GOAL SKETCHES IN FRACTION LEARNING

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To examine how conceptual knowledge about fraction magnitudes changes as students' learning progresses, 5th and 7th-grade students were asked to solve fraction magnitude problems that entailed finding a fraction between two given fractions and then to evaluate solutions for similar problems that were modeled for them. When the given fractions share a common denominator or numerator, a simple strategy is to keep the common value and choose an intermediate value for the other component. 5th graders used this strategy on both common-numerator and common-denominator problems, and judged it "very smart" when it was modeled. 7th graders typically converted common-numerator fractions to a common denominator and often judged the strategy of picking an intermediate denominator "not smart."

INTRODUCTION

While many U.S. students, and often even their teachers (c.f., Ma, 1999), think of mathematics learning as primarily a matter of learning computational procedures, educational psychologists and mathematics educators agree that mathematics learning needs to be conceptually grounded (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997; Kilpatrick, Swafford, & Findell, 2001). Indeed, contemporary research in cognitive science indicates that the growth of mathematical understanding and the growth of computational skill are mutually facilitative (Rittle-Johnson, Siegler, & Alibali, 2001; Kilpatrick et al., 2001). In particular, conceptual knowledge appears to play a fundamental role in the development of problem-solving strategies (Siegler, 1996; Siegler & Crowley, 1994). For example, there is evidence that students can make sound judgments about the merits of a new strategy even before they themselves have begun to use it. When Siegler and Crowley asked students to rate a variety of strategies that were modeled for them, the students rated as "very smart" strategies that were more advanced than the ones they themselves used. Siegler (1996) postulated that knowledge structures called goal sketches, “specif[y] the hierarchy of objectives that a satisfactory strategy in the domain must meet" and thus provide a conceptual basis for evaluating possible strategies and for generating new ones (p. 194).

On this account, students’ judgments about alternative problem-solving strategies are a potentially rich source of information about their conceptual knowledge of the problem domain for which the strategies are proposed. The research to be reported here applied this reasoning to the examination of students’ developing knowledge about fraction magnitudes. This domain was selected for study because there is abundant evidence that students have difficulty understanding the magnitude relations between fractions with different denominators (e.g., Peck, Jencks, & Chatterley, 1980). Thus, the objective of the present research was to examine student’s goal sketches about fraction magnitudes, particularly in relation to fractions with different denominators, and to examine how goal sketches change with age and years of schooling.

To separate conceptual knowledge from learned computational rules, problems on which students were not likely to have been directly instructed were used. On each problem, two fractions were presented and the task asked of the child was to generate a third fraction that was in between the two in its value. After completing all the problems, the student was asked to make judgments about different solution methods modeled by the experimenter on a similar problem set.

We focus here specifically on problems in which one of the two components of the fractions—numerator or denominator—was the same across the two fractions presented (e.g. 3/13 & 4/13, a common-denominator problem, or 1/4 vs. 1/5. a common-numerator problem). These are of particular interest because it is not necessary to do precise calculations in order to identify with certainty a fraction

that is of intermediate magnitude. A straightforward solution strategy is to preserve the common numerator or denominator while generating an intermediate value for the component that differs across the two given fractions (e.g. 2/9, for a fraction between 1/9 & 4/9). This type of strategy (which shall henceforth be termed the "intermediate-component strategy") is interesting, particularly in relation to common-numerator problems, because it uses an understanding of the problem constraints to avoid the need for the relatively complex computational procedure of finding a common denominator for the fractions.

