A FRAMEWORK FOR THE STUDY OF INTUITIVE ANSWERS TO RATIO-COMPARISON (PROBABILITY) TASKS.

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ABSTRACT. This paper describes a framework for the study of strategies used in ratio-comparison problems, which was constructed for the analysis of adults' responses to double urn probability tasks. This framework involves two systems, one for the interpretation and classification of answers (strategies), and one for the planning of the numbers involved in ratio-comparison questions (situations). It was applied in an experiment with university students. Some results are reported, which refer to the relative occurrence of strategies, the difficulty levels of different situations and a classification of the subjects according to their performance.

The purpose of this article is to describe a framework for the classification of answers given in ratio-comparison tasks and for the categorization of the numbers involved in questions for such tasks. It was constructed in the frame of a research aiming at understanding how Mexican university students solve the "quantification" problems designed by Piaget and Inhelder (1951) in the context of a double urn probability task. In this research it became clear: a) that many of these adults cannot correctly solve what Piaget and Inhelder claimed to be solvable by the age of 15, and b) that the Piagetian categories were ineffective for the explanation of their behaviours.

Although the research was based upon the classical definition of probability, its main purpose was not to see whether the participant subjects use it or not, but to understand how they deal intuitively with the tasks. Piaget and Inhelder's (1951) work concerning children's conceptions of chance is at the foundations of the study. Fischbein's (1975) definition of intuition is used: it is a form of immediate knowledge which can appear to a person as obvious; it responds to a biological need for action and for certainty, from where it gets its characteristic reliability, stability and coercing nature. Tversky's et al (1982) concept of heuristics is also considered, as is their notion that people use a limited amount of heuristic principles to simplify the complex tasks of estimating probabilities and predicting values, but which lead frequently to serious and systematic errors. In the construction of the framework several references were considered, among which Noelting's (1980) classical orange juice experiment, and Falk et al's (1980) and Maury's (1986) researches on probability learning.

The adult subjects who participated in this study were shown two sets of cards, A and B, that were black or white on the obverse and red patterned on the reverse. When the subject had clearly seen all the cards, they were turned upside down and each set was shuffled separately. He or she was then asked: "Suppose there was a big prize if you took out a black card, but you only have one chance of choosing one card from one of the two sets. From which set would you choose a card: A, B or is it the same?" (No actual prizes were to be given, however). The subject had to make a decision and to justify it. When this was done, the number of black and white cards in each set was changed and the same question was posed. After some such experiments with actual cards, the subject was just shown drawings representing the sets as they were before the cards where turned upside down and shuffled, such as the one in figure 1.

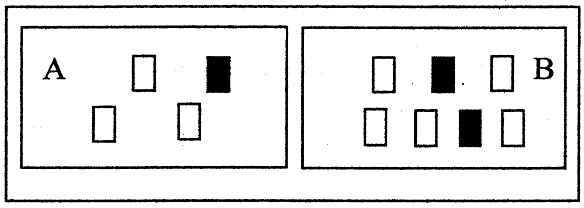

Figure 1: Array (1,3)(2,5)

A framework was constructed, which consists of two ways of envisaging these problems. One of them focuses on the answers given by the subjects; this generates a system of strategies. The other one focuses on the particular quantities involved in the questions; this generates a system of situations. Although different in nature, both systems have a correspondence with each other. The two sections of this article are dedicated to the presentation of the framework constructed and to the description of some results obtained from its experimental application.

THE FRAMEWORK

Each question is defined as an *array*, which is a pair of sample spaces S1 and S2; each of them is in turn characterized by a pair of *favourable* (f: black cards) and *unfavourable* (u: white cards) cases. An array is thus an expression of the form $(f_1,u_1)(f_2,u_2)$. Also defined for each array are the *total cases* ($n = f + u$), the *differences* ($d = f - u$) [1] and the *probabilities* ($p = {}^{f}/_{n}$) [2]. For instance, the question in figure 1 has $f_1=1$, $u_1=3$, $n_1=4$, $d_1=-2$, $p_1={}^{1}/_{4}$; $f_2=2$, $u_2=5$, $n_2=7$, $d_2=-3$, and $p_2={}^{2}/_{7}$. These definitions are used in the constructed framework, which will be described here: the systems for strategies and situations, as well as the correspondence between them.

Strategies

Strategies are understood here as solution mechanisms used to solve problems, that have a certain structure or logic and that can be reproduced for other problems of the same type. They are structured heuristics in the sense of Tversky et al (1974) [3]. Strategies are classified as simple or composed; simple strategies are in turn grouped in centrations and relations [4].

