PROBING STUDENTS' UNDERSTANDING OF VARIABLES THROUGH COGNITIVE CONFLICT PROBLEMS : IS THE CONCEPT OF A VARIABLE SO DIFFICULT FOR STUDENTS TO UNDERSTAND?

Toshiakaira Fujii
Tokyo Gakugei University, Tokyo, Japan

This talk will report on a study of students' understanding of school algebra from two aspects. The first presents research which I carried out in order to probe students' understanding of literal symbols. The resulting analysis shows that many students in junior high school appear to have a very poor grasp of what literal symbols denote and how they are to be treated in mathematical expressions. In the second part, an attempt is made to show how the curriculum of the elementary school can offer better opportunities for young people to think algebraically. Utilizing the potentially algebraic nature of arithmetic is one way of building a stronger bridge between early arithmetical experiences and the concept of a variable. In this paper I use the terms generalisable numerical expressions or quasi variable expressions to make a case for a needed reform to the curriculum of the elementary school. Videotape records and written evidences are presented to show students' understanding of algebra and then we seek to an alternative way of teaching of school algebra.

INTRODUCTION

Understanding of algebra in school mathematics is one of the most important goals for secondary mathematics education. On the other hand, algebra has been a critical wall for students. In fact, many reports identify specific difficulties of learning of algebra: cognitive obstacles (Herscovics, 1989), lack of closure (Collis, 1975), name-process dilemma (Davis, 1975), letter as objects (Kuchemann, 1981), misapplication of the concatenation notation (Chalouh & Herscovics, 1988), misinterpretation of order system in number (Dunkels, 1989) and so on. Matz (1979) also has identified inappropriate but plausible use of literal symbols in the process of transforming algebraic expressions.

In Japan, we are facing with the same problem that many students in junior high school are still confusing unknown numbers and variables. However we need to be careful of diagnosing of their nature of understanding, simply because students seem to be good at solving conventional school type problems. Although ratios of correct answers in mathematics achievement tests such as IEA results and PISA results are high, Japanese mathematics educators suspect that limited understanding may coexist with this apparent success story. We need therefore to devise an instrument that can probe the understanding lying behind students' apparent procedural efficiency. To this end, the author has been developing cognitive conflict problems as tools to elicit and probe students' understanding. The first part of this paper will focus on the function of cognitive conflict problems and survey data collected by the author himself to illustrate Japanese students understanding of algebra.

The second part of this paper focuses on some ways of laying foundations for algebraic thinking from the early years of schools by attempting to bridge to the divide which exists between arithmetic and algebra. Some researchers in the past, for example Collis (1975), have tended to suggest that the notion of variable is linked to an extended abstract thinking - a conclusion that is not surprising given that many students in junior high schools show an incomplete understanding of a variable. This conclusion may not be so clear, and that the concept of a variable number may be accessible to students at a much younger age. Many currently used approaches to early algebra appear to focus exclusively on introducing frame words and literal symbols as devices for solving simple number sentences. Essentially, these problems require students to supply a missing or unknown number to a mathematical sentence, such as 7 +   = 11. Sentences such as this are often called "missing number sentences", which we suspect some students solve by trial and error or guesswork. Number sentences of this type may be quite effective in promoting knowledge of simple number facts, but they are quite limited in developing algebraic thinking. Algebraic thinking necessarily involves students in patterns of generalization. In the second half of this paper, I will present some approaches to introducing algebraic thinking in the elementary and junior high school curriculum using generalisable numerical expressions based on a concept of a quasi-variable. I argue that the problem we are facing might be more related to curriculum than to any supposed cognitive level.

A FRAMEWORK OF PROBING STUDENTS' UNDERSTANDING OF ALGEBRA

Algebra in secondary school mathematics can be described as learning how to use symbolic expressions. These symbolic expressions are composed of numerals and mathematical signs together with alphabetical letters. We can represent the process of using symbolic expressions in terms of a mathematical modeling process. That is, starting from a situation, we express the situation in terms of mathematical expressions, then transform them to get a mathematical conclusion. Finally we need to read or interpret the mathematical conclusion into the original situation to get insight or new interpretation or discoveries. T. Miwa (2001) has illustrated the process as the scheme of use of symbolic expressions as shown below:

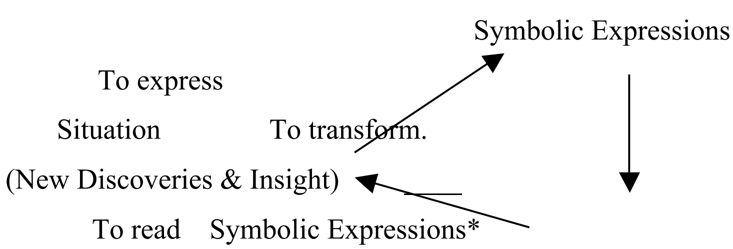


Fig. 1 Scheme of Use of Symbolic Expressions

In this paper, the scheme of use of symbolic or mathematical expressions regarded as a framework of probing students' understanding of algebra. Let me start with the introduction of letter x in early algebra.

