

# REMEDYING SECONDARY SCHOOL STUDENTS' ILLUSION OF LINEARITY: A TEACHING EXPERIMENT

Wim Van Dooren<sup>1,2</sup>, Dirk De Bock<sup>2,3</sup>, An Hessels<sup>2</sup>, Dirk Janssens<sup>2</sup>  
and Lieven Verschaffel<sup>2</sup>□

<sup>1</sup> Research Assistant of the Fund for Scientific Research – Flanders (F.W.O.)<sup>2</sup> University of Leuven and <sup>3</sup> European Institute of Higher Education Brussels; Belgium

*Previous research has shown that many secondary school students improperly apply linear models when solving non-linear problems involving lengths, area and volume of similar plane figures and solids. This phenomenon is called the “illusion of linearity”. This paper presents a teaching experiment in which we developed and tested a learning environment to help students overcome the illusion of linearity. The experiment was successful in improving students’ performance on non-linear problems, but this improvement was disappointingly small. Moreover, there was a decline in students’ performance on linear problems. So, the experiment was not successful in developing in students a profound conceptual understanding of proportional and non-proportional relations, which includes the disposition to distinguish between situations that can and cannot be modelled linearly.*

## THEORETICAL AND EMPIRICAL BACKGROUND

According to Freudenthal (1983, p. 267), “linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear.” This phenomenon is often referred to as the “illusion of linearity”. It has been exemplarily reported in students of different ages and in different domains of mathematics and science education, such as elementary arithmetic, algebra and physics, and has been systematically studied in geometry (for an overview, see De Bock, 2002) and recently also in probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2002).

In the domain of geometry, a systematic line of research by means of paper-and-pencil tests has shown a very strong tendency among secondary school students aged 12-16 to overgeneralise the linear (or proportional<sup>1</sup>) model to problems about the relationship between lengths and areas/volumes of similarly enlarged or reduced geometrical shapes (see, e.g., De Bock, 2002; De Bock, Verschaffel, & Janssens, 1998). These studies have shown that even with considerable support, such as self-made and ready-made drawings or metacognitive hints, only very few students made the shift from incorrect proportional to correct non-proportional reasoning. In a recent study with individual interviews (De Bock, Van Dooren, Janssens, & Verschaffel, 2002), information was obtained on the problem solving processes and explanatory factors underlying this tendency to produce linear answers. First of all, the results showed that the majority of the students use the linear model in a spontaneous, almost intuitive way, while some students really are convinced that linear functions are applicable “everywhere”. Second, students have particular shortcomings in their geometrical knowledge (e.g., the belief that irregular figures have no area, or that a similarly enlarged figure is not necessarily enlarged to the same extent in two dimensions), prohibiting them from discovering the correct solutions for some non-linear problems. Third, many students have inadequate habits, beliefs and

attitudes towards solving problems in (school) mathematics, leading to stereotyped and superficial mathematical modelling.

The next stage of the research program – which will be the focus of the current paper – involves the design, implementation and evaluation of a learning environment aimed at overcoming the illusion of linearity, more specifically in the context of the enlargement/reduction of plane figures and solids, and the effect on lengths, (surface) areas and volumes. We aim at developing in students a deep conceptual understanding of proportional and non-proportional relations and situations, the adequate geometrical knowledge base to solve this type of problems, and a more mindful approach towards mathematical modelling.

### **ORGANIZATION OF THE TEACHING EXPERIMENT**

A series of 10 one-hour experimental lessons – including all teacher and learner materials – was developed for use with 13-14-year old students. With respect to the purposes of the lesson series, the results and the conclusions of earlier studies (e.g., De Bock, 2002; De Bock et al., 1998, 2002) discussed above were taken into account. The development of the learning environment was moreover strongly inspired on the principles of realistic mathematics education (see, e.g., Gravemeijer, 1994). First, the lessons were interspersed with various realistic problem situations aimed at challenging students' mathematical (mis)conceptions, beliefs and habits that lead to stereotyped and superficial modelling. Second, the problem situations and tasks allowed rediscovering of the required mathematical notions, building on students' own productions and informal knowledge. Third, the learning environment relied on a combination of instructional techniques that have proven to be successful in enhancing students' deep understanding and higher-order thinking skills (e.g., articulation and reflection) (see also Collins, Brown, & Newman, 1989). A fourth characteristic was that multiple representations of the learning contents (such as drawings, schemes, tables, graphs, formulas, and words) were used and their reciprocal relationships were accentuated to enhance deep-level learning (NCTM, 2000).

