GROWTH IN STUDENT MATHEMATICAL UNDERSTANDING THROUGH PRECALCULUS STUDENT AND TEACHER INTERACTIONS

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This paper investigates the role of teacher interaction in the development of mathematical understanding of five students who worked together on a math-modelling task. The dialogue between the teacher/researcher and students is analyzed. Preliminary findings suggest that where the mathematical thinking of the students was understood, interventions helped develop students' thinking. The Pirie-Kiernen model of mathematical understanding guided interpretation of student mathematical thinking and understanding. [1]

INTRODUCTION

The students, engaged in conversation with the teacher, often give explanations for their ideas. A question arises as to what influence, if any, the teacher's response to those explanations have on student progress. This report examines dialogue between teacher and students and seeks to investigate the effect on students' growth in mathematical understanding.

The data come from a two-week summer institute that was a component of a longitudinal study on the development of proof making in students [2]. The students worked in groups on precalculus level mathematics problems. This paper focuses on one group of students and one of the problems they examined.

THEORETICAL FRAMEWORK

Communication is an essential part of the mathematics classroom providing a means for students to express their ideas and explain their thinking (NCTM 2000). Through communication students can share ideas and discoveries about the mathematics on which they are working. Since the communication process helps students create meaning for their ideas, NCTM (2000) includes communication as one of the standards in *Principles and Standards for School Mathematics*. Because of the need for communication in the classroom, an environment can be created where teachers and students engage in very important dialogue. Dialogue is important because it helps teachers assess the student's mathematical understanding and allows the students to clarify and express their ideas. Towers (1998) developed several themes to describe teacher interaction with students and illustrated how these interactions occasion the growth of students' understanding. A teacher should be skilled in interacting with students in order to gain access to students' mathematical understanding. Teacher questioning can help students justify and extend ideas, make connections, and generalize their conjectures (Dann, Pantozzi, Steencken. 1995). The development of these skills is not immediate for the teacher, but once gained the teacher has an effective way to facilitate the growth of a student's understanding (Martino and Maher, 1999).

Being a participant in the classroom discourse, the teacher has an important function. In describing a classroom where students are working in small groups on a task, Maher, Davis, and Alston (1991) indicate that the teacher plays many roles: listening to children, offering suggestions, asking questions, facilitating discussions, drawing out justifications. When students discuss with their teachers the meaning of mathematical notions, students are expected to think about concepts, their meanings and their interrelations (Vinner 1997). If students do think about concepts, they are in a conceptual mode of thinking (Vinner 1997). If students do not think conceptually, but still produce answers which seem to be conceptual, then Vinner (1997) states the students are in a pseudo-conceptual mode of thinking. The teacher must continuously assess whether or not the students have learned the mathematical concept, truly understands the reasoning behind their problem solving approach, and can adequately support and defend their conclusions using their previously learned mathematical knowledge. One way to determine what a student knows is to use a model of understanding. While many models exist, this research analyzes student understanding using the Pirie-Kiernen model of mathematical understanding (Pirie & Kiernen, 1994). This model is "a theory of the growth of understanding which is based on the consideration of understanding as a whole, dynamic, levelled but non-linear process of growth" (Pirie & Kiernen, 1994, p. 83). This theory shows that student understanding is an organization of ones knowledge not an acquiring of categories of knowing.

In a regular classroom, it is not always possible to observe what a student does after an interaction with the teacher. Because this observation is not always possible, it is difficult for a teacher to determine if the interaction was beneficial to the student. Videotape data that follows the student when the teacher leaves make possible gaining a better understanding of a student's actions. Interacting with students is a challenging task for the teacher, who has to make instantaneous decisions. The researcher, who has the benefit of studying and referring to videotape data, however, can learn from the interaction after the fact. What the researcher learns from the interaction can be shared with the teachers, who can reflect on their actions, and help facilitate a growth in students understanding.

METHODOLOGY

Participants. Five students seated at the same table (four males and one female) and one teacher/researcher were subjects in this study. All of the students were entering their fourth year of high school. The teacher/researcher involved in the interaction is an experienced professor of mathematics and mathematics education at the university level.

Task. The students were given a picture of a fossilized shell called Placenticeras. The first part of the task was to draw a ray from the center of the shell in any direction. Then with polar coordinates as a way to describe the spiral of the shell, the students were to make a table of r as a function of theta. After creating the table, the students were asked what they could say about r as a function of theta. The students had graphing calculators, transparencies, rulers, and markers at their disposal for completing this task.