STUDENTS' UNDERSTANDING OF LITERAL SYMBOL X: EXPRESSING AND INTERPRETING OF LITERAL SYMBOLS

In the process of learning and teaching of algebra, many misconceptions have been identified by teachers and researchers. Here I focus on the conventions or rules in the expression and interpretation of literal symbols. One of the well-documented misconceptions is the convention of interpreting letters, namely a belief that different letters must represent different values. This misconception is illustrated by students' responses of "never" to the following question:

When is the following true – always, never or sometime?

$$L+M+N=L+P+N$$

Kuchemann (1981) reported in the CSMS project that 51% of students answered "never" and Booth (1984) reported in SESM project that 14 out of 35 students (ages 13 to 15 years), namely 40%, gave this response on interview. Olivier (1988) reported that 74% of 13 year olds also answered "never". He suggested that the underlying mechanism for not allowing different literal symbols to take equal values stems from a combination with other valid knowledge, that is, the correct proposition that the same literal symbols in the same expression take the same value. In other words, some students who are aware of the proposition that the same letter stands for the same number, they tend to think that the converse of this proposition is also correct. The author claims that the convention, the same letter stands for the same value, is not grasped well by students, based on a survey conducted with Japanese and American students (Fujii, 1993, 2001). In some situations, students conceive that the same letter does not necessarily stand for the same number. Focusing on this incorrect convention, this section of the paper aims to clarify students' understanding of literal symbols in algebra through two studies: a preliminary written survey identifying interview subjects and a subsequent clinical interview with students.

Preliminary written survey aimed to identify interview subjects

The written survey task is aimed at identifying students' understanding of literal symbols in order to pair students with different understandings. Specifically, "different" in this context means that the paired students held inconsistent conceptions. The interview context created a conflict that allowed students to express their ideas explicitly to each other. The methodology of this careful and purposeful identification of subjects for interview is one of the characteristic features of the study. The written survey problem tasks are shown below.

Problem 1

Mary has the following problem to solve:

"Find value(s) for x in the expression: x + x + x = 12"

She answered in the following manner.

a. 2, 5, 5
b. 10, 1, 1
c. 4, 4, 4

Which of her answer(s) is (are) correct? (Circle the letter(s) that are correct: a,b,c)

State the reason for your selection.

Problem 2

Jon has the following problem to solve:

"Find value(s) for x and y in the expression: x + y = 16"

He answered in the following manner.

a. 6, 10

b. 9, 7

c. 8, 8

Which of his answer(s) is (are) correct? (Circle the letter(s) that are correct: a,b,c)

State the reason for your selection.

Results of the Written Survey

Initially, the author intended to analyze separately data from these two problems.. However, results showed that problem 1 and 2 are related and need to be considered as a related set. For Problem 1, some students chose only the same value item c (4,4,4) and in Problem 2 they chose only the different value items a (6, 10), b (9, 7). The reason for this kind of response appears to be that "The same letter stands for the same number" in Problem 1, and "Different letters stand for different numbers" in Problem 2. Based on this conception, some students had Problem 1 correct, but Problem 2 incorrect. We call this type of response Type A.

On the other hand, there were other students who selected all items in Problem 1 and also selected all items in Problem 2. The reason for this kind of selection appeared to be that " All add up to 12" for Problem 1 and " All add up to 16" for Problem 2. These students seem to ignore differences in the letters and seem to consider that letters can stand for any numbers. Based on this conception, they had Problem 1 incorrect, but Problem 2 correct. We call this type of response Type B.

In summary, the written survey identified Type A and Type B responses as described below:

Type A: Holding the misconception that different letters stands for different numbers.

Student had Problem 1 correct.

Student had Problem 2 incorrect by rejecting (8, 8).

Type B: Holding the misconception the same letter does not necessarily stand for the same number.

Student had Problem 1 incorrect by accepting all items.

Student had Problem 2 correct.

