RECOGNISING EQUIVALENT ALGEBRAIC EXPRESSIONS: AN IMPORTANT COMPONENT OF ALGEBRAIC EXPECTATION FOR WORKING WITH CAS

Lynda Ball*, Robyn Pierce† and Kaye Stacey*

*University of Melbourne and †University of Ballarat

The ability to recognize equivalent algebraic expressions quickly and confidently is important for doing mathematics in an intelligent partnership with computer algebra. It is also a key aspect of algebraic expectation, the algebraic skill that parallels numeric estimation. The progress in this ability of 50 students learning mathematics with CAS over two school years was monitored using a novel instrument called the Algebraic Expectation Quiz. Students began with very low facility and confidence and made moderate progress, confirming the importance of addressing this obstacle to using CAS explicitly in the curriculum. We offer conclusions about test items, students' strengths and weaknesses and suggest possible teaching actions. The results demonstrate that recognizing equivalence, even in simple cases, is a significant obstacle for students.

INTRODUCTION

The purpose of this paper is to report on students' ability to recognize the equivalence of algebraic expressions and their confidence in their judgments, obtained from a novel instrument called the Algebraic Expectation Quiz. We report the results of using this instrument with 50 students over 21 months, during their final two years of secondary schooling learning mathematics with CAS, drawing conclusions about test items, students' strengths and weaknesses and finally suggesting possible teaching actions. The results demonstrate that recognizing equivalence is a more significant obstacle for students than we and their teachers had expected and that students lack confidence.

The work in this paper is motivated by the need to develop and monitor students' algebraic thinking when computer algebra systems (CAS) are available for doing, teaching and learning mathematics. It has been carried out as part of a project (see CAS-CAT Research Project website) where students at three secondary schools studied a new tertiary-preparation mathematics subject for which CAS calculators (graphics calculators that additionally have a symbolic manipulation facility) were available at all times, including for examinations. Students learning and doing mathematics using CAS will not need the same mastery of algebraic routines as do other students. They will be able to use CAS where other students need a by-hand technique. However to use CAS effectively and to attain other educational goals, students will still require conceptual understanding and technical facility with algebra, the exact nature of which is a topic of debate and concern in the literature (see, for example, Drijvers (2000); Herget, Heugl, Kutzler & Lehmann (2000); Pierce (2002)). Artigue (2001) suggests that there is a complex dialectic between conceptual and technical work in algebra. Passing technical work over to CAS means that it still needs attention, although possibly different attention, in the curriculum. Controlled use of technology (such as CAS) almost certainly requires specific competencies that are not covered by the standard curriculum.

Equivalent forms and Algebraic Expectation

The ability to recognise equivalent forms of algebraic expressions, which is the focus of this paper, has been identified as a central part of working with CAS, and one that is likely to take on new importance in future curricula. Many researchers, for example Heid (1989), Lagrange (1999) and Drijvers (2000), comment on the importance of this ability for a CAS user. Firstly, when students are entering expressions with complex syntax into a CAS they need to be able to recognise whether the expression entered is equivalent to the one they are working from on paper. For example, in order to enter into a CAS the expression sin $2x$, which requires no brackets in by-hand mathematics, students need to appreciate that brackets are required in (most) CAS and then that sin $(2x)$ is not the same as $(\sin 2)x$. To enter $(a + p)/q$ (possibly written with a vinculum instead of brackets), students must appreciate that this is not the same as $a + p/q$. Secondly, students need to be able to convert output into a standard form since CAS does not always present results in the manner which is conventional in a given educational setting. The roots of the quadratic equation x^2-10x-16 = 0, for example may be given as $\sqrt{41} + 5$ and $-(\sqrt{41} - 5)$, whereas teachers in our local schools would almost invariably give these as $5 + \sqrt{41}$ and $5 - \sqrt{41}$. Thirdly, students need knowledge of equivalent expressions to be able to deal with the automatic simplification feature of CAS. This feature means that an input is often processed to an equivalent form immediately on entry. When an expression such as $2f - g + 3f - g$ is entered into a CAS such as the TI-89, the input is automatically simplified and appears as $5f - 2g$, the input $12x/6x$ appears as 2, $(b + a)^2$ is reordered alphabetically and appears as $(a + b)^2$ and $6 + (4a + 2b)/2$ appears as $2a + b + 6$. Lagrange (1999) reported that students reflected deeply on events like these, when teachers wanted them to consider deeper phenomena. Effective CAS users need to be able to quickly recognize that the inputs have been made correctly, even though the form appearing on the screen is different. Fourthly, on-going monitoring of the progress of a calculation requires a general alertness to the appropriateness of intermediate results and quick, confident decisions. At all stages of their work, CAS users need to recognise simple equivalences quickly and confidently, so that they are not derailed by the trivial but are alerted to the significant unexpected. Guin and Trouche (1999) noted that a surprising CAS result did not necessarily induce questioning in students who could not compare expressions.

