Definitions and Images for the Definite Integral Concept

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Definitions and images, as well as the relation between them of the definite integral concept, were examined in 41 English high school students. A questionnaire was designed to explore the cognitive schemes for the definite integral concept that are evoked by the students. One question aimed to check whether the students knew to define the concept of definite integral. Five others were designed to categorize how students worked with the concept of definite integral and how this related to the definition. The results show that only 7 students out of 41 of our sample knew the definition.

All mathematical concepts except the primitive ones have definitions. Many of them are introduced to high school or college students. However, the students do not necessarily use the definition to decide whether a given idea is or is not an example of the concept. In most cases, they decide on the basis of their concept image, that is, all the mental pictures, properties and processes associated with the concept in their mind. (Tall & Vinner, 1981; Rasslan & Vinner, 1997).

The concept of the definite integral is a central in the calculus. In many countries, including the UK, it is taught in the last two years of school to students aged approximately 16–18. The students in this study followed a curriculum based on the School Mathematics Project A-level. In the current version of the textbook (SMP, 1997), integration is introduced through activities to estimate the area between a graph and the x-axis using pictures and numerical methods. After this experience the notion of integral is defined as follows (in the form of a description rather than a formal Riemann sum):

> The symbol $\int_a^b f(x)\,dx$ denotes the *precise* value of the area under the graph of f between $x = a$ and $x = b$.
> It is known as the integral of y with respect to x over the interval from a to b.
> The integral can be found *approximately* by various numerical methods.

Figure 1: The definition of the integral concept (SMP, 1997, p.143).

This is followed by ten pages of experience with numerical approximations including functions with positive and negative values before algebraic integration is introduced in the next chapter. Here the student is encouraged to build up the relationship between polynomials and their (definite) integrals, before the fundamental theorem is introduced using local straightness as a visual form of derivative in the following terms:

> For any differentiable function f, $\int_a^b f'(x)\;dx = f(b) - f(a)$.

Figure 2: The fundamental theorem of calculus (SMP, 1997, p. 169).

Several studies have highlighted difficulties with the integral concept. Tall (1993) remarked on conflicts and contradictions that arise as students study the calculus. For instance, in the 'function of a function rule' $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, the student is told not to cancel the dx—it has no separate meaning. Later in $\int f(x)\,dx$, its meaning changes to 'with respect to x'. Orton (1980) observed student difficulty with the integral $\int_a^b f(x)\,dx$ when $f(x)$ is negative or b is less than a. Mundy (1984) reported student problems with integrals slightly beyond their experience, such as $\int_{-3}^{3} |x+2|\,dx$.

Students also experience difficulties with communication. Rather than responding conceptually, they may exhibit 'pseudo-conceptual behaviour' by using minimal effort to respond in a way thy hope will satisfy the teacher (Vinner, 1997; Rasslan & Vinner, 1997). Examples of such responses will arise in this study.

The empirical data collection is based on a questionnaire designed to seek the students' definition of the definite integral and to categorise their responses to selected problems for their use of the definition or image. The research questions are as follows:

- What definitions of the definite integral are given by high school students?
- What images of the definite integral do students use in various problems?
- What misconceptions do they exhibit relating to the definite integral?

METHOD

Sample

Our sample comprised 41 students in four classes of final year ('upper sixth form') English high school students. All these students had access to graphical calculators and had encountered all the concepts on the test. The average A-level score of the school is 20.2, which is above the national average of 18.5.

The Questionnaire

The Questionnaire in figure 3 was administered to all subjects in the sample. Questions 1 to 5 were designed to examine aspects of the respondents' concept images revealed through *doing* integration, whereas Question 6 was designed to examine their definitions. Question 1 was designed to examine how students apply the definite integral to integrals when the integrand becomes infinite. Question 3 examines whether the students understand an integral when the function changes its sign.

Questions 2, 4, and 5 test the student using functions that are not simple formulae. They are more easily answered by drawing the graph and calculating the area directly from the picture. Are the students able to see the integral as 'the *precise* value of the area under the graph' as given in the definition in the text, or do they feel a need to carry out symbolic integration attempting to extend the techniques at their disposal?

