YOUNG CHILDREN'S UNDERSTANDING OF EQUALS: A LONGITUDINAL STUDY

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This paper examines the change in young children's understanding of equals as equivalence over a three-year period and their ability to express this understanding in real world problems. Seventy-six children participated in the longitudinal study. The results indicate that approximately one third of the sample had an understanding of equals as equivalence and the difficulties and misunderstandings they experienced in Year 3 persisted through to the end of Year 5. Many of those who appeared to have an understanding of equivalence could not express this understanding as real world problems.

INTRODUCTION

In spite of the wealth of research in the algebraic domain, including student learning, teaching strategies, and the use of technology, it still remains a barrier for participation in high levels of mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Many students at both the high school and tertiary level continue to experience difficulties. In an answer to this continuing problem current research has turned to the elementary years with a particular focus on arithmetic as a key access to algebra (Carpenter & Levi, 2000; Carraher, Schielmann, & Brizuela, 2001; Kaput & Blanton, 2001; Warren & Cooper, 2001). It is believed that one of the most pressing factors for algebraic reform is to develop in elementary students the arithmetic underpinnings of algebra (Warren & Cooper, 2001), and to extend these to the beginnings of algebraic reasoning (Carpenter & Franke, 2001). The aim of this reform is to allow elementary school students access to powerful schemes of thinking about mathematics (Carpenter & Franke, 2001) that assist them in participating in algebra at later grades. The arithmetic underpinnings of algebra include understanding of operations, arithmetic properties and equals, the focus of this paper (Boulton-Lewis, Cooper, Atweh, Pillay & Wills, 1998).

EQUALS AND EQUIVALENCE

Freundenthal (1983) delineated a number of roles for the equal sign. These included: (a) indicating a task or a question (e.g., 3+4 = ?: 3+? = 7). In this instance the = sign suggests that an answer needs to be found; (b) representing equivalent situations (quantitative sameness), the symmetric quality of the equal sign where the left and right of the sign mean the same thing (e.g., 18 ÷ 3=24 ÷ 4); (c) stating something is true for all values (e.g., a+b=b+a) and, (d) introducing a new variable (e.g., a+b=c). In the elementary school the primary focus seems to be on the first of these roles as many children interpret equals as a sign to do something (Behr, Elwanger & Nicols, 1980; Carpenter & Levi, 2001). Most children do not have an understanding of equals as representing quantitative sameness (Carpenter & Levi, 2001), or stating something is true for all values, in particular recognising examples of the commutative law as being true. Warren (2001) found that when presented with examples such as 2+3=3+2 a significant number of children stated that *it is not true because the equal sign is meant to go last and the plus first,* or offered 2+3=5+2=7. The second instance is an example of linearity of thinking where the children simply work from left to right (Saenz-Ludlow & Walgamuth, 1998). It was conjectured that for these children the role of the equal sign as indicating a question and an answer needed to be found, was so strong that it interfered with other understandings of the equal sign. Past research with young children has also tended to focus on the symbolic representations of equivalent situations (e.g., 4+5=6+3). Few studies have explored young children's ability to apply this understanding to real world contexts.

The dichotomy between representing mathematical procedure or applying the knowledge of arithmetic is referred to as contrasting solving by purely formal processes according to formal rules with the application of 'real knowledge' to solve real world problems. The first is characterised as task and performance, where the child is given a task and is simply asked to perform. In this instance the 'fixed words are well-shaped utterances of arithmetical language, …. and are simply automatic linguistic utterances" (Freundenthal, 1983, p464). For example, in the case of 7-4 saying '"takeaway four from seven" is an instance of using arithmetic language to echo the process. This is seen as the most primitive relation. The second requires recognising the underlying structure of the relationships between the quantities (MacGregor & Stacey, 1998), and applying this understanding to create problem situations, changing between the symbolic register to natural language register. Duval (2002) referred to this as a mathematical transformation involving a conversion, the most difficult transformation in mathematics. In this instance language is not simply the formal language of mathematics but entails the coordination of natural language with correct mathematical language, a difficult process. It is suggested in the literature that incorporation of 'real world' language adds to the difficulty of the task. Impacting on this complexity is the belief held by Pririe and Martin (1997) that language only serves problems where the equal sign appears just before the answer, thus reinforcing the equals indicating an action.

