ANALYZING GEOMETRY PROBLEMATIC LEARNING SITUATIONS BY THEORY OF PERCEPTION

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Abstract

In this paper we suggest an analysis of 'elementary' difficulties and PLS (problematic learning situations) encountered in geometry classes in junior high school using theories of perception and perception-based knowledge representation. Our purpose is to illustrate a neglected aspect of the difficulties encountered in geometry studies. The demonstrations in this paper are based on and taken from a longitudinal research study using methodology described in previous works (e.g. Gal & Linchevski, 2000).

FOREWORD

Over an extended period of time we observed student teachers that taught geometry to slow and average junior high school students. The focus was on locating, identifying, and analyzing Problematic Learning Situations - PLS. (We use the terminology of PLS for situations arising in the course of teaching where teachers find it difficult to help students with problems in learning. See also Gal & Linchevski, 2000). We were interested in both the student's and teacher's point of view in an attempt to explain difficulties encountered (e.g. Gal & Vinner, 1997, Gal, 1998, Gal & Linchevski, 2000).

While analyzing the PLS we found that perception and perception-based knowledge representations, among other cognitive theories, provide explanations for many of the difficulties. Therefore, introducing these theories to teachers and showing their relevancy help teachers uncover and understand student difficulties that they were not previously aware of, and propose coping strategies.

As part of our research, we have planned a yearly academic course that aims to enhance teacher understanding of the ways that students think. Teachers participating in this course are presented with theory (general topics - e.g. concept and concept image - and specific topics included mainly topics relevant to understanding geometrical concepts - e.g. visual perception, Van- Hiele theory etc.) and actual student difficulties in geometry (using video clips of PLS, reports of class discussions, etc.) We found that teachers' ability to analyze the reasons for student and teacher difficulties increased significantly during the course. Later in this paper we will present some examples of difficulties that were recognized and explained by these teachers. Describing the method used and results of the involvement are beyond the scope of this paper.

In this paper we wish to focus on theories of perception and demonstrate their use to explain and analyze difficulties in geometry class.

THEORIES ABOUT PERCEPTION AND PERCEPTION-BASED KNOWLEDGE REPRESENTATION

Visual perception

Visual perception involves processing information that comes through our eyes (e.g. Anderson, 1995). There are two phases:

1. Registering the sensory information - visual information processing.
2. Interpreting the identified shapes and objects - visual pattern recognition.

Visual information processing

In the first stage of visual perception, shapes and objects are extracted from the visual scene. In order to do this, we need to know "what goes with what" to form the object. The set of principles according to which objects are organized into groups is called the Gestalt principles of organization (e.g. Anderson, 1995).

a. Principle of proximity: elements close together tend to organize into units.
b. Principle of similarity: objects that look alike tend to be grouped together.
c. Principle of continuation: we perceive lines with continuous turns better than lines with a sharp turn.
d. Principle of closure and good form: we tend to see shapes as closed rather than open, and with a regular shape rather than an irregular one.

Visual pattern recognition

In the second stage of visual perception, shapes and objects are recognized. Recognition is the result of feature analysis, in which the object is segmented into a set of sub-objects, the output of early visual processing (first stage). Each sub-object is classified, and when the pieces out of which the object is composed and their configuration are determined, the object is recognized as a pattern composed of these pieces (Anderson, 1995; Barsalou, 1992).

Recognition can be the result of either 'bottom-up' or 'top-down' processing (Barsalou, 1992). Bottom-up processing uses information from the sensory physical stimulus for pattern recognition. When context or general world knowledge guides perception, we refer to the processing as top-down processing, because high-level general knowledge contributes to the interpretation of the low-level perceptual units (Anderson, 1995). The context has an important role in pattern recognition, tuning a specific interpretation in top-down processing.

Attention is another parameter that affects information processing. There are automatic processes that require no attention or conscious control, and there are controlled processes.

Perception-based knowledge representation

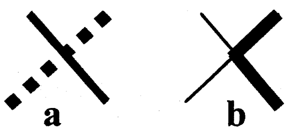
There are two ways for visual information to be represented in the cognitive system (Anderson, 1995): perception-based knowledge representation and meaning-based knowledge representation (which is beyond the scope of this paper). In perception-based knowledge representation, there are separate representations for verbal and visual information. Some visual information, such as the shape of geometric objects, is stored according to spatial position, while words are stored in linear order. Moreover, memory for pictorial material is superior to memory for verbal material.

