TEACHERS' MATHEMATICS: CURIOUS OBLIGATIONS[1]

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W e explore the nature and consequences of teachers' problem solving through an example of a teacher's mathematical problem solving as it was occasioned by a student's mathematics. This illustration demonstrates the value of an interpretive framework that points to the mathematics of the classroom as a collective. Arising from this exploration is our core assertion, that the mathematics teacher has an obligation to be curious about mathematics. Our research has significant implications for teacher education as it points to the significance of the teacher's mathematics within the classroom collective and the possibilities for the growth of the teachers' understandings within that collective.

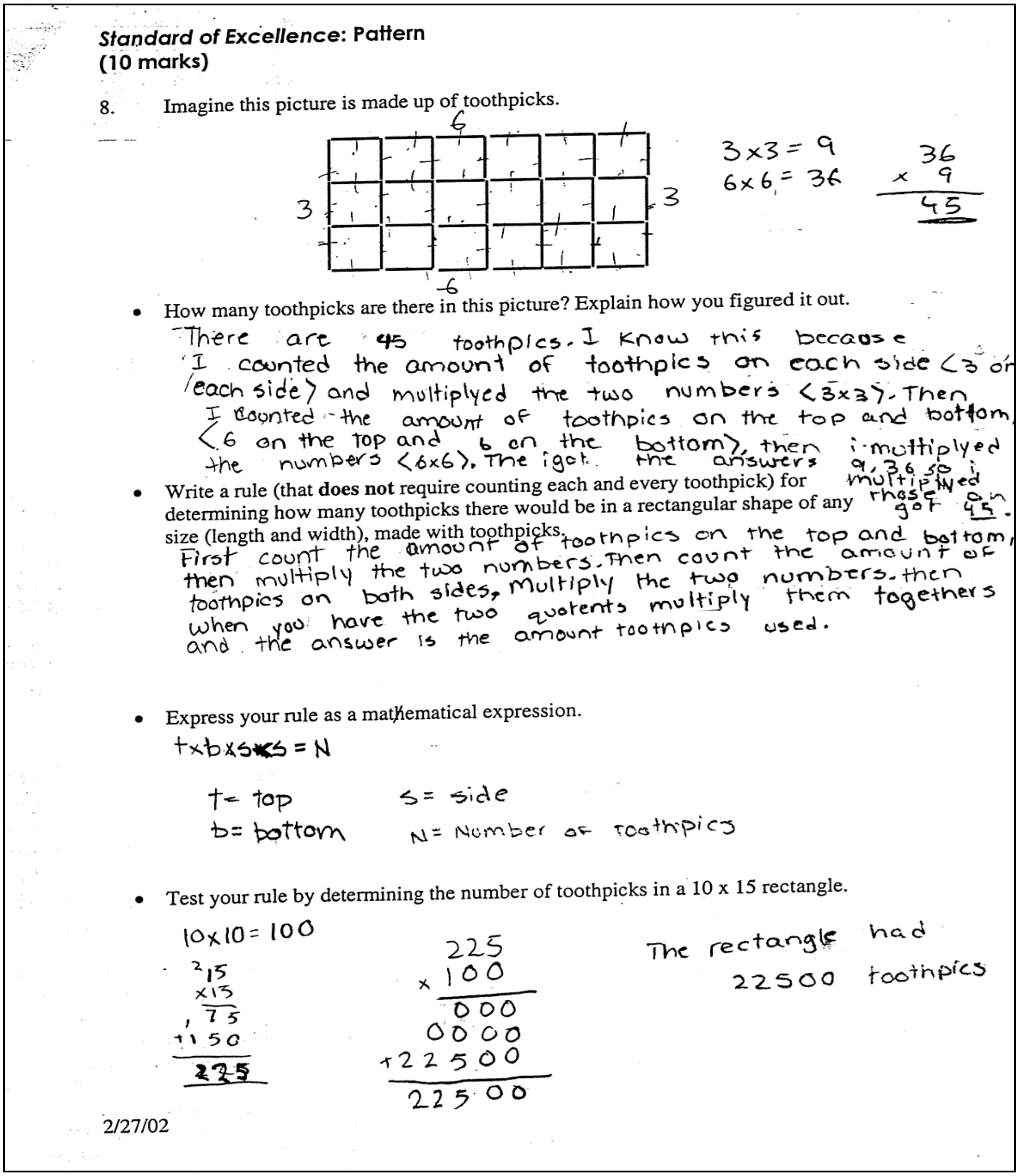
THE WAY TEACHERS NEED TO BE WITH MATHEMATICS

