

# AN ALTERNATIVE MODEL FOR THE DESCRIPTION OF COMPUTATIONAL ESTIMATION STRATEGIES

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*In this study we have analyzed the estimation strategies employed by preservice elementary teachers. It has been used an alternative model for the description of computational estimation strategies. In this model, strategies are integrated by approximation skills, mental computation algorithms, cognitive processes (reformulation, translation and compensation) and metacognitive processes (as the assessment of the outcome). The adoption of this model allows making the identification and characterization of estimation strategies and to complete a systematical classification of the strategies, attending to the estimation processes intertwined with them.*

In the analysis of strategies used in computational estimation tasks, Reys, Bestgen, Rybolt, and Wyatt (1982) identified three high-level cognitive processes that are intertwined with these strategies: reformulation, translation and compensation. Reformulation is "the process of altering numerical data to produce a more mentally manageable form. This process leaves the structure of the problem intact" (p. 187); translation is "the process of changing the mathematical structure of the problem to a more mentally manageable form" (p. 188); it is considered that there is a change in the mathematical structure of the problem (and, in consequence, a translation process) when the substitution of the initial data produces a change in the computation algorithm employed to find the result; this change can be produced in the operations effectuated or in the order in which the operations are executed. Finally, compensation is manifested in the "adjustments made to reflect numerical variation that came about as a result of translation or reformulation of the problem" (p. 189).

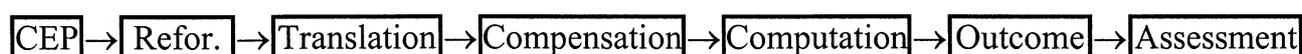
In most of the investigations devoted to the identification and characterization of computational estimation strategies (Dowker, 1992; Hanson and Hogan, 2000; LeFevre, Greenham, and Waheed, 1993; Lemaire, Lecacheur, and Farioli, 2000; Levine, 1982; Reys and et al., 1982) it has been considered that the "specific strategies" are the approximation skills used to substitute the initial data by other more mentally manageable (rounding, truncation, substitution of a decimal number by a fraction, use of compatible numbers, etc.). However, they have also been considered "specific strategies" the computational algorithms ("proceeding algorithmically" and "incomplete partial products"), the properties of the operations (distributive strategy), the general estimation processes (intermediate compensation, final compensation), and even the performances of the students in which is manifested that they are not accomplishing an estimation (trying to make an exact

mental computation, or the cited “proceeding algorithmically”) or those in which there is not a genuine mathematical procedure (attempting to guess the result or refusing to solve). This absence of criterion when making a systematical classification of the estimation strategies leads to the fact that each author uses a different classification and causes that it is difficult to make a coherent synthesis with the results of these studies on strategies.

On the contrary, this study is different from the previous one in the sense that the term "strategy" is taken here in a broader sense. We consider the approximation skills as constitutive elements of the strategies, but within them it can also be included the estimation processes, the mental computation algorithms and the assessment of the outcome. Adopting this position enables us to include –within estimation strategies– metacognitive processes as those described by Sowder (1994). This authoress asserts that: “individuals who are considered to be successful at computational estimation are usually characterized as flexible, self-confident, tolerant of error in estimates and [...] seek the reasonableness in results” (p. 142). Thus, the subject that accomplishes an estimation must be capable of choosing in a flexible way a strategy adapted for the estimation problem and assessing the process (modifying it if it would be appropriate) as well as the result (examining its reasonableness). Sowder (1994) considers the flexible choice of strategies and the valuation of the process and the result as examples of "self-regulation" and "self-monitoring" that constitute metacognitive processes.

The present study analyzes the computational estimation strategies employed by preservice elementary teachers through a model described in Segovia, E. Castro, Rico and E. Castro (1989). In this model, all the estimation processes identified in previous studies are collected. In the approach adopted in this work, producing an estimation consists basically in: Substituting the original data by approximations that enable to reduce the complexity of the calculations, maintaining the necessary proximity to the exact result; applying a mental computation algorithm to these approximations; effectuating a compensation (previous or subsequent to the computation); and making a valuation of the obtained result. As has been exposed, it depends on the type of substitution that is made with the initial data, and if this implies (or not) a change in the computation algorithm, we will face a process of reformulation or one of translation.

The characteristics of this model for the estimation strategies can be summarized in figure 1, taken from Segovia et al. (p. 152). In previous pages some general strategies that stem from this model are presented in various schemas. An example of them is the following diagram:



The main objective of this research is to describe the general strategies employed by preservice elementary teachers in the accomplishment of computational estimation

tasks without context, and proving if these strategies can be explained through the model described in the previous paragraph.

