SOLUTION OF WORD PROBLEMS THROUGH A NUMERICAL APPROACH. EVIDENCES ON THE DETACHMENT OF THE ARITHMETICAL USE OF THE UNKNOWN AND THE CONSTRUCTION OF ITS ALGEBRAIC SENSE BY PRE-UNIVERSITY STUDENTS

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In this document, a didactical proposal based in a method to solve arithmetic – algebraic word problems which uses a numerical approach is showed. The method serves as a mediator between the arithmetical and algebraic methods revealing three different uses of the unknown, namely: the "arithmetical unknown", the "numerical unknown", and the "algebraic unknown." Evidences obtained in a case study are shown, and such evidences present the need to make pre-university students face problems conveying to the use of a "numerical unknown." Then, students can achieve the detachment of the arithmetical use of the unknown, construct its algebraic sense and move forward in the competence of solving word problems through algebra.

This investigation is part of a more than 10 years project in which a problem solving approach to pre-university algebra it has been fostered. Some others results can be observed in (Rubio, 1990; Filloy & Rubio, 1993; Rubio, 1994; Filloy, Rojano & Rubio, 2001).The evidences presented hereby are linked to the performance of 15-16 year-old students in tests, clinic interviews and their work in the classroom by using a didactical proposal based in a successive explorations method to solve word problems where numerical values are assigned to the unknown. The purpose of the use of numbers is that they serve the student to: **a)** facilitate the analysis of some families of problems; **b)** detach the sense of the arithmetical use of the unknown that students tend to use when solving word problems; and **c)** advance in the construction of the algebraic use of the unknown through the use of a method that allows the student to give sense to the operation of the unknown of a problem, following a process in which it is operated with a numerical representation of the unknown in the first place, and then with a letter, as specified by the algebraic method to solve problems. It is considered that an unknown has an algebraic use sense when it is operated with the letter it represents.

The numerical approach used by the successive explorations method does not have as central objective that one student must solve a problem with numbers, but to facilitate its analysis, help to construct and give sense to the operations between the elements of the problem, and achieve its representation and solution algebraically. The method was conceived as a didactical device from which the student is detached as it acquires greater competence in more abstract uses of the algebraic language. In fact, it can contribute with elements to close the gap existing between the syntactical development of the algebraic language – to which one generally tends in teaching – and the semantic one.

REFERENTIAL AND THEORETICAL FRAMEWORK

In order to structure the didactical proposal, central aspects of the false position method were taken into account (Bruño, 1939; Vallejo, 1835), as well as the stages established by Piaget (1979) in relation to the assimilation process of the real facts to the mathematical-logical structures in the development of theories constructed from the physical experiences verified by such theories. Besides, the successive explorations method in which the didactical proposal is based, the trend observed in many students regarding the spontaneous use of non-algebraic methods were taken into account, such as the trial-and-error method to undertake several families of arithmetic/algebraic word problems.

The numerical approach of the method is connected from an epistemological point of view with: a) the two first stages described by Piaget (1979) and; b) the first indications about the actions to be made in order to apply the false position method (*Regula Falsi).* Piaget argues that those stages precede the data translation to the system equations (stage 3). The first one, the establishment of facts or data from the real world is not independent of mathematics modeling such as: classification, relationships, correspondences, measurement, etc. The second one is oriented to the search of an explicative model of the facts or data of the actual world starting from the construction of an intuitive and qualitative scheme. The numerical interpretation of the method captures both stages and its main purpose is to facilitate the settlement of the relationships between the elements of the problem, before translating the data, and its relationships to the equation representing the problem.