Every now and then, the mathematics teacher is compelled to engage in mathematical problem solving. We use one such event to frame this paper. In a recent classroom-based study, a test question, a student's response to that question, and the teacher's response to the student's response prompted us to rethink some of our own assumptions about the way teachers need to be with mathematics.

We borrow this idea from Ball and Bass (2002) who argue that the teacher's knowledge of mathematics is of a different sort than that of the research mathematician. Briefly, the research mathematician's work might be characterized in terms of the formulation of increasingly powerful generalizations. One might say, such efforts are oriented toward the compression of ideas. By contrast, the grade school teacher's responsibilities are more about decompressing mathematical ideas and student responses to those ideas. The teacher, that is, must have well-honed abilities to "pull apart", "unpack", and otherwise interpret the mathematizations that she encounters.

The focus of this writing is another aspect of the way a teacher needs to be with mathematics. We argue the teacher also has an obligation to be curious about the subject matter that she teaches—which, although clearly an issue in psychology, is not a topic that is often addressed an the contemporary psychology of mathematics education literature. The idea does find support in related literatures, however. For example, the importance of curiosity in research mathematics is a common theme (e.g., Burton, 1999a). As well, as Damasio (1994) develops from a neurological and cognitive scientific perspective, curiosity is among the many emotions that are necessary to the development of the rational faculties.

As a route into our discussion, we invite you to review, first, a seventh grade student's response to an exam question that was posed after a unit on patterns and relations (Figure 1), and then the teacher's log entry in response to Arlene's answer.



Standard of Excellence: Pattern
(10 marks)

8. Imagine this picture is made up of toothpicks.

6

3 3

6

$3 \times 3 = 9$
$6 \times 6 = 36$

36
× 9
45

- How many toothpicks are there in this picture? Explain how you figured it out.

There are 45 toothpics. I know this because I counted the amount of toothpics on each side (3 on each side) and multiplyed the two numbers (3x3). Then I counted the amount of toothpics on the top and bottom, (6 on the top and 6 on the bottom), then i multiplyed the numbers (6x6). The i got the answers 9, 36 so i multiplyed these an got 45.

- Write a rule (that **does not** require counting each and every toothpick) for determining how many toothpicks there would be in a rectangular shape of any size (length and width), made with toothpicks.

First count the amount of toothpics on the top and bottom, then multiply the two numbers. Then count the amount of toothpics on both sides, Multiply the two numbers. then when you have the two quotents multiply them togethers and the answer is the amount toothpics used.

- Express your rule as a mathematical expression.

t×b×s×s = N

t= top s= side
b= bottom N= Number of toothpics

- Test your rule by determining the number of toothpicks in a 10 x 15 rectangle.

10x10 = 100

15
×15
75
150
225

225
× 100
000
0000
+22500
22500

The rectangle had 22500 toothpics

2/27/02

Figure 1: Arlene's Response

THE TEACHER'S JOURNAL ENTRY

In marking this exam paper, I scanned the student's response and found myself looking at a "correct number" with an equation for finding that number that I did not recognize. It was neither like my solution nor any of the student solutions that I had already marked. My immediate response to Arlene's work was to wonder, "Is her computation correct?" Within her solution there are some errors; however, it is the case that $6^2 + 3^2 = 45$, the correct number of matchsticks for the case given.