It is interesting to note that both Japanese and American students showed a similar tendency (Fujii, 1992, 2001). It is also important to note that it is rare for students to get both problems correct, which was also consistent with the data for both countries. Let me select the Athens (GA) 6th, 8th and 9th graders from the American data, simply because these students have a common educational environment. The percentages of correct answers for 6th, 8th, and 9th grade are 11.5%, 11.5% and 5.7% respectively. For Japanese students, the correct response from 5th, 6th, 7th, 8th, 10th and 11th grades are

0%, 3.7%, 9.5%, 10.8%, 18.1% and 24.8% respectively (Fujii, 1993). For both countries, the percentages of correct response are disturbingly low and the percentages do not dramatically increase according to the grades as we may expect. Mathematics educators from both countries may have to reconsider this fact seriously.

Students Interview Tasks and Procedures

Paired students for the interview were chosen one each from the two groups: Type A and Type B. The interview context was designed to include conflicting points of view in the hope that students would express their ideas explicitly to each other. Here, I am going to show the U.S. data, one group from 6th grade consisting of, as it happens, three students, one from Type A and two from Type B.

While the written survey task such as problems 1 and 2 were used in the interview, an additional task was used in interviews by modifying the task used in the study conducted by Takamatsu (1987). Takamatsu reported that some 6th grade student expressed the relation between the sides and perimeter of a square by using x, as x+x+x+x=x. In the first stage of the interview, subjects were introduced to this expression with a square, both were written on a paper, and an explanation as follows:

A Japanese student expressed the relation between the sides and perimeter of a square by using x as x+x+x+x=x. Is this a correct or incorrect expression?

In the second stage of the interview, subjects were asked about any inconsistencies between their responses in the interview and those in the preliminary survey task results. For instance, if a student identified the expression x+x+x+x=x as incorrect, then his/her responses on the expression x+x+x=12which had been interpreted as 2+5+5=12, 10+1+1=12 besides 4+4+4=12 were critically examined. On the other hand, if a student identified the expression x+x+x+x=x as correct by saying, for example, that the letter x can be any number, then his/her responses on the expression that the expression x+x+x=12 which had been interpreted 4+4+4=12 were critically examined.

RESULTS OF THE INTERVIEW

Analysis on the Same Letter: On the expression x+x+x+x=x

Asked about the correctness of the expression x+x+x+x=x, the Type B (boy) stated "correct" and gave this reason:" Because x is a variable."

The other type B (girl) recommended that the right hand side x could be 4x. Then she tried to substitute number 4 into x. At this stage the Type A student become articulate and stated her idea as follows:

> But, this is, in that sentence x has to be the same number, doesn't it?

Based on this comment the Type B (girl) suggested to replace x into a or y, who was trying to be consistent with the Type A (girl). The Type B (boy) seemed to think that it was not necessary to do that. Eventually the three concluded as follows:

> Type A (girl): "Because x is supposed to be the same thing in whole sentence."
>
> Type B (boy): "It doesn't have to be the same thing. It's a variable."

Their final comments on the correctness of the expression x+x+x+x=x are shown below:

> Type A (girl): "No, because x has to be the same thing."
>
> Type B (boy): "I think its right."
>
> Type B (girl): "I think its right."

Analysis on the expression x+x+x=12

Through interpreting the letter x in the expression x+x+x=12, students' ideas became more explicit by expressing their own words. In fact, the Type B (boy) gave a reason why he thought items (2,5,5) and (10,1,1) were acceptable which was:

> x is unknown so it could be anything.

The type A (girl) responded as follows:

> I think that since in this sentence there are 3 x's, all of the x's have to be the same number, even though they are unknown, so that would have to be just the three numbers that add up to 12.

The Type B (boy) insisted that whether we would replace x+x+x into 3x depend on what x stands for as saying below:

> It can, but it can also be wrong. It depends on what x equals, which, because x can equal 10, the first x, and then second x can equal 2.

The type A (girl) disagreed with it and stated that:

> I think that all the x's are the same number and so you can write 3x.

She added an explanation as follows:

> I will say that x is a variable and if it is in the same problem with another x then it has to be the same number."

Although the Type B (boy) used same word "variable" and saying that "Because x is a variable", he meant x could be any number in the same problem.

Analysis of different letters in the expression x+y=16

Concerning the different letter, the Type A (girl) stated clearly that:

> They have to be different numbers because they are different variables, and so the first two fit that and the last one doesn't.

The type A (girl) did not accept the item (8,8) for x+y=16, because, she said, x and y are different. This explanation is a typical for Type A students. On the other hand, the Type B student accepted the item (8,8) without hesitation by saying that" I think all three of them are right."

