Darstellung* just change the *Darstellung* to clarify if the misunderstanding originates from the *Vorstellung* or from the *Darstellung.* Of course also non-standard *Darstellungen* can be used. Clinical interviews with experienced interviewers - and also experienced class room teachers - can identify the student's conceptual understanding. Written tests - like TIMSS or PISA - usually cannot help to prove conceptual understanding[3].

Skemp distinguished between instrumental understanding and relational understanding. ***Instrumental understanding*** is characterized by selecting and applying appropriate rules to solve the problem without knowing why ("rules without reasons"). In our terminology specific *Darstellungen* are expected and it is necessary to (re-)produce them: "Tell me what to do and I will do so". There is no adequate mathematical *Vorstellung* behind.

An understanding with more *Vorstellung* behind it, was termed a relational understanding by Skemp (1978): "knowing both, what to do and why". Here an adequate *Vorstellung* leads to an expected *Darstellung.* We want to call this a ***communicable understanding***. It combines both, the conceptual understanding with the ability to communicate in a wanted or given format. A person with communicable understanding is able to communicate flexibly in those *Darstellungen* which are expected or - according to the specific problem - in *Darstellungen* which fit best to the problem. We will give some examples to illustrate these various aspects of understanding[4].

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[2] In earlier papers we called this a relational understanding (which is different to SKEMP's relational understanding).

[3] HOSPESOVA and TICHA found through interviews examples for both, "good conceptual understanding, but no expected TIMSS-Darstellung" and "correct TIMSS-Darstellung, but no adequate conceptual understanding".

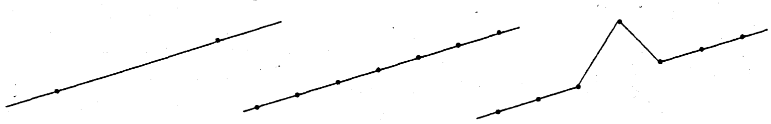
[4] For more details on the role of communication and communicational conflicts see Anna Sfard and Gerald Goldin in this Forum-paper.

Example 1 (Sorting objects)

A teacher and about 25 students (age ~9) are sitting around a set of about 40 geometrical solids. Teacher: "I have a rule in my mind to sort these solids. These two solids follow my rule. Who can find other solids which follow my rule?" Only directed by the teacher's YES or NO without any further explanations the set gets sorted into the set of solids which follow the rule and a second set (of counterexamples). This non verbal process of sorting objects by guess and test may lead to an intuitive concept (of *polyhedra* or *rectangular solid* or *rectangle* or ...), i.e. an (partly unconscious) conceptual understanding develops. But during the process of sorting in the classroom also discussions start to guess the "rule", to verbalize the situation. The concept becomes more conscious and a communicable understanding develops.

Example 2 (Linear functions, discuss possible *Vorstellungen*)

To draw the graph of $1.5x - \sqrt{8}\,y = -\sqrt{3}$ we get the following *Darstellungen:*



Example 3 (Interpreting functions, analyze the conceptual understanding)

Student A draws the correct graph of a given function and determines correctly by computation the *maximum* at x=5. Student B draws the same graph and determines the *minimum* at x=5 (by applying the correct algorithms with a computational mistake).

Example 4 (Procepts)

A keyword or symbol or tag as a *Darstellung* can serve as a stimuli to evoke proceptual thinking (Gray & Tall, 1994). Here the *Vorstellung* involves both, a procedural and a conceptual aspect. E.g. y=f(x) may be seen as an assignment (process) or a function (concept).

Connecting Understandings from Mathematics and Mathematics Education Research

Gerald A. Goldin, Rutgers University

The perspective I bring to this discussion may be a controversial one, but I shall start noncontroversially by building on Hartwig Meissner's accompanying presentation. Meissner highlights differences among the notions of *Einstellung* (attitude), *Vorstellung* (internal representation), and *Darstellung* (external representation) as descriptors of processes in mathematical learning and understanding, and takes mathematics to be something that "exists independently" of these. Before considering aspects of the nature of mathematics, let me continue with two further, important ideas about representation.