We see the ability to recognize equivalent expressions as one indicator of a more general ability of *algebraic expectation* (Pierce, 2002; Pierce & Stacey, 2002). An important component of the algebra needed when CAS is used is ‘symbol sense’, rather analogous to the ‘number sense’ that is needed for doing arithmetic with a four function calculator. Arcarvi (1994) outlines what ‘symbol sense’ might be and how it is involved in all aspects of solving problems with algebra, including formulating problems algebraically, technical work within the mathematical world and interpreting algebraic answers in a real context. This describes the understanding of algebra required for working in partnership with technology. Within this overarching symbol sense, Pierce and Stacey (2002) have identified *algebraic expectation* as the algebraic parallel to numerical estimation. It is the thinking process that takes place when a mathematician considers the nature of the symbolic result expected as the outcome of some algebraic process. Its essence is making a quick assessment of the expected characteristics of algebraic results. Pierce and Stacey

(2002) define and analyse algebraic expectation in terms of several different elements: knowing conventions and basic properties of operations, and being able to identify the structure and key features of algebraic expressions. Algebraic expectation is the general ability needed to monitor the succession of expressions appearing on a CAS screen, making on-going rough checks for mathematical sense.

Because of the centrality of the ability to quickly recognize algebraic equivalence to the more general ability of algebraic expectation, we call the instrument described below the Algebraic Expectation Quiz. However, adequate monitoring of algebraic expectation requires other assessments, so in the CAS-CAT project (CAS-CAT Research Project) the Algebraic Expectation Quiz was supplemented by two tests that are not reported here: a 6-item Constructed Response Test and a separate test of both multiple choice and constructed response items assessing the ability to link representations.

THE ALGEBRAIC EXPECTATION QUIZ

Design of instrument

Outside the CAS experience, one of the daily experiences students have of needing to recognise equivalence of algebraic expressions is in checking their work from the back of a textbook. Very often, a student's answer is not in the same form as the answer in the book and so the student must decide whether their answer is correct or not. This situation was used to supply a realistic context for the Algebraic Expectation Quiz. Students were told that they would be presented with a series of slides, each showing two expressions. One expression was the answer given by a mythical student (Julie) to a problem and the other was the textbook answer. For example, the first item (item 1 of Table 1) presents on one slide $x \div y$ (Julie's answer) and $y \div x$ (the textbook answer). The students' task was to decide if Julie was definitely wrong, probably wrong, probably right or definitely right. Students unable to decide could choose the option 'no idea'. This mechanism enables a measure of confidence in the decision to be obtained simultaneously with accuracy. To reduce confusion of right/wrong and correct/incorrect in reporting results, we will refer to the items where Julie's answer matched the textbook answer as true items and the other items as false.

The Algebraic Expectation Quiz tries to capture the essence of making quick, real-time decisions, mirroring the way in which algebraic expectation provides constant and almost unconscious on-going monitoring of algebraic results. Hence it was important to present items with a restricted time to respond. To do this, the test was designed to be administered using Microsoft PowerPoint with a fixed time of 10 seconds for each item (time based on prior trialing by Pierce (2002)). Because of the need for unbroken concentration, the test time cannot exceed about 5 minutes, so that approximately 25 items can be used.