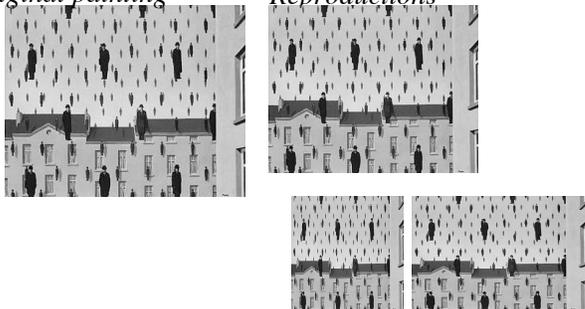
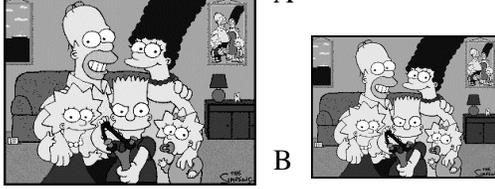
During the 10 lessons, the following topics were successively addressed: recognizing and constructing similar figures/objects, proportional relations and their properties, linear growth of the lengths and perimeter in similar figures, quadratic growth of the area and cubic growth of the volume. The lesson series ended with a project about the “Life and Work of the Gnomes” (Poortvliet & Huygen, 1976), in which the students were engaged in attractive, challenging and authentic problems involving the combined application of all learnt contents. Examples of learning activities and exercises from the experimental lessons are given in Figure 1.

### **RESEARCH METHOD**

Two comparable groups of secondary school students (8<sup>th</sup> graders, aged 13-14) were involved in the study. The experimental group of 18 students followed the experimental lesson series, while the control group of 17 students followed the regular lessons (in which none of the contents under consideration was explicitly treated). All lessons in the experimental group were videotaped and all student notes were collected. Moreover,

during the lesson series students' perceptions and evaluations of the lessons were registered by means of a questionnaire after the fourth and eighth lesson.

The learning gains in both groups were assessed by means of a word-problem test consisting of 2 proportional items (about the perimeter of an enlarged square or circle) and 4 non-proportional items (about the area or volume of an enlarged square/cube or irregular figure). Table 1 gives an example of a proportional and a non-proportional item. Three parallel versions of this test were constructed, which were, respectively, administered to the experimental group before the intervention (pretest), after the intervention (posttest), and three months afterwards (retention test). For practical reasons, the control group received only the pretest and the retention test.

<p style="text-align: center;"><b>Similar figures/objects (lesson 1-2)</b></p> <p>Which reproductions are similar to the original painting?</p> <p><i>Original painting</i>      <i>Reproductions</i></p>  <p>Afterwards: examination of similarity of real objects (cans, envelopes, bottles, ...)</p>	<p style="text-align: center;"><b>Proportionality/Linearity (lesson 3)</b></p> <p style="text-align: center;">A</p>  <p style="text-align: center;">B</p> <table border="1" data-bbox="885 787 1388 940"> <thead> <tr> <th></th> <th>picture A</th> <th>picture B</th> </tr> </thead> <tbody> <tr> <td>width of picture</td> <td>... mm</td> <td>... mm</td> </tr> <tr> <td>height of picture</td> <td>... mm</td> <td>... mm</td> </tr> <tr> <td>height of the cupboard</td> <td>... mm</td> <td>... mm</td> </tr> </tbody> </table> <p>Put the data from the table in a graph and explore that graph. (e.g. "an object with a height of ... mm on picture A is ... mm high on picture B")</p>		picture A	picture B	width of picture	... mm	... mm	height of picture	... mm	... mm	height of the cupboard	... mm	... mm
	picture A	picture B											
width of picture	... mm	... mm											
height of picture	... mm	... mm											
height of the cupboard	... mm	... mm											
<p style="text-align: center;"><b>Perimeter: linear growth (lesson 4)</b></p> <p>Is a big cola bottle of 1.5 litres similar to a small cola bottle of 0.5 litres?</p>  <p>If you strip the labels of the bottles, the label of the small bottle is 5 cm high by 20 cm wide. The label of the big bottle is 7.3 cm high. When both bottles are similar to each other, what should be the width of the label of the big bottle?</p>	<p style="text-align: center;"><b>Area: quadratic growth (lesson 5)</b></p> <p>It's Anne's birthday and her mother is going to make pancakes, using three pans of different sizes. Anne asks her friend: "You may choose between two big pancakes (30 cm diameter), four regular ones (20 cm diameter) or six small ones (15 cm diameter)."</p> <p>Her friend reasons as follows: "You better choose six small pancakes because <math>2 \times 30 \text{ cm} = 60 \text{ cm}</math>, <math>4 \times 20 \text{ cm} = 80 \text{ cm}</math> and <math>6 \times 15 \text{ cm} = 90 \text{ cm}</math>"</p> <p>What do you think about the reasoning of Anne's friend?</p>												

