COMPARING REPRESENTATIONS AND REASONING IN YOUNG CHILDREN WITH TWO-YEAR COLLEGE STUDENTS

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Problem solving and justification of a diversified group of two-year college students was compared with approaches of younger pre-college students working on the same task. The students in this study were engaged in thoughtful mathematics. Both groups found patterns, justified that their patterns were reasonable and, utilized similar strategies for their solution and methods of justification. The findings support the importance of introducing rich problems to pre-college and college students, under particular conditions.

Students enrolled in college level mathematics are expected to have developed effective reasoning skills. Unfortunately this is often not the case. This may be explained, in part, by a history of mathematics instruction in settings that devalue thinking and focus on procedural learning. From a perspective of conceptualizing reasoning in terms of solving open-ended problems, it was of interest to learn whether two-year college students enrolled in a liberal-arts college mathematics course could be successful in providing arguments to support their reasoning in a problem-solving based curriculum.

Considerable data have been collected showing pre-college students' success in solving open ended problems, over time, under conditions that encouraged critical thinking and building arguments to support their solutions (Maher & Martino, 1996, 1997, 1998; Muter, 1999; Maher, 1998; Maher & Speiser, 1997; Kiczek & Maher, 1998; Muter & Maher, 1998). These studies with younger students raise the question if similar reasoning and justifications are achievable by liberal-arts college students within a well-implemented curriculum that includes a strand of connected problems to be solved over the course of the semester. Specifically, this paper reports on one aspect of a larger study of two-year college students enrolled in liberal arts mathematics. It will describe, in the context of combinatorics, (1) how college students solve non-routine mathematical investigations and (2) how college students' representations and level of reasoning contrast with those of younger children from a longitudinal study engaged in the same investigations.

THEORETICAL FRAMEWORK

The growth of mathematical knowledge is the process whereby a student constructs internal representations and connects these representations to each other. Understanding is the process of making connections between different pieces of knowledge that are already internally represented or between existing internal connections and new knowledge. (Hiebert and Carpenter, 1992) Students build their understanding of concepts by building upon previous experience, not by imitating the actions of a teacher or being told what to do. (Maher, Davis & Alston, 1991) Learners who first learn procedures without attaching meaning to them are less likely to develop well connected conceptual knowledge. When students encounter new problems they are more likely to retrieve prior knowledge that is well connected than to retrieve loosely connected information. (Hiebert and Carpenter, 1992). The students in this study were encouraged to think about their solutions to the problems that they were given, develop understanding of the mathematics and justify their answers.

BACKGROUND FOR PRE-COLLEGE STUDENT RESULTS

Researchers have documented children's thinking as they investigate problems in the area of combinatorics to determine how they think about the problems and justify their solutions (Maher & Martino, 1996, 1998, 2000; Muter, 1999; Maher, 1998; Maher & Speiser, 1997; Kiczek & Maher, 1998; Muter & Maher, 1998; Martino, 1992). One of these problems is the Towers Problem, which invites a student to determine how many different towers of a specified height can be built when selecting from two different color cubes, and to justify that all possibilities have been found. In a Rutgers University longitudinal study, students first encountered the Towers Problem in the third grade when they worked on towers that were four tall. Researchers found that these pre-college students were able to invent strategies that they used to solve the problems. Most of the pre-college students started the Towers Problem by using guess and check methods to create towers, and check for duplicates. They then became more organized and created local organization strategies This resulted in the use of patterns such as "opposites", two towers with the colors reversed, "cousins", two towers that were vertical inversions, staircase patterns, and elevator patterns. They later developed global organization strategies as they realized that their local organization strategies created duplicates and could not be used to account for all possibilities (Maher & Martino, 1999, 2000; Muter, 1999; Martino, 1992). These students were also able to create convincing arguments that they had accounted for all possible combinations (Maher & Martino, 1996, 1998, 1999, 2000; Muter, 1999; Muter & Maher, 1998; Martino, 1992). The initial method of justification of many of the third and fourth grade pre-college students was to state that they could not find any more towers (Martino, 1992; Maher & Martino, 1996). For example, fourth grade Milin justified that he and his partner had found all possibilities because they had gone a long period of time without finding another one (Alston & Maher, 1993). Some of the children developed a proof by cases and others developed an inductive argument. For example, Milin started to build a proof by cases, but then developed an inductive argument when he looked at simpler cases of the Towers Problem (one-tall, two-tall and three-tall) and noticed the doubling pattern. He was able to explain how he could build up the larger towers from the smaller towers by placing either one or the other of the available colors on the top of the smaller towers

(Alston & Maher, 1993). Within this same time period, fourth grade Stephanie was able to present a proof by cases for the towers that were five tall. During a small group assessment, Stephanie presented a proof by cases for towers that were three tall and Milin presented his argument by induction (Maher& Martino, 1996; Alston & Maher, 1993). About a year later, in grade five, Stephanie also developed an inductive argument (Maher & Martino, 1996).