DISCUSSIONS

The relationship between the same letter and different letter

Based on the written survey and the following interview, students who consider that the same letter stands for the same number appear to think that different letters must stand for different numbers. The type A (girl) stated that:

> I am not so absolutely positive that I am right, it just makes more sense, (be)cause if there are two different variables, they probably (re)present two different numbers.

It is interesting to note that this tendency was common to both American and Japanese students (Fujii, 1992, 2001).

The misconception with the same letter

The written survey and the interview revealed that many Japanese and American students tend to have a misconception that the same literal symbol does not necessarily stand for the same number. This misconception has not been explicitly reported by English speaking researchers. However, we could identify the tendency that appeared in the past research that students consider the same letter does not necessarily stand for the same number. For instance, in the context of solving equations such that: $x + x/4 = 6 + x/4$, Filloy, E & T. Rojano(1984) reported that the student considered that the x on the left hand side must be 6 and the x expressed in the x/4 on both sides could be any number. Similarly, given the equation: $x + 5 = x + x$, students interpreted that x in the left side can be any number, but the second x on the right side must be 5.

The rule that the same letter stands for the same number is a basic one in the process of interpreting letters in mathematical expressions. These studies show that this basic convention has not been grasped by students in the USA and in Japan. Understanding the convention that same letter stands for the same number is crucial for both American and Japanese students.

The levels of Understanding of Literal Symbols

The concept of variable has been discussed for a long time in mathematics education community. The definition of variable given in the SMSG (School Mathematics Study Group) Student's Text was "the variable is a numeral which represents a definite through unspecified number from a given set of admissible number" (School Mathematics Study Group, 1960, p.37). Although the ideas *definite* and *unspecified* appear to be in tension, the concept of variable needs to include these different aspects (Van Engen, 1961a, b). Let me now consider the survey and interview results from these aspects.

Data from two surveys are evidence that students appear to lack one or both aspects. The "definite" aspect of the concept of variable is most clearly embodied in the convention that the same letter stands for the same number. Students' misconceptions described as "x can be any number" emphasizes only the "unspecified" aspect of a variable. This misconception is not likely to be revealed in expressions that contain only one literal symbol. Students' responses that $x+x+x+x=x$ is correct, and their interpretation of $x+x+x=12$ as $2+5+5=12$ appear to result from considering only the "unspecified" aspect of the concept of variable.

On the other hand, the misconception, different letters stand for the different numbers, could be characterized as an unduly strict interpretation of the "definite" aspect of variable by students who persistently reject substituting the same number for different literal symbols. Although the domain of variable does not depend on the literal symbol itself, the interview revealed that students tend to focus on the surface character of literal symbols, such as differences in letter, within the domain of variables.

In the analysis of the written and interview survey, four responses were identified: "both problems are correct", "Type A", "Type B" and "other". These four groups appear to show levels of understanding of literal symbols. These levels can be described as follows: Level 0, which is "the other" responses in the survey, where students have a vague conception of literal symbols. There are no rules to interpret literal symbols, or no rules for substituting numbers into literal symbols. We could not identify an explicit rule for choosing items in the problem 1 and problem 2 in the written survey.

On the other hand, in Level 1, Type B, there is some logic behind students' responses. At this level the "unspecified" aspect of variable is dominant, but the "definite" aspect is missing.

In Level 2, Type A, the "definite" aspect of variable appears to become dominant, and items are chosen by the convention that the same letter stands for the same number. However, there are misconceptions in dealing with the different letters based on the premise that different letters must stand for different numbers. These students focus on the "definite" aspect of variable but they are not able to consider the "unspecified" aspect at the same time.

Level 3, students are able to attend to both aspects of variable, which, as I remarked before, have to be seen in some tension with each other. The students can consider that the same letter stands for the same number, and also that different letters do not necessarily or always stand for different numbers.

These four levels of understanding of literal symbols may serve to help teachers see clearly the diverse conceptual demands of teaching school algebra from its beginnings. In particular, teachers may have to consider how best to promote students' progress in understanding from Level 2 to Level 3. This seems especially important given that the American and Japanese surveys both show that moving from Level 2 to Level 3 is hard for many students. This evidence raises the question of what teaching approaches might bring a more substantial change of levels of understanding. It is important for teachers to use teaching approaches that help to integrate the "definite" and "unspecified" aspects of variable.

