ABSTRACTING THE DENSITY OF NUMBERS ON THE NUMBER LINE – A QUASI-EXPERIMENTAL STUDY

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The starting point for this study was the resistant nature of prior knowledge in conceptual change from natural numbers to rational numbers observed in our previous study. Thus, in this study the effects of deliberately teaching the abstraction of the density of numbers on the number line was tested in a quasi-experimental study at the beginning of students' first course in calculus. The results suggest a significant and stable effect in the post test between the test group and control group. This effect had an important relation to the level and quality of abstraction of the number line, which was found in the short essays written by the participants of the test group during the intervention.

INTRODUCTION

In the discussion of conceptual change in science learning (e.g. Vosniadou, 1999) the term "conceptual change" has referred to a radical change or reorganization of prior knowledge which is typical to learning of concepts in science. Thus, there are plenty of empirical research from the viewpoint of conceptual change for example in the field of biology (e.g. Ferrari & Chi 1998; Hatano & Inagaki, 1998) physics (e.g. Vosniadou, 1994; Ioannides & Vosniadou, 2002) and some also of mathematics (e.g. Merenluoto & Lehtinen, 2002a and b; Vamvakoussi & Vosniadou, 2002). The most characteristic results in these studies were the resistant nature of students' prior knowledge, and the fact that the students' difficulties primary seemed to be due to the quality of prior knowledge of the student than to the complexity of the concept to be learned.

Results from above mentioned studies in the field of mathematics refer to mistaken transfer of the properties of natural numbers to the domains of more advanced numbers. In the midst of this mistaken transfer is the fundamental discrete nature of natural numbers. For the learner, it poses a cognitive conflict with the compact nature of rational numbers and the continuum of real numbers. By the term discrete is meant the instinctive feeling connected to numbers that there is always the "next number" or to be precise: that every number has a successor and no two numbers have the same successor (e.g. Landau, 1960). This intuition of a "next" number seems to have the same self-evident, self-justifiable and self-explanatory characteristics as primitive intuition described by Fischbein (1987) thus easily leading to overconfidence. As such it seems to act as an obstacle for conceptual change in more advanced domains of numbers: revision the thinking of numbers does not even come to mind for the students.

In our extended survey (reported e.g. in Merenluoto & Lehtinen, 2002 a and b) a majority of the students on upper secondary school level still spontaneously used the abstraction of discrete natural numbers familiar to them in describing the density of rational and real numbers on the number line. There were, however, students who referred to the infinite density of numbers using operational explanations like the possibility to add decimals. This kind of explanation seems to be based on an abstraction, which is very similar to the abstraction of continuity of natural numbers: you can always add more numbers. Thus, according to Vosniadou (1999) this kind of learning seems to be based more on enrichment of prior knowledge than on a radical change. Any indications to a more radical conceptual change in thinking of numbers were found only in very few explanations where the students stated how the “next” number is not defined in the domain of rational and real numbers.

In order to gain more understanding of supporting the conceptual change from discrete numbers to the density of numbers, a teaching experiment was planned according to the suggestions of Ohlsson and Lehtinen (1997) who claim that abstraction is a prerequisite of learning; contrary to the traditional thinking where learning is seen as a procedure from concrete to generalization or abstraction. This means that for the learner two objects or relations are seen as similar to the extent that they fit the same abstraction. But to fit something to an abstraction the learner must already possess that abstraction, and moreover, what is needed, is that the teacher deliberately focuses the attention of the students to the essential features of the abstraction (see Dreyfus 1991).

AIM OF THE STUDY

The aim of the study is to describe our experiment and its results, especially how the effect was related to quality of abstraction of the number line the participants presented in the beginning of the intervention.

METHOD

Participants and design

Two groups of students on upper secondary school level and studying advanced courses in mathematics was selected for the study. The selection of the groups based on the results from our previous study: both groups used the same textbook in mathematics and both schools were scored above median in our previous survey, but they were not the same students.

A quasi-experimental design with pre, post and delayed tests for both groups was planned. The pre test was carried out at the beginning of students’ first course in calculus, the post test about six weeks later, and the delayed test again about six weeks later. A short intervention period (2 hours) described below, was organized for the test group after their pre test, while the control group proceeded in the normal way. The test group consisted of 21 students (9 boys and 12 girls, age 16-17 years) and the control group of 25 students (15 boys and 10 girls, same age) who participated in both pre- and post tests.