<b>Volume: cubic growth (lesson 6-7)</b>	<b>Integrative project (lesson 8-9-10)</b>
<p>An apple grower sells two sizes of apples: The first one has an average diameter of 6 cm and costs 10 eurocent and the second one has an average diameter of 9 cm and costs 20 eurocent.</p> <p>→ Compare both sizes of the apples. What is the enlargement factor (<math>k</math>)?</p> <p>→ The apples have a similar shape. How much more weighs a big apple compared to a small apple?</p> <p>→ Which apple size is the most economical to make apple sauce?</p> <p>Another apple grower has other prices for the two kinds of apples: “The apples with an average diameter of 6 cm costs 1/kg, those with an average diameter of 9 cm costs 1,20/kg”. Which apple is the most economical now?</p>	<div style="display: flex; justify-content: space-around;"> <div data-bbox="889 216 1105 499">  <p><i>true height: 15 cm</i></p> </div> <div data-bbox="1154 216 1370 499">  <p><i>a gnome weighs 300 grams</i></p> </div> </div> <p>Assuming that a gnome is similar to a human being, is it possible than that a gnome with length 15 cm weighs 300 grams?</p> <p>How long is the belt of a gnome?</p> <p>What is the area of the sole of a gnome?</p> <p>How much coffee is there in a cup for gnomes?</p>

Figure 2. Examples of materials used in the lesson series

Table 2. Examples of word problems used in the test

<i>Proportional item (perimeter)</i>	<i>Non-proportional item (volume)</i>
<p>Steve needs 10 minutes to dig a ditch around a square sandcastle with a side of 50 cm. How much time will he approximately need to dig a ditch around a square sandcastle with a side of 150cm?</p>	<p>In his toy box, John has dice in several sizes. The smallest one has a side of 10mm and weighs 800 mg. What would be the weight of the largest die (with a side of 30 mm)?</p>

All problems were offered as open questions, and students had to write down their answer as well as their calculations. Answers were scored either as correct or as incorrect, and incorrect answers were further categorized in terms of one of the following categories, based on a scrutinized analysis of the students’ solution steps: application of proportional methods to non-proportional items, application of non-proportional methods to proportional items, other errors.

### RESEARCH QUESTIONS AND HYPOTHESES

The goal of this study was to test whether a learning environment with the above-mentioned characteristics could cause a substantial reduction in students’ tendency to produce linear answers in situations where they are not correct. Based on our previous studies (De Bock, 2002; De Bock et al., 1998, 2002) we expected that on the pretest, experimental and control group students would generally respond correctly to the proportional items and incorrectly on the non-proportional items, because of their tendency to apply proportional strategies for these latter items too. Due to the learning experiences in the experimental lessons, we expected a significant progress in the performance of the experimental group on the posttest – more specifically on the non-proportional items – and that this progress would largely persist on the retention test. For

the control group, no significant evolution from pretest to retention test was expected, because these students were not involved in any learning activities specifically addressing the linearity misconception.

## RESULTS

A 2 × 2 × 3 repeated measures ANOVA was conducted with ‘group’ (experimental vs. control), ‘item’ (proportional vs. non-proportional) and ‘test’ (pretest vs. posttest vs. retention test) as independent variables and the performance on the word problems as the dependent variable. Based on our hypothesis, we expected a significant ‘item’×‘group’×‘test’ interaction effect. The results of the ANOVA confirmed this expectation,  $F(1,488) = 4.80, p = 0.0290$ . An overview of the percentages of correct answers is given in Table 2. Because of the significant three-way interaction effect, all pairwise differences in this table were statistically tested by means of post-hoc Tukey tests (correcting for multiple comparisons).

