EXPLORING THE RELATIONSHIP BETWEEN SIMILAR SOLUTION STRATEGIES AND ANALOGICAL REASONING

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Analogical reasoning is a powerful problem solving strategy that exploits the isomorphic relationship between two problems. The result of this exploitation is the production of similar solution strategies for the two problems. However, the presence of similar solution strategies is not a result exclusive to analogical reasoning. This study examines the use of similar solution strategies as an indicator of analogical reasoning in pre-service elementary school teachers' attempts to solve repeating pattern problems. Findings show that an awareness of problem similarity is not a necessary requirement for the production of similar solution strategies. As a result, the use of similar solution strategies as a measure of analogical reasoning needs to be modified.

INTRODUCTION

Making connections and utilizing similarities between problems is at the core of mathematical reasoning (English, 1998; National Council of Teachers of Mathematics, 2000). There are several proposed mechanisms by which learners weave a thread between what is known and what is new. One such mechanism, referred to as analogical reasoning, builds this connection by exploiting the tension between the similarities and the differences of two situations (Jardine & Morgan, 1987). However, a "subjects' conceptions of similarities between situations are analyzed relative to normative criteria with respect to predetermined mapping between the situations, built into the experimental set-up by the researcher." (Greer & Harel, 1998, p. 11)

This article examines the use of similar problem solving strategies as an indicator of analogical reasoning. I make the argument that strict adherence to similar solution strategies as an indicator of analogical reasoning causes improbable results to emerge from the data. An alternate form of reasoning is proposed that can be used to explain situations in which similar solution strategies are utilized in the absence of analogical reasoning.

ANALOGICAL REASONING

Mathematical reasoning in general, and analogical reasoning in particular are most closely associated with how students solve mathematical problems. Polya (1957) acknowledged reasoning by analogy as an explicit part of problem solving with such strategies as "think of a related problem", and "use a simpler (but similar) problem". These are very general recommendations to problem solvers as to how to proceed when faced with a challenging problem. Reasoning by analogy is a more specific description of the mechanism that forms the underpinnings of these strategies. As it applies to problem solving, analogical reasoning involves mapping the relational structure of a known problem (that has been solved previously, referred to as the *source*) onto a similar problem (referred to as the *target*) and using this known structure to help solve the similar problem (English, 1997, 1998; Novick, 1990, 1995; Novick & Holyoak, 1991). If the source problem and the target problem are almost isomorphic (that is, the source problem is isomorphic to part of the target problem or vice versa) then some adaptation or extension of the solution strategy may be required (English, 1998; Novick & Holyoak, 1991)

ANALOGICAL REASONING: A FRAMEWORK FOR ANALYSIS

Studies examining students' abilities to utilize analogical reasoning can be classified by what I refer to as their degree of openness (ranging from closed to open). Closed studies are ones that direct students, through the experimental set up, to use analogical reasoning skills. Experiments involving problem sorting, grouping tasks, or possible source problem identification from a list of problems are examples of such studies (e.g. English, 1998). These studies do not determine whether or not a student uses analogical reasoning skills but how well they use it as indicated by correct selection of source problems or normative sorting of problems.

Open studies on analogical reasoning abilities are ones in which the experimental design allows the students the freedom to invoke whatever problem solving strategies they see fit and then to analyze the results for evidence of analogical reasoning skills as indicated by similar problem solving strategies. In order to assure that participants are utilizing analogical reasoning it is necessary that the problems presented to the participant have many possible solution strategies (Bassok & Holyoak, 1989; Novick & Holyoak, 1991).

A particular study's openness is dependent on the impact that the experimental set-up has on the participants' choice of problem solving strategy. A partially open study would be one in which the participants are operating in a problem solving situation but the experimental structure may guide them into using analogical reasoning. Studies using a hint/no hint paradigm (e.g. Novick & Holyoak, 1991) to measure the impact of retrieval on successful solution strategy transfer are an example of a partially open study. Successful use of analogical reasoning is again indicated through the use of similar problem solving strategies.

With open, or partially open studies, however, analysis of data operates on two assumptions. The first is that the use of different problem solving strategies on similar problems indicates a lack of analogical reasoning, and the second is that the use of similar problem solving strategy indicates analogical reasoning. Analogical reasoning hinges on a solver's explicit identification of the similarity between two problems. This is a description (or a definition) of the mechanism of analogical reasoning. Similar solution strategies are a product of this mechanism. The question is, is the product an indicator of the process? Is the presence of similar solution strategies an accurate indicator of analogical reasoning?

