THE CONSTRUCTION OF COMMENSURATE FRACTIONS

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This research took place in the context of a 3-year constructivist teaching experiment. The report indicates how Joe and Patricia, two 9-year old children, were able to construct what we are calling a commensurate fractional scheme, whereby composite fractions (e.g. 6/24) of a composite unit can be renamed as fractional quantities in their simplest form (and vice-versa). During the second year of the experiment both children constructed an iterative fractional scheme with which they could produce both common and improper fractions as iterations of a unit fraction. They also constructed an equi-partitioning scheme for composite units (a "partitive division" scheme). Commensurate fractions emerged through a coordination of these two schemes and recursive partitioning.

THE CONSTRUCTIVIST TEACHING EXPERIMENT[1]

The research reported in this paper is part of an on-going retrospective analysis of videotaped data from a three-year constructivist teaching experiment with 12 children (Steffe & Olive, 1990; Steffe, 1998). A team of researchers began working with the children at the beginning of their third-grade and continued through the end of their fifth grade year in a rural elementary school in the southern United States.

More than 600 video-taped teaching episodes were conducted during the three years of the teaching experiment. Pairs of children worked with a teacher/researcher using specially designed computer tools (TIMA) (Olive, 2000a). The major hypothesis to be tested was that children could reorganize their whole number knowledge to build schemes for working with fractional quantities and numbers (the rational numbers of arithmetic) in meaningful ways. This reorganization hypothesis (Olive, 1999) contrasted with the prevailing assumption that whole number knowledge is a "barrier" or "interferes" with rational number knowledge (Behr et al., 1984; Streefland, 1993).

Previously Reported Results.

In my report for PME-25 (Olive, 2000b) I presented evidence for Joe's construction of an *iterative unit fractional scheme* that enabled him to construct common and improper fractions as iterations of a unit fraction of a designated whole. I indicated in that paper how Joe used his multiplicative reasoning with whole numbers and his *equi-partitioning scheme* to establish fractions of composite units.

Subjects of this Report.

During the second half of Joe's second year in the teaching experiment he was partnered with Patricia. We hypothesized that she would make a good partner for Joe based on our analysis of Patricia's first year in the teaching experiment. We hypothesized that she had constructed at least an *Explicitly Nested Number Sequence* (Steffe and Cobb, 1988) and possibly had constructed iterable composite units (Olive, 1999) prior to our work with her in the second year. Patricia was able to establish recursive partitioning operations that led to her construction of composite fractions not in simplest form. Patricia provided evidence of being able to project a partition into the elements of a partitioned stick (using our TIMA: Sticks software – see Figure 1 below) and maintain the relations between the different levels of partitioning. Joe evidenced a similar ability while working with Patricia during the episodes that took place in April and May of year 2. The teacher/researcher working with these two children during year 2 was a doctoral research assistant named Azita.

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We choose to use the term "commensurate fractions" rather than "equivalent fractions" so as not to imply that the children had constructed equivalence classes for rational numbers, a far more abstract mathematical construct than that with which the children and the researchers were grappling in this experiment. Commensurate fractions are fractional numbers that provide measures for the same quantity.

Provoking Recursive Partitioning by Taking a Fraction of a Fraction.

In a teaching episode that took place in April of the second year, we introduced composition of fractions as a problem situation that might bring forth recursive partitioning and an awareness of the inverse relation between the resulting fraction and the original whole. The context was established using our computer environment TIMA: Sticks to represent pizzas that could be cut into so many slices. In the course of the session, Joe explained how he worked out 1/2 of 1/3 of a pizza as 1/6 of the pizza: "It'll be two, umm two of those (pointing to the half of one third) in each one (pointing to the 3 parts of the pizza stick), and just count them up and it'll be six." This explanation explicitly indicates recursive partitioning: mentally inserting a partition into the results of a prior partition in order to solve a non-partitioning problem. In this same episode Patricia spontaneously named twice 3/4 as 6/4. Joe renamed 9/4 as "2 whole pizzas and a fourth left over." These responses indicated that both children were comfortable with fractions greater than one and could produce them through iteration of non-unit fractions.

Generating a Fractional Number Sequence for Twelfths.

