MANIPULATIVE HELP IN VERBAL SHARING OUT OF CONTINUOUS AND DISCRETE WHOLES PROBLEMS SOLVING

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Abstract. *This is the report for the study of a teaching program consisting on activities related to the use of fractions. The study was conducted on a small group of second grade primary education students at a state school. Students had received no previous teaching of fractions. In this investigation, we will show the feasibility of manipulative material and of the collective game as a type of tool to assist the student in building an initial understanding of sharing out (although it can not be take as the only tool). A questionnaire was given to the students as well, both before and after the teaching program. In order to prove that the results reported in the final questionnaire were accurate, various individual interviews were carried out on three children that participated in the teaching program.*

Introduction

In Mexico[1], like in other countries, fractions make up one of the most difficult areas in the teaching of math at the basic educational levels (particularly regarding primary and secondary school). Therefore, many students who complete their secondary education and go on to a higher level find themselves with limited knowledge of fractions, even though they have studied fractions since primary school. This problem has become evident from the results obtained from some recent studies (Fredenthal, 1983; Pitkethly and Hunting, 1996; Kieren, 1980, 1998; Valdemoros, 1993; Figueras, 1998; among others), and from the teaching experience of those that participated in this study. It is possible that one of the many factors that fall into this process is the teaching method, which is considered not only determinant but also a process through which the child acquires knowledge.

As we face this problem of complexity when students acquire fraction knowledge, we must ask: What teaching activities can a teacher design in order to promote or facilitate a rich understanding of fractions among students? This question has served to delimit and concentrate on the problems presented in this study, which focus on how the systematic school teaching influences a child's previous knowledge[2] regarding the sharing out and elementary notions of fractions.

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[1] Mexico: Fractions are first taught in the third grade of primary school and the didactic strategies are taught between the third and sixth grades. During this time, a broad semantic overhaul of the relationship between equivalences and order is developed along with an elementary introduction of addition and subtraction.

[2] Previous knowledge is made up of an organized set of notions, social representations, and hypotheses and/or theories that put into play the understanding of scholastic components and that function as true, assimilated instruments of such components.

Our hypothesis is set out as follows: *The manipulative material will provide the third grade primary school student with assistance in order to build a formal understanding of the sharing out upon his previous knowledge.*

Therefore, the objective of the present investigation is to identify and analyze the effects on students in the third year of primary school when solving sharing out problems in continuous[3] and discrete[4] models with the help of manipulative materials.

Theoretical Framework

Freudenthal (1983) states that the dividing into equal parts is a link to the mental construction of all types of magnitudes. While observing children from seven to eight years old, he realized that they are capable of estimating the half or one third of an irregular shaped area. Freudenthal considers this ability to divide as an important component in the mental area object.

On the other hand, Kieren (1983, 1984, 1988) recognizes that the division, the equivalence, and the formation of a divisible unit (constructive mechanisms) constitutes the cognitive basis for fraction language of the part-whole relationship. In their studies, researchers like Kieren, Nelson and Smith (1985), Figueras (1996), Piaget and Inhelder (1966), Hunting and Korbosky (1990), Pitkethy and Hunting (1996), and Pothier and Sawada (1989) have separated the strategies children use to solve partition and sharing out problems.

Freudenthal (1983) states that the formation of mental objects should come prior to the acquisition of concepts because the mental objects lay the foundations for the acquisition of concepts. The phenomenon are first seen as mental objects and then are transformed into concepts. Using specific material, the mental objects are formed of phenomena and not of concepts. Comparing mental objects will lead to the concept. The specific material holds a temporary significance because, once a student has formed long-lasting and permanent mental objects, the specific material is no longer necessary.

Researchers like Behr and Post (1988), Behr, Wachsmuth and Post (1998), Kaplan, Yamamoto and Ginsburg (1989), Bezuk and Cramer (1989), and other researchers, have used manipulative material when teaching fractions. They have suggested as well that it be used with primary school children and have given teachers both orientations and suggestions for correct usage.

