FUNCTIONS, MANY YEARS AFTER SCHOOL: WHAT DO ADULTS REMEMBER?

Ronnie Karsenty, Weizmann Institute of Science, Israel

Shlomo Vinner, Ben Gurion University, Israel

This paper presents findings from a qualitative study regarding adults' recalled definitions of function, and their attempts to draw simple linear functions. The subjects in this research were 24 men and women between the ages of 30 to 45, all engaged in successful careers. Findings support the principal conclusion of Bahrick and Hall (1991), who claimed that retaining of high school mathematical material strongly depends on the length of acquisition period and the amount of practice. Also, some of the findings exemplify Bartlett's theory (1932) concerning reconstruction vs. reproduction in the mechanism of recalling.

Introduction

Consider the phrase "curricular consensus"; sounds like an oxymoron, does it not? So many different views exist about what should be taught in schools, that it seems a great achievement if an agreement can be reached within a single school. Nevertheless, we might find a consensus in the shared implicit expectation of all curricula, which is – as we dare to assume – that graduates should be knowledgeable to a certain degree about the contents they have learned. Such an expectation evokes a cardinal question: What do adults remember from their past studies? This intriguing question was scarcely addressed by research until two decades ago, much to the explicit disappointment of cognitive psychologists specializing in memory research (Neisser, 1978, Bahrick, 1979). Since 1980, some progress has been made in this area (see a review by Semb et al., 1993), but the number of relevant researches is still relatively low, probably due to unresolved methodological problems. In regard to mathematics, a unique work is that of Bahrick and Hall (1991), a large-scale quantitative study designed to identify variables that affect losses in recall of high school algebra and geometry contents. A major finding of this work was:

> "When the acquisition period extends over several years, during which the original content is relearned and used in additional mathematics courses, the performance level at the end of the training is retained for more than 50 years, even for participants who report no significant additional rehearsal during this long period. In contrast, those whose acquisition period is limited to a single year perform at near-chance levels..." (p.30).

The performance of Bahrick and Hall's research participants (about 1700 in number) was measured by psychometric means (that is, correct/incorrect answers); indeed, a statistical study of such a large scale does not usually involve cognitive analysis of answers to open-ended questions. Since cognitive analysis could potentially reveal phenomena that remain unnoticed by psychometric analysis (Karsenty & Vinner, 1996), it seems that a qualitative study, scrutinizing adults' attempts to recall mathematical material, could add to the general picture in the issue of maintenance of high school mathematical contents. In this article we chose to present attempts to recall the definition of function and graphical representations of linear functions.

Theoretical background

Mechanisms of recalling. The cognitive aspect of recalling contents has long been an attractive subject for researchers (mostly referring to contents acquired in research laboratories). The well-acknowledged classical work of Bartlett (1932) laid the foundations of this field. Bartlett suggested that recalling is a mechanism of *reconstruction* rather than *reproduction*. In other words, when requested to repeat a story heard or an event experienced, people are likely to produce an interpreted version, though they may be unaware of doing so. Thus, some details might be omitted, others emphasized or even added. In this article we present examples of attempts to recall the graphical representation of a linear function, in a way that suggests that a similar mechanism of reconstruction takes place. Moreover, we intend to show that such reconstructions, idiosyncratic as they might be, sometimes follow a certain inner logic that is recruited in the absence of accessible relevant details.

The concept to be recalled: What is a function? Math educators and researchers have debated on proper ways of introducing the concept of function, one of the most difficult and complicated concepts that secondary school mathematics students encounter (Buck, 1970). One of the core questions in that matter concerns the kind of definition that should be used. Roughly speaking, we can divide definitions into two groups, based on the historical development of the notion of function (see Kleiner, 1989). The first group is what we might call "old definitions" – definitions in the spirit of 18^{th} century mathematicians Euler and Lagrange. For them, a function was "an analytic expression representing the relation between two variables with its graph having no corners" (Malik, 1980, p. 490). The second group of definitions can be referred to as "modern". In these definitions the key word is correspondence, as in the Dirichlet-Bourbaki definition. This major change in the definition of function was reflected in most of the mathematical secondary school curricula all over the world. Examination of ten 20^{th} century Israeli textbooks, dating from 1923 to 1995, reveal a shift from "old" definitions to "modern" ones, around the end of the sixties and the beginning of the seventies (Karsenty, 2001). The question of whether this curricular shift contributes to a better understanding of functions, stands at the heart of the debate mentioned above. Malik (1980) criticizes the use of modern definition of function in courses intended for average students:

> "The necessity of teaching the modern definition of function at school level is not at all obvious and most of the instructors feel that pedagogical considerations were ignored while designing the course content and the mode of presentation" (p.490)

Vinner and Dreyfus (1989) doubt the appropriateness of a modern definition for certain courses, and suggest that an intensive example-based introduction should precede and accompany the formal definition. Markovits et al. (1986) suggest ways to overcome some obstacles and enable a successful use of the modern definition.

In light of these opinions, it was interesting to look into the following question in regard to our research subjects. Considering the fact that they had attended high schools around the decade of change (see methodology section below), what kind of definitions will they use when requested to define a function in an open-ended manner? Some results follow the methodology description.

Methodology

The data and analyses reported here are part of a study, whose aim was to investigate cognitive and affective characteristics of adults' recollections in regard to their mathematical experience and knowledge (Karsenty, 2001). The study was defined as a "collective case study" (Stake, 1994), and focused on adults with considerably high level of education. The subjects' selection procedure was described in detail in a previous article (Karsenty & Vinner, 2000), and will be repeated here briefly.

12 men and 12 women were selected for the study from an upper-middle class community whose residents came from all over the country. At the time of the interviewing process the subjects were between the ages of 30 to 45, and their high school graduation year ranged from 1968 to 1984. All of them were graduates of college or university, but they varied in the level of math they had taken in high school[1] and the type of their current profession[2]. Each participant attended an individual session, devoted mainly to an interview, which was semi-structured and lasted about 2 hours[3]. The first part of it intended to explore personal affective components of learning mathematics (see Karsenty & Vinner, 2000, for some findings). In the second part of the interview, subjects were asked to "think aloud" about fourteen mathematical tasks involving basic concepts and procedures. The interview was intentionally held in a moderately-intervening manner. Thus, when a subject persisted that he or she "doesn't remember anything", small hints were suggested as triggers for eliciting any kind of respond associated with the question at hand. All interviews were recorded and transcribed.

In the next section we present results in regard to two interview tasks, introduced to the research participants as follows:

What is a function?
Can you sketch a graphical description of the function $y=2x$?[4]

[1] In Israel mathematics is a compulsory subject throughout high school, and can be studied in three levels, herein referred to as high level, medium level and low level.
[2] Professions were categorized by The Standard Classification of Occupations, a scale of 10 categories published by the Israeli Central Bureau of Statistics.
[3] Subjects also filled two questionnaires, findings from which will be published elsewhere.
[4] In some cases the given function was $y=x$ or $y=x+1$ or $y=2x+1$.

Results and Discussion

I. Responses to the interview question "what is a function" were grouped into six categories, which will be described and illustrated here in short. The categories are partly adapted from Vinner and Dreyfus (1989).

Category A. The subject's response includes, as a dominant component, one of the following expressions: "relation", "dependence", "influence", in regard to variables or number values.
Example:

"A function is, I think, a relation or a connection between two variables"

(Nurit (female), 38, psychologist, studied math in a high level track)

Category B. The subject's response includes, as a dominant component, one of the following words: "equation", "formula", "operation".
Examples:

"A function is an equation. An equation that describes some kind of a line"

(Gadi (male), 41, architect, studied math in a medium level track)

"A function is... using a... a general formula, and then you can plug in different numbers to reach certain goals. That is, it's some kind of a general formula that afterwards you can use each time with variables that are appropriate in a certain situation"

(Irit (female), 33, senior banker, studied math in a medium level track)

Category C. The dominant aspect in the subject's response is visual; the notion of function is mainly identified with a graphical representation.
Example:

"Ah... What is a function? A function is that thing with the... inside the graph, which has all kinds of shapes, and it looks differently according to its elements. [...] I remember that there are all kinds of functions [draws 3 different sketches of graphs]. Some are like this, some are like this, it depends on how the x relates to the y."