Items were selected based on previous research on learning algebra or using CAS (see for example Drijvers, 2000; Kieran,1992; MacGregor and Stacey,1997)), Pierce's quiz (2002) and the researchers' experience of working with students at this level. To confirm appropriateness and validity, items were reviewed by experienced mathematics teachers, who were asked whether they expected that students in Years 11 and 12 could quickly establish equivalence (or otherwise) of the expressions without any written steps. Whether or not a student can be said to have "good algebraic expectation" depends on their age and stage, so this heavily influences the creation of items. With few exceptions, we chose items from curriculum topics in Year 10 and below, using the four basic operations and square root. Informative items will canvas topics that a teacher assumes students already confidently know as well as what they need to know for their coming studies. Links between the items and skills required for using CAS can be seen by comparing the items with the examples used in the Introduction to this paper.

Table 1 lists 22 items, ordered according to item facility at the first test time. All items involve knowledge of conventions and basic properties of operations (for example that addition is commutative, the square root of a sum is not the sum of square roots), but this is not always the source of challenge for an item. Analysis of the first results of administering the Algebraic Expectation Quiz (Ball, Pierce & Stacey, 2001) showed that the most effective items required at most two intermediate steps which were simple enough to be carried out mentally. For example, Item 12 requires canceling the negatives and splitting the fraction. More complicated items were discarded.

Data collection

Along with other assessments, the Algebraic Expectation Quiz was administered to students in three project schools four times during the 2001 and 2002 school years to monitor their developing algebraic knowledge and skills. The complete cohort contained students learning with CAS and without it. However, the results in this paper are from the fifty students who completed the test in both February 2001 (beginning of Year 11) and September 2002 (just before the end of Year 12 and their final external examinations) and who undertook the new mathematics subject that permitted CAS. The results presented in this paper are for the 22 items common to the February 2001 (Feb 01) and September 2002 (Sep 02) administrations of the quiz. Information about the three project schools indicates that the 50 students may have academic ability better than the state average.

The quiz was administered by two of the present authors and one other researcher. They began by outlining the purpose of the quiz, explaining that each slide would only be shown for ten seconds and that there would not be enough time to do any written working but that students were expected to make a quick judgment based on their mathematical experience. Two sample items were demonstrated before the timed sequence began and they were used to stress that the students needed to judge the algebraic equivalence of the expressions rather than nicety of form. It was also explained that the intention was not to look for special cases, but to consider general expressions. For example, $x \div y$ is indeed equal to $y \div x$ when $x = \pm y$, but not in general, so Julie's answer in Figure 1 is wrong (i.e. Item 1 is a false item). The Algebraic Expectation Quiz took less than 5 minutes followed by other assessments not reported here. Students completed the Algebraic Expectation Quiz without use of pen and paper or technology except to record answers.

RESULTS

For each item Table 1 reports (i) the *facility* (% of students answering correctly i.e. *definitely right* or *probably right* on true items, and *definitely wrong* or *probably wrong* on false items) and (ii) percentage of students with well-placed certainty (both certain and correct i.e. answering *definitely wrong* for a false item or *definitely right* for a true item) and (iii) misplaced certainty (certain and incorrect i.e. *definitely wrong* for true or *definitely right* for false) in Feb 01 and Sep 02.

Item no	Item	Item Facility Feb 01	Item Facility Sep 02	Well-placed certainty Feb 01	Well-placed certainty Sep 02	Misplaced certainty Feb 01	Misplaced certainty Sep 02
21	$\frac{2a^3+5}{a^2} \quad 2a+\frac{5}{a^2}$	10	40	4	36	38	16
12	$\frac{s}{t}+\frac{p}{t} \quad \frac{\square(s+p)}{\square t}$	16	36	6	22	30	12
16	$\frac{4+b}{4} \quad 1+\frac{b}{4}$	16	56	8	32	32	12
20	$\sqrt{2}+\sqrt{y} \quad \sqrt{2+y}$	18	34	10	22	42	44
11	$a+p \div q \quad \frac{a+p}{q}$	24	52	18	36	58	34
4	$\frac{12x}{6x} \quad 2x$	24	58	20	42	54	30
7	$\sqrt{16x \square 4y} \quad 2\sqrt{4x \square y}$	26	48	14	24	26	16
3	$\frac{2+x}{y+2} \quad \frac{x}{y}$	28	40	20	20	32	42
15	$2x^2 \square y^2 \quad (2x \square y)(2x+y)$	28	82	18	42	38	10
17	$6+(4a+2b) \div 2 \quad 6+2a+b$	34	28	16	14	12	8
10	$\frac{1}{3}+\frac{1}{y} \quad \frac{2}{3+y}$	34	62	14	36	28	10
9	$\frac{a \square b}{d \square c} \quad \frac{a}{d} \square \frac{b}{c}$	38	38	34	26	16	26
14	$sin(2x) \quad (sin\,2)x$	54	64	24	36	8	6
8	$h=x^2 \quad x=\pm\sqrt{h}$	54	90	28	76	12	2
5	$\frac{12x}{6x} \quad 2$	56	70	36	58	34	20
23	$(b+a)^2 \quad a^2+b^2+2ab$	58	94	42	78	28	2
2	$n^2(n^3+1) \quad n^5+n^2$	60	88	32	78	32	4
1	$x \div y \quad y \div x$	72	96	52	92	22	0
24	$\sqrt{xy} \quad \sqrt{x+y}$	82	96	66	92	6	2
13	$2f \square g+3f \square g \quad 5f \square 2g$	88	94	64	72	4	2
22	$\square 2y+6 \quad \square 2(y \square 3)$	92	94	78	82	6	0
6	$5m \quad m^5$	92	100	78	94	6	0
	Average over items	46	66	31	51	26	14