A critical historical analysis of the use of the *Regula Falsi* to solve problems in old text books was useful to incorporate its central ideas to the phases of the successive explorations method, which central aspect is the analytical intention of the use of numerical values to unchain the solution of a problem. The study of some of the problems solved with the *Regula Falsi* led us to conclude that its importance is due to the fact that it has an objective very similar to that of the algebraic method, which is to facilitate the analysis of problems and treat all of them in a similar way, as if they were the same problem (different to the arithmetical method in which each problem is analyzed case by case, in a particular way). The *Regula Falsi* (e.g. Bruño, 1939) starts by considering the problem as solved, but instead of using a letter to represent to unknown as in the algebraic method, a numerical value is proposed as a possible solution, using it later to make all the operations indicated in the problem; finally the *Regula Falsi* ends with the settlement of a proportion that leads to the solution of the problem. This last aspect was not incorporated in the explorations method.

METHODOLOGY

The general project contemplated in its first stage an initial study during one school year in which the use of the method was observed in the classroom with 15-16 year-old students and through the clinic interview of one girl student of the same age and school, but from another group. The main objective of this study was to improve the teaching model based on the successive explorations method. In the second stage of the project, a Case Study was carried out composed by six students whose clinical interviews were videotaped. Such students were chosen among 35 (15-16 years old) through a classification in classes based on their performance in three tests linked to three areas of competence: arithmetical problems; algebraic problems and solution of equations.

THE STUDY

The empirical investigation was focused in the study of meaning processes given by pre-university students to the unknown when they face some families of arithmetic / algebraic problems. The case study tried to find evidences regarding the changes of the use that students give to the unknown when facing problems which nature is "more arithmetic" than algebraic or vice-versa. The empirical work revealed the need to distinguish three different senses of use of the unknown: "arithmetical unknown" when given an arithmetic use, that is, when not operating with it; the "numerical unknown" when it is assigned a numerical value and when operations involving this value are accomplished and the "algebraic unknown" when it is represented with a letter and it is operated with the latter.

It was observed that in order to construct the algebraic meaning of the unknown it is required that the student is successively detached from the senses of the arithmetic and numerical uses of the unknown, considering that there is a detachment in the sense of the arithmetical use, when in the solution of a problem it is used a representation of it (either a number or a letter). The study proved that in order to propitiate such a detachment, it is required to face the student to problems which arithmetical solutions are complex or need a competence in arithmetic at an expert level. On the other hand, the detachment from the numerical use of the unknown is induced by the whole structure of the successive explorations method, where one starts operating with the "numerical unknown" and ends with the algebraic solution of one equation where the letter representing the unknown of the problem is used.

Briefly, in the study we try to evidence that the use of a "numerical unknown" can propitiate that the student detaches from its arithmetical use and begins the construction of the "algebraic unknown". It is thought that this is due since the "numerical unknown" facilitates the analysis of problems, where the relationships between their elements are established between numbers and where it can be seen that the numerical relationships between the elements of the problem not only have more semantic load than the algebraic ones, but also that one can use the unknown in them, although such unknown is "numerical", within an analysis process "moving forward" as in the algebraic method where one operates with a letter.

The observation of the solution processes in the case study allowed us to obtain evidences about the moments and situations conveying the student to leave the arithmetical use of the unknown and those situations in which the student constructs a pre-algebraic or algebraic use. The latter was achieved by facing the student to problems of such nature that their arithmetic solution is complex, but by also giving him/her a way that leads him/her to the algebraic use. In the study can be seen that the use of a "numerical unknown" helps to fulfill the aforesaid since it facilitates the analysis of the problems and the settlement of relationships between their elements by operating first with numbers and then with the "algebraic unknown" when solving the equation obtained in the last phase of the successive explorations method.

It is also evidenced that in order to achieve that one student be competent in the algebraic method it is not enough that he/she detaches from the arithmetic and algebraic uses of the unknown, and that even he/she represents a problem with an equation, but that it is also required that he/she progressively advances in the competence to solve algebraically the equations that arose by following the phases of the successive explorations method and then, progressively, of the algebraic method.

