EXPLORATION OF AN ART VENUE FOR THE LEARNING OF MATHEMATICS: THE CASE OF SYMMETRY

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This study examines an art venue for abstract mathematics thinking, specifically, students' understanding of symmetry groups in the context of a course for art students. The findings suggest that when working with repetitious geometric art designs students' approach of various symmetries often go beyond the one-to-one correspondence between specific pieces of art and particular transformations, but they also develop a correspondence between types of art designs and abstract types of symmetries. Further, familiarity with symmetry groups allows students to use abstract mathematical thinking as a tool in their artistic creations.

The idea of viewing mathematics through an art lens can be appealing at many levels. People who find joy in doing mathematics advocate the natural connections between mathematics and art (e.g., Huntley, 1970; Kappraff, 1991; Linn, 1974). Convincing examples are provided by the complexity of Islamic mosaics in the Alhambra, the Greek canons of harmony and beauty, and the magnificence of many architectural designs. Renaissance artists like Leonardo da Vinci, Michaelangelo, Durer, and Piero della Francesca were not only master painters, but skilled geometers as well. In later years, mathematicians such as Kepler, Euler, Gauss and Fermat, studied problems related to art, including tilings, and 3-D solids. More contemporary designs, for example, the works of M C. Escher and A. Fomenco, are coming from the perspective of artists drawing inspiration from complex mathematics rules, and mathematicians who use art as a means to express certain mathematical abstractions, respectively, and they illustrate the strong bond between mathematics and art. An exploration of the natural connection of math and art is now being proposed in the service of education. The idea of "learning mathematics by making art" comes to advance our thinking on mathematics and on pedagogy. Mathematics educators suggest that the integration of arts with mathematics may liven our classes and may also be motivating to some students for whom the "traditional" curriculum and pedagogy has not proven fruitful. This vision is also presented by the National Council of Teachers of Mathematics (2000) who calls for the introduction of extended projects, group work, and discussions among students, as well as the integration of mathematics across the curriculum – elements of a learning environment that seems more similar to a studio art course rather than to a traditional mathematics class.

In recent years, there have been a few examples of interventions that demonstrate the power of the vision that integrates art and mathematics. Yet, little of this work has been done at advanced levels; the bulk of the research in this area has been conducted in elementary and secondary school classrooms. For example, in a quantitative study at the elementary school level, Willett (1992) demonstrated that mathematics learning can be very effective in the context of arts-based lessons. Similarly, Loeb's visual mathematics curriculum (Loeb, 1993), though it has not been studied formally, provides substantial anecdotal evidence which supports the study of the formal mathematics of symmetry through a design studio as an effective learning environment for undergraduate students. The "Escher World" project (Shaffer, 1997) explored successfully an open learning environment created by combining mathematics and design activities.

Here, we explore one more example of a learning environment created by combining mathematics and art activities in an art studio-like environment. We designed a course for students interested in arts and design, with the emphasis on understanding mathematical concepts by self-explorations in art (Grzegorczyk, 2000). Instead of introducing theoretical concepts, the course created a vast reservoir of art-related examples and hands-on experiences. The course contained a wide variety of accessible yet challenging problems, and served as an introduction to systematic and complex mathematical thinking, experiences and discovery used in contemporary and classic arts. One of the main themes in the course was symmetry in mathematics.

The focus of this article is to examine one specific aspect of this art venue for mathematics learning, specifically to document students' understanding of symmetry groups. While our broader goal is to study the process by which mathematics learning takes place in environments where it is approached through art, and to further develop a model for this type of teaching, here we focus on the learning of symmetry and the implications that this may have on the overall mathematics learning and attitudes of students towards mathematics. Specifically, our goal was to examine the ways, strategies, and approaches college students use art as a gateway in their attempts to develop an understanding of design classification and symmetry (through crystallographic groups).

METHOD

Participants and course activities: Twenty-five undergraduate students who major in art studies participated in a one-semester mathematics course. Students were asked to work on short problem investigations, or extended projects with increasing sophistication and complexity. These included either the study of existing artwork to develop mathematical insights, or the use of mathematical ideas to develop original art and design works. Students worked on these projects and then presented their work to their peers for discussion, questions, comments. The class was equipped with commercial and shareware drawing and image-manipulation programs as well as geometry and symmetry software.

