THE SPONTANEOUS EMERGENCE OF ELEMENTARY NUMBER-THEORETIC CONCEPTS AND TECHNIQUES IN INTERACTION WITH COMPUTING TECHNOLOGY[1]

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We present in this paper five case studies of instrumental genesis of number-theoretic concepts and techniques involving multiples, divisors, and numerical decomposition. The pupils, who were 12 to 15 years old, used the multi-line screen display of the graphing calculator to explore numerical tasks related to the "Five steps to zero" problem. The paper begins with a mathematical task analysis of various techniques that could be used to determine divisibility by 9. Findings show that the use of the technological tool permits students of this age to spontaneously develop variants of these techniques within tasks that simultaneously afford the co-emergence of related conceptual notions.

A body of current research in computer-supported mathematics learning environments is oriented toward the theoretical development and empirical study of the interactions among the following concepts: artefact, instrument, technique, task, and theory (e.g., Artigue, 2001; Guin & Trouche, 1999; Guzman & Kieran 2002; Lagrange, 2000; Mariotti, 2002; Trouche, 2000; Varillon & Rabardel, 1995). In particular, these studies discuss the role played by the instrument in the process of instrumental genesis within the context of specific tasks and with pupils of different ages and technological experience. In a paper presented last year at PME (Guzman & Kieran, 2002), we reported, for example, on the perceptions expressed by case-study students of the role played by the calculating tool in the evolution of their numerical conceptual schemas. The present paper, which is quite different, analyses the nature of the mathematical techniques that emerged spontaneously in representative participants of the study. By means of two case studies from each of Secondary 1 and 2, and one from Secondary 3, we illustrate the number-based interactions between students and their calculators and show how, within a given set of tasks, the students developed numerical techniques that were afforded by the technology-supported environment in which their numerical explorations occurred. The emergence of these techniques provides further evidence of the intertwining nature of the development of procedural and conceptual learning in mathematics.

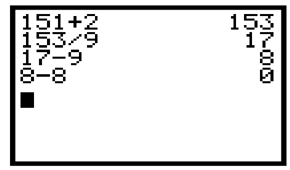
THE RESEARCH STUDY

The study was carried out in Mexico and Canada during the two school years of 1999-2000 and 2000-2001. Its aims were to investigate the role that calculating technology can play in the emergence and development of numerical thinking in students during their first three years of secondary school. Teaching activities were designed that focused on the mathematical content of factors, divisors, and prime and composite numbers. One of

[1] We thank the Social Sciences and Humanities Research Council of Canada (Grant # 410-99-1515) and CONACYT of Mexico (Grant # I32810-S) for their support of the research described in this paper.

these activities was a week-long sequence based on the “Five steps to zero” problem (Williams & Stephens, 1992) given below (along with an example done in four steps—one operation per step). Students were initially pretested on their knowledge of the related mathematical material. The aim of the short pretest was to see if students had the basics that were needed to deal with tasks involving factors and multiples. During the week’s activities that are reported in this paper, students worked alone or in groups—as they wished. From time to time, individual students presented their work to the class by means of the classroom view-screen that was hooked up to their calculator. There were occasions on which students were questioned about the approaches they were using, but no classroom teaching as such took place. At the conclusion of the week’s activities with this problem, four students from each of the six participating classes were individually interviewed.

"*Take any whole number from 1 to 999 and try to get it down to zero in five steps or less, using only the whole numbers 1 to 9 and the four basic operations +, -,* x, *÷. You may use the same number in your operations more than once.*" (based on Williams & Stephens, 1992)



MATHEMATICAL ANALYSIS OF TECHNIQUES

The task of bringing a whole number down to zero in the minimum number of steps (the constraint of minimization was added at the start of the second year of the project) would suggest the use of the largest divisor possible, which according to the rules of the game is 9. The technique of prime decomposition, which a more experienced student of mathematics might turn to, would in many cases involve primes outside the rules of the game and would thus not be admissible. Our “expert” would then likely turn to the criterion for divisibility by 9 and possibly proceed in the following way(s) if presented with, for example, the number 989.

