BUILDING NEWTON–RAPHSON CONCEPTS WITH CAS

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In this study the computer algebra system (CAS) Derive in TI-92 calculators was used to encourage an inter-representational approach to the study of the Newton-Raphson method. Students in both New Zealand and Korea were given the opportunity to use CAS to build an understanding of how and why the method works. In addition they were able to construct some of the key concepts of function underpinning the method. The results of the tests and questionnaire suggest that students can integrate CAS into an overall strategy in a way that assists them in their conceptual understanding of this method.

Background

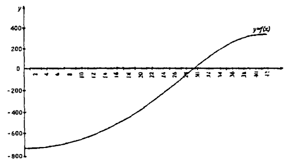
A deep understanding of mathematics involves an appreciation of how different representations contribute in often subtle ways to concept formation. A number of aspects converge in building this type of understanding for students: forming conceptual links between different representations; the ability to translate between representational systems (Lesh, 2000; Hong, Thomas, & Kwon, 2000); and the qualitative nature of the interactions with each representation (Thomas & Hong, 2001). None of these aspects is straightforward. For example, as Greer and Harel (1998) point out, students may have a surface ability to form links between procedures in two representational systems without forming an understanding of the deeper, conceptual links which are imbedded in the transformations between representations. In the case of interactions, Thomas and Hong (2001) have described up to nine qualitatively different types of interaction with representations which students may engage in. These are divided into those which involve procedural interactions, such as calculating the gradient of a function at a point from an algebraic formula, table or a graph, and those which require either a conceptual process (CP) or a conceptual object (CO) perspective of the representation. Examples include investigations of whether a function has points where the gradient is the same (CP), or conservation of properties under transformations of the graph of a function (CO).

CAS calculators provide an opportunity to promote investigation and discovery through a multi–representational approach to problem solving, enabling manipulation of mathematical concepts both within and between different representations. However, it is becoming increasingly apparent that using CAS is not as straightforward matter as it might once have appeared. First there is the question of what prerequisites students need in order to use CAS effectively. Pierce and Stacey (2001) have described a framework to analyse algebraic insight, which they believe is crucial in advance of CAS work. Secondly, others have expressed doubts about the extent to which students generally form inter-representational thinking (e.g. Tall, 1996). Finally, there is the issue of the choices facing teachers and students in the implementation of CAS calculators. Students have to be able to form an overall strategy to solve a given problem and to decide where, and how, to integrate CAS use. They may be faced with a considerable number of choices with regard to such implementation. This involves the issue of instrumentation (Trouche, 2000), the change from calculator as tool to its use as an instrument, with the instrumentation process being separate from the conceptualisation process. This instrumentation process involves the nature of student interaction with CAS, and the integration of pen and paper methods with CAS use (Lagrange, 2000). In the former case the CAS can be used in a procedural way, as a form of 'black box' which simply produces results of calculations on demand, or as a window onto concepts, or a combination of these. In the latter students have to form a partnership with the CAS, making individual decisions within an overall strategy on when thinking with pen and paper is appropriate and when and how to use the CAS.

Method

This study considers the above issues in the context of New Zealand and Korean students using CAS in the Newton–Raphson (NR) method. We investigated whether CAS use improved students' conceptual understanding of this method of calculating approximations to roots of equations; what factors were influencing this; what kind of partnerships with CAS students formed; and whether there were any cultural differences (based on the different languages and diverse cultures).

The diagram below shows the graph of $y=f(t)$ for $0 \le t \le 42$.



3. Use an initial estimate of $x_0=15$. Find the first iterate, x_1, of the Newton-Raphson method to solve the equation $f(t)=0$. Round your final answer to two decimal places.

4. Use an initial estimate of $x_0=15$. Use the above to show geometrically how the first iterate, x_1, and then the second iterate x_2, are found using the Newton-Raphson method. Clearly indicate on the graph x_0, x_1 and x_2.

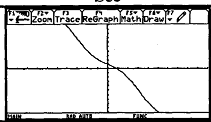
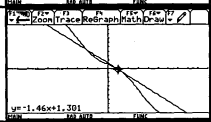
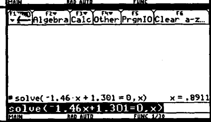
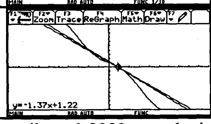
Figure 1. A Newton–Raphson examination question requiring geometric understanding.