(Amira (female), 31, museum director, studied math in a medium level track)

Category D. The subject recalls the concept of function mainly in a context of exercises that he or she used to work on.
Example:

"I recall some story of substituting points, of a formula that you check. There's a question or instruction to plug in points and then every point to put on a graph [...] I remember the requirement, say in an exercise, of the peak point, something like that, and the lowest point, such things I remember. Or questions that dealt with the relation between the y-axis and the x-axis. Below zero, above zero, such things I recall, very vaguely. It's been some 20 years ago, you know. If you ask me what happens with this and what is it used for, I won't be able to tell you"

Eli (male), 40, principal of a post-secondary institution, studied math in a low track)

Category E. The subject can only elicit blurry associations related to the process of sketching graphs.
Example:

> "I remember that function was... we used to set up the... we used to set up on the graph there, on the x-axis and the y-axis, points... what was it? x-axis, y-axis, the points, and construct the function accordingly"
>
> (Arie (male), 38, insurance representative, studied math in a low level track)

Category F. The subject can neither define, nor exemplify, or even elicit associations to the word 'function'. It seems like the concept has been "deleted".

The distribution of subjects within the six categories is shown in table 1. Several observations can be made in regard to the data presented. First, it should be noted that none of the 24 subjects has defined a function in a way that could be regarded as modern. Generally it can be said that half of the subjects remembered the concept of function and defined it mostly in the spirit of old definitions (relation, dependence, equation, formula, operation, or via smooth graphs). The fact that the modern definition was not preserved in subjects' memory might suggest that it was not well assimilated, in spite of the favorable learning circumstances mentioned before (i.e., when the modern definition became popular in textbooks). It could be claimed, however, that this state of affairs is due to teachers' dissatisfaction with the use of the modern definition, as noted by Malik (1980) (see also Cha & Wilson's report (1999) concerning the inclination of prospective math teachers to define functions as equations or machines rather than by way of sets). One might suspect that, as a result of such dissatisfaction, the modern definition of function has played a lesser role in the implemented curriculum than it did in the intended curriculum.
Second, a connection can be noticed between high school math level tracks and the category distribution: Most of the subjects who participated in a high level track were classified to categories A-B, expressing familiarity with the concept of function. All

Category	No. of subjects assigned to this category, distributed by level of math taken in high school			
	High level:	Medium level:	Low level:	Total in this category:
A: Relation, dependence, influence	3	-	2	5
B: Equation, formula, operation	2	2	1	5
C: Visual response	-	1	1	2
D: Context of exercises to solve	1	2	2	5
E: Blurry associations	-	-	3	3
F: The concept has been "deleted"	1	-	3	4

Table 1. Distribution of subjects within the six categories of function definition (N=24).

medium-level graduates were classified to categories B-D. Most of the low-level graduates (8 of 12) were classified to categories D-F, which can be characterized by vague recollections, if any at all, about the definition of function. This finding is in accordance with the general conclusion of Bahrick and Hall (1991) in regard to lifetime maintenance of mathematical material, which is basically that the rate of loss is most affected by variables concerned with the amount and distribution of practice.

Finally, attention should be drawn to the other half of the subjects, those assigned to categories D-F. Our general impression was that for them, the concept of function was absent from what might be referred to as an "available concept pool". Using Vinner's (1983) terms, it can be said that the cell of "concept definition" became devoid; any associations, if elicited at all, were based on random pieces from the "concept image" cell. This finding is noteworthy, considering the fact that these people are educated adults who are well positioned in modern society. One can only conjecture on the extent of loss, in regard to this important concept, among the wider population of high school graduates.

II. Subjects' attempts to draw graphs for simple linear functions were classified into seven categories. We will not describe these categories here, due to space limitations. Instead, we present three examples of drawings made by subjects. However, we would like to point out that of the 24 adults participating in the study, only 7 could sketch, without intervening clues, a graph that was correct in its general idea.