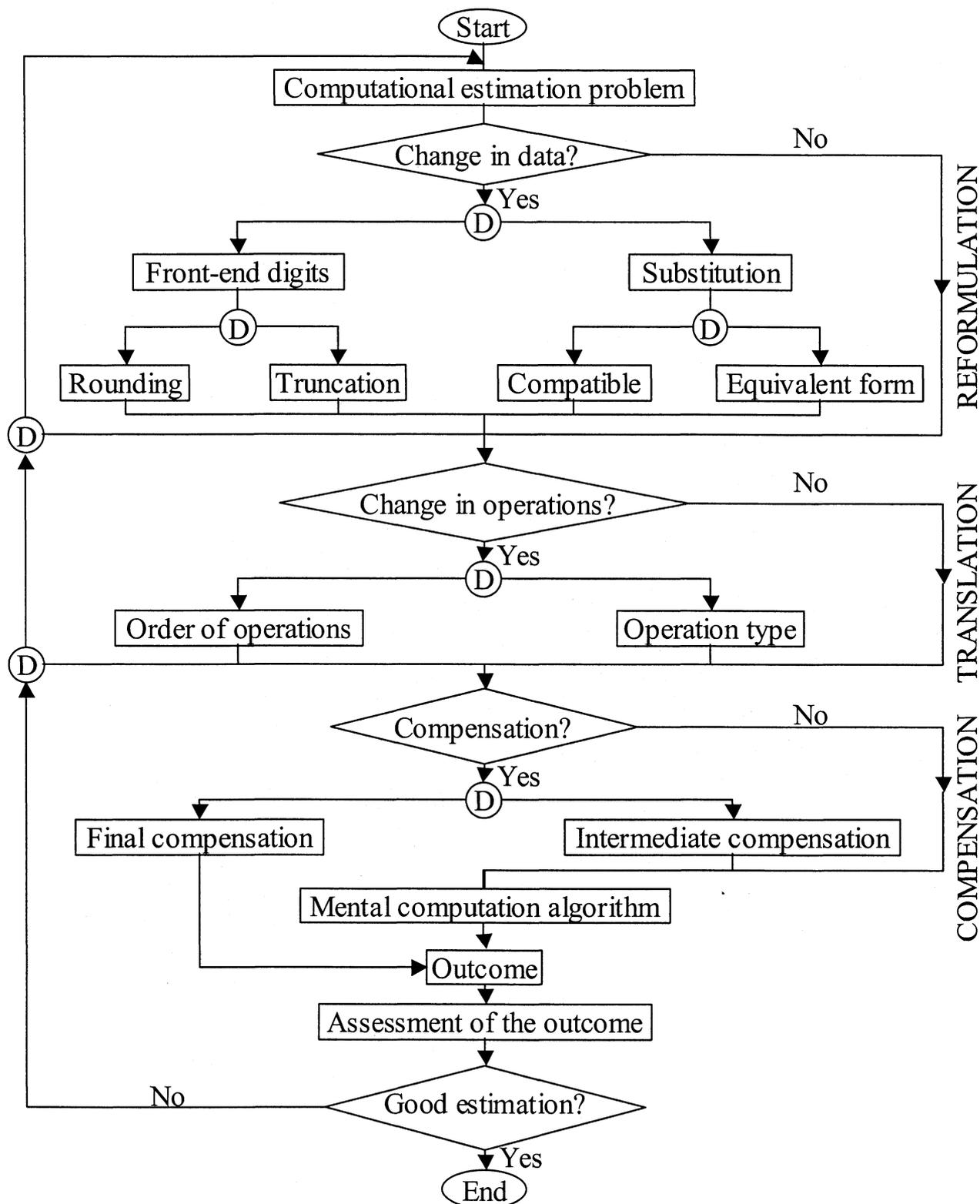


Figure 1. Flowchart of computational estimation strategies. The symbol D represents the points of the procedure where one must to take a decision

## METHOD

This investigation belongs to a most general work (Castro, 2001) in which it is studied the influence of number type in computational estimation tasks. With this objective it has been used the test TEA (Test of Estimation Ability) (Levine, 1982) that consists of ten multiplications and ten divisions and combines different types of numbers: whole, decimal greater than one and decimal less than one. In this section, it is given the description of the methodological part corresponding to the strategy analysis.

