FROM COGNITIVE SCIENCE TO SCHOOL PRACTICE: BUILDING THE BRIDGE[1]

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The paper is focused on recent researches in neuroscience and developmental psychology regarding mathematical abilities of infants. A model that tries to explain these findings is developed. The model underlies the mental operations that could be systematically trained to generate efficient school learning. The model is built from a cognitive constructivist perspective on learning, is transferable into teaching-learning materials and its development has direct implications in structuring the curriculum and reorienting teaching practice.

HOW MUCH IS THE NEWBORN BRAIN SHAPED FOR FURTHER DEVELOPMENT?

In the early childhood, through the comprehensive learning of maternal language, the child is crossing an extremely dense period of mental accumulation. Synthetically described, this is a recursive categorical learning; the child acquires category knowledge through a few-steps-inductive process and he/she is able to retrieve prototypes or simply exemplars on a similarity-based analysis. For example, the young child is able to identify a table even this table is made from wood or metal, rectangular or oval, covered with food or not, present in a 3D space or just painted on a piece of paper; he/she internalized a category the table belongs to. This process of learning, which commonly has a single mentor – the family environment, and a single "mental medium" – the maternal language, is time dependent and self-regulating. From this view, perception is equivalent to categorizing, with its double process: identifying the class a specific object belongs to, and recognizing that object as a prototype of that class. Category perception is a complex mental process, which supposes recognizing and classifying and it is learned, even in early childhood, through numerous crossings in between various levels of concreteness and hierarchies that implies abstracting.

The last three decades research in cognitive science, neuroscience, and developmental psychology has been significantly concentrated on understanding children's capabilities to learn in the first year of life. In many researches, the perception of grouping and separating similar objects has been interpreted in relation with the number sense. For example, Wynn (1990, 1992) showed that 5-months infants seam to be able to compare the cardinals of two sets of objects up to three and to react when the result of putting together or taking away is not the right one. Infants showed longer looking at arrays presenting the wrong number of objects, even when the shapes, colors, and spatial location of the objects in the displays were new (Simon, Hespos and Rochat, 1995). Other experiments with older infants using different response systems (manual search/locomor choice) (Feigenson, Carey and Hauser, in press) led to the same

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conclusion. In addition, infants successfully discriminated 16 from 32 dots or 8 versus 16, but failed to discriminate 8 versus 12, or 16 versus 24 dots. These results have been confirmed using various sensory modality (visual or auditory) or format (spatial versus temporal) (Xu and Spelke, 2000). It seams that the sense of approximating natural numbers found in adults appears to be present and functional in 5 to 10-months-old infants. To measure perception, experimenters used the Weber's law: as numerosity increases, the variance in subjects' representations of numerosity increases proportionately, and therefore discriminability between distinct numerosities depends on their difference ratio. Representations of large approximate numerosities show a variation of ratio in Weber law between 1.5 and 2 for infants (Spelke, 2002) and 1.15 for adults (van Oeffelen and Vos, 1982).

A PROMISING HYPOTHESIS FROM THE EDUCATIONAL PERSPECTIVE

How could we explain these findings? Although the theories differ in important ways, they share en emphasis on considering that infants possess amazing competencies that biologically predispose them to learn. As far as we are navigating among hypothesizes, we should focus on the less restrictive ones and the most productive ones from the perspective of mathematics learning. The claim I argue for is that the mind is generically endowed with some categories of basic mental operations. The extensiveness, the deepness, the ramifications of these operations are subjects to further development through education, but the roots of these operations are inherited and they permit to infants to contact the real world and to think about it. I will call these roots *inner operations*. In the early childhood, the inner operations permit to build classes of objects on a similarity base, and to develop the extensions of these classes to more abstract mental constructions.

The categories to which the inner operations belong to will be nominated in operational terms in order to underlie their dynamics. These categories are: *Associating, Relating, Algebraic operations, Logical operations, Topological operations, Iterating, Generating*. The names used to identify them are just labels – they do not express by themselves the complexity of each operational category. A synthetic description of these basic mental operations, based on clustering, is presented below.

