

THE CONTEXT SENSITIVITY OF MATHEMATICAL GENERALIZATIONS

Joseph F. Wagner

University of California, Berkeley

Many theoretical explanations for knowledge transfer or generalization assume that such processes are rooted in the acquisition of abstract rules, principles, or schemata applicable in context-independent ways. This case study is part of a larger research program examining how what often appears to be generalized knowledge or performance is, in fact, supported by systems of context-sensitive knowledge. A microgenetic analysis of an undergraduate student's solution to two mathematically isomorphic probability problems demonstrates how he perceived the two problems as structurally and phenomenologically different despite his ability to apply a single, correct, schematic solution to both. The means by which he finally perceived the problems as structurally "alike" was influenced by both the relevant solution principle and the problem context.

INTRODUCTION

Many theoretical attempts to understand and explain what has been traditionally called the "transfer of knowledge" make a distinction between the underlying structure of a problem or situation and its "real world" context or surface features (see, for example, Anderson, Reder, & Simon, 1996; Gick & Holyoak, 1987; Reed, 1993; Singley & Anderson, 1989). From this perspective, the perception of two problem situations as supporting the same means of solution requires the problem solver to notice sufficiently the "objective structural similarity" of the problems, although the saliency of such similarities may depend on the individual's expertise (Gick & Holyoak, 1987). Thus, while problem contexts have long been understood to influence the types of structure perceived in a situation, successful transfer of an abstract mathematical solution or principle is said to depend on a presumably objective, context-independent structural similarity across problem situations.

The "surface features" of a problem are generally understood to refer to objects or object attributes non-essential to the problem's solution. Structural similarity across two situations depends on their sharing relational properties as well. One of the most widely cited theoretical frameworks for modeling the perception of structural similarity is Gentner's structure-mapping model (Gentner, 1983; Gentner & Markman, 1997), which posits the representation of situations as systems of object nodes, object attributes, and hierarchies of relations among nodes. The perception of a particular hierarchy of nodes and relations in one problem situation permits it to be mapped onto another problem of similar structure whose solution is known; or, perhaps, the problem's structure may be directly associated with an abstract solution method, principle, or schema for problems of such structure (Reed, 1993). Ultimately, abstract rules, principles, or schemata serve to explain how it is that knowledge is transferred and applied across contexts.

While numerous researchers have critiqued the very cognitive foundations of the traditional transfer paradigm from the perspective of situated cognition (see, for example,

Bransford & Schwartz, 1999; Greeno, Moore, & Smith, 1993; Lave, 1988), recent attempts have been made to reconsider the problem of transfer, and the surface/structure distinction in particular, while maintaining a focus on the individual and cognitive dimensions of the phenomenon (Lobato & Siebert, 2002; Wagner, 2002). The case study analyzed in the present research demonstrates that the successful application of abstract rules can depend upon the acquisition of context-sensitive ways of perceiving structure within a problem situation. Most significantly, such structure is not located objectively in the problem situation itself; nor is it determined by the problem situation alone. Rather, it arises through the interaction of the problem solver's understanding of both the problem context and the relevant mathematical principle. From this perspective, the application of abstract rules or principles may depend on acquiring a network of supporting knowledge that is often sensitive to the variety of situations and contexts in which those rules or principles may apply.

THEORETICAL FRAMEWORK

The theoretical perspective of this analysis originates in diSessa's (1993) epistemological framework for the learning of concepts of science and mathematics. From this perspective, the acquisition of such concepts depends on the organization and systematization of prior knowledge that is often highly sensitive to context. Wagner (2002) offered the beginnings of how diSessa's epistemology could be used to address transfer phenomena. In his case-study analysis, descriptive and explanatory statistical ideas initially used by a student only in isolated situations coalesced into associations of knowledge more likely to be cued in a wider variety of contexts in which they had not been used before.