Mental objects are dealt with like physical objects. Mental comparison of visual properties involves difficulties similar to those involved in visual perception. Complex figures composed of hierarchical decompositions (Anderson, 1995).

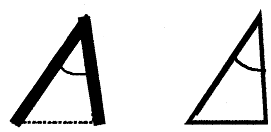
IMPLICATIONS FOR THE ANALYSIS OF STUDENTS' DIFFICULTIES IN GEOMETRY

Analysis by the Gestalt principles

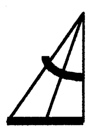
Identifying an angle formed between two straight lines. Many students, when talking about perpendicular lines, cannot point out where the right angle lies. They point at the area near the point of intersection. Or if the perpendiculars are part of a complex shape, such as the diagonals of a square, the student sometimes points at other right angles (see Gal & Vinner, 1997). This difficulty can easily be explained by the principle of continuation (third Gestalt principle mentioned before): the tendency to perceive lines which continue in the same direction rather than lines with a sharp turn causes the student to see two straight lines (figure a) instead of two rays emanating from one point, forming an angle (figure b).



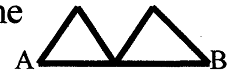
Mistaking an angle for a triangle. There are students who draw a triangle whenever they are asked to draw an angle. Why is that so? Generally, students encounter angles as parts of triangles and not as 'basic angles' i.e. two rays (or ray segments) emanating from one point. In such a case, separating the angle rays from the triangle's third side, in order to identify the basic angle, contradicts the principle of closure and good form. It also explains why the angle is frequently interpreted as a closed shape (triangle).



Identifying common parts (segments or angles) in complex shapes. A frequent task is pointing out shared parts of two (or more) triangles with a common side. Very often, students point out the wrong parts as shared parts: the side intersecting with the shared side may be identified as a shared side, or a combination of the two angles may be pointed out as a shared angle (see the darkened side/angle in the drawn figure). In order to find the right answer, one needs to separate the configuration into two triangles and check their parts. But the need for separation contradicts the principle of proximity as well as the principle of closure (of the "big" triangle).



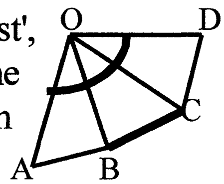
A junior high school teacher reported [1] problems with the drawing shown here, where students indicated AB as a common side for the two triangles. The teacher also explained that this was due to the principle of continuation.



Using the principle of similarity. The principle of similarity can be used to overcome difficulties based on the principle of proximity or the principle of continuation. Coloring parts we want to focus on can make use of the tendency to group together objects that look alike. (We can use a transparency placed on the figure, and color the relevant parts). To further demonstrate, in the first three examples presented above, we actually used this principle: the relevant parts of the figures were widened, darkened or drawn as a dotted line to group them together through similarity.

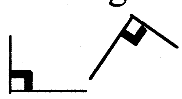
Analysis by top-down and bottom-up processing

1. Using the Gestalt principles to explain difficulties in *identifying common parts in complex shapes* we actually presumed a bottom-up processing which uses sensory information. Another explanation considers top-down processing, i.e. when a wrong context impairs recognition. The segment BC can be considered as a "common" segment to all three "small" triangles by means of 'in between', 'amongst', as in the case of a common yard or premises of several buildings. The same for the angle AOD, which can be considered as "common" in context of 'every one contribute a part' to create 'altogether'.



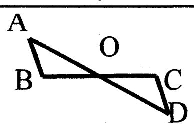
2. It is possible to recognize a right angle by bottom-up processing using sensory stimulus, which thus recognizes horizontal and vertical segments as a pattern of a (prototype – e.g. Rosch, 1978) right angle or a (prototype) right triangle:

Alternatively, recognizing a right angle by top-down processing occurs when using the conventional sign of a small right angle (instead of an arc). In this case, the special sign context is used to supplement feature information in recognition of the figure.



3. The following task requires the solver to identify two triangles in the configuration (in order to prove that they are congruent). This can be done by decomposing the figure into two triangles.

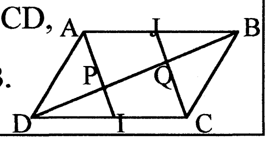
Given that:
AO=OD, BO=OC,
Prove that AB=CD



Such decomposition fits the principle of closure and good form. Therefore, information based on the stimulus is sufficient to enable the perceiver to recognize the triangles. This is an example of how bottom-up processing occurs.