The theme of baking pizzas continued in the next teaching episode that was conducted three days later. Our goal was to provoke the children into thinking about different fractional names for quantities of pizza, based on the number of slices in a pizza. The children chose to make pizza with 12 slices (sticks partitioned into 12 parts). Azita asked them to name all the different fractions of that pizza. Patricia began by naming 1/12, 2/12, etc. all the way to 12/12. Joe spontaneously went beyond the whole (13/12, 14/12 etc.). Patricia then realized that this naming process could go on indefinitely and chimed in with "infinity twelfths"! Twelfths had become units of a fractional number sequence for these two children that was on a par with their whole number sequence. They reasoned with fractions now using their whole number operations. Joe found the stick that was 1/3 of the original stick by finding a 4-stick because "three times 4 is 12." He was, however, able to rename the stick as a 4/12-stick when asked what he meant by "4". Being able to switch back and forth between reasoning with whole numbers and naming the fractional number is further indication that the children's fractional and whole number sequences had achieved similar levels of abstraction.

Later in this same episode Patricia offered the 3-stick as another possible fraction of the original 12-stick. Joe renamed this stick "one fourth" when asked by Azita because "Three four times...to make a 12-stick." In demonstrating what he meant by this statement, Joe attempted to repeat the 3-stick four times to make a stick the same length as the original pizza stick. He inadvertently clicked one extra time when repeating the 3-stick, creating a stick that was five iterations of the 3-stick. Both children named this new stick as 5/4 of the pizza stick. Patricia explained that 3/12 was one fourth based on the classroom procedure she had been taught for reducing fractions:

> We did this in math. What's it called? You reduce. If you reduce 3/12. Like, how many... You divide by 3. Three will go into 3 and 3 will go into 12. So 3 divided by 3 is 1 and 3 divided by 12 is 4. So it's reduced to 1/4. (Patricia drew the numerals on the table with her finger as she talked).

Joe's way of demonstrating that the 3-stick was 1/4 of the 12-stick was to iterate the 3-stick four times to make a stick the same length as the 12-stick (an application of his iterative unit fractional scheme). Repeating the 3-stick one time too many was serendipitous for us as it gave evidence that both children could reason with their commensurate fractions beyond the original whole. Patricia counted in triplets to establish the 15-part stick as 5/4 of the original 12-stick. Later in this same episode, Joe indicated that he might be able to reason with commensurate fractions that were not unit fractions of the original whole. When they were asked to find a twelfths fraction that was the same as 3/4 of the pizza stick, Joe appeared to solve the problem by finding what 3/4 of 12 would be (9) and then looking for the stick that was composed of 9 parts (9 twelfths). It is unclear how Joe found 3/4 of 12. He may have remembered that 1/4 was 3 and then multiplied 3 by 3. If so, this would indicate a decomposition strategy that was lacking in previous episodes.

Finding Commensurate Fractions of a 24-Part Pizza Stick.

In the next teaching episode one week later, we decided to use a 24-part pizza stick with which the children could work. Both children quickly named fractions in terms of twenty-fourths. Joe spontaneously offered a half of the 24-part stick and correctly renamed it as 12/24. The encouraging surprise came when Joe established 1/4 of the 24-part stick by pulling 6 parts out of his 12/24-stick (see Figure 1). This indicates that he regarded each part of the 12-part stick as still being 1/24 of the original stick.

He maintained his 1/4-relation through the intervening half-stick. I hypothesize that he knew implicitly that 1/2 of 1/2 was 1/4 of the original whole.

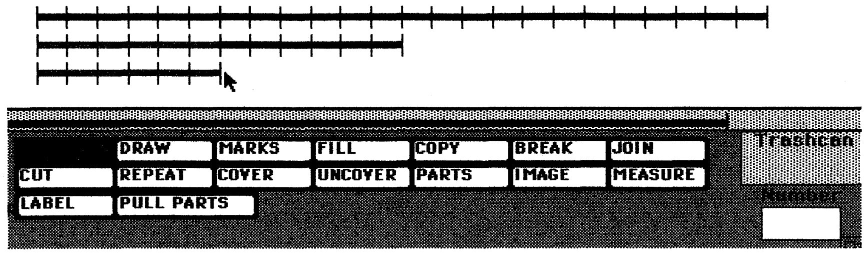

Figure 1: Pulling out 6 parts from a 12/24-stick in TIMA: Sticks

Joe was also able to spontaneously rename the 1/4-stick as 6/24, again indicating that he was relating the 6 parts he pulled out of the 12-part stick back to the original 24-part stick. Patricia agreed with Joe that one fourth was 6/24 because "Six times four is 24." Joe demonstrated the one-fourth relation by iterating the 6/24-stick four times to make a stick the same length as the original 24-part stick, an application of his iterative unit fractional scheme. In the continuation of this episode, however, Joe (J) appears to associate twelfths with his 12-part stick rather than with two parts out of the 24-part stick. The first protocol begins as Azita (A) asks Patricia (P) to think of a fraction.