STUDENTS' UNSERSTANDING UNDERLYING PROCEDURAL EFFICIENCY

Algebra embodies a critical difference from other language, in that it can be transformed according to certain rules without changing connotations. This feature makes algebra a powerful tool for mathematical problem solving. Because of this feature, teaching and learning of procedural efficiency in algebra are highly valued, and students need to be trained up to a certain level of skills. In Japan, a country where students face high-stakes exams to enter upper secondary schools or universities, students have no choice about mastering these skills to solve problems within a certain fixed time. As an outcome, Japanese students seem to be good at solving mathematic problems presented in school algebra. But is this really any indication that students have a deep understanding of the subject matter or is it only superficial understanding? R. Skemp (1976) called this "Instrumental Understanding". Instrumental understanding means knowing what to do but without knowing why. On the other hand; the "Relational Understanding means knowing what to do and why (Skemp, 1976). Although the instrumental understanding is shallow, it can still work effectively in almost all conventional school mathematical problems.

The author has been developing set of cognitive conflict problems, where cognitive conflict is regarded as a tool to probe and assess the depth and quality of students' understanding (Fujii, 199*3*). Problems on linear equations and inequalities were developed. In solving linear equalities and inequalities in which the solution set contain all numbers, clearly the 'disappearance' of x was expected to provoke cognitive conflict in students. By analyzing how students went about resolving this conflict, it was possible to identify the nature of their understanding behind procedural efficiency.

The Problems

Problems on linear equations and inequalities were given to the 7th and 8th graders. Here is one of the inequality problems (other problems are quite similar).

> Mr. A solved the inequality $1 - 2x < 2(6 - x)$ as follows:
>
> $1 - 2x < 2(6 - x)$
> $1 - 2x < 12 - 2x$
> $-2x + 2x < 12 - 1$
> $0 < 11$
>
> Here Mr. A got into difficulty.
> 1 Write down your opinion about Mr. A's solution.
> 2 Write down your way of solving this inequality $1 - 2x < 2(6 - 2x)$ and your reasons.

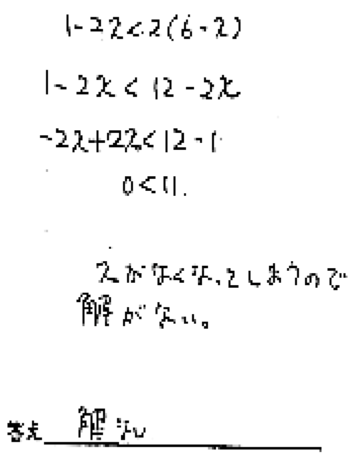
The problem was designed to include "the disappearance of x", with the verbal expression "Here Mr. A got into difficulty", and the mathematical expression "$0 < 11$" to highlight the nature of the problem. The expression could have been written as "$0x < 11$. Whether the students had been provoked or not could be determined by examining their reactions to the problem. Students' conflicts regarding Mr. A's difficulty caused were evident in the following responses: "I also got stuck here", and "At the moment I have no idea what to do". However, students' comments such as "I do not know why Mr. A got into difficulty here" was identified as a sign for not being provoked by the conflict. Unprovoked responses were found in only 3.5% of students, while most students, 96.5%, seemed to be genuinely provoked by the conflict. Almost all students wrote some conclusion in their papers. Whether these conclusions were correct or not, they were considered a necessary condition for resolving the conflict.

Analysis of Students' Answers

Students' responses were further classified into five categories. Category A (13%) consisted of responses where the conflict was able to resolve by giving the correct answer. Among lower secondary second graders (n = 123), very few were included in this category. Other students' rationales reflected two ways of resolving the cognitive conflict produced by the disappearance of x. The first was exhibited in the students' persistence of coming up with an answer that contained x. This group comprised Category B(34%). Category B was further sub-divided into two groups B1(26%) and B2(8%). Students in B1group, persisted in having x in the final answer by using irrelevant procedures, while students in B2 who expected to get an answer containing x but couldn't retain an x finally give up by concluding that "there is no solution". Category C (18%) consisted of students who reached a final answer not containing x. Category D (3%) gave no answer or solution (Fujii, 1989).

For students in Category B, the goal of solving an inequality was intended to obtain a form such as x > a. Though one such student knew that –2x + 2x = 0 is true, but in this instance the students claimed that a final answer without x is not possible. Thus, the student wrote x>18/11.

Students in Category C seemed to consider that solving equations and inequalities needed to follow the rules of equations and inequalities, and whatever the last expression was, even if it did not contain x, it should be the final answer. Students in Category C seemed to accept a final expression without x believing that to solve equations and inequalities means transforming the expre ssion into its simplest form. Category C students showed only a vague understanding of the meaning of the solutions of equations and inequalities. These students consider x to be no more than an object in transforming the expression. It is likely that these students have been successful in solving the equations with procedural efficiency without any understanding of what the solution means or should look like.