The aim of paper is not only to examine how young children's understanding of equals as equivalence changes over a three year period but also to ascertain whether their understanding simply represents a mathematical procedure or can they express this understanding in real world contexts.

METHODS

Sample

The sample was comprised of 76 children from four elementary schools in low to medium socio-economic areas in Australia. The children were all participants in a three-year longitudinal study investigating early literacy and numeracy development. By the conclusion of the study the children had completed Year 3, Year 4, and Year 5 of their elementary schooling. The average age of the sample at the beginning of the study was 8 years and 6 months and at the conclusion of the study was 10 years and 6 months. Prior to commencing the study, all had completed the first three years of formal education.

Instruments

At the end of each year all children completed a written test. After completion of the written test, each child was also interviewed. These interviews served to illuminate the responses on the written tests. A number of items were also common across the three years.

At the end of year 3 children were asked in the interview if the number sentences on the following cards were true or not true and to explain why they were true or not true.

2 + 3 = 3 + 2	**31 + 16 = 16 + 31**
2 - 3 = 3 – 2	**31 - 16 = 16 - 31**

Figure 1 Cards used for Task 2

The results of this activity indicated that children's understanding of "=" impacted on the responses given. Many of the responses in category 3 stated that *2+3 doesn't equal 3* and offered either 2+3=5 or 2+3=5+2=7, confirming the claim that many children in elementary grades generally think that the equal sign means that they should carry out the calculation that proceeds it and the number following the equal sign is the answer to the calculation (Warren 2001).

In order to further probe this misconception, the following task was developed.

Find the missing number : 7 + 8 = □ + 9

Figure 2 Written question for the Year 4 and Year 5 test.

At the end of Year 4 and Year 5 the children were asked to solve the above question on a written test. In the interview that followed the completion of the written test, they were given their responses and asked to explain how they obtained their answer and to give, where possible, a word story for 7 + 8 = □ + 9. They were all familiar with 'word stories' as this is a significant component of the Queensland syllabus. The Year 4 results indicated that most could provide a story for the problem □ - 15 = 41. A typical response to this problem was *I had some lollies and gave away 15. Now I have 41 left.*

RESULTS

The results for this task fell into four broad categories. The categories and frequency of responses for each are presented in Table 1.

Table 1 Frequency of responses for 7 + 8 = □ + 9

Interpretations for equals	Year 4	Year 5
1. Correctly identified the unknown (□ =6)	20 (26%)	28 (37%)
2. Interpreted equals as indicating the answer (□ = 15)	23 (30%)	20 (26%)
3. Applied linear thinking to the number sentence (□ = 24)	19 (25%)	18 (24%)
4. Other	14 (19%)	10 (13%)

An analysis of the data using a Chi Squared ($\chi^2_3 = 2.23$) test indicated that there was no significant difference between the responses across the two years. Over half the children persisted in seeing the role of = as indicating an action, in finding the answer. They either simply focused on the 7 + 8 = $\square$ component of the question, providing 15 as the unknown, or calculated 7+8=15+9=24, exhibiting linearity of thinking. This misconception persisted throughout Year 4 and Year 5, indicating a certain robustness to this understanding over an extended period of time. In most instances, where children were capable of provide a word story to illustrate the problem, these stories reflected their thinking. For example, for those who gave the answer 15 tended to give word stories such as *I had 7 lollies and someone gave me 8 more. How many do I have altogether?* The following section examines the responses of the students who correctly identified the unknown (Category 1).

Correct responses

Thirteen of the 20 children who correctly responded to this question in Year 4 also provided correct responses in Year 5. Of the other 7, in year 5 two believed that it should be 15 and two responded with 24, indicating that they had changed their perception of equal to one of indicating the answer. The other three respectively responded with 7, 8 and 13.