Likewise, Kamii (1985) introduces three teaching principles: autonomy, daily situations, and collective games. She points out that if the teacher considers autonomy as the main purpose of teaching, he must foster the exchange and coordination of the students' points of view, allow them to express their thoughts freely, to make their own decisions, encourage them to think on their own instead of

[3] Continuous models: Fractions are a way to relate one part to the whole and are put into context with geometric figures in that the fraction symbolizes a part of the unit (Hart, K. 1981).
[4] Discrete models: Fractions are a way to relate a subset of a set in which it is included and it is put into context in sharing out situations.

reciting correct answers, let them debate his answers. Moreover, he suggests that everyday situations happening in the classroom should be taken advantage of by the teacher, since they are great opportunities for arithmetic. In the same manner, Kamii says (1985, 1994) that collective games are an essential part of constructive teaching. Games provide a way of organized playing through which, children are encouraged to think in numerical combinations and remember them. They also allow children to develop their ability to govern themselves, foster social interaction, and provide feedback among participants.

Methodological Aspects

In the Mexican school, as mentioned in the beginning of this text, the teaching of fractions starts officially in the third grade of primary education; however, at the time of this study, we became aware of the fact that this subject had already been taught to third graders. Consequently, this study is carried out on a second grade class during the last period of the school year (may – June). This period was chosen because it would be certain that there be no previous teaching of fractions. This study is of a qualitative kind, due to the fact that an analysis of the progress of a small group of children in their natural environment (the classroom) is enclosed in this study, in order to recognize the effects produced in their thinking during the process and termination of the application of a brief teaching program focused on sharing out.

The study began with an exploratory exam given to 28 students, with the purpose of getting information about the previous knowledge on fractions the children had. Moreover, the questionnaire allowed the selection of 10 children who participated in the study and whose spontaneous and intuitive knowledge on the subject was recognized before the start of the teaching program.

The questionnaire included 13 tasks organized in five blocks, with sharing out situations represented by pictograms[5]. In the first block, all are distributed continuously among several objects or several people. In the three following blocks, all are distributed discretely among several people, the result of the sharing out may be less than one or more than one. The last block focuses on the comparison of different sharing outs and on the reconnaissance of equivalence. The following is a task from the last block: From this cookie drawing, Julie will eat one fourth, and Michael will eat two eighths. Who will eat more, Rosa or Luis?

The study's second instrument was the teaching program, with a constructivist focus, which basically considers that children learn and develop as they build accurate concepts of their study subjects. This building includes the child's active attitude, his availability, his previous knowledge in an interactive situation where the teacher's role is that of a guide and mediator between the child and the culture (Solé and Coll, 1999).

[5] Pictograms: drawing representations of qualitative and quantitative aspects of a certain problematic situation involving the use of fractions. Operations can also be made with the use of pictograms (Valdemoros, 1993).

The didactics considered for the teaching program is Kamii's proposal (1985), presents three teaching principles: autonomy, everyday situations, and collective games. From this point of view, the purpose of the teaching program was to create a proper environment where the child could adequately develop the activities proposed in the work sessions and that would allow him to acquire experiences in which he could establish different types of relations that would help him building basic knowledge on the subject of fractions.

The tasks that make up the teaching program were designed taking into account the results obtained from the initial questionnaire (the child's previous knowledge on the learning content) and the purposes of our research. The activities included are related to the context of everyday situations and collective games.

It was carried out in twelve 45 minute sessions[6] every three days. The activities developed in the work sessions were: covering a figure's surface with other equal figures, building a figure with other equal figures, figure-folding to obtain two, four, and eight equal parts, partition of figures in two, four, and eight equal parts, comparison of a figure's fractional parts, identification of a figure's fractional parts, writing the fraction that represents each part of the fractioned unit in halves, thirds, quarters, and eighths, the game used for the exchange of ideas and coordination of points of view for the debating of answers as a feedback process of acquired knowledge, and the solving of verbal sharing out problems.