1. Find, if you can: (a) $\int_0^6 \frac{1}{(x-4)^{2/3}}\,dx$ (b) $\int_{-1}^{2} \frac{1}{x^4}\,dx$.

If you can, please explain the sign of the answer.

2. The function $f(x) = x - [x]$ is given. Find the area directly below the graph and above the x--axis between $x = 0$ and $x = 3$.

3. Find the area bounded between the function $y = \sin x$ and x-axis over $[0, 2\pi]$.

4. The following function is given: $f(x) = \begin{cases} 2x, & x \le \frac{1}{2} \\ 2x-2, & x \ge \frac{1}{2} \end{cases}$. Find: $\int_0^1 f(x)\,dx$.

5. The following function is given: $f(x) = 1 - |x-1|$. Find $\int_0^2 f(x)\,dx$.

6. In your opinion what is $\int_a^b f(x)\,dx$ (the definite integral of the function f in the interval $[a, b]$).

Figure 3. The Questionnaire

Procedure

The questionnaire was administered to the students in their classes. They were not asked to fill in their names, only their background information. It took them 40-50 minutes at most to complete the questionnaire. All the questions in the questionnaire were analysed in detail by the two authors in order to determine the answer categories.

RESULTS

The Definition Categories

We categorized the student's answers according to methods described elsewhere (Vinner & Dreyfus, 1989; Rasslan & Vinner, 1997). We illustrate each category with a number of sample responses.

Question 6: The definition of $\int_a^b f(x)dx$

Category I: The area between the graph and the x-axis between $x = a$ and $x = b$. (4/41).
Example: The area between the x- axis and the graph $f(x)$ between the limits $x = a$ and $x = b$.

Category II: A procedure of calculation; $\int_a^b f(x)\,dx = F(b) - F(a)$. (3/41).
Example: $\left(\int fx \text{ where } x = a\right) - \left(\int fx \text{ where } x = b\right)$.

Category III: Students substitute specific formula in the definite integral. (3/41).
Example: $\left[x^2/2 + c\right]_a^b = (b^2/2 + c) - (a^2/2 + c)$.

Category IV: Answers based on pseudo-conceptual mode of thinking or wrong answers. (5/41).
Examples: 1. $[f(x)]_a^b$. 2. $f(x)$ increases.

Category V: No answer or missing answers. (26/41).

In the above categorization only seven students out of 41 (categories I, II) gave a definition of the definite integral concept. The three category III students show that they can *use* the concept of integration in specific cases. The majority are in category IV and V, with five erroneous responses and twenty six not responding. The students were not directed to memorise definitions and the majority do not appear to be able (or willing) to explain the definition of the definite integral.

The Concept Images. Questions 1-5

Various aspects of the definite integral concept, as conceived by the students, were expressed in their answers to questions 1 to 5. Some of these aspects are given below:

Question 1.a

Category "Zero": Students with correct theory. (0/41).

Category I: The definite integral is the area between the function and x-axis in $[a, b]$. (7/41)

Category I_a: As above with the correct calculation of the definite integral. (3/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = \left[3(x-4)^{1/3}\right]_0^6 = 3.78 - -4.76 = 8.54$. The graph is all above the x-axis from 0 to 6, and so the sign will be positive.

Category I_b: As above, but only giving the final answer without showing any working.(3/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = 8.542$. The area indicated by the integral is above the x-axis.

Category I_c: As above, with wrong calculation of the definite integral. (1/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx \Rightarrow (x-4)^{-2/3} \Rightarrow (x-4)^{1/3} \Rightarrow \left[3(6-4)^{1/3}\right] - \left[3(0-4)^{1/3}\right] = 2 - -4 = 6$.

If the sign is positive, the area calculated it from the x-axis and above (i.e. for positive values of y).

Category II: Answers without explanation: the definite integral is a procedure of calculation. (27/41)

Category II_a: As above, but with right calculation of the definite integral. (8/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = [3(x-4)^{1/3}]_0^6 = 3.78 - - 4.76 = 8.54$.

Category II_b: As above, but with wrong calculation. Incorrect use of algorithms. (19/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = \left[\ln\ (x-4)^{2/3}\right]_0^6 = (\ln(x-4)^{2/3}) - (\ln(0-4)^{2/3}) = 0.46$.