I turned to her written work and could not find anything in it that helped me understand why summing the squares of the length and the width would produce the correct result. I speculated that Arlene's generalization certainly would not work for the 10 x 15 case and went about testing it only to find out that it does lead to a correct answer. (Her work was not of much help to me given the mistakes and apparently incorrect reasoning.) Finding that the 10 x 15 case could also be calculated by adding the squares of the dimensions was highly bothersome for me since I had carefully (or thought I had carefully) selected the "test" case. I was deliberate to select a second example that was unrelated to the first case—unrelated in the sense that I worried about it being a multiple of the first case for example, thus making the second one a case of the first.

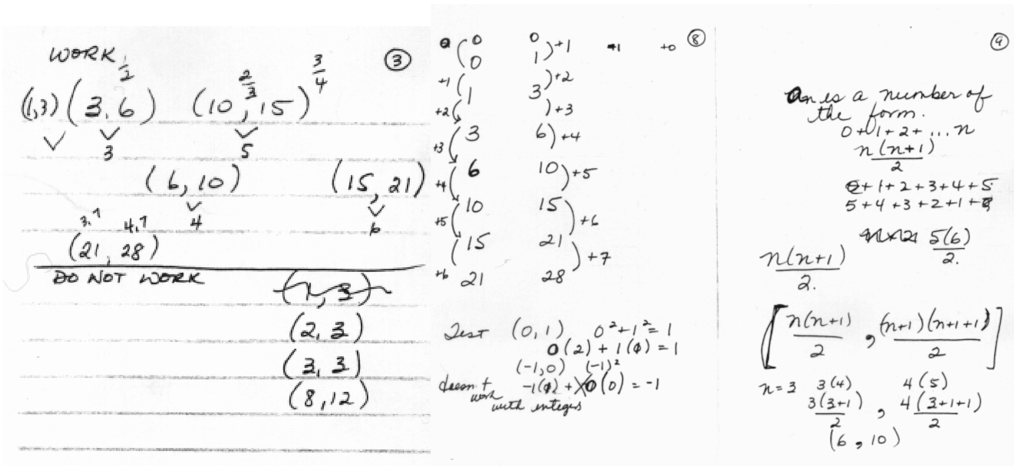


Figure 2: Extracts from the teacher's specializing and generalizing

Seeing two cases that both could be solved with a model that I could not yet "see" related to the problem I realized that in order for Arlene's method to work that it must be equal to the expression I wrote to model the problem. I wondered, "Is $l^2 + w^2 = l(w+1) + w(l+1)$ in general?" With only two lines of algebraic manipulations it was clear this is not the case. Hence, it could only be true under special circumstances and clearly a 3 x 6 rectangle and a 10 x 15 were two of those special cases.

