

TOWARDS A REDEFINITION OF THE MATHEMATICS CULTURE IN THE CLASSROOM

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The questions addressed in this paper are: how do students interpret the non-solution problems as a mathematical practice in the classroom? What kinds of arguments do the students offer? To deal with these questions the utterances of students working in the classroom with problems in which they had to look for general relations in order to argue such things as “there is no least positive fraction” or “it’s impossible to tessellate a rectangle with some kind of pieces” are examined and discussed.

INTRODUCTION

The mathematics program for the secondary school in Mexico suggests the teaching of the discipline should be based on problem solving approaches. Among its objectives are that students develop their discovering abilities, recognize and analyze the components of a problem and that they formulate conjectures, communicate and validate them (Alarcón, et al., 1994).

To achieve these objectives, and also many of the objectives the NCTM (2000) has put forward, it’s necessary to change radically the roles teachers and students usually enact in those classrooms where teaching and learning goes under the School Mathematics Tradition (Cobb et al., 1992)¹. For instance, the students now explore open problems or situations in which they must pose questions, make conjectures and argue or criticize mathematical ideas.

In such situations they must recognize if it is possible or not to find or do what they are supposed to do. In this paper we discuss student’s utterances that emerge when faced with what we call *non-solution problems*, that is, problems in which they must answer that it’s not possible to find or do what they are asked to do, so they must find general relations within the problematic situation in order to argue such things as “the number 123 doesn’t belong to that sequence”, “there is no least positive fraction” or “it’s impossible to tessellate a rectangle with that kind of pieces.” The questions addressed here are: how do the students interpret the non-solution problems as a mathematical practice in the classroom? What kinds of arguments do the students offer?

COMPONENTS OF A CONCEPTUAL FRAMEWORK

The learning of mathematics is considered an individual process of construction and, also, an acculturation process into the mathematical practices of a wider society². The use

¹ Roughly speaking, this type of teaching is based on a teacher transmitting information and by the scheme eliciting-answer-evaluation (Mehan, 1979).

² We mean those practices accepted by math teachers, researchers in mathematics education, professional mathematicians, etc. Even though this *wider society* is not homogeneous, each

and meaning of the symbols and those mathematical practices are negotiated in the process of constituting a mathematical community in the classroom. (Cobb, Jaworski, & Presmeg, 1996; Yackel & Cobb, 1996).

The focus in this paper is on the argumentation process in the classroom. In this context, the notion of argumentation is related to offering an intentional explanation of the reasoning behind the solution. Several authors have reported the enormous difficulties students experiment when faced to the task of arguing (Fischbein, 1982; Balacheff, 2000). Beyond the social factors that may inhibit the student's processes of argumentation, Fischbein (1982) y Balacheff (2000) claim that the cognitive belief constitutes a major obstacle because students find no need to argue when they have intuitively seen the solution. In this perspective, the type of tasks implemented during the development of the study might function as a vehicle to promote students' presentations of arguments or convincing explanations.

Balacheff (2000) has differentiated two kind of proofs the students offer: *pragmatic proofs*, those in which students show the actions they performed finding the solution and *intellectual proofs*, based on the formulation of relations and characteristics of the problematic situation, for instance when they use a *generic example*, that is, when they refer to a particular case in order to show general relations. Pragmatic proofs imply the realization of a material task. This suggests an exploration theme for instruction; facing students with problematic situations where that material realization is not possible so that the student is being forced to search for general relations.

Topic	Problems
I Numerical patterns	<input type="checkbox"/> What's the position of number 311 within the sequence 3, 6, 9, ...? <input type="checkbox"/> What's the position of number 123 within the sequence 2, 5, 8, ...?
II Tessellations	<input type="checkbox"/> There are plenty pieces alike, formed by four 1-side squares as shown in the figure on the right. Make a 3 \times 8 rectangle and a 5 \times 10 rectangle using these pieces, without overlapping them or leaving holes.  <input type="checkbox"/> Make a rectangle using pieces like the one on the figure on the right. 
III Arithmetic	<input type="checkbox"/> Find the least positive fraction <input type="checkbox"/> Which number has the greatest number of multiples? <input type="checkbox"/> Which number has the greatest number of divisors?

teacher acts in his classroom according to his own interpretation on which of those mathematical practices are the accepted ones.

Table 1. Non-solution problems

METHODS AND GENERAL PROCEDURES

We worked with a seventh grade group of 30 middle class students in a school located in Mexico City. The students come from different primary schools (grade 1-6), which can be considered within the School Mathematics Tradition.