Centrations. The subject concentrates on only one class of elements of the array: the total, the favourable or the unfavourable cases. In each instance, the subject can decide to choose the side where there are more elements of the class (*positive centration*), or where there are less elements (*negative centration*), or he or she can answer "it is the same" if in both sides there is the same amount of the considered class of elements (*equality centration*). There are nine different centrations, grouped in three families:

{CN}. Centrations on total cases: Choosing the side where the total amount of cards is smaller ({CN–}) or larger ({CN+}), or saying "it is the same" because in both sides this amount is the same ({CN=}).

{CF}. Centrations on favourable cases: Choosing the side where the amount of black cards is smaller ({CF–}) or larger ({CF+}), or saying "it is the same" because in both sides this amount is the same ({CF=}).

{CU}. Centrations on unfavourable cases: Choosing the side where the amount of white cards is smaller ({CU–}) or larger ({CU+}), or saying "it is the same" because in both sides this amount is the same ({CN=}).

Relations. The subject considers simultaneously two classes of elements, and after establishing two relationships between them, he or she compares the results. These relationships can be based on order comparisons, or they can have an additive or a proportional nature [5].There are ten different relations, grouped in three families:

{RO}. Order relations: Choosing the side where the black cards prevail over the white ones, whereas in the other side either the white cards prevail ({ROwl}: win-lose) or there are as many black cards as white ones ({ROwd}: win-draw). Or choosing the side where there are as many black cards as white ones, whereas in the other one either the white cards prevail ({ROdl}: draw-lose) or the black ones prevail ({ROd}: draw). Or saying "it is the same" because in both sides the black [white] cards prevail ({RO=}: win-win [lose-lose]).

{RD}. Difference relations: Choosing the side where the difference of black minus white cards is the largest ({RD+}) or the smallest ({RD–}), or saying "it is the same" because the difference is the same in both sides ({RD=}).

{RP}. Proportionality relations: Choosing the side where the quotient $^{f}/_{n}$ or the quotient $^{f}/_{u}$ is the largest ({RP+}), or saying "it is the same" because the quotient is the same in both sides ({RP=}).

Composed strategies. Two or more simple strategies can be linked in a logical juxtaposition; each of them may be dominant or dominated. If *X* and *Y* are two strategies, there are four possible compositions between them:

{*X* & *Y*}. Conjunction: Both *X* and *Y* lead to the same decision (S1, S2 or "same") and they support each other. Both *X* and *Y* are dominant.

{*X* ¬ *Y*}. Exclusion: *X* leads to the election of one side or to the decision "it is the same", and *Y* leads to the election of the other side, but *X* prevails. *X* is dominant and *Y* is dominated.

{*X* * *Y*}. Compensation: *X* leads to the election of one side, and *Y* leads to the decision "it is the same", but *X* prevails. *X* is dominant and *Y* is dominated.

{*X* ⊥ *Y*}. Counterweight: *X* and *Y* lead to the election of different sides and they cancel each other, causing the decision to be "it is the same". *X* and *Y* are dominated.

There are also multiple compositions: compositions of strategies, one (or both) of which is in turn a composed strategy.

Examples. Table 1 shows the most common strategies that would be applicable in figure 1, the decisions they lead to, justifications that could have been prototypically given by subjects, and the mathematical description of simple strategies.

Centrations	{CN–}	A	A has fewer cards.	4<7
	{CF+}	B	B has more black cards.	2>1
	{CU–}	A	A has fewer white cards.	3<5
Relations	{RO=}	Same	In both sides there are fewer black cards than white ones.	1<3 & 2<5
	{RD+}	A	If I take out couples of black and white cards, less white cards remain in A than in B.	1–3 > 2–5
	{RP+}	B	A has three white cards for each black card, and B lacks one white card for that, so it has "fewer" white cards	$^2/_7>^1/_4$
Compositions	{CN– & CU–}	A	A has fewer cards altogether *and* it has fewer white cards.	
	{CF+ ¬ CU–}	B	B has more black cards, *although* A has fewer white cards.	
	{CF+ * RO=}	B	B has more black cards, *and* (or *although*) it has, as A does too, fewer black cards than white ones.	
	{CN– ⊥ CF+}	Same	*On the one hand* A has fewer cards altogether *but on the other hand* B has more black cards.	