Criterion for divisibility.

Our expert first notices that 989 is not divisible by 9 (without obtaining a remainder) for its digits do not sum to a multiple of 9—they sum to 26. So she has two choices if she still wants to try division by 9. She can add 1 to or subtract 8 from 989. The former leads to a quotient of 110 and the latter, to its predecessor 109. As neither 110 nor 109 is immediately divisible by 9, the entire process of reaching zero will require six steps:

This try: 989 + 1 = 990, 990 / 9 = 110, 110 – 2 = 108, 108 / 9 = 12, 12 / 6 = 2, 2 – 2 = 0

Chaining factors.

Undaunted by a six-step solution, our expert has other techniques to try. As the number 989 is odd, division by 8 is not feasible. And a quick mental calculation lets her know that division by 7 is equally impossible. So, to save some time, our expert—aided by a calculator--decides to try to chain factors as a speedy way in which to find two factors that will work. Chaining three 9s, which yields 729, does not get close enough to 989. So she next tries dividing 989 by implicit chains of two factors to see if she can get to a quotient, that when truncated or rounded, will work back up to a product that is within 9 of 989. When she divides 989 by 49 (i.e., 7 x 7), she is rewarded by a quotient having a small-enough decimal portion that, when truncated, works back to a number within 9 of 989 and, thus, a solution.

Trials: 9x9x9=729, 989/81=12.2, 989/72=13.73, 989/64=15.4, 989/49=20.18
Subsequent trial to work back to a number within 9 of 989: 7x7x5x4=98 (5 and 4 being the factors of the truncated quotient of 20.18)
Solution: 989-9=980, 980/7=140, 140/7=20, 20/5=4, 4-4=0

The division algorithm.

A person unaware of the criterion for divisibility by 9 might approach the task from the perspective of the division algorithm whereby any whole number can be expressed as the product of two whole numbers plus remainder, that is, $a = b \times c + d$ (in this case, 989 = 9 x 109 + 8). Operationalizing this theorem into a solution could take many forms as the value of c is obtainable by various means. The example below, for instance, illustrates the carrying out of a trial division by 9, followed by the multiplication of the truncated quotient with 9 in order to see how far the product is from the initial number, 989—an approach that we will call the "division algorithm invoking trial division."
Example: 989/9=109.8888889, 9x109=981, 989-8=981, 989/9=109, and so on
A variant of this approach, which we call the "division algorithm invoking trial multiplication," involves carrying out, perhaps several, trial multiplications in order to find the value of c. For example, 9x106=954, 9x108=972, 9x109=981--the latter trial clearly bringing the solver into the interval that is within 9 on either side of 989.
However, both of the solution paths suggested by the division algorithm will, for the given number 989, engender a six-step solution. Dividing by 9, after adjusting the given number by subtracting, will not produce a five-step solution for 989; other divisors less than 9 must be found (as in the "chaining factors" approach above). In actual fact, all of the whole numbers from 1 to 1000, with the exception of 851 and 853, can be reduced to zero in five or fewer steps. These two exceptions require six steps--851 being the product of the two primes 23 and 37, and 853 being itself prime. All of the numbers in the vicinity of these two, that is, within 9 on either side of 851 and 853, require five steps to reach zero, thus necessitating six steps for these two.
Without knowing in advance the minimal number of steps required to bring such numbers down to zero, the expert finds herself searching for multiples of, especially, 8 or 9, in the immediate vicinity—being careful to avoid, if possible, those composite numbers whose prime decomposition includes primes beyond the range of the acceptable divisors, as each of these requires an extra step in order to be converted to some divisible number. But while the non-expert may share the same general goals as the expert, he or she may lack the techniques of the expert for generating fruitful solution paths and may thus have to rely on a great deal more trial-and-improvement. Despite the importance of elementary concepts of number theory to the field of mathematics, we know very little about the ways in which students at the lower levels of secondary school develop such content knowledge. A few related studies have been carried out with preservice elementary school teachers (e.g., Lester & Mau, 1993; Martin & Harel, 1989; Zazkis & Campbell, 1996); however, as pointed out by Zazkis and Campbell, such concepts have received scant attention in mathematics education research. The upcoming section describes the techniques developed by the 12- to 15-year-old students of our study (none of the case-study examples included below knew the criterion for divisibility by 9) and

suggests how their increasingly instrumented[2] use of the technology within these tasks was an indispensable tool for the growth of their number theoretic concepts.