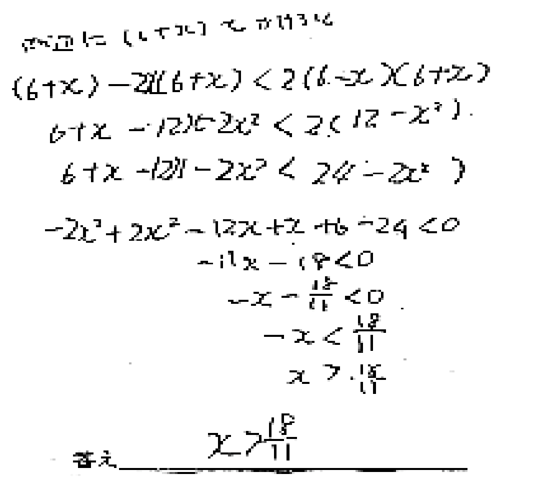


Fig.2 Typical Example in Category C Fig. 3 Typical Example in Category B1

On the other hand, students who can think of x as a variable can come up with the correct answer by interpreting x to take a definite but unspecified value. Student I wrote the expression: 1–2x < 12–2x, replacing –2x with , then re-expressing the original expression as 1 + <12 + . Student I explained as follows: "The sign of the inequality remains the same even if we add the same number to, or subtract it from both sides of the expression. Any number will do for ; hence the same applies for x." Note that this student focuses on the calculation of adding –2x to both sides without seeing any need to find a concrete number for –2x or x. By re-expressing the original expression, this student seemed able to pay more attention to the operation itself and to the structure of the expression than to the objects of calculation such as –2x, 1–2x and 12–2x. This approach is clear evidence of understanding of x as a variable.

CREATING A BRIDGE BETWEEN EARLY ALGEBRA AND ARITHMETIC

Any improvement in the teaching of algebra must focus on how children are introduced to express quantitative relationships that focus on general mathematical relationships, how they read or interpret algebraic expressions, and how they can calculate algebraic expressions based on the attributes of equality. The remainder of this paper focuses on how children from a quite young age can be introduced to algebraic thinking through generalisable numerical expressions. The aim is to show that this fundamental aspect of algebraic thinking should be cultivated systematically at all stages of schooling.

There is a reluctance to introduce children to algebraic thinking in the early years of elementary school where the focus for almost all teaching of early number is on developing a strong foundation in counting and numeration. Yet Carpenter and Levi (1999) draw attention to "the artificial separation of arithmetic and algebra" which, they argue, "deprives children of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades" (p. 3)..In their study, they introduced first and second-grade students to the concept of true and false number sentences. One of the number sentences that they used was 78 – 49 + 49 = 78.When asked whether they thought this was a true sentence, all but one child answered that it was. One child said, "I do because you took away the 49 and it's just like getting it back".

It was never the intention of Carpenter and Levi to introduce first and second-grade children to the formal algebraic expression, a – b + b = a. These children will certainly meet it and other formal algebraic expressions in their later years of school. What Carpenter and Levi wanted children to understand is that the sentence 78 – 49 + 49 = 78 belongs to a type of number sentence which is true whatever number is taken away and then added back. This type of number sentence is also true whatever the first number is, provided the same number is taken away and then added back. Fujii (2000) and Fujii & Stephens (2001) refer to this use of numbers as quasi-variables. By this expression, we mean a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are. Used in this way, our contention is that generalisable numerical expressions can assist children to identify and discuss algebraic generalisations long before they learn formal algebraic notation. The idea behind the term "quasi-variable" is not a new one in the teaching of algebra. In his history of mathematics, Nakamura (1971) introduces the expression "quasi-general method" to capture the same meaning.

We argue that the use of generalisable numerical expressions can provide an important bridge between arithmetic and algebraic thinking which children need to cross continually during their elementary and junior high school years. The concept of a quasi-variable provides an essential counterbalance to that treatment of algebra in the elementary and junior high where the concept of an unknown often dominates students' and teachers' thinking. As Radford (1996) points out, "While the unknown is a number which does not vary, the variable designates a quantity whose value can change" (p. 47). The same point is made by Schoenfeld and Arcavi (1988) that a variable *varies* (p. 421). The use of generalisable numerical sentences to represent quasi-variables can provide a gateway to the concept of a variable in the early years of school.

