SEMIOTIC ANALYSIS OF DREYFUS' POTENTIAL IN FIRST-YEAR CALCULUS

Carl Winsløw, Danish University of Education

This paper presents a semiotic analysis of the (often claimed) potential of computer algebra systems (CAS) for enabling mathematical activity on a higher conceptual level than usual. Theoretical points are illustrated by an example from a development project in the context of a first year university course on calculus. We also discuss how they may be used in 'a posteriori' didactical analysis.

DREYFUS' POTENTIAL.

Broadly speaking, there are two main types of issues relating to the use of standard CAS, such as *Maple* or *Mathcad*, in undergraduate mathematics teaching:

- *Pragmatic issues*, concerning the competencies that students need in present and future mathematics-related practice, where CAS is or could be a relevant tool,
- *Didactical issues*, concerning the actual or potential impact of CAS-use on the students' learning of mathematics.

In the first case, we are talking about *needs for actual practice*, such as solving concrete mathematical problems, while in the second case, CAS is viewed as a vehicle for *learning*.

We shall discuss, in this paper, the didactical aspects of *Dreyfus' potential*, defined as the possibility that CAS may serve as follows:

> The idea is for students to operate at a high conceptual level; in other words, they can concentrate on the operations that are intended to be the focus of attention and leave the lower-level operations to the computer. [Dreyfus (1994) p. 205]

The didactical interest of Dreyfus' potential is rather obvious, in particular it is likely to reflect the ideal of most CAS-using university teachers. However, as it stands, it is exactly 'an idea' that may or may not 'appeal' to their colleagues, depending on their experience and personal preferences. In order to rationally discuss, plan, implement and document realisations of this idea, we need to formulate it in terms of a clear theoretical framework. In particular we need to clarify the notion of a 'high conceptual level' in its (potential) relation to CAS use in university education. This is where the analytic tools of *semiotics* come in; specifically, we draw mainly on Duval's semiotic analysis of mathematical cognition (cf. the next section). Of course, we must also relate the theoretical model with actual practice; at the end of this paper, we give examples of how it may be used in didactical analysis and design based on semiotic variables.

SEMIOTICS AS AN ANALYTIC TOOL FOR DIDACTICS.

Duval (2000) points out three major cognitive functions related to the semiotic registers used in mathematics: *representation* (e.g. a graph representing a function), *processing* (e.g. computation with number symbols) and *conversion* (change of register of representation). Even though students do not acquire mastery of these functions separately, their separate analysis proves useful for our purposes.

Semiotic *representations* in mathematics do not 'represent' in the naïve sense: *the entire discourse refers to...nothing other than its own signs* (Rotman, 1988). Mathematical objects are created *and* invoked through semiotic representations (Sfard, 1999). Yet in many (if not all) mathematical contexts it is crucial not to confuse an object with a particular representation (Duval, 2000), e.g. not to identify a 'triangle' with a concrete drawing. In fact, particular representations are less important than the changes (*processing*, *conversion*) of such representations that might be effected, while retaining a representation of the object. A simple semiotic description of an individuals' conceptual image of an object can thus be said to be a class of representations invariant under some class of transformations (processing, conversion). These classes will typically be implicit for the learner, and will develop gradually through actual instances of representing, processing and converting, briefly, through *semiotic activity*. This provides a theoretical basis for the need that students engage in such activity using a wide range of relevant registers and transformations. It may, in particular, challenge the modern contempt of 'training exercises' to the extent these exhibit such variation – even if variation pertains only to processing. However, as pointed out by Duval (1995, pp. 45-59), many of the most interesting and persistent learning difficulties arise in the context of conversion. Hence the 'degree of freedom' of the individual with respect to conversion (coordination of registers pertaining to the same conceptual object) is essential for the individual's conceptual development (*opus cit.*, p. 69).

In university level mathematics, patterns of *relations* among *different* conceptual objects are at the heart of the learning enterprise. To establish and develop such relations coherently, very specific forms of discourse – involving more or less formal uses of natural language in coordination with 'simple' semiotic registers – are needed. These forms may be classified as *apophantic*, *expansive* and *reflective* functions of discourse (*opus cit.*, chap. II). While a 'high conceptual level' may be indicated in semantic terms, it is usually accompanied by (at least implicit) semiotic and discursive complexity, and so its realisation in undergraduate mathematics education is intrinsically linked to developing the students' degree of semiotic freedom within this larger discursive framework. This implies the need for a careful design of learning situations centred on specific discursive and semiotic functions, where the latter includes pertinent forms of processing and conversion. It points to semiotic activity and discursive functions as crucial variables in didactical engineering (Artigue, 1989 and Brousseau, 1997, pp. 24f) at the undergraduate level.