Data Analysis. The data come from a two-hour videotape session during the third day of a two-week Institute. The interactions were coded to consider perspectives of the teacher and the students. For the students, the following codes were developed and used: S(i): Student ignores the suggestion made by the teacher; S(c): Student asks the teacher for clarification of a statement; S(a): Student attempts the teacher's idea or suggestion; S(e): Student engages in conversation for the purpose of explaining their own views. For the teacher: T(r): Teacher restates the problem or returns to an old idea; T(f): Teacher follows the student's idea or suggestion; T(n): Teacher introduces a new idea; T(c): Teacher asks the student to clarify their statements or idea. The codes were used to follow the choices of the teacher and the resulting action by the student. When students became engaged in a conversation, their words were examined for evidence of their understanding.

FINDINGS

The students' own words demonstrate where mathematical understanding occurs, and where their growth about a solution to this problem appears. This dialogue follows a discussion between the teacher and students regarding their solution to the task.

Student 1: S(e) I think it does. I mean if you look, if you look at the regression. It's just like a parabola. And uh your data.

Student 2: It is a parabola.

Student 3: S(e) It is a parabola. A very nice parabola. And like you know. I mean you can't use anything behind past zero on the x obviously because it can't have negative growth. That doesn't make sense. So you can't do that. But I mean the way, the way it goes up and the reason why it goes sharply up is just the fact that. I mean even from here to here like say the distance is 6 then all of a sudden it is 40. It's not going to keep on going little by little. Eventually it's getting wider like this. And that's why it's jumping so high up. It's not the fact that it's off or it's not predicting anything. It's just the numbers are getting larger and larger. It has to go higher and higher. So that's why it goes that steep angle like that.

Teacher: T(f) Ok well. I am still interested in that earlier part because. Are you saying that this animal really started growing where we're saying theta equals zero is.

Student 3: Um hmm.

Teacher: How do you know that?

Student 3: S(e) It's gotta start somewhere. And it doesn't start. You can't start. You can't start.

Student 2: You can't start with anything negative.

Student 3: anything past nothing.

Student 2: Yeah.

Student 3: You know.

Student 2: Cause then it doesn't exist. In which case it's not there.

The students do not look further into the data beyond a visual fit of a scatter plot and their curve. The teacher/researcher returns to the idea about how the model describes the start of the growth of the shell.

Teacher: T(r) See then I am wondering about that fourth power model cause if you go to the left on it. You are sort of going inward on the shell right. You are going backward in time.

Student 3: Yeah.

Teacher: But then suddenly as you keep going left it goes up.

Student 3: S(e) Oh but there is nothing there though. That is the thing. Like you have to set limitations somewhere because some things are just physically impossible you know.

Teacher: I think we're beginning to understand each other.

Student 3: Yeah.

Teacher: Ok, Umm.

Student 3: S(e) I mean its like. I guess its like certain things like if you figure out like differences with like electricity or something or like in physics. Like you can't have things that are. Sometimes you can't have things that are negative. There are things that are just physically impossible to have. And that to have something, to have an animal or a living thing that is a negative distance would mean that it isn't there. So it's not physically possible to have that anything past that zero. You know. It just wouldn't be there. This animal would not be there if there was a negative number. Basically.

The teacher/researcher continues to question the students about why their model does not work for certain values.

Teacher: T(c) Oh, so there's a place. Okay then you are agreeing that there's a place where the regression doesn't model the animal.

Student 2: S(e) You can put it so that the restriction has to be greater than zero.

Student 3: S(e) Yes, but that's necessary for other things too. There's limitations.

Teacher: Okay, Okay.

Student 3: S(e) like like the first graph we did with uh with the running thing, with the uh, with the thing you had to put limitations on it cause there were certain things that went past a certain time.

The teacher/researcher and the students continue the discussion by focusing on the accuracy of the model outside the range of their collected data. The question of what a model would look like if the data were collected again moves the conversation topic to the model's general shape. After this discussion, the teacher/researcher returned to the left side of the student's model. By the left and right side, the teacher/researcher and students are using the origin of the co-ordinate plane as their reference point. Therefore the left side would refer to negative values of time, and the right side would refer to positive values of time. When the teacher/researcher returned to the left side of the student model, the teacher/researcher and students revisited discussing the model's accuracy during negative values of time.

Teacher: T(r) So it looks like we are making sense on the right and then we got questions on the left. Is that fair?

Student 3: Sure, why not.

Teacher: Okay.

Student 2: S(c) What possible questions could you have on the left. It's dead. It doesn't exist.

Teacher: Well I just don't.

Student 3: S(e) Not even that. It's not even born yet.

Teacher: T(r) It's very hard for me, yeah. It's very hard for me to believe that this at some point in the distance past.

Student 2: It doesn't exist.