A second aspect of diSessa's work relevant to generalization and transfer is the theory of coordination classes (diSessa & Sherin, 1998). diSessa and Sherin argued that understanding at least some concepts (coordination classes) requires the acquisition of particular *readout* and *coordination* strategies, often dependent on context. In a prototypical example, the ability to perceive a single concept such as a force in different situations might require a student to learn to attend to ("read out") information about different features of the situation and coordinate this information in different ways in different contexts. While students may well be able to learn and state abstract principles or solution schemata, the use of those ideas across contexts depends on the development of increasingly sophisticated networks of context-sensitive knowledge. Such knowledge might include acquired means of perceiving different types of structure by attending to and coordinating different problem features in different situations.

The analysis presented here reveals how Philip, an undergraduate student, demonstrated himself capable of correctly applying normative probabilistic reasoning to two mathematically isomorphic problems while nevertheless *denying* that such reasoning applied in one of the two cases. Despite his ability to recognize and coordinate all the essential features of both problems to obtain a correct solution, the structure he perceived in the two problems differed substantially, and he resisted my extensive efforts to convince him of their normative isomorphism. Using fine-grained methods of microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1993), I analyzed transcripts of Philip's interviews to reveal instances of useful, normative reasoning that he had used in

other situations, but had not yet applied to the problem offering him difficulty. I will argue that I was able to use that analysis to construct a reformulation of the problem that convinced Philip of its normative solution by cueing his good reasoning strategies used in other contexts. Moreover, the means by which Philip learned to perceive new structure in the problem arose not from the problem situation itself, but through his understanding of both the relevant mathematical principle and the particular context. Philip's learned means of perceiving structure is thus shown to apply to *some but not all* situations in which the statistical principle applies. This behavior is not readily explained by those who posit that transfer takes place through the recognition of "objective" situational structure, or that problems supporting a particular abstract principle are understood to be alike because they all support a common structural interpretation.

METHODS

Philip was one of fourteen undergraduate students who participated in this research during the summer of 2001 or 2002. The students were enrolled in an introductory course in statistics at a large, public, university in the United States, and each agreed to meet with me for two hours each week in one-on-one sessions for the duration of their eight-week course. During the first half of each interview, I offered myself as a personal tutor, and the use of our time depended largely on the questions and concerns each student brought to the meeting. During the second half of each interview, students engaged in a variety of activities including think-aloud problem-solving sessions, computer simulations, experiments with simple randomization devices such as dice or spinners, and interviews and discussions about their understanding of probability and statistics. These interviews were audio- and video-taped, and salient portions of them were later transcribed for fine-grained examination. All work done during these interviews was directly related to the material the students were studying in their course, so most subjects indicated that they found their participation in this research both useful and motivational.

Among the problems given to the students to examine were collections of problems deemed mathematically similar or isomorphic because their solutions involved the same mathematical principle. The problems examined in the present analysis were presented to students after the mathematical principles relevant to their solution had been covered in their course. When students offered different or non-normative solutions to problems that I perceived to be similar, I asked the students to compare them and explain how they saw them as similar or different. I engaged in a deliberately instructive role only if students found such comparisons unhelpful in assisting their normative reasoning. The data permitted detailed analyses of how students succeeded or failed in offering normative solutions to problems deemed mathematically "alike," and how they came to identify problems as similar or isomorphic after failing to do so in their initial solutions to them.

THE CASE OF PHILIP

For reasons that will become clear as this section unfolds, the case of Philip will be told as a chronological narrative. The interviews relevant to this analysis took place during the final two weeks of Philip's eight-week course. Philip appeared to be highly engaged in all of our activities, and he showed himself to be successful in his coursework by earning an A in the course from his instructor (not the researcher). During the seventh of our interviews, I asked Philip to solve the following problem, which was accompanied by a

picture of a circular spinner divided into ten sectors of equal size, seven of which were colored blue and three of which were colored green:

Suppose someone spins the spinner at the right ten times in a row. Of the following possible outcomes, which is the most likely to occur?

- a) The spinner will land on blue five times and on green five times.
- b) The spinner will land on blue seven times and on green three times.
- c) The spinner will land on blue all ten times.
- d) All of the above are equally likely.
- e) It is not possible to answer this question.