Table 1: Simple strategies and some compositions applicable in the array of figure 1

Situations

A *situation* is a set of arrays such that the different simple strategies lead to the same decisions in all the arrays. Each situation is characterized by two subsystems, which define a crossed pattern; the two subsystems are combinations and locations.

Combinations. A combination is the succession of results obtained when an order relationship is established between the following pairs of an array: the total cases n, the favourable cases f, the unfavourable cases u, the differences d, and the quotients p. For instance, the array $(f_1,u_1)(f_2,u_2) = (1,3)(2,5)$ of figure 1 has $n_1<n_2$, $f_1<f_2$, $u_1<u_2$, $d_1>d_2$ and $p_1<p_2$, which gives the combination (<<<><). Seventeen such combinations exist [6], which are the following: K0 (=====): identity array; K1 (<<><<), K2 (<><>>): arrays without discrimination (most strategies lead to the same side: S2 in K1 and S1 in K2); K3 (=<><<), K4 (<<=<<), K5 (<=<>>): equalities of total, unfavourable or favourable cases; K6 (<<<<<), K7 (<<<>>): {RD} and {RP} lead to the same side; K8 (<<<><), K9 (<<<<>): {RD} and {RP} lead to different sides; K10 (<<<=<), K11 (<<<=>): equality of differences; K12 (<=<>=), K13 (<<=<=), K14 (<<<>=), K15 (<<<<=) and K16 (<<<==): equality of probabilities, with probabilities respectively equal to p=0, p=1, p<½, p>½ and p=½. (In K6 through K16, centrations lead to the election of different sides). Thus, the array of figure 1 is K8 [7].

Locations. A location is a non-ordered pair of the following alternatives for both probabilities of the array: *c:* certainty (p=1); *w:* win (1>p>½); *d:* draw (p=½); *l:* lose (½>p>0); and *i:* impossibility (p=0). There are also seventeen possible locations [6]: *c=c* $(1=p_h=p_k)$, *cw* $(1=p_h>p_k>½)$, *cd* $(1=p_h>p_k=½)$, *cl* $(1=p_h>½>p_k>0)$, *ci* ($1=p_h>p_k=0$); *w=w* ($1>p_h=p_k>½$), *ww* ($1>p_h>p_k>½$), *wd* ($1>p_h>p_k=½$), *wl* ($1>p_h>½>p_k>0$), *wi* ($1>p_h>½>p_k=0$); *d=d* ($p_h=½=p_k$), *dl* ($½=p_h>p_k>0$), *di* ($½=p_h>p_k=0$); *l=l* ($½>p_h=p_k>0$), *ll* ($½>p_h>p_k>0$), *li* ($½>p_h>p_k=0$); and *i=i* ($p_h=p_k=0$). Location *c=c* is called "double certainty", *dl* is "draw-lose", *ww* is "win-win", etc. For instance, the array (1,3)(2,5) of figure 1 is *ll* ("lose-lose") [8].

Not all locations exist in all combinations; there are 86 intersections of both subsystems of categories, and thus 86 different possible situations [6, 9].

Correspondence between the systems of strategies and situations

One of the most important features of these systems lies in the correspondence between them, which makes it possible to know, for each problem posed, which strategies may be occurring and which lead to each of the three different decisions (S1, S2 or same). The combinations account for the possible decisions induced by all simple strategies except the {RO} family, and the locations account for the ones induced by the {RO} family. For instance, the applicable strategies, the decisions and the justifications depicted in table 1 could be the same for all K8-*ll* arrays.

The correspondence also permits a classification of all strategies according to their correctness in each situation. A *correct strategy* is one that coincides algebraically with the formal probability, and an *incorrect strategy* is a behavioural pattern associated with an inadequate intuition, one that may eventually lead to a decision different from that prescribed by the formal probability. Three groups can be defined:

- Correct strategies: always correct when applicable. Relations {RP+} (applicable in K1–K11), {RP=} (K0, and K12–K16); {ROwl} (locations *wi, ci, cl, wl*), {ROdl} (*di, dl*), and {ROwd} (*wd, cd*). Ten composed strategies such as {CF+ & CU–} (K1–K3) and {CF+ * CN=} (K3). Compositions with correct dominant and incorrect dominated strategies, such as {RP= ¬ N–}, are also considered correct.
- Eventually correct strategies: correct only in certain situations. Centrations {CF+}, {CF=} (only correct in *i* locations), {CU–} and {CU=} (only correct in *c* locations). Four composed strategies such as {CU– * CF=} (only correct in K5, but also incorrectly applicable in K12, which only exists in *i=i*).
- Incorrect strategies: always incorrect when applicable. Families {CN} and {RD}. Strategies {CF–}, {CU+}, {ROd} and {RO=}. Also, most composed strategies.

