ALGORITHMIC MODELS: ITALIAN AND ISRAELI STUDENTS' SOLUTIONS TO ALGEBRAIC INEQUALITIES

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This paper describes a study regarding Israeli and Italian students' solutions to algebraic inequalities. Fischbein's notions of intuitive and algorithmic knowledge are used to analyze the data. The findings presented here show similarities in students' correct and incorrect solutions, in both countries. The findings indicate that students intuitively considered the balance model, saying that it is always permitted to "do the same thing on both sides" of a given inequality. Most students were intuitively drawing analogies to the solutions of related equations, either by excluding only zero values when dividing both sides of an inequality by a not-necessarily positive value, or when dividing by an expression without the exclusion of zero values as well.

There is a wide call for using students' correct and incorrect ways of thinking in teaching (e.g., NCTM, 2000). While this recommendation seems to be "speaking for itself", any attempt to take it from theory to practice, shows how complex this is. Among the prerequisites for such teaching, are familiarities with students' correct and incorrect reactions to various related tasks, with possible reasons for students' errors, and with available teaching approaches to be considered under specific circumstances. All of the above are needed, but do not guarantee that the student whom we teach will gain mathematical understanding. In this paper we use Fischbein's notions of intuitive and algorithmic knowledge as a theoretical framework to analyze students' solutions to inequalities, and consequently, we suggest possible implications for teaching.

Fischbein's theory deals with the formal, the intuitive and the algorithmic components in one's mathematical performance (e.g., Fischbein, 1993). According to Fischbein, formal knowledge is based on propositional thinking. It relates to rigor and consistency in deductive construction, being free of the constraints imposed by concrete or practical characteristics. Intuitive knowledge is a kind of cognition, which is accepted directly and confidently as being obvious, imparting the feeling that no justification is required. Algorithmic knowledge is the ability to use theoretically justified procedures. Each of the three components plays a vital part in students' mathematical performance, but since they are usually inseparable, the relations between them are not less significant. Fischbein explained that "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures" (ibid. p. 14). Consequently, he identified and investigated with his colleagues a number of algorithmic models related to various mathematical operations, such as subtraction of natural numbers, and methods of reduction in processes of simplifying algebraic expressions (e.g., Fischbein & Barash, 1993).

They explained, for instance, that the distributive law serves as a prototype in the algorithmic models of simplifying algebraic expressions, triggering students to present $3(a+b)^2$ as $3a^2+3b^2$. Here, the algorithmic model is expressed in the shift from $(a+b)^2$ to a^2+b^2 to if these are equivalent expression, like $2\cdot(a+b)$ and $2\cdot a+2\cdot b$. This paper discusses Italian and Israel secondary-school students' intuitive ideas and algorithmic models when solving algebraic inequalities.

Usually, publications in mathematics education journals present instructional suggestions with no research support. They recommend, for instance, the sign-chart method (e.g., Dobbs & Peterson, 1991), the number-line method (e.g., Parish, 1992), and various versions of the graphic method (e.g., Dreyfus & Eisenberg, 1985; Parish, 1992). Only little attention has been paid to students' conceptions of inequalities (e.g., Bazzini, 2000; Linchevski & Sfard, 1991; Tsamir, Almog & Tirosh, 1998; Tsamir & Bazzini, 2001). These studies pointed, for instance, to students' difficulties in grasping the role of the sign, and in using logical connectives.

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities (see also Tsamir & Bazzini, 2001). In this paper we focus on the question: What intuitive ideas and what algorithmic models can be identified in Israeli and Italian secondary school students' solutions to algebraic inequalities?

METHODOLOGY

Participants

One-hundred-and-ninety two Italian and 210 Israeli high school students participated in this study. All participants were 16-17 year old who planned to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

In both Italy and Israel, algebraic inequalities usually receive relatively little attention and are commonly presented in an algorithmic way by discussing various algebraic manipulations. That year, the participants in this study had studied the topic of algebraic inequalities, including linear, quadratic, rational and absolute value inequalities. In both countries, the participants were taught different methods for solving the different types of inequalities, such as, graphic methods to solve quadratic inequalities, and "multiplying by the square of the denominator" for the solutions of rational inequalities.