THE CASE STUDIES

Secondary 1: The cases of Marianne and Mara Marianne and Mara were mathematically strong, according to both their teacher and their pretest results. From the very beginning of their work with the activity sequence being reported herein, they showed a preference for trying to find the largest divisors possible for the given numbers, or for first transforming the given numbers by addition or subtraction into numbers that could potentially be divided by 7, 8, or 9; but all was by means of trial-and-improvement. However, towards the end of the week-long sequence, a new technique emerged for Marianne, that of the use of the “division algorithm invoking trial multiplication.” This became evident when she was invited to come to the front of the class, where a classmate threw her the challenge of bringing 971 down to zero in the minimum number of steps possible. Figures 1a and 1b show each calculator entry she tried—her approach being demonstrated on the classroom view-screen.3

L2 : 9 × 86	774
L3 : $^{(2)}$ 9 × 76	684
L4 : 9 × 97	873
L5 : 9 × $^{(3)}$ 1	
L6 : 9 × 99	891
~~[L1, L2,…, L6]~~	
L7 : 9 × 105	945
L8 : 9 × 110	990

Figure 1a (Marianne)

L9: 9 × 107	963
L10 : $^{(2)}$ 9 × 106	954
L11 : 9 × 108	972
L12 : 971 + 1	972
L13 : 972/9	108
L14 : $^{(3)}$ 108/6	18
L15 : 18/9	2
L16 : 2 – 2	0

Figure 1b (Marianne)

Having found the value of c to be 108 in $a = b \times c + d$, where $a = 971$ and $b = 9$ (see L11 of Figure 1b), Marianne generated a five-step solution (L12-L16).

During the same class session, Mara was invited to come forward to the view-screen and was given the number 731 to tackle. A new technique seemed to emerge for her during the very moment that she worked on this problem (see Figures 2a and 2b). Her first tries consisted, up to line 16 (see Figs. 2a & 2b), of what we would call “the unsystematic search for another number that is a multiple of 9, in the vicinity of the given number.” But then in L17, we notice the evolution from the “unsystematic search for a multiple of 9” technique to one that was new for Mara, that of the “division algorithm invoking trial division.” She seemed to have taken note, perhaps for the first time, of the quotient produced in L16, and then used the rounded-up quotient as a trial factor in L17. The product of 9 and 82 (i.e., 738) suggested to her the exact adjustment necessary to be made

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[2] According to Varillon and Rabardel (1995, p. 85), a tool becomes an instrument for the subject when “the subject has been able to appropriate it for himself—has been able to subordinate it as a means to his end—and, in this respect, has integrated it with his activity.”

[3] Legend for calculator-screen transcriptions: Ln—refers to the line of the calculator display screen. A set of screen lines that is crossed out denotes those that the student has deleted from the screen. The small number in parentheses indicates the time taken by the student before entering the number that follows—5 blinks of the screen cursor being equal to one unit in parentheses.

to the given number 731 in order to have a number (738) that would be divisible by 9. It did not lead to a five-step solution; but it represented an advance in the development of her mathematical thinking.

