Extending example spaces as a learning/teaching strategy in mathematics

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Abstract *This paper gives an illustrated theoretical account of some roles for student-generated examples in the teaching and learning of mathematics. We give instances which show that the development of example spaces contributes to conceptual engagement, and that such spaces are personal and situated.*

Introduction

In this paper, learning is seen as growth and adaptation (*extension*) of personal, situated, example spaces; teaching involves providing situations in which this can take place. We find this metaphor useful in characterising many teaching strategies which promote active engagement with concepts, and also in characterising experiences of conceptual learning. This theoretical approach has been developed over a number of years through examination of our own mathematical work, through the responses of groups to mathematical tasks we have offered them, through our observations of teachers and their students, and through some of the literature on exemplification.

We recognise the importance of interaction, discourse, ethos and enculturation in the learning of mathematics, but we find ourselves unable to adopt a solely social constructivist or socio-culturalist position when thinking about learning and doing of mathematics. When we do mathematics for ourselves, we do not recognise our experience as anything other than internal struggle with the unknown to achieve what feels like a self-contained sense of understanding. As institutional learners, personal struggle for sense was supplemented by a need to satisfy an external authority (the teacher). We acknowledge that these descriptions, our behaviour and our judgements are subtly guided by social practices, of which we are a constituent part, but that has little bearing on our *sense* of relationship with the mathematical problem or concept once we have engaged with it. The questions which guide us are:

How can engagement in mathematics be encouraged?

How can construction of mathematics be supported?

We see mathematics as a structure of agreed quantitative and spatial conventions which can be extended in agreed ways into abstract worlds, about which we communicate in ways which imply agreements. Learning mathematics involves making personal sense of experiences of quantitative, spatial and abstract objects and relationships, and matching these senses to further experiences and conventions. The interactive aspects of these experiences are crucial, because language (verbal, non-verbal, symbolic) is the medium through which a teacher can construct situations in which learners meet abstract concepts and compare the implications of their constructions with the implications of conventions. Thus we are interested in 'goodfit' relationships between language and active construal, that is, relationships which are effective in leading learners to construe mathematics in useful ways.

Put more simply, we ask what tasks teachers can set to help students learn conventional mathematics, given that learning involves constructing meaning in the environment created predominantly by the teacher.

The role of examples in learning mathematics

It is has long been acknowledged that people learn mathematics principally through engagement with examples, rather than through formal definitions and techniques. Indeed, it is only through examples that definitions have any meaning, since the technical words of mathematics describe classes of objects or relations with which the learner has to become familiar.

We use 'example' to cover a broad range of mathematical genres, including examples of classes, examples illustrating concepts, worked examples demonstrating techniques, examples of problems and questions which can be resolved, examples of appropriate objects which satisfy certain conditions, examples of ways of answering questions, constructing proofs, and so on. Thus learning mathematics can be seen as a process of generalising from specific examples: learning to add involves generalising a process which works for given examples so that it can be applied to examples one has not met before; learning about quadrilaterals involves understanding what types are possible, and what their properties are. The broader the range of examples, the richer the possibilities for generalisations and connections to be made, so the extent of the set of familiar examples is influential in construction of conceptual understanding.

Most examples come to students from authorities (by whom we mean teachers, textbook and examination writers), but there is a contrast between a single example seen as paradigmatic, or generic, by the expert but as something merely to learn by a student (Mason and Pimm, 1984). For instance, a teacher might multiply some numbers by 0.3 to show, generically, that multipliers between 0 and 1 make 'smaller' products, but are students aware that this is an example of a range of numbers which act similarly, or do they perceive merely a set of practice exercises?

It is useful, then, for students to have several examples from which to get a general sense of what is being taught. Firstly, it is important to compare examples to see what features are common, and hence to appreciate generality. This helps avoid students' fixations with figural concepts which have unhelpful features. Choice of examples is important in helping students develop generalisations of structures rather than surface features (Bills and Rowland, 1999). The process of comparing examples can also highlight the conceptual, semantic and structural similarities between problems rather than superficially seeking cues in syntax, habit or context (Reimann &Schult,1996). Learners may however only use worked examples as templates to be followed using different inputs, rather than as demonstrations of how to manipulate representations, to transform relationships, or to synthesise facts to achieve solution (Anthony 1994). For example, the processes of elimination and substitution essential to solving simultaneous equations may be lost in the detail of "multiplying and subtracting".

There has been much debate about the usefulness of including counter-examples in students' experience. In some studies, counter-examples appear to be helpful in focusing students on what is relevant and what is irrelevant; in other studies, the role of counter-examples appears to confuse students who do not understand how or what it refutes (Zaslavsky and Ron, 1998).

The extent and extension of example spaces

There is a special kind of counter-example, referred to by Askew and Wiliam (1995): 'The ideal examples to use in teaching are those that are only just examples, and the ideal non-examples are those that are very nearly examples' (p. iii). Our searches for such examples have led us to develop the more general notion of boundary examples. For example: if a general straight line is given as $y = mx + b$, then lines of the form $x = a$ may be excluded from students' experience because they are not expressed in the general form; consequently teachers have to introduce these deliberately in some way. We call '$x = a$' a *boundary* example for straight lines expressed as $y = ax + b$.