In said activities several manipulative materials were used, some were acquired, others adapted and others were designed. The manipulative materials were used in the first ten work sessions but not in the last two. Below, we enumerate some of them.

Pattern Blocks is made of prisms in different colors, shapes, and sizes that can fit together to build other figures. It was used in activities to cover and build figures. Geometric figures made from sheets of paper in different colors and sizes (squares, rectangles, and circles), they were used to carry out activities of partition in half, quarters, and eighths. They were also used to represent objects such as: chocolates, cakes, cookies, jellies, etc., in sharing out situations.

The "figure resaque" was taken from the Montessori didactic material called "metallic resaque", with several modifications. It was used for comparisons of fractional parts, which fraction is larger than or equal to another.

A domino set was designed in plastic, with the characteristics of a common domino. The difference lies on the fact that it is related to the fractions: ½, 1/3, ¼, 1/5, 1/6, and 1/8, and it was used to bring about the exchange of ideas and coordination of points of view among the children, in order for them to debate their answers as a feedback process of acquired knowledge from previous activities.

[6] Sessions: the number of sessions developed in teaching is equal to the amount of time which is officially dedicated to these topics.

After applying the teaching program, the initial questionnaire was applied again to the ten children who participated in the working sessions. The period of time which went by between the first and second application was of five weeks, which makes the solution of the second one based on the first one impossible. The purpose was the assessment of the advances achieved during the teaching program. Based on the results obtained in the last questionnaire and the participation of the teaching program, three students were chosen to carry out an interview with each one of them.

The purpose was to verify if the given results were solid in the last evaluation, as well as to search for an answer to some situations manifested in said evaluation, which weren't fully appreciated. Likewise, the interviews allowed the strengthening if the manipulative material is required by the student when developing his/her resolution strategies of the given tasks. The interviews were semi structured (according Cohen and Manion, 1990), due to the fact that there was a previous protocol and the needed material for each case. The general nature of the interviews may be considered as an evaluation instrument, which means that it was carried out *a priori* of the last questionnaire application so as to verify that the answers to the tasks presented by the students in said evaluation were sound. The purpose of this interview evaluation is different from Piaget's exploratory one (1978).

The interview's design consisted of the presentation of five semi structured tasks. Pictograms were not shown on them so that the children could solve the problems using different strategies. The three first tasks are presented in sharing out situations, the result of the first and the third ones are less than one and the result of the second one is greater than one. The fourth task presents sharing out situations in which relations of order and equivalence between ½ and ¼ of a whole are established. The last task establishes the equivalence between ½ and 2/4 of a whole. Next, we present the fourth task: from a whole pie, Mrs. Linda bought a half and Mrs. Rosa bought a fourth. ¿Who bought more pie?

In order to validate this study, the results obtained in the questionnaire were considered. Such questionnaire included the one of the interviews and the observations of the research responsible and the one of another observant. The compiled data in the questionnaire , the observations in the teaching sessions, and the observations and data obtained in the interviews were compared in the previous analysis.

Results

In the analysis of the initial questionnaire, the children resorted to their previous knowledge (acquired from experiences of their family and the society) in order to solve the tasks. They did not come across with any obstacles to carry out equitable partitions in two and four parts of a continuous quantity. Nevertheless, most of them presented problems when fractioning in three equal parts a continuous whole, manifesting non-equitable partitions; despite the fact that the task's sentence specifies that "all the persons must get the same". They also had problems when solving sharing out activities of a discreet whole among x number of persons.

The children who participated in this evaluation showed greater difficulty in the activities when establishing the relation of order and equivalence between the obtained parts in two different situations of sharing out. It is safe to state that the few experiences which with the second grade students count, in regards of the equal division of continuous wholes limit their skills to develop appropriate sharing out strategies. These students also showed lack of vocabulary in order to name the fractional part in the distributed whole, which reveals that the cultural contents of their previous knowledge was really low.