### Subjects

The sample used is constituted, in the general framework of the investigation, by 53 preservice elementary teachers of different first course specialties of the Escuela Universitaria La Salle, assigned to the Universidad Autónoma de Madrid. From this sample 10 subjects were selected for the strategies exploration. All the students had a period of instruction on computational estimation of 10 hours, during which they received teaching of computational estimation strategies and estimation was practiced in direct and applied computations.

### Instruments and application

After the administration of the Test of Levine (1982) to all the participants, the results were used to select subjects for the phase of strategy analysis that is described in this work. The analysis of the strategies used by the subjects was accomplished through an interview in which it was requested to the subjects to give an estimation for a proposed calculation and, afterwards, they had to explain the procedure used to produce their estimation. In the interview they used the items of the test of Levine (1982) in which appear decimal numbers less than one ( $187.5 \times 0.06$ ;  $64.5 \times 0.16$ ;  $424 \times 0.76$ ;  $0.47 \times 0.26$ ;  $66 \div 0.86$ ;  $943 \div 0.48$ ;  $0.76 \div 0.89$ ). The subjects were selected for the interview when their estimations –for the cited items– were all compatible (or all incompatible) with the adequate knowledge of the relative effect of the operations. We consider that an estimation for the calculation  $187.5 \times 0.06$  is compatible with an adequate knowledge of the relative effect of the operations, if it is within the interval (0.06, 187.5); in other words, if the student does not incur in misconceptions such as: "multiplication makes bigger" or "the division makes smaller". The interviews have been accomplished individually. They were registered using a recorder and transcribed to paper for their subsequent analysis.

## RESULTS

The strategies produced by the subjects have been classified taking into account the estimation processes appearing in them. Thus, there are strategies in which only it is given a process of reformulation; in other strategies, the reformulation is produced together with a compensation process (intermediate or final); a third group of strategies is formed by those in which we can detect processes of reformulation and translation; and finally, in some strategies we found all the estimation processes

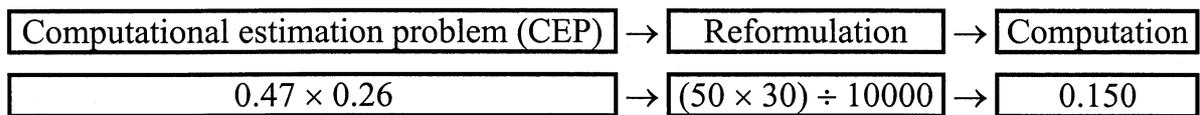
(reformulation, translation and compensation) considered by Reys et al. (1982). Now, we are going to present some examples of analysis of strategies belonging to each one of the cited groups. The analysis of each strategy begins with a fragment of the transcription of the interviews accomplished by the participants of the study, that comes accompanied with a diagram of the strategy.

### Reformulation

In some strategies, it has been only identified a process of reformulation. We propose the following example to illustrate this situation:

Interviewer:  $0.47 \times 0.26$

Student 41: 0.3... [Rounding the second factor]. I have multiplied 50 by 30, 0.150.



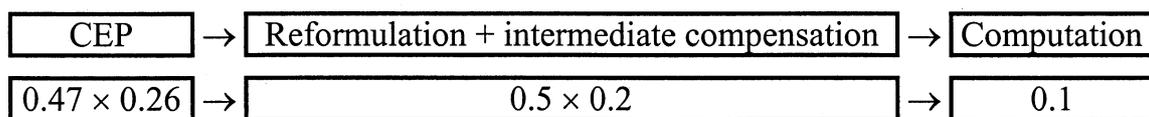
In this case, the reformulation has been carried out through the rounding of both numbers. Taking into account that both roundings have been accomplished upward, it would be suitable to make a final compensation "downward". However, it has been observed that some subjects refuse to accomplish any type of compensation. Maybe because of the complexity of this process, some subjects prefer not using it instead of taking the risk of committing an error, even in clear situations as the previous ones.

### Reformulation and compensation

In the following examples, besides a process of reformulation, we can find intermediate and final compensations accomplished for trying to "repair" the error produced in the substitution of the initial data in the reformulation.

