CONSTRUCTION OF KNOWLEDGE AND ITS CONSOLIDATION: A CASE STUDY FROM THE EARLY – ALGEBRA CLASSROOM

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Abstract

The process of abstraction is central to construction of knowledge. It has been discussed intensively, but only seldom studied experimentally. The following study exemplifies a way for tracing processes of knowledge construction and its consolidation. In doing so, we extend the nested model of abstraction elaborated by Hershkowitz, Schwarz, & Dreyfus (2001), to study two 7-graders, collaborating to investigate algebra problem situations in successive activities, along the year, in technological learning environment. The analysis demonstrates the construction of knowledge in an ongoing dialectical process, between construction and consolidation, which took place along three activities, when pieces of knowledge incrementally accumulate from one activity to the other.

This study is part of a longitudinal study, which examined the collaborative work done by two 7-grade students, in five activities of investigating problem situations along the academic year. One activity was analyzed in Tabach, Hershkowitz & Schwartz (2001). Here we will analyze selected episodes from other 3 activities, where all of them deal implicitly with the exponential growth phenomenon.

THEORETICAL FRAMEWORK

Mathematical activity, like any other human activity, is embedded in a socio-cultural environment (e.g., Voigt, 1995). This view is increasingly accounted for by the mathematics education community, which sees mathematics learning as a culture of mathematisation in practice. Such approach gains from combining collective with individual activities, analytic with reflective stages, and integrating intra with inter-processes that are at the root of mathematical development (Hershkowitz & Schwarz, 1999a, see related ideas of scientific enculturation in classrooms in Woodruff & Meyer, 1997).

Abstraction is at the heart of mathematisation (Freudenthal, 1991; Gravemeijer, 1995). To study abstraction experimentally, Hershkowitz, Schwarz, and Dreyfus (2001) gave an operational definition of abstraction: *an activity of vertically reorganizing previously constructed mathematical knowledge into a new structure.* They suggested a model, which is based on three observable epistemic actions, nested one in each other: Constructing (C) is the central action of abstraction. It consists of assembling knowledge artifacts to produce a new structure with which the participants become acquainted. The action of Recognizing (R) a familiar mathematical structure, occurs when a student realizes that the structure is relevant to the problem situation in which participants are engaged. The Building-With (B) action consists of combining existing artifacts in order to comply with a goal such as exploiting a strategy or justifying a statement. The term Consolidation denotes (according to Hershkowitz, Schwarz, and Dreyfus, 2001; Dreyfus & Tsamir, 2001) the progressive familiarization through observable recognizing and building-with actions, in four types of situations: Reconstruct the new structure or actualizing it by recognizing it in various contexts. Use it with increasing facility for building-with in various contexts. Use it in the construction of higher structures for which it is a necessary prerequisite. Verbalize about it – possibly during or after an activity of reflection, such as reporting or summary discussion in class. The RBC model of abstraction will be used in this article to trace the construction and consolidation of new mathematical knowledge along various activities.

THE STUDY

We focus here on the work of two Grade 7 students who participated in a one-year algebra course. The basis for choosing of these two students was their high verbal ability. Three activities, which deal with exponential change phenomena, were chosen out of the algebra course of the CompuMath project. The approach adopted by the development team of the CompuMath project is a function approach to algebra (Hershkowitz et al, in press). The activities goals are the construction of some generalizations of growth phenomena patterns within problem situations, and the use of these generalizations to build the phenomena numerically and graphically, using spreadsheet program (Excel). The problem situations, which are the mile stones of the algebra course, were designed to give opportunities to students' construction of new knowledge structures concerning mathematical concepts (algebraic variables and models) and of various mathematical processes (hypothesizing, making generalizations, testing hypotheses, interpreting representational information, solving and justifying). In the present study, we examined three aspects, in which these constructions take place. That is, we observed (a) the types of interactions while collaborative work is taking place (due to space shortage it will not discussed here). (b) The construction of a shared knowledge of the pair and, (c) The contribution of each participant, as well as what is left of it in the individual. All these aspects are examined within and between the three activities as one continuum along the academic year.

All activities were open - no guidance for solution was provided to students and no instruction if and how to make use of Excel was given.

The class work of the pair was videotaped and written works were collected. The videotapes were transcribed. Following Chi (1997), the protocols were divided into "cognitive segments".

In this presentation we will analyze selected parts from these 3 activities (They were the first, third and forth activities of the research sequence of activities). In all of them creation of new structures of knowledge, concerning exponential change phenomenon takes place, while students collaborate together. The designers of the Algebra Course intention was that students will be involved in investigating such a phenomenon, but not via the explicit algebraic formulae of exponential growth.