COLLEGE STUDENTS

Data Collection and Analysis

Nine classes ranging in size from 6 to 25 students were studied from 1998 – 2000. Two groups from each class were videotaped as they worked on the Towers Problem. Following the class sessions each student was required to submit a write-up of the problem. In addition, videotaped, task based, interviews were conducted.

Eleven students, representative of the larger population were selected for a case study analysis. The criteria included student willingness to be videotaped, fully participate in regular class problem investigation sessions and participate in follow-up interviews. Videotapes from class sessions and interviews were transcribed, coded, and analyzed for methods of problem solving and justification. The written work of students was coded and analyzed. The coding schemes for problem solving strategies and methods of justification are in table 1 and table 2.

S_r – random checking	S_p – looked for patterns	S_b – worked backwards
S_a – thought of a similar problem	S_s – thought of a simpler problem	S_{al} – used an algebraic equation or formula
S_i – used inductive method	S_c – conjectured	S_v – controlled for variables
S_{sp}– divided the problem into sub-problems	S_f – applied a previously learned procedure	

Table 1: Code of Problem Solving Strategies

A_o – none	A_c – cases approach	A_i – inductive approach
A_a – checked with another member of the class	A_e – everyone in the group agreed	A_f – can't find any more
A_s – same answer as a previous problem	A_l – applied previously learned formula or procedure	A_n – numerical argument
A_v – created a visually appealing pattern	A_{ci}– challenges instructor to show missing combination	A_g – geometric argument
A_p – discovered a pattern	A_{pe} – explained why pattern worked	

Table 2: Code of Methods of Justification

Setting

The study was conducted at a small community college in a mathematics course for liberal arts majors. Sections of the course met for two seventy-five minute classes each week or three fifty minute classes each week for fifteen weeks. The students spent approximately half of all class time working on various non-routine problems in a small group setting. The Towers Problem was given during the ninth week of the semester. Two groups from each class were videotaped as they worked on the problem. The students began by working on towers that were four cubes tall. They then were asked to consider towers that were five cubes tall. Some groups also worked with towers where three different color cubes were available.

Results

Most of the college students used patterns or some other form of local organization immediately and some immediately imposed a global organization scheme. One group started the problem by randomly generating towers using a build and check method. A student in the group, Jeff, soon suggested, however, that they should organize the towers that they had built by cases so that they could check more easily for missing combinations. After another pair of students, Stephanie and Tracy, built all sixteen towers using opposite pairs, they stated that they had to impose some type of organization to convince themselves that they had found all combinations. They first used staircase patterns, but rejected this because it did not account for all towers. They then arranged the towers by cases (Glass, 2001). While working on the five-tall Towers problem one student, Wesley, built his towers five tall by adding a red cube to the top of each of his towers of four and then building the opposites of these towers to find all towers with a yellow cube on top. Another student, Errol used an inductive argument that he had developed while working with towers four tall to predict that there would be thirty-two towers of five. He produced a list of all towers five tall on his written assignment using an inductive method (Glass, 2001).

Six of the profiled college students justified that they had found all possible towers four tall by using a cases approach. Each of these students used an elevator pattern to justify the towers with three cubes of one color and one cube of the other color. The students used a variety of methods to justify that they had found all towers with two cubes of each color. (Glass, 2001). Two groups, Melinda's group and Donna's group justified that they had found all towers because they couldn't find anymore. As these groups spoke to the instructor they began to organize their towers and moved toward a proof by cases. Both groups still justified that they had all towers with two of each color because they could not find anymore.

Five of the profiled college students did a proof by cases for the five-tall towers. Each of these proofs by cases referred to opposites. They also all used an elevator pattern to account for the towers with one cube of one color and four cubes of the other color. They used a variety of methods to justify that they had all towers with two cubes of one color and three cubes of the other color (Glass, 2001). One student, Wesley built his towers five tall by adding a red cube to the top of each of his towers of four. He then built the opposites of these towers to find all towers with a yellow cube on top. He justified that he had found all towers five tall with an inductive argument. He was unable at this time to extend this reasoning to predict how many towers that he would get six tall. During an interview seven weeks later, Wesley correctly extended the doubling pattern beyond the case that went from four tall to five tall. Another student, Jeff applied the fundamental counting principle to predict that there would be thirty-two towers that were five tall. After Jeff's group had produced the thirty-two towers he used an inductive argument to show that they had found all possible towers by pairing each of the five-tall towers with the corresponding towers that were four tall. Errol, who did not wish to build the five-tall towers in class, used an inductive argument on his written assignment to justify that he had listed all the possibilities. Penny, a student in Errol's group who was absent the day that the class worked on the Towers Problem, completed the problem at home. She invented a tree diagram strategy to produce an inductive argument for the towers four and five tall. Tim used a binary coding system to justify that he had found all possible combinations. This is the same method that tenth grade Michael from the Rutgers longitudinal study used to justify that there were thirty-two different pizzas with five available toppings (Muter, 1999).