The NR method was considered a suitable subject for research because: CAS use is beneficial in calculus (e.g. Heid, 1988; Hong & Thomas, 1997; Trouche, 2000; Drijvers, 2000); it is an area where teaching can concentrate on the procedural symbolic, with little attention to visualisation; and CAS can be used in a 'black box' mode without understanding of the process. For example, for New Zealand university entrance examination questions such as that shown in Figure 1, students often fail to demonstrate any geometrical understanding of the NR method. Typical comments on the geometric aspects of questions like this from New Zealand examiners' reports over a number of years, include: "Very poorly done. It is apparent that many candidates had no geometrical appreciation of the Newton Raphson method. For such students, this topic becomes nothing more than an exercise in evaluating a number of arithmetic expressions." A case study methodology was employed in each country with a partial qualitative and quantitative comparative analysis superimposed.

Subjects: Two high schools were used in this study, one in Auckland, New Zealand (NZ) and one in Seoul, Korea. The NZ school is an independent co-educational school with a relatively high socio–economic intake. The seventeen 16–17 year old students in this advanced Form 6 class had previously covered all the university entrance work, including the calculus and the NR method, but without using the CAS calculators. The twenty 16–17 year old Form 6 Korean students were all volunteers especially interested in mathematics, attending a co-educational school with an average socio-economic intake. The Korean students had no background in calculator use and none had ever used a graphic or a CAS calculator. In contrast, while the NZ school has a positive view of technology use, its students had not previously used CAS.

Since the NR method is not included in the Korean curriculum, after their normal classes on differentiation the procedure was first taught without the calculators. This was to mirror the experience of the NZ students in order to be able better to make comparisons. Also Korean Form 6 students do not learn about exponential function and differentiation of sine and cosine functions, so one short lesson on these was added before the pre-test. The research project was carried out in New Zealand during 16th to 26th July, and in Korea from the 12th to 20th November of the same year.

Example. Solve the equation $\sin x = 2x - 1$ using the Newton-Raphson method. Give the answer accurate to 4 d.p.
The first step is to define the graph of $y = \sin x - 2x + 1$ and sketch the graph:

Press	See	Explanation
Method 1. [♦] [Y=] [SIN] x [)] [−] $2x$ [+] 1 [ENTER] [♦] [GRAPH]		The function is defined using $y1 = \sin x - 2x + 1$
F5 A:Tangent [ENTER] Tangent at? 1 [ENTER]		The first tangent line starts at the point $x_1 = 1$. The equation of the tangent is given as $y = -1.46x + 1.301$. This will be used to find x_2 (which should be closer to the root than x_1), by seeing where it crosses the x-axis.
[♦] [HOME] F2 1 [ENTER] [(-)] 1.46 x [+] 1.301 = 0, x [)] [ENTER]		We can see that the next point $x_2 = 0.8911$
Return to [GRAPH] F5 A:Tangent [ENTER] Tangent at? 0.8911. [ENTER]		The equation of the tangent is given as $y = -1.372x + 1.218$ at the point $x_2 = 0.8911$. This will be used to find x_3 (which should be closer to the root than x_2).

This process was then repeated until $x = 0.8882$ was obtained.

Figure 2. A section of the module showing the layout and calculator screens.

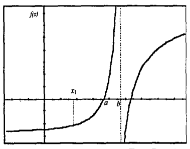
Instruments: A module of work using the TI–92 calculator was prepared. It addressed key mathematical concepts and ideas required for understanding NR, including: variable, expression (function), equation, tangent, graph, limit, differentiation, and the NR method, doing so in an inter-representational manner. Each section of the module was set out according to a 'Press', 'See', and 'Explanation' format (see Figure 2 for an example). Two NR methods were presented. The first comprised an intuitive approach seeing visually how we can often get nearer to a zero α of a function by drawing successive tangents and using the CAS to find the equation of the tangent in the graphical mode and its zero on the algebraic home screen. The second method used two derivations of the standard algebraic formula: rearranging the equation of the tangent: $y - f(x_1) = f'(x_1)(x - x_1)$ with $y = 0$ when $x = x_2$,; and a more visual approach, using the triangle created by the tangent and equating the gradient written as the quotient of the two sides and the derivative, i.e. $f'(x) = \frac{f(x_1)}{(x_1 - x_2)}$. A section from the module illustrating one method for solving the equation

$\sin x = 2x - 1$, is given in Figure 2. Two parallel tests using different numerical values were constructed for the pre-and post-tests. After the post-test the students were also given a questionnaire.