First, I would emphasize that individual representational configurations, whether external or internal, cannot be understood in isolation. Rather they occur within *representational systems.* The latter are not mere collections of representations, but have complex structures that in practice may be ambiguously defined or context-dependent (Goldin, 1998). Thus words and sentences occur within natural language systems, having conventional grammatical and syntactic structural features that can be characterized as *external* to any one cognizing individual. *Internal,* verbal representational configurations also occur in each individual, within a personal system of linguistic competencies encoded in the brain that has its own structural features. All these depend on context in various ways. The "communicational approach" in Anna Sfard's accompanying presentation, at least tacitly, involves such structural features of language. Likewise in mathematics, we have conventional, external systems of representation including base ten numeration, rules for arithmetic operations, ways of denoting rational numbers, Cartesian graphs, a system of algebraic notation, etc., with accompanying verbal descriptions, all usefully regarded as external to the individual. And we have the visual imagery, notation-images, kinesthetic encodings, and so forth, occurring within personal systems that may be partially-developed and embody misconceptions, contradictions, and idiosyncratic structural features. We might use the terms *Darstellungsysteme* and *Vorstellungsysteme* to refer respectively to external and internal representational systems.

In my work I have found it useful to distinguish five different types of *Vorstellungsysteme* that come into play in learning and doing mathematics: (1) verbal/syntactic systems, referring to internal natural language competencies, (2) imagistic systems, including visual/spatial representation, kinesthetic representation, and auditory/rhythmic representation, (3) formal notational systems, referring to internal procedural/structural competencies associated with the conventional representations of mathematics, (4) a system of heuristic planning and executive control, including configurations for strategic decision-making that govern problem-solving activity, and (5) an affective system including rapidly-changing emotional states as well as more stable, multiply-encoded constructs such as attitudes, beliefs, and values. A psychologically adequate description of mathematical learning, development, and problem solving requires that we take account of all five types of *Vorstellungsysteme* in interaction with each other and with *Darstellungsysteme.* Here I think Meissner's term *Einstellung* usefully distinguishes certain more stable aspects of affect and related cognition that individuals bring to mathematical situations.

The second idea I want to emphasize is the strong psychological role that the initial assignment of meaning, or semiotic step, plays in the individual's developing *Vorstellungsysteme.* Understanding this is important not only to education, but to grasping how mathematics itself has evolved.

For example, children frequently learn that multiplication of natural numbers ("times") is an abbreviation for repeated addition: i.e. '3 x 5' *means* '5 + 5 + 5' (three fives). The formal notational and imagistic representational subsystems associated with the operation of multiplication then develop structurally, making use of this 'meaning'. The usual multiplication tables are constructed, and patterns found in them. The commutative and associative properties of multiplication, and the distributive property of multiplication across addition, are verified and illustrated. As more structure is built on the initial meaning, its psychological persuasiveness increases. Repeated addition becomes for the learner what multiplication *really is.* But the moment comes when the meaning fails! A child may interpret '3 x ½' as '½ + ½ + ½', but '½ x 3' is problematic—what does it mean to 'add three one half of a time'? The structural commitment to the commutative property suggests a value for ½ x 3, but the absence of the original meaning leaves a gap in understanding—a cognitive obstacle. We have well-documented related misconceptions, such as the idea that "multiplication always makes larger," which may persist until some reconceptualization has occurred.

Similar obstacles occur not only in individuals learning mathematics, but in the history of the mathematical field. They have their origins in the structural extensions of mathematical systems that require *relinquishing* the necessity of the original semiotic connections—so that the mathematical structures are abstracted, and the ways in which the mathematical notations function as representations of imagistic configurations are generalized. Mathematicians of earlier eras struggled mightily with the concepts we today call irrational numbers, imaginary numbers, negative numbers, and non-Euclidean geometries, due in part to the psychological difficulty of abandoning the initial, "real" meanings attributed to numbers, to points and lines, and so on. The necessity for such reconceptualization is well understood now, and has influenced our evolving notions of "mathematical existence" and "mathematical truth."

This brings me to the major point in my presentation—the notion of mathematical truth, its recent unfortunate downplay in mathematics education, and the consequent widening chasm between the fields of mathematics and of mathematics education. I view the gulf that has developed as both damaging and unnecessary. Although the divide has reached a depth that seriously impedes our common educational goals, I do not think the sociological reasons for rivalry are so strong as to generate inevitable conflict. For both communities, and for the next generation of students, the value of achieving meaningful improvement is extremely high, providing a powerful incentive for real collaboration. I want to focus on what I perceive to be *intellectual* reasons for division, intending my comments to be strongly critical.

At the root of the problem in each community is a willingness *to deny or dismiss the very integrity of the knowledge* generated by the other. It is not always apparent to mathematicians when they do this in relation to mathematics education research, nor is it always apparent to non-mathematicians when they do this in relation to mathematics. Sometimes, however, it seems to be done consciously and opportunistically, as a way of inviting attention and gaining a following—with a kind of wilfully-maintained ignorance of the other discipline.