Tools

Italian and Hebrew versions of a 15-task questionnaire were administered to the students. Here we focus on three tasks dealing with "dividing an inequality by a not-necessarily-positive factor". Two tasks ask to judge statements regarding parametric inequalities, and the third task, poses a parametric "solve" inequality.

Research findings indicate that when solving rational inequalities, students frequently multiplied both sides of given inequalities by a negative number without changing the direction of the inequality (e.g. Tsamir, Almog & Tirosh, 1998). It was also reported that students encounter difficulties when solving mathematical tasks, presented in a way different from the way they are used to. For example, when having to deal with parametric equations and inequalities that are commonly not discussed in class (e.g., Furinghetti & Paola, 1994; Ilani, 1998). Taking into account these data, we constructed tasks I, II, and III.

Task I: Examine the following claim: for any a in R, $a \cdot x < 5 \implies x < 5/a$

Task II: Examine the following statement: for any $a \neq 0$ in R, $a \cdot x < 5 \implies x < 5/a$

Task III: Solve the inequality: $(a-5) \cdot x > 2a-1$ x being the variable and 'a' a parameter.

The tasks were designed to provide information regarding students' distinction between the sufficiency of mentioning a=0 as a counterexample in Task I, and the insufficiency of the '$a \neq 0$' condition given in Task II, a condition, which is sufficient in the case of equations. Our aim was to see whether students regarded this limitation as sufficient for the inequality as well. While Tasks I and II asked the students to examine the claim regarding the equivalence of parametric inequalities, Task III dealt with the same issue in a different manner, asking the students to solve a similar given parametric inequality.

Procedure

The students were given approximately one hour, during mathematics lessons, to complete their written solutions. In order to get a better insight into the students' ways of thinking, forty-five students were individually interviewed, usually being asked to elaborate on their written solutions. Each interview lasted 30 to 45 minutes.

RESULTS

The results will be presented in the following order. First, we present an analysis of students' responses to Task I, Task II and Task III. Then, we present the intuitive ideas and the algorithmic models we identified in the students' reactions to the three tasks. It should be noted that no significant differences were found between the Israeli and Italian students' solutions.

Students' Reactions to Task I

Most of the participating students correctly responded to this task (see Table 1). About three- quarters of the students accompanied their correct judgement with an acceptable justification – 18% of them elaborated on the role of the sign of the value substituted for a in determining the direction of the ">", and 54% mentioned only the "zero case" as a counterexample to the given statement.

In their explanations, students who *related to the sign of a* wrote, for instance, "for $a>0$ this is a correct statement, but for $a \leq 0$ it is not"; or "there are three cases: when $a>0$ the statement is correct, for $a = 0$ it is impossible to divide by zero, and when a is negative the conclusion is that $x<5/a$". Others explained more briefly that, "this statement is correct only for positive 'a's"; or "when a is negative the direction of the sign changes".

A number of students provided specific *counterexamples*, such as, "if a=(-1) the statement is not correct"; or "if –5x<5 the conclusion is that x>(-1), instead of x<(-1)". A few added "I gave one example, but a single counterexample is sufficient for proving that the statement is false."

Table 1: Frequencies of students' solutions and explanations to Task I (in %)

Judgement Justification	**ISRAEL** N=210	**ITALY** N=192	**TOTAL** N=402
False*			
Relating to the sign of a	20	16	18
Relating only to $a \neq 0$	59	48	54
True	6	2	4
Other	15	34	24

* Correct solution

Most prevalent was the students' tendency to use *only the 'a=0' case,* as a *counterexample* to refute the statement. They wrote, for instance, "this statement is false when a equals zero"; or "the statement is false, because of the case of a=0". Many added "division by zero is undefined, therefore the statement is not *always* correct."

In their oral interviews these students' typically commented,

> Sophia: the statement here refers to any number. BUT, since it is false for a=0, the statement is not true for ANY number. It is, therefore, false.
>
> Gabby: It is enough to show that the statement is false in one case, and zero is such a case.
>
> Dana: One counterexample is sufficient for proving that the statement is incorrect. Zero, that is a=0 is such a counterexample.