Table 3. Percentage correct answers (and standard deviations) of the experimental and control group on the proportional and non-proportional items at the three test moments

	Proportional items						Non-proportional items					
	Pre		Post		Retent		Pre		Post		Retent	
	%	SD	%	SD	%	SD	%	SD	%	SD	%	SD
<b>Experimental</b>	83.3	7.8	58.3	7.8	52.8	7.8	29.2	6.2	61.1	6.2	50.0	6.2
<b>Control</b>	85.3	8.1			73.5	8.1	13.2	6.4			16.1	6.4

As expected, on the pretest there was a significant difference between the performance on the proportional items (which were solved very well) and the non-proportional items (which were mainly solved incorrectly), both in the experimental group,  $t(488) = 17.56, p = 0.0001$ , and in the control group,  $t(488) = 8.48, p = 0.0001$ . Again, this is evidence for students’ overwhelming tendency to produce proportional answers in non-proportional situations. Moreover, at the pretest there was no significant difference between the experimental and control group, indicating that both groups were indeed comparable. Both groups performed almost the same on the proportional items, and the difference for the non-proportional items was also not significant<sup>ii</sup>.

We will first discuss the results of the control group. Afterwards, we will examine more closely how the performances of the experimental group evolved from pretest to retention test.

With respect to the control group, we did not expect a significant evolution from pretest to retention test. In line with this expectation, we observed a very small, non-significant increase in the performance on the non-proportional items (from 13.2% to 16.1% correct answers) and a non-significant decrease in the number of correctly answered proportional items (from 85.3% to 73.5%) from pretest to retention test. A qualitative analysis of the protocols in this group showed that, first, as expected, the percentage of answers resulting from an improper application of linearity on the non-linear items on the pretest and retention test was about 80%. Second, an increase in the number of overgeneralisations of

non-linear strategies to linear items could be observed (from about 11% to 18%). Probably as an effect of retesting, some students started to apply non-proportional solution methods to the proportional problems they solved correctly before. As will be explained below, this is similar to observations made in earlier studies (De Bock, 2002; De Bock et al., 1998, 2002).

In the experimental group, there was a significant improvement in the performances on the non-proportional items from pretest (29.2%) to posttest (61.1%),  $t(488) = 3.09$ ,  $p < 0.0001$ , followed by a non-significant decrease in the performances from posttest to retention test (to 50.0% correct answers) This means that students in the experimental group made a significant progress in their performance on the non-proportional items, and this progress persisted over several months. However, this improvement in performance was not as high as we had expected. Contrary to the results for the non-proportional items, the score of the experimental group on the proportional items decreased from 83.3% correct answers on the pretest to 58.3% on the posttest,  $t(488) = 2.62$ ,  $p < 0.0090$ , and went further down from posttest to retention test (although not significantly) to 52.8%. Apparently, in line with our earlier studies, when these students discovered that some problems can not be solved by applying proportions, they started to apply non-proportional solution schemes to proportional problems too (De Bock, 2002; De Bock et al., 1998, 2002). An additional qualitative analysis of the answers of the experimental group revealed first of all that on the pretest, about 70% of all the solutions on the non-proportional items could indeed be characterized as linear. This number of unwarranted linear answers strongly decreased in the posttest to about 18%, while in the retention test, the percentage raised again to about 30%. But students who no longer applied linear solutions to solve non-linear problems, did not always perform better than before: in the posttest and retention test they made errors in applying non-linear solutions on these non-linear problems (such as confusing area and volume, just taking the square of one of the given numbers, etc.). The qualitative analysis also confirmed the overgeneralisation effect: while on the pretest only 13% of all the solutions to linear items could be characterised as an application of non-linear strategies, this number raised to 36% on the posttest and retention test.