In the next part we will describe shortly each problem situation, and the work done by the pair of students Avi&Ben on it. In our analysis here we will present selected utterances from the full transcript of their work in these activities, as evidences to our claims.

THE PROBLEM SITUATIONS

Efrat's Savings activity

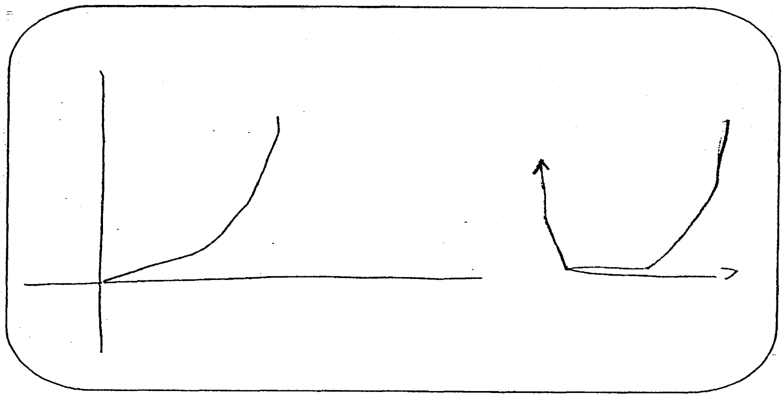
> *Efrat received her weekly savings as follows: at the end of the first week two cents, (which are 0.02 $), and in every weekend she received the same amount that she had in her saving box in the last week. Efrat saved all the money.*

The students were asked to hypothesize how much money will be in Efrats' saving box by the end of the year, in comparison with other four linear ways of savings, which they explore during the week before (for more details, see Freidlander & Tabach, 2001). Then the students were asked to investigate the phenomenon in the computer laboratory with the help of Excel, and to check if their hypothesis was correct. It is important to note that it was their first meeting with the exponential growth.

The students' initial hypothesis is that the amount of money in Efrat's saving box will be the lowest among all other linear ways of saving. In the computer laboratory Avi&Ben try to find an algebraic generalization that will help them to construct the numerical representation of the phenomenon in the computer. Avi says: *Because she started from two. What, one times one, one times one, one times one, what, we should start from two* (A31). Avi is trying to develop an algebraic generalization for that change. Avi feels that this phenomenon has some thing to do with repeating multiplication, and maybe he is starting to see the generalization as an expression, which includes powers. Ben doesn't understand what Avi is saying, but both are aware to the need for an Excel formula. Avi says: *We should write a formula* (A36), and Ben reacts: *Exactly, so I will do A2* (B37). And then Ben continues: *Oh, yes, =B2+B2* (B39). Ben suggests a 'translation' of the verbal representation of the situation to a local connection, $\boxed{\text{=B2+B2}}$. Ben ignores Avi's suggestion to consider powers: *No, no, just a second, if we have here powers it will be good* [pointing to the screen] (A40), and drags "his Excel" formulae until B20, then the whole phenomenon appears numerically. They are both quite surprised from the large numbers they receive, and react: *Until 20. Yooo!* (A45).

The explicit algebraic generalization of Efrat saving, $\boxed{0.02*2^{x}}$, were x stands for the week number, is beyond 7 graders knowledge. And yet, Avi tries to reach such generalization. However, as we can see, Ben easily creates the whole phenomenon by recursion relation between the savings of two successive weeks, and the "dragging" operation in Excel. That is, by writing in cell B2 $\boxed{0.02}$, and in cell B3 $\boxed{\text{=B2+B2}}$, and by 'dragging' this formula to the next cells in the same column. In that way Avi&Ben get the whole phenomenon in its numerical representation.

Next, they are asked to sketch a graphical hypothesis of the same phenomenon, and then to check their sketch on the computer screen. They accomplish it quite quickly, with no evidential difficulties. Yet, if we compare their sketches, we can see that Avi's is much more accurate in his graphical hypothesis then Ben's.



Ben's sketch Avi's sketch

In this activity Avi&Ben constructed some new knowledge regarding the exponential growth: at the beginning they underestimated the exponential growth, but when they received its numerical representation, they were surprised by its rapid growth. Their graphical representation sketches were quite close to graphical representations of the exponential change. And yet, we can't be sure what kind of knowledge concerning exponential growth was emerged.