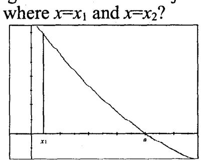
B3. If, in the Newton–Raphson method, for a function $y = f(x)$, $f(x)>0$, $x_1 = 2$ and $f'(x_1)>0$, where x_1 is the first approximation to the root, is $x_2>x_1$ or is $x_2<x_1$, where x_2 is the second approximation to the root? Explain your answer.

B7. How could you use the Newton–Raphson method to find the x-value of the intersection of the graphs of $y=2e^{-x}+\cos x$ and $y=2$? Explain your method clearly.

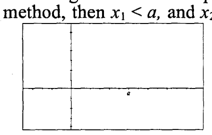
B4. a) Explain why x_1 in the diagram alongside is an unsatisfactory first estimate in the Newton-Raphson method for the root $x=a$ of $y=f(x)$.
b) When would x_1 be a satisfactory first estimate?



B5. For the function $f(x)$ shown below the 2[nd] approximation x_2 to the root $x=a$ is exactly 0.8 closer than the first approximation x_1. What is: (a) the relationship between $f'(x_1)$ and $f(x_1)$? (b) the gradient of the chord joining the points where $x=x_1$ and $x=x_2$?



B6.b) Draw a continuous function below where, if x_1 and x_2 are the 1[st] and 2[nd] approximations to the root $x=a$ using the Newton–Raphson method, then $x_1 < a$, and $x_2 > a$.



c) What determines whether x_2 and x_3 etc. are less than a or greater than a?

Figure 3. Some section B post–test questions on the Newton–Raphson method.

The tests were divided into sections A and B, with the former containing essentially procedural skills but the latter demanding greater conceptual understanding, especially geometrically. We wanted to know if they knew how and why the method worked and could apply this. We were also interested in when and how they would make use of the CAS. In B3, B4 and B5 (see Figure 3) a general function $f(x)$ rather than an explicit function was given, discouraging immediate use of a procedure, instead necessitating understanding and application of relationships.

Procedure: The module, written in English, was given to the New Zealand teacher, and a Korean translation was given to the Korean teacher. Once the teachers were comfortable with the calculator and the material they taught their class for four lessons covering basic facilities of the calculators and describing ways to find a limit, a gradient function, and roots of an equation. Each student had access to their own TI–92 for the whole study, including time at home. In each country the teacher stood at the front of the class, who sat in the traditional rows of desks, and demonstrated each step using a viewscreen while the students followed in the module and copied the working onto their calculator. Afterwards the students worked while the teacher circulated and assisted with any problems.

Results

Comparing the overall results of the Korean and New Zealand students on the pre- and post-tests there was a significant difference on both section A ($m_{\text{Korea}}=3.15$, $m_{\text{NZ}}=5.35$, $t=3.51$, $p<0.005$) and B ($m_{\text{Korea}}=2.7$, $m_{\text{NZ}}=5.6$, $t=3.14$, $p<0.005$) at the pre-test. It was not surprising that the New Zealand students were performing better initially since they had studied the topic previously. However, at the post-test an interesting difference emerged. As expected, for the section A skills questions, the Korean students improved considerably, and although there was no significant difference between the scores on this section ($m_{\text{Korea}}=7.15$, $m_{\text{NZ}}=6.35$, $t=1.17$, n.s.), there was a highly significant difference in the gain scores in favour of the Korean students (see Table 1). In contrast both groups improved about the same on section B (see Table 1), and so the New Zealand students remained significantly better than the Korean students here (m_{Korea}=5.90, m_{NZ}=8.53, t=1.97, p<0.05).

Table 1

A Comparison of the Pre– to Post–test Gain Scores

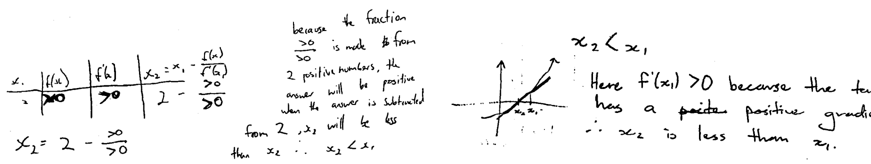
Max score=29 (Sec A=10; Sec B=19)	Mean Korea	NZ	t	p
Total Score (Post–Pre)	7.25	3.94	2.32	<0.05
Section A (Post–Pre)	4	1	5.03	<0.00005
Section B (Post–Pre)	3.25	2.94	0.26	n.s.

Thus it seems that the use of CAS calculators was effective for both procedural and conceptual understanding in both countries, for those familiar and not familiar with the method, but for the former the main benefit was in conceptual understanding.

