# UNKNOWNS, ARITHMETIC TO ALGEBRA: TWO EXEMPLARS

Elizabeth Warren - Australian Catholic University

*This paper examines young children's ability to solve sentences with unknowns. A semi-structured interview was conducted with 87 children who had just completed their first three years of formal schooling. The purpose of this interview was to ascertain their thinking when solving problems with unknowns and draw implications for early years algebra. The results of the interviews indicated that many children are experiencing difficulties in solving for unknowns.*

## INTRODUCTION

The transition from arithmetic to algebra has been a major focus of recent research into the learning and teaching of algebra (Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 2000). A focus of this research is the gap between the knowledge required to solve arithmetical equations by arithmetical methods and the knowledge required to solve algebraic equations by operating on or with the unknown (Booth, 1988; Herscovics & Linchevski, 1994; Filloy & Rojano, 1989). Student's inability to operate with or on the unknown has been central to delineating arithmetic from algebra, where algebra traditionally follows arithmetic. Early algebra research is moving towards the integration of arithmetic and algebraic reasoning in the elementary grades. Thus the importance of how young children deal with unknowns is growing.

## ARITHMETIC IN THE EARLY YEARS

A variety of addition and subtraction problems exist in the real world. Fuson (1992) identified four broad categories that addition and subtraction problems seem to fall into. For each, the use of the unknown varies. The first is referred to as *Change-Add-To* and *Change-Take-Away*. This involves beginning with a single collection and changing the initial collection either by adding or removing something from it. This category covers the majority of problems children solve in the elementary school (Baroody & Standifer, 1993). In this instance, the unknown is the outcome of the change. Its manifestation in algebra (e.g., $4+3=x$) is trivial and represents limited challenges in developing meaning for unknowns.

The second category focuses on considering arithmetic situations as comprising three components, two parts and a whole and arithmetic operation as joining the parts or removing one part. In this understanding, in order to find the unknown it is simply a matter of working out which component is missing and applying an appropriate solution strategy. For example, if we have 3 + ? = 7 or 7 - ? = 5 then we have the whole and one of the parts. To reach a solution we need to work out the other part. For problems such as 3 + 4 = ? or ? - 4 = 7 we have the two parts and need to work out the whole. This thinking not only assists in classifying problems but also suggests ways of reaching appropriate solutions for number sentences that fit the *Part-Part-Whole* framework. It also represents many of the unknown situations covered on the way to algebra. However, its limitations lie in its inability to model equivalent situations with two or more terms on both sides (e.g., 2+5=1+?). Thus, while it serves an understanding of arithmetic processes, its usefulness for algebra is uncertain.

The third and fourth categories are similar. The third, *Equalize*, involves removing the difference between two collections, for example, Jill has 5 cars and John has 8 cars, how many more does Jill have to buy in order to have the same amount of cars as John? In this instance, the unknown is the difference. The last category, *Compare*, considers the difference between two numbers, and is similar to *Equalize*, but does not entail any action and the difference between the two numbers persists. For example, John has 8 cars, he has three more than Jill, how many cars does Jill have? In both these instances the unknown represents change or difference.

## EARLY ALGEBRA AND UNKNOWNS

Early algebra is not about introducing formal algebra in the early years but is about developing arithmetic reasoning in conjunction with algebraic reasoning. Thus exploring problems with unknowns in the early years is important. Slavitt (1999) suggested that a key to early algebraic competence is the ability to abstract computation to more structural realms, commonly referred to as generalised arithmetic. In this generalising process it is the numbers that map to variables and the operations remain the same, although the meaning of the operations could change.

Recent research has begun to focus on the development of young children's algebraic thinking (Falkner, Levi, & Carpenter, 1999), with a focus on children's understanding of equals as equality. Most students do not have this understanding; rather they have a persistent idea that the equals sign is either a *syntactic indicator* (i.e., a symbol indicating where the answer should be written) or an *operator sign* (i.e., a stimulus to action or "to do something") (Behr, Erlwanger & Nichols, 1980; Filloy & Rojano, 1989). Warren (2001) indicated that (i) young children are capable of reaching generalisations, (ii) classroom experiences can interfere with them reaching valid generalisations, (iii) in some instances teaching materials act as cognitive obstacles to abstracting the underlying mathematical structure, and (iv) new learning in mathematics can be incorrectly used to solve old problems. Incorporating 'real' world language can further complicate many situations. Pririe and Martin (1997) claimed that language only serves problems where the equal sign appears just before the answer, thus reinforcing the equation as an action and privileging equations with the unknown occurring directly after the equal sign.

