THE EFFECTS OF NUMERICAL AND FIGURAL CUES ON THE INDUCTION PROCESSES OF PRESERVICE ELEMENTARY TEACHERS

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In this report, we address the following questions: What aspects of information do preservice elementary teachers rely on when performing inductive reasoning? What contexts enable them to perceive the inherent invariant relationships from a finite sample and, thus, formulate viable generalizations? To what extent are they able to justify inductive results noninductively? Our responses are based mostly on inductors' ability to perceive similarity between compared entities that they compute over numerical and figural cues. We tested a model of similarity and examined predictions of the model in the case of two problem tasks. In the model, we assume that both numerical and figural cues contribute to similarity leading to induction. An analysis of the induction processes of the 42 participants suggests that even if relationships among numerical values have had greater contribution to similarity than did figural ones, those who induced figurally acquired a better understanding of the generalizations they constructed.

PURPOSE

The question concerning how concepts are formed is central in almost all work in cognition and learning. Since inductive reasoning plays a significant role in the study of patterns, including object categorization and classification, there has been, and still is, a need to systematically explore processes that enable successful induction. Inductive reasoning, or generalizing knowledge from a finite sample of particular instances, is a common activity in school mathematics. All students need to acquire proficiency in performing inductive reasoning because its predictive function helps reduce the amount of time and energy that would be needed if all cases were to be investigated one at a time. Further, induction promotes generalization and abstraction, two key processes that are necessary and highly valued in mathematical discourse (Schoenfeld & Arcavi, 1988; Skemp, 1971).

In this research report, we deal with the following issues: What aspects of information do preservice elementary teachers rely on when performing inductive reasoning? What contexts enable them to perceive the inherent invariant relationships from a variety of particular instances and, thus, formulate (viable) symbol-based generalizations? Further, to what extent are they able to justify inductive results noninductively, that is, by other means of explanation that have not been drawn by force of surface appearances or by mere arbitrary speculation? Our responses are based mostly on inductors' ability to perceive similarity between compared entities that they compute over numerical and figural (e.g. by way of illustrations) cues. For example, when prospective teachers look for a pattern in order to describe in symbols the number of regions formed by connecting points on a circle, some might compare from the numbers or values they generate from specific cases rather than establish them geometrically through relationships that could be drawn from the figures.

In this work, we developed and tested a model of similarity and examined predictions of the model in the case of problem tasks in which inductive relations were, in the final stage, expressed algebraically. In the model, we assume that both numerical and figural cues contribute to similarity leading to induction.

CONCEPTUAL FRAMEWORK: SIMILARITY AND INDUCTION

Recent psychological research has focused on understanding mechanisms that underlie children's use of similarity in induction processes because they relate to tasks that involve, at the very least, classification and categorization of objects, the formulation of common properties, and the establishment of word meanings (Davidson & Gelman, 1990). The underlying principle behind similarity is that it is not static: it changes based on the context in which it is used (e.g., the type of task influences the judgment made about similarity (Hahn & Ramscar, 2001)). Furthermore, children and adults' ability to engage in similarity continues to evolve as a consequence of intellectual maturation, personal experiences, and "increases" in "domain knowledge" (Gentner & Rattermann, 1991, p. 226).

While much work has focused on children and adults' symbolic abilities to use induction in natural cases (such as breathing practices of different kinds of fish), little attention has been paid on how they would perform induction within the context of mathematical tasks which are governed mostly by symbol systems that are artifactual in nature (i.e., human-made with specific rules of engagement). The conceptual framework that we have developed in this study is used primarily to ascertain the extent to which insights on similarity and inductive reasoning in natural kind categories also apply to mathematical situations that involve induction.

Gelman (1988) shares Simon's (1981) claim that "natural kinds are susceptible to scientific study, whereas artifacts have traditionally not been. … (I)t is possible to study artifacts scientifically …. However, the scientific study of artifacts differs fundamentally from the scientific study of natural kinds" (p. 69). Gelman highlights the significance of "properties that emerge from the interaction between an object and its environment" (ibid) which may affect an individual's ability to induce. Studies have yet to determine if "expertise" in the use of artifact obeys the same structure as inducing from natural kinds. Gelman astutely points out that while natural kinds are "constrained by their genetic or molecular structure" (p. 70), artifacts are characterized by "functions" that change or could be modified. What this means in the context of this study deals with the possibility that inducing constructs in mathematics (treated as artifacts) may not be the same as inducing constructs in natural settings. Attempts to induce attributes in mathematics may likely be constrained by domain-specific factors such as artifactual relationships borne of the technical language and symbols that enable the construction and existence of the attributes which may not be perceived by an individual to be of the natural kind. That is, attributes that are not found in nature and, hence, possess a structure that is strange to common sense and intuition and may require more elaboration (Tall, 1986) and reflective abstraction (in Piaget's sense).