Several groups also had time to work on the Towers Problem with three available colors. Mike's group and Rob's group worked on four-tall towers, while Jeff's group worked on three-tall towers. Rob and his partners applied the inductive method that they had developed for towers with two colors to quickly solve the problem. Jeff used the fundamental counting principle to calculate the number of towers, but did not use an inductive method to build the towers. Mike's group divided the problem into the three problems of finding towers with two available colors and the problem of finding towers that contained all three colors. They then built all the towers in a systematic fashion.

CONCLUSIONS, IMPLICATIONS AND LIMITATIONS

The college students used many of the same strategies for solution that the pre-college students from the longitudinal study had used when they solved the same problems. However, unlike the third and fourth grade children they did not generally rely on random checking as a primary strategy when they began to solve the problems. In general, the college students were able to solve the problem more quickly than the third and fourth grade children working on the same tasks. All were able to solve the four-tall Towers Problem and start the five-tall Towers Problem within a fifty-minute or a seventy-five-minute period. Most also finished the five-tall problem within the same period. Several students also had time to work on extensions of the problem within the class period. The college students also showed less inclination to think about problems for extended periods of time. Many stopped thinking about the problem after they had arrived at an answer, even when the instructor asked them to think more about the problem in order to reveal their thinking.

The college students also demonstrated methods of justification that were similar to those of the pre-college students. The method used by one of the college students, Robert, to show that he had found all towers with two cubes of one color and three cubes of the other color was the same as that used by Michael and Ankur from the Rutgers longitudinal study in one of their tenth grade after school sessions (Muter, 1999). Two of the profiled college students, Stephanie and Lisa used a method in which they fixed one of the two cubes on the top of the tower and moved the second cube into all possible positions. They then fixed one cube on the bottom and moved the other cube into all possible positions. This is similar to what fourth grade Stephanie from the Rutgers longitudinal study had done with towers six tall. Lisa and Stephanie from this study accounted for towers that did not have one of the two cubes on either the top or on the bottom of the tower while fourth grade Stephanie did not pursue this idea (Maher & Martino, 1996; Glass, 2001).

After another student, Rob and his group had built their towers by organizing them by cases they noticed the doubling pattern and developed a proof by induction, which Rob used in his written assignment. This is similar to what fourth grade Milin had done. College-aged Mike also noticed that the number of towers was doubling, but was unable to think of a reason for the doubling pattern (Glass, 2001). It is interesting to note that both Stephanie and Milin from the Rutgers longitudinal study had noticed the doubling pattern in the Towers Problem as fourth graders. Stephanie's progression from pattern recognition to development of an inductive argument took about eight months while

Milin's understanding developed more quickly (Alston & Maher, 1993; Maher & Martino, 2000). Perhaps college-aged Mike would also have recognized the reason for the doubling pattern if, like fourth grade Stephanie, he had been given an extended time frame in which to develop his ideas. Mike, however, was limited to about eight weeks to develop his ideas about the problem.

Although some of the conditions of the Rutgers University longitudinal study such as extended classroom sessions and revisiting the same problem several times within an extended time frame, could not be replicated because of time constraints within a college classroom, many of the conditions that enabled the pre-college students to become thoughtful problem solvers were duplicated. Both groups were given rich mathematical tasks and were encouraged to explain their reasoning and methods of solution and justify their solutions to the problems. Both groups of students were engaged in thoughtful mathematics. They found patterns, justified that their patterns were reasonable, and developed methods of proof. While it cannot be disputed that the students in the Rutgers longitudinal study benefited from exposure to rich mathematical experiences over an extended period of time, the students in this study, who had previously experienced a variety of traditional mathematics instruction, demonstrated that it is not too late to introduce rich mathematical experiences in a collegiate level mathematics class. The level of reasoning that these students demonstrated provides evidence that it is possible to experience thoughtful mathematics within a traditional fifteen-week semester. The findings support the importance of introducing rich problems to college students and giving them opportunities to work together toward a solution.

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