L1 : 731 (1) + 1	732
L2 : 732/9	81.33
…	
L10 : 731 – 8	723
L11 : 723/9	80.33

Figure 2a (Mara)

L16 : 731/9	81.22
L17 : 9 □ 82	738
L18 : 731 + 7	738
L19 : 738/9	82
L20 : 82/2	41

Figure 2b (Mara)

Secondary 2: The cases of Pablo and Nicolas Pablo was an average student in mathematics, according to both his teacher and his pretest results. During the first few days of the activity sequence designed around the "Five steps to zero" problem, Pablo had developed the technique involving the "division algorithm invoking trial division." This was seen in his immediate use of this technique when his classmates proposed to him that he try the number 931 (see Figure 3). On one of his earlier worksheets he had described his technique as follows: "I divide the given number by 9, and the whole number that I obtain, I multiply it by 9. After, I subtract in order to have a multiple of 9 and afterwards divide by 9." What is of interest, however, is that he did not continue with this technique all the way down to zero. There was a lengthy pause (see L4 of Fig. 3), followed by the unsystematic search for another number, in the vicinity of 103, that would be a multiple of 7, or 8, or 9 (see L7). We conjecture that he had developed the awareness that, each time it was used, his initial technique involved two steps--the adjustment up or down of the given number, followed by the division by 9. Thus, continuing with it would not lead to a five-step solution when the quotient after performing the second division by 9 was greater than 9. He was therefore searching for an alternate technique to apply to the second round of his solving process, one that would permit him to bring numbers in the vicinity of 100 down to zero in three or fewer steps.

L1 : 931/9	103.44
L2 : 9 □ 103	927
L3 : 931 – 4	927
L4 : ans/9	103 (5)
…	
L7 : 107/7	14.71

Figure 3 (Pablo)

L5 : 9□9□4	324
L6 : 9 □ 9□5	405
L7 : 9□8□5	360
L8 : 362 – 2	360
L9 : 360 / 9	40
L 10 : 40 / 8	5

Figure 4 (Nicolas)

Because the case of Nicolas was described in detail in Guzman and Kieran (2002), it will suffice to mention here that Nicolas, a bright student of mathematics, developed over the course of the week's activity with the "Five steps to zero" problem a technique that combined the "chaining of factors" approach with the "division algorithm invoking trial multiplication." It took the form: $a = b \times c \times d + e$ (where a is the given number; b, c, and d are the factors whose product is within the interval of 9 on either side of a; and e is the remainder) (see Figure 4 for his work with the given number 362).

Secondary 3: The case of David David was mathematically strong, according to both his teacher and his pretest results. From the very beginning of his work with the activity sequence, he showed a preference for trying to "find the largest divisors possible (e.g., 9, 8, 7, 6) for the given number." As was indicated during the interview with him at the end of the week's activities, when he was presented with, for example, the number 546, he was quite consistent with the application of this technique (see Figures 5a and 5b). When, however, the given number was not divisible by one of these factors, as was the case with 323 which was presented to him later in the interview, we noted that he had developed the technique of the "division algorithm invoking trial division"—but with a twist. Rather than rounding or truncating the quotient after dividing by 9, he successively tried to eliminate the decimal portion of the quotient by adjusting the given number up or down in a quite systematic way (see Figures 6a and 6b).

L1 : 546/9	60.66
L2 : 546/8	68.25
L3 : 546/7	78
L4 : ans/9	8.66
L5 : 78/8	9.75
L6 : 78/6	13

Figure 5a (David)

L7 : 546/7	78
L8 : ans/7	11.14
L9 : 78/6	13
L10 : ans – 4	9
L11 : ans – 9	0

Figure 5b (David)

L1 : 323/9	35.88
L2 : 326/9	36.22
L3 : 324/9	36
~~[L1, L2, L3]~~	

Figure 6a (David)

L4 : 323 + 1	324
L5 : ans/9	36
L6 : ans/9	4
L7 : ans – 4	0

Figure 6b (David)

DISCUSSION

Mathematics educators have, for decades, tended to view conceptual and procedural knowledge as two quite different entities—the latter being clearly associated with the development of techniques, skills, and mathematical processes; the former with the understanding of the objects being manipulated (see, e.g., Hiebert, 1986). The notion that a learner could be fostering conceptual learning at the same time as perfecting his/her expertise with mathematical procedures has, until now, never really been elaborated. However, the growth of new theoretical perspectives, related to the use of technology in the learning of mathematics, currently furnishes us with the tools to think about conceptual and technical work in terms of their interaction and co-emergence rather than as completely separate activities.