Research into Children's Thinking

Currently Fujii and Stephens are working together with children in Year 2 and 3 in Australia and Japan using an interview-dialogue based on a method actually used by a student called Peter in subtracting 5. The purpose of the interview is to see how readily young children are able to focus on structural features of Peter's Method. In other words, can they engage in quasi-variable thinking as outlined in this paper and in Fujii & Stephens (2001), and how do they express that thinking?

The interview-dialogue starts with Peter subtracting 5 from some numbers..

> 37 – 5 = 32
>
> 59 – 5 = 54
>
> 86 – 5 = 81
>
> He says that these are quite easy to do. Do you agree?
>
> But some others are not so easy, like:
>
> 32 – 5
>
> 53 – 5
>
> 84 – 5
>
> Peter says, "I do these by first adding 5 and then subtracting 10, like
>
> 32 – 5 = 32 + 5 – 10..Working it out this way is easier."

Does Peter's method give the right answer? Look at the other two questions Peter has. Can you use Peter's method? Rewrite each question first using Peter's method, and then work out the answer.

Some children have difficulty re-writing the questions in a form that matches Peter's Method. They go straight to the answer. When asked how to explain why Peter's method works, they say it works because it gives the right answer. The interview does not point children in one direction or the other. But if children follow this kind of thinking, where their focus is on following a correct procedure for subtraction, the interview does not continue any further.

On the other hand, Alan (8 years and 10 months, at end of Year 2) gives a quite different explanation when he says:

> Instead of taking away 5, he (Peter) adds 5 and then takes away 10. If you add 5 you need to take away 10 to equal it out.

This explanation appears to attend more closely to the structural elements of Peter's Method, and suggests that Peter's Method is generalisable. Those children who give an explanation which attends to the structural features of Peter's Method are asked to create some examples of their own for subtracting 5 using Peter's Method, and are then asked to consider how Peter might use his method to subtract 6. The interviewer asks:

> What number would Peter put in the box to give a correct answer?
>
> 73 – 6 = 73 +   - 10

If students answer this question successfully, they are asked to create some other examples showing how Peter's Method could be used to subtract 6. Finally, students are told: “Peter says that his method works for subtracting 7, and 8 and 9.” They are then asked to show how Peter’s Method could be used to re-express subtractions, such as.

83 –7,

123 – 8, and

235 – 9.

The final part of the interview invites students to explain how Peter’s Method works in all these different cases. Alan, who was quoted above, said:

> For any number you take away, you have to add the other number, which is between 1 and 10 that equals 10; like 7 and 3, or 4 and 6. You take away 10 and that gives you the answer.

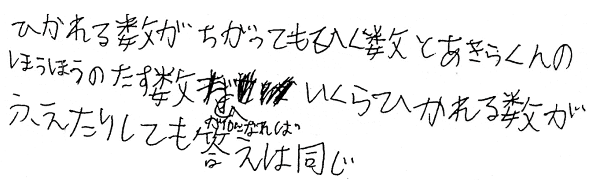
Alan’s thinking seems very clearly to embody quasi-variable thinking. He sees that Peter’s Method does not depend in any way on the initial number (83, 123, or 235). Alan’s explanation also shows that Peters’ Method can be generalised for numbers between 1 and 10..Zoe, aged 8 years and 4 months, gives a similar explanation:

> Whatever the number is you are taking away, it needs to have another number to make 10..You add the number to make 10, and then take away 10. Say, if you had 22 – 9, you know 9 + 1 = 10, so you add the 1 to 22 and then take away 10.

Another student, Tim, (age 9 years and 1 month at the start of Year 3) says:

> Here is an explanation for all numbers. Whatever number he (Peter) is taking away, you plus the number that would make a ten, and you take away ten. The bigger the number you are subtracting, the smaller the number you are pulsing. They all make a ten together.

Japanese student, Kou, (age 9 years and 6 month at the start of Year 3) says:” It does not matter what number is taken way, when (the) adding number makes a ten the answer is always the same whatever the subtracting number is increasing or decreasing.”



All these students are able to ‘ignore’ for the purposes of their explanation the value of the ‘starting number’. They recognize that it is not important for their explanation. In this sense, they show that they are comfortable with “a lack of closure”. Their explanations focus on describing in their own language the equivalence between the expressions that experts would represent as a – b and a + (10 – b) - 10 where b is a whole number between 1 and 10. These children show algebraic thinking in so far as they are able to explain how Peter’s Method always works “whatever number he is taking away” (Tim), “whatever the number is you are taking away” (Zoe), “for any number you are taking away” (Alan), “there is always a number to make ten” (Adam), “whether the subtracting number is increasing or decreasing” (Kou).

