

SOCIAL TRANSFORMATION OF STUDENTS' CONCEPTUAL MODEL: ANALYSIS OF STUDENTS' USE OF METAPHOR FOR DIFFERENTIAL EQUATIONS¹

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This research applies the discursive approach to investigate the social transformation of students' conceptual model of differential equations. The analysis focuses on the students' use of metaphor in class in order to find kinds of metaphor used, their characteristics, and a pattern in the use of metaphor: Based on the analysis, it is concluded that the students' conceptual model of differential equations gradually becomes transformed with respect to the historical and cultural structure of the communal practice of mathematics. The findings suggest that through participating in the daily practice of mathematics as a historical and cultural structure, a learner becomes socially transformed to a certain kind of a cultural being with historicity. This implies that mathematics education is concerned with the formation of historical and cultural identity at a fundamental level.

INTRODUCTION

What does a student learn in mathematics class? How does a student change as s/he participates in mathematical practice in class? This paper aims to address these questions by applying social practice theory to analyze mathematics classroom discourse in order to investigate students' social transformation in mathematics class. The notion of social transformation is the core of social practice theory. In social practice theory, knowledge such as mathematics is the historical and cultural structuring of social existence in a mathematics community. Learning is a process of legitimate peripheral participation through which a learner renegotiates his/her epistemological standpoint with respect to the cultural organization of a community (Lave & Wenger, 1991; Wenger, 1998).

Although the notion of mathematics as an abstract, disembodied, universal, value-free and transcendental truth has traditionally been the most dominant discourse in shaping teaching and learning mathematics at schools, the newly developing perspective on social practice has influenced theory and practice in mathematics education in a fundamental way. Indeed, recently, the notion of community and practice has become increasingly popular in theoretical discourse of mathematics education and has become the basic unit for analysis of classroom interaction. From this perspective, cultural and social processes are integral to mathematical activity. The mathematics class is considered of as a practice community where participants negotiate their mathematical meanings and ways of doing mathematics to create their own mathematical culture through daily practice of mathematics (Cobb & Bauersfield, 1995; Voigt, Seeger, & Waschescio, 1998).

This research applied socio-cultural perspectives to the analysis of mathematical practice in the classroom in order to investigate the social transformation of students. Social practice theorists argue that the cultural epistemological standpoint historically

¹ This research was supported by Korean Research Foundation Grant (KRF-2002-037-B00053, KRF-2002-041-B00468).

constructed by a mathematics community is mediated through mathematics. As a consequence of the mediation, a learner's epistemological perspective becomes socially transformed. Then, how does social transformation happen? How is the process like? This research seeks to answer these questions by analyzing students' discursive practice in mathematics class.

METAPHOR AS A CULTURAL MODEL

Contemporary cognitive scientist has shown that mathematical cognition is fundamentally metaphorical (Lakoff & Nunez, 2000). It has been shown that human beings are born with innate arithmetic ability such as subitizing. In addition, recent brain research has discovered that some mathematical abilities are associated with certain areas of the brain (Lakoff & Nunez, *ibid*). This represents conceptual metaphor through which the primitive innate mathematical abilities become elaborated into higher level of mathematical abilities. Metaphor as a cross-domain mapping bridges between the source domain of physical sensorimotor experience and the target domain of an abstract concept enables people to reason about an abstract concept following the inferential structure of a source domain of tangible sensorimotor experience (Lakeoff & Nunez, 2000).

Based on the theory of metaphor, Lakoff and Nunez (2000) argue that mathematics as we know it is brain-and-mind-based. That is, mathematics is a product of the function of the brain and of the body, and our experience in the world. Primitive innate mathematical abilities become elaborated into a more sophisticated kind of mathematical knowledge through experience. In the process, it is conceptual metaphor that mediates the experiential world to mathematical knowledge. On one hand,, the existence of innate arithmetic ability explains the universality of mathematical concepts such as numbers. On the other hand, the universal innate arithmetic ability becomes transformed to a culturally specific kind of mathematics by conceptual metaphor through which the culturally specific inferential structure of lived-in experience is transferred to the domain of mathematics. In this regard, mathematical metaphor represents a cultural model of mathematical concepts.

Cultural models are defined as presupposed, taken-for-granted models of the world that are widely shared by the members of a society and that play an enormous role in their understanding of that world and their behavior in it (Quinn & Hollan, 1987). When considering that mathematics is a product of metaphor reflecting a specific cultural structure share in a society, conceptual models of mathematics are cultural models. Cultural knowledge such as mathematics is organized into a culturally structured model that provides a culturally shared basis for seeing, reasoning, and in general, doing. Metaphor plays an important role in the construction of the cultural model since it allows for knowledge to be mapped from known domains of the culturally structured lived-in world onto conceptualizations in the social and psychological abstract domains. For instance, Ju (2001) found that while professional mathematicians creatively applies metaphor in mathematical arguments, the use of metaphor should be legitimized by peer mathematicians. The notions of legitimacy and taken-to-be-shared in the use of metaphor imply that mathematical metaphor is related to the conceptual model culturally shared in a mathematics community.