METHODOLOGY

This study was designed to probe the relationship between the students' explicit identification of problem pair isomorphism and the use of similar solution strategies. Participants in this study were preservice elementary school teachers enrolled in a "Foundations of Mathematics for Teachers" course. Twelve of the students enrolled in the course volunteered to participate in clinical interviews. The tasks, along with some of the questions they were asked to solve were:

The Calendar Problem.

> I've chosen a calendar page, October 2000, and I'm going to place a red marker on the 1, a blue on the 2, a green on the 3, and a yellow on the 4. Now, I'm going to repeat this pattern; red on the 5, blue on the 6, green on the 7, and yellow on the 8.

a. What colour will number 13 be? What colour will 28 be?
b. If the calendar continued on forever, what colour would 61 be? 178? 799?
c. If there were five colours (red, blue, green, yellow, and black), what colour would 799 be? If there were 6 colours, what colour would 799 be?

The Sequence Problem

a. Consider the sequence 1, 5, 9, ...What will the next few numbers in the sequence be?
b. Will the number 48 be in this sequence? Will 63?
d. Can you give me a big number that you know for sure will be in the sequence?
e. Consider the sequence 5, 12, 19, ... Is 96 going to be in this sequence?
f. Can you give me a big number that you know for sure will be in the sequence?
g. Consider the sequence 8, 15, 22, ... Can you give me a big number that you know for sure will be in the sequence?
h. Consider the sequence 15, 28, 41, ... Is 1302 going to be in this sequence?

Comparison of the Two Problems

a. Consider the two problems you have worked on here: the calendar problem and the sequence problem. Is there any similarity between the two problems?
b. Is there a similarity between the strategies you used?

The interviews were conducted in the later part of the course. Students had studied the topic of 'arithmetic sequences' in the earlier part of the course.

The calendar problem and the sequence problem were chosen in order to present students with two situations that were almost isomorphic. The questions in the sequence problem were chosen to be 'twist' or 'inverse' problems (Zazkis & Hazzan, 1998). Rather than ask standard questions of finding the n^{th} element or which term a given element is, these questions 'twist' what is sought and what is given. Similar problems were used in Zazkis and Liljedahl (2002) and have been shown to be useful in getting students to examine the situation rather than simply use established algorithms.

Specific questions within each of the two tasks were removed or added (at the discretion of the interviewer) in order to help establish solid understanding of the problems before proceeding with the interview.

The two main tasks, the calendar problem and the sequence problem, were chosen for their implicit isomorphic relationship. Comparing the two it becomes clear that arithmetic sequences can be viewed as a special subset of the calendar problem. That is, the sequence 1, 5, 9, ... represents the sequence of red markers in the calendar problem.

RESULTS AND ANALYSIS

The transcripts of the clinical interviews, written work, and the audiotapes themselves, were used to analyze the problem solving strategies utilized by the students. A simple partitioning of the data using solution strategies as a gauge produced two groups: those who did not use similar solution strategies, and those who did.

Initial analysis revealed that five of the participants utilized different strategies, six utilized the same strategy, and one used multiple strategies for both questions. Of specific interest to this study were the six students who used similar solution strategies to solve both the calendar problem and the sequence problem.

Although the use of similar solution strategies for the almost isomorphic problems could be taken as indication of the use of analogical reasoning in solving the sequence problem, the responses to the probing questions indicated otherwise.

When asked to comment on the similarity between the problems both Deanna and Helen have difficulties identifying any.

Deanna: *They're kind of hard to connect right away, or at all. [...] But it's hard to see that they're similar, for me at least. It's hard to recognize that and see that, it's very hard to play around with numbers and to kind of see the relationships between the different numbers.*

Helen: *I used multiples of 4 for all of them, . . .*
Interviewer: *Okay, anything else?*
Helen: *Um, they all continue in a pattern. . .*
Interviewer: *Okay, anything else?*
Helen: *No. . .*

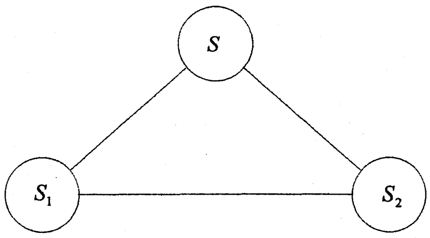
John, on the other hand, has no difficulty identifying the similarity between the problems when prompted to do so. However, probing for his views on the similarity of his strategies reveals that he cannot recall the strategy he used in the calendar problem.