METHOD

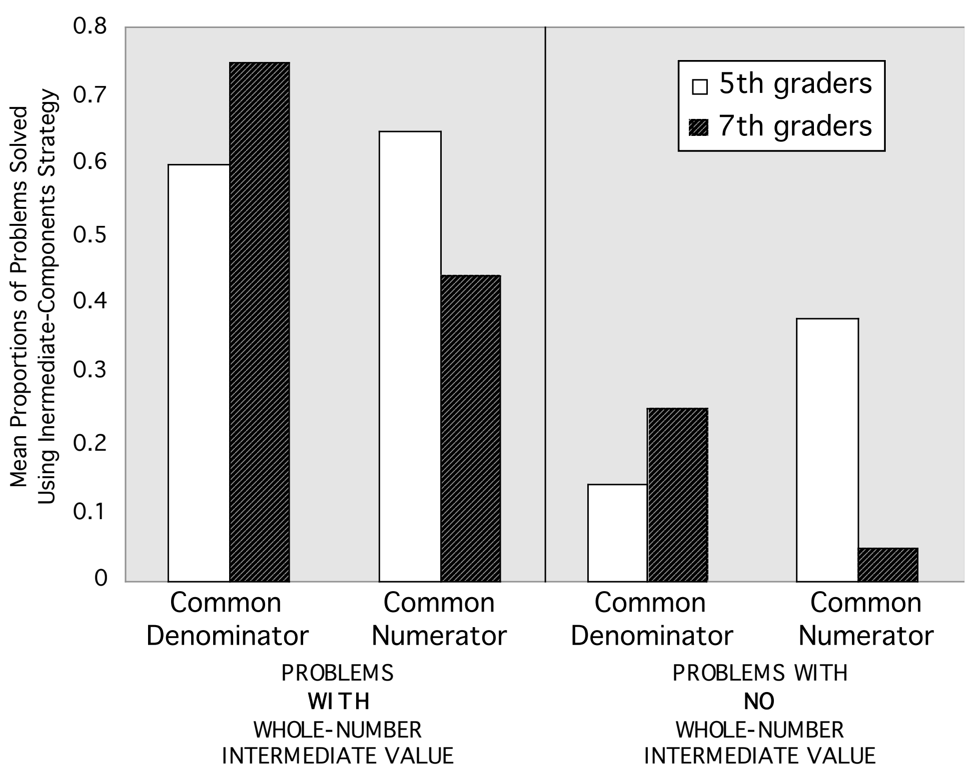
Twenty-four 5^{th}-grade children (11 female, 13 male) and 10 7th-grade children (7 female, 3 male) have participated in the study to date (the samples will be equal in size when data collection is completed). A female experimenter tested students individually in a single session lasting about 20 minutes. First, students received 12 problems on which they were asked to find a fraction whose value was in between the two fractions they were given. Four of these were common-denominator problems and four were common-numerator problems; the rest involved pairs of fractions that differed in both their numerators and their denominators. Within the common-denominator and common-numerator problems, half the problems involved numerators or denominators (whichever differed) that differed by 3 (e.g., 2/9 vs. 5/9), while the other problems involved numerators or denominators that differed by only 1 (e.g., 1/4 vs. 1/5). This distinction is significant for the intermediate-component strategy because, in the latter case, there is no whole-number intermediate value. It is therefore necessary, if an intermediate-component strategy is to be used, either to use a fractional value for numerator or denominator or to convert the fractions to equivalents that contain larger (and more widely-spaced) numbers, e.g., by multiplying numerator and denominator by 2.

In the second part of the experiment, the experimenter modeled solutions for 10 problems: three common-denominator problems, three common-numerator problems, and four problems for which the given fractions differed in both their numerators and their denominators. On the common-denominator and common-numerator problems, three strategies were modeled by the experimenter: (1) an intermediate-component strategy resulting in a whole-number value (when the component that was not in common differed by more than one; e.g., e.g. 2/7 was found for a fraction between 2/6 vs. 2/9), (2) an intermediate-component strategy resulting in a fractional value (for problems on which the component that was not in common differed by only one; e.g. 1 over 4-1/2 was found for a fraction between 1/4 vs. 1/5), and, (3) an intermediate-component strategy involving conversion to equivalent fractions (e.g. 2/9 was found for a fraction between 1/4 vs. 1/5). Each of these strategies was modeled once on a common-denominator problem and once on a common-numerator problem. After each problem the child was asked to judge whether or not the experimenter's answer was correct and also to rate it as "very smart," "kind of smart," or "not smart."

RESULTS & DISCUSSION

Figure 1 summarizes students' use of the intermediate-component strategy in their own problem solving. There was a sharp drop in use of the intermediate-component strategy between the problems for which a whole-number intermediate value was available (plotted in the left panel of the figure) and those for which only a fractional intermediate value could be generated without converting to equivalent fractions (plotted in the right panel). Clearly, the availability of a whole-number intermediate value made it much more likely that students would adopt the intermediate-component strategy. Additionally, usage of the intermediate-component strategy changed with grade level, as can be seen

