"What if not?" Problem Posing and Spatial Geometry - A Case Study

Ilana Lavy, Emek Yezreel College
Irina Bershadsky, Technion - I.I.T, Haifa

Abstract

The aim of the study was to try to find out what kinds of problems are posed by prospective teachers on the basis of complex spatial geometry tasks using the "what if not?" strategy, and what is the educational value of such an activity. Analyzing the posed problems revealed a wide range of problems ranging from problems including a change of one of the numerical data to another specific one, to a proof problem. We also discuss the educational aspects of problem posing in spatial geometry using the "what if not?" strategy, which could encourage the learner to rethink spatial geometry concepts and make connections between them.

Background

Generally the term "problem posing " refers to three kinds of cognitive activities: (a) posing sub-problems within the process of solving a complex problem; (b) posing new problems on the basis of a solved problem; (c) posing problems regardless of the referred problem solution (Technai, 2001). Three types of problem posing can be distinguished in regard to posing new problems on the basis of a given problem: problem posing on basis of free, semi-structured and structured problems (Southwell, 1998). The term "problem posing" in the literature usually refers to an activity in which the problem posing itself is the focus of attention and not as a problem-solving tool.

This study focuses on problem posing on the basis of a given structured problem using the "what if not?" strategy (Brown & Walter, 1993). According to "what if not?" strategy, we examine each component of the problem and manipulate it through the process of asking "what if not?"

The problem posing role in students' mathematics learning

During the study of school mathematics, students experience problem solving. Usually they get the problems from the math teacher or from text books and only rarely are they asked to pose problems of their own. Mathematics educational researchers emphasizied the important educational value of problem posing by students and suggested incorporation of activities of problem posing within the mathematics sessions in school (Goldenberg, 1993; Leung & Silver, 1997; Mason, 2000; NCTM, 2000; Silver, Mamona-Downs, Leung & Kenney, 1996). The importance of an ability to pose significant problems was recognized by Einstein and Infeld (1938), who wrote:

"The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old questions from a new angle, require creative imagination and marks real advance in science"(p.92) (in Ellerton and Clarkson (1996).

Students who engage in problem posing activities become enterprising, creative and active learners. They have the opportunity to navigate the problems they pose to their domains of interest according to their cognitive abilities (Goldenberg, 1993; Mason, 2000). Studies show that problem posing might reduce mathematics anxiety (Brown & Walter, 1993; Moses, Bjork & Goldenberg, 1990). Including activities of problem posing might improve the students' attitude towards mathematics and make them more responsible for their learning (Brown & Walter, 1993; Silver, Mamona-Downs, Leung & Kenney, 1996). Researchers emphasized the inverse process in which developing of problem solving skills can be helpful to the developing of problem posing skills (Brown & Walter, 1993).

The problem posing role in teachers' mathematics learning

Developing skills of mathematical problem posing is important for mathematics teachers as well (Silver, Mamona-Downs, Leung & Kenney, 1996; Southwell, 1998). Southwell (1998) found that posing problems based on given problems could be a useful strategy for developing the problem solving ability of prospective mathematics teachers. Integrating problem posing activities in their mathematics lessons enabled them to get to know their students' mathematical knowledge and understanding better. Since teachers have an essential role in the use of problem posing activities in their mathematics classrooms, they should develop their own problem posing skills (Leung & Silver, 1997).

The study

In the present study we explored two questions:

1. What are the kinds of problems posed by prospective teachers on the basis of complex spatial geometry tasks using the "what if not?" strategy?
2. What is the educational value of posing problems using the "what if not?" strategy referring to spatial geometry?

Eighteen prospective teachers participated in the present research.The participants were participating in the "teaching methods for secondary school mathematics" course in which they were introduced to methods and activities referring to central topics studied in school.

The participants were asked to pose as many problems as possible on the base of the following problem:

Find the distance from the center of the base of a regular triangular pyramid to the pyramid's lateral face given that the pyramid's height is 10 cm and the dihedral angle (the angle between the two faces) is 67^0.

Our data comes from two sources:
(1) written protocols submitted by the participants.
(2) clinical interviews with several participants.