Mathematics of the course: Central to this course was the development of the concept of various types of two and three-dimensional symmetries, a topic that is significant both in art and in nature. This was motivated by discussions on designs produced by individual artists and traditional designs from various cultures around the world. Students entered the course with a vague understanding of the concept of symmetry, one that was often disconnected from mathematics. As their discussions on works of art and the complexity of their designs progressed, they were encouraged to use precise language and to clarify the term "symmetric." It became apparent that the class would benefit from a systematic and deeper study of this broad topic. Students were invited to study symmetries as *rigid motions of the plane*. They identified the symmetry motions of small (finite) designs using polygons as a main example, i.e. they studied various types of rotations and reflections, and their compositions – a result of one motion followed by another. The classification of designs with various symmetries and the development of symmetry group tables, led to the underlying definition of the *dihedral groups* and *cyclic groups*, and, finally, to the definition of an *abstract group*. Students verified the associative, identity, and inverse properties of the transformations for various symmetry groups, and re-visited the art designs that initially motivated the study of group symmetry and discussed how their new understanding of group theory may assist them in gaining new insights in art. They generated their own designs using symmetric transformations, and discussed the utility of group theory with respect to the generation of repetitious designs.

The symmetries of strip patterns were analyzed next to discover that all seven symmetry groups for repetitious (periodic) strip patterns. The natural next step was to study plane patterns such as wallpaper and quilting patterns, that is, designs where the repetition of the theme occurs in more than one direction. It was noted that there are exactly 17 plane symmetry groups, which correspond to 17 types of patterns and could be analyzed or generated using various symmetries. The sophistication of the students and complexity of the patterns studied was increasing. The notion of the fundamental region for a pattern lead to the study of tilings – when is it possible to cover the plane with a single tile or combination of tiles. The most intriguing and interesting tilings of Escher especially lent itself for symmetry investigations on tiling. Finally, the study of symmetric groups was extended to three-dimensional solids (mostly Platonic solids) and their tiling.

Data collection and analysis: For this study we collected students' sketches and designs from the workshops for review and analysis. Further, students were administered a pre- and a post-test that focused explicitly on their understanding of various symmetries and were asked to complete a survey which focused on the students' attitudes towards mathematics and the class.

Responses to each problem were first coded for mathematical correctness. When responses included images, then these were coded for the presence of mathematical principles and their correctness. When responses included verbal descriptions, these, too, were coded for the extent to which mathematical ideas and concepts were used and, subsequently, for the correct use of mathematics. In this case, we differentiated between explicit, formal use of mathematics concepts and language and general, or vague references to mathematics. We also differentiated between different strategies that students employed when they approached tasks or designs that involved the use of symmetry in various forms. Finally, when coding was complete, frequencies were tallied for each code and pre- and post-interview totals were compared.

RESULTS

Recognizing and using symmetry. Our first coding scheme concerned the overall recognition of the existence of symmetry in designs and students' subsequent attempts (or lack thereof) to utilize symmetry principles. To code students' understanding of symmetry we used the criteria suggested by Gardner, i.e. the "ability to use ideas in appropriate contexts, to apply ideas to new situations, to explain ideas, and to extend ideas by finding new examples" (Gardner, 1993; Shaffer, 1997). Using this definition, our results indicate that all students were able to make designs using mirror symmetries and rotational symmetries (Table 1). In fact, students reported that they regularly used symmetry in their designs, both in and out of the course. By contrast, early in the of the course (pre-test), when asked to make a design which incorporates mathematical ideas, only 2 students used symmetry. The remaining 23 students used, or made references to mostly measurement concepts (dividing areas in parts, measuring) and, in a few cases, attempted to construct polygons.