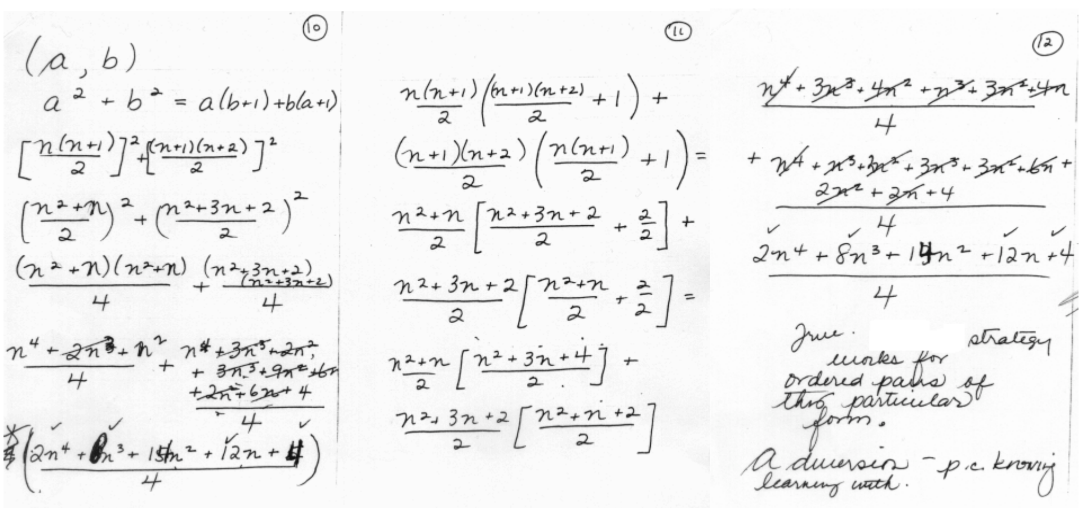


Figure 3: The teacher's proof

Muddling my way through this new problem and trying to make sense of the situation I was now in I specialized and put the dimensions back into the two equations and saw where they

> came together but it did not help much. So, I tried yet another case. This time I tried a 2 x 3 rectangle. I now had a case that did not work with Arlene's model. Now my curiosity was fully engaged. I needed to know: "Under what conditions does $l^2 + w^2 = l(w+1) + w(l+1)$?"
>
> After 4 pages of work in which I moved from specializing to generalizing and back again I learned that $l^2 + w^2 = l(w+1) + w(l+1)$ when l and w are two numbers, n and n+1, such that n is the sum of the first x whole numbers and n+1 is the next such number. When shown the problem, a colleague noticed these were triangular numbers; only then did I recall that fact.

As I researcher, I reflect back on my response to Arlene's mathematics and wonder, How was I with the mathematics?

A TEACHER'S OBLIGATIONS

One of the major themes to emerge in the mathematics education literature over the past few decades is the matter of teacher attendance to students' efforts to represent their emergent understandings. The topic of teachers attending to learners has been particularly prominent within the constructivist literature, as radical and social constructivists alike have highlighted the issue. See, for example, Pirie and Kieren's (1994) model for observing the growth of mathematical understanding and the Cognitively Guided Instruction Group's (Franke & Kazemi, 2001) strategy of engaging teachers in careful observation of students' mathematical activity.

More recently, Ball and Bass (2002) have tied this issue of teacher attendance to the issue of teacher knowledge of mathematics. This research focuses mainly on the sorts of conceptual competencies and interpretive abilities that are necessary for one individual to make sense of another's understandings. In the interpretations that follow, we concern ourselves more with the classroom collective than with individual understandings. Our intention is to examine the contribution to the collective of teachers' responses to student work. (Note that, as demonstrated in the example presented, we are not limiting the discussion to teachers' responses to students themselves. Our more general concern is the matter of their responses to student work, whether or not the substance of those responses are represented to students.) As elaborated below, we view the individual student as just one of a number of nested complex learning systems, and we wonder if the individual learner is the learning system that should be the primary focus of the teachers' attentions. In effect, we are trying to ask, if we change our assumptions about the individual as the locus of mathematics learning in school classrooms—and instead focus on the classroom collective as the learning system of interest—then what becomes the obligations of mathematics teachers?

In the case presented, the teacher's mathematizations were neither brought back to the student nor represented to other members of the class. At first pass, then, it might seem odd that we have framed our core assertion—that mathematics teachers have a sort of obligation to be curious about the subject matter—with this particular narrative. After all, the most common rationale for the teacher to engage in mathematical inquiry is so that she or he can better model for students what it means to engage with mathematical problems and processes. For us, the rationale of modeling is hinged to some deeply engrained but problematic assumptions. In particular, the notion that teaching is a modeling activity seems to be rooted in the assumption of radical separations among persons in the classroom. The teacher models, the learner mimics, but their respective actions are seen to be separable and to spring from different histories, interests, and so on.