In the mid-1990s, in France, when Computer Algebra Systems (CAS) started to make their appearance in high school mathematics classes, researchers (Artigue et al., 1998) noticed that teachers were emphasizing the conceptual dimensions while neglecting the role of the technical work in algebra learning. However, this emphasis on conceptual work was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection (Lagrange, 1996). From their observations, the research team came to think of techniques as a link between tasks and conceptual reflection. Lagrange has argued that a technique "plays an epistemic role contributing to an understanding of the objects that it handles particularly during its elaboration; it offers also an object for conceptual reflection when comparing it with other techniques or discussing its consistency" (Lagrange, in press, p. 2). While this theoretical perspective has been elaborated within the context of CAS, the findings of the present study point to its relevance for other technology-supported learning environments in school mathematics.

The case studies reported in this paper support the thesis that students' conceptual development with respect to elementary number-theoretic notions is enhanced by their technical work on tasks involving computing technology. Secondary 1 and 2 students' work during the early part of the week-long activities on the "Five steps to zero problem" tended to be based on much more primitive techniques, such as converting the given number into one that ended in 5 or 0 in order to carry out a division by 5 (Guzman, Kieran, & Squalli, 2001). Very few students at these two grade levels initially used the technique of increasing or decreasing the given number so as to make it divisible by 9 (in contrast, this tended to be the dominant technique employed by Secondary 3 students right from the start). However, without any instruction, most of these students evolved toward variants of this latter technique. The case studies described herein show the different ways in which this technique developed and the forms that it took. The case of Mara illustrated in particular how her specific technique emerged spontaneously while she was in the process of attempting to find a solution. As was seen with all of the cases presented above, it was primarily the division algorithm, involving some version of either trial division or trial multiplication, that served as the basis for exploring divisibility by 9, and related notions of multiples and numerical decomposition.

None of the cases presented here were aware of the criterion for divisibility by 9, and space constraints did not allow for including extracts of the work of students who were so aware. Students not knowing this rule needed to develop their own techniques for finding out if a number was divisible by 9, if they wished to divide by the largest number possible in the "Five steps to zero" activity. Some educators may ask, why not immediately teach them the criterion for divisibility by 9? Suffice it to say that the students, who at the outset of this activity knew the rule of adding up the digits of the given number to determine whether the total was a multiple of 9, did not develop the repertoire of number-theoretic techniques or conceptual notions described above. In fact, when their use of the criterion for divisibility by 9 did not bring them to zero in five steps, they floundered—their subsequent work tending to be unsystematic. They behaved as if there was no need to struggle with developing a technique, as they already had one—even if it did not always lead to a solution. Thus, the richness that was observed in the approaches that emerged in the cases presented above was not, in general, seen in those who had begun this study with the criterion for divisibility by 9.

Salomon, Perkins, and Globerson (1991) have distinguished "effects with technology obtained during intellectual partnership with it, from effects of technology in terms of the transferable cognitive residue that this partnership leaves behind in the form of better mastery of skills and strategies" (p. 2). The students who knew the criterion for divisibility by 9 were relatively successful working with the technology. But because they did not have to work to find a technique for determining if a number was divisible by 9, the numerical explorations that were engaged in by the case-study students and others were absent in them. Thus, it was those who did not have the prior awareness of this criterion who could be said to have benefitted most from the effects of the technology. In their search for techniques for handling the task at hand, their number sense with respect to factors, divisors, and numerical decomposition evolved in a manner that was not evident among their criterion-aware peers.

In 1995, Verillon and Rabardel, the pioneers of instrumental theory related to technological artefacts and instruments, posed the questions: "How are instruments, derived from modern technology, associated by subjects with their actions and, as such, inserted in their activity? What influence does this have on activity? How is it modified?" (p. 84). We hope that the case studies presented herein, against the backdrop of the mathematical analysis that was offered with respect to the divisibility-by-9 part of the task, have been able to partially respond to these questions by illustrating the interaction that occurred for these subjects between their use of the calculator as a means for developing new techniques and its role in the emergence of new ways of thinking about number and its structural decomposition.

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