Category III: Wrong explanations, which based on pseudo-conceptual mode of thinking. (2/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = 4000$. The graph has their area above x-axis, so the sign is true.

Category IV: No answer. (5/41)

From the above categorisation only 3 students out of 41 (category I_a) give evidence that they understand the definite integral concept. For eleven students of our sample (categories I_b, II_a) we cannot claim it but we cannot claim the opposite. The remaining 27 students (categories I_c, II_b, III, IV), do not seem to be able to handle the concept of definite integral with an infinite discontinuity.

Question 1. b

The categories of this question were intended to be the same as in question 1. a. This was true in the majority of the cases (see Table 1). However, the three students adjudged correct in Question 1.a gave a faulty answer to question 1.b which diverges, indicating that they may have simply ignored the question of convergence in both cases.

Category	"zero"	I_a	I_b	I_c	II_a	II_b	III	IV
Distribution	0	0	0	5	0	26	4	6

Table 1: Distribution (Number of respondents) of the Categories to Question 1.b (N=41)

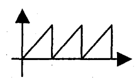
When we analyzed Question 1.a we added category "zero" which refers to the correct theory. It turns out that no one of our sample knew the correct theory according to the improper integrals. This fact is true also for Question 1.b.

Question 2

Category I: Numerical answer. (14/41)

Category I_a: As above with indication of knowledge how to calculate the area correctly (the student draw a graph of the function and calculate the area). (9/41)

Example: 0.5 + 0.5 + 0.5 = 1.5 (with right graph).



Category I_b: Right answer without explanation. (5/41)

Example: 1.5.

Category II: Wrong answers based on overgeneralization of symbolic method. (13/41)

Example: $\int_0^3 (x-[x])\,dx = \left[\frac{1}{2}x^2 - \frac{1}{2}(x)^2\right]_0^3 = 1.5$.

Category III: Incorrect use of algorithms. (4/41)

Examples: 1. Area = $[1-[x]]_0^3$. 2. 0.5.

In the first example the students uses differentiation and not integration. On the other hand the student does not know what to do with [x], so leaves it as it is.

Category IV: No answer. (10/41)

From the above it turns out that for only 9 students out of 41 (category I_a) give evidence that they know how to find the area between the graph of the function and the x-axis. As for other 5 students (category II), we cannot claim that, but we cannot claim the opposite. The remaining 27 students of our sample (categories I_b, III, IV), either make errors or do not respond.

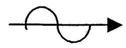
Question 3

Category I: The area is the definite integral. (5/41)

Example: area = $\int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} \sin x dx = [-\cos x]_0^{\pi} + [-\cos x]_{\pi}^{2\pi} = 2 + 2 = 4$.

Category II: Right answer without reasoning. (1/41). Example: 4.

Category III: Positive/negative area means positive/negative y respectively. The area is 0 because the positive area and the negative area cancel each other. (11/41)

Examples: 1. 0. (See figure). 2. Area = 0 units^2. 

Category IV: The area is the definite integral $\int_a^b f(x)dx$. The student does not refer to the sign of the function in $[a, b]$. (15/41)

Example: $\int_0^{2\pi} \sin x \, dx = [-\cos 2\pi] - [\cos 0] = 2.$

Category V: Pseudo-conceptual or seemingly nonsensical answers. (3/41)

Example: $\int_0^{180} \sin x \, dx = \left[\dfrac{\cos x^2}{2}\right]_0^{180} = 114.54.$

Category VI: No answer. (6/41)

From the above categorization, 5 students out of 41 (category I) may be claimed to understand the definite integral concept. For one student we cannot claim that, but we cannot claim the opposite. The remaining 35 students of our sample (categories III, IV, V, VI), either make errors or do not respond.

In our analysis of the results in Question 6, we hypothesised that the students not necessarily know how to calculate the area when the function change its sign. From the above categorization, it follows also that 15 students out of 41 (category IV) do not explicitly evoke a change in sign when it occurs in a given interval $[a, b]$.