A normative solution to this problem would require an understanding of a principle that might be stated as follows:

The most likely (expected) result of a series of n observations of a binomial random variable with probability p of success on each observation is np successes.

For convenience, this principle will be referred to in this paper as the *binomial principle*. Neither Philip nor his classmates would have been expected to learn this principle in so specific a form. It is an idea that they would have been more or less expected to deduce from their study of the expected value of a random variable. It has, however, an intuitive appeal, since it predicts that the most likely outcome to occur from a series of ten spins of a spinner is the one that most closely resembles the proportion of blue and green sectors on the spinner, namely, seven blues and three greens.

Philip took very little time before offering a normative solution to the Spinner Problem:

Philip: Well, it's still a small group of spins, so you could get all blue or about 50-50 green and blue, um, more often than if you were to spin it a hundred times. Because, obviously, I guess it would get closer to a 70-30 split. But, um, it's only ten spins, but I still think it would be close to about a 7-3 split, just because, it's kind of like a problem where, if you have an average of a box, and they say after 400 draws, what do you expect the average to be? You expect it to be right around the average of the box. And even for five draws I would expect it to be around the same. So, that's why I, I picked B. I mean, it's most likely it would be seven and three.

Not only was Philip's answer correct, his explanation was virtually picture-perfect as he correctly noted the role of sampling variability in samples of such a small size. He nevertheless recognized that the most likely outcome was a seven-to-three ratio, regardless of the size of the sample. In referring to "the average of the box," Philip made use of his classroom experience of using box models to simulate random draws. The average of the box was the expected value of the draws (in this case, 70% blue).

Immediately after solving the Spinner Problem, I presented Philip with the Box Problem. In this problem, he was asked to imagine a box containing 500 tickets, 350 labeled "B" for blue and 150 labeled "G" for green. If ten tickets were drawn at random from the box, what would be the most likely outcome? The five multiple-choice solutions offered to Philip corresponded precisely to those accompanying the Spinner Problem. From a mathematical perspective, the two problems are isomorphic, both asking for the most likely outcome of a series of ten draws of a binomial random variable with probability (of

drawing blue) 0.7. Only a “surface feature,” the means of making the random selection, has been changed. Philip, however, perceived something very different:

Philip: If I were to keep the same logic, I'd say seven are B and three are G, and that would be the most likely. But the, the difference in this is there are so many, now, that are blue. There are a lot. I mean, in this case [indicates the Spinner Problem] there's only four more that are blue. In this case [indicates the Box Problem] there's, like, two hundred more that are blue, and, um, I'm pulling ten. OK, I'm gonna, I'm actually gonna s-, I'm gonna say C, that all ten of the slips say B, is, is most likely....

Philip's response is particularly striking because he demonstrated that he both remembered his reasoning from the Spinner Problem and could apply it to the Box Problem, but he nevertheless denied that such reasoning was appropriate. Thus, it cannot be argued that Philip simply did not recall the correct rules or that he did not learn them at a sufficiently abstract level to apply them. In extensive follow-up discussion, Philip maintained that draws from the box differed from spins of the spinner because the large number of blue tickets “overwhelmed” the green ones in a way that the few extra blue sectors of the spinner did not. In short, the structure Philip perceived in the problems that permitted him to recall and apply the binomial principle was not the same as the structure that would permit him to perceive both problems as appropriate instances of the principle.

Following Philip's initial solution to the two problems, he and I engaged in a 23-minute conversation during which I tried to no avail to convince Philip of the two problems' normative isomorphism. A detailed analysis of that conversation is beyond the scope of this paper, so I summarize here only particularly salient highlights. During our conversation, Philip repeatedly acknowledged the 70% proportion of blue in both problems, but found that correspondence insignificant compared to the difference in absolute numbers. While Philip's “problem” might be seen simply as a failure to realize that only the percentage of blues and not their absolute number was relevant, it became clear that for me as an instructor merely to *tell* him this was insufficient. Philip spontaneously acknowledged that the “mathematical” answer was “probably” the same for both problems, but he did not believe that such a mathematical solution had anything to do with what would really happen in the two situations. Even had he accepted the normative answer on my authority, it would not have changed his perception of the situations—the structure he saw—that led him to consider the problems differently.