Procedure of intervention

The intervention was planned by us, but carried out by the mathematics teacher of the test group, who was following our plan. We wanted to involve the teacher to the planning period, but he was not willing to. The intervention period was, however, tape recorded and observed. It had three phases. In the first phase the teacher began with the elements of everyday knowledge of the students, and started a discussion on numbers, next numbers and continuity of natural numbers. During the second phase the students were given an exercise, where they were asked to mentally add numbers on the number line with the help of given questions (Table 1) and then write with their own words how they think of it. This was done in order to activate students’ prior knowledge on the subject, to make them aware of their own mental conception of numbers and to define the quality and level of their abstraction of the density of numbers on the number line in the beginning of the experiment. These short essays were qualitative analysed.

> In this exercise we ask you to consider, how you "see" or think the numbers on the number line and write it down. It is likely that everyone "sees" them a little differently. Try to be as exact as possible. You can use following questions to assist your thinking:
>
> How do you "see" the number line? – Explain in your own words.
>
> Add the numbers 1, 2, 3 … on the line in your mind. How long it is possible to continue?
>
> Continue then with fractions $^1/_2$, $^1/_3$, $^1/_4$, $^1/_5$ … How long it is possible to continue?
>
> On what interval on the number line are these numbers placed?
>
> How many irrational numbers it is possible to add between the numbers given above?

TABLE 1. The instruction for the exercise, given to the test group in the second phase of the experiment.

At the third phase, the teacher guided the students into a discussion of different domains of numbers, where their attention was deliberately focused on gradual adding the density of the numbers on the number line with rational and real numbers. Then, the teacher presented one number on the blackboard and asked if anyone could give a number, which is close to it. He asked the students if anyone could give a number which could be still closer to the given one. He even made a bet, and promised a small sum of money to the student, who could give a number, so close to the given number that the teacher would not be able to find a number still closer. Some of the students, especially boys, had their expressions brightened of the hope for some extra money. But during the process they experienced, how it is always possible to find a number which is closer to the given number than any given positive number. Finally the teacher presented the concept of limit as an abstraction of infinite divisibility, where the "next" number is not defined.

Pre-, post and delayed tests

The questions pertaining to the density of numbers on the number line in the tests were from the same questionnaire which we used in our previous survey, reported before (Merenluoto & Lehtinen, 2002a and b), namely: *"How many fractions/ real numbers there are between two given numbers (3/5 and 5/6; .99 and 1.00) on the number line*"; and the questions posing a cognitive conflict: *"Which one is the 'next' after 3/5 or 'closest' to 1.00"* (four items, alpha .828). The answers were scored on scale from zero to four, thus the maximum score was 16 points.

RESULTS

In the test group students' grades in mathematics had a mean of 7.5 (on a scale 4 - 10) and the respective measurements for the control group were 7.1, the difference was not statistically significant. Likewise there were no statistical differences between the groups in the scores of the pre test. The overall results run parallel with our previous survey: the formal understanding of the different domains of numbers was quite low.

Effect of the intervention

The scores in post- and delayed tests had a significant difference between the test group and control group. MANOVA (repeated measures) revealed a significant main effect of the intervention, F (1, 44) =10.46; p=.002; with an effect size _^{2}=.192 and power .885, between the pre-and post tests. In the delayed test the loss in the control group was notable, more than half of the students did not want to bother in answering the same questionnaire third time. But there was difference between groups also in the delayed test, (_^{2}= 3.60, p=.05). The results (Fig.1) refer to a quite stable effect.

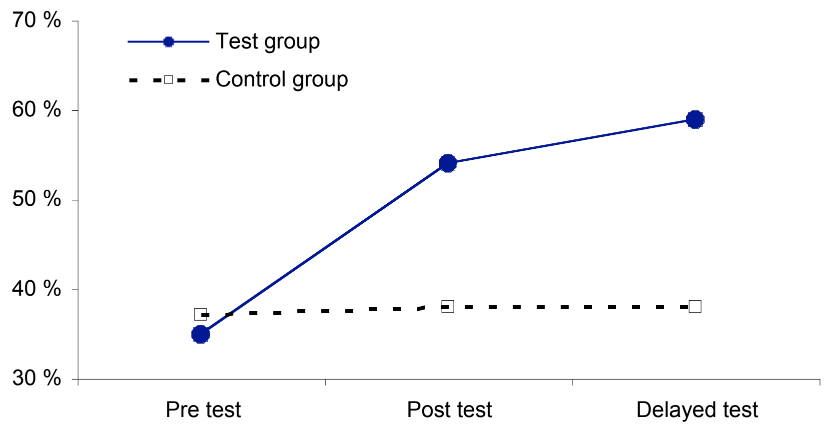

Figure 1. Means of scores as percentages of maximum for both groups in pre, post and delayed tests