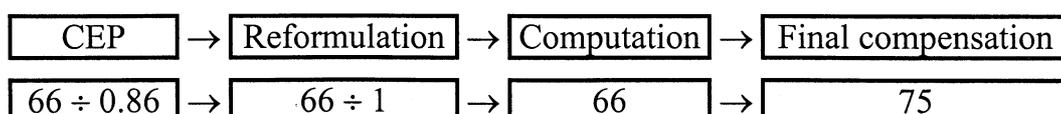
Interviewer:  $0.47 \times 0.26$

Student 2: 0.1. Rounding this [pointing 0.47] up and this [pointing 0.26] down.  $0.5 \times 0.2$ .



Interviewer:  $66 \div 0.86$

Student 42: Let's see. When we are dividing, the number increases ... This number [pointing to 66], if we divide it by one it doesn't change. But it's smaller and then this number makes bigger. Then it would be... 0.86 [the student returns to check the size of the divisor for adjusting the compensation] ... it would be almost ... 75.



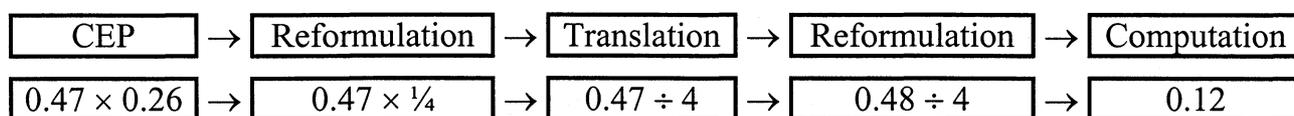
In the first case, 0.47 is rounded to 0.5 and, in order to compensate the increase in one of the factors, the other factor is rounded down. Thus, we have an intermediate compensation (previous to the calculation). In the second case the compensation is subsequent to the accomplishment of the calculation and is produced at the end of the process.

### Reformulation and translation

In other strategies the processes of reformulation and translation are combined. These strategies usually have a greater conceptual wealth, manifested in the abundance of relationships that are established (especially those which emphasize connecting decimal numbers with fractions).

Interviewer:  $0.47 \times 0.26$

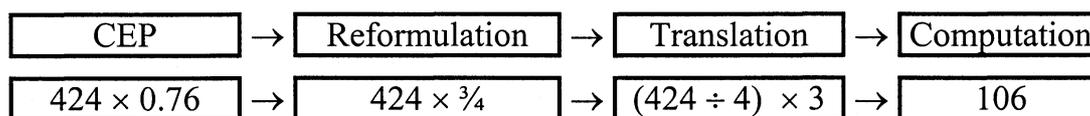
Student 53: Multiplying by 0.25 is approximately multiplying by  $\frac{1}{4}$ . Then, I divide 0.47 by 4, well... 0.12.



This strategy begins rounding 0.26 to 0.25. Then 0.25 is substituted by  $\frac{1}{4}$ . This substitution supposes a change of operation (multiplication by division) that implies a translation process. After translation, it has been realized another reformulation to simplify the calculation, substituting the dividend by a multiple of the divisor (use of compatible numbers). The first reformulation carries implicitly an "intermediate compensation" that has not been indicated in the diagram because it does not seem intended. Another example of translation is the following:

Interviewer:  $424 \times 0.76$

Student 53: Multiplying by 0.76 is approximately multiplying by  $\frac{3}{4}$ . Then, this is approximately... First I divide by 4 and get 106, then it is 106 by 3, 318.



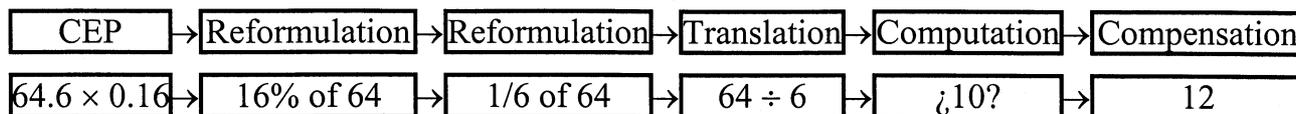
In this case, instead of multiplying by 76 (or by 7 or 8 in the case of using only one significant digit) and then dividing by 100 (or by 10), 424 is divided by four and then multiplied by three. Thus, in addition to changing the numbers in the operations, there is a change in the order of the operations and, by so much, changes the mathematical structure of the problem.

### Reformulation, translation and compensation

Only in some few cases, the three computational estimation processes –described in Reys et al. (1982)– have been combined in the same estimation strategy. In the following examples it is produced a substitution of a decimal by a fraction that thereafter leads to a translation process (because in both cases is produced an operation change).

