ALGEBRA IN ELEMENTARY SCHOOL[1]

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Increasing numbers of mathematics educators, policy makers, and researchers believe that algebra should become part of the elementary education curriculum. Such endorsements require careful research. This paper presents the general results of a longitudinal classroom investigation of children's thinking and representations over two and a half years, as they participate in Early Algebra activities. Results show that 3rd and 4th grade students are capable of learning and understanding elementary algebraic ideas and representations as an integral part of the early mathematics curriculum.

BACKGROUND AND EARLY RESEARCH QUESTIONS

Early research about algebraic reasoning highlighted shortcomings such as students' (1) limited interpretations of the equals-sign (Booth, 1984, 1988; Kieran, 1981, 1985; Vergnaud, 1985); (2) misconceptions about the meaning of letters standing for variables (Kieran, 1985; Kuchemann, 1981; Vergnaud, 1985); (3) refusal to accept an expression such as "3a +7" as an answer to a problem (Sfard & Linchevski, 1994); and (4) difficulty in solving equations with variables on both sides of the equals sign (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Many researchers originally attributed such findings to developmental constraints and the inherent abstractness of algebra.

Over time, however, data from innovative classroom activities began to support a new view. Davis (1985, 1989) gave convincing examples of how algebra could be introduced in 5th grade mathematics classrooms. Successful work in the former Soviet Union (Davydov, 1991, Bodanskii, 1991), with even younger children, came to the attention of researchers. Mathematics educators began to find common ground between arithmetic and algebra (Bass, 1998; Carpenter & Franke, 2001; Carpenter & Levi, 2000; Carraher, Schliemann, & Brizuela, 2000, 2001, 2003; Davis, 1985, 1989; Kaput & Blanton, 2001; Schifter, 1999; Schoenfeld, 1995; Schwartz, 1995). Approaching the introduction of algebra in the early grades from various, occasionally overlapping, perspectives (generalizing arithmetic, moving from particular to generalized numbers, focusing on mathematical structures common to sets of algorithms, introducing variables and co-variation in word problems, focusing on the concept of *function*, to tie together isolated mathematics topics, etc.), they identified previously overlooked opportunities to explore the algebraic character of early mathematics. These recent studies suggest that shortcomings of instruction may have had a decisive role in the gloomy results from early

studies of algebraic reasoning among adolescents (Booth, 1988; Schliemann & Carraher, 2002).

In this paper we describe the general results of a longitudinal study we developed in 2nd to 4th grade classrooms. We will argue that, given the proper conditions and activities, elementary school children can reason algebraically and meaningfully use the representational tools of algebra. We hope that our empirical research on Early Algebra will help make advances regarding issues that most researchers in mathematics education recognize as important, such as development versus learning, the role of contexts, and the role of representational systems.

OUR APPROACH

Our approach to the introduction of algebraic concepts and notations in elementary school was guided by the following ideas about learning:

(a) Cognitive deficits and cognitive difficulties with algebra may result from the limitations of the mathematics curriculum elementary school children have access to; (b) Mathematical understanding is an individual construction that is transformed and expanded through social interaction, experience in multiple meaningful contexts, and access to mathematical symbolic systems and tools; and (c) Children need to be socialized into the symbolic systems but also need to make them their own. To this goal, students benefit from opportunities to begin with their own intuitive representations and gradually adopt conventional representations as tools for representing and for understanding mathematical relations.

Furthermore, it rests on the following ideas about mathematics:

A. Concerning arithmetic, algebra, and their interrelations

(a) Opportunities to explore the algebraic character of elementary mathematics are present throughout existing curricula, though rarely seized upon; (b) Algebra is both a notational system and a field of mathematics devoted to the study of mathematical structures; (c) Arithmetic is a part of algebra, namely, the part that deals with number systems, the number line, numerical functions, and so on; (d) generalizing lies at the heart of algebraic reasoning; and (e) Arithmetical operations can be viewed as functions.

B. Concerning symbolic representation

(a) Mathematical concepts (algebra included) are closely identified with four *key symbolic systems* (natural language, number, geometry, and algebraic-symbolic notation); each of these has important roles to play already in early mathematics education; (b) Each of these systems has its own expressive rules and internal logic; and (c) A central problem in mathematics consists in moving back and forth between diverse representations, often across these key symbolic systems.

Our approach focuses on algebra as a generalized arithmetic of numbers and quantities. This highlights the shift from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations among variables. This requires engaging students in specially designed activities, so that they can begin to note, articulate, and represent the general patterns they see among variables.