The sessions developed in a cyclic mode: 1) It starts with a set of different kinds of problems discussed within small groups of students (2 to 4 students). Each group writes down a report and one of them is audio recorded; 2) the next two or three sessions are dedicated to a plenary discussion of these problems, promoting the students to present different interpretations and solutions to the task; 3) Based on the analysis of the reports and the audio recording, a set of new problems is designed and a new cycle is initiated.

Within the school year, students worked with *non-solution problems* (Table 1). These problems were mixed with other kind of problems so they couldn't know, in advance, which type they were dealing with.

Two weeks after the students had worked with the last of these problems they were asked to write down, individually, the following items: *Describe non-solution problems and give examples. How come do they appear? Are they important?*

RESULTS

We discuss in separate sections each of the questions posed in the introduction of this paper.

Interpretation of non-solution problems

The teacher systematically promoted the acceptance and valorization of the mathematical practice of dealing with non-solution problems making explicit comments of how natural is to pose questions and problems in which we cannot know, in advance, whether they can or cannot be solved, and that recognizing that it's not possible to do or find something is an important form of knowledge. However, each student interpreted this practice in different ways.

The first time students worked with a non-solution problem, we observed, as we expected, difficulties in assuming that it was not possible to do what the teacher asked them to do; only one fourth of the groups answered that 311 is not in the succession 3, 6, 9, ... The audio-recorded group made the division $311 \div 3$, obtaining 103 with remainder 2 and reported 103 as an answer. Few minutes later, a student from another group *revealed* (whispering) "The first one is wrong":

Pam: He is right, 'cause it's not a sequence

Jim: Why is it not a sequence?

Pam: Because 311 is a multiple of 3, so it can't be in a sequence that goes 3 plus 3, plus 3...

It's remarkable that Pam accepts without hesitation that their answer was wrong, as if she was not comfortable with 103 as an answer, but she didn't even consider the possibility of saying no to what the teacher had asked them to do. Only after the *revelation* she did establish, using a correct argument, that 311 is not in the succession.

Once the possibility of a negative answer became part of the students' answers repertory, it was observed that some students gave up the explorations after a few tries, concluding, "It's not possible." For instance, working with tessellation problems the answer "we tried and it can't be done" was relatively frequent³. The audio-recorded group, trying to make a 5×10 rectangle with the L-shape pieces, failed for the second time:

John: There are two squares left.

Daniel: Two left! It can't be done.

Mary: Let's write, "There are always two squares left" ... but why are there always two left? We have to write that too.

They quickly conjecture that it can't be done but they also recognize that they have to explain why it is so (they did explain it later). Something similar occurred when students searched for the number with the greatest number of divisors; after a few tries some groups concluded that there is no such number although they didn't offer any argument supporting their claim. Others didn't easily accept the legitimacy of that kind of answer; for instance, when trying to tessellate a rectangle with U-shape pieces (this was the fifth non-solution problem he had met) a student asked the teacher: "is it possible that it is not possible?"

This behavior is similar to the one observed by other authors: Schönfeld (1992) concluded that students don't make long searches because they believe that "*you know the answer or you don't*" so there is no use in doing long searches. Also, as it has been reported (Balacheff, 2000), students usually make generalizations after analyzing a few particular cases. This author refers to this process as *naive empiricism*.

The analysis of the individual reports about what they think of non- solution problems made clear that, for most of the students, non-solution problems are viewed as anomalous or as a tool used by the teacher to make them aware of what they were doing, stopping them from acting automatically. Only a few considered that "it's normal that they appear when you ask questions and they help you to find new ideas and theories."

Most of the students showed correct examples of non-solution problems similar to those they had worked with. Even an example as "find the transformation represented by 3×5 , 5×9 , 9×5 ", which is wrong, mathematically speaking, because there is an infinite number of transformations that satisfy the requested conditions, it is a good example in the context they've been working, that is, transformations of the type $ax + b$, $x^2 + a$ and $x^3 + a$. Others used inappropriate examples, again thinking in operating terms: " 3×2 is not equal to 15."

Arguments offered by the students

Number Pattern. Some groups operated with the numbers they had at hand, applying an algorithm that showed to be successful in another situation and paid no attention to the fractional quotient. This compulsory need to operate with the numbers in the problem, even in the most absurd situations, has been widely documented (IREM, 1980; Schubauer-Leoni, & Ntamakiliro, 1998).

³ In his interactions with the students the teacher systematically used the phrase "is it that *it can't be done* or that *you can't do it?*" trying to problematize their experience.