RESEARCH DESIGN

In this research, language is taken as a unit of analysis under the assumption that “Facets of cultural values and beliefs, social institutions and forms, roles and personalities, history and ecology of a community may have to be examined in their bearing on communicative events and patterns” (Hymes, 1974, p.4). Metaphor is one of the important communicative patterns shared in a mathematics community (Ju, 2001; Lakoff & Nunez, 2000; English, 1997; Pimm, 1987). More importantly, as discussed in the above, metaphor can be seen as a significant indicator in depicting the cultural model of a mathematical concept. Therefore, by investigating the patterns in students’ use of metaphor, especially, how it changes through students’ participation in mathematics practice in class this paper aims to describe how students’ model of a mathematical concept – particularly differential equations in this research – becomes transformed and in what way the transformation is social. For this purpose, the discourse analysis specifically focuses on the following questions: what kind of metaphor did students use in representing their mathematical reasoning? What are the characteristics of the metaphors used? How did the students’ use of metaphor change through the semester?

Data was collected through participatory observation in a differential equations course at a university in Korea through a fall semester in 2002. All class sessions were video recorded and transcribed for later detailed discourse analysis. The instructional design of the differential equation course is based on the philosophy of Realistic Mathematics Education (RME). The core of RME approach is progressive mathematization, that is, mathematizing both everyday-life experience and mathematical subject matter itself (Treffers, 1987). Through progressive mathematization, students do not merely seek solutions for specific tasks, rather they renegotiate their mathematical meaning and a taken-as-shared mathematical reality emerges gradually. For progressive mathematization, context problems are essential. They provide grounds for students to develop context-specific solutions through horizontal mathematization. This turns into the object of mathematical analysis in the following stage of vertical mathematization to emerge into a taken-to-be-shared reality (Gravemeijer & Doorman, 1999).

In this research, a set of context problems was developed for the students. This leads to a series of processes of horizontal and vertical mathematization, which became the basis for the mathematical reinvention that the instructor intended. The context problems were constructed to reflect mathematical phenomena experientially realistic to the students. The context problems were also of historical significance. For instance, one of the worksheet problems was concerned with spring-mass motions which historically motivated the invention of differential equations (Kline, 1972). In the class, the students formed small groups of three or four to work on the context problems. While the students worked in a small group setting, the instructor interacted with the students. After the small group discussion, the students joined together for the purpose of a whole class discussion. Several students came to the front to share their results and the instructor facilitated the sharing of their mathematical ideas.

FINDINGS

Kinds of Metaphor Used and Characteristics

Mathematically, solutions of different equations have dual aspects. On one hand, they are treated as objects: something satisfying a given equation. On the other hand, differential equation solutions are functions representing phenomena changing with respect to the independent variables assigned. These dual aspects of differential equation solutions reflected in the students' use of metaphor, can be largely categorized into two types: machine metaphor and fictive motion metaphor.

“A Function Is A Machine” is one of the grounding metaphors for functions (Lakoff & Nunez, 1997). In the machine metaphor, functions are regarded as machine processing inputs to produce outputs. The notion of algorithm is based on the machine metaphor in the sense that an algorithm as a metaphorical machine operates on input to yield output. Followings are examples from the class:

Since the time interval decreased to 0.5..I mean as Δt decreased to $1/2$, the coefficient of the rate of the change decrease ..so we made it $0.1P$. Then what we computed last time was P ..uh PO begins with $3P$.. $P1$ is 3 .. $P1$ is 1 there but 1 ..because (it) means 0.5 , $3+$ we add there what we put 3 into P (in the equation). As a result, 0.342 came out. We add it to get 3.342 .

In this example, the students worked on a context problem concerning prediction of population. In the previous session, the students predicted an animal population for each year with a given differential equation $dP/dt = 0.3P(1-P/12.5)$. In this session, the problem required that students predict the animal population for every six months. In the above transcript, the student presented her solution of how to make a prediction from the differential equation. In her explanation, she treated the given differential equation as a machine. With the changed time interval, she changed the coefficient as if modifying the machine to fix a new purpose. Then she entered a number to operate the machine and a number sprung out of it.

With the machine metaphor, the students regarded differential equations as a device for calculation, so they needed to enter a number to acquire a numeral value. As a result, their approaches were quantitative and their reasoning focused on the product of change identified by differential equations. Due to the focus on the product, motion metaphor did not highlight the process of change, in particular the rate of change in the differential equation. Also, since they had to input definite numbers into differential equations, their approaches were discrete. In the above example, the student constructed the approximate solution graph of a differential equation by computing values at discrete points.