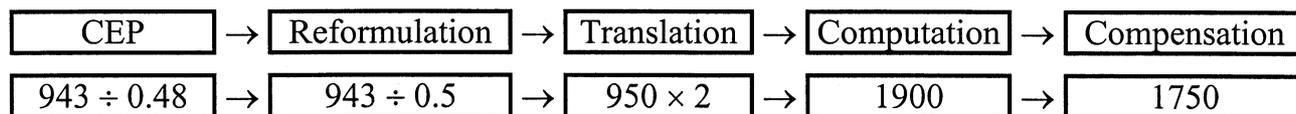
Interviewer:  $64.6 \times 0.16$

Student 42: This is 16% of 64, which becomes the sixth part; a little more... 12 would be my estimate.



Interviewer:  $943 \div 0.48$

Student 42: 943 divided by ... 1750. If we multiply this [pointing to 943] by 0.5 this becomes the double, it is like multiplying by two. And this [multiplied] by 2 is almost ... 1900, then would be a little less 1800 or 1750.



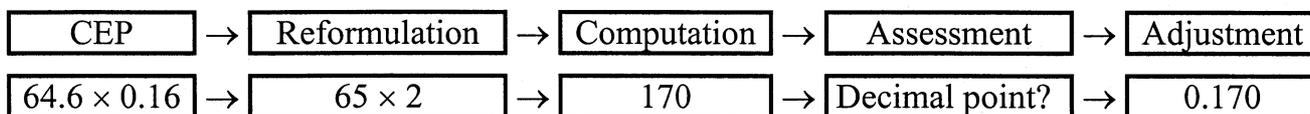
Concerning the two final compensations, there have been errors in the direction given to them.

### Metacognitive processes

Now we present examples in which it is manifested the presence of metacognitive processes as those described by Sowder (1994).

Interviewer:  $64.6 \times 0.16$

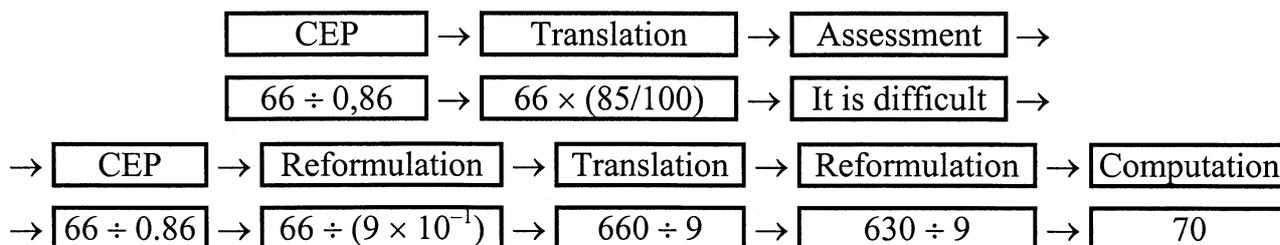
Student 45: Here I make... 64.6... I put 65 and 0.16... I put 2 and then they are 170. Then I put 168... the point? ... No, because it isn't the same 65 for 2 than... [The student adjusts the decimal point and gives the estimation] 0.170.



There have also been estimation processes where the subject has made an assessment conducive to a change in the strategy:

Interviewer:  $66 \div 0.86$

Student 53: 66 [divided] by 0.86 is like multiplying by 85 hundredths. Then I divide by 100 and remains... Oh! It's very difficult. I multiply this [pointing to 66] by... more or less... I'm going to approximate it to 0.9. Then  $9 \times 10^{-1}$ , it is a division, then this would remain me approximately 660 [divided] by 9 and will be... [9 multiplied] by 7... 63, is approximately 70.



In the second attempt, the subject uses the substitution through the use of exponents and this supposes a translation process. This substitution is complemented –as in the

two previous cases– with the use of compatible numbers, replacing 660 by 630 (that it is multiple of 9).

## CONCLUSIONS AND IMPLICATIONS

In this work, we have used an alternative model based on a "global" conception of the estimation strategies. In this model, strategies are integrated by approximation skills, mental computation algorithms, cognitive processes (reformulation, translation and compensation) and metacognitive processes (as the assessment of the outcome). The adoption of this model has allowed us to accomplish a classification of the computational estimation strategies (attending to the cognitive processes appearing in them). We think that this approach enables us to make the description and the analysis of the strategies produced in the performance of computational estimation tasks, in a more systematic way than the approach provided by previous theoretical frameworks. The results obtained in this study have permitted to validate the flowchart of strategies analysis shown as a comprehensive theoretical model for the estimation processes accomplished by the subjects of the sample.

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