Students also developed an ability to apply the concept of symmetry to their analysis and discussions of various designs and art works. Before the course, students made references to symmetry an average of 0.36 times (total of 9 references made by 25 students) while describing two images (see Figure 1). Note that all references to symmetry were made with respect to mirror symmetry only; no mention was made to either translations, rotations or glides. As the course progressed, all students mentioned symmetry, and the mean rose to 3.16 references over the same 2 images (79 references). These included references to mirror reflections (59 references) and rotational symmetries (16 references), as well as to the lack of symmetries (see Table 1). But, perhaps, more striking is the qualitative change in students' responses to the designs they were discussing. Students used a richer, more formal and mathematical vocabulary to describe images. In fact, 4 students made direct references to symmetry groups (i.e., dihedral and/or cyclic groups). Indicative of this qualitative change in students' discussions are the comments of one student, Julie, whose pre- and post-course discussion of the designs appears below:

> Pre-test comments: "You can fold this design either up or down in the middle and each side is the same, like a mirror. There's lots of symmetry."
>
> Post-test comments: "This has 2 symmetry lines; they divide it in 4 regions. There are four pieces in the square. When you divide it in half you can rotate [each piece] by 180^0. Or each piece, each fourth, can be rotated 90^0. It's like a D_{11} group [Dihedral 2]. All dihedral groups have both symmetry lines and rotations. The other one has 11 regions. No symmetries. It's a cyclic group. C_{11}. All cyclic groups just have rotations. No mirror symmetries..."(note that Julie used the word "symmetry" to denote mirror symmetry)

Julie's first mention of symmetry was made in general terms – "there's lots of symmetry". At the end of the semester though, Julie made specific references to both mirror and rotational symmetries and connected these to a more formal understanding of symmetry using group theory terms.

Symmetry use	Pre-test	Post-test
Use of symmetry in doing art		
- mirror reflections	2	25
- rotations	--	25
Use of symmetry in analyzing art		
- reference to mirror symmetry	9	59
- reference to rotation symmetry	--	16
- reference to formal symmetry concepts	--	4

Table 1: Frequency in symmetry use

Approaching Symmetry via Art	Approaching Art via Symmetry
(a) Use of a local, one-to-one correspondence between one specific piece of art and an instance of a mathematical concept - recalling design examples when asked to discuss the mathematics - e.g., "Cyclic group C_4 is like *that* four-leaf rosette"	*(a) Use of local mathematical concepts related to symmetry to understand art designs* - recalling a collection of symmetry concepts when trying to understand the structure of an art design, or to create one. - e.g., Escher in *Pegasus* made use of glides only, so the entire print looks like it is made of strips
(b) Use of a global correspondence among types of objects - connecting broader math concepts to broad art concepts - e.g., "cyclic groups are like rosettes"	*(b) Use of global, abstract concepts related to symmetry to understand and create art* - deriving specific facts and concepts from abstract symmetry concepts to aid in art creation.

Table 2: Types of symmetry use

Task 1: Discuss the mathematics of the two designs shown below.

Task 2: Fill in the circle with a design and explain the mathematics of your design.

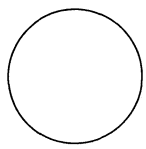

Figure 1: Two of the pre- and post-test tasks: Analyzing and creating symmetric designs

Towards a formal understanding of symmetry. All students developed their understanding of symmetry groups in the sense of recognition of symmetry in designs and in their ability to use symmetry to generate their own designs. Our coding suggested that students approached symmetry in two distinct ways: A number of students approached the mathematics of symmetry using art. Others, however, gave clear indications that, after a certain point, they used abstract mathematical objects as a gateway to art.

The former approach, math-through-art, did not surprise us; it was a clear expectation that students in this class engage in mathematical thinking and problem solving at a certain level, and art could be used as a tangible tool to help explain or apply the abstract concepts of mathematics. Since all of our students were well-versed in the language of art, we helped them build bridges to mathematics via art, so that underlying mathematics also becomes tangible and real. For example, students were expected to understand that cyclic groups have the characteristics of an asymmetric rosette – no mirror symmetries, but with a rotation symmetry, while the angle of the smallest rotation determines the type of the underlying cyclic group. This math-through art approach was manifested in two ways:

(a) recalling design examples when asked to discuss the mathematics (e.g., "Cyclic group C_4 is like *that* four-leaf rosette") – a local one-to-one correspondence between a specific piece of art and an instance of a mathematical concept, and

(b) connecting broader mathematical concepts to broad art concepts – a global correspondence among types of objects in art and in mathematics (e.g., "cyclic groups are like rosettes").