Figure 1

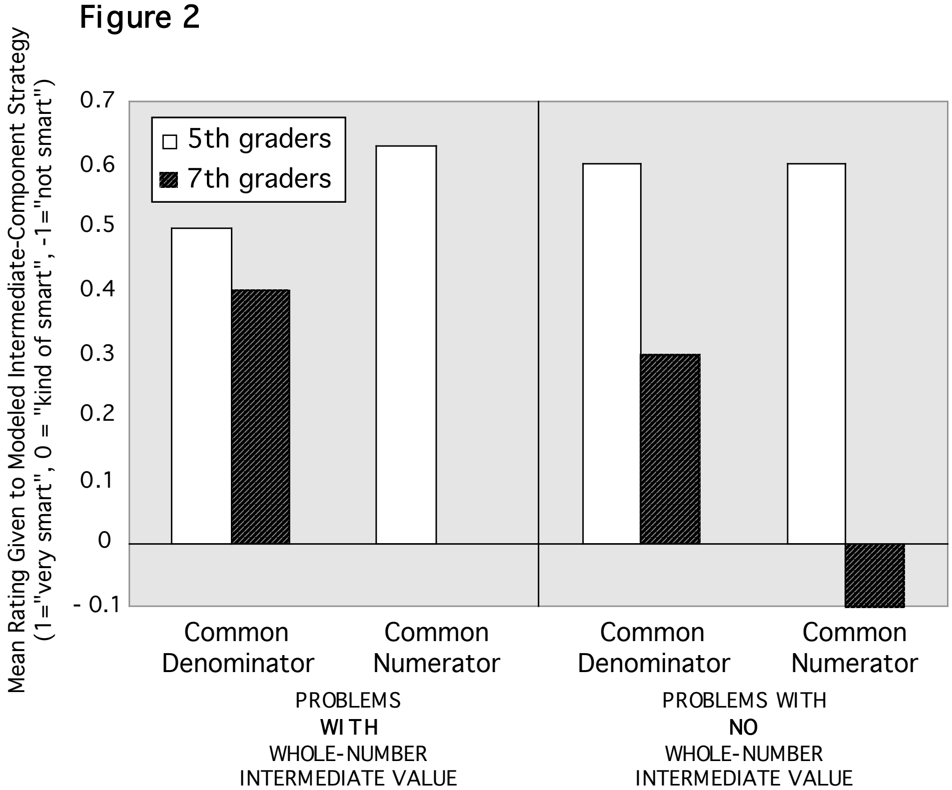


by comparing the white versus shaded bars within each panel. The 7th graders used the intermediate-component strategy more often than the 5th graders did on common-denominator problems, but less often than the 5th graders on common-numerator problems. Instead, they solved 45% of the common-numerator problems on which there was a whole-number intermediate value between the two denominators, and 55% of those on which there was not, by converting the fractions to common denominators. In contrast, none of the 5th graders even once attempted to convert different-denominator fractions to a common denominator in order to solve the problems. Thus, the 7th graders appear to have extended the school-taught procedure of finding a common denominator for different-denominator fractions to the problems presented here—displacing a simpler solution strategy that was widely used by 5th graders.

Figure 2 summarizes the judgments students made about the intermediate-component strategy when it was modeled by the experimenter. 5th graders gave much more positive ratings overall than 7th graders did—in part because the younger students were reluctant to judge anything the experimenter did as "not smart". The pattern of ratings within each grade level, however, illuminates more telling differences between the groups.

Among the 5th graders, the pattern of judgments diverges from the pattern of strategy use in that applications of the strategy that result in fractional numerators or denominators are judged no less smart than ones that result in more conventional whole-number values. This is the only aspect of the data in which we see the kind of divergence between

Figure 2



students' own strategy use and their judgments about modeled strategies that provided evidence for goal sketches in research in other mathematical domains. Although the 5th graders tend not to use the intermediate-component strategy when it results in a fractional numerator or denominator, they acknowledge it to be a fairly smart way to solve the problems when the experimenter models it.

Among the 7th graders, the pattern of judgments closely resembles the pattern of strategy use observed in the first part of the study. Thus, just as the 7th graders were more likely to use an intermediate-component strategy on common-denominator problems than on common-numerator ones, they also deemed it "smarter" when the experimenter applied it to common-denominator problems than when she applied it to common-numerator problems. Likewise, they deemed it "smarter" when it yielded a whole-number value for the intermediate numerator or denominator than when it yielded a fractional value for that component. The close correspondence between the strategies 7th graders use in their own problem-solving and their judgments about strategies modeled by the experimenter suggests that there is no longer a gap between their goal sketches and their actual problem-solving, at least with respect to the strategies studied here.

Insofar as goal sketches guide strategy development, then, it appears that acquisition of the computational algorithm of finding a common denominator has derailed rather than stimulated the process of strategy development. In learning the procedure of converting fractions to common denominators, students apparently came to believe that that is the only correct way to work with different-denominator fractions. That 7th graders were able to extend the algorithm of converting to a common denominator to new problems can be seen as positive in that it indicates the generalizeability of their learning. However, the fact that this strategy for solving the present problems displaced simpler but equally effective alternative strategies, so that they were not even judged positively when modeled for the students, underscores concerns about the dominance of procedural over conceptual aspects of student learning. In focusing on common denominators, they failed to recognize the possibility of drawing conclusions about magnitude relations among different-denominator fractions—surely an important element of understanding fraction magnitudes.

Thus, while procedural learning has been found to be facilitative of conceptual understanding in previous research (e.g., Rittle-Johnson, et al., 2001), the present findings underscore concerns that it can also have an adverse effect. Clearly, a fundamental problem for educators and psychologists alike is to clarify the conditions under which its impact is positive versus negative. To clarify this issue, research on students' goals sketches should be combined with detailed examination of the instruction those students are receiving.

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