Key			
Key	<40%	40-75%	>75%

Table 1: Algebraic Expectation Quiz results Feb 2001 and Sept 2002 (N=50)

The facilities in Table 1 show that the items covered a wide range of difficulty. For example, Item 6 (the comparison of $5m$ with m^5) is a false item, and in Feb 01, 92% of students answered correctly *definitely wrong* or *probably wrong*. By Sep 02, this had increased to 100% suggesting that students had good algebraic expectation for items of this type for the duration of the study. The table also indicates that most of the students were certain of their correct result (78% in Feb 01, 94% in Sep 02). If an erroneous answer of this type was obtained from CAS work, we would expect students to identify it. Notational items (items 6 and 24) were well done as was linear factoring (item 22) and collecting like terms (item 13). Simple reorganizations such as the slightly unusual presentation of $(a+b)^2$ (item 23) were unexpectedly difficult although there was a significant improvement in both facility (Wilcoxon signed ranks test comparing median item scores for Feb 01 and Sep 02: W=228.5, p=0.000) and well-placed certainty (W=226.5, p=0.000) in Sep 02. Almost all fraction items (for example, items 12 and 21) had very low facility on both occasions. The relatively high percentage of students with misplaced certainty shows that many students remain unaware of their errors. On average, however, the facility of items improved by 20% over the two years, as did the well-placed certainty, whilst happily the misplaced certainty decreased (W=21, p=0.001).

Figure 1 shows that certainty (total of well-placed and misplaced certainty) and item facility are moderately related (correlation Feb 01=0.60, Sep 02 =0.76). The scattergrams show that items with highest facility in Sep 02 generally had high certainty, whereas in Feb 01 there was lower correlation. There are many interesting individual items. For example, in Sep 02 confidence for item 15 remains low, despite high facility. On this item, in Feb 01 students displayed misplaced certainty; in Sep 02 more are correct but they are unsure of their response. Items 4, 11 and 20 have low success and high certainty. In Feb 01, about half of the students showed misplaced certainty, and still about one third in Sep 02. These items correspond to persistent misconceptions (see MacGregor et al, 1997, for example). We conclude that many students are unlikely to identify errors of this nature in their work. In contrast, the three most difficult items (12, 16 and 21) have low facility, but moderate certainty. By Sep 02, only 13% showed misplaced certainty on these items. Although they were not sure of the correct answers, most students had at least become aware that these items represented areas where care is needed.