On the one hand, some in the mathematical sciences community insist on imposing—with unwarranted confidence—tacit but naive models of what it means to learn mathematics. It is straightforward for mathematicians to focus on building powerful competencies in *formal notational* systems, as these are culturally agreed to be part of mathematics, and competencies in them can usually be tested straightforwardly. Powerful problem-solving strategies and *heuristic planning* techniques that work in various mathematical domains (Polya, 1954, 1962, 1965), including proofs, are likewise generally agreed to be part of, or closely related to, mathematics, though techniques such as "work backward from the goal" or "solve a simpler, related problem" are difficult to test. But for many mathematicians the traditional view of mathematics as consisting of abstract systems encoded formally accords only casual or unimportant status to all but the most standard of representations. The power of formal, logical reasoning when applied to abstract mathematical entities, together with the fact appreciated in mathematics that visual intuitions can mislead, creates a reluctance to place a high, explicit value on *imagistic* representation, especially non-standard representation with accompanying differences in individual learning styles, or on *affect.* Visualization, metaphor and metonymy, emotions, and the relation between feeling and mathematical imagination, are dismissed or relegated to incidental status, despite growing empirical evidence for their fundamental roles in the learning of mathematics.

One extreme position is to discretize, take as "given", and value very highly in defining the curriculum a collection of standard mathematical material, in disregard of the complexity of the processes through which mathematical understanding develops in students of diverse abilities and motivations. This view has energized the "traditionalist" side of the recent "math wars" in the United States. Skills are seen as prerequisites to conceptual understanding, and are thus to be taught first. Mathematical achievement manifests itself through speed and accuracy in answering test questions. Mathematical ability is seen as an innate, unitary characteristic of individual students, describing the rapidity with which they acquire formal notational competencies when trained in them. Of course I am *not* saying that all or most mathematicians adopt such dismissive positions, though some educators have sought to establish this stereotype. *Some* mathematicians do, and the fact that they do offers a convenient rationale for counterpart dismissive fashions in education.

Fundamental to the integrity of mathematical knowledge is the notion of "truth", which has evolved significantly by virtue of mathematical insights achieved over millenia (Kline, 1980). Let me use this term in a certain way that mathematicians typically use it. The field of mathematics has been characterized by many as the study of pattern (e.g. Sawyer, 1955). This includes pattern detected in the natural world, and pattern in systems invented by human beings. To study patterns, mathematicians seek to characterize them as precisely as possible. One way this is done is to formulate definitions and axioms or postulates that describe a system or class of systems incorporating a pattern. We then have a collection of mathematical statements taken from the outset as true. Further propositions (called theorems), often not at all obvious, can be proven from the axioms by means of well-defined rules of inference and are thus demonstrated to be true. Truths in mathematics occur within systems of assumptions. In developing such systems, our concepts change and evolve. Some lines of reasoning turn out to be valid, while others are demonstrated to be invalid. Often our initial conceptions turn out to be too limited, or even self-contradictory. Sometimes imagistic thinking guides us to mathematical truths, and sometimes it misleads us. In short, there exist essentially *objective* answers to important mathematical questions. Furthermore the system we create is *abstract*, and not necessarily restricted to apply only to the original, motivating conceptual domain—other, unexpected models are likely to exist! And there are fundamental, logical limits—*proven* limits—to the possibility of demonstrating the completeness or consistency of mathematical axiom systems.

The fashionable but dismissive intellectual trends influencing mathematics education research have in the past two decades been *ultrarelativist.* Such views are ideological (in the sense of being nonfalsifiable), since a contrary argument can never be more than an alternative viewpoint. They include radical constructivism, radical social constructivism, and variations of postmodernism, each in its own way *denying the very possibility* of objective truth, knowledge, or validity, and thus dismissing from the outset the central construct of mathematical inquiry. These have energized extremes on the "reform" side of the "math wars." Most recently we have the grand claim that mathematics consists entirely in "conceptual metaphors" (Lakoff, G. & Núñez, R., 2000), predictably attracting favor among some mathematics education researchers. Here there are only conceptions, no misconceptions. Ideas and visualizations (familiar to mathematicians) that underlie and motivate abstract constructions are renamed as metaphors, presented as if newly-discovered, and taken to *be* the mathematics—with those mathematicians who might disagree caricatured as Platonists, naive realists, or empty formalists. In this view mathematics cannot possibly "exist" independently of human metaphors, so the initial point in Meissner's presentation is be rejected entirely.