Question 4

Category I: Numerical answer. (24/41)

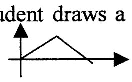
Category I_a: As above with right integration and calculation. (17/41)

Example: $\int_0^{1/2} 2x dx + \int_{1/2}^{1} (2-2x)dx = [x^2]_0^{1/2} + [2x - x^2]_{1/2}^1 = 0.25 + 0.25 = 0.5.$

Category I_b: As above with wrong integration / calculation. (7/41)

Example: $\int_0^{1/2} 2x \, dx = [x^2]_0^{1/2} = 0.25.$

Category II: Visual answers: the student draws a correct graph of the function and calculates the triangle area. (4/41)



Example: 0.5·1/2 + 0.5·1/2 =0.5.

Category III: Answers without explanations. (2/41). Examples: 1. 0.25 + 0.25 = 0.5. 2. 0.5.

Category IV: No answers. (11/41)

For 21 students out of 41 (categories I_a, II) we can claim that they are able to apply the definite integral to a split domain function. For 2 other students (category III) we cannot claim that, but we cannot claim the opposite. The remaining 18 students (categories I_b, IV), either make errors or do not reply.

Question 5

The categories of this question were supposed to be the same as in question 4. This was true in the majority of the cases (see Table 2), however, the results show that for only 2 students out of 41 (categories I_a, II) we claim that they know the application for the definite integral for such a special function. For 6 students of our sample (category III) we cannot claim that but we cannot claim the opposite. About the rest (categories I_b, and those who did not answer the question) 33 students of our sample, we can claim that they do not know to apply the definite integral for the above function.

Category	I_A	I_b	II	III	IV
Question 4	17	7	4	2	11
Question 5	0	14	2	6	19

Table 2: Distribution (Number of respondents) of the Categories to Questions 4 and 5 (N=41).

When we analyzed the tasks in Questions 2, 4 and 5 we suggested that the students might be expected to give answers involving visualization (categories I, IV, IV respectively). Table 3 provides this information about our sample. It turns out that the visualization thinking is very weak in our sample.

Question	question 2	question 4	question 5
Number	14	4	2

Table 3: Distribution (respondents) with correct visualization answers to Questions 2, 4, 5. (N = 41)

When we analyzed Question 6 we mentioned that it is interesting to compare the results there to the results of other questions; especially the 26 students who did not respond (category IV) in Question 6. Table 5 provides information about our sample.

Question	1.a	1.b	2	3	4	5
Number	0	0	5	8	9	0

Table 4: Distribution (of respondents) of students who did not answer Question 6 and answered correctly Questions 1-5. (N = 26)

Table 4 shows the responses of those who did not answer Question 6 but gave correct answers to Questions 1-5. The conclusion is that these students know what to do but they do not know to explain at the general level. However, our conclusion is not true when they face improper integrals or definite integral of modulus function (Questions 1 and 5).

Discussion

One of the goals of this study was to expose some common images of the definite integral of a function held by A-level high school students. This has a direct implication for teaching. If one wants to teach the definite integral of a function to a group similar to our sample, it is important to know the starting point of the students (Rasslan & Vinner, 1997). Taking into account the difficulties mentioned in this study and also in Tall (1993) at least some doubts should be raised whether the given approach to the definite integral is the most effective way for teaching such a concept. If improper integrals, definite integrals of more general functions such as the modulus function or the integer-value function, are needed, we suggest that they should be introduced as cases extended the students' previous experience. The pool of examples introduced to the students should include a variety of examples and students should be encouraged to express their ideas in ways which help them to build a more insightful concept. A similar conclusion was mentioned by Rasslan and Vinner (1997) according to other concepts, such as even/odd function.

The strategy applied by the School Mathematics Project is to introduce conceptual ideas through class discussion and then to experience them in use. The using of the ideas is a major part of the activity in a manner reminiscent of the 'tool-object dialectic' (Douady, 1986). The concept definition is essentially an incidental part of the process which is far more concerned in practice with developing experience and images of the concepts themselves. Students learn implicitly what they do. Ferrini-Mundy & Guardard (1992) have already illustrated that students who essentially practice routines in High School Calculus learn procedural techniques which may even be prejudicial to later developments at College. Here we have investigated students whose examination results are above average who are following a curriculum intended to be more experiential and conceptual. The majority do not write meaningfully about the definition of definite integral, and have difficulty interpreting problems calculating areas and definite integrals in wider contexts.

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