We use the word 'boundary' because we see students' experiences of examples in terms of spaces: families of related objects which collectively satisfy a particular situation, or answer a particular mathematics question, or deserve the same label. Such spaces appear to cluster around dominant central images.

For example, some mathematics postgraduates all offered $|x|$ as a function which is continuous but not everywhere differentiable but they needed considerable prompting to extend their images and provide other examples. When asked for a second example most translated $|x|$; on being asked for a third example some commented (excitedly!) that a vast number could be generated from $|x|$; others searched for very different examples and reported looking for other central images from which to work, such as an image of a curve becoming a straight line. None of them tried to construct anything solely from definitions. A group of mathematics graduate teacher trainees nearly all drew segments of circles when asked to give an example of an image for a quarter; a few gave sections of rectangles; one wrote (1/4); none gave a point on a number line. Only after coaxing, critique and discussion did other images arise.

Student-generated examples

Waywood (1992) encouraged a metacognitive shift by asking students to write journals which included collections of examples. He detected a range of responses from the inclusion of examples copied from other sources, through the use of examples to demonstrate an application or use or illustration, to the construction of annotated examples to summarise aspects of a topic or idea. Over time, his students progressed towards the personal construction of examples. In the two incidents described just before this section, students worked on extending their example spaces by generating more examples for themselves. In the process, they had to think about the meaning of concepts and draw on experiences which had not readily come to mind.

There is a growing body of work in which the use of student-generated examples is advocated. Mainly these are for assessment purposes, to motivate interest in a topic, or to give students the opportunity to pose their own problems in the hope that this will make them better able to solve problems posed by authorities.

Contextual questions created by students are motivating for the producer and also likely to relate more closely to the experience of other students. Shifting the responsibility for test construction and problem posing from teachers to students can have clear affective gains, and necessarily involves some review of material used. Ellerton (1988) comments that the questions and examples produced by students generally reflect what they are used to from their teachers' styles. Further, experience of problem-posing enabled some students to 'reason by analogy' when presented with similar questions (English, 1999).

But the ability of students to create and answer questions which are *dis*similar from those previously experienced would be a more powerful indicator of mathematical learning. Silver and Cai (1996) working in urban middle schools with ethnically and linguistically diverse populations in economically disadvantaged communities, found that their students could indeed create complex problems involving several relationships or challenging given constraints if they had appropriate support. In their study, active extension of the example space of possible questions took place, moving students away from the centrality of the teacher's example as dominant template.

Use of SGEs for concept development

A few writers use SGEs to promote conceptual development of new mathematics rather than primarily for motivation or assessment. For example, Sadovsky (1999) challenges her students to exemplify division operations which give a dividend of 32 and a remainder of 27. She asks 'How many are there? If you think there are less than three write them all down, and explain why there are no other ones. If you think there are more than three write down at least four of them and explain how other solutions can be found' (p. 4-147). One outcome of her study, whose main aim was to explore algebraic shifts, was her conclusion that '... these problems are simultaneously a chance to find the limits of arithmetic practices and to enrich the conception of Euclidean division'. But concept development through exemplification need not be an incidental effect; it can be an explicit pedagogic aim. Sowder (1980) reports using the prompt 'Give me an example, if possible, of ....', with the teacher taking responsibility for guiding students towards peculiar examples as an integral component of concept construal. Dyrszlag (1984) suggests asking learners to give examples as a way of expressing their own understanding of a concept, and thus affecting future learning[1]. Zaslavsky (1995) asked students to 'find an equation of a straight line that has two intersection points with the parabola $y = x^2 + 4x + 5$.' Attention was thus directed to features of a parabola and a straight line, rather than to algebraic or trial-and-error techniques for finding intersections. Each strategy led to further questions such as 'find an equation of a straight line which does not intersect twice with the parabola....'. Students encountered most of the analytical geometric syllabus, engaging with the structures and equations of straight lines and parabolae through their own examples.

A particularly interesting account of the use of exemplification for conceptual understanding is given by Dahlberg and Housman (1997). They asked their tertiary students to give examples and counter-examples of analytical concepts as well as their own explanations, having previously been given only formal definitions. Students who consistently employed example generation as an integral part of their learning strategy, even when not specifically prompted to do so, underwent more shifts of concept image, could give better explanations, developed broader example-spaces and hence had a more complete understanding of the taught concept. "Example usage, particularly example generation and verification, is crucial for understanding a new concept." (p. 284). These reports fall far short of the practices described by Brown and Coles in several accounts (e.g. 2000) in which students raise mathematical questions and examples as a natural part of their classroom discussions.