Results and discussion

The problems posed by the participants can be divided into two main categories: changing one of the given problem components; changing the problem question. Each one of the above categories was refined into subcategories, which we describe below. The changing of one of the components given in the problem can be divided into two subcategories: 1.Changing of the numerical value of the data; 2. Changing of the kind of data. The subcategory "Changing of the numerical value of data" refers to:
1.1. A change of a specific numerical value, given in the problem, to another one. For example, the pyramid height was changed from 10 cm into 12 cm.
1.2. A change of a specific numerical value, given in the problem, into a range of values. For example, the angle between the lateral faces was changed from 67^0 into an angle between 67^0 and 90^0.
1.3. Negation of the specific numerical value given in the problem. For example, what if the hieght is not 10?
1.4. Implied generalization of the given numerical value of data. For example, the height of the pyramid is not 10 cm (for example, 12, 20 and so on).
The subcategory "Changing of the kind of data" refers to:
2.1. A change of a specific kind of data, given in the problem, into another kind. For example, from a triangular base pyramid to a square base pyramid.
2.2. Negation of the given data kind. For example, what if not a pyramid?
2.3 Generalization of the given data kind. This sub-category can be divided again into implied and formal generalization. The term "implied generalization" refers to the participants' answers in which they offered a number of alternative examples to the negated component of the problem finishing their answer by the words: "and so on". This term is taken from a similar classification of students' utterances in Lavy (1999).
2.3.1. Implied generalization of the given data kind. For example, a pyramid does not have a triangular base (for example, quadrangular, pentagonal and so on).
2.3.2 Formal generalization of the given data kind. For example, the pyramid has an n regular polygon base.

The second category " Changing of the problem question" refers to another specific question. For example find the pyramid base area. It can be referred also to the changing of the given problem into a proof problem, for example, proof that sin $\alpha/2 = 5/8$ while the relation between a lateral edge of the regular triangle pyramid to the base edge is 5/9.

The sub-categories related to the negation of one of the problem data components could be considered as an uncompleted new posed problem, since the participants did not offer an alternative to the negated data component.
To visualize our findings, we draw the following diagram:

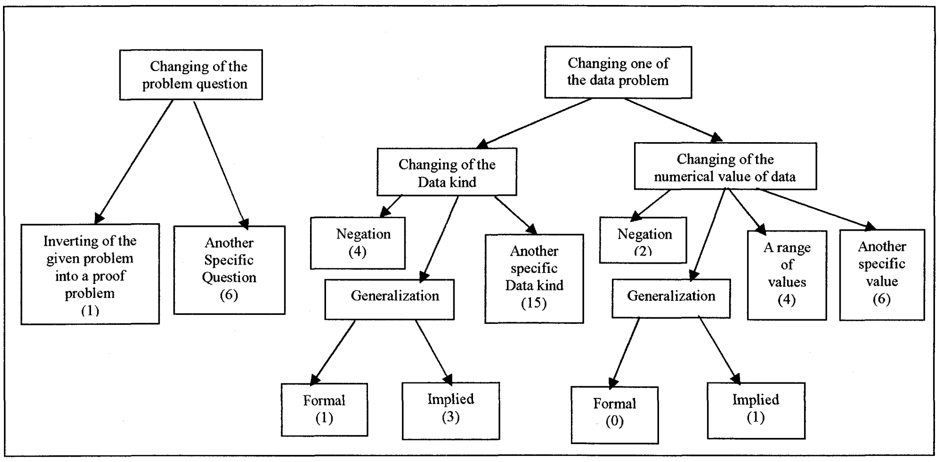

Diagram 1:
Classification according to data and question change

In cases where we had difficulty deciding to which category a suggested posed problem belongs, we were aided by the participant's interview. For example, the participant's utterance "what if not a triangle?" was interpreted as an implied generalization. Since in other posed questions of the same participant when he or she indicated another specific shape, he or she wrote so explicitly.

Classification of the posed problems reveals a wide range of problems ranging from easily phrased problems to the inverting of the original problem to a proof problem. The term "easily phrased" problems, refers to the change of one (or more) components in the original numerical data to another specific one.
The study revealed that the number of posed problems with another numerical value had a greater frequency than generalized posed problems (9 versus 3). It could be explained that posing another numerical example is a common activity in mathematics lessons in school and students are more familiar with this kind of activity than with posing generalized problem on the base of the numerical data. In other words, it could be said that when students are asked to pose a new problem on a basis of a given numerical data problem, the technical and the immediate widespread strategy is to pose a similar problem with another numerical value. Analyzing the solution of numerical data problems during school mathematics, is not a common activity (Polya, 1981; Schoenfeld, 1985). The solution analysis

including the testing of different numerical values could result in problems with another type of solution (if any) and might lead eventually to problem generalization. In addition, posing a generalized problem on a basis of a given problem is a complex cognitive activity rather than a technical change .

Another issue refering to posing problems with different numerical data emerged from Maria's interview. Observing her posed problems revealed that none of them included similar problems with new numerical values.

> Maria:"What if the height is not 10 cm? If I had the problem solution I would have seen how the change of height influences it. Without the solution it is difficult for me to decide what to take instead of 10. Maybe there is a number that doesn't influence the solution, but may be there exists a value that will change the problem to another one with a different way of solution".