Protocol I: Establishing 1/12 of a 24-part pizza

A: Now, Patricia, can you think of another you would like to do?

P: A twelfth.

A: You want to do a twelfth.

P: (to herself) A twelfth of 24, let's see.

J: (Looks quizzically at P) I already did that one.

A: Which one?

J: Twelve. It's right there (points to his 12-part stick).

A: (To Joe) What's a twelfth of 24?

J: Twelve (he has his head on his arms). A half of 24.

P: I don't know if it's called that, but this is what I meant.

(Patricia pulls out a 2-part stick from the 24-part stick. Joe sees this and then won't look at the screen. She repeats the 2-part stick 12 times making a stick the same as the 24-part stick and then lines them up one above the other.)

A: So what did you make?

P: I pulled 2 out like this (she pulls 2 parts out of the 24-part stick) and I repeated it 12 times and it made a 24-stick.

J: Yep, yep.

A: So what is that (pointing to the 2-part stick)? What fraction of the whole is that?

P: One twelth of 24.

A: That's 1/12 of 24. That's really good.

(Azita asks Patricia to get rid of the two 24-part sticks and pull the 2-part stick underneath the 12-part and 6-part sticks.)

A: (Pointing to the 2-part stick) Is there another name for that guy, Joe?

J: (After 5 seconds) 2/24.

A: That's really good. Why is it 2/24?

J: There's only 2 of those twenty-fourths, umm stick.

Patricia apparently chose 1/12 as a fraction to make before she knew how much of the 24-part stick she would need. She may have chosen 12 as a possible divisor and then figured out that she would need 2 parts twelve times to make 24. Joe associated a twelfth with his 12-stick rather than as a fraction of the 24-stick. He seemed to be responding to his interpretation of Patricia's problem as to make a 12-stick from the 24-stick, and knew that this would be half of the 24-stick and that he had already done that one. When he saw Patricia pull out the 2-part stick, he realized his mistake and was able to accept the stick as not only 1/12 but also 2/24 of the original stick. In the second Protocol, both Joe and Patricia confirm their construction of a scheme for generating commensurate fractions for unit fractions of a 24-part whole.

Protocol II: Establishing a commensurate unit fractional scheme

(Next problem: Joe pulls an 8-part stick out of the 24-part whole.)

A: What are you trying to make?

J: A third... of 24.

A: Why is it a third?

J: Because, umm, it goes into 24 three times.

A: What fraction of the 24-part whole is that?

J: 8/24.

A: O.K.

J: Or a third.

A: Or a third. Excellent! You're next (to Patricia). Joe, how many 1/24 do we have in 12/24?

J: 12.

A: How many 2/24 do we have in 4/24?

J: Two.

In the first part of this protocol Joe used his *equi-partitioning scheme for composite units* to explain why an 8-stick would be 1/3 of a 24-stick: "Because, umm, it goes into 24 three times." He had partitioned his composite unit 24 into 3 equal parts, with 8 parts each. He also knew that this one third was also 8/24. He used the two names as referring to the same quantity. They were *commensurate fractions* for Joe.

In responding to Azita's question that there would be two 2/24 in 4/24, Joe was reasoning with 24ths as composite units in much the same way as he reasoned multiplicatively with whole numbers. This is another indication that Joe had constructed a *fractional number sequence* (Olive, 2000b) at a level of abstraction similar to his explicitly nested whole number sequence.

While Joe was responding to Azita's final question in the above protocol, Patricia pulled three parts out of the 24-part stick. Joe immediately named this 3-part stick "one eighth." Patricia agreed with Joe and demonstrated the one-eighth relation by repeating the 3-part stick 8 times to make a stick commensurate with the original 24-part stick. Joe offered "3/24" as another name for the fraction "because it's three of those 24ths." Azita then asked Joe if he could make 1/6 of the original stick. Joe was about to pull out 4 parts when Azita asked him to stop. She covered the four parts he had selected and asked the children for another name for 1/6. Both responded with "4/24". Joe commented "I already knew that. If I didn't know that I wouldn't know how many to pull parts." This comment indicates that he was aware of his numerical operations and their results ahead of his actions in the microworld.