In our longitudinal study, we worked with 70 students in four classrooms (three mainstream and one bilingual education classroom), as they learned about algebraic relations and notation, from grade 2 to 4. Students were from a multiethnic community (75% Latino) in Greater Boston. From the beginning of their 2nd semester in 2nd grade

to the end of their 4th grade, we implemented and analyzed weekly activities in their classrooms. Each semester, students participated in 6 to 8 activities, each activity lasting for about 90 minutes. The activities related to addition, subtraction, multiplication, division, fractions, ratio, proportion, and negative numbers. The project documented, in the classroom and in interviews, how the students worked with variables, functions, positive and negative numbers, algebraic notation, function tables, graphs, and equations.

In our classroom work we generally began not with a polished mathematical product, such as a number sentence or a graph, but rather with an open-ended problem or a statement such as "Maria has twice as much money as Fred." After holding an initial discussion about the situation, we ask students to express their ideas in writing. We discuss their representations and introduce a conventional representation, such as two parallel number lines drawn on the floor (on which Maria and Fred's amounts correspond to positions); the new convention is often foreshadowed in students' own drawings. We may ask them to show how the representations (their own and the conventional one) must be updated to account for new information, such as amounts changing while some properties remain invariant (the relation "twice as much" continues to hold). In the specific case of a Cartesian coordinate graph and of the problem involving Fred and Maria, we would then rotate one of the number lines by 90 degrees and ask the students to plot themselves as points at the intersections of ordered pairs for Maria and Fred's amounts. A "Human Graph" is thus built and becomes the object of discussions. We explore the relations between the mathematical representation and the problem-situation in terms of correspondences between the graph and the underlying situation. What do you notice about the way in which students/points are patterned? Why are they falling on a straight line? (When each point is expressed through coordinate notation) what do you notice about the relation between the two numbers? If I choose a position on Fred's line/axis, can you predict where the Fred-Maria point will land? If Fred has a really large amount of money can you say anything about where the Fred-Maria point will be located? (See Schliemann & Carraher, 2002; Schliemann, Goodrow, & Lara-Roth, 2001a; and our website--earlyalgebra.terc.edu--for detailed information and partial results of our Early Algebra activities).

RESULTS

Partial results of our longitudinal study have been reported in detail at PME meetings and published elsewhere. Here is a general description of our analysis of classroom activities:

> Young students (9-10 years of age) can learn to think of arithmetical operations as functions rather than merely as computations on particular numbers (Carraher, Schliemann, & Brizuela, 2003; Schliemann & Carraher, 2002; Schliemann, Carraher, & Brizuela, 2003). They can also work with mapping notation, such as n $\rightarrow$ 2n −1, and realize that the algebraic expression constitutes a rule according to which one set of input numbers maps onto another, output set (Carraher, Schliemann, & Brizuela, 2003).

> For young learners, the number line can be a meaningful tool for representing numbers and operations and to solve problems (Carraher, Brizuela, & Earnest, 2001; Carraher, Schliemann, & Brizuela, 2001; Peled & Carraher, 2004 in preparation).

> Even young students can easily learn to accept the idea of negative numbers. However, operations involving negative numbers pose special challenges when dealing with word

problems, number lines, graphs, and other contexts (Peled & Carraher, 2004 in preparation).

The idea of "difference" (corresponding initially to the expression, |a-b|, and later to (a-b) is important for appreciating the algebraic character of additive structures; yet it takes on subtle differences in meaning across the contexts of number lines, measurement, subtraction, tables, graphs and vector diagrams (Carraher, Brizuela, & Earnest, 2001).

Certain large-scale activities where children enact mathematical objects and relations, can play an important role, helping to introduce new algebraic concepts. Such activities provide meaning for children to interpret mathematical notation and solve algebraic problems (Schliemann & Carraher, 2002; Schliemann, Goodrow, & Lara-Roth, 2001a).

Children often complete function tables by treating each column as a number sequence task to be solved in a downward fashion regardless of the values contained in the other column. This led us to seek and successfully implement alternative ways to pose problems so as to highlight the functional relations across columns (Schliemann, Goodrow, & Lara-Roth, 2001b).

There is an inherent ambiguity in how letters represent quantities in algebraic expressions. Students must recognize that letters may refer to particular values or instances, but also to sets of possible values or variables. This shift in meaning has generally not been addressed in curricula. We found that, given appropriate activities, 3rd graders can grasp the meaning of variables, as opposed to instantiated values (Carraher & Schliemann, 2002; Schliemann, Carraher, & Brizuela, 2002).

Graphs of linear functions are within reach of 3rd grade students, contributing to their initial understandings of function and of multiplicative structure concepts (Schliemann & Carraher, 2002; Schliemann, Goodrow, & Lara-Roth, 2001a).

After participating in our activities, 4th grade students were able to solve algebraic problems using multiple representation systems such as tables, graphs, and written equations with variables on both sides of the equality (Brizuela, 2002; Brizuela & Schliemann, 2003).