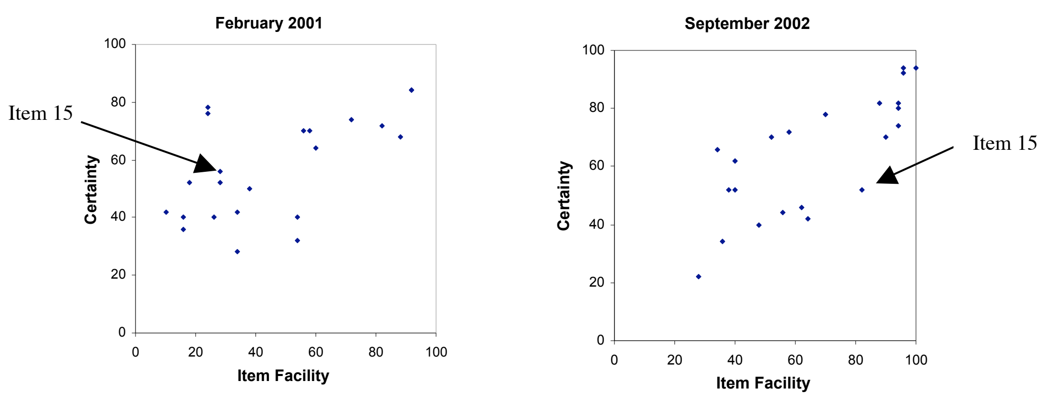

Figure 1: Scattergrams of Certainty versus Item Facility for Feb 01 and Sep 02

DISCUSSION

This discussion of the usefulness of the Algebraic Expectation Quiz as an instrument, information about students' algebraic expectation, and implications for teaching, is based on the results, together with feedback from the researchers who administered the quiz.

The Algebraic Expectation Quiz format was very successful. Students understood the scenario of comparing student and textbook answers and enjoyed the fun, fast Powerpoint presentation. The opportunity to indicate certainty reduced pressure on students and it provided deeper insight into their algebraic thinking. Since the quiz is quick to administer and easy to assess, the format is likely to be useful. A disadvantage is that the quiz format requires continuous concentration and therefore the number of items is limited.

The quiz results provide a partial but useful indicator of students' algebraic expectation, specifically their ability to recognize equivalent algebraic form. Overall, the results demonstrated both low facility and low levels of confidence. Improvement over two years was moderate, given that algebra becomes more important in the curriculum during years 11 and 12. Improvement in well-placed certainty was also moderate, although accompanied by a pleasing reduction in misplaced certainty. It is likely that students were increasingly aware of areas where they needed to be careful. The identification of areas of misplaced certainty (such as items 21, 12, 20, 3) provides information to teachers about firmly entrenched student misconceptions. Lack of certainty may also lead to excessive reliance on the machine: we have sometimes observed students who insist on copying machine output absolutely precisely, including factors of 1 or $ln(e)$, for example.

Overall, the low facility and low confidence demonstrate that students may benefit from the support of CAS in more situations than teachers expect. At the same time, it is likely that they will find CAS harder to use than might be expected, because of the difficulties outlined above related to identifying equivalent algebraic forms. This is especially so, given that participating schools were of above average academic attainment.

As noted, to develop controlled use of technology, it is likely that the standard curriculum will need to change. The results show that giving more emphasis to algebraic expectation and considering a wider range of algebraic forms is a vital direction for change. Targeting items to known CAS peculiarities may be useful e.g. changing the notation of the hand-written vinculum for division to the one line calculator format using a division sign or negative power. Pierce, who studied tertiary students learning with CAS extensively and documented their progress (Pierce, 2002) suggests that teachers discussing expressions may routinely ask about structure and key features. This can take little time, and become a routine for teacher and students, which develops algebraic expectation. Lagrange (1999) and Guin and Trouche (1999) have also been concerned to develop strategies.

Managing the tensions created by differences in algebraic form between by-hand and by-CAS algebra was always a concern for teachers working in the CAS-CAT project. In an interview at the end of the second year of teaching with CAS Lucy, one teacher in the CAS-CAT project, commented that her students were increasingly better able to deal with differences in form and she identified a teaching strategy developed to help them:

> "I'm finding increasingly that when the kids have done it by-hand and then they use the CAS, they're actually better prepared for the differences in form. And I notice one other thing.

When I'm doing symbolic procedures more often at the end now, I'll write three or four answers and then I'll ask the kids, 'Which one did the calculator give you?' . . . Sometimes the way that it will express something that has surds or has fractions is not consistent [with by-hand conventions] so … you have to be able to be confident to say that's just the same thing."

As teachers like Lucy experiment with teaching with CAS, experience of how to develop better algebraic expectation will grow. Algebraic knowledge is required to identify equivalent forms (or non-equivalent forms if errors are made) quickly, with well placed certainty, for on-going monitoring. The data presented in this paper has demonstrated that students may have greater needs in this area than have been appreciated.

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