We favor a different interpretation of human interaction, one that posits more profound intertwinings of identities and intentions. Framed by ecological theories (e.g., Bateson, 1979) and complexity science (e.g., Johnson, 2001), we argue that the reason for the teacher to engage in mathematical inquiry—or, more specifically, the reason the teacher is obligated to be curious about mathematics—has to do with her or his role in the emergence of a mathematical community, not with modeling. In terms of classrooms, whereas the imperative for the teacher to model mathematical engagement appears to be rooted in the assumption that the classroom is a collection of learners, we believe that the teacher needs to be mathematically curious because he or she is a part of the collective learning system of the classroom. This shift in phrasing is more than a rhetorical gesture. Academically speaking, it corresponds to a recent elaboration of established interests in the psychology of learners toward an interest in the psychology of social systems. (See, e.g., Burton, 1999b.) This conception of a classroom, in terms of a collective character rather than a collection of individual characters, is consistent with insights from the emergent field of complexity science.

Elsewhere we have discussed some of the common ground and some of the divergences of complexity science (and related discourses, such as enactivism) and many of the theoretical perspectives that currently figure prominently among mathematics education researchers, including radical constructivism and social constructionism (see Davis & Simmt, in press; Gordon Calvert, 2001; Towers & Davis, 2002). As such, we do not address that topic here. Instead we use complexity as a window into the role of teachers' mathematical curiosities in the project of school mathematics.

"SITUATED IN" VERSUS "PART OF"

In the main, when matters of individual learning and collective groupings are both addressed, the discussions tend to be framed in terms of what it means for learning agent(s) to be situated in particular social context(s). The locus of learning, that is, is generally assumed to be the solitary human who is cast as a sort of fundamental particle of cognition. Complexity science challenges the deeply engrained cultural assumptions that underpin this habit of interpretation. For the complexivist, learning is a broader notion, coterminous with the idea of evolutionary adaptation. Whenever a coherent system undergoes transformations in a manner that enable it to maintain its coherence within its dynamic circumstances, in complexity terms, it can be said to have learned. Events of learning thus include such diverse phenomena as the formation of the European Union, adjustments in stock market values, the rise of life on the earth, and the emergence of consciousness in a species. In terms of the project of modern schooling, some relevant learning systems include societies, mathematics (understood in terms of the intertwined activities of a mathematical culture), schools, and classrooms—in addition to individuals. Culturally speaking, this recent assertion of complexity science might be interpreted as a remembering of an ancient intuition. There has long been a tendency to discuss and describe each organization's level in this range of phenomena in terms of bodies (e.g., a body of knowledge, a student body) that grow and adapt.

This shift in frame is what prompts us to speak differently about individuals and classrooms—not as agents in situations, but as coherent forms that are parts of coherent forms (that are parts of coherent forms, and so on). The classroom collective unfolds from and is enfolded in learners and teachers. This frame undercuts many of the binary oppositions that are so often used to characterize learners and schooling—most obviously, perhaps, the common contrast of teacher-centered and student-centered instruction. There are no centers to complex systems.

There is a problem with this manner of characterization, especially when applied to something as deliberate as mathematics teaching. In our experience, it is rare and unusual that a classroom collective emerges around the generation of mathematical knowledge. Rather, the collective project (and when humans gather together, there is always some manner of collective project, even if is mutual destruction) seems most often to be organized around matters of social positioning. For us, the interesting question—and the common feature of our varied research efforts over the last decade—has to do with the emergence of collective possibility around the mathematics itself. With regard to the topic at hand, it is here that the issue of a teacher's personal engagement with mathematics takes on a particular relevance. The example of Arlene and the teacher provides a good example of what we mean by the expression, "a classroom collective emerges around the generation of mathematical knowledge". As we examine the 11 pages of the teacher's mathematizing, we encounter several mathematical claims that are new to us. To mention one, in terms of a theorem, for every pair of consecutive triangular numbers, a and b—but only for pairs of consecutive triangular numbers—the following is true: $a^2 + b^2 = 2ab + a + b$.

