FLEXIBLE MENTAL COMPUTATION: WHAT ABOUT ACCURACY?

Ann Heirdsfield

Centre for Mathematics and Science Education, QUT, Brisbane, Australia, 4059

Flexibility in strategy choice in mental computation is considered to be a component of number sense. This paper reports on an investigation into cognitive, metacognitive, and affective factors that support both flexibility and accuracy in mental addition and subtraction in Year 3 students. While some factors appeared to be essential for flexibility, additional factors were necessary for accurate employment of strategies. Further, there were qualitative differences between the mental strategies employed by the students who were accurate and those who were inaccurate.

While standard computational algorithms (both written and mental) are still taught in many classrooms around the world, there is reduced emphasis on the importance of these algorithms and increased emphasis on "number sense" (e.g., National Council of Teachers of Mathematics, 1989). Several interpretations exist for number sense; however, "flexibility" and "inventiveness" seem appropriate ones in this discussion (Anghileri, 2001). It has been recognised that the development of flexible mental computation strategies are a component of number sense (e.g., Klein & Beishuizen, 1994; McIntosh, 1998; Reys, Reys, Nohda, & Emori, 1995), and that when children are encouraged to formulate their own mental computation strategies, they learn how numbers work, develop number sense and develop confidence in their ability to make sense of number operations (Kamii & Dominick, 1998).

It would appear that the purpose of the inclusion of mental computation in any mathematics curriculum would be to develop flexible computational strategies, and thus promote number sense. Some teaching experiments have focused on the successful development of students' flexible computational strategies (e.g., Buzeika, 1999; Kamii, 1989). Other literature reports that children have the ability to develop their own efficient mental strategies, even without instruction (e.g., Heirdsfield, 1999). While flexibility might be one of the foci of this research, proficiency (defined here as both flexibility *and* accuracy) would appear to be important as well. It is posited that the study of proficiency in mental computation extends beyond the development of flexible mental strategies, but also encompasses the study of associated factors that might contribute to both flexibility and accuracy.

Associated factors have been reported elsewhere (e.g., Heirdsfield, 1996; Kamii, Lewis, & Jones, 1991; Sowder, 1994; Van der Heijden, 1994). These include (a) number sense (number fact knowledge, estimation, numeration, effects of operation on number), (b) affective factors (beliefs, attributions, attitudes, self efficacy, social context), (c) metacognitive processes (strategies, beliefs, knowledge), and (d) memory (short term memory, long term memory – knowledge base).

This paper reports on a study of Year 3 children's mental computation (addition and subtraction), which was conducted in three classrooms (2 schools, A and B). In all of these classrooms, students were taught traditional pen and paper algorithms (mental computation is not mentioned in the existing Queensland Years 1-10 mathematics curriculum document and is not treated consistently in Queensland schools). Mental computation appeared to only refer to number facts and extended number facts (e.g., 30+40=70, because 3+4=7). However, in one of the classrooms (School B), students were also encouraged to consider alternative strategies, and would have been permitted to use them. Of interest here is those students who were identified as being flexible in their mental addition and subtraction strategies. While it appears that the students developed their own flexible mental strategies, not all were successful (i.e., accurate) in applying them. It could also be argued that not all were "efficient" either.

THE STUDY

Participants. Seven flexible Year 3 students were selected from three classes in two Brisbane Independent schools (Schools A and B) that served high and middle socioeconomic areas. The students were selected on the basis of their responses to a structured addition and subtraction mental computation selection interview. All students were identified as being flexible in their employment of mental strategies, 4 were accurate (2 from School A and 2 from School B) and 3 were inaccurate (School A). Accurate computers were those who were more than eighty percent correct in their responses on both the addition and subtraction selection items. Inaccurate students generally attained between thirty and eighty percent accuracy on either the addition and subtraction items (more errors were made on the subtraction items).

Procedure and instruments. All students from the three classes were withdrawn from class and interviewed individually in a structured mental computation interview. Students who employed a variety of strategies were selected to participate in further interviews. These interviews constituted a series of videotaped semi-structured clinical interviews in a quiet room in the school. The interviews addressed mental computation strategies, number facts, computational estimation, numeration, effect of operation on number, metacognition, affect, and memory. These have been described in more depth elsewhere (Heirdsfield, 2001; Heirdsfield & Cooper, 1997).