In both problems the students recognized the pattern $+ 3$ governing the succession, but not all took into account the starting number and argued that 123 is the 41st number of the succession 2, 5, 8, ... because $41 \square 3 = 123$. Other students used a proportional approach stating that 123 is the 43rd number of the succession because 20 is the seventh, so 120 is the 42nd ($120 = 20 \square 6$ and $42 = 7 \square 6$); some students simply generated the succession using a calculator until number 122 concluding that 123 is not in the succession, finally, some used a generic example as an argument: “Number 123 is not in the succession, we know it because $7 \square 3$ is 21 and because we started with 2 it’s 20 (this is a way of saying that numbers in the succession are of the form $3n - 1$) and 124 is not a multiple of 3.”

Tessellation problems. The problem of tessellating a rectangle using U-shape pieces turned out to be quite a complex one. All of the groups concluded that it’s not possible to do so but none could find a valid argument. On the other hand, the problem of tessellating rectangles with L-shape pieces proved to be adequate. All the students could construct the $3 \square 8$ rectangle. The following discussion took place within the audio-recorded group just after they found there were two squares left, in their second try to construct the $5 \square 10$ rectangle:

Mary: What were the dimensions of this one?

John: $3 \square 8$

Daniel: And this one? Five times ten... I think that it’s only possible when... three times eight, forty-eight, even number, five times ten, fifteen⁴, odd number... Hey! I know why it can’t be done (very excited). Because it’s four (he refers to the four squares in the L-shape pieces), four and forty-eight are even numbers... that’s why it can be done, or something like that, right?

John: Five times ten... 50, 50 isn’t it an even number?

Daniel: Yeah, why?

John: **Five times ten**

D: Ay! Ay!

Daniel’s strategy is to find a property that holds in the $3 \square 8$ rectangle but fails in the other one. He first conjectured that the rectangle’s area should be of the same parity as the piece’s area, which is true but this property holds in both rectangles and he missed this point because of the mistake he made multiplying. He began conjecturing about multiples of three but this exploring line is interrupted by the (correct) argument offered by John that can be resumed as: the rectangle’s area must be a multiple of the piece’s area.

Daniel: Maybe it’s because of the multiples of three... Yeah, three times eight is 24 a multiple of 3...and 50 isn’t. Let’s try with...

John: 24 divided by 4? Would it be **exact**?

⁴ In Spanish the words three (*tres*), six (*seis*) and ten (*diez*) rhyme and maybe this auditive association can explain why Daniel answered 48 ($6 \square 8$) as the result of $3 \square 8$ and 15 for $10 \square 5$. Three days after this episode took place, the teacher unexpectedly asked the student “ $3 \square 8$?” and he quickly answered 48. A few days later, the teacher asked him again and he could answer correctly only after a brief pause.

Mary: Yeah.

John: 50 divided by 4? That is not exact. Four because of those 4-pieces

Mary: It can't be done, of course, because fifty, that is, five times ten...

John: And it isn't a multiple of four.

Arithmetic Problems. The three arithmetic problems can be argued by *absurdum reductio*. Searching for the least positive fraction the students realized that they can always find a smaller one: "if we think we have found it we add a zero to the denominator and we get a smaller one." Also a fraction with a denominator like 999... or 1000... was reported.

Trying to find the number with the greatest number of multiples, students noticed that the process of finding the multiples is potentially infinite. They also realized that not every number has the same number of multiples: 0 has only one and some of them claimed that 1 has more multiples than any number, arguing that, for instance, "multiples of one go one by one and the multiples of two go two by two." This claim led to an interesting discussion; many students argued that all, except zero, have the same number of multiples because they are generated multiplying them by 1, 2, 3, ...⁵

The problem of finding the number with the greatest number of divisors turned out to be a more complex one. This time the search for the divisors doesn't lead naturally to the idea that you can always construct a number with more divisors. Only four out of 12 groups could argue using a generic example: "There is no such number with the greatest number of divisors. For instance, 36, if we multiply it by 2, 72 has the same divisors as 36 and also 72 as a divisor and we can continue multiplying."

CONCLUSIONS

Students are used to show the solution of a problem applying a known algorithm. The algorithm is constituted as the main generator of answers. But when they have to make an argument about the impossibility of something they can no longer hold on any known algorithm. Most of the non-solution problems proved to be adequate for promoting the search for general relations. In particular the problem of tessellating a rectangle with L-shape pieces turned out to be a fruitful exploring situation that can be easily generalized changing both the dimensions of the rectangle and the shape of the pieces.

Negotiation of new mathematical practices in the classroom is a process that takes time, especially when they contradict strongly attached beliefs of the students. Although most of the students got relatively quickly used to work with non-solution problems, they don't think of them as common and natural tasks when learning mathematics but only as a didactical trick.

⁵ Even older students *clearly see* that the succession 1, 2, 3, ... has more elements than, for instance, the succession 2, 4, 6, ... and it's hard to convince them that it's not the case, but the way these successions emerged made some of the students *clearly see* they have the same number of elements.

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