All the correct strategies other than {RP+} or {RP=} may be thought of as Vergnaud's (1981) *théorèmes en acte*: the subject using them may be perceiving and using properties of the relationships between elements of the array without necessarily being able to make them explicit or to justify them. A particular case are the correct composed strategies, such as {CF+ * CN=} in K3, because when a subject uses {CF+} in K3 the question could arise that he or she might be considering {CN=} but is not stating it, because it is too obvious. In such cases {CF+} is considered a *potentially incomplete expression of a correct justification* (abbreviated PIECJ).

AN APPLICATION OF THE FRAMEWORK; SOME RESULTS

In order to prove its effectiveness, the framework was applied in six experiments in Mexico City involving 64 university students who had not previously taken probability courses [10]. A test was designed in each experiment [11]; each test had between 20 and 40 items of the exposed form, all with n_1 and $n_2 \leq 10$, and covering a variety of situations. Some of the tests were applied in paper and pencil form, and some in taped interviews. A total of 1630 answers was obtained, of which 1144 (70%) were interpreted, 405 (25%) had a decision (S1, S2 or same) but were non-interpretable (mainly because of poor or none justification), and 81 (5%) had to be cancelled [12].

The evaluation of the experimental results is carried out from three viewpoints: strategies, situations, and subjects. Firstly, the frequency of each strategy or group of strategies is calculated. Secondly, the amount of correct responses in each situation permits to define levels of difficulty for situations. Thirdly, each subject can be classified according to his or her performance in each of those levels.

Occurrence of strategies. In the analysis of the 1144 interpreted answers, each strategy's occurrence was calculated as a quotient of the times it was observed over the times it could possibly be observed (for instance, {ROwd} could have been observed only in items with locations *wd* or *cd*, which were 200). Table 2 shows the relative occurrence of the most common strategies in simple or composed forms.

Correct strat.	*N*	S	C	Eventually correct strat.	*N*	S	C	Incorrect strat.	*N*	S	C
{RP+}	*813*	16	*	{CF+}	*925*	25	10	{CN–}	*911*	6	7
{RP=}	*331*	28	5	{CF=}	*219*	26	15	{CN=}	*233*	1	8
{ROwl}	*44*	50	11	{CU–}	*859*	11	5	{ROd}	*122*	5	3
{ROwd}	*200*	15	1	{CU=}	*285*	12	12	{RO=}	*396*	17	11
{ROdl}	*227*	21	2					{RD+}	*945*	2	1
								{RD=}	*199*	6	2

Table 2: Possibilities *(N)* and relative occurrence (%) as simple (S) or composed (C) forms (including dominant and dominated) of the most common strategies [13]

Among these results, the following may be highlighted. The high occurrence of {CF} among these young adults contradicts many authors' assertion that the centration in favourable cases disappears by the age of 11 (e.g. Falk et al, 1980). Also noticeable is the high occurrence of the incorrect {RO=}, greater than the additive {RD}. The proportionality reasoning was easier when the ratios were equal: {RP=} was more frequent than {RP+} (to these, a 3% of arithmetically incorrect attempts at {RP} may be added).

Overall correctness distribution. Of the 1144 interpreted answers, 37% had incorrect justifications (12% with and 25% without a correct decision), 53% had correct justifications and 10% were PIECJ. Among the correct justifications, 38% came from the simple proportionality strategies, 16% from simple correct order relations, 44% from *théorème en acte* strategies (simple or composed) and 2% from other correct compositions. It is noticeable that *théorème en acte* strategies, to which the PIECJ could be added, amount for approximately one half of the answers.

Evaluation of situations. The distribution of correct justifications varies among situations. All 1549 non-cancelled answers were used to analyse the correctness distribution of each situation, and six difficulty levels were defined according to the percentage of correct justifications [14]: I: Situations with impossibilities (including K12 and K13) (93%); II: Situations with certainties and identity situations (77%); III: Situations around p=½: locations *wl, dl* and *wd* (59%); IV: Proportionality situations: combinations K14 to K16 (35%); V: Combinations K1 to K5 in *ll* or *ww* locations (35%); VI: Combinations K6 to K11 in *ll* or *ww* locations (13%) [15].