“Fictive motion metaphor” is another dominant kind of metaphor that the students used, in which the solution of a differential equation is represented as a trajectory of a moving point. In general, the fictive motion metaphor, variables are travelers moving along axes and the function relating the variables is conceptualized as a traveler moving across the plan or space. The following examples of the fictive motion metaphor from the class:

Whatever P is, if it becomes 12.5 , it keeps going just this this

When P is smaller than 12.5 , the rate of change is positive so it keep increasing. Then as it gets closer to 12.5 , uh to the continuously changing increment..uh..if it becomes 12.5 , it does not increase anymore.

In these examples, the students were dealing with the same population problem discussed in the above. However, the students approached the context in a different way. In the

analysis, the students thought of a solution in terms of trajectory of a point continuously moving along. As a result, the students viewed problems situations as continuously changing. For instance, when the student used the verb “become” instead of “be” in the first example to represent a continuous change. Because the fictive motion metaphor represents a solution as a trajectory of a moving point, the students focused on the entire patterns of a solution curve. So the students qualitatively reasoned about differential equations based on the holistic pattern of solutions instead of precise values.

SOCIAL TRANSFORMATION IN THE USE OF METAPHOR

So far, kinds of metaphor used and their characteristics have been identified. Now, what is the pattern in the students’ use of metaphor in the class? In what spect is the pattern social? In other words, does the pattern indicate the path of the students’ transformation with respect to the cultural organization of communal practice of mathematics? In this regard, one of the salient patterns in the use of metaphor is concerned with the switch between the two types of metaphor. Generally, there was an observed tendency for students to rely more often on fictive motion metaphor than one machine metaphor through out the semester.

In the beginning of the semester, the students were more likely to interpret their mathematical tasks as algorithmic and discrete and as a result, they applied the machine metaphor more often. This is partly due to the nature of mathematical tasks. For instance, in the beginning of the semester, the course provided context problems to lead the students to reinvent Euler’s method. In this way, discrete and algorithmic approaches were intended by the course designer. However, the tendency was also the product of the students’ own conception of “what is mathematics?” In the beginning of the semester, the students often expressed the view that law-like-principles or algorithms were the only “real mathematics”. This kind of mathematical belief is the product of their previous mathematical practice such as grade-making in mathematics class. However, machine metaphor rarely matched a given mathematical situation of change, as it was not efficient in revealing the changing aspects of the phenomenon under inquiry. Through meaning negotiation, the students realized the weakness of the machine metaphor and switched to the fictive motion metaphor, which is more appropriate to highlight the continuity of a changing situation. In this section of the paper, a sequence of examples will be presented to illustrate the way that the students proceeded from the machine metaphor to the fictive motion metaphor. The examples are taken from a session in which the students investigate a context of population growth to decide which change in the population is determined, the initial population or time.

The students began their discussion with the machine metaphor, as illustrated here:

Student A: Before..in high school we learned Fibonacci sequence. Like there are two rabbits and they reproduce once a year and the babies reproduce in a year..I thought that this (problem) is similar to that.... When I thought about this problem in terms of Fibonacci sequence, we have to multiply by the number of fish to get this number so the equation is concerned with P. When we make a Fibonacci sequence, the degree is t. So t seems to be involved too.

This student A began her discussion with a discrete model of population growth. She applied the notion of sequence to the situation and thought only about the population at each generation without taking account for the process of population growth between

generations. She tried to explain how to obtain population at a certain moment. Thus, she developed an algorithmic expression to determine which variables were involved with population change.

However, the algorithmic discrete model was not quite persuasive and another student B disputed:

Student B: I still believe that only P is involved with the population growth. In the beginning, I thought that time might be involved..(drawing a graph)..For instance, starting from an initial value it seemed to be related to time, if it starts from here, then it will merge from this point. So the population can be expressed in terms of only P regardless time.

Student A: But then...the parents and the babies reproduce together.

Other students: Should they reproduce just once?

In this argumentation, Student B introduced fictive motion metaphor by representing the population growth as a process changing along a curve. With the metaphor, she highlighted the continuity in the population growth. As a result, she could see the context as an integrated whole and the insight enabled her to make a more meaningful claim.

Following Student B's dispute, Student A replied by trying to perpetuate her discrete model based on the machine metaphor. In listening to their argument, several students began to question the appropriateness of the model presented by Student B. through further meaning negotiation, the class ended with more elaborated argumentation based on fictive motion metaphor. In particular, after the discussion, the class introduced software producing slope fields of differential equations. The technology visualizes slope fields in a very dynamic way, which enables the students to manipulate slope marks and see the continuity of the change. Through the experience, the students came to appreciate the appropriateness of fictive motion metaphor as language to represent contexts concerning differential equations. Moreover, their use of fictive motion metaphor became more sophisticated.