Gentner and Rattermann (1991) distinguish between relational similarity (analogy) and object-based similarity (mere appearances) resulting from studies which claim that individuals perform similarity based on either the attributes observed in objects or the relational attributes that structure the objects. In one of Gentner's (1988) works, he demonstrates the "relational shift" phenomenon that takes place among children: young children tend to perform similarity on objects while older children and adults tend to perform similarity on relations. There is as well a developmental view which makes a distinction between lower- and higher-order "relationship commonalities" (Gentner & Rattermann, 1991, p. 228), whereby the acquisition of formal operations (in Piaget's sense) marks the shift from the lower to the higher order (Inhelder & Piaget, 1964).

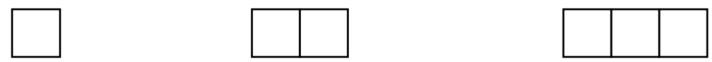
Markman (1989) and Gelman (1988) both advance the notion that "homogeneity" is a property that can very well explain (successful) categorization of objects. Homogeneity pertains to traits shared by category members (such as the striped character of all zebras or family resemblances from within classes of objects). If categories are hierarchically organized based on complexity and specificity, children are found to be capable of inducing "basic-level" homogenous categories first (Rosch, Mervis, Gray, Johnson, & Boyes-Braem, 1976). Then they learn superordinate categories, albeit with relative difficulty (for e.g., "chair" is basic while "furniture" is superordinate (Markman & Callanan, 1984); see also Skemp, 1971, pp. 19-34). Basic level objects in a category are "overdetermined" in the sense that they share common features or common parts or common functions (Markman, 1989). Superordinate objects in a category possess few common properties and "are more inclusive, with greater perceptual dissimilarity" among objects (Markman, 1989, p. 73), making them difficult for younger children to discover. Comparing the manner in which younger and older children categorize, Horton and Markman (1980) point out that facility and efficiency in language use are likely to affect the way basic and superordinate categories are acquired. Further, Callanan (1985) claims that parents tend to use basic level terms with young children, while Gelman (1988) finds that adults who induce depend on factors other than homogeneity. This idea of homogeneity is significant in the construction of mathematical tasks that involve induction because it provides a structure for classifying tasks based on the level of homogeneity, including the kinds of tasks that are explored by students in actual instruction.

METHOD (PARTICIPANTS, DESIGN, AND PROCEDURE)

Participants in this study included 42 undergraduates (34 women, 8 men) who took the test for extra credit. They were enrolled in an introductory course for elementary mathematics teachers in a public university in northern California. Their ages ranged from 19 to 55, with a mean age of 23.42. Racial profile is as follows: 15 Caucasian Americans, 4 African Americans, 11 Asians and Asian Americans, and 12 Hispanic Americans.

Four induction tasks were prepared with each task consisting of three figures accompanied by three numerical values. Each triad was constructed so that the second and third figures and numerical values were related to the first and second figures and numerical values, respectively. Each task required all participants to either draw or compute values for two additional cases before they were asked to obtain a generalization of the task. Due to limitations in time and space, we discuss results obtained from the two tasks given in Table 1 below.

1. Consider the problem below.



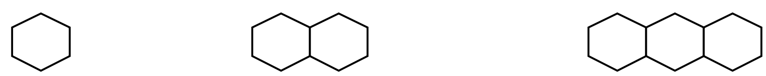
Number of squares	1	2	3
Number of matchsticks	4	7	10

How many matchsticks are needed to form 4 squares?

How many matchsticks are needed to form 5 squares?

How many matchsticks are needed to form n squares?

2. In the figures below, one hexagon takes 6 toothpicks to build, two hexagons take 11 toothpicks to build, and 3 hexagons take 16 toothpicks to build.



Number of hexagons	1	2	3
Number of toothpicks	6	11	16

How many toothpicks are needed to form 4 hexagons?

How many toothpicks are needed to form 5 hexagons?

How many toothpicks are needed to form n hexagons?

Table 1: Two Induction Tasks

Each participant was tested individually by the presenting author. Each interview lasted between 25 and 45 minutes and was audiotaped. Each participant was asked to read the problem and to simultaneously think aloud and write down what they were thinking.

RESULTS AND DISCUSSION

The primary task that this study was aiming to accomplish is to articulate structures of prospective teachers' induction processes in the case of mathematical problems whereby the inductive results are all expressed in variable form. In particular, when aspects of information have been provided, we wanted to find out if they induced from the figures (i.e., figural similarity) or from the values (i.e., numerical similarity). In the case of successful inductors, we were interested in determining the contexts that enabled them to see through features that remain unchanged and how they employed induction to capture the invariance. These contexts, we contend, would play a significant role in the manner in which generalizations are justified inductively and noninductively as well. Tables 2 and 3 present the data according to success level and the type of similarity employed. At the outset, the data confirmed our claim that the participants induced numerically rather than figurally, with a mean of 64.5% (counting correct and incorrect responses). Both quantitative and qualitative data (i.e., interview transcripts) also confirmed an assumption we made in our model of similarity, that is, prospective teachers performed similarity from among the already known and computed numerical values and paid little attention to the figures in which the numbers were actually derived.