Understanding concepts: Our analysis of the NR method indicated that students needed to form a partnership with the CAS interacting with its representations (Thomas & Hong, 2001) in a way that would help them to construct an inter-representational understanding of at least four key ideas:

1. The sign of f and $f'(x)$ (gradient of the tangent) affects whether the second estimate is > or < the first estimate.
2. The size of $f'(x)$ (gradient of the tangent) affects the existence and value of the second estimate.
3. The sign of $f''(x)$ (concavity of the curve) at the first estimate affects whether the second estimate is > or < the root a.
4. A geometric understanding of how and why the method works ie tangent intercepts getting progressively closer to a root and a sense of where the algebraic formula comes from.

Solutions to question B3 (post-test mean 0.996 out of 2) gave some insight on student thinking on point one. For example, Figure 4 shows the post-test solutions of two students, NZ1 and NZ2, who were both unable to answer at the pre-test.

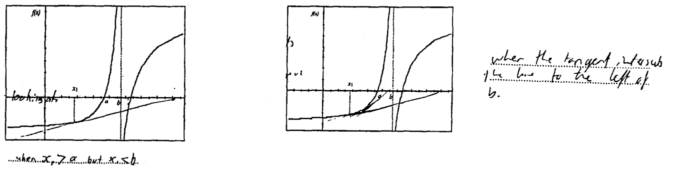


NZ1 NZ2

Figure 4. Student working on question B3 showing appreciation of the sign of f'.

NZ1's solution involved interacting with the algebraic formula representation rather than visualising the graph and the tangent. It was based on general principles of the sign of the two functions $f(x)$ and $f'(x)$, showing that for f and f' >0 x_2<x_1. Evidence that he was also able to think visually about them in the graphical representation was apparent from his comments in the questionnaire. Showing that he had understood the way the tangent influences the solution process he responded to the question 'How does the Newton–Raphson method work?' with "Tangents on the graphs bring you closer to the root of the graph." Thus while he had an inter-representational perspective he chose not to use it on this occasion. This is an important point. Student NZ2, however, connected the symbolic representation of the gradient function, $f'(x)$, with the tangent in the graphical representation and interacted with this representation to demonstrate the influence of the sign of the function (drawn as positive), and the gradient, on the position of the second approximation.

Point two, learning the role of the size of f' can be illustrated by the post-test working of students NZ3 and NZ4 on question B4 (post-test mean 0.947 out of 3, see Figure 5). Interacting with the graph both have appreciated that the first estimate has to be sufficiently close to a for the gradient to be large enough for the tangent to cut between $x=a$ and $x=b$.



NZ3 NZ4

Figure 5. Student NZ3 and NZ4's working showing appreciation of the role of the size of f'.

Further evidence of this understanding was provided in questionnaire comments. When asked 'What is important about the choice of the first approximation to a root when using the Newton–Raphson method?', students showed an understanding that a gradient which was too small or zero would cause problems:

NZ1: You can't choose a max or min point or else you won't cut the x-axis. Also the tangent could go towards the wrong root.

NZ5: It is very important that the approximation is close enough the root and not on a turning point. Otherwise you might be finding the wrong root.

NZ6: It must be close to the root so the tangent gives you the nearest value. Also you can't choose a stationary point as a first value.

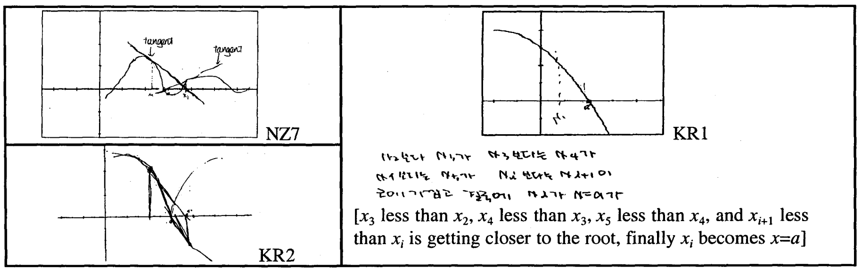


Figure 6. Student NZ7, KR1 and KR2's working showing appreciation of the role of concavity.