Several studies have investigated student's equation solving strategies. The various methods used by algebra students have been classified as follows; using number facts, using counting techniques, cover-up, undoing (or working backwards), trial-and-error substitution, transposing from one side of the equation to the other, and performing the same operation on both sides (Kieran, 1992). Commonly, it is assumed that students bring with them from their experiences in arithmetic in the elementary school some understanding of the first two methods, number facts and counting techniques. Both these methods assist young children to find the 'missing addend' in number sentences. Counting techniques consist of counting on, counting up, or counting back (Fuson & Fuson, 1992). The questions are: do these techniques assist children in solving missing addend problems in a range of situations or are there problems that are beyond these techniques, how are the methods aligned with preparing young children for algebra, and are young children capable of using other strategies in meaningful ways in the early years?

The specific aims of this paper were to investigate young children's strategies for finding the unknown in first degree equations with one unknown, and to delineate the thinking that supports the development of algebraic understanding.

## METHODS

The sample comprised of 84 children from four elementary schools in low to medium socio-economic areas. The children are all participants in a three year longitudinal study investigating early literacy and numeracy development. The average age of the sample was 8 years and 6 months and all had completed the first three years of formal education. In Queensland children spend 7 years in the elementary school. Five tasks were developed for the semi-structured interview. Two tasks were developed to probe young children's ability to ascertain the unknown in equations (see Figure 1).

| **Task 1** | **16 + □ = 49** | **Task 2** | **54 = □ - 12** |
|---|---|---|---|

*Figure 1* Tasks presented for the interview

All children had completed their formal introduction to the concepts of addition and subtraction and could add and subtract numbers involving tens and ones. The two tasks chosen for this segment of the interview were believed to probe children's ability to operate with equations in differing formats and to understand the inverse relationship between addition and subtraction. The format reflected formatting commonly used in algebra, that is, the equations were represented horizontally, the unknown was on different sides of the equation, and more than one term followed the equal sign. Both tasks were presented in a symbolic format. Task 1 represented format commonly presented in most classrooms with the unknown on the left hand side and one number of the right hand side. Task 2 was atypical with the unknown

on the right hand side of the equation and the equal sign followed by two terms. Two digit numbers were deliberately chosen for the two tasks. It was believed that this would force children to use equation solving strategies other than number facts and counting on. Throughout the interview students had access to a calculator and were encouraged to use it if they so wished.

The script for this segment of the interview was: *What is the card asking you to do? How can you find the missing numbers? What is the missing number? How did you find it?* The interviews were audio-taped and the scripts transcribed for analysis.

## RESULTS

### Task 1

An examination of the transcripts indicated that the responses to the first task (**16 + □ = 49**) fell into four broad categories, namely, counting up, counting back, subtraction, and no response, each reflecting the general approach taken to reach a solution. Each category consisted of a number of differing solution paths. The next section describes each of the categories and includes some typical responses.

Category 1 **Counting up**

Counting up consisted of three strategies, counting up in ones, counting up in tens then ones, and using a trial and error method on a calculator. A typical response for counting in 1's was:

> *What is the card asking you to do?* Well it doesn't have a number in there and it is not a turn around and it equals 49. I have to try and figure out the answer in there. *What are you thinking about?* I was just counting in my head 17,18, 16, 17, 18, 19, 20 ... ... ... ...33, 34, ... ...41, .....49. [While counting up the student kept track of the number of tens with her fingers.] That is 34.

A typical response for the trial and error strategy was:

> *What is the card asking you to do?* 16 plus something equals 49. *How do you find the something?* You find it by well is should be something else so you can figure it out. *So can you work it out for me?* [Long silence] 16 + 16 equals [the child enters the two numbers into a calculator] No. 16+17. No. 16+29. No 16 +34. No. 16+33. It is 33.

Category 2 **Counting back**

A typical response for this category was:

> *What is the card asking you to do?* Put the number in there. *How can you find the missing number?* Well you go from 49 and count back 16 and whatever you land on is your answer. 49, 48, 47, 46, 45, ... .....33 So you put 33 in the box.

Category 3 **Subtraction**

Subtraction consisted of two strategies, using a pen and paper algorithm (either 49-16 or 16-49), or reaching the answer by mental computation processes.A typical response for the pen and paper algorithm was:

*What is the card asking you to do?* Something plus 16 equals 49. *How would you do it?* Take 16 away from 49. [The child proceeded to write the sum 49 - 16 in the vertical format and obtained the answer of 33]

Each response was coded according to the four categories. Table 1 summarises the frequency of responses for each category.