Analysis. For the purposes of identifying flexibility in mental computation, mental computation strategies were identified using the categorisation scheme (based on Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995; Thompson & Smith, 1999) that divided the strategies into the following categories: (1) *separation* (e.g., 38+17: 30+10=40, 8+7=15=10+5, 40+10+5=55); (2) *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55); (3) *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55); and (4) *mental image of pen and paper algorithm* – following an image of the formal setting out of the written algorithm (taught to almost automaticity in the schools the students attended).

Mental computation responses were analysed for strategy choice, flexibility, accuracy, and understanding of the effects of operation on number, numeration, computational estimation, and number facts. Analysis of the interviews investigating these individual factors was also undertaken, with the intention of exploring connections with mental computation. Students' responses were also analysed for metacognition and affects, and scores and strategies were recorded for the memory tasks. Each student's results for aspects of number sense, metacognition, affects and memory were summarised. These summaries were combined for each of the computation types: accurate/flexible and inaccurate/flexible, so that comparisons could be made between the two types. The knowledge shown by the students of each type were analysed for commonalities and these commonalities were used to develop a *composite* picture of a typical student of that type. The two resulting knowledge structures, one for accurate and one for inaccurate, were depicted by networks.

RESULTS

Both the accurate and inaccurate students spontaneously employed a variety of strategies (*separation*, *aggregation*, and *wholistic*) although the inaccurate students tended to have less variety, using predominantly *separation* strategies. When encouraged to access different strategies, both accurate and inaccurate students were able to do so but with different outcomes. The accurate students were successful in their use of the new strategies while the inaccurate experienced difficulties in completing the strategies (although they had sufficient understanding to access the strategy).

Accurate students. Although both accurate and inaccurate students were identified as flexible, there was little in common between the two groups. The students who were accurate showed in their responses to the interviews that they possessed well-integrated knowledge bases. The composite picture of their knowledge is depicted as a network in Figure 1.

As can be seen in the figure, the accurate mental computers were fast and accurate with their number facts, used efficient number facts strategies (e.g., 8+6=14, because double 6 and 2 more make 14) when facts were not known by *recall*, and had extended their number facts strategies to efficient mental computation strategies (e.g., 9+6=10+6-1=15 is similar to 246+99=246+100-1). Although it might have been expected that estimation would contribute to mental computation, only one of the accurate students exhibited proficiency in estimation. This student also employed estimation in mental computation to get a feel for the answer and check the solutions.

The accurate students used good numeration understanding (particularly canonical, noncanonical, multiplicative, and proximity of number) and some understanding of the effect of operation on number to support their efficient use of a variety of strategies. This was particularly so for the *wholistic compensation* strategy (e.g., 246+99=246+100-1) for which numeration understanding (particularly proximity of number) and understanding of the effect of operation on number appeared to be essential. The accurate students had accurate perceptions of their ability to solve the mental computation tasks (metacognitive beliefs), and they used metacognitive strategies (e.g., monitoring, reflecting, regulating, and evaluation). Beliefs in self seemed to be associated with a belief about the place of the teacher in the student's learning; for instance, accurate students tended to have confidence in self-initiated strategies (c.f., teacher-taught strategies). Although there was not always evidence of the belief that mathematics makes sense, that belief was strong in one student.