Is it realistic to expect *all* students to be able to generate examples of newly-met concepts? Students in a low-attaining year 9 class were nearly all able to produce examples of a variety of mathematical ideas when asked, having become used to being asked for these by a teacher during only three lessons. Sometimes, unexpected examples would be produced which had been constructed rather than recalled from previous lessons. On one occasion, a student responded to the question 'do you know what a prime number is?' by offering a non-example with its factors and saying 'you can't do that with prime numbers'.

For a few years we have been working on ways to characterise teaching actions which engage students in active reorganisation of knowledge structures or creation of mathematical objects. The purpose of such characterisation would be to give teachers access to a range of strategies which they may adapt and develop, while recognising that every individual teacher's practice is particular to them (Watson and Mason, 1998). Many of the teaching approaches we have observed can be described as offering opportunities for extending personal example spaces.

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[1] A translation of Dyrszlag's suggestions, provided for us by Anna Sierpinska, can be found in Watson and Mason (1998 p31)

The individuality and situatedness of example spaces

When we consider a concept, a paradigmatic image might come to mind and, as we have already suggested, this image might have special features which make it rather less general than we would hope. For instance, a rectangle with sides parallel to the edges of the paper appears when someone says "quadrilateral", a unit fraction appears when someone says "fraction", a decimal number including lots of 9s appears when someone asks for a number very close to 1. Some of these dominate because they have been presented frequently to learners through texts, others because of recent exposure. For some learners, central images have an affective dimension: they might be associated with a favourite teacher, or a particular memory of a page of a textbook. Hence, although a significant number of a group of teachers drew 'page-parallel' rectangles with sides roughly 2:1 when we asked them to draw a quadrilateral, because it is a dominant image in (and outside of) mathematics texts, not everyone did. For some, other influences were brought to bear. Their experience of mathematics, and of us, led them to be more creative or devious in what they drew. Some drew a square, because their interpretation of the task was that something very special is required; some felt too insecure to draw anything because they did not know the success criteria. We then asked them to draw further quadrilaterals which satisfied more ands more constraints, structuring our request so that eventually they had to abandon their rectangles and construct shapes from properties, rather than just searching through their existing categorisations and figural examples.

What comes to mind for individual learners when working on known concepts, or when asked for examples of mathematical objects, can relate to dominant or figural images in the topic, but can also be influenced by past experience, preferences, interpretations of what is required and what is valued. The teacher trainees who drew segments of circles as fractions may have been interpreting the task as about offering images to their students; or believing that an image had to be a picture. The post-graduates offering $|x|$ were giving it as a starting point for whatever was to come, not as an illustration of their total knowledge.

Similarly, how dominant images provide access to other examples and images varies from person to person. We asked a group to find families of quadratics which have the same 'inter-rootal' distance[2] One, a secondary mathematics educator, imagined a family of quadratic curves, all the same 'shape', related to each other by lateral translation and then paused, hoping to find something else as well. Another, who worked much of the time with dynamic images, saw these as essentially the same curve. For him, the family of translations was one example, not infinitely many. His focus was on generating curves which were different shapes but were all 'pinned' at two fixed points on a notional x-axis. Both of these knowledgeable mathematicians knew about all the curves that would have this property, but their senses of what

[2] We define 'inter-rootal distance' as the distance between the roots along the x-axis

could be varied and how to vary it were different because of their different ways of working in that moment. Their examples were the same, but they were in differently structured spaces in terms of relations between examples.

These stories from our experiences of working with others illustrate the potential for variety in the examples, and their structuring, which are evoked in response to teachers' prompts. From these and many similar experiences we deduce that example spaces come into being in response to prompts, and are differently structured, with different contents, for different learners, at different times and in different contexts.

Extending example spaces

Individual spaces, situated by time, place, people and prompt, provide the starting point for students' work. Even with a new topic, there are connections with previous knowledge which will bring images and examples to mind. Whatever the teacher says, students will work with their own example spaces, and they will try to fit what is said into them. Sometimes learning involves restructuring the contents of such a space; sometimes learning involves extending the space to include new items. Consider this prompt: *Find two numbers such that the square of the smaller is larger than the square of the larger.*

For some students this may well be a mystery, because their example spaces of numbers only include positive integers; in order to answer this they are going to have to begin extending it to include some other numbers – but what other numbers? Responding to this prompt might lead to positive numbers between 0 and 1, but it might also raise the question of what is meant by 'smaller' and 'larger' when negative numbers are considered. The search for two appropriate numbers, probably aided with a calculator and number line or graph, encourages an extension of the example space which may be lasting for some students, or ephemeral for others.

The prompt to find functions with the same inter-rootal distance can lead people to try to restructure their example spaces, some wishing to remove the limitations of fixed axes from their spaces, others wishing to see curves as representing an infinite number of translated versions of themselves from now on.

Conclusion

We have illustrated how seeing teaching/learning mathematics as creation and extension of personal example spaces can inform the construction of tasks in which students can work directly on their own mathematical structures and relationships. Achieving competence in mathematics can be seen as the development of complex, interconnected, accessible example spaces.

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