A number of them added,

> Gabby: It is similar to equations.
>
> Dana: I am sure, because I know it from equations.

The few students, who regarded this statement as correct, explained, "we divided both sides by the same thing". One student based his incorrect "true" judgment on an example: "this is correct. For example, if 2x<5 then x<2.5".

In their oral interviews these students added,

> Naomi: I divided both sides by a [pause]. It's OK. I operated in the same way on both sides.
>
> Jonathan: It's OK to do the same thing on both sides. When doing the same operation on both sides, the equivalency is preserved.
> Interviewer: What do you mean?

Jonathan: In equations, it is allowed to add, subtract, and multiply [pause] to do any operation with the same number on both sides.
Interviewer: What do you mean by "allowed"?
Jonathan: [thinking] it does not change the solution…
Interviewer: You related to equations, but here we have an inequality…
Jonathan: It's the same…

Students' Reactions to Task II

Only 30% of the participants correctly responded that the claim is false and accompanied their response with a valid justification (Table 2). All of them related to the role of the sign in their decision. They explained, for instance, "if a is negative then $x>5/a$"; or "the claim is correct only for positive a". Some students added specific examples, "It is false, because it holds only when a is positive. For example, if $a=(-2)$, then $-2\cdot x<5 \Longrightarrow x>(-2.5)$". Students were usually satisfied with a single counterexample, occasionally explaining, "one counterexample is sufficient in order to show that the proposition is false".

Table 2: Frequencies of students' solutions and explanations to Task II (in %)

Judgement *Justification*	**ISRAEL** N=210	**ITALY** N=192	**TOTAL** N=402
False*			
Valid explanation	37	23	30
True			
Relating to $a \neq 0$	50	57	53
Other	13	20	17

* Correct solution

Most prevalent in both countries (over 50%) was the incorrect response that the statement is true, which was explicitly based on the given that $a \neq 0$. Students wrote, for instance, "It is correct because of the given condition that $a \neq 0$." In the interviews, these students' elaborated explanations pointed to connections they made between equations and inequalities. In the interview Daniel explained,

Daniel: I can show how this works in an easier case [pause]…
Interviewer: What do you mean?
Daniel: I am looking for an example of a similar, but easier, task… like $2x=6$. Yes, here, x simply equals 6/2 [pause], then, $ax=6$ means that $x=6/a$, but this is true *only* if a is not zero. If $a=0$ than it is impossible to divide… by zero.
Interviewer: Your example is an equation, and here we have an inequality.
Daniel: It's the same [pause]. In a way inequalities are a certain type of equations. Just that equations are easier, so I use simple examples of equations when I have a difficult inequality.

Several students gave explanations, similar to the following one given by John,

John: In equations and inequalities, dividing by zero is problematic. But if we deal with this problem, operating with identical numbers and by means of the same operation on both sides is not only permitted, it is actually *the way* to solve the given tasks.

Students' Reactions to Task III

Fewer than 15% of the participants provided a comprehensive analysis of the various (positive, zero and negative) options for 'a' (Table 3). About 50% of the participants did not solve this task, and more than a quarter used an "equation-like" approach, writing that $x>(2a-1)/(a-5)$ for $a\neq 5$. In their interviews, a substantial number of them clearly mentioned drawing analogies to equations. Bettina, for instance, explained,

Bettina: I divided both sides by the same expression, but I had to make sure that it is a non-zero expression. So, I wrote that a cannot be 5, because then a-5 equals zero...

Interviewer: Are you sure that five is the only problematic value here?

Bettina: [confidently] sure. I have done that a million times when solving equations.

