TRANSLATION OF A FUNCTION: COPING WITH PERCEIVED INCONSISTENCY

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A horizontal translation of a function is the focus of this study. We examine the explanations provided by secondary school students and secondary school teachers to a translation of a function, focusing on the example of parabola $y=(x-3)^2$ *and its relationship with* $y=x^2$. *The participants' explanations focused on attending to patterns, location of zero of the function and pointwise calculation of the function values. The results confirm that the horizontal shift of the parabola is, at least initially, inconsistent with expectations and counterintuitive to most participants. We articulate possible sources of this perceived inconsistency and describe a pedagogical approach aimed at resolving it.*

Imagine the graphs of (a) $y=x^2$ and (b) $y=(x-3)^2$. Check your image with that of a graphing device. If you're not surprised, it's probably because you've already explored the relationship between the two parabolas in the past. Most people conjecture that graph (b) will appear 3 units to the left of graph (a). The surprising result is that (b) is actually 3 units to the right of (a).

As students, we have memorized this result as one of the counterintuitive facts of mathematics. As teachers, we have attempted to explain this phenomenon to our students, often creating a conflict between their intuition and our explanations. As researchers, we became interested in how teachers and students deal with the phenomenon and what explanations they provide to themselves and to others. Moreover, we wished to understand possible sources of students' "wrong" intuition and sought pedagogical solutions.

BACKGROUND

Teaching, learning, or understanding of functions has been an important focus in mathematics education research in the past decades. However, within a variety of research reports focusing on functions, only minor attention has been given to the transformations of functions. A common treatment of transformations of functions in pre-calculus courses involves a consideration of a graph of a function $f(x)$ on a Cartesian plane. Functions $f(ax)$, $f(-x)$, $f(x)+k$ and $f(x+k)$ correspond to a dilation, reflection, vertical translation and horizontal translation of $f(x)$ respectively.

Eisenberg and Dreifus (1994) conducted an extensive exploration on students' understanding of function transformations, focusing on visualization. They recognized the difficulty in visualizing a horizontal translation in comparison to a vertical one, suggesting that "there is much more involved in visually processing the transformation of f to $f(x+k)$ than in processing the transformation of f to $f(x)+k$ " (p.58). Baker, Hemenway and Trigueros (2000) have investigated understanding of transformations of functions and confirmed that vertical translations appear easier for the students than horizontal translations. Further, students' difficulty with function transformation was attributed, at least in part, to their incomplete understanding of the concept of function.

Though the difficulty presented by a horizontal transformation of a function has been acknowledged in prior research, there has been no attempt to investigate how students and teachers cope with this difficulty and how it is possible to overcome or at least to reduce this difficulty; these are our goals. We examine the understanding of horizontal translation of functions in general, and of parabolas in particular. We explore how participants cope with the difficulty presented by a horizontal translation and suggest a pedagogical approach to transformations of functions that addresses the "mystery" of graphs "moving in unexpected directions".

METHODOLOGY

Participants in this study were preservice secondary teachers (n=15), practicing secondary teachers (n=16) and grade 11/12 students (n=10). The teachers were volunteers attending courses at the University in which this study was conducted. The students were volunteers referred by the participating teachers, all classified as having "above-average" ability. All the participants were asked to predict, check and explain the relationship between the graph of $y = x^2$ and the graph of $y = (x - 3)^2$. In addition, if the issue did not come up naturally in the conversation, the participants were asked to discuss the graph of $y = x^2 - 3$ and compare it to $y = (x - 3)^2$. The analysis of responses attended to (1) common trends in explanations, and (2) attitudes towards perceived inconsistency.

RESULTS

The fact that the shape of $y = (x - 3)^2$ is a parabola that is congruent to the canonical parabola y=x2 was taken for granted by teachers and students alike. The fact that the graph of $y = (x - 3)^2$ is a horizontal translation of $y = x^2$ was also mentioned as requiring no further explanation. Therefore, we turn our focus now to the direction of this horizontal translation.

1. Students' responses

The participating students had learned about transformations of quadratics (conic sections) before this study was conducted. Nevertheless, half of the students predicted the spatial location of $y = (x - 3)^2$ incorrectly. There was no sincere attempt on students' part to explain the phenomenon, though all admitted that the observed location of the parabola was counterintuitive. The following conversation with Mitch is rather representative of students' responses. The interview took place after Mitch incorrectly sketched the graph of $y = (x - 3)^2$.

Int: Would you please check this.

Mitch: No problem. (pause) Yes problem. I should have remembered this. It moves in the wrong direction.

Int: What do you mean, wrong direction?

Mitch: It's minus 3, so you sort of expect it to move left. But it moves the other way. It moves right.