All of the children when in Year 4 could describe the procedure they used to find the unknown, procedural mathematical language. An example of this is utterances such as "7 plus 8 is 15 and 9 plus something is 15 so it must be 6". One during this process actually changed his mind and now believed that he was wrong on the test and the answer should be 24. Eleven could not provide an everyday story using natural language for the equation (No story). Of the remaining nine, one provided an inappropriate story using all the numbers (e.g., *I had 8 apples on a tree and 7 more grew that made 15 and then 9 died*) (Inappropriate story). Four gave a story for each side of equation (e.g., *I had 7 lollies. Mark gave me 8 and my friend had 6 lollies and Mark gave him 9)* (Story for each side). Two others, when telling their stories insisted on finding the answer for the whole equations, indicating a tendency towards linearity of thinking (e.g., *There were 7 monkeys in a tree and 8 more came and then there were 15 there. Then another 6 monkeys came and another 9 monkeys came and altogether there were 30 monkeys in the tree)* (Story for each side with closure). Only one child gave a story that represented the problem (Appropriate story). James said *I had 7 lollies and then I got 8 and that equalled 15 and then my best friend got 6 lollies and then he got 9 lollies and he got the same as me*. In nearly all instances the children needed to calculate the unknown ($\square$ = 6) before they could provide a story.

Similar trends existed in the Year 5 results. Table 2 presents the frequency of responses for each of the above trends for Year 4 and Year 5.

The results indicate that as children moved from Year 4 to Year 5 many more found the correct answer for the unknown and many more could also give appropriate stories for the equality. One of the Year 5 stories started with the two groups of 15 and then broke them into their components.

Table 2 Frequency of responses for word stories

Types of Stories	Year 4	Year 5

1. No Story	12	11
2. Inappropriate Story	1	5
3. Story for each side of the equation only	4	1
4. Story for each side with closure (e.g., *that makes 30)*	2	2
5. Appropriate story	1	10
Total	20	29

He said,

> There was two lots of 15 animals in two paddocks and the farmer wanted to split them up but he couldn't use the same amount 'cause he wanted to split them up into four separate paddocks but he couldn't do that because – well he could but he had to have different numbers because they were all small. Well one was small and then it goes a bit bigger, medium, like that. And they had to be at least one or two apart the numbers, or three. So what ones would he have? 7 plus 8 or 6 plus 9.

With regard to the 13 children who correctly responded in Year 4 and Year 5, 7 were incapable of providing any story for the problem in their Year 4 interview. Of these 7, by the end of year 5, 4 could still not provide an appropriate story of the equality. The remaining three attempted to provide a story. Of these three, one provided a story for each side of the equation and two provided an appropriate story. Of the remaining 6 children who correctly identified the unknown in Year 4, 4 provided an appropriate story in Year 5 and the remaining 2 provided inappropriate stories. The following figure summarises how their stories changed over the intervening period.