THE SUCCESSIVE EXPLORATIONS METHOD

The method consists in six phases, each one with an objective of their own. The first three are basically pre-algebraic and have as paradigm the false position method (Bruño, 1939; Vallejo, 1835). With them, it is intended that the student: a) recognizes that in the problematic situation exists the presence of something unknown that can be determined by considering the limitations of the problem; b) uses a "numerical unknown" to settle the relationships between the elements of the problem through a numerical operations; and c) compares two amounts representing the same thing in the problem in order to construct the sense of the algebraic use of the equal sign (Kieran, 1981). The objective of the other phases is the construction and solution of the equation representing the problem.

To represent its phases we will use an episode of Maribel's interview, a student that was not competent in the solution of problems with the arithmetical method, but that was competent in the solution of some families of problems with the algebraic method and others with the successive explorations method. It was observed that she was using this method when she was not able to understand the relationships between the elements of the problem and, regarding this, Maribel says in one part of the interview: "....when I find it difficult to understand things...I start to use ...to use the method that I took... thus I find out what is happening in the problem...".

The episode begins by showing an algebraic problem, the Problem of the Bottles: *"350lt. of liquor must be bottled in 500 containers, some of them of 3/4 lt. and others of 1/2 lt. How many bottles of each type were used?"*

Phase 1 demands that the unknown must be explicit. This is illustrated in the next:

Maribel: ...Yes...now I am taking out...which are our unknown quantities...[She writes one below another of the unknown quantities: No. of 3/4 lt. bottles and No. of 1/2 lt. bottles].

Phase 2 asks for the assignation of a numerical value to the unknown as a hypothetical solution of the problem, which allows the generation of the problem's operations. Let's see this:

Maribel: ...Now I am going to...suppose...a value...to see if from there...I can go to one equation.

Interviewer: ...What are you going to do?

Maribel: There are 500 containers...I guess we have 200 here... [She uses this number as a "numerical unknown" and points out where she wrote "No. of 1/2 lt. bottles].

The use of a "numerical unknown" is now helpful to Maribel to construct and give sense to the relationships between the elements of the problem as follows:

Maribel: ...I multiply 200 containers by 1/2lt...and I have ...100 litres...[This represents the litres in the 200 bottles]. Then she adds: ... I have 500 containers...then...there must be here...300 of...[She points out the other unknown: No. of 3/4 lt. bottles]

Interviewer: That's it...How did you get it? ...

Maribel: I got it from...subtracting 200 from 500...then multiply 300 containers by 3/4 lt. ...that would be...75lt...[This value representing the litres in 300 3/4 lt. bottles, but she forgot multiply by 4].

Phase 3 prepares the path of the equation. It requires the comparison between two amounts representing the same thing in the problem. In the interview,

Maribel: ...They are 175 litres...which is not the same to 350... [Maribel compares two amounts representing the total of litres to be bottled in 500 containers].

Interviewer: ...Isn't it the same?

Maribel: ...No...then I get from there my comparison ... [She writes: 175lt. = 350lt? and below each of these numbers their meaning in words.]

Phase 4 indicates that it is necessary to work backwards starting from the comparison: A=B? recovering the operations made with the purpose of getting a "numerical pattern". Let's illustrate it:

Interviewer: Now...What do you do?

Maribel: ...I recover operations...the 175 came out...of the addition of 100 plus 75...the 100 came out of...200...which is the value I supposed by...1/2 ... and the 75 came out of ...300 by...3/4...[While speaking, she wrote the operations and obtained the numerical expression: $200\times1/2+300\times3/4=350$?]. After reading what she wrote Maribel adds: But the 300 came out of...the subtraction of 200...minus...well...500 minus 200... [When saying this, the student wrote the numerical pattern: $\underline{\mathbf{200}}\times1/2+(500-\underline{\mathbf{200}})\times3/4=350$?]

Phase 5 requires the use of a letter in order to represent the unknown, as well as the construction of the equation of the problem from the numerical pattern obtained in the previous phase, as follows:

Maribel: In order to get the equation...we say that "x" is equal to the 1/2lt. bottles that...are used...and we substitute the supposed value... [We can see that the letter is used here as a name]

Interviewer: That is…the "x" is the same as the supposed value?...