Example 1. Figure 1 presents a drawing of the function y=x, made by Dov (male, 37, government official, studied math in a low-level track).

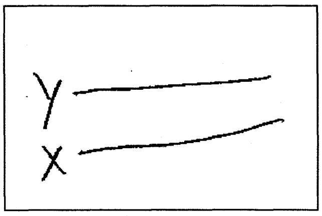

Figure 1. Drawing of the function y=x, made by Dov.

As can be seen, Dov does not recall the Cartesian system. In the absence of this frame of reference, he tackles the task in an idiosyncratic way. In other words, since Dov's knowledge of linear graphs is inaccessible, and thus can not be reproduced, Dov re-constructs the idea of y=x based on his present interpretation. Apparently, y=x is seen as an equality between quantities, and the two identical line segments illustrate this understanding. Two other subjects gave quantity illustrations; one described y=x as two identical circles, the other described y=2x as two small circles and a bigger circle.

Example 2. Figure 2 presents drawings of the functions y=x and y=2x, made by Tamar (female, 42, high school humanities teacher, studied math in a low-level track).

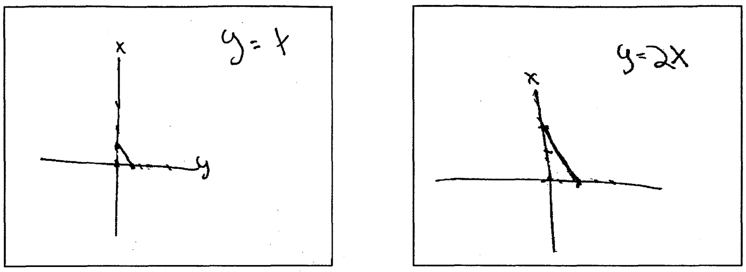

Figure 2. Drawing of the functions y=x and y=2x made by Tamar.

Tamar recalls the axis system, but not the Cartesian representation of points. Like Dov, she reconstructs the idea of linear graphs on the basis of her current common sense. While drawing y=x she says: "Well, anyway, there's got to be something equal here". Thus, she allocates equal segments of one unit on both axis. She then joins the two endpoints by a line. This last action might be ascribed to vague residues of the notion of function, elicited earlier by Tamar: When requested to define a function, she said - "There is a horizontal axis and a vertical axis, and there are points. You join the points and you get a function". Note that Tamar persists with the same logic when drawing y=2x. This time the segment allocated on the x-axis is twice as long as the one allocated on the y-axis, thus reflecting a common proportional misconception. Similar descriptions of linear functions were given by two other subjects.

Example 3. Figure 3 presents a drawing of the function y=3x+2 made by Judy (female, 35, lawyer, studied math in a medium-level track).

It is the second graph drawn by Judy. She first drew a correct graph of the function y=x+1, by calculating values of several points and joining them with a straight line. However, while doing so, she remarked: "It's a straight graph because it's very simple, it only has 1 added, and no multiplication". This remark led the interviewer to ask her to draw y=3x+2. This time, although Judy correctly calculates five point values, she joins the points with a curved line, as can be seen in Figure 3. She claims that due to the multiplication of x by 3, "what happens is that it's going to be higher and higher".

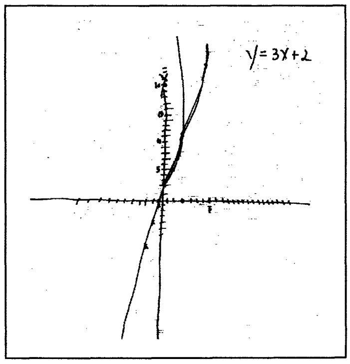

Figure 3. Drawing of the function y=3x+2, made by Judy

Again, we can see that a mechanism of reconstruction of knowledge takes place, even when the basic ideas of function graphical representation can be recalled. As a final comment we would like to emphasize, that the examples given above are merely illustrations of interesting phenomena of recalling mathematical material, yet to be explored.

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