***Aunt Berta* activity**.

Five months later, the students are asked to solve the Aunt Berta activity. (It is important to mention that during this five month, they didn't meet any situation regarding exponential change).

> *Yosi received a letter from his rich aunt Berta.*
>
> *Dear Yosi!*
>
> *I have reached the age of 65, and my life is comfortable. I would like to give you some of my money. You can choose one of the options:*
>
> *One. I will give you this year 1,100 $, next year 1,200 $, and so on. Each year 100 $ more then the last year.*
>
> *Two. I will give you this year 2,000 $, next year 1,900 $, and so on. Each year 100 $ less then the last year.*
>
> *Three. I will give you this year 100$, next year 150$, and so on. Each year 1.5 more then last year.*
>
> *Four. I will give you this year 8 $, next year 16 $, in two years 32$, and so on. Each year 2 times more then last year.*
>
> *The agreement will go on while I am still alive. Let me know your decision soon, yours, Berta.*

The students were asked to help Yosi to make the best choice, meaning to make hypotheses, taking into consideration that aunt Berta will live at least until the age of 80. Then they are asked to investigate the phenomenon in the computer laboratory with the help of Excel, and to check whether their advises to Yosi (their hypotheses) are correct.

Avi&Ben's advice to Yosi is to take the forth option. Ben estimates that the amount of money will grow quickly after it will reach 100$, and Avi estimates that the amount of money will grow in several millions.

They try to construct the generalization with the help of Excel. For the third option Ben starts to build a recursion generalization: *100, equals D2, ahh...* (B40). Avi helps by saying: *times 1.5* (A41). They drag the generalization $\boxed{\text{=D2*1.5}}$, were in D2 they put $\boxed{100}$. In investigating the forth option, Ben takes the lead again by starting to build a recursion formula. Avi interferes by saying: *Make Power* (A43). Ben objects: *No, No, I will just double it, don't you think so?* (B44). Avi hesitates: *no,...try it on* (A45). So Ben writes $\boxed{\text{= E2*2}}$, drag it, and they both agree on the numerical results. Avi's reaction to the numerical data is: *I was wrong here* (B49). He explains that his estimation was for 80 years, and he tries to drag the generalization more, up to 80 years. But, since the column width is limited, he does not get a proper reaction from the computer, and he remains frustrated: *irritating, O.K.* (A69).

In Aunt Berta Activity, Avi&Ben had no doubt that the forth option is the best. How can we explain the over estimation they gave, especially Avi? It seems that the source of this over-estimation is the surprise they had in the first activity, concerning the large numbers, which were received. This over-estimating may serve as evidence that during the first activity they indeed construct the knowledge that exponential growth is very fast. This knowledge is consolidated here, as evidenced by the correct choice they made, and their over-estimation. Moreover, this time Avi has no doubts that he can use powers for the forth option. The way Avi tries to check his estimation, shows that he understands intuitively that the exponent is responsible for that rapid growth.

***The Crazy Candy* activity.**

Lets assume that in the next years the rate of inflation will be 10% per year.

Write the name and price of your favorite candy.

What will be the price of that candy when you will reach the age of 120?

Suppose we could do the following changes:

Cut down by half the price of your candy, and increased the inflation rate to 11% (and not 10%).

Is this deal profitable?

The students are asked to make hypotheses concerning the prices of their candy, and then they are asked to investigate the phenomenon in the computer laboratory with the help of Excel, and to check their hypotheses.

The current price of Avi&Ben's candy is 3.5$. Ben's estimation is that in 100 years it will be ~100,000. Avi's estimation is about 11,000,000. They try to build the right expression, and Ben's suggestion is: *oh, B2/10+B2. Is it?* (B52). They are both surprised by the "low price" of the candy after 107 years. Avi tries again to "drag" the expression for some more years, and is disappointed.

The generalization of the price in an explicit algebraic formula, $\boxed{A1*1.1^x}$, were x stands for the year number and A1 is its present price, is again far beyond the knowledge of 7 graders. However, finding a local recursion connection between the prices of two consecutive years is quite easy. If we write in cell B2 the current price, and in cell B3 we write $\boxed{=B2*1.1}$, and "drag" the formulae to the next cells in the same column, we will receive the whole phenomenon numerically.

Here it is interesting to note that: (a) both students knew that the phenomenon has a rapid change. As they dealt with 1.1^x and not with 2^x, they over-estimated this change. (b) The generalization Avi&Ben used is of additive nature $\boxed{=B2/10+B2}$, which might be explained by the fact that this time the verbal representation of the situation involved percentages, a topic in which their knowledge is quite poor.