The symbolic systems used in algebra are an inherent and important part of learning algebra; children need both access to these systems as well as opportunities to constantly represent algebraic concepts in multiple ways, both conventionally and idiosyncratically.

At the end of 4th grade, three to four weeks after our last class, the students were individually interviewed and asked to solve problems. A control group (CG) of 26 children, at the end of their 5th grade, who had been taught by the same teachers in the school, were also interviewed and compared to our experimental group (EG). Here are some preliminary results:

Question 1: Is 6 + 9 = 7 + 8 True or False?

Here we explored students' understanding of the equivalence in an equation, treating both sides symmetrically, instead of holding an "input – output" conception and thinking that they should do an operation on the left side and get a result on the right.

We found that 85% of the 4th grade students (EG) responded correctly that the equation was true while only 65% of the fifth graders (CG) did so.

Question 2: Below [A drawing of two boxes and another box plus 9 candies was shown under the question.] there are boxes of candy [each containing an unspecified amount]

and loose candies. Each box has the same number of candies. Which would you rather have? Two boxes of candy? Or one box of candy and 9 loose candies? Why?

Here we explored whether the students could relate and operate on an unknown quantity. Could they generalize and integrate their part-whole schema with a comparison schema in a situation with two piles, each composed of different parts and involving unknown amounts? Could they handle situations in which there was no "one answer", but rather a set of solutions, or a variable that depends on the changing value of an amount in the situation (the independent variable)?

We found that a larger proportion of children in the EG (44%) could handle situations with unknown quantities, answering that their choice would depend on the number of candies inside each box, while 31% of the children in the CG did so.

Question 3a: The children were asked to complete a function table representing the following problem: Mary has three times as much money as John. Column 1 was labeled 'John' and column 2 was labeled 'Mary'. After completing the table, we continued: If we don't know how much money John has, we can say that he has N dollars. If John has N dollars, what could you write to say how much money Mary would have?

How would the students fill up the function table? Would they continue focusing on isolated columns as was the case in 2nd grade? Or would they focus on the functional relationship between the two variables?

We found that 65% of the EG filled the table working with functional relations and 50% of the CG did so. To represent that Mary had three times as much money as John, 70% of the EG accepted to represent John's amount of money as N and Mary's amount as John's amount times three. In the control group, only 29% of the children did so.

Question 3b: Which of the graphs [three linear function graphs were shown as possible answers] shows that Mary has three times as much as John? How do you know you chose the correct graph?

Would the children be able to identify a specific linear function relationship in a graph. And if so, how would they justify their choice? Would they focus on isolated points or would they identify the general properties of the function depicted in the graph?

Here, 78% of the EG chose the correct line while only 46% of the CG did so. Note that the CG, had also received instruction on drawing graphs by their regular teachers. Of the children choosing the correct line, 39% in the EG provided general justifications that took into account any possible pair of numbers (e.g., "Because when you times John's money by 3 it tells Mary's number of money" Or "Because if I would times 3 all the bottom numbers it would be on that line."). In the CG 25% adopted this general approach.

Question 4: In the last part of the interview, children were asked to represent in writing and to solve the following problem: "Harold has some money. Sally has four times as much money as Harold. Harold earns $18.00 more dollars. Now he has the same amount as Sally. Can you figure out how much money Harold has altogether? What about Sally?" Each step in the problem was presented gradually.

In this problem, we wanted to see if children could accept to work with an unknown amount, how they would represent the unknown amount, and whether they would use equations and the syntax of algebra to find a solution to the problem.

Of the 63 EG children who were interviewed, 56% represented Harold's initial amount as N, X, or H and 49% represented Sally's amount as Nx4. For Harold's amount after earning 18 more dollars, 35% of the children wrote N + 18. 17% of the children wrote the full equation N + 18 = N x 4 and 27% of the children correctly solved the problem. However, only 6% (four children) systematically used the algebra method to simplify the equation. Two children, when prompted, correctly explained the algebra method. Apparently, as the children worked in their written representations, they easily inferred that Harold's starting amount was 6, without the need to use the algebra method. As Albert stated, "I thought about six because it just popped in my head." In the CG, 23% of the children solved the problem but no one wrote an equation or used algebra methods.

DISCUSSION

During the last few years, we have made certain strides forward in expanding students' mathematical reasoning and in helping them develop and use algebra notations and tools to solve problems. However, we did not explore the limits of children's capabilities regarding algebra. As we focused on discrete quantities, linear functions, and graph spaces in Quadrant I, we may have underestimated children's potential to learn algebra.

The issue of sustainability of learning is also still open. In our research we worked with the students in their classrooms, but met with them for only six to eight times per school term. We believe, and this is what we want to test in our next study, that much more can be achieved if children participate in early algebra activities on a daily basis, as part of their regular curriculum.

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