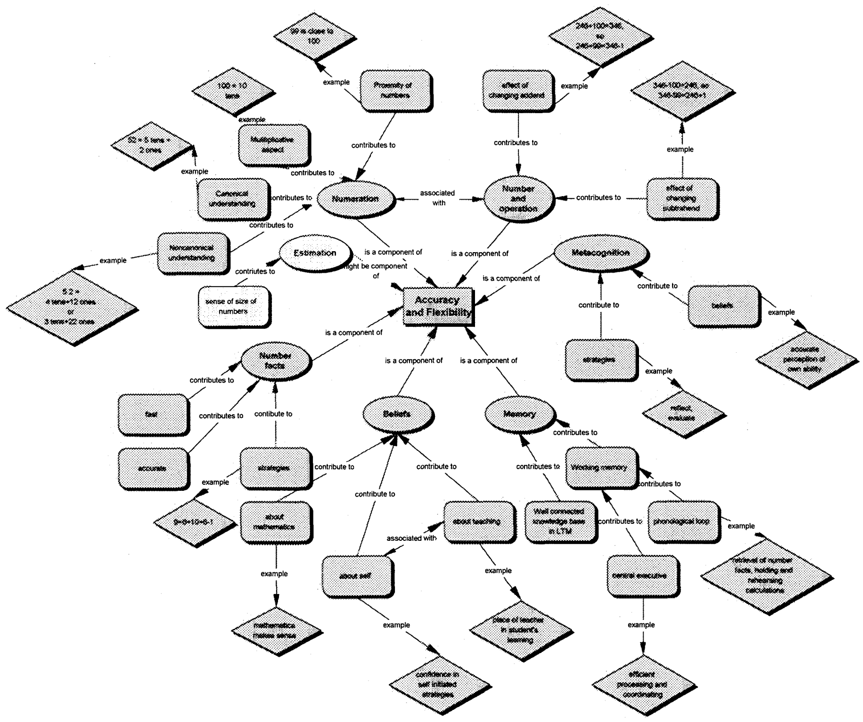

Figure 1. Network showing knowledge for accurate/flexible mental computation (shaded - present, speckled - partially present)

Accurate students showed reasonable short-term memory (STM) and central executive functioning. This would have provided them with efficient retrieval of number facts from long-term memory (LTM), effective holding and rehearsal of interim calculations, and efficient processing and coordinating of strategies. However, the study also showed that STM was not as important as might be predicted as efficient mental strategies place less demand on STM and require fewer interim calculations.

In summary, the composite accurate/flexible mental computer was shown to have a rich integrated network of cognitive, metacognitive and affective components.

Inaccurate students. Although the inaccurate students in this study were categorised as flexible, they did not exhibit the same degree of flexibility as the accurate students. They did employ a variety of strategies, but they tended not to be high-level strategies (e.g., *wholistic*), and there was very little in common between the two groups. The composite picture of their knowledge is depicted as a network in Figure 2. It shows that the inaccurate students had much less knowledge and fewer connections between factors than the accurate students.

All knowledge exhibited by inaccurate students seemed to be at a threshold level, rather than at an optimum. The inaccurate students exhibited some flexibility and efficiency (although not always speed and accuracy) in number fact strategies. However, these strategies did not always support interim calculations in mental computation, as the students often calculated interim calculations by counting, rather than by employing more efficient *derived facts strategies,* which they used in the number facts test. Similarly, numeration understanding was evident at a threshold level, particularly, canonical and noncanonical. A further attribute of numeration, proximity of number appeared to be at a threshold level, as the students attempted to use this when accessing the *wholistic compensation* strategy. However, their knowledge of the effect of operation on number did not support high-level strategies and their estimation was poor. There was evidence of some metacognitive strategies, such as reflection, evaluation, and checking solutions. However, unlike the accurate students, metacognitive beliefs were poor.

Beliefs, in general, were difficult to elicit from the inaccurate students, and when elicited, were inconsistent. There might have been several reasons this. These students might not have held any strong beliefs about themselves, about mathematics (e.g., whether mathematics should make sense), or about teaching. Also, they might have been unaccustomed to verbalising their beliefs.

Finally, as with the accurate students, the inaccurate students had reasonable STM and central executive functioning (e.g., planning and allocation of attention). However, these abilities were little help to the students because number facts were not sufficiently well known to be retrieved by STM, interim calculations were completed so slowly that they placed a heavy load on STM, and the students' knowledge base was so poor that the central executive could not successfully retrieve information.

The question remains: Why were the inaccurate students flexible? The answer might lie in what appeared to be the lack of understanding of taught procedures. When these students were unable to use these procedures, they compensated by inventing strategies. These strategies tended to be lower level (i.e., *separation*) and their use unsuccessful, as the inaccurate students' knowledge (particularly of numeration and effect of operation on number) was insufficient to enable higher-level strategies to be attempted and any calculation to be completed accurately.