**Table 1** Frequency of response for each category to 16 + □ = 49

| Category | Strategy | Frequency | Correct |
|---|---|---|---|
| 1.Counting up | Counting up in 1's | 26 | 8 |
| | Counting up in 10's and 1's | 25 | 22 |
| | Trial and error | 6 | 3 |
| 2.Counting back | | 2 | 1 |
| 3.Take away | Correct algorithm (49-16) | 7 | 7 |
| (inverse operation) | Incorrect algorithm (16-49) | 2 | 0 |
| | Mental computation | 6 | 6 |
| 4.No response | | 10 | 1 |

The most commonly chosen strategy was counting up. Not surprisingly many of those who chose counting up in ones experienced difficulty in reaching the correct solution. They either included the starting number in their solution or simply lost track of where they where up to. Only 15 children (18%) used the inverse relationship between addition and subtraction to ascertain the unknown with all of those who chose the correct algorithm (49-16) reaching the correct solution.

**Task 2**

An examination of the transcripts indicated that the responses to the second (**54 = □ - 12**) fell into four broad categories, namely reformatting the original problem, interpreting = as 'the answer must follow' and thus the unknown must be 54, reaching the solution by addition, and no response

Category 1 **Reformatting the problem**

Three different strategies were used for reformatting the original problem. Each resulted in a format where there was only one 'thing' after the equal sign, either the unknown or one of the numbers in the problem. The first simply entailed 'flipping the format over' (**12 - □ = 54**). The second involved swapping '-' and '=' (**54 - □ = 12**). and the last strategy placed the unknown after the equal sign (**12 - 54 = □ or 54 - 12= □**). Some typical responses were

*What is the card asking you to do?* 54 take something = 12 no 54 = something take 12. You have to put the missing number in the box...but it is going backwards. *How would* *you find the missing number?* Count on. *What way would you like to see the problem?* 12 take something = 54. *Can you work out the number?* No you can't do it

*What is the card asking you to do?* 54 minus something = 12. 54 equals something minus 12... I don't know what that means. *So what is wrong with it?* Because the = is before the 12 and it shouldn't be. *What should it look like?* Well 54 - 12 = something. The box is after the equals it should be at the end. *How would you work that one out?* I don't know how to .... I will try 26 [ The child entered 54 -26 into the calculator]. I'll try 35 Oh I am doing it wrong it is 42

Category 2 **Interpreting = as 'the answer"**

*What is the card asking you to do?* 54 = something take away 12. *What is that asking you to do?* 54 equals 54. *Why? Can you explain?* The answer is 54 because it must be 0 plus 54 equals 54.

Category 3 **Solving by addition**

*What is the card asking you to do?* 12 take away.. is it going that way?... 66 goes in there.*How did you work that out?* Because the 12 needs a higher number... the 54 is the answer the 12 has to have a big even number and it was a 6 and then the other one needed a 6. *Why did the 12 need a 6?* To make that number and it was 66 and that takes away a 10 then it has to be 6 and then the 12 take away the two and it equals 54

Each response was coded according to these categories. Table 2 summarises the frequency of response for each category.

**Table 2** Frequency of response for each category of responses to 54 = □ - 12

| Category | Strategy | Frequency | Correct |
|---|---|---|---|
| 1. Reformatting the problem | **( 12 - □ = 54 )** | 18 | |
| | **( 54 - □ = 12 )** | 3 | |
| | **( 12 - 54 = □ )** | 7 | |
| | **( 54 - 12= □ )** | 20 | |
| 2. '=' as the answer | | 3 | |
| 3. Addition (inverse) | | 21 | 20 |
| 4. No response | | 12 | |

Over half the sample (58%) needed to reformat the problem before they could even attempt to reach a solution, with most insisting on placing the unknown after the = sign. As compared with Task 1, the number of children who reached a solution by using the inverse relationship increased from 15 to 21 children and all but one of these indicated that the unknown must be 66. Only 8 of these 21 children used the inverse relationship to solve both tasks. The rest either solved Task 1 by counting on in ones (4 children) or counting on in tens then ones (9 children).