The latter approach, however, that is, using abstract mathematical concepts to understand art, deserves further attention. In this case, students used their understanding of a cyclic group to convince themselves that, once a design is identified as equivalent to a cyclic group, there is no need to look any further for mirror symmetries – they do not exist. Students were able to use *abstract* mathematical thinking in generating their own designs and to make *short cuts* in designing repetitious patterns using certain group properties. The art-through-math approach was also manifested in either a local or global perspective:

(a) Use of a collection of mathematical concepts related to symmetry to understand and create art designs – a local use of mathematics as a way to approach art. For example, a student may look at a given art design and look for the existence of mirror reflections, translations or rotations and attempt to understand the overall "behavior" of the art work, or attempt to reproduce the design using, partially, these symmetry principles. We often observed this behavior when students were exposed to the work of M. C. Escher – students examined the tessellation designs by looking for symmetries and, based on that understanding, attempted to reproduce Escher's work, or to create their own tessellation designs.

(b) Use of abstract concepts related to symmetry to understand and create art designs – a more global way to approach art via mathematics. In this case students derived information regarding a design by thinking of symmetry as a collection of organized and relatively abstract objects. This was a process of deduction. Students who developed an understanding of the structure of dihedral or cyclic groups systematically derived the characteristics of a group that were of use to them in their designs, or helped them justify their claims. The effect of this approach to the understanding or creating designs was obvious when asked to produce wallpaper patterns; a few students attempted to make use of their understanding of dihedral and cyclic groups – some students recalled that cyclic groups have no mirror symmetries and used this fact to produce wallpapers with rotational themes.

DISCUSSION

One of our goals for this study was to examine the potential of a learning environment created by combining mathematics and art activities, and, specifically to document students' understanding of symmetry. Furthermore, we aimed to examine the ways in which students come to understand symmetry via the use of art, their strategies and approaches. Willett (1992), Loeb (1993), Gura (1996) and Shaffer (1997) among others argued that an art studio can facilitate the learning of mathematics and the mathematics of symmetry can be a meaningful organizing principle when teaching a course in this setting. Indeed, we found that our students over the course of the semester developed their ability to detect the existence of symmetry in its various forms in art designs, and to classify various designs using familiar properties (in this case properties of symmetry groups). Furthermore, students regularly incorporated symmetry in their own designs. This involved not only the use of relatively elementary concepts such as mirror lines, but they also gave evidence of a thoughtful use of specific types of symmetries in order to achieve certain visual effects (e.g., students often explicitly mentioned that they *chose* to use a cyclic or rotational design to avoid having mirror images). Students' overall behavior suggests that they learned to appreciate an abstract approach to symmetry as a way to create designs and had begun to appreciate abstraction in mathematics in general.

Our second goal was to examine the *ways* in which undergraduate students use art as a gateway in their attempts to develop an understanding of symmetry. The findings suggested that students developed the ability to approach symmetry beyond the one-to-one correspondence between a specific piece of art and a mathematical concept, but they also develop a correspondence between types of art designs and abstract types of symmetry concepts. Further, familiarity with symmetry concepts allowed students to use abstract mathematical concepts as a tool in their art creation.

Finally, we focused on the learning of symmetry and the implications that this may have on the overall mathematics learning and attitudes of students towards mathematics. Students' comments suggested that they developed a positive attitude and, often for the first time, they could see a role for mathematics in their lives and their creations.

Overall, students who participated in our study learned about the mathematical idea of symmetry and discovered a new meaning for mathematics. An important question to ask is what facilitated the learning of mathematics in an art studio. Shaffer (1997) in a study involving middle school students in a similar environment suggested two venues to explore as factors in students' learning: the issues of control, that is the freedom to make decisions in one's own learning (Dewey, 1938) and expression (Parker, 1984). Shaffer suggested that expressive arts-based activities put students in control of their own learning. Observational evidence suggested that in our course, the freedom for students to choose the means in which to apply the concepts they find most useful in the context of art with which they feel familiar, also facilitated the learning of mathematics. Students often talked about using ideas of symmetry out of class in their own project and the transfer of ideas of symmetry to their art projects. Our results suggest a framework for thinking about the teaching of mathematics in the context of other courses. We believe symmetry to be only one of the many possible principles that can be used as powerful vehicles to build mathematical bridges for students that are often difficult to reach.

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