From the above excerpt it follows that Maria added another constraint to the given task. She strove to pose problems that will have another mathematical solution, different from the original one and not just a cosmetic solution change. If she had solved the problem first, it could have helped her to estimate new numerical values that could lead to different ways of solution.

> Maria: "It was difficult to me at the beginning. I did not know where to start. When I solve the problem I know what I am using, and what can influence the solution. Without the solution it is difficult for me to visualize influential changes, that will lead eventually to a reasonable solution. I felt I needed to solve the problem first. I tried to pose a problem, but I could not come up with anything".

According to Maria's interview the fact that the participants were not asked to solve the problem first prevented her from posing meaningful new problems.

Julia, on the other hand, said that it was easy for her to pose many problems on the basis of the given one, emphasizing that if she had to solve the given problem first, she would have difficulties in posing many new problems.

> Julia: "Because you do not see the spatial problem, you are in the computing process. You become technical and after that you cannot disconnect yourself".

Julia needed the overview of the problem to be able to pose many new problems on a basis of the given one. For her, getting into the solution process diverted her attention from having the global viewpoint that enabled her to focus on the activity of posing problems.

Students differ in their view as to the assistance that a familiarity with a solution provides in problem posing.

One of the interesting phenomena is the lack of posed problems including formal generalization and the relatively small number of posed problems including implied generalization. As to posing problems including implied generalization, there is a difference between the number of posed problems including changing of numerical data versus posed problems including generalization of the given geometrical shape. The process of generalizing the numerical value of data in geometric problems leads to the analysis of extreme cases such as zero. Most of other numerical data changes cause mainly a cosmetic change of the problem, while the change of the data kind usually offers a new problem with a totally different solution.

Observing carefully the data reveals also that in "data kind change" posed problems there were more changes of plane data kind than spatial data kind (9 versus 2). There was a small scope of spatial shapes than of plane shapes in the posed problems. The most common plane shape was a square and it can be explained that the common students' example for a quadrangle is a square (Hershkowitz et al., 1990).

Considering the educational aspect of problem posing in spatial geometry using the "what if not?" strategy, such an activity makes the learner rethink different spatial geometry concepts. For example, questions such as "what if not a pyramid?" or "what if not a regular pyramid?" require the learner to know and understand the relevant geometrical concepts appearing in the problem and at the same time to think of the meanings and outcomes emerging from the negation of them. For example, if the concept in the given problem is a pyramid, the learner should know and understand the pyramid definition and think of the potential alternative concepts that might replace this concept, such as a cube, in the process of posing problem using the "what if not?" strategy. Concerning the question: "what if not a regular pyramid?" we will refer to the following paragraph from Maria's interview:

Interviewer: "Let's have a look at your posed questions. Could you clarify some of them? You wrote, what if not a pyramid? What did you mean?"

Maria: "Another shape, cone or rectangular parallelepiped. What if not a triangular pyramid: quadrilateral, pentagonal or any other polygon. What if not a regular pyramid: regular means that all the triangles are equilateral triangles...no ...[mumbling] only the base, the others are isosceles. What if not equilateral triangular base, but isosceles triangle or anything."

Observing Maria's posed problems during the workshop made us decide to interview her, since Maria posed, in comparison to others, many problems which had a foggy formulation and we thought it would be helpful for our analysis to get some clarifications. We asked Maria to tell us what she meant by her posed problem: "what if not a pyramid?" and her answer appears in the first part of the quoted paragraph. As to the question "what if not a regular pyramid?", Maria was thinking aloud about the meaning of a regular pyramid: a pyramid composed of equilateral triangles which later on she corrects herself to the definition that only the pyramid base has to be equilateral and the pyramid faces should be isosceles triangles. Maria's rethinking creates a new problem, which negates the regularity property of the pyramid. ... "What if not equilateral triangular base, but isosceles triangle or <u>anything else</u>."

Maria posed a new problem using the negation of the pyramid base regularity property. By doing so, she created a non-regular pyramid since its base is not an

equilateral triangle anymore. As an alternative to the given pyramid base, she offered an isosceles triangular base or non particular triangular base.
In order to be able to raise problems using the "what if not?" strategy, Maria deconstructs the term "regular pyramid" into a list of properties which define this term, negates one of them and builds an alternative geometric entity. The act of thinking reflectively about the properties of a certain geometric shape and the geometric implications emanating from the negation of one of the properties is a cognitive activity leading to deeper understanding of the relevant geometric terms. If we continue the same line of thinking, several other problems could be raised such as: what if the pyramid is not right? What if the pyramid faces are also equilateral triangles? What if the pyramid consists of scalene triangles? etc. Such a discussion could contribute to strengthening the connections between the various geometric terms and as a result to deepen the understanding of them.