The semiotic activity of students serves *pragmatic* as well as *didactic* purposes.

Pragmatic, because mathematical competencies are articulated – and can only be observed and evaluated – through discursive and semiotic performance (e.g. Sfard, 1999). Didactic, because an individual's conceptual development depends on his effective range of semiotic activity, as explained above. Notice that this is by no means a denial of the social nature of the learning enterprise, as semiotic activity to a large extent derives its meaning from a social context (interchange with other agents). Moreover, semiotic activity is influenced by the social context in the forms of agents, media and codes (the latter being a matter of *consensus* but also a *condition* for semiotic activity). Only codes may be independent of the context, that is, they belong to a large extent to a *global* context of mathematical discourse (Duval, 1995, p.225). The shared perception – largely implicit – of codes among participating *agents* is, on other hand, decisive for their potential for engaging in shared semiotic activity.

SEMIOTIC VIEW OF DREYFUS' POTENTIAL.

In a semiotic interpretation, the basis of Dreyfus' potential is that a CAS may provide a higher degree of semiotic freedom, primarily by facilitating *processing*, and – in a few cases, such as plotting – *conversion*. To 'operate on a high conceptual level' implies, in functional terms: to engage in discourse involving complex semiotic activity. The 'lower level operations' – carried out by the CAS – consist mainly of the *processing* parts of this activity. Typically, the explicit complexity of semiotic activity is reduced by the use of CAS, as the CAS tends to leave out several intermediate steps that may, or may not, be made explicit at wish. In fact, the 'black box issue' (Dreyfus, 1994) arises to the extent this is the case.

It follows from the above interpretation that the use of a CAS – at least, a priori – facilitates neither *coordination* of registers nor the main *discursive* functions. The simple *representation* of objects and transformations is not simplified, either. On the contrary, we have an extra medium (the computer), an additional special code (depending on the CAS) for semiotic activity, and a kind of 'automatic semiotic agent' with a potential influence on discourse (Winsløw, 2000). These additions may be particularly disturbing for novice users of CAS, but their influence remains important even in a context with experienced CAS-users.

An example.

Our key example is an event observed during a development project (Solovej et al., 2001) in the context of a freshman calculus unit. It occurred towards the end of a two-hour class session, in which students had presented and discussed a number of exercises on linear differential equations. In this unit, *Maple* was generally used by students in their homework, and it was demonstrated and used as illustrating device during lectures. In class, the computer was mainly used as a medium for presenting homework (students bring floppy disks to the class) and, occasionally – but with increasing frequency – as a semiotic agent in 'whole-class' exploring. There was one PC in the classroom, connected to a screen projector; both students and teacher used this PC to present their work and ideas. At the point we shall consider, all exercises set for that day have been treated. Then a student – let us call him Peter – raises a question. He has been working on an exercise from the book (in the same section, but not among those given as homework), concerning the initial value problem

$$(*) \qquad u''(t) + u(t) = \sin \nu t, \ \ u(0) = 1, \ \ u'(0) = 0$$

where ν is an undefined parameter. Using *Maple*'s routine (*dsolve*) for solving ordinary differential equations, he had obtained the output [copied from *Maple*]:

$$(1) \qquad \mathrm{u}(t) = \cos(t) + \frac{\nu \sin(t)}{-1+\nu^2} - \frac{\sin(\nu\, t)}{-1+\nu^2}$$

and then wondered how this should be interpreted in the case $\nu = 1$. The teacher asks Peter to show his solution, as outlined above, to the class (which he does). The teacher then proposes to let *Maple* solve the problem (*) with ν substituted by 1. Peter copies the first input to a separate input line, and makes the proposed change. This results in the following new output:

$$(2) \qquad \mathrm{u}(t) = \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\, t + \cos(t)$$

which, on the face of it, seems quite different from the first answer. The students are confused. Peter comments: 'That looks strange'. The teacher says: 'Clearly, one cannot simply substitute ν by 1 here [points to the first output, then hesitates for a while]. But how about taking a *limit* of it as ν tends to 1?' The student at the PC copies (1) to a new command line and ads the said limit. This gives the second output (2). The teacher says a few words about the possibility of studying the 'continuity of solutions' with respect to a parameter before the lesson is over.