Student C: In the cases that the number of fish is 3 and 5, respectively, if we look at the relation on the graphs, the slope was steeper when an initial value is larger..since they reproduce to the same degree..so if you put the cursor here, the curve is identical to that curve starting from here. Like this if it start from 2, doesn't it become 3 at a certain moment? Right? Because the same species reproduces to the same degree.

In the above example, Student C represented the population growth as a curve made by a traveling point (e.g., "starting from here", "doesn't it becomes 3"). By saying "to the same degree", she emphasized the notion of rate of change in differential equations. Also, she explains the change in the slope in terms of the change in the steepness of the travel route. These aspects of fictive motion metaphor extends it from an ordinary metaphor for function to a metaphor specifically representing differential equations and revealing the notion of "rate of change" in these equations.

The above examples were taken from a session in the early stage of the semester and the emergence of the fictive motion metaphor out of the machine metaphor had taken through out an entire session. However, students applied fictive motion metaphor increasingly and more efficiently throughout the semester. At the end of the semester, it was observed that students promptly switched between the two metaphors depending on their appropriateness for a given mathematical task. This implies that the machine metaphor is not redundant. Rather, the machine metaphor provided an intermediary stage

for the students to transgress to the model of differential equations based on fictive motion metaphor.

The change in use and efficiency is interpreted as evidence that students' conceptual model become transformed according to the historical and cultural organization of practice in the mathematics community. In this regard, the transformation is social. It is social transformation in the sense that it reflects the instructional design of the course as communal practice. For instance, in the first session, the professor contrasted two notions of differential equations: its algorithmic aspect and its meaning as language for describing changing contexts. However, the instructional design does not merely reflect the instructional designer's private belief. It is related to the historical and cultural meaning of differential equations grasped by him/her at a fundamental level. The subject of differential equations was historically invented as a language describing motion, that is, continuously changing situations such as the two-body problem in astronomy, spring-mass, and elasticity (Kline, 1972). In this aspect, fictive motion metaphor is the historically and culturally legitimate model of differential equations shared within the mathematics community. This suggests that change in the use of metaphor indicates a renegotiation of the students' conceptual model with respect to historical and cultural organization of the mathematics community.

CONCLUSION

The purpose of this research was to describe the social transformation of students in a differential equations course based on the analysis of mathematical metaphor that students used in class. The analysis shows that machine metaphor and fictive motion metaphor were the most dominant kinds of metaphor used by the students. Through meaning negotiation, the students realized their weakness and strength as a language for differential equations. Throughout the semester, the students gradually came to apply fictive motion metaphor more frequently and efficiently. This tendency in the use of metaphor is interpreted as indicating the social transformation of students' conceptual models of differential equations in the sense that the students came to grasp the cultural legitimacy of fictive motion metaphor.

Although metaphor used in the class were classified into two types, each participants used metaphor creatively based on one's own understanding of the problem context. As a result, diverse perspectives evolved and through meaning negotiation, they emerged to form shared mathematical meaning in the class. As a consequence, the participants develop their own culture of mathematics in the class. In this regard, the mathematics class can be seen as a community of practice. It has its own unique culture which is distinct from the culture of the larger mathematics community but not entirely.

This suggests that the meaning negotiation in mathematics class is not an arbitrary process but firmly grounded to the history and culture of a larger mathematics community. Through participating in the practice of mathematics mediating a historical and cultural structure of a mathematics community, a learner becomes transformed to a certain kind of a cultural being with historicity. Then, what creates a link between the two communities? Although the culture of a mathematics class is the product of co-engagement among the participants, it is important to note that a mathematics teacher plays a critical role in the process. Even though the professor did not instruct in the class, she insinuated her mathematical perspective by her use of metaphor and in terms of her

instructional design. For instance, context problems played an important role in the class in the sense that they affected the students' choice of metaphor. Also, it was observed that technology contributed to the formation of cultural models of differential equations in the class by affecting the process of metaphorical switch. However, it is fundamentally a mathematics teacher who selects context problems and technologies in order to infuse one's own mathematical perspective of legitimate practice of mathematics. In this regard, there is asymmetry in the relation between a teacher and a student in terms of whose notion of mathematics is to be legitimized. However, a teacher's intervention is critical when it lays the foundation of a future possibility instead of confining students' potentiality.

This description of social transformation in mathematics class ultimately reveals the intricateness of learning mathematics. Learning mathematics is experience of liberation but at the same time of enculturation with respect to a specific history and culture. How do these two poles of experience lead to the creation of a learner as an integrated whole? What is the role of a mathematics teacher in the process of social transformation? What resources does a mathematics teacher rely on to support the process? Further investigation of social transformation will help to provide answers to these questions, based on the historicity and culture of mathematics.

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