The results of the experimental group on the posttest and retention test revealed the fragile and unsteady nature of these students' emerging non-proportional reasoning scheme. A careful analysis of the videotapes of the experimental lessons supported these conclusions. Certain fragments indicated that non-linear relations and the effect of enlargements on area and volume remained intrinsically difficult and counterintuitive for many students. For example, there were students who at the same time understood that the area of a square increases 4 times if the sides are doubled in length (since the enlargement of the area goes "in two dimensions"), while they had difficulty in understanding why this does *not* hold for the perimeter (which also increases in two "directions", according to a student).

## CONCLUSIONS AND DISCUSSION

In general, the results of this study confirmed our hypothesis. Initially, both the experimental and the control group performed well on the proportional items but often failed on the non-proportional items, due to the application of linear methods. After the

experimental lesson series, the experimental group applied linear solution methods less often on the non-linear items on the test. Apparently, the illusion of linearity was broken in these students. However, a considerable part of the non-linear items on the posttest and retention test were still solved erroneously, due to linear reasoning or to mistakes in the application of non-linear strategies. Moreover, at the posttest and retention test, the experimental group made more errors on the proportional problems, because they started to overgeneralise the newly learnt non-proportional strategies to the proportional problems they previously solved very well. It seems that after the lessons, the students still experienced serious difficulties in knowing which model they had to use in which situation.

Therefore, we can hardly argue that the lesson series has reached its goal. The experimental lessons were unable to develop in the students a deeper understanding of proportionality and non-proportionality, and a disposition to distinguish between situations that can and cannot be modelled proportionally. The findings indicate that the experimental students' emerging non-proportional reasoning scheme remained fragile and unsteady.

A first possible reason is that our intervention involved 13-14-year olds, i.e. students at an age where the proportional reasoning scheme was already well established and practiced – explaining why non-proportional relations were experienced as counter-intuitive by some students. It seems, therefore, important that, at the first time when students meet proportional relationships in their mathematics curriculum, they are also confronted with counterexamples (situations where linearity does not work). As a second reason, it seems that an intervention of only 10 hours – that was moreover separated (and considerably different from) the regular mathematics curriculum – was not satisfactory to change the students' habits and beliefs contributing to a superficial modelling process, while these habits and beliefs are an important facilitating factor in the occurrence of improper linear reasoning (see De Bock et al., 2002).

The results on the tests and the (ongoing) analysis of the videotapes offer many valuable indications for the development of an improved version of the current learning environment, which will be implemented and tested on a larger scale in the near future.

## References

- Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing and mathematics. In L. B. Resnick (Ed.), *Knowing, learning and instruction. Essays in honor of Robert Glaser* (pp. 453–494). Hillsdale, NJ: Lawrence Erlbaum Associates.
- De Bock, D. (2002). *The illusion of linearity: An empirical analysis of secondary school students' improper proportional reasoning in geometry problems*. Unpublished doctoral dissertation, University of Leuven, Belgium.
- De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, 50(3), 311–334.
- De Bock, D., Verschaffel, L., & Janssens, D. (1998). The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics*, 35, 65–83.

- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel.
- Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: Freudenthal Institute, University of Utrecht.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Poortvliet, R., & Huygen, W. (1976). *Leven en werken van de kabouter*. [Life and work of the gnomes]. Houten: Holkema & Warendorf.
- Van Dooren, W., De Bock, D., Depaepe, F., Janssens, D., & Verschaffel, L. (2002). Secondary school students' illusion of linearity: expanding the evidence towards probabilistic reasoning. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> Conference International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 305–312). Norwich: University of East Anglia.

---

<sup>i</sup> The terms *linear* and *proportional* are here used as synonyms, referring to relations of the form  $f(x) = cx$ , for which the properties  $f(a + b) = f(a) + f(b)$  and  $f(ka) = kf(a)$  hold, graphically represented by a straight line through the origin.

<sup>iii</sup> Despite the non-significant outcome of the Tukey test, the difference in both groups' performances on the non-proportional items seemed to be meaningful. Therefore, an additional analysis of covariance (ANCOVA) was performed, predicting the performances on the non-proportional items on the retention test on the basis of the group (experimental/control) *correcting for the performances on these items on the pretest* (and thus cancelling out any differences between the groups at the pretest). The corrected means of both groups at the posttest (44.3% and 22.3% for the experimental and control group respectively) were still statistically different,  $F(1,32) = 6.42, p = 0.0164$ .