This appears to be a fairly clear-cut example of how the *processing* powers of CAS may be used to realise Dreyfus' potential. The teacher suggests the *coordination* of the (for the students at this level) separate registers of 'ODE solving operations' and 'limit operations', in order to enrich the concept of 'solution' with a relation to 'con-tinuity'; the *didactic intention* of the teacher is served by CAS as 'processing agent'. While the students had some hands-on experience with using the standard procedure for solving inhomogeneous ODE's, the discussion – that occurred during the last 5 minutes of a lesson – can be 'lifted' to a higher conceptual level (discussion of solu-tions in terms of a parameter) only because semiotic *processing* is left to the CAS.

THE ROLE OF TEACHER CONTENT KNOWLEDGE.

In the classroom example outlined above, the discussion (discourse beyond simple semiotic activity) is entirely governed by the teacher. However, Dreyfus' potential talks about *students* operating on a higher conceptual level, while in the situation just described, these operations are only formally performed by the student (assisted by

CAS). The students do not actively participate in the 'higher level' *discourse* about 'continuity of solutions', and it is not clear to what extent they are informed by attending it. In a pessimistic evaluation, we may thus have a CAS version of the *Jourdain effect* (Brousseau, 1997, p. 26): students are led to perform certain CAS-assisted semiotic actions, and are then told 'what they have done' in terms of a higher-level discourse that is essentially beyond their reach.

In the example, this does not quite seem to be a fair evaluation of the situation. The discourse was indeed initiated by the student's question on the special case $\nu = 1$ of the solution. The students understand the solution of the special case, but Peter's comment ('That looks strange') certainly calls for an explanation; one input (ODE) is a special case of the other, yet the two outputs (solutions) look different. Given the impending end of the class, the teachers' choice of a quick explanation cannot surprise, except perhaps that it is both improvised and correct. However, in retrospect, the question could be made useful in many different ways, and this might be a starting point for CAS-based didactical engineering.

In another class, *Maple* was *only* used by the teacher as an alternative way of processing (within one register, except for standard plots of functions), usually demonstrated just after 'blackboard presentation' of the same problem. Discussions of an abstract nature never involved use of CAS. Students in this class primarily saw CAS as a *quicker* and *easier* means of reaching a pragmatic goal (processing), and their expressed motivation for trying to use it was almost exclusively the advantage they supposed to achieve for the upcoming written exam. Their conceptual understanding was most likely not enriched by the repetitive demonstration that a CAS can do in seconds what they had struggled to do on paper. One may say that, in this case, CAS was allowed as a semiotic agent only in cases where it 'echoed' other agents. The teacher repetitively used expressions like 'finally, it's nice to see these calculations in reality' (meaning, on the blackboard rather than on the screen).

While the teacher of the first class happens to be an eminent researcher of mathematics with a deep and flexible knowledge of mathematics, the teacher of this second class was a TA with his main occupation in high school teaching, and a strong focus on formal aspects (procedures, 'official' definitions etc.) of the topics taught. Throughout the project, he maintained a pessimistic view of the effect of CAS use on students' performance and understanding. The importance of *teacher content knowledge* – and its flexibility – is often found to be a main determinant for the development of CAS-based pedagogy (see Kendal et al., 2001). It may, in our framework, be interpreted as the importance of the teachers' own freedom of semiotic action within the discourse that he should help students to engage in.

TOWARDS CAS-BASED DIDACTICAL ENGINEERING.

We now return to the key example from the first class in order to sketch how this didactical situation could be improved by didactical engineering based on semiotic and discursive variables, in order to more fully realise Dreyfus' potential. This discussion is partially based on a conversation with the teacher, a few days after. It is important that in this *a posteriori* analysis, the situation must be conceived in its total discursive context, which we can only describe here to the extent it enters our discussion. It must also be stressed that this kind of analysis and design depends crucially on collaboration with teachers who, based on their own agency in mathematical discourse, are aware of the importance of the variables mentioned. This awareness is, initially, likely to be mainly non-explicit, but it is an important part of collaboration to change that.