Interviewer: *Um, the strategy you used here for solving those types of questions, would you say it's the same as the strategy you used here?*
John: *(pause) You would ask me, so if I was going to say. . .*
Interviewer: *Do you remember the strategy you used here?*
John: *Uh, I would like to go over one example just so [...] So yeah they are the same.*

After some time to refresh his memory, John indicates that he sees the strategies as being the same. This delay, however, shows that he was not exploiting the similarity of the problems and utilizing his strategy for the calendar problem while he was working on the sequence problem.

Each of the six participants who produced similar solution strategies either failed to explicitly identify the similarities between the problems or gave indication that, although they now saw the similarity, they failed to capitalize on it during their efforts to solve the sequence problem.

Greer and Harel (1998) take a broader view of the analysis of solution strategies of isomorphic problems. They classify the isomorphic relationship into three cases. The first two are identical to the processes of analogical reasoning (see English, 1998 for more details). The third is a case which Greer and Harel refer to as 'mediated isomorphism'.

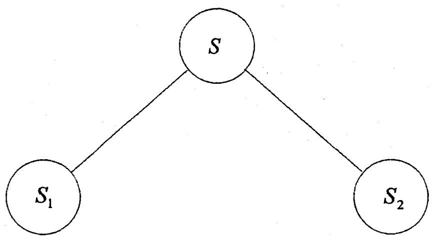


Mediated Isomorphism (Greer & Harel, 1998, p. 12)

Figure 1

Mediated isomorphism relies on the solver first identifying the similarities of the two problems S_1 and S_2 to the more general problem S, and then using these similarities to help establish an isomorphic relationship between the two problems S_1 and S_2. This mechanism has many of the same markers as analogical reasoning: two similar problems, similar solution strategies, an awareness of problem similarity. However, it is a mechanism different from analogical reasoning in that it lacks the willful exploitation of problem similarity for the purpose of solving the second problem. In fact, problem similarity is more a byproduct of the solutions than a mechanism towards the solution. Two of the participants showed evidence of this mediating effect as they became aware of the problem similarity while answering the comparison questions.

However, this mechanism fails to explain the results displayed by Deanna. Deanna's solution to both the calendar problem and the sequence problem indicated that she treated both problems as special cases of repeating pattern problems. Yet, she was not aware of the inherent relationship that existed between them. As an isolated case it could be classified as an incomplete execution of mediated isomorphism. However, the prevalence of this result in this study necessitates the recognition of this process as a different form of reasoning. I call this mechanism *non-mediated generalization*, where, as shown in *figure 2*, there is no direct connection between S_1 and S_2.



Non-mediated Generalization

Figure 2

All six of the participants who produced similar solution strategies for the calendar problem and the sequence problem indicated (through their responses to the comparison questions) that they were unaware of the similarity between the problems while they were actually engaged in the solving of the problems.

CONCLUSION

There is no denying that reasoning by analogy is a powerful problem solving strategy. As mathematicians we have come to rely on its strength. As teachers we make it our goal to give students the experiences and guidance necessary to improve this skill. As researchers we try to measure its use and effectiveness and try to identify the factors that aid or impede its success. However, there are other forms of reasoning, valid and invalid, available to the student in an analogical problem-solving situation. Carefully constructed instruments will be able to guarantee the willful use of analogical reasoning, but only in very controlled environments. Tasks such as problem sorting and source/target pairing direct participants to utilize reasoning by analogy through the experimental set-up. Studies using such tasks will, therefore, be able to discern the sorts of features that are attended to and allow analysis as to how well analogical reasoning is invoked. But if we utilize the same sort of analysis in less controlled environments we are making the assumption that solvers are, indeed, intending to utilize analogical reasoning. If true problem solving efforts are to be evaluated for the application of analogical reasoning then there must be a contingency for alternate forms of reasoning in any subsequent analysis.

This study has shown that even seemingly successful cases of analogical reasoning as indicated by similar solution strategies must be treated carefully. The theory of analogical reasoning is constructed on the explicit awareness of the similarity between the source problem and the target problem and the willful exploitation of this similarity in solving the target problem. Although a similar solution strategy is a good indicator of this form of problem solving it is not an exclusive outcome of analogical reasoning. Mediated isomorphism also relies on the explicit awareness of the similarity. However, because the similarity between the problems is not identified through direct comparison of the two problems as suggested by the theory of analogical reasoning, the purposeful use of the similarity is absent. Non-mediated generalization is a theoretical refinement of mediated isomorphism in that it lacks not only the willful use of the problem similarity, but more fundamentally, it lacks the very awareness of the similarity. However, both mediated isomorphism and non-mediated generalization have been shown to be present in cases in which similar solution strategies were used. Researchers need to be conscious of these subtleties both when considering experimental design and when analyzing data in order to avoid misattributing student's efforts to analogical reasoning.

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