Evaluation of subjects. These levels also permit the classification of subjects, with the following criterion: A subject is assigned level L if he or she can correctly solve and justify at least 50% of the items of level L and all levels prior to it, but cannot reach 50% in the subsequent levels. Of the 64 subjects, 23% were in level I, 28% in level II, 14% in level III, 16% in level IV, 11% in level V, and 8% in level VI.

Thus, half of the adult subjects who participated in these experiments could only correctly solve and justify the items corresponding situations with impossibilities and certainties; the "one variable" problems (level V), which according to Piaget and Inhelder (1951, Ch VI-4) can be solved by the age of 8, could only be solved by 19%, and the "two variables" problems (level VI), supposedly solvable by the age of 15 (*ibid,* ChVI-6), could only be solved by 8%. A question that remains to be answered is whether university students of this kind encounter also difficulties in solving other ratio-comparison tasks, or if the difficulties were enhanced by the probabilistic nature of the task, as happens with younger subjects (Cañizares et al, 1997).

NOTES

1. The term *difference* is used in this paper as f–u and not as |f–u|, thus possibly having positive or negative values.
2. For the sake of non-repetition, $n_1 \leq n_2$ and if $n_1 = n_2$ then $f_1 \leq f_2$. Thus, in figure 1, side A is S1 and side B is S2.
3. Other unstructured heuristics were also found in the experimentations with adults, but they were rare and were not considered in the construction of the system of strategies. Some examples: attraction ("It is more fun to choose A") and graphic presentation ("The way in which they are put begins with a black card..."). Usually, subjects who used them did so only at the beginning of the experiment, and then settled into the strategies considered in the system.
4. Centrations and relations have also been respectively called "single variable" and "two variables" strategies (Cañizares et al, 1997). Positive centrations correspond to the "more A-more B" intuitive rule identified by Stavy et al (2000) and equality centrations to the "same A-same B" intuitive rule.
5. All relations are described and exemplified in this paper in their "within" forms. However, all of them can also happen in a "between" form (See Noelting, 1980).
6. The proof of these assertions requires simple (albeit long) algebra; they can be consulted in Alatorre (1994).
7. Some other examples: KO: (3,5)(3,5); K1: (1,5)(4,3); K2: (2,3)(1,5); K3: (4,3)(5,2); K4: ((2,4)(3,4); K5: (3,1)(3,2); K6: (1,2)(4,3); K7: (2,1)(3,4); K8: (1,2)(4,6); K10: (1,2)(2,3); K12: (0,1)(0,2); K14: (1,2)(2,4); K16: (2,2)(3,3).
8. Some other examples: *c=c*: (2,0)(3,0); *cw*: (2,1)(3,0); *cl*: (1,0)(2,3); *ci*: (1,0)(0,2); *w=w*: (2,1)(6,3); *ww*: (2,1)(4,3);

wd: (2,2)(3,2); *wl*: (2,1)(2,3); *d=d*: (1,1)(2,2); *di*: (2,2)(0,5); *l=l*: (0,1)(0,3); *ll*: (1,3)(2,5); *li*: (0,3)(2,4).

9. Some examples: combinations K1, K2 and K3 have arrays in all locations except *c=c, w=w, d=d, l=l* and *i=i*. K12 – K16 only exist in one of these = locations (which only exist in them and in K0). K8 and K10 only have *ll* and *li*.
10. See Alatorre (1994, 1999). For an application in other non-probabilistic contexts, see Alatorre (2000).
11. In each experiment, some items were repeated identically in different parts of the test, and some were used as controls for A and B sides of the same array (see note 2). There was a general inconsistency, such as observed by Koch (1987), although not more in the latter than in the former.
12. Of the cancellations, 51 were due to a misunderstanding of the task, 18 to the use of heuristics such as described in note 3, and 12 to other reasons.
13. Not included in table 2 are {CN+}, {CF–}, {CU+} and {RD+}, because they had very low occurrences (S+C<2%).
14. These percentages were calculated in each group of situations as the mean of the percentages of only correct justifications and correct plus PIECJ plus non-justified-correct-decision answers.
15. These results generally coincide with those of authors like Falk et al (1980), Maury (1986) and Noelting (1980), although the comparison is difficult because of the differences in subjects' age, settings and framework (for instance, all of Noelting's K3, K5 and K7 items were in *wl, dl* or *wd* locations). Also, it has been shown that there is a strong influence of context in the results obtained in ratio-comparison tasks (Karplus et al, 1983).

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