Most of my attempts to convince Philip of the isomorphic nature of the two problems involved presenting him with reformulations of the spinner. At first, for example, I took the ten-sectored spinner and began sketching in extra lines as though to increase the number of sectors to 500, corresponding to the number of tickets in the box. Philip denied the relevance of this move, however, pointing out that by increasing the number sectors I did not increase the overall *amount* of blue present on the spinner. This stood in contrast to increasing a set of 10 tickets to 500 tickets, which would introduce an “overwhelming” amount of blue to the box not initially present. I countered by asking Philip to imagine my taking the 500 tickets and laying them on the spinner, as though to reconstruct the face of the 10-sectored spinner with the 500 tickets. Philip again denied the correspondence, noting that each sector of the spinner should correspond only to one ticket in the box. Thinking I finally had him, I suggested that we take all 500 tickets and

lay them around the circumference of the spinner, with blue and green tickets lined up along the edge of their corresponding sectors. Then, I said, the needle would point to only one ticket at a time. Philip acknowledged that this “put a hole” in his argument, but he then surprised me by suggesting a reformulation of his own. In my suggested reconsiderations of the spinner, I never changed the placement of the original ten blue and green sectors, so even after subdividing them into a larger number of sectors, the actual distribution of color on the spinner’s face remained unchanged. Philip asserted that if the spinner were subdivided into 500 sectors and the blue and green sectors were randomly scattered on the spinner’s face, then the spinner and the box would match and the most likely outcome of ten spins of the spinner would be ten blues! He stated that the large number of blue sectors would overwhelm the intermingled green ones, and the needle would have a more difficult time picking out the green from among the blue.

In response to my questioning, Philip suggested that it would take a demonstration of draws from a box to convince him of the normative solution. Not only did I not have the requisite supplies to carry out such a demonstration, I was also skeptical of the wisdom of doing so. What if we had drawn ten blues on the first try? Even if a demonstration had supported my argument, would it have offered Philip any new understanding of the situation to convince him that the normative solution *should* be the correct one? After 23 minutes of debate, Philip and I agreed to take up the issue again during our next meeting.

In the five days before our next meeting, I closely examined Philip’s reasoning about the Spinner and Box Problems, hoping to devise a means of convincing him of the problems’ isomorphism. I looked for reasoning strategies Philip used successfully in contexts other than the Box Problem, and I noted two in particular. Philip twice defended his correct reasoning about the ten-sectored spinner by noting a one-to-one correspondence between the number of sectors on the spinner and the size of the sample: “There’s only ten slots. And seven are blue and three are green. And you’re only spinning it ten times, so you have to get one thing on, on each spin.” Also, Philip reasoned normatively about spinner and box situations in which the number of blues and greens were equally likely. So, for example, he acknowledged the likelihood of five blues and five greens in ten spins of a spinner colored half blue and half green, as well as in ten draws from a box containing 250 blue and 250 green tickets. I predicted that a reformulation of the Box Problem designed to cue these reasoning strategies would enable Philip to reason normatively about draws from the box.

When we next met, Philip indicated that he remained firm in his non-normative expectations about draws from a box. In reply, I asked him to imagine a box of balls numbered 1 through 10, and a separate box of 500 balls, with 50 balls each numbered 1 through 10. Philip took a short time to acknowledge that there should be no discernable difference in drawing (with replacement) from either box, emphasizing his awareness that there were “the same number” of each class of ball in each box, and thus drawing on his good reasoning about equally likely outcomes. Finally, I asked him to imagine that the balls numbered 1 through 7 were colored blue and those numbered 8 through 10 were colored green. Philip immediately responded, “Yeah, that would convince me, then.” After some probing on my part, Philip explained his change of reasoning:

Interviewer: You're, you're not just going along with me? [laughs]

Philip: No, it's a good way, it's a good way of looking at it. Because, I, I think about it with just the numbers on it, one through ten, and I'm thinking, well, now there's so many different choices in there, it's not just blue and green. It's, uh, it's any numbers. But, uh, it, but then when you throw the catch in, on the back, one through seven were blue, then it, it makes perfect sense, then. Because you're gonna pull some eights, and you're gonna pull some ones, and, the numbers between, you're gonna get seventy percent of the numbers one through seven. Yeah. And then, you throw the clause in that those are colored blue, it sheds light on it. [laughs] Yeah.