Whose insight is this? Following the conventions of contemporary research culture, and assuming it hasn't already been published, it clearly belongs to the teacher. However, an attention to the events that surround the emergence of the insight reveals that, in fact, the idea arose in the cogitations in a few overlapping communities, which themselves are hooked into still broader communities. For instance three key events that contributed to the emergence of the idea, and without which the insight might never have arisen, are Arlene's erroneous response, the teacher's accidental selection of two pairs of numbers that satisfy the theorem, and a colleague's casual mention that the numbers generated by the teacher are triangular.

An error by a student, a coincidental choice by a teacher, and a comment from a colleague. Such elements are not the typical fare of mathematical progress. Or are they? A close examination of the strew of interests represented in current mathematics research suggests that the there is something troublesome about the classic definition of 'progress' as a linear movement toward a perceptible goal. Progress seems to be neither linear nor directional, but more about the pursuit of interests that unfold as individuals and collectives negotiate their ways through constantly shifting interpretive backgrounds. We might further highlight the recursively elaborative nature of the event to emphasize the nonlinear, nondirectional nature of mathematical insight. Culminating in this paper, this writing might be described as the interpretation of a group of mathematics education researchers to an interpretation of a teacher-researcher to the interpretation of a colleague to the interpretation of the teacher-researcher to the interpretation of a student's interpretation of a test question. These qualities of complex intertwinings, recursive elaborations, and unforeseeable ends are what prompt us to argue that the teacher must be mathematically curious. This curiosity cannot be framed in terms of causal influence in one's teaching. It is more a matter of necessary contribution. To underscore this point, we would hazard the claim that, in our combined recollected experiences as researchers, teachers, and learners, every "teachable moment" that we've encountered has been dependent on (but, of course, not determined by) the teacher's expressed curiosities.

Teachable moments, we believe, are moments of complex emergence—that is, moments in which diverse agents cohere into collectives with shared purposes and insights. And although we argue that teacher curiosity is a critical element, we would not want to diminish the significance of student's individual interpretations, their private interactions, the texts and other artifacts that are made available, and so on. However, we do feel that teacher curiosity stands out as a critical element in the mathematics classroom. As the above example demonstrates, it compels teacher attendance to student articulations, it opens up close-ended questions, and (as described elsewhere, see Davis & Simmt, in press) it can trigger similar contributions from learners. Indeed, we would go so far as to suggest that one of the biggest problems facing contemporary school mathematics is that, generally speaking, mathematics teachers are not curious about the subject matter.

TEACHER EDUCATION

Many of our preservice teacher education students arrive to our classes with a genuine curiosity about the subject matter, particularly at the secondary level. At the other extreme, many arrive with a fear of the subject matter, especially at the primary levels. A few arrive neither curious nor fearful, having opted into the route of a mathematics teacher because the subject matter seems so easy to teach. It is, after all (and in their opinions), unambiguous. Over the years, it has been curious to observe that, contrary to prominently expressed opinions in the mathematics education literature (see, e.g., Ernest, 1991), it seems the aspects that most frame these diverse students' engagements with the subject matter in our classes are not their beliefs about the nature of mathematics, but the extent to which they can become curious about the subject matter. Platonist and social constructionist alike can be profoundly engaged or frustratingly detached.

Our experience has also shown that curiosity is not an innate proclivity, but can be learned, to some extent at least. Shifts in attitudes can be occasioned as prospective teachers take part in mathematical activities—and, in particular, in those sorts of activities that require them to be participants in a learning collective. (Such activities are not difficult to design. For instance, it can be done by extending almost any mathematical activity with the simple task of formulating a new question to pose to—and hopefully stump—classmates.) A key seems to be that such activities are neither teacher-centered nor learner-centered, but mathematics-oriented. The agents are brought together by common activity with shared purposes—which, psychologically- and sociologically-speaking, is a critical element in transforming a collection of me's into a collective of us (see Johnson, 2001). We thus support the current and widespread practice of structuring courses for preservice mathematics teachers around mathematical activities, and would advocate for an elaborated practice of framing those activities in terms of collective or joint inquiries. This suggestion stems from our belief that, like mathematical knowledge, curiosity is a collective phenomenon, even when it is expressed in private pursuits.