A different task is presented by Duval (Duval, 1998, p. 41). In this case, bottom-up processing is generally not enough to get to the solution. That is because many sub-configurations can be seen. The Gestalt principles encourage decomposing into two "big" triangles (ABD, DBC), or six "small" polygons (APD,

Given a parallelogram ABCD,
I, J - midpoints of CD, AB.



AJQP, JBQ, etc.) In order to find the relevant decomposition (ABP, DQC) one needs to be aware of the context of *middles* explicitly recalling its theorem. This helps in finding the correct decomposition of the configuration. This is top-down processing.

Analysis by automatic and controlled processes

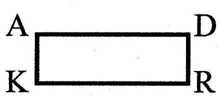
Many controlled processes are problematic for geometry students. Typical examples of a concept (prototypes) are easier to recognize (Rosch, 1978). The occurrence of an automatic process can explain this. A controlled process is needed to recognize less prototypical members of the category, e.g. recognizing a right angle in various orientations, an angle in a composite configuration, the height of triangle (oriented towards a not horizontal base), a square as a rectangle etc. Turning recognition into an automatic process can be part of the solution. Repeatedly exposing the learner to a wide set of examples can contribute to automation.

Another difficulty results because the organization of the prevailing field determines perception of its components (i.e. "field dependent/independence", Witkin et al., 1977). Again, turning recognition of geometrical figures in "conflicting" surrounding frameworks into an automatic process can help to overcome the difficulty.

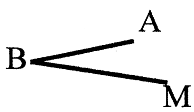
Analysis by perception-based knowledge representation

Some difficulties may be explained by the fact that we have different representations for verbal and visual information:

Naming polygons by indicating their vertex names in a clockwise fashion (according to the spacial path) is an accepted practice. For example, this rectangle is named ADRK. Unfortunately, students sometimes label such a rectangle according to its verbal representation. Therefore, instead of naming it ADRK it will mistakenly be called ADKR because of the linear representation of verbal information (reading from left to right, from top to bottom).

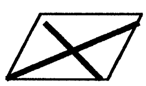


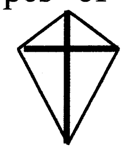
The three-letter notation for angles causes difficulties for many students. Many of them mistakenly call the angle in the figure ∠BAM. However, it may also be explained by the fact that the students consider the letters as conveying verbal information, and so they read them from left to right, and from top to bottom, as they would read English.



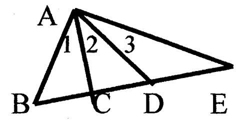
The following example demonstrates the mental comparison of visual properties:

A student who needs to use the properties of diagonals in different types of quadrilateral will probably check a mental image of the prototype of the specific quadrilateral.
In such a case, it may be easier to mentally analyze the kite's (rhombus) diagonals' properties rather than analyzing that of the parallelogram's. This is because of the greater difference in the size of the diagonals in the prototype of the kite than in the parallelogram.





The following is one of the examples brought by the teachers during the course (see box). It demonstrates the hierarchical decomposition of complex figures. The teacher reported that the students could not see that the angles were equal.

> Given: AC, AD bisectors of angles in ABD, ACE.
>
> <BAE=66°
>
> Find: <A1, <A2, <A3
>
> 
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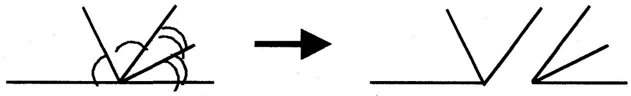
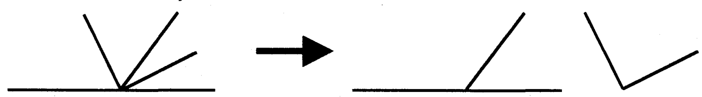
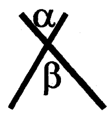
Another teacher, one of the course participants, explained this as resulting from the hierarchical structure of the drawing, which prevents the students from seeing angle A2 as part of both BAD and CAE, simultaneously.