Reading the second part of the problem, Ben immediately said that this deal is not profitable. However, when trying to generalize it, Ben offered the expression =B2/11+B2, which is obviously a wrong one, but it supports the assumption about their poor knowledge concerning percentages.

Their right hypothesis shows us, the researchers, that another piece of knowledge was constructed and consolidated here: intuitively, they understand that the exponent basis, weighs more in the rate of change of the exponential phenomenon.

CONCLUDING REMARKS

Tracing the action of abstraction (including the construction of new structures of knowledge and their consolidation) is a complicated task. The evidence for the construction of knowledge is sometimes by actions that took place in the same activity, as was shown in Tabach, Hershkowitz & Schwarz (2001). In other cases, the experimental evidences for the construction of knowledge took place in later activities, while the constructed knowledge is consolidated, in the sense it was described in the theoretical framework section. Therefore, longitudinal studies like the one that was described in this paper are needed, in order to trace such evidences for construction and consolidation of knowledge.

The knowledge of Avi&Ben regarding the exponential change is not yet fully formalized – for example, they do not know the explicit expression of it. And yet, we can see from one activity to the other how the knowledge incrementally accumulated: At first they approached the exponential growth with an under estimation, and they were surprised.. As a second step in this activity we observed their graphical hypotheses which were quite close to the right exponential graph. Whether they constructed some knowledge concerning the exponential growth or whether they succeeded to "translate" from table to graph is not clear yet. In the second activity they over estimated the same phenomenon, but this time Avi was sure that this change has some thing to do with powers. Avi than reflected critically on his own hypothesis in the light of the findings he got, and dragged the generalization down to the eighty row. This action evidences a kind of consolidation leading to further construction (reconstruction). In the third activity they again over estimate the growth, but this time is due to the small basis of the exponent (1.1^x and not 2^x). In the second part of the third activity they have to choose between 1.1^x and 1.11^x, and surprisingly they made the right choice. This is an additional evidence for the dialectic process of construction and consolidation of incrementally accumulative knowledge between and within activities.

Reference

Chi, M. (1997). Quantifying qualitative analyses of verbal data: a practical guide. *The Journal of the Learning Sciences*, *6*(3), 271-315.

Dreyfus, T. & Tsamir, P. (2001). Ben's consolidation of knowledge structures about infinite sets. Technical Report, Tel-Avi, Israel, Tel-Aviv University.

Freudenthal, H. (1991). *Revisiting Mathematics Education*, Kluwer Academic Publishers, Dordrecht.

Friedlander, A. & Tabach. M. (2001). Promoting Multiple Representations in Algebra. (pp. 173-185). In A. A. Cuoco & F. R. Curcio (Eds.) *The Roles of Representation in School Mathematics. 2001 Yearbook*. The National Council of Teachers of Mathematics; Reston, Virginia.

Gravemeijer, K.P.E. (1995). *Taking a Different Perspective*, Freudenthal Institute, University of Utrecht.

Hershkowitz, R. & Schwarz, B. B. (1999a). Reflective processes in a technology-based mathematics classroom. *Cognition and Instruction, 17*(1), 65–91.

Hershkowitz, R., Schwarz, B. B, & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *The Journal for Research in Mathematics Education 31* (2), 195-222.

Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., Tabach, M. & Schwarz, B. B. (in press). Mathematics curriculum development for computerized environments: A designer-researcher-teacher-learner-activity. To appear in L.D. English (Ed.) *Handbook of International Research in Mathematics Education*. Lawrence Erlbaum Associates, Pub.

Hershkowitz, R. Schwarz, B. B, & Dreyfus, T. (2001). Abstraction in context: Consolidation of constructed knowledge in the course of successive activities. Technical Report, Weizmann Institute, Israel.

Tabach, M., Hershkowitz, R. and Schwarz, B.B. (2001). The struggle towards algebraic generalization and its consolidation, In M. Van-den Heuvel (ed.), *Proceedings of the 25th International Conference for Psychology of Mathematics Education, Vol. 4*, pp. 241-248, Utrech, The Netherlands, OW&OC.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *Emergence of Mathematical Meaning: Interaction in Classroom Cultures* (pp. 163-201). Hillsdale, NJ: Erlbaum.

Woodruff, E., & Meyer, K. (1997). Explanations from Intra- and Inter – Group Discourse: Students Building Knowledge in the Science Classroom. *Research in Science Education*, *27*(1), 25 – 39.