Maribel: You can say so, yes, because…with the "x" we are giving…a supposed value

Interviewer: The "x" is two hundred?

Maribel: No…

Interviewer: So? ... the "x"...What is it?...

Maribel: The "x" is the exact 1/2 lt. bottles used to bottle one part of the 350 litres …

When saying the abovementioned and writing: $1/2\ x + (500 - x) \times 3/4 = 350$, Maribel is detached from the sense of the numerical use of the unknown that fosters the successive explorations method.

Phase 6 asks to solve algebraically the equation obtained. In the interview:

Interviewer: …What do you do to find out the value of "x"? She says:

Maribel: The "x"…I solve the "x" to know its value…that is to say…I solve the equation

THE "ARITHMETICAL UNKNOWN" AND "NUMERICAL UNKNOWN" IN EDGAR'S INTERVIEW

The use of an "arithmetical unknown". This student was chosen because he showed competence to solve some families of problems with the arithmetical method and others with the algebraic one. Besides, it was observed that even though he didn't use the successive exploration method's phases systematically; he supported himself above all in the numerical exploration to unchain the analysis of complex problems. An episode of Edgar's interview is shown as follows, where we can see the solution of a problem, through the arithmetical method and the arithmetic use of the unknown, ("arithmetical unknown").

The episode begins showing the Perfume Problem: *"How many millilitres of perfume essence must be added to 60 millilitres of alcohol to get a lotion with 70% of perfume?*. Once Edgar has read it he says:

Edgar: …We have 60ml. of alcohol…and we need it to have 70% of perfume…then the other 30% missing here…it is supposed to be covered by alcohol…then…because there are 60ml…they are divided by 3… [Edgar writes: 10% — 20ml] …each 10% of the mix…they will be 20ml…then…a 30% is 60ml….a 10% will be…20ml… Then…I need them to be 70% of perfume…I multiply 20 by 7 …it will be 140….millilitres…of perfume [This is the value of the arithmetical unknown. It is observed that Edgar does not operate with such unknown].

The use of a "numerical unknown". In the following episode of the interview it is observed how Edgar detaches from the arithmetical use he was giving to the unknown in the Perfume Problem presented to him a few minutes earlier by posing him a problem which complexity leads him to use one "numerical unknown" to facilitate its analysis and where he uses numbers with this representation of the unknown to obtain the relationships between the elements of the problem. Then, the student supports himself on the numerical representation of the problem to give an algebraic sense to the unknown and to get the equation representing the problem.

The episode begins by presenting the student the Problem of the Perfumes' Factory: *"In one perfumes' factory are two containers, one with 18% concentration of Chanel and the other with 43% concentration of Chanel. How many millimetres from each container must be used if a 12ml. bottle is needed and which must have a concentration of 36% of pure Chanel?"*

In the study of cases, it was observed that when the student always expressed explicitly the unknown quantities of the problem, it was the first signal that he/she was not going to use an "arithmetical unknown" to solve it. This is evidenced when Edgar draws three containers and says:

Edgar: ... How many millilitres...will be taken out from each container to fill this one...Right?...[He points out the 12ml. bottle]...once again we do not know how many millilitres are here...[He points out the containers with 18% and 43% of perfume]. Then, Edgar adds: ...We have to pour from both containers ...but they are in different concentrations...then...they are 12ml. ...let us suppose...if we pour half of it there will be ...18% from the first one and 43% from the second one ...then ...out of this 12ml....we will take 6ml. of each container ...right?...

It can be observed that at the moment of supposing a numerical value, Edgar detaches from the sense of the arithmetical use of the unknown using this value ("numerical unknown") to get the relationships between the elements of the problem:

Edgar: Yes ...half and half...then...of the first container, from the six millilitres ["numerical unknown" value] we took ... we know that...one 1.08 [He multiplies the "numerical unknown" value by 0.18] ...will be perfume and from the second container ... 2.58ml. [He multiplies 6 by 0.43] ...will be perfume...then ...once putting together what was taken from both containers ...there will be ...3.66ml. of perfume...in the 12ml. bottle we are going to fill...