Int: Oh, and why is that?

Mitch: That's the way it is. It's always like that. I really knew that. Just remember to do the opposite. But not always. ... Only if your number is in the brackets.

Int: Please explain.

Mitch: OK. When you do something to x in the brackets, like x–3 or x+2 your graph moves the opposite way. For x+2 it moves left, 2 times. For x–3, the one you wanted, it moves right, 3 times. And once you remember that, it works for all the graphs.

Int: You mentioned something about brackets.

Mitch: Yes, it moves the opposite way only when there are brackets. Without brackets, say for $y = x^2 + 3$, written this way, it moves the way you want it to move. This one will move up, and with negative-3 it will move down. This one is doing exactly what it should.

Int: So the other one is doing something it shouldn't.

Mitch Not really, just not what one would guess, but the other way around.

Int: Does it bother you?

Mitch: Not really. You just see on the calculator what it's doing and you know it will do it all the time.

Int: I wonder why the graph [of $y = (x - 3)^2$] moves the way it moves. It's not what I expected, it's not what you expected, at least initially. Can you help me understand this?

Mitch: (pause). Maybe try small steps. Try $y = (x - 1)^2$. You see, it goes right one step. And try $y = (x - 2)^2$. It will go right 2 steps, you see them together here [demonstrates with graphing calculator]. So after seeing this you do not expect (x–3) to do something different, do you?

Int: Is this how your teacher explained it to you?

Mitch: I'm not sure he explained this at all. But graph it once and you know how it works.

In examining Mitch's response we note the initial confusion, which is rapidly corrected based on the feedback from the graphing calculator. The claim "I should have remembered" demonstrates that the phenomenon is not new to Mitch, he has encountered this behavior of graphs earlier and expects to base his knowledge on this recollection. The claim "It moves in the wrong direction" and further clarification on what is meant by "wrong direction" presents a conflict between the result and the expectations; however, these expectations have been adjusted based on experience. Further, Mitch generalizes his experience of observing translation of the parabola ("once you remember that, it works for all the graphs") and also limits the scope of this observation ("only if your number is in the brackets"). Mitch explains that while the vertical translation matches his expectations, as "it moves the way you want it to move," the case of the horizontal translation in his approach is "just remember to do the opposite". However, Mitch is not troubled with the apparently acknowledged counterintuitive behavior of $y = (x - 3)^2$. His main concern appears to be with remembering this behavior, rather than with understanding it. Responding to the interviewer's quest to understand the location of the function, Mitch suggests the consideration of a pattern. Observing graphs of $y = (x - 1)^2$ and $y = (x - 2)^2$ implies the location of the graph of $y = (x - 3)^2$. This explanation is consistent with his strong belief in internal consistency of this pattern – "graph it once and you know how it works".

Remembering "to do the opposite" was a common strategy for coping with perceived inconsistency. There was no desire on the part of student-interviewees to understand or explain the phenomenon. When an explanation was requested, it relied on rules to be memorized or on generalizations from attending to patterns.

2. Teachers' responses

All the teachers participating in this study have sketched the graph of $y = (x - 3)^2$ correctly. However, for the practicing teachers it was an immediate and effortless recall from memory, the way one would recall, rather than derive, a multiplication fact 7x9. It was evident that for some preservice teachers the horizontal translation of a parabola was not in their immediate repertoire of knowledge, but the location of the graph was derived correctly and without major effort.

Compared to uniformity in students' tendency to rely on memorized rules, there was more variety in teachers' responses to the interviewer's request to explain the move of the parabola. We summarize below several themes that emerged in teacher's responses.

Citing rules. About one half of the teachers interviewed referred to the "rule of horizontal translation". According to this "rule", $y = (x - 3)^2$ has the same shape as $y = x^2$ but is located 3 units to the right. Thus, teacher's reliance on rules is indicative of students' reliance on memorizing these rules.

Pointwise approach. Plugging numbers into the equation and then plotting the points seemed more convincing for teachers than simply accepting what the computer or graphing calculator was showing. Some teachers explained that they saw advantage in using the point-by-point creation of $y = (x - 3)^2$ as an explanatory tool for their students. Interestingly though, a pointwise approach was not mentioned by any of the students. It appears that the utility and availability of graphing calculators and the lack of extensive experience with creating graphs manually, point by point, influences students' perception of graphs and suggests that the convincing power of teachers' explanations needs to be reexamined. Moreover, the influence of graphing calculators and graphing software on participants' explanations deserves the attention of further research.