Teacher: T(r) That that it was very large as the fourth power, as that fourth power curve suggests.

Student 2: S(e) Alright fine, we'll do a third power curve, it'll be very small. It'll be gone.

Student 3: S(e) No, you know what. You know what you got to do. You set a limitation on that graph so there is no left side and then we won't have this problem. Can we do that?

The students have provided a way to adjust the model so that it does not show a large shell when time is negative. Though the two students believe the left side of this model does not accurately portray the growth of the shell, their methods of correcting the inaccuracy are different. Student two wants to change the regression curve to the third power model, which would result in a new equation that models a different rate of change, and continues the inaccuracy of the model before the shell began to grow. Student three explains that he wants to remove the left side and keep the model's representation of the right side.

CONCLUSIONS

The transcript provides evidence of two areas of understanding: the rate of the growth of the shell and the model that represents the growth of the shell. Regarding the first area, the students' understanding of the rate that the shell is growing did not grow during this interaction. For the second area, student three exhibits a growth in understanding from *primitive knowledge* to *property noticing* of the Pirie-Kiernen model of mathematical understanding.

The dialogue showed the students used a fourth power regression to represent the rate of growth of the shell. However, the students used the word parabola to describe the curve. Student one stated, "I mean if you look, if you look at the regression. It's just like a parabola." Student two and three both followed with "It is a parabola". Their early classification of the graph as parabolic demonstrates an *image having* level of understanding. Since parabolic and quartic curves represent different rates of growth, the students are just using the visual image of the function and not the properties of their fourth degree regression equation. Later when student two and three recommended changes for their model, they provided different methods for a correction. Student two suggested changing their regression to a third power, and student three suggested restricting the left side of the model. Since student two's correction used a different regression model, he did not make a connection between the rate of growth and the type of curve needed to model that growth. Neither of the students provided evidence as to why the model is quartic. The students' earlier understanding about the rate of growth did not grow during this interaction because the have not moved beyond a visual inspection of the model's shape.

Despite this misunderstanding, the teacher/researcher did not correct or criticize their comments. Rather the focus of the teacher/researcher was to discuss the students' model for the growth of the shell. By focusing on student three, the transcript shows a growth in his understanding. When questioned by the teacher about the negative values of the fourth power model getting larger, the student responds that there should not be any values because "you have to set limitations somewhere because some things are just physically impossible". Student three's explanation showed a *primitive knowing* level of understanding about their model around zero. The student has demonstrated basic knowledge about when a situation requires limitations. When the teacher/researcher returned to the growth of the shell around zero, the student's level of understanding grew through engagement in the conversation. Without further prompting, the student connected the limitations on the model to other physical situations. This exhibited an *image making* understanding by using previous knowledge about specific situations that need limitations.

Despite a growth in understanding to *image making*, the teacher/researcher asked the student to clarify his ideas about where the regression function does not model the shell. As a response, student three referred to "the running thing", an earlier problem from this workshop. This response moved him to an *image having* level of understanding. He used a mental construct of that activity to further justify his ideas about the need for a limitation. The teacher/researcher returned to the idea of the left side of the model after a discussion of the general shape of the model. During this engagement, student three explained how to change the model so the left side did not exist. He suggested, "You set a limitation on the graph so there is no left side and then we won't have this problem". The student demonstrated another growth in understanding to *property noticing*. The student recognized that the model should be altered to a model that has the same property as other limitation situations. He suggested changing the model so it contains the property of having no values on the graph for negative time. Previously the student explained why the shell could not exist for negative values, but has now moved forward to provide a possible method for representing the limitation on the model.

Through interaction between the teacher and students, the students made public their level of mathematical understanding. By examining the episodes presented, one can see that the teacher/researcher consistently returned to the idea of how the model demonstrated the growth of the shell over the entire domain of the students' model. Additionally, the students' ideas are followed or they are asked to clarify their statements. Using this method of questioning, the students were given the chance to make connections and reorganize their thoughts about their model. By reorganizing his thoughts, student three's understanding grew from *primitive knowledge* to *property noticing*. The opportunity for growth occurred because the teacher continually returned to old ideas. As a consequence, the students had multiple possibilities to become engaged in conversations and articulate their understanding of the mathematics.

This research provides a foundation for continuing a dialogue about the affects of teacher and student interactions in the classroom. These preliminary findings imply that teacher interaction helps the student to express their mathematical understanding. Further research can help to indicate whether the teachers/researchers can learn from their choices during interactions to see if they are constructively contributing to students' progress. More research is needed to provide a better understanding of how teacher interventions, particularly questioning, can contribute to students' mathematical understanding.

NOTES

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