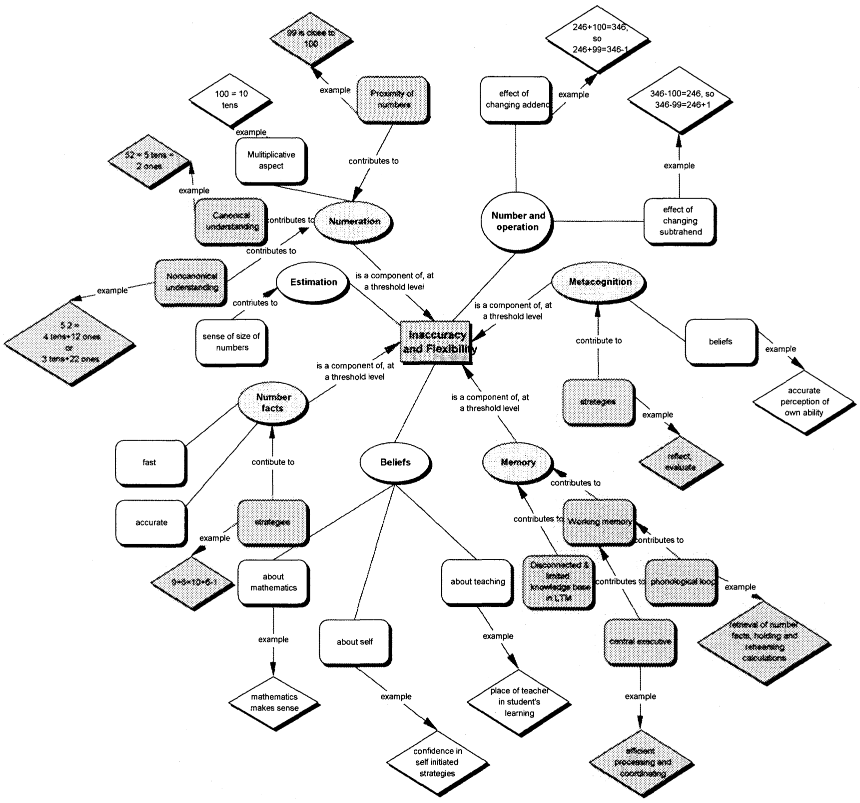

Figure 2. Network showing knowledge for inaccurate/flexible mental computation (shaded - present, speckled - partially present, clear - absent)

In summary, the composite inaccurate/flexible mental computer was shown to have knowledge at a threshold level that was insufficient for employment of advanced mental strategies and accurate use of other strategies.

DISCUSSION

Although flexible use of mental computation strategies is an important component of the development of number sense, this study shows that it is not sufficient for accurate computation. A well-connected knowledge base (where number facts, numeration, number and operation, and to less extent, estimation were part of that knowledge base), metacognitive strategies and beliefs, and an efficient central executive (to coordinate retrieval from LTM and allocation of strategies and facts for short-term storage and manipulation of numbers) supported accuracy and flexibility in mental computation. Without this knowledge base, students were inaccurate.

It seems that with a strong connected knowledge, accurate students had more options available for mental strategies. With a less connected and weaker knowledge base, inaccurate students' use of strategies was an attempt to compensate for their lack of knowledge. The inaccurate students compensated by inventing strategies when the teacher-taught strategies could not be followed. However, although STM was sufficient, these students' knowledge base was so minimal and disconnected that the use of the strategies was not efficient, and resulted in errors. Further, the knowledge base did not support high-level strategies.

This demonstrates the need for teaching practices to go beyond developing flexible use of strategies in mental computation. The practices should not focus on the strategies in isolation; they have to focus on the development of an extensive and integrated knowledge base to support the strategy use. This means covering number facts, numeration, effect of operation on number, and estimation. Other factors that need to be addressed are metacognition and affects.