Attending to zero and "making up". Another common explanation was to find the zero (x=3) of the parabola and imply that the rest of the points are "symmetrically determined" around it. Those teachers were prompted to explain in what way the location of zero would determine the location of the rest of the points. In most cases preservation of shape and symmetry were put forward as justifications. However, several teachers presented variations on the following explanation: "to get to a particular y-value, the x-value must be 3 greater because 3 is being subtracted". This was usually an extension of attending to the zero of the function and an attempt to explain "what happens to other points on a graph is the same as what happens to zero", without an explicit calculation. It was noted that explanations similar to these can help in justifying what happens after the result of translation is presented, rather than help in predicting the result.

A search for consistency. It has been acknowledged that the direction of translation of parabola is inconsistent with intuitive expectations. This creates a dissonance for both students and teachers. This dissonance is further aggravated by the fact that the vertical translation operates "as expected", that is, the graph of the function + 3 is a vertical upward translation by 3 units of the graph of $y = x^2$. Therefore, the learners face not only a counterintuitive behavior of functions, but also inconsistency in the fact that some translations act according to their expectations while others do not.

In summary, teachers provided a variety of explanations, most common of which were citing the rules, considering the function pointwise and attending to the zero of the function. Most teachers were not completely satisfied with their explanations, but claimed to "have never seen a better one". Preservice teacher's responses differed from responses of practicing teachers on two accounts. The first is the ease of recall, acknowledged early in this section. Automatic and fluent retrieval is considered to be one of the indicators of expert knowledge (Bransford, Brown & Cocking, 2000), and therefore it is not surprising that practicing teachers exhibited better expertise in the subject matter they taught than preservice teachers. Second, as expected by assuming expertise not only in content but also in a pedagogical content knowledge, practicing teachers, had a better understanding of the perceived inconsistency and of the problematics that a horizontal translation presents to a learner. However, there was no significant difference between the attempts of practicing teachers to explain the translation and the attempts of preservice teachers.

ANALYSIS: COGNITIVE AND PEDAGOGICAL OBSTACLES

The results indicate that the ability to determine correctly the direction of horizontal translation does not imply understanding of this function transformation and the ability to justify it. From the perspective of a learner, the behavior of functions is counterintuitive. From the perspective of an observer, the learners' intuition is misleading and it presents an obstacle to their understanding. In order to understand the sources of this intuition we turn our attention first to the notion of an obstacle, cognitive and epistemological.

The notion of an epistemological obstacle was introduced by the French philosopher and scientist Gaston Bachelard. Bachelard described the development of scientific knowledge as constrained by intrinsic, and at times unrecognized, factors associated with the process of understanding, rather than by the complexities of the phenomena. Herscovics (1989) reserved the term "epistemological obstacles" for obstacles that are encountered in the development of knowledge in a discipline, while denoting the obstacles in the conceptual development of an individual learner as "cognitive obstacles". Sierpinska (1994) considered a broader interpretation of epistemological obstacles and described them as obstacles to some change in the frame of mind, that can be attributed to the historical development of knowledge in a discipline, as well as to the knowledge development of an individual learner. One of the constraints identified by Bachelard as a potential epistemological obstacle is possibly deceptive intuitions (Norman & Prichard, 1994).

Turning to translation of functions, learners' experiences with numbers suggest that adding 3 results in moving in the positive direction (right), while subtracting 3 (or adding negative 3) results in moving in the negative direction (left). On a surface, such a view is an overgeneralization of prior experience of adding and subtracting numbers on the number line. Examining the issue closer, we believe that a possible source of an obstacle is in trying to see (b) $y = (x - 3)^2$ as a transformation of (a) $y = x^2$, and in doing so creating confusion between the function defining parabola and the function of transformation. Students' focus is on the algebraic representation of functions. Comparing the algebraic representation $y = x^2$ and $y = (x - 3)^2$, students notice, correctly, that (b) is derived from (a) by substituting (x–3) in place of x. As mentioned earlier this interpretation is extended, mostly intuitively, to the view of the number line: for any number x, x–3 is located 3 units to the left of x. We suggest that the main source of difficulty here is in seeing this algebraic replacement as a transformation (x moves to x–3) and trying to imply the geometric transformation, the movement of the graph, from the algebraic substitution. That is to say that the transformation $f(x) \mapsto f(x\text{-}3)$ is simplified to be viewed as $x \mapsto (x\text{-}3)$. Such a view is in accordance with what Hazzan (1996) described as reducing abstraction, which is a strategy used by learners to cope with complexity. In our example students' attention is on an object of a variable (x) rather than on a more abstract object of a function f(x).