## DISCUSSION AND CONCLUSIONS

As expected, most children found Task 1 easier than Task 2. Some conjectured reasons are that Task 1 meets student's understanding of equals and it can be easily represented in everyday language in all four categories (Change-Add-to, Part/Part/Whole, Compare & Equalise). By contrast, Task 2 seemed not to meet student's understanding of equals, and cannot easily be represented in everyday language in any of the categories. Arazello (1998) believed that natural language is crucial for developing an algebraic way of thinking. So how do young children deal with problems that cannot be easily translated into real world contexts? Does continually situating problems in real world contexts privilege some types of problems over others, as suggested by Pririe and Martin (1997)?

Early algebra involves a reconceptualisation of arithmetic in the elementary school. One expectation is that children will be able to solve for the unknown/unknowns in a wide variety of contexts and formats, including number sentences such as 12+?-1=23+5. As we move towards asking young children to be able to solve an array of problems with unknowns that are not commonly presented in our elementary schools, the role of natural language in developing an algebraic way of thinking takes on new meaning. Is it necessary condition to represents problems in real world contexts or is it sufficient for children to simply to be able to translate problems into spoken language?

The format of the tasks seemed to evoke differing solution strategies. For Task 1, the inclusion of addition seemed to suggest counting up rather than using the inverse relationship between addition and subtraction, whereas Task 2 did not suggest counting back but rather either reformatting the problem to an easily recognised form, or using addition, the inverse operation. This conjecture is supported by the increase in the number of children who chose to use the inverse relationship to solve Task 2. The use of the inverse algorithm for Task 2could also reflect young children's unease with counting back (Fuson, 1992). The need to reformat the problem supports previous findings with regard to interpretation of equal as a *syntactic indicator*, that is, a symbol indicating where the answer should be written.

Young children are using a variety of strategies for finding the unknown. While the majority used counting strategies, some used an undoing strategy (recognising addition as undoing subtraction and vice versa). In fact, all the children who were successful in Task 2 used this strategy. The tasks did not seem to elicit an understanding of balance to reach the solution. This is reflected in their lack of success in rewriting Task 2, with all maintaining the '-' sign in their transformations.

Trial and error was also a strategy used by some. These children already have equation solving strategies beyond number facts and counting techniques.

As we broaden the exemplars involving unknowns in the early years, the nature of categories, the role of natural language and equation solving strategies need further investigation, as all seem to play some part in successfully reaching solutions.


Arzarello, F. (1998). The role of natural language in prealgebraic and algebraic thinking. In H. Steinbring & M. Bussi & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 249-261). Reston, Virginia: National Council of Teachers of Mathematics.

Baroody, A., & Standifer, D. (1993). Addition and subtraction in the elementary grades. In R. J. Jensen (Ed.), *Research ideas for the classroom: Early childhood mathematics.* New York: National Council of Teachers of Mathematics, Macmillan.

Behr, M. et al (1980). How children view the equal sign. *Mathematics teaching, 2,* 13-15.

Boulton-Lewis, G., Cooper, T.J., Atweh, B., Pillay, H., & Wilss, L. (2000).Readiness for algebra. In T. Nakahra and M.Koyama (Eds.), *Proceedings of the 42nd International Conference for Psychology of Mathematics Education* (Vol 2, pp. 89-96).Hiroshima University: Program Committee for PME24.

Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching children mathematics*, December, 232-236.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19-25.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: MacmillanSlavit, D. (1998). The Role of Operation Sense in Transitions from Arithmetic to Algebraic Thought. *Educational Studies in Mathematics, 37*(3), 251-274.

Fuson, K. C., & Fuson, A. M. (1992). Instruction Supporting Children's Counting on for Addition and Counting up for Subtraction. *Journal for Research in Mathematics Education, 23*(1), 72-78. Herscovics, N., & Linchevski, L. (1994). A Cognitive Gap between Arithmetic and Algebra. *Educational Studies in Mathematics, 27*(1), 59-78.

Kieran, C. (1992). The learning and teaching of school algebra (Chapter 17). In D. A. Grouws (Ed.), *Handbook of Research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics.* (pp. 390-404). New York: Macmillan Publishing Company.

Pirie, S. E. B., & Martin, L. (1997). The Equation, the Whole Equation, and Nothing but the Equation! One Approach to the Teaching of Linear Equations. *Educational Studies in Mathematics, 34*(2), 159-181.

Slavit, D. (1998). The Role of Operation Sense in Transitions from Arithmetic to Algebraic Thought. *Educational Studies in Mathematics, 37*(3), 251-274.

Warren, E. (2001). Algebraic understanding and the importance of operation sense. In M. van den Heuvel-Penhuizen (Ed.), *Proceedings of the 25th conference of the international group for the psychology of mathematics education.* 4., 399-406. Utrech, Netherlands