Table 3: Frequencies of students' solutions to Task III (in %)

	ISRAEL N=210	ITALY N=192	TOTAL N=402
Full Solution*	13	12	12
$x>(2a-1)/(a-5)$	18	15	16
$x>(2a-1)/(a-5)$, $a \neq 5$	29	26	27
Other	41	48	46

* Correct solution

About 15% of the students gave the $x>(2a-1)/(a-5)$ solution (with no mentioning of the $a\neq 5$ limitation), and in the oral interviews they frequently mentioned the use of equation-ideas. All of them explained that it is allowed to "do the same thing on both sides of an inequality". Many, like Anna, related to equations and to inequalities,

Anna: I divided both sides by a-5?

Interviewer: Is it OK to divide both sides by a-5?

Anna: Yes. I have done the same thing on both sides. If you do the same thing on both sides of an equation [pause], I mean an inequality [pause], actually both, you reach an equation or an inequality that has the same solution as the given one.

Interviewer: Always?

Anna: It is not only allowed, it is necessary to do that in order to solve the problem.

DISCUSSION

We opened this article by referring to Fischbein's notion of algorithmic models, rooted in the application of intuitive ideas and/or formal overgeneralizations to rigid algorithms (e.g., Fischbein, 1993). These models are usually coercive, used with confidence and grasped as being self-evident, even though they frequently lead to erroneous solutions. According to Fischbein, algorithmic models evolve when students' intuitive ideas manipulate their formal reasoning and/or their use of algorithmic procedures. In this spirit we posed the question: What intuitive ideas and what algorithmic models can be identified in Israeli and Italian secondary school students' solutions to algebraic inequalities?

An initial examination of the Italian and Israeli participants' written solutions to Task I revealed almost no intuitive, erroneous ideas. Most of the students correctly rejected the statement: $\forall a \in R \ a \cdot x < 5 \implies x < 5/a$, and based this rejection either on a comprehensive analysis of the sign of a, or on a single counterexample. In the latter case, many students used zero as their counterexample, and most of them, correctly stressed that one counterexample is sufficient for the refutation of a proposition.

Had we stopped here, we might have assumed that most students have a good formal understanding of such parametric inequalities. However, an examination of their responses to tasks II and III revealed that this was not the case. The students' solutions to the latter two tasks, and their explanations in the interviews, clearly pointed to their intuitive grasp of inequalities as being "similar to", "the same as", "a certain type of" or analogous to equations. That is to say, equations were found to serve as a prototype in the algorithmic models of solving inequalities. This algorithmic model had two appearances: (1) "do the same operation with the same numbers on both sides"; or (2) "exclude the possibility of zero in the denominator, and then, do the same operation with the same numbers on both sides".

Those who applied the concise version of the algorithmic model (1), could easily be identified, since they incorrectly solved Tasks I and II by judging the statements to be true, or erroneously wrote in Task III that $x>(2a-1)/(a-5)$ is the solution to the inequality $(a-5)\cdot x>(2a-1)$. These students used the balance model when incorrectly solving equations, assuming that "doing the same thing" on both sides of an equation *always* leads to an equivalent equation, and consequently to the solution. When solving inequalities, they based their solutions on the equation-model, and thus, made no restriction when dividing (same operation) both sides of the inequality by the same expression. They were so sure that equations and inequalities follow the same set of rules, that they occasionally confused the two terms when explaining their solutions (See interviews with Anna, Dana and Gabby).

Most participants applied the second version of this algorithmic model. That is to say, they knew that when solving equations they should be careful not to divide by zero, and since they held the equation-model for solving inequalities, they imposed the same condition. These students usually answered Task I correctly, providing the '$a=0$' case as a counterexample to refute the given proposition. However, they frequently, incorrectly regarded the proposition of Task II as valid or suggested $x>(2a-1)/(a-5)$ $a\neq5$ as the solution for Task III. The use of the equation-model by these students could not be identified by their written solutions to Task I only.

We have seen three, parametric inequalities that may be helpful in triggering students who hold different equation-based models to answer differently, and occasionally also incorrectly. We would like to suggest such tasks to be presented and discussed in class, in order to promote students' awareness of their intuitive ideas and the resulting algorithmic model they intuitively use. How to implement such instruction and their impact on students' performance should further be investigated.

In the presentation we will show, how the equation-model was identified in students' solutions to additional inequalities in our study.

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