Some students gained understanding of the role of concavity in the NR method. Figure 6 shows the post–test working of three students NZ7, KR1 and KR2 on question B6 (post-test mean 1.13 out of 4). In each case they have managed to construct a function whose graph has the required concavity (concave down to the left of a) to put the first and second estimates on opposite sides of the root, as requested (although KR2 has an error - drawing a function which is concave up on the right and mistakenly trying to show the third estimate as greater than a). Interestingly NZ7 has constructed a function with 2 zeros, which was not required, and has arranged the gradient of the tangent at x_2 to be small enough so that it crossed the axis near $x=a$, rather than close to the nearer zero. This shows a good grasp of principles in this conceptual interaction with the graphical representation. KR1's written explanation shows that she understands that all further estimates of the root will be greater than a for her function. Student understanding of the way in which the NR method works (point 3 above) was provided in answers to the question 'How does the Newton–Raphson method work?' Some of the responses were:

KR3: Take some point x_1, draw the tangent line at the point. Find the other intersection point x_2 on the graph. In the same way, finally find the intersection point x on the x-axis.

NZ2: It finds a root of a graph by finding out where the tangent of a close guess cuts the x-axis.

NZ7: Start with the initial point. Draw the tangent line. Take the approximation closer to the root.

These (one translated from Korean) and others, refer to tangents, and they clearly have the idea of how NR uses these to get closer to the root. They are developing representational fluency (Lesh, 2000) and are no longer limited to working only within an algebraic representation.

CAS Partnership: One of the methods used in the study for integrating CAS into the method of estimating the roots using NR required the students to have a good overview of the strategy and to see clearly when and how to employ the CAS and when pen and paper techniques (point 4 above).

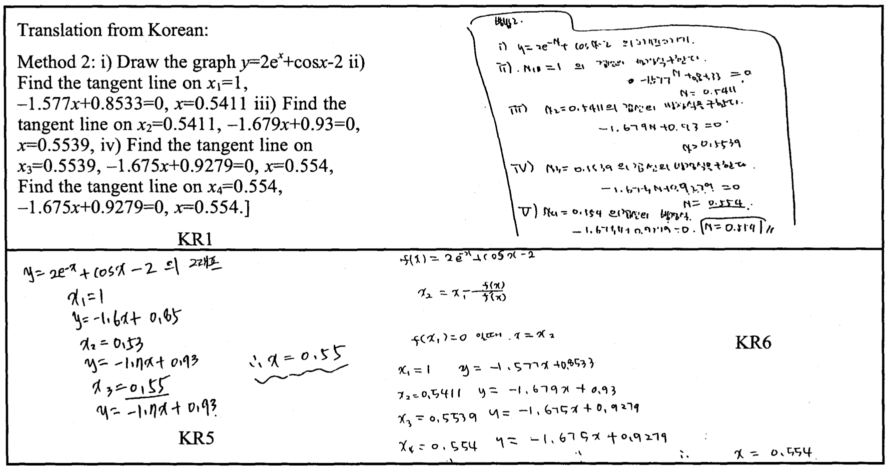


Figure 7. Students KR1, KR5 and KR6's pen and paper working for the CAS method.

Figure 7 shows work from three students on question B7 (post-test mean 0.729 out of 2), which asked them to solve $2e^{-x}$+cos$x$=2, without specifying a method. All have correctly identified the function to use and KR6 has also written the standard algebraic formula for the NR method. In each case they have chosen to integrate CAS into an overall strategy, as encouraged in the module. First they select a reasonable initial estimate, x_1=1 in all three cases. Moving to the CAS they enter the function using the [Y=] facility and then plot the graph, making sure that the viewing window is satisfactory. Next they use the CAS graphing screen to find the equation of the tangent at 1, and write this down using pen and paper. They have to know what the strategy requires them to do with this tangent equation, namely using the CAS algebraic home screen to find its zero. Appreciating the role of this value, 0.5411, as the next estimate in the overall method they can return to the graphical representation and use the CAS to find the equation of the tangent at the point x=0.5411. Finally they have to be able to see that this cycle is repeated until the accuracy required is obtained. This is a fairly sophisticated integration of CAS into a strategy, and shows that these students had become competent users of CAS for the NR method, able to form a good partnership with the technology. Of course, once students have a good overview of the method and its concepts then there is no reason why they should not use the CAS in 'black box' mode and enter $solve(2e^{-x}+\cos x =2, x)$ to obtain the root.

There were some differences between students in the two countries. For example, the Korean students were caused some anxiety by the calculator commands, since these appeared only in English. The teacher translated them into Korean for the students but some confusion ensued until they became more accustomed to the English. The psychological anxiety caused by this could have affected their self-confidence and hence their ability to build mathematical understanding. In spite of this the students from both countries have improved in their understanding of concepts associated with NR method. They have increased the versatility of their representational interactions (Thomas & Hong, 2001) and begun the process of instrumentation of the CAS calculator. This has enabled them to be more comfortable with overall strategies integrating CAS with pen and paper methods.

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