While developing the teaching program, teamwork qualified children to exchange their ideas, carry out decision-making, discuss their answers, as well as accept their own mistakes and their knowledge feedback. The manipulative material was required by the children so they can solve problems; sometimes it was used as a feedback support when they had doubts of what they were doing or when they wanted to prove their answers.

The individual work brought about development in discovery and creativity skills when the children solved the tasks by themselves.

The result analysis given by the second application of the questionnaire, confirms the advances achieved by the children within the teaching program. The difference between both evaluations is self-explanatory, not only due to the number of right answers per student, but also because of the processes of significance, representation, and partition which they develop in order to solve the tasks.

Regarding partitions, the children carried out equitable partitions in two, three, four, and eight parts of continuous and discrete. Wholes in matters of sharing out, they made exhaustive equitable situations of sharing out in two, three and four equal parts. They established the order and equivalence relations between two different situations of sharing out. The students used symbolic fraction expressions to name the fractional part which came as a result of distributing the whole.

The capacity shown by the children when trying to solve these tasks by means of the appropriate procedure in the second application of the questionnaire, may be considered as a background for a development in their mental structures. To seek and find the answer to a problematic situation by themselves is a sign of improvement. With the aforementioned information, we may state that the teaching program fostered the usage of more appropriate partition and sharing out strategies.

After assessing the results obtained in the last questionnaire, we carried out the interview of three children: Manuel, Ana y Lizeth in order to enquire about the answers to some situations which manifested themselves in the evaluation and which were not possible to analyze in depth.

Manuel solved the tasks mentally, providing the correct answer for each and everyone of the tasks given to him. We believe that the child effortlessly constructed the mental objects related to the fraction, which allowed him to do without the manipulative material. He also learned the meaning of fraction as related to with

sharing out. This was evident when he mentally manipulated the fractional parts of a unit obtained by using a partition. He also established the order and equivalence relations between two different situations of sharing out made in one same unit.

Ana had to use the manipulative material to represent each problematic situation in order to visualize it first and then continue to construct the mental strategy. The educational assistant helped her in developing the correct process for each activity in order to get the correct answer. She was able to identify the existing relation between the numerator and the parts given to a person. She also found the correspondence between the denominator and the parts into which the unit was divided. The generalization of the concept was made in an intuitive fashion. But she was not able to recognize the equivalence relations between two different situations of sharing out done for one same unit, despite using the manipulative material to develop strategies. There's no doubt she will require more teaching time to attain a solid learning of the relations of equivalence between fractions.

Lizeth used pictograms to develop her strategies and to solve the tasks. The representations of the problematic situations by means of pictograms, made getting the right answers easier. She also managed to mentally establish the relations of equivalence of two different situations of sharing out made for a single unit . Such pictographic representations proved to be a good aid when explaining her thought.

Conclusions

This teaching program fostered and developed the basic semantic contents related with fractions, helping the children to construct their own knowledge based on daily life experiences. During the development of the teaching program , the students successfully handled diverse geometric shapes with the aid of manipulative material, involving several modes of sharing out in: halves, fourths and eighths. In the final stage of the teaching program when solving sharing out problems, the students were able to use the technical-symbolic language with clear support in diverse significance processes.

By using the manipulative material, the children managed to establish the relations of order and equivalence between fraction couples. The manipulative material, geometric shapes, divided into halves, thirds and eighths were useful when the child wanted to identify and write the fraction represented by each fractional part of the unit. The educational assistant, "domino", fostered debate and exchange of ideas between the students for them to have a feedback in their acquired knowledge.

After building some basic fraction notions, the students broadened their knowledge when solving sharing out problems using different strategies without the aid of manipulative material. This research shows that the study of fractions in primary schools may be taught in a very propitious fashion through partition, sharing out and equivalence activities for which the manipulative material have proven to be extremely useful.

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