On the other hand, other students needed to close the sentence, by first deciding to calculate the results of 83 –7, 123 – 8, and 235 – 9, and then tried to calculate the number to place in the + on the right hand side. Eventually, some came up with a correct number, but interestingly, none could answer the question which asked them to explain how this method *always* works. Those who first calculated the left side of the equal sign seemed unable to ignore the 'starting number' and unable to leave the expression in unexecuted form. There were clear differences between these students and those who were comfortable with "a lack of closure". The present elementary school curriculum does little to shift students who are inclined to "close" away from this thinking.

IMPLICATIONS FOR THE REFORM OF ELEMENTARY SCHOOL MATHEMATICS

A conclusion of our research is the importance of recognizing the potentially algebraic nature of arithmetic, as distinct from trying to move children from arithmetic to algebra. Specific algebraic reasoning opportunities need to be engineered for use in the primary grades. These are needed to assist teachers and students to see numbers algebraically.

Quasi-variable or generalisable numerical expressions can be developed in many settings of elementary and junior high school mathematics, and allow teachers to build a bridge from existing arithmetic problems to opportunities for thinking algebraically without having to rely on prior knowledge of literal symbolic forms. These expressions are usually written in uncalculated form in order to disclose the relationships between the numbers involved. When a student explains the truth of the expression or statement by reference to its structural properties, then *quasi-variable thinking* is shown. This kind of reasoning appears to be quite different from that shown by students who rely on calculating the numerical values of expressions in order to determine their truth. Quasi-variable thinking, as we are investigating it, does not require the use of algebraic symbols. Further research is needed to show how young children identify and explain these relationships..

This is not an easy task when teachers' vision has for so long been restricted to thinking arithmetically. In the elementary school, this means attending to the symbolic nature of arithmetic operations. Research suggests that many of today's students fail to abstract from their elementary school experiences the mathematical structures that are necessary for them to make a later successful transition to algebra. As Carpenter and Franke (2001) point out: "one of the hallmarks of this transition from arithmetic to algebraic thinking is a shift from a procedural view to a relational view of equality, and developing a relational understanding of the meaning of the equal sign underlies the ability to mark and represent generalizations"(p. 156). Here are three suggestions for ways to smooth this transition:

- Describing and making use of generalisable processes and structural properties of arithmetic, generally; and of quasi-variable expressions in particular.
- Generalising solutions to arithmetic problems that assist students to develop the concept of a variable in an informal sense.
- Providing opportunities for students to discuss their solution strategies to these problems in order to highlight fundamental mathematical processes and ideas.

Blanton and Kaput (2001) remark, teachers in the elementary school, especially, need to grow "algebra eyes and ears" (p. 91) in order to see and make use of these opportunities.

This is not an easy task when teachers' vision has for so long been restricted to thinking arithmetically. In a mathematics curriculum for the primary school of the 21st century, teachers and students need to explore the potentially algebraic nature of arithmetic. This can provide a stronger bridge to algebra in the later years of school, and can also strengthen children's understanding of basic arithmetic. Any reform of the arithmetic curriculum in the elementary school must address these two objectives.

FINAL REMARKS

Three processes - expressing, transforming, and to reading - are all important elements of mathematical activity, and need to be related each other in how mathematics is described in curriculum documents and in how it is taught and learned . Particularly, the process of transformation needs to connect with the expressing and reading process. The research data in this paper have illustrated students' tendency to transform literal symbols without reading them carefully. This appears also to be true for numerical expressions. When students are dealing with generalizable numerical expressions or quasi-variable expressions as I have called them, teachers have to assist students not to read these expressions as commands to calculate. Identifying the critical numbers and the relational elements embodied in these expressions requires students to focus especially on expressing and transforming the underlying structure. This has important implications for teaching and learning.

Many reports have confirmed that school algebra is difficult for students to understand. The problem should not be construed simply in terms of the cognitive demands that pertain to algebraic thinking as opposed to arithmetical thinking. Important as those cognitive elements are, there is also a serious problem in the way that algebraic thinking and arithmetical thinking have been separated in the school curriculum, especially in the elementary school. In a mathematics curriculum for elementary and secondary schools of the 21st century, we need to develop teaching approaches to connect these three processes of mathematical activity. Starting in the elementary years, this can be achieved by exploring the potentially algebraic nature of arithmetic. Any reform of the curriculum of the elementary and secondary school must consider the role of algebra as a tool for mathematical thinking about numerical expressions long before children are introduced to formal symbolic notation. The latter particularly can provide a stronger bridge to algebra in the later years of school, and can also strengthen children's understanding of basic arithmetic.

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