Of course, the definitive characterization of mathematical truth and the validity of mathematical reasoning are far from a solved philosophical problem. We have a lot to learn about it, and many unanswered questions remain. However, we must distinguish between the assumptions, definitions, conventions, and rules of inference chosen to characterize some visualized or imagined patterns (socially constructed, subject to negotiation in their framing, and possibly "metaphorical") and their logical consequences (now "true" in an important, "objective" sense). Denying or dismissing the very construct—replacing "mathematical truth" by "social consensus" or "stability of human metaphor", replacing "validity" by "viability", and so on—makes *no contribution* to our mathematical understanding. Rather it seems to make deeper mathematical understanding unnecessary. Some version of ultrarelativism may be a tempting response to closed-minded or "absolutist" views among mathematicians. It may seem to justify our being open to students' various ways of thinking mathematically, to our emphasizing in education the *ideas* of mathematics, imagery and metaphor, open-ended problem solving, discovery processes, social and cultural environments, and various systems of belief—all that I strongly favor! But ultrarelativist "isms" undermine what should be central goals of mathematics education—conveying the nature of mathematical truth and the power of valid, objective mathematical reasoning, bringing learners to experience the processes of abstraction and proof, and helping students to identify the same abstract mathematical constructs in a variety of different conceptual domains.

In my studies of mathematics and of students' processes of learning and problem solving in mathematics, I have never found what we learn validly as mathematicians and what we learn validly as researchers in the psychology of mathematics education to contradict each other. Both sets of understandings are needed. Mathematicians who are "absolutists" nevertheless offer important mathematical insights. "Ultrarelativist" educational researchers have designed and reported on ground-breaking studies. Progress is made when mathematicians and educational researchers communicate effectively and learn from each other, so that our understandings of difficult notions are enhanced—not when we erase distinctions or dismiss centrally important constructs.

To conclude, then, it is time that mathematics education researchers exercise far greater discernment than we have in the past. Let us *knowledgeably and thoughtfully abandon* the dismissive fads, fashions, and "paradigms" in mathematics education, in favor of a unifying scientific and eclectic approach.

Concluding Points

Lyn D. English, Queensland University of Technology

The papers of this Forum have presented a range of perspectives on the nature of mathematics as viewed from mathematics education research. The authors have raised controversial and, at times, opposing viewpoints on the issues presented in the introudction. The ideas expressed by each of the authors provide a rich basis for tackling the numerous debates emerging within and beyond our discipline, debates that are being fuelled by the increasing scrutiny of mathematics education by the public, by governments, by mathematicians, and by school systems. Such debates include the nature of the mathematical content we should teach, how and when we should introduce this content, how we can provide all students with access to powerful mathematical ideas, and how we can encourage more students to undertake courses in mathematics. We face many challenges as we attempt to deal with these concerns. The question remains as to how effectively we are meeting these challenges.

Although not denying the importance of diversity in our perspectives and the richness this brings to mathematics education, I believe we have become too divided and too insular in many of our beliefs, theories, and ideologies. We seem to be addressing only those questions that fall readily within our particular ideological stance while ignoring other important issues. While following a particular social-constructivist perspective in exploring children's mathematical learning, for example, we often overlook the inherent structure of the mathematical tasks or at least give it superficial treatment. So we might argue for the richness of children's learning through their classroom interactions, while failing to recognize the mathematical inadequacies of the tasks being explored.

As no one perspective can provide a satisfactory answer to all of our research and teaching issues, we need to be more cognizant of alternative viewpoints and incorporate the best of our own ideas with those of others. This Forum represents an important step in this direction. There still remain, though, many issues in need of attention as we continue to foster the growth of our teaching and research communities. Some of these issues are listed below for further debate.

- To what extent are our theoretical bases addressing the mathematical needs of society in the new millennium?
- How are our existing ideologies impacting on students, teachers, and on the community at large?
- How can we ensure a closer match between what we believe about mathematics and mathematics education and the goods we deliver?
- To what extent are we dismissing one another's perspectives and philosophies, both within our own mathematics education community and with our neighbours, the mathematicians?
- Mathematics continues to grow rapidly. How is this growth changing our views on the nature of mathematics and what we consider important to know and understand?

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[i] Needless to say, this extremely concise exposition cannot possibly count as a proper introduction of the comprehensive conceptual framework. The interested reader may turn for elaboration to Harre & Gillett (1995), Edwards (1997), and Sfard (2000 a, b; 2001a).

[ii] Two discursive sequences will be regarded as being instances of the same discursive routine if they comply with the same set of *meta-discursive rules*. This latter term refers to principles that help the observer to account for the regularities she spots in the behavior of the interlocutors. Rather, than being prescriptions which the speakers follow in a conscious way, these are propositions that help the analyst encapsulate the discursive flow the way the formula of free fall helps physicist to encapsulate the movement of falling bodies (cf. Sfard, 2000b; Sfard & Kieran, 2001a).