Interviewer: Is there anything else you have to do?

Edgar: Now...we have to see if this ... [He points out the 3.66ml. of perfume] is the 36% of this amount... [He points out the 12ml. bottle]...now...we will get the 36% of 12...we see that...it is 4.32...which is incorrect...because it must be 3.66 ... [3.66ml. is compared to 4.32ml. which are the two amounts representing the same in the problem, phase 3 of the method].

After doing this, Edgar assigns other values to the "numerical unknown" and gets the pattern: $\underline{\mathbf{8}}\times18/100 + (12-\underline{\mathbf{8}}) \times 43/100=12\times36/100$ and as of such pattern he constructs the following equation: $x\times8/100 + (12-x) \times 43/100)=12\times36/100$. It is observed that in a first moment, the "x" is used as a name, but already being under tension with an algebraic use since the letter arose from a "numerical unknown" which has been operated. The tension ends with the algebraic solution of the equation.

CONCLUSIONS

As it can be observed in the episodes of Maribel's and Edgar's interviews, the numerical approach to solve algebraic-arithmetic word problems proposed can be used as a bridge between the arithmetic method which is tended to be used even in the university level (Ursini & Trigueros, 1997) and the algebraic method. The use of the successive explorations method revealed several aspects: a) the need to distinguish and characterize three senses of use of the unknown ("arithmetic unknown", "numerical unknown" and "algebraic unknown"); b) the need to pose problems propitiating the student to detach him/herself from the use of an "arithmetical unknown"; c) the possibility to achieve this by using one "numerical unknown", since this unknown is less abstract than the algebraic one; d) the possibility to unchain the analysis of some families of complex problems by making easier the settlement of the relationships between the elements of the problem with the use of numbers instead of letters; and e) the possibility to prepare the construction of the algebraic use of the unknown and of the algebraic method by connecting them to the syntax of the algebraic language.

REFERENCES

Bruño, G. M. (1939). '*Elementos de Aritmética'*. Imprenta de la Vda. de C. Bouret. París/México, pp. 329-333.

Filloy, E. and Rubio, G. (1993). 'Didactic Models, Cognition and Competence in the Solutions of Arithmetic & Algebra Word Problems'; *XVII Annual Conference for the Psychology of Mathematics Education.* Vol. I, pp. 154-161. Tsukuba, Ibaraki, Japan.

Filloy, E.; Rojano T. & Rubio, G. (2001). 'Propositions concerning the resolution of arithmetical – algebraic problems? in *Perspectivas on School Álgebra.* Sutherland, R.; Rojano, T.; Bell, A. & Lins, R. (eds.), Dordrech/Boston/London: Kluwer, pp. 155- 176.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics,* 12, pp. 317-326.

Piaget, J. (1979). 'Los problemas principales de la epistemología de la matemática". Tratado de lógica y conocimiento científico'. Ed. Paidos. Buenos Aires, pp. 170-182. (Original version: Logique et connaissance scientifique. Gallimard, París).

Rubio, G. (1990). 'Algebra Word Problems: A Numerical Approach for its Resolution (A teaching Experiment in the Classroom)', *XIV Annual Conference for the Psychology of Mathematics Education.* Oaxtepec. México. Vol II, 125-132.

Rubio G. (1994). 'Modelos Didácticos para Resolver Problemas Verbales Aritmético-Algebraicos. Tesis Teóricas y Observación Empírica'. PhD Tesis unpublished. Departamento de Matemática Educativa. CINVESTAV. México.

Ursini, S. & Trigueros, M. (1997). 'Understanding of different use of variable: A study with starting college students', *XXI Annual Conference for the Psychology of Mathematics Education.* Vol IV, pp. 254-261. Lahti, Finland.

Vallejo, J. M. (1835). 'Compendio de Matemáticas'. Tomos I y II, primera edición. Rosa y Bouret, Imprenta Waldor, París.