Students can and do formulate their own strategies and this should be encouraged because of the learning that results with respect to number sense (e.g., Reys, Reys, Nohda, & Emori, 1995). However, if accuracy in mental computation is one of the aims of computation, more has to be done than encouraging students to formulate their own strategies. While research (e.g., Buzeika, 1999; Kamii, 1989) has reported success with teaching experiments that encourage students to formulate their own strategies, it is obvious that other cognitive, metacognitive and affective factors come into play. In this study, it was shown that accurate (and flexible) mental computation was supported by a complex interaction of cognitive, metacognitive and affective factors. Further research is warranted as to teaching practices that can develop flexibility and the supporting knowledge necessary for accuracy and flexibility, possibly following the lines of Cognitive Guided Instruction (Carpenter, Fennema, Franke, Levi, & Empson, 1999), but including children's affects and metacognition.

REFERENCES

Anghileri, J. (2001). What are we trying to achieve in teaching standard calculating procedures? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education Vol 2.* (pp. 41-48). Utrecht: Freudenthal Institute, Utrecht.

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education, 24*(4), 294-323.

Buzeika, A. (1999). Invented algorithms: Teachers face the challenge. In J. M. Truran & K. M. Truran (Eds.), *Making the difference. Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 128-134). Sydney: MERGA.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics. Cognitively guided instruction.* Portsmouth, NH: Heinemann

Cooper, T. J., Heirdsfield, A., & Irons, C. J (1996). Children's mental strategies for addition and subtraction word problems. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 147-162). Adelaide: Australian Association of Mathematics Teachers, Inc.

Heirdsfield, A. M. (1996). *Mental computation, computational estimation, and number fact knowledge for addition and subtraction in year 4 children.* Unpublished master's thesis, Queensland University of Technology, Brisbane.

Heirdsfield, A. M. (1999). Mental addition and subtraction strategies: Two case studies. In J. M. Truran & K. M. Truran (Eds.), *Making the difference.* (pp. 253-260). Sydney: The Mathematics Education Research Group of Australasia Inc.

Heirdsfield, A. M. (2001). Integration, compensation and memory in mental addition and subtraction. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education Vol 3,* (pp. 129-136). Utrecht: Freudenthal Institute, Utrecht.

Heirdsfield, A. M., & Cooper, T. J. (1997). *The architecture of mental addition and subtraction.* Paper presented at the annual conference of the Australian Association of Research in Education, Brisbane, (http://www.swin.edu.au/aare97pap/HEIRA97395.htm).

Kamii, C. (1989). Young children continue to reinvent arithmetic - 2nd grade: Implications of Piaget's theory. New York: Teachers College Press.

Kamii, C., & Dominick, A. (1998). The harmful effects of algorithms in grades 1-4. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 yearbook.* (pp. 130-140). Reston, VA: NCTM

Kamii, C., Lewis, B., & Jones, S. (1991). Reform in primary education: A constructivist view. *Educational Horizons. 70*(1), 19-26.

Klein, T., & Beishuizen, M. (1994). Assessment of flexibility in mental arithmetic. In J. E. H. Van Luit (Ed.), *Research on learning and instruction of mathematics in kindergarten and primary schools.* (pp. 125-152). Doetinchem, The Netherlands: Graviatt Publishing Company.

McIntosh, A. (1998). Teaching mental algorithms constructively. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics, 1998 yearbook.* (pp. 44-48). Reston, VA: NCTM.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

Reys, R. E., Reys, B. J., Nohda, N., & Emori, H. (1995). Mental computation performance and strategy use of Japanese students in grades 2, 4, 6, and 8. *Journal for Research in Mathematics Education, 26*(4), 304-326.

Sowder, J. (1994). Cognitive and metacognitive processes in mental computation and computational estimation. In R. Reys & N. Nohda (Eds.), *Computational alternatives for the twenty-first century.* Reston, Virginia: NCTM.

Thompson, I., & Smith, F. (1999). *Mental calculation strategies for the addition and subtraction of 2-digit numbers.* Final report. University of Newcastle, Newcastle upon Tyne.

Van der Heijden, M. K. (1994). A Vygotskian approach on mental addition and subtraction up to hundred; Handy arithmetic for primary school children. In J. Van Luit (Ed.), *Research on learning and instruction of mathematics in kindergarten and primary school.* (pp. 108-124). The Netherlands: Graviant Publishing Company.