The above episode indicates that both children could now produce unit fractions *commensurate* with composite fractional parts of a partitioned whole, as long as the number of partitions was factorable. The episode continued with Azita asking the children to produce non-unit fractions of the 24/24. Protocol III begins with the problem of making 3/4 of the 24-part stick.

Protocol III: Establishing commensurate fractions for common (non-unit) fractions

A: Can you make me 3/4 of the whole?

J & P: Three... fourths.

P: Oh! I know.

J: I've got it! I've got it!

(Patricia is counting along the 8-part stick. She then moves the 6/24-stick to the middle of the screen.)

A: Patricia, do you know the answer?

J: Yes, I know. Tell me the answer first! Tell me the answer (to P).

P: Three of these, three of these (waiving the 6/24-stick around). Six, six, six.

J: Well, what is that? Tell us how long it will be.

P: Oh! Three times 6 -- 18. 18/24.

J: (Claps his hands) Yeah!

A: How did you know that? How did you know that was 18/24?

Both: Because 6 times 3 is 18.

In contrast to the preceding teaching episode just one week prior to this one, both children now had an immediate strategy for finding 3/4 of the 24-part stick. Patricia realized that it would be three of the 6/24-stick because that stick was 1/4 of the 24-

part stick. Her explanation that it will be "Six, six, six" and her response to Joe's request for how long that would be: "Three times 6 -- 18. 18/24." indicate that Patricia could use her iterative unit fractional scheme to generate composite units with commensurate fractions, thus extending her commensurate unit fractional scheme into a scheme for generating fractions commensurate with common fractions. Joe was asking for the length of Patricia's iterated 6-part stick in terms of 24ths. Joe affirmed her result, indicating that he had already worked it out.

In the continuation of this episode, Azita asked Joe to find 5/8 of the whole pizza stick. Joe immediately responded with "15" and verified his response by iterating the 3/24-stick five times. This is further evidence that Joe had extended his commensurate fractional scheme to include common fractions by decomposing the quantity 5/8 into five of 1/8 of 24. It is also evident that he was working with the 3/24 as a composite unit of 3 units. In response to Azita's request for another name for the 5/8, Joe exclaimed "Oh! It's 15/24."

This episode indicates that both children could iterate a composite-unit fraction (6/24) three times to construct a non-unit fraction (18/24) as the quantity 3/4. Further, they had constructed the operations necessary to *transform* a partitive fraction such as 3/24 (3 parts out of 24) into the quantitative relation 1/8 (of 24/24) and use the transformed fraction to create the quantity 5/8 of 24/24 by iterating the 3/24 five times. These are the constitutive operations of a *commensurate fractional scheme*. This episode also indicates that Joe was now able to decompose and recompose non-unit fractions. He saw 5/8 (of 24/24) as 5 of 1/8 of (24/24), and substituted the commensurate fraction 3/24 for the 1/8, thus obtaining 5 of 3/24 to give him 15/24.

DISCUSSION

Rather than interfering with their construction of commensurate fractions, the children's whole number multiplicative schemes were instrumental in the construction of their fractional schemes. The decision (on our part) to use partitioned sticks as the referent unit for the children's fractional reasoning created situations which were assimilated into both their multiplicative schemes and their iterative fractional schemes. This dual assimilation provided the children with powerful ways of operating whereby the same fractional quantities could be named in different ways. Their iterative unit fractional schemes enabled them to interpret 3/4 as three of 1/4; their multiplicative schemes enabled them to find 1/4 of the 24-part stick as 6/24 because "six times four is 24." Combining these two constructions provided the children with the insight that 3/4 was three of 6/24, and that would be 18/24 "Because 6 times 3 is 18." Using their multiplicative operations with whole numbers in this way is very different from Patricia's learned classroom procedure: "Three divided by 3 is one and 3 divided by 12 is 4 – so it's reduced to 1/4."

Unit fractions were now unit items on a par with the children's whole number units and the children could apply all the operations and complex unit structures (units of units of a unit) of their explicitly nested number sequence to these fractional units.

This transformation of their iterative unit fractional schemes is necessary for constructing commensurate fractions that generate quantitative equivalence.

NOTES

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