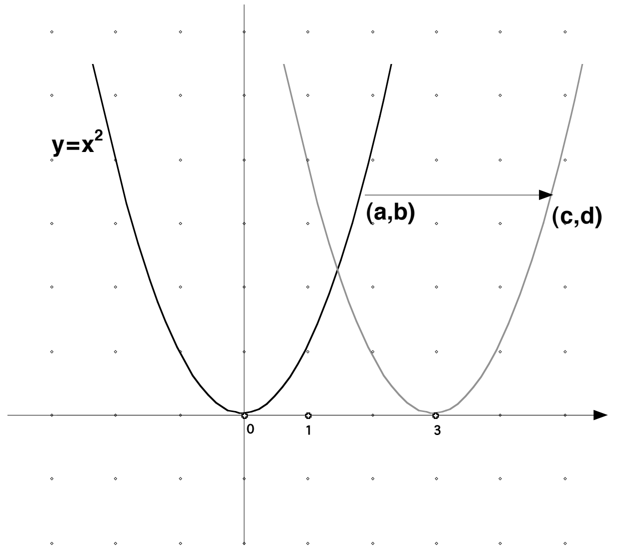
Though the studies of Eisenberg and Dreifus (1994) and Baker et. al. (2000) agree on the difficulty presented by horizontal translation, they differ in a way of explaining this difficulty. Eisenberg and Dreifus see the difficulty in visual processing of information. Baker et. al. attribute the difficulty to the complexity of mental construction needed to process a horizontal transformation. We described above how students' difficulty can be attributed to a cognitive-epistemological obstacle and to human tendency to reduce the abstraction level when facing complexity.

However, there is another possible source of the difficulty, and it is in the instructional sequence in which the discussion of transformation of functions takes place. Students' and teachers' focus on algebraic representation is not their fault and may not be their choice. This choice is prescribed by a traditional curricular approach to function transformations. This approach embeds the discussion of transformations of parabolas, quadratic relations, or polynomial functions in general in the context of functions, rather than in the context of transformations. This standard pedagogical approach could be reinforcing the obstacle, rather than trying to overcome it. Our belief is that the problem can be avoided, or at least minimized, by a pedagogical approach that situates transformations of functions in the context of transformations. We outline this approach below.

PEDAGOGICAL APPROACH

In situating transformations of function in the context of transformations, rather than functions, we focus on translations. Translation on a plane is defined by a vector (or directed segment) that specifies the direction of the motion and the distance. It is our experience that students tend to identify translations by breaking this motion into its horizontal and vertical components. This view leads to the natural introduction of the formal notation for translation $(x, y) \mapsto (x+a,\ y+b)$ or $T((x,y)) = (x+a,\ y+b)$, where a and b are horizontal and vertical components of the motion respectively. This function notation for a transformation is often referred to as a mapping rule.

Connecting mapping rules and resulting translations, students will identify positive and negative values of 'a' with motion to the right or to the left respectively; positive and negative values of 'b' with motion up and down respectively. Furthermore, they will associate the strict horizontal motion with b=0 and strict vertical motion with a=0. Since any set of points can be translated according to the mapping rule, this set of points can be a parabola. In particular, applying $T((x,y)) = (x+3,\ y)$ on a canonical parabola $y = x^2$ represents a horizontal translation by 3 units to the right.



$T((x,y)) = (x+3,\ y)$ applied to graph of $y = x^2$

The task now becomes connecting the translated image to its algebraic representation. Recall that the set of points of the source is described by $y = x^2$. Without loss of generality, focus on a point (a,b) of the source set that was translated to the point (c, d) of the image set. According to the specific translation performed, $d=b$ and $c=a+3$. We wish to connect c and d in an equation. Relating c to d we obtain the following: $d=b;\ c=a+3$ which implies $a=c-3$. However, $b = a^2$, as (a,b) is a point on the source parabola. Substitution leads to $d = (c - 3)^2$. Since the above is true for every point of the image set, the image of the translation is described by the equation $y = (x - 3)^2$. (Of course, one can work directly with x's and y's, but switching to a's b's and c's can be helpful for students). This explains the "unexpected" appearance of "–3" in the horizontal translation to the right.

Consideration of a geometric transformation as a starting point is not limited to translations. We envision an approach in which students explore a variety of transformations, making a connection between the conventional definition of a transformation and its effect on specific sets of points on the plane. Such sets could be simple geometric shapes in the initial stages of exploration and graphs of specific functions or relationships in later stages.

The goal of our pedagogical approach is not to change students' minds about the direction of horizontal translation. This is done successfully with the use of graphing devices, either electronic or manual. Our goal is to balance the dissonance presented by the discord between initial intuition and graphical image.

Future research will determine whether the intervention or instructional sequence presented above is helpful for students' understanding of the translation of parabola in specific and of any function in general. So far, we explored such intervention on a smaller scale, presenting it to teachers as an alternative pedagogical approach. The participating teachers referred to this view of transformations as "an eye opening clarification" or a "pedagogical AHA!".

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