The operational category generically called ***Associating*** includes operations described as connecting two entities based on a one-to-one correspondence. The capacity of building correspondences one-to-one evolves from its primitive form of *matching* objects one-to-one, to associating through isomorphisms various representations (see Singer, 2001, for a detailed presentation). As an inner operation, associating permits to infants to discriminate between one and two objects. The infants automatically match one or two objects "one-to-one" and are surprised when "the partner" is missing, but they could only exceptionally do this for three or more objects, due to limited carrying capacity of memory at this age. The replication of some experiments confirms this. For example, utilizing the looking time technique, Uller and Leslie (2000) showed that 12-month-olds understand what "exactly two" means, but they have difficulties to pass over 3 when adding objects. Uller (2002) concluded that this capacity of differentiating up to four objects (sometimes called subitizing) should be considered primitive and foundational, perhaps at the core of cognition.

The operational category generically called ***Relating*** contains operations described as connecting an entity to one or more others, based on a relationship. As an inner operation, *relating* permits to infants to compare one specific object to others around them in order to assess their similarities and differences. Within this inner operation, the child realizes that there is a difference between the mother's face and the father's face; there is a difference between one object and two similar objects. From another perspective, while associating emphasizes symmetry, relating could emphasize asymmetry.

The first operational category described above (*Associating*, which could be modeled by bijective functions) is included in the second one (*Relating*, which could be modeled by mathematical relations), but there is no meaningful to study this inclusion; this is why the two categories are separately listed. While associating suppose *matching* (bilateral connection) as a representational task, relating suppose *mapping* (network connection).

The category of ***Algebraic operations*** contains operations dealing with quantities that are combined in a specific well-defined way and the result of this combination is analyzed from a quantitative perspective. In school, one studies the binary operations (defined on a Cartesian product with two factors), such as addition and multiplication. As inner operations of the algebraic category, *Pre-arithmetical operations* refer mainly to a list of general operations that, quantitatively expressed, lead to addition, subtraction and so on, such as: grouping, taking away, magnifying, reducing, adding, combining, etc. These permit to infant to distinguish between the situation when an object is added onto a scene and the one in which an object is taking away. The inner operations of this category assure also a sense of increasing and decreasing quantities.

While the first two operational categories (associating and relating) act by building relationships between two or more entities without modifying their nature, the algebraic operations suppose an intervention to obtain from the given sets of entities another one, whose cardinal (measure) is to be determined.

The category of ***Logical operations*** is completing the category of algebraic operations by giving a formal meaning to it. In the young child mind, rudimentary elements of logic are present as inner operations; they consist in relating two facts through conjunction or disjunction and perceiving the result of the two as a third fact. For example, when mother *and* father are coming, the child perceives that they are coming *together*, comparing to the situation when mother and father appears separately in space/time and the child is expected to see mother *or* father. Moreover, very early in life the child is able to react to the "*don't*"-s. Another logical operation refers to inferring in the format of "If *p*, then *q*". This type of reasoning appears in the early years mostly in simple causal inferences, associated to conditioned reflexes.

The category of ***Logical operations*** is extending the current mathematical meaning of logical reasoning to the general capacity to formulate logical inferences for different types of reasoning. For example, different patterns are involved in deductive reasoning and non-deductive reasoning. In addition, different patterns are involved in different types of *deductive reasoning*, which could be: *conditional, consecutive, causal, modal, normative, procedural.* Different patterns are also involved in the following types of non-deductive reasoning: inductive reasoning (in which, based on examination of a number of cases from a class of objects, a conclusion is formulating about the entire class) and analogical reasoning (in which, a conclusion about an individual case is formulated as a consequence of observing one or more individual cases.). The daily reasoning and argumentation actually mix many of these types, but when analyzing the mechanisms that underlie understanding, it is important to describe them as precisely as we could, in order to develop appropriate training.

The category of ***Topological operations*** has a pervasive presence on development in the first years of life. The topological operations permit to identify boundaries, to relate them with discrete components, to globally perceive objects, to pass the frontier between discrete and continuous, to have an intuition of infinity. They account for the convergence of thinking and for global perception. The term *topology* usually refers to a domain of mathematics that started from geometry and has as central idea that continuous geometric phenomena can be understood by the use of discrete invariants. One of the strengths of algebraic topology has always been its wide degree of applicability to other domains. Nowadays that includes physics, differential geometry, algebraic geometry, and number theory. In the context here, I use an extension of the mathematical term. In a wider sense, in this category are included mental operations dealing with the idea of infinity (convergence to infinite). The topological operations permit to associate to a thing those properties that are in a geometrical sense the most permanent - the ones that will survive distortion and stretching. From a topological perspective, the geometrical distances and angles are not relevant, the geometrical shape neither, what it is important is conserving the continuity. The primitive topological property of mind permits to infants to discriminate numerosities when they are significantly different and permit later to make numerical approximations with different orders of magnitude and various degrees of precision. They also conduct to globally perceive continuous surfaces. This type of perception was identified in some researches, while checking the subitizing paradigm. For example, Clearfield and Mix (2001) reported that, when systematically manipulated contour length or aria, 6-to 8-month-old infants dishabituated to a change in either contour length or aria, but not to a change in number. They concluded that infants might actually be using continuous quantity rather than number to discriminate between displays.