	Figural	**Numerical**
Correct Response		
3n + 1	5	5
4 + (n – 1)3	1	
Partially Correct Response		
4 + 3n	4	1
n (or x) + 3	5	16
Incorrect Response		
4n		1
n + (n – 1) = #n – 1		1
n x 4 - 1	1	
n + 3 = 16		1
Unable to Generalize		1

Table 2 Summary of Responses from Problem 1 (n = 42)

	Figural	**Numerical**
Correct Response		
5n + 1	4	6
6n – n + 1		1
6 + (n – 1)5	1	
Partially Correct Response		
6 + 5n	4	1
n (or x) + 5	5	17
Incorrect Response		
n + (n – 1) = #n – 1		1
26 + n = 31		1
Unable to Generalize		1

Table 3 Summary of Responses from Problem 2 (n = 42)

ANALYSIS OF RESPONSES

Prospective teachers who gave correct and partially correct responses (mean of 56%) to the two induction tasks saw that numerical values provided a better clue to similarity than the figures that accompanied the numbers. This result is not surprising considering the fact that they accommodated new knowledge in accordance with their prior experiences which oftentimes asked them to obtain formulas from sequences of numbers using algebraic methods such as finite differences. In obtaining a variable expression for each of the two problems, the most frequent response that used numerical similarity was, however, partially correct. On average, 42% of the teachers responded by saying that the similarity relation among the numbers in Problems 1 and 2 could be expressed by the formula "n + 3" and "n + 5," respectively. When prompted to justify the viability of the expressions, 69% of them pointed to common differences which they obtained from the sequences without linking the numerical differences to the figural differences between any consecutive pair of figures. Further, due to lack of notational fluency, the variable "n" in "n + 3" and "n + 5" was defined to mean "the number of sticks before it." What they meant, of course, were the following two recursive relations: "$a_n = a_{n-1} + 3$" and "$a_n = a_{n-1} + 5$."

Another partially correct response using numerical similarity involves the expressions "4 + 3n" and "6 + 5n." In the case of "4 + 3n," because the participants established numerical similarity from among the sequence of dependent values (4, 7, 10, 13, 16) without having considered how those values were related to the squares being formed, the variable "n" has been assumed to take on values beginning with 1. A similar reasoning holds in the case of the expression "6 + 5n."

Among the correct responses that used numerical similarity (with a mean of 15%), all the participants used "guess and check" to generate the expressions "3n +1," "5n + 1," and "6n – n + 1." In the case of "3n + 1," for instance, some teachers constructed a two-column table showing number of squares in the first column and number of sticks in the second column. They then obtained the common difference between two successive values in the second column, wrote down "3n" and observed that each term was "always 1 more than 3 times n." The same process was made in the case of "5n + 1." A second numerical similarity strategy for obtaining "3n + 1" involves the painstaking process of trial and error. One of the participants, Jose, started out with the expression "4n - 1" and computed the value for n = 1. Because the value obtained was 3, he then tried "4n – n" and evaluated this expression for n = 1. Seeing that he needed 1 more to obtain the first term, 4, he added 1 to "4n – n." Once again he evaluated "4n – n + 1" and saw that it worked for n = 2 and 3. He then simplified the expression to "3n + 1" and checked to see if the numbers 4, 10, 13, and 16 did obey the rule. Jose employed the same method in figuring out the expression "6n – n + 1" for the second problem. Another participant, Raina, used a different numerical similarity method in obtaining the expressions "3n + 1" and "5n + 1." First, she used the given table of numbers and column by column computed the difference between the independent and dependent values in the same column. Because a pattern emerged from the differences which she computed for the first three cases, she then extended the table by simply following the pattern. This enabled her to determine the number of matchsticks for the next two cases. However, in establishing the formula, she simply relied on the first difference, 3, and guessed that by adding 1 (pointing to the first term in the first row of values) to 3n, she would be able to generate all the dependent terms of the sequence. Prospective teachers who induced using numerical similarity methods such as common differences, guess and check, and trial and error were unable to justify how their formulas were related to the problems they were solving.

In the case of prospective teachers whose induction processes involved figural similarity, we observed that the formulas they generated reflected the manner in which they interpreted the figures drawn, including the ones they were asked to construct. The generalized formula was an indication of the process of construction which for them remained uniform and invariant throughout. In thinking about the expression “3n + 1,” Shelly, for instance, started out by computing the common difference, 3, and explained that 3 was the number that determined the “difference between one figure to the next” since forming a new square meant adding 3 new sticks:

> You’re trying to make ahm a full square with four matchsticks and if you already have one side, then you would be adding 3 more on to it depending on the number of squares that you wanna make ‘coz that’s how many you’re gonna put, that’s how many 3s you’re gonna add on.