One more teacher reported difficulties in proving that the bisectors of adjacent angles form a right angle. She proposed two possible hierarchies for the configuration explaining, the difference between the desirable decomposition (needed for the proof) and the actual decomposition of the configuration:

Actual decomposition: 

Desirable decomposition: 

Finally, one teacher was asked if she can apply the different representations of visual and verbal information, and the superiority of memorizing pictorial material in comparison to that of verbal material to geometry instruction:

> Of course these things can be applied to geometry instruction. In tests, before proving problems, I ask for definitions and phrasing of theorems. Though I'm doing it for long time, it's the first time I understand why most of the pupils draw a picture before the verbal definition, if they add the verbal definition at all.
>
> For example, if I ask what are vertex angles, the common answer is to draw: $\beta=\alpha$ 
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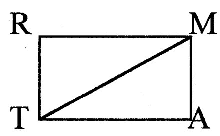
ANALYZING PLS

Here is an example of a protocol of a PLS. We shall give a detailed analysis involving theories of perception.

> During a lesson on quadrilaterals in a ninth-grade class, the students were solving in class a problem they had been given for homework.
>
> > Given: a quadrilateral RMAT
> > RM=TA, RT=MA
> >
> > Prove that sides are parallel
> >
> > 
> >
>
> The students show that the triangles are congruent. Some claim that they can use the fact that there is a right angle because they see it. They find it difficult to explain why the sides are parallel.
>
> The teacher asks if they remember what they have learned about parallel lines.

Their answers compel her return to parallel lines. She draws two lines and a third one that intersects them, reminding them about the different types of angles.

Student A: But angle A has nothing to do with that thing!

The teacher extends RM and TA in both directions and also extends TM.

Now the students are unsuccessfully trying to figure the alternate and corresponding angles in the original drawing.

Student A suggests erasing RT and MA so that it will be possible to see the connection with the teacher's drawing of the parallel and intersecting lines.

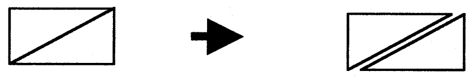
The teacher accepts the idea, praising the student. The discussion goes on until they conclude that the sides are parallel.

Analysis

This PLS involves two kinds of difficulties: Difficulties of visual pattern recognition (feature analysis and object recognition), and difficulties concerning Van Hiele's theory. In this paper we discuss only the first one.

The configuration appearing in the assignment has two possible "natural" decompositions in accordance with the principle of closure and good form:

or

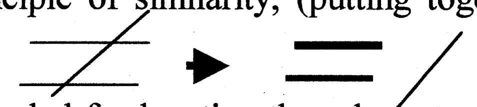


In order to recognize the applicability of the parallel theorem to this problem, however, it is necessary to decompose the configuration as follows:

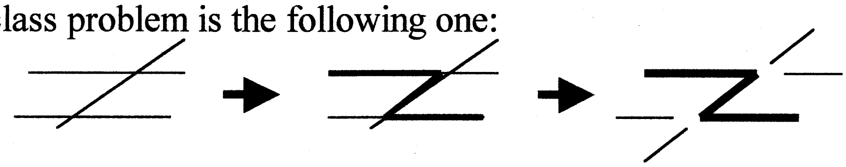
(1)

Such decomposition contradicts the principle of closure and good form.

Moreover, the drawing demonstrating the parallel theorem is decomposed in accordance with the principle of similarity, (putting together the two parallels with same orientation):



But the decomposition needed for locating the relevant angles in the above discussed class problem is the following one:



(2)

This decomposition contradicts the similarity principle. And even if the proper decomposition is found, the phase of pattern recognition remains: the two patterns, (1) and (2) need to be recognized as being the same.

SUMMARY

Theories of perception, though not new, are rarely introduced to teachers in general, or in the context of geometry teaching in particular. These theories can be a powerful tool for explaining a wide range of difficulties. In this paper, we have presented a few of the examples we encountered in order to exemplify the potential of these theories for teaching geometry. Being familiar with these theories can help teachers cope with Problematic Learning Situations by helping them to plan their instruction and making decisions during instruction. However, presenting this knowledge to teachers is not enough. Making it relevant, presenting actual class situations and inviting teachers to look carefully at their own teaching experiences, would likely improve their ability to recognize, analyze and cope with PLS. These methods were implemented in the course we suggested. Our findings regarding these intervention methods are the subject of a future paper.

[1] We include examples reported by teachers attending the above-described course in order to enrich ours. These examples may give an idea about the awareness and capability of the teachers participating the course to analyze PLS. Describing the changes in teachers' ability to identify and analyze difficulties and cope with PLS during the course is beyond of the scope of this paper.

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