While algebraic and logical inner operations are dealing with finite and discrete quantities, topological operations are addressing to infinite and continuous properties. The nature and mechanisms of the inner operations in the topological and algebraic categories are completely different; and from this perspective, the results found by Daehene, Spelke and others during the last decade are completely predictable and they constitute verification for this theory.

Iterating is described as perceiving regularities in variability; this operation allows initially for imitation, but further for identifying and developing patterns. In recent psychological research, mimicking (Fisher and Bidell, 1998) is seen as an ability to overcome the skill level by manifesting a behavior analogous to the next level; even it is less consistent comparing to the advanced level, this ability stimulates progress in learning at the early ages. In this way, imitation is seen as a primitive form of *perceiving and developing patterns*. Iterating as an inner operation accounts for the recursive property of mind.

Generating could be described as an operation-category whose elements create new entities, previously unknown, starting, eventually, from entities already known. A special element in this category is the unconditioned generating: *Grasping*. To be more specific, *Grasping* could be defined as perceiving an entity or its essence instantaneously, without proceeding discursively in space or time (i.e. by passing from one bit of information to another). As a basic mental operation, *Grasping* plays also an important role in the emergence of spurts and other types of discontinuities recorded in the experiments done in micro-development (Fisher and Bidell, 1998). *Generating* category assures a kind of readiness to start. This operational category help the mind to build the leap to learning, it creates *the innate dimension of the motivation to learn.*

Therefore, on short, the operational-categories identified as foundational for learning are: *Associating, Relating, Algebraic operations, Logical operations, Topological operations, Iterating, Generating*. To have a shorter reference in the following, I will use the initials and I will call this list the ARALTIG model. Apparently, most of the operations on the list look to be mathematically defined. This is true only in the measure in which mathematics conceptually offers powerful tools to accurately define and build categories.

In the newborn mind the ARALTIG operations are less differentiated. Through training, they reproduce and ramificate while internalizing and self-structuring new knowledge. They are incorporated in and act for each part of this knowledge. The specific operations of a class share, as representatives, the property that defines the class. That means they are functioning on the same mental scheme, no matter the level of abstraction involved by the entities the operations apply to. This invariance creates conditions to foster patterns. The ARALTIG operations naturally combined in various ways predispose the human brain to learn language and to develop through language. These categories of operations are necessary and sufficient to nest and sustain the brain-mind development. They are the mold that permits and assures architecturing the development; the other operations the mind is processing result from the ARALTIG model through relating the basic operations in various ways.

STRUCTURING PREDISPOSITIONS – A WAY TO OPTIMIZE LEARNING

The born mind starts building category representations from the beginning, and for doing this, it is endowed with operations (or properties) that are self-developing in interaction with the environment. The ARALTIG model is built in terms of operations and not in terms of properties for three reasons: the elements of these categories ***act*** as operations, ***could be trained*** as operations and ***constitute the basis*** for complexes of operations. The strength of this model consists in its connectivity to an effective and creative learning. If until now I tried to build arguments, I will try next to explain the consequences. I will use, as before, conclusions of various experiments, then I will try to schematize some principles for an operations-centered training, principles which have been applied for building dynamic structures of thinking (Singer, 1996; 2001).

First of all, a short insight into a question: why arithmetic learning is so difficult? Though usually children begin verbal counting in their second or third year of life, they arrive to understand the meaning of the counting routine late, after two or more years. The experiments done by Wynn (1990, 1992) provide evidence that children understanding of counting develops in four steps. In the first stage, the child is able to understand “one” as referring to an object, but he/she is not able to associate another numeral with a number of objects. He/she can only dissociate one and more. At this moment, one, two, three,… are just words-label, not words-meaning. After 9 months of counting experience, Wynn's children entered the second stage of comprehension, in which they were able to correctly identify "two" with various arrays of two objects and to make comparison between one, two and more. After three further months of training, on average, children showed they learned the meaning of "three". Finally, in the fourth stage, they proved comprehension for all the ten numbers in the counting routine. In this experiment, children needed about 1 to 1 and 1/2 year of training/practice to associate the counting routine with its meaning. Is this a long term? Considering the complexity of the task, it is not.