Prospective teachers who induced by figural similarity clearly understood how symbols played out and what they meant in explicitly expressing generalized relationships. In the transcript below, Chuck explained how he established the expression “4 + (n – 1)3” for the first problem.

> How many matchsticks are needed to form four squares? So ahm I’m looking for a pattern. For every square you add 3 more. So let’s see. So that would be four plus 3 for 2 squares. Plus three more would be for 3 squares. So it’s ten matchsticks. So you have 4. So there would be 13. So 13 plus 3 more is 16. … So for three squares, it would just be two 3s. So there’d be two 3s, three 3s is for four squares, and four 3s for five squares. For n squares, it would just be ahm n minus 1 3s.

GENERAL DISCUSSION

In this study, prospective teachers analyzed induction tasks that contained both figural and numerical cues. Relying mostly on their prior mathematical experiences, their induction processes suggest a preference for numerical similarity strategies. It seems that, on average, six out of ten participants saw invariant attributes in numbers rather than in figures. An analysis of their written responses also indicates that it did not matter for inductors who employed numerical similarity to obtain a large number of cases because they would usually make an attempt to generalize only from the known cases. Although the evidence is still weak, we observed that inductors who employed numerical similarity seemed less capable in justifying their results noninductively, while those who employed figural similarity provided sufficient noninductive justifications due, in part, to the manner in which they connected the symbols and variables they used to the patterns that were generated from the figures. Further, it seems that those inductors who used numerical similarity employed processes and established results that contained fallacies and contradictions. Raina’s induction process, for instance, only made sense within the context of the two problems that she actually solved and not for other problems. Overall, inductors who employed figural similarity were more relation-oriented, while those who employed numerical similarity were more object-oriented since the generalizations they developed were justified solely in terms of how well they fit the information already known and available to them.

The question concerning contexts that enable prospective teachers to determine invariant characteristics from a finite sample of particulars is addressed by considering (1) the level of homogeneity of the numerical values or figures, (2) the nature of the property of invariance being induced, (3) the typicality of the induction tasks being performed, and (4) especially in the case of prospective teachers, the kind of mathematical knowledge that they bring with them to induction. Our data shows how important it is for prospective teachers to have a proper understanding of symbols and variables because they affect the manner in which invariant features are expressed. The two problems presented in this report could be classified as comprising basic-level objects, that is, the figures and numbers were homogenous in that they shared many common perceptual attributes which, thus, encouraged similarity leading to induction. Our data suggest that the prospective teachers found it easier to perform similarity and induction from basic-level homogenous objects than superordinate ones. They experienced tremendous difficulty in determining invariant characteristics of figures and/or numbers when they became too diverse from each other and when the induction tasks became rather untypical.

References

Davidson, N. & Gelman, S. (1990). Inductions from novel categories: The role of language and conceptual structure. *Cognitive development, 5*, 151-176.

Gelman, S. (1988). The development of induction within natural kind and artifact categories. *Cognitive psychology, 20*, 65-95.

Gentner, D. & Rattermann, M.J. (1991). Language and the career of similarity. In S. Gelman & J. Byrnes (Eds.), *Perspectives on language and thought: Interrelations in development* (pp. 225-277). New York, NY: Cambridge University Press.

Gentner, D. (1988). Metaphor as structure mapping: The relational shift. *Child development, 59*, 47-59.

Hahn, U. & Ramscar, M. (2001). Introduction: Similarity and categorization. In U. Hahn & M. Ramscar (Eds.), *Similarity and categorization* (pp. 1-12). Oxford, OX: Oxford University Press.

Horton, M. & Markman, E. (1980). Developmental differences in the acquisition of basic and superordinate categories. *Child development, 51*, 708-719.

Inhelder, B. & Piaget, J. (1964). *The early growth of logic in the child*. New York, NY: Norton Books.

Markman, E. (1989). *Categorization and naming in children: Problems of induction*. Cambridge, MA: MIT Press.

Markman & Callanan (1984). An analysis of hierarchical classification. In R. Sternberg (Ed.), *Advances in the psychology of human intelligence*, vol. 2. Hillsdale, NJ: Erlbaum.

Rosch, E., Mervis, C., Gray, W., Johnson, D., & Boyes-Braem, P. (1976). Basic objects in natural categories. *Cognitive psychology, 8*, 382-439.

Schoenfeld, A. & Arcavi, A. (1988). On the meaning of variable. *Mathematics teacher, 81*(6), 420-428.

Skemp, R. (1971). *The psychology of learning mathematics*. New York, NY: Penguin Books.