Philip's account that "it's a good way of looking at it" is precisely that: a learned way of looking at the situation and perceiving a new kind of structure in the problem. Whereas Philip had earlier attended only to the absolute number of blue, the numbering scheme encouraged him to attend to equal classes of balls, and his earlier reasoning about one-to-one correspondences between the spinner sectors and the sample now informed his reasoning about the box as he saw that "you're gonna get seventy percent of the numbers one through seven," thereby highlighting the proportional nature of the situation.

DISCUSSION

The means by which Philip arrived at a normative understanding of the Box Problem is of theoretical importance. First, the structure he came to perceive in the situation was by no means suggested merely by the situation itself. Rather, it was mutually influenced by the nature of the binomial principle and the problem context that permitted Philip to imagine the population under an imposed classification system. Thus, learning this "way of looking at it" is inherently tied to one's understanding of the binomial principle and should not be imagined as a structure that one *first* perceives in a situation that *then* reminds one of an appropriate rule or solution. Second, while Philip learned a powerful way of perceiving some situations, the means by which he perceived the normative solution to the Box Problem is not useful in all instances of the binomial principle. More specifically, Philip learned a way of perceiving structure in drawing from an available population, and such structure is not readily perceived in instances of the binomial principle for which no such population is available. Finally, the structure Philip learned to perceive in the Box Problem enabled him to reason directly that a draw of seven blues and three greens was the most likely outcome of ten draws; he did not need to "apply" an abstract rule or principle. While it is quite likely that students learn in the long run to apply such reasoning schematically (as Philip, in fact, initially did), this analysis suggests that the similarity of problems governed by a common mathematical principle lies not in a single structure shared by all, but in learned, context-sensitive ways of structuring them that enable problem solvers to perceive them as instances of that principle.

References

- Anderson, J. R., Reder, L. M., & Simon, H. A. (1996). Situated learning and education. *Educational Researcher*, 25(4), 5-11.

- Bransford, J. D., & Schwartz, D. L. (1999). Rethinking transfer: A simple proposal with multiple implications. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of Research in Education* (Vol. 24, pp. 61-100). Washington, DC: American Educational Research Association.
- diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition & Instruction*, 10(2-3), 105-225.
- diSessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change? *International Journal of Science Education*, 20(10), 1155-1191.
- Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science*, 7, 155-170.
- Gentner, D., & Markman, A. B. (1997). Structure mapping in analogy and similarity. *American Psychologist*, 52, 45-56.
- Gick, M. L., & Holyoak, K. J. (1987). The cognitive basis of knowledge transfer. In S. M. Cormier & J. D. Hagman (Eds.), *Transfer of learning: Contemporary research and applications* (pp. 9-46). San Diego, CA: Academic Press, Inc.
- Greeno, J. G., Moore, J. L., & Smith, D. R. (1993). Transfer of situated learning. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 99-167). Norwood, NJ: Ablex.
- Lave, J. (1988). *Cognition in practice*. New York: Cambridge University Press.
- Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior*, 118, 1-30.
- Reed, S. K. (1993). A schema-based theory of transfer. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 39-67). Norwood, NJ: Ablex.
- Schoenfeld, A. H., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. IV, pp. 55-175). Hillsdale, NJ: Erlbaum.
- Singley, M. K., & Anderson, J. R. (1989). *The transfer of cognitive skill*. Cambridge, MA: Harvard University Press.
- Wagner, J. F. (2002). Constructing generalizations: An analysis of one student's progress toward a generalized understanding of the law of large numbers. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 1335-1338). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.