To return to the example that we used to frame this paper, we cannot say how the teacher's response to the student's response affected the course of activity in the classroom. But it is precisely the fact that we cannot specify the consequences that prompts us to argue that curiosity around the subject matter is an obligation, not an option for the teacher. Complexity does not tell us how a teacher's attitudes and activities contribute to the collective, only that they do.

Note

1. This paper is based on data collected in a year-long teaching experiment (funded by the Social Sciences and Humanities Research Council grant 410-2000-0500) in which Simmt taught a grade 7 mathematics class. That teaching experiment was part of a collaborative research project in which Towers, Gordon and Simmt are exploring the implications of high activity and interaction rich mathematics classes.

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LOGICO-MATHEMATICAL ACTIVITY VERSUS EMPIRICAL ACTIVITY: EXAMINING A PEDAGOGICAL DISTINCTION

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I present a theoretical distinction that may prove useful in conceptualizing mathematics teacher education (and graduate education) and research on mathematics teacher education. Further, the distinction can contribute to developing frameworks on the design of mathematics curricula. The distinction between empirical activity and logico-mathematical activity focuses on the nature of a mathematical concept and how that concept develops, key issues in the quest to teach mathematics for understanding.

A primary goal of the mathematics education reform in North America during the last 15 years has been to promote students' learning of mathematics with understanding. This goal is in response to a widespread perception that too many mathematics students learn mathematics as a collection of disconnected and meaningless (to the learner) facts and procedures. This reform effort has been fueled by and has continued to require re-conceptualization of the nature of mathematics, what it means to do mathematics in school, how mathematical concepts are learned, and how mathematical concepts can be taught. In this article, I explicate a pedagogical distinction that could prove useful in conceptualizing the design of mathematics lessons and the education of mathematics educators. The theoretical distinction presented is grounded in a Piagetian empirical framework. Examples of data and author-generated lessons provide the basis for examining this distinction.

Over the last 6 years, my colleagues and I have been engaged in a research project, the Mathematics Teacher Development (MTD) Project[i]. The purpose of the project has been to understand the mathematical and pedagogical development of K-6 teachers (inservice and preservice) as they participated in a comprehensive reform-oriented teacher education program. This research has resulted in a set of distinctions about the pedagogical thinking that underlies the practice of teachers participating in the reform (cf., (Heinz, 2000; Simon, 2000; Tzur, Simon, Heinz, & Kinzel, 2001). In this article, I explore another distinction, deriving in part from the MTD research, that involves conceptualization of the nature of mathematical concepts, what it means to do mathematics in school and how mathematical concepts are learned.

One characteristic of classrooms and curricula guided by participation in the reform is an emphasis on students' active involvement in the development of new (to them) mathematical ideas. Different modes of active involvement have often been articulated (e.g., problem solving, looking for patterns, representing, explaining, justifying, finding counter examples). In the two lessons that follow, the first from MTD data and the second from one of the recent NSF-supported curricula, a similar lesson structure is used that makes use of pattern recognition. After describing these lessons, I will make a case for what I consider to be problematic aspects of the pedagogical conceptions underlying these lessons. I will then exemplify and briefly describe a contrasting framework for conceptualizing mathematics concept development and lesson design.

IVY'S LESSON ON AREA OF TRIANGLES

The MTD data that I describe in this section were included in a detailed analysis of Ivy's practice (Heinz, 2000). That analysis focused on the underlying structure of Ivy's practice. Subsequent observations, including situations that were not part of the MTD project, have led to a re-examination of these data and articulation of a new distinction.

Ivy, a sixth grade teacher (students age 11 years), was in her sixth year of teaching when she designed and taught this lesson on the area of triangles.

Ivy wanted her students to

> find the formula . . . I really believe that they forget what we just tell them and that they will remember what they figured out. And if they don't remember it, they can figure it out again and maybe faster the next time.
> . . . I want them to understand it.