An additional educational benefit of problem posing using the "what if not?" strategy is connected with the following issues: Whether the data change of one (or more) of the given problems leads to a solvable or unsolvable problem? Whether the created problem solution is totally different from the original one or are there only cosmetic changes in it? These issues appear in diagram 2.

The second classification we made was the influence of data change on the solvability of the problem and its solution.

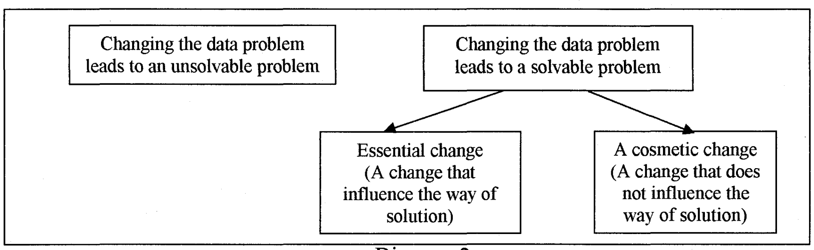

Diagram 2:
Classification according to data influence on the problem solution (if any)

Although our study is a case study, we found interesting phenomena with educational and learning benefits. The present case study raises some questions, which could be the basis to further studies, such as:

1. How can we cause learners to "produce" new correct mathematical problems?
2. How can we explain the phenomena of raising many problems (or few) and the reasons for it?
3. Does solving the given problem first influence the posing of problems based on it?

The activity of posing new problems on the basis of a given spatial problem made the students interact with their geometrical knowledge and use it as a basis for both a widening and deepening of their understanding.

References:

Brown, S. & Walter, M. (1993), Problem Posing in Mathematics Education. In S. Brown & M. Walter (Eds.) *Problem Posing: Reflection and Applications*, Hillsdale, New Jersey: Lawrence Erlbaum Associates. pp. 16-27

Ellerton, N.F. and Clarkson, P.C. (1996). Language Factors in Mathematics Teaching and Learning. In Bishop, Clements, Keitel, Kilpatrick and Laborde(Eds.). International Handbook of Mathematics Education. Kluwer Academic Publishers. Pp. 987-1033

Goldenberg, P.E.(1993). On Bulding Curriculum Materials that Foster Problem Posing. In S. Brown & M. Walter (Eds.), Problem Posing: Reflections and Applications, Hillsdale, New-Jersey: Lawrence Erlbaum Associcates. pp. 31-38

Hershkowitz, R., Ben-Haim, D., Holyes, C., Lappan, G., Mitchelmore, M. and Vinner, S. (1990). Psychological Aspects of Learning Geometry. In Nesher, P., and Kilpatrick, J. (Eds.) Mathematics and Cognition, ICMI Studies Series, Cambridge University Press, pp. 70-95

Lavy, I. (1999). Understanding basic concepts in elementary number theory: explorations in an interactive computerized environment. Unpblished doctoral research thesis, Technion.

Leung, S.S. & Silver, E.A. (1997). The Role of Task Format, Mathematics Knowledge and Creative Thinking on the Arithmetic Problem Posing of Prospective Elementary School Teachers. Mathematics Education Research Journal. 9(1), pp. 5-24.

Mason, J. (2000). Asking Mathematical Quations Mathematically, International Journal of Mathematical Education in Science and Technology, 31(1), pp. 97-111.

Moses, B., Bjork E. & Goldenberg, P.E. (1990). Beyond Problem Solving: Problem Posing. In T. J. Cooney & C.R. Hirsch (Eds.), Teaching and Learning Mathematics in the 1990's, National Council of Teaching of Mathematics. pp. 82-91

NCTM - National Council of Teachers of Mathematics (2000), Principles and standards for school mathematics. Reston, VA: NCTM.

Polya, G. (1981). Mathematical discovery: On Understanding, learning, and teaching problem solving. (2 vols. ; combined ed.). New York: John Wiley & Sons.

Schoenfeld, A. (1985). Mathematical problem solving. Orlando, Fla. : Academic Press.

Silver, E.A., Mamona-Downs J., Leung S.S. & Kenney A.P. (1996). Posing Mathematical Problems: An Exploratory Study, Journal for Research in Mathematics Education, 27 (3), pp. 293-309.

Southwell, B.(1998). Problem Solving Through Problem Posing: The Experience of Two Teacher Education Students. Paper presented in MERGA conference.

Tachnai, R. (2001). Problem Posing in Mathematics – pupils' ways of coping with the task. . Unpblished final project research, Technion.