In what consists the complexity of this task? The children have to internalize and automatize a double invariance: invariance of the class the particularly object belongs to (the specific physical elements of the set of which the cardinal is determined) and the invariance of the class of classes the object belongs to (conservation over nature and order of objects and successive inclusions). The counting routine adds a complex association to this: associating to the ordered sets their cardinals, and a recursion (transforming cardinals into ordinals). To be more explicit, a suggestive non-standard representation for this 4-levels of complexity (double invariance, association and recursion) is given in Fig. 1. When "2" appears, it is restructuring the relationship between "1" and its set; the same, when "3" appears, it is restructuring the relationship between "1" and "2" and their set; the process is going on recursively, for 4, 5 and so on.

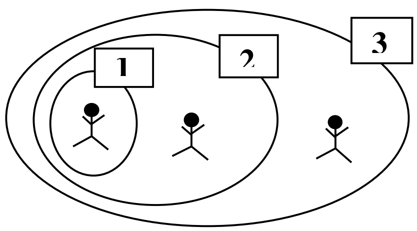

FIGURE. 1: RESTRUCTURING

In terms of computational representations in the brain, as far as the cardinal is increasing, an already created connection has to be successively interrupted (disconnected) and replaced by another, different connection. The recursion of this process: is essential, is difficult to be learned because the process continuously restructures connections, and constitute the fundamental difference from the nature of the subitizing process. At the same time, this seams to be possible due to the human mind innate ability of iterating and identifying patterns.

This short insight gives just a snapshot about the nature of difficulties to face when describing the process of learning. The ARALTIG model has the advantage of showing the few meaningful directions on which the training could be focused with the purpose to foster the natural predispositions and to avoid low significant redundant information (the so-called noise in complex systems).

TRAINING FLEXIBILITY – A WAY TO OPTIMIZE UNDERSTANDING

I will repeat a truism: a consistent training should be based on a strong understanding of the initial conditions. In this context, the initial conditions consist in *mental operations* applied on *information* that have different *degrees of complexity* and involve various *levels of abstraction*. In describing the information complexity, I used three independent vectorial dimensions: *the nature*, *the structure*, and *the procedure*. Concerning the *nature* of information, the criterion was the *level of generalization*: information could be organized on a scale from general (entities, laws, theories, etc.) to particular (examples, case-studies, etc.). The *structure* of information is described by the *connections among concepts*, which organize information on different stages from unstructured (non-systemic – poor connections, no hierarchies) to structured (systemic – strong connections and hierarchies). Concerning the *procedure*, the criterion was the *dynamics of connections* and from this perspective, information could be organized from reproductive to creative procedural actions.

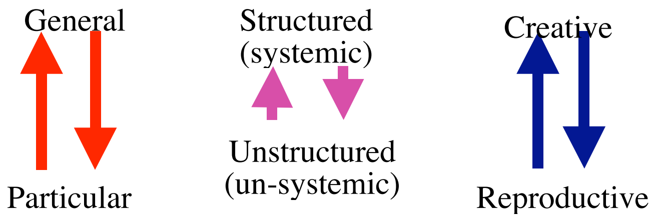

Figure 2: Structuring the information degrees of complexity

Differentiating these dimensions is significant for organizing training. Thus, to make abstractions incorporable in mental structures, a lot of passages from particular to general and vice versa need to be practiced. It is also important to pass through as many organizational stages as possible, as well as to face both algorithmic and creative tasks. The schema in Fig. 2 stress that, as far as we identified significant stages on each of the three scales, the training needs to focus on the transfer in-between various stages, more than to focus on each one separately. This is removing the didactical approach from a "horizontal" way of perceiving teaching: islands of information to be transmitted, to a permanent process of "vertically" restructuring students' knowledge by incorporating the new elements into a dynamic structure. Particularly, this implies systematically practicing the basic operations by creating patterns of variability. This kind of training supposes identifying and developing optimal individual pathways in a multidimensional network. These pathways could be incorporated in a learning technology of the future, aiming at creating fluent thinkers. Here, technology is used in a very broad sense, as strategic mean to achieve a purpose; the didactical technology I am referring to includes printed materials for teachers and students and eventually software, but its most pervasive component is didactical intelligence.

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