Mathematical relationships that Ivy was aware of were the basis for her lesson design.

> We are building off those right triangle ideas because that is where the formula builds from, which is actually from rectangles. So I am trying to take them from rectangles to right triangles to non-right triangles to see how it is all related to the rectangle itself.

Following is an outline of Ivy's lesson:

1. Ivy led a review of how to find the area of a rectangle on a geoboard.
2. Students worked in small groups to find the area of a 2x3 right triangle.
3. The whole class discussed their strategies and results for step #2.
4. Students worked in small groups to find the areas of all of the right triangles they could make on their geoboards and recorded the measures of the base, height, and area for each triangle.
5. Students shared their data from step #4 with the whole class while Ivy recorded the information in a 3-column table
6. Students examined the table to come up with a formula.

Ivy's instructions for step #6 were:

> Look at how these numbers are in this chart with our areas . . . and see if you can figure out a pattern that you can use every time using the numbers [measures of base and height] to come up with the area. . . . There is something that you can do to these [measures of] the bases and the heights to get the area.

A PUBLISHED LESSON ON EQUIVALENT FRACTIONS

In the United States, mathematics educators often consider the state of the art in reform-based mathematics education curricula to be represented by recent National Science Foundation supported curricula.[ii] It is my experience that the lessons within each curriculum, although generally superior to those found in preexisting curricula, are uneven in quality. One explanation for this phenomenon might be the multiple authors involved in writing each of the curricula. However, I would argue that a more important reason is the lack of or inadequacy of explicit frameworks for guiding lesson design. This latter point suggests work to be done in mathematics education. The pedagogical distinction that I explicate in the next section may prove useful in curricular design efforts.

I include the first 7 steps of the "At a Glance" (*Math trailblazers: A mathematical journey using science and language arts (K-5)*, 1999) that summarizes the lesson on equivalent fractions.

1. Ask students to use their fraction chart from Lesson 3 to find all of the fractions that are equivalent to 1/2. List these on the board or overhead.
2. Ask students to compare the numerators and the denominators of the equivalent fractions in order to look for patterns.
3. Ask students to suggest other fractions that are equivalent to 1/2.
4. Write number sentences on the board or overhead showing the equivalencies.
5. Students look for patterns in the number sentences.
6. Students use the patterns (multiplying or dividing the numerator and the denominator by the same number) to find fractions equivalent to 3/4, 1/3, and 2/5.
7. Students use the patterns to complete number sentences involving equivalent fractions.

ANALYSIS OF THE TWO LESSONS AND DEVELOPMENT OF DISTINCTIONS

The two lessons, just described, have similar goals and structure. The goals involve the generation of a computational strategy (generalization) or formula with "understanding."[iii] The structure involves generating a set of examples, finding the numerical pattern (relationship) among the parts of the examples, and establishing that pattern as a generalization for computing the missing number in further examples.

Lessons of this type, if criticized, are generally criticized on the basis of issues of justification. That is, although examining a set of examples to find a pattern is appropriate for generating a conjecture, it does not constitute mathematical proof that the relationships involved are true for all cases of the type being considered. There remains a need for deductive justification. This is an important mathematical issue, but not the one that I focus on here.

Let us consider what students might learn from these lessons. In Ivy's lesson, students are likely to learn that there is a fixed relationship among the base, height, and area of a triangle and that it can be represented as A=bh/2. Similarly, in the lesson on equivalent fractions, students might learn that there is a numerical relationship among equivalent fractions. To produce an equivalent fraction, one can multiply the numerator and denominator by the same number (not zero and not necessarily an integer). Is this what we mean by "understanding" in mathematics? I argue that it is not.

Understanding is a broad term, and a single definition is unlikely to capture all significant meanings (cf., Piaget, 2001; Sierpinska, 1994; Simon, 2002). However, for the purpose of analysis and contrast with the lessons described above, I offer the following characterization of understanding. *Mathematical understanding is a learned anticipation of the logical necessity of a particular pattern or relationship(s)*[iv].