Processing and the 'black box' issue

In order to achieve an inclusive mode of discourse, where students participate actively, it is clearly desirable to maintain *coherence* with previous (not too distant) elements of discourse that included the students. In the class situation considered, the previous discourse focused on the problem of *solving* linear differential equations with constant coefficients (in the sequel, abbreviated LODE). In particular, the focus had been on: the solution of *homogeneous* LODE, partial solutions to *inhomogeneous* LODE (obtained by 'informed guessing'), and the principle for finding the complete solution to inhomogeneous LODE (sum of two previous). The black box issue was partially mitigated in the case of homogeneous LODE's, by reading off the characteristic polynomial from the coefficients of the equation, and then use the *solve* routine to find its roots. This way, the form of the solution of the LODE could be related to intermediate steps of the solution process. After identifying the 'homogenous part' of the general solution of an inhomogeneous LODE, the partial solution could be motivated as the result of an appropriate 'guess' based on the form of the right hand side. A strategy for dealing with Peter's question might thus be to suggest a comparison between the two *problems* that result from (*) in the special cases $\nu = 1$ and (for example) $\nu = 2$, rather than between the two solutions (1) and (2), directly. Using the same patterns of analysis as previously, the students would find the crucial difference between the problems: in case $\nu = 1$, the right hand side, $\sin \nu t$, is a homogenous solution, while it is not in case $\nu = 2$. Then the different forms of (1) and (2) may be explained by differences in the 'good guess' for a partial solution. And this might bring forward *the point of knowing (in principle, and perhaps a choice of) the intermediate steps of processing* involved in the solution process – *and* their function in discourse – for the task of evaluating and comparing solutions, and otherwise reflect on their status. Notice that while this does not prevent usage of *Maple* as a semiotic agent, it requires a more flexible and informed usage than simply asking for final solutions.

One might also proceed with the original suggestion of taking a limit, and then use the reaction of Peter ('That looks strange') to bring out the point above, by asking the question: *how could one proceed to justify this limit?* Incidentally, the students learned about l'Hospital's rule 6 weeks before this event, and about partial derivatives just 3 weeks ago. This might, for instance, lead to a discussion involving the following two events of *Maple* processing:

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> diff(nu*sin(t)-sin(nu*t),nu)/diff(-1+nu^2,nu);
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$$\frac{1}{2}\frac{\sin(t)-\cos(\nu\, t)\, t}{\nu}$$

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> limit(%, nu=1);
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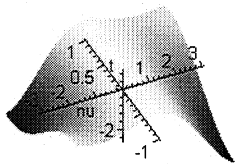
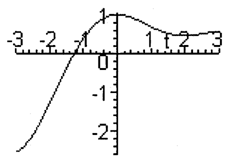
$$\frac{1}{2}\sin(t)-\frac{1}{2}\cos(t)\, t$$

(Here, the first line is *Maple* input code for: $\frac{\partial(\nu \sin t - \sin \nu t)/\partial\nu}{\partial(-1+\nu^2)/\partial\nu}$.)

Conversion and coordination

Continuing with the idea of simply exploring the relation between (1) and (2), one might profit from the fact that the students had studied functions of several variables 4 weeks before, in particular they were familiar with the conversion of *algebraic* to *graphical* representations of functions of two variables. In order to connect with this idea, one might suggest that the right hand side of (1) be regarded as the algebraic representation of a function of the two variables t and ν, and see if one might use the corresponding graph to relate it with (2). After a little experimentation with the settings, one might then produce the following two graphs of (1) and (2), respectively, understood as above, and with the domains [-3, 3] for t and [-1,1] for ν:





In *Maple* (not here!) the left image may be rotated in space, but even in the above 'static' form, it is not difficult to see the similarity between the 'upper edge' (corresponding to fixing $\nu = 1$) and the graph to the left. The discussion that could evolve from this would have, as its main point, to provide an exercise in the (difficult) coordination of algebraic and graphical registers that is motivated by a concrete problem. It is facilitated by CAS as an agent of *conversion.*

Expansive and reflective discursive functions

In order to avoid the Jourdain effect described in the previous section, the students must be acquainted with – and ideally control – the discursive contexts of the proposed semiotic activities. Notice that the actual *devolutions* have yet to be devised (for an example, see Brousseau, 1997, 33-35). Typically, students must be in the presence of (or have access to) a number of discursive units (apophantic sequences of phrases, including semiotic elements) on which they will, in the adidactical situation, have to *expand discursively*, using their observation of *similarity* (semantic or semiotic) among these elements, or to produce *reflective* discourse concerning the status (logical, semantic) of these elements. In the case of the example discussed above, the units present *a priori* may not suffice for the students to expand on (1) and (2) in order to relate them, and we have only outlined possible extensions of this fragment of the 'given' discursive inventory. The talk will present results (in terms of realised discursive expansion) from planned classroom experiments with more concrete extensions of this discursive basis of evolution.

But, in terms of discursive functions, the above expansions do not quite cover what the teacher, in the example, had in mind. Namely, once it has been established that (2) may be seen as the limit of (1) – and one has done similar investigations with other LODEs as well – one might try to help students proceed to a new kind of expansion: to develop (perhaps also prove) hypotheses in terms of more general forms of the *problem* (*) and its solutions. We then approach the professional mathematicians' (pragmatic!) version of Dreyfus' potential: studying 'concrete' examples using CAS as a 'semiotic slave' in order to generate hypotheses at a 'high conceptual level'.

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