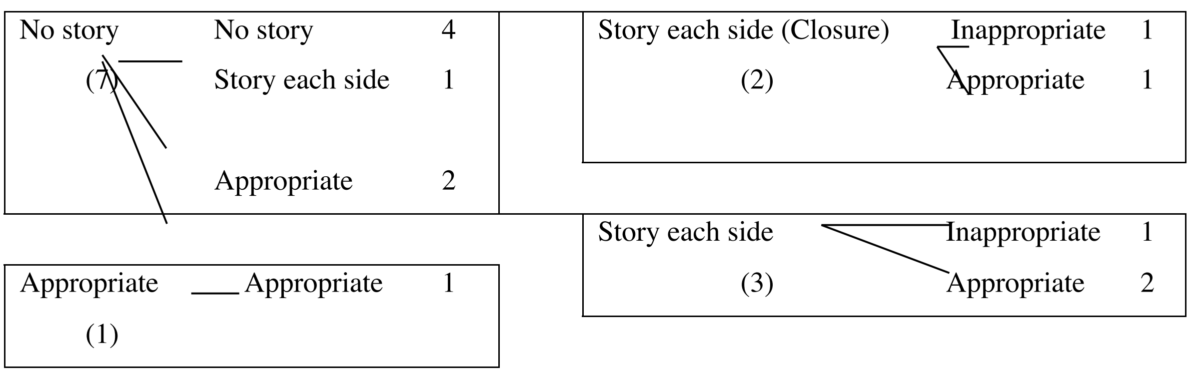

Figure 1 Frequency of thirteen children's story from Year 4 to Year 5

In both instances, the stories considered as inappropriate were such that the answer to them was the unknown (6). For example, Jan said *there were 15 seats in the circus and 9 were taken how many seats were left*. From the above results it seems that children can arrive at the ability to tell appropriate stories from different pathways. Even though the categories delineated in Table 2 suggest levels of development (from no story to story for each side to appropriate story) it seems that children can simply go from no story to appropriate story without passing through the other levels of story telling. Interestingly none of these thirteen children provided stories with closure, and the child who could tell an appropriate story in Year 4 provided an appropriate story in Year 5.

DISCUSSION AND CONCLUSION

First, from the responses given in Table 1, it can be seen that many reflect common misconceptions identified in the literature (e.g., equal as indicating the answer and computing from left to right to find an answer). This research adds to the literature in that it seems that once these misconceptions are formed they seem to remain fairly stable. This could reflect the types of problems elementary children are commonly presented with where the answer occurs after the equal sign. Further research needs to occur to ascertain (i) the robustness of these misconceptions impact on understanding equal as equivalence in the later years, and (ii) the types of activities that would assist young children to challenge these misconceptions.

Second, many of the children were capable of language utterances that reflected the processes they used to find the answer. Even when they were incorrect their utterances matched their misconceptions. The majority found difficulties in expressing the symbolic representation in a real world language context. Although in many instances their real world representations mirrored their misconceptions. For example, with the answer 15 some children gave examples such as, *I had 7 pencils and my friend gave me 8 more. How many do I have altogether [15].* As young children are negotiating mathematical understanding what roles do mathematical utterances and real world language play in the negotiation of meaning? How is their interpretations of symbols influenced by their ability to represent the symbols in real world terms. In other words, do they interpret the above expression as 7+8=15 simply because that is the expression they can express in real world language. Duval (2002) suggests that mathematical utterances of the mathematical processes entail staying in the one mathematical register. This is considered as the easiest mathematical activity, and thus is commonly the activity that occurs in many classrooms. On the other hand expressing symbols in real world language entails changing mathematical registers. Duval (2002) refers to this as a conversion, one of the most difficult mathematical activities. It involves a mapping from one register to another as compared to a mapping within the same register. Comprehension in mathematics commonly involves the coordination of at least two registers of semiotic representations. The results of this research support Duval's (2002) claim that such coordination of registers does not come naturally, but it is in this coordination that mathematical thinking occurs.

Third, while there is no evidence that supports a sequence in growth of ability to use language to provide real world stories, it seems that a significant number of children (10) could pose relevant word problems for this context by the end of Year 5. Also, many of those who were unsuccessful on the task provided language problems that mirrored their responses. In most of these instances the problems provided were structured such that the answer appeared just after the equal sign, the type of word problems commonly privileged in many classrooms (Pririe and Martin, 1997). Further research needs to be carried out in order to explore the role of 'real world' language on the interpretation of symbols, does it indeed favour certain interpretations.

This research indicates that some young children are not only capable of correctly interpreting equivalent situations but also can recognise the underlying structure and express this in an appropriate real world context. But in most instances they needed to assign a value to the unknown before they could create a problem. There was also a reluctance to use words such as 'some' for the unknown. Most past research has tended to focus on how young children can represent word problems in symbols and not the reverse process. The influence of this capability and the need to find the unknown before being able to pose a problem on their ability to represent word problems in symbols needs further investigation. The research also indicates that children's narrow conception of the equal sign not only occurs on early in their development but also persists as they progress through their elementary years.

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