In an analysis of <u>change between the scores in pre- and post tests</u>, the students were divided into four distinctive groups: (1) three students (14%), with low scores in both tests; (2) ten students (48%) with a small change (better answer only in one of the density questions), (3) four students (19%), with a prominent change: low scores in the pre test and high scores in the post test, (4) four students (19%), who already had high scores in the pre test. The respective grouping in the control group yielded to a statistically different distribution respectively: 56%, 20%, 4%, and 20% (Mann-Whitney, z =-2.89; p=.004)

Essays of the density of numbers on the number line

In most of the descriptions of the students from groups one and two, it was easy to identify the model of number line presented in the text books on mathematics.

<u>The students in the first group</u> were girls, who described their thinking of the number line in a very concrete way, or had mistakes in their descriptions, like follows:

> My number line is so long that nobody is able to measure it, I can not even image which is the smallest or largest number and if they even exist. When I think the numbers 1, 2, 3… it is long but when I think the numbers _, $^1/_3$ … it still gets longer… they all go between numbers 1 and 2 or between numbers 2 and 3. – Lisa[1]

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[1] All the names have been changed.

> My number line is a line with small cross lines on it. I'm able to think numbers until number 10, but then the rest are so far away that I'm not able to see them. I'm not able to put all those fractions on it, because there is not enough space – may be I could fit a couple of irrational numbers on it, though.– Lena

Their explanation had a very low level of abstraction. In both pre- and post tests they gave answers based on the abstraction of discrete numbers, for example: "*the fraction 4/5 is the next number after 3/5, because it just popped into my mind – Lena."*

<u>In the second group with a slight change between the tests</u>, there were nine girls and one boy. To this group it was typical that they explained in their essay of the number line, how it is possible to endlessly add fractions between numbers. In the post test they, however, used the familiar abstraction of discrete numbers in describing the density of numbers. The only change in their answers in the post test, was that they explained, how there are an infinity of real numbers between .99 and 1.00, because it is possible to add decimals. Typical to these students was that they had hardly any explanations to their answers in the tests. In their essays they used a little higher level of abstractions than those the previous group. For example, Karen explains about the possibility to change the scale of the number line:

> My number line just continues… and it is possible to change its' scale if needed. There are whole numbers on even intervals and a lot of fractions between them. My number line is like a world and you can use it to examine the space, globe, but also the tiniest bacteria with it.

Maria refers to the possibility to add decimals:

> My number line continues without an end… It is possible to continue as long as you want to. It is possible to write endlessly fractions depending on how many decimals you use. First I examined only integers, then my picture enlarged and the numbers were closer to each others… finally their space between them disappeared when I got into smaller numbers.

Sara described her number line between concrete and imaginary associations, but stated how she felt able to imagine infinity of numbers if needed:

> My number line continues without an end, it has small cross lines on it with numbers above. … I associate it to a thermometer… It is so long that it goes around the world…There are fractions also and it is possible to write them until knowledge and skills come to their ends. Every number has its place on the line and so have the fractions also. It's impossible to see the irrational numbers, but if needed I feel able to imagine them on it.

<u>The students in the third group </u>were boys, with a prominent change between the tests. They gave answers based on abstraction of discrete natural numbers in the pre test, but in the post test they referred to infinity of numbers between rational and real numbers and to the impossibility to define the "next" one, reasoning these with the possibility of adding decimals. The prominent change might have a relation to their somewhat more elaborative explanations in their essays. One of them had frustrated expressions also in his essay.

Eric, for example, had identified the logic of numbers which for him was the same for natural numbers and fractions and told of about his versatile ability to use numbers:

> I see numbers in a different way depending on the situation and I'm able to operate with them on versatile ways, only very large and very small numbers are difficult to handle. The

> numbers have their logic order on the number line and while it is easiest to think with whole numbers, it is as easy to think the fractions because the logic is the same, they fit between _ and zero.

Tom's explanation had an advanced abstraction how all that is needed are the numbers between zero and one. Especially in the post test, he seemed to be a little frustrated while he explained: *"by adding decimals you are always able to find a larger number, because mathematics has its own incomprehensible world of peculiar concepts"*. Indications of this kind of frustration and of thinking numbers mostly at school situations were, however, visible already in his essay at the beginning of the intervention:

> I can see it as an endless horizontal line, and I can think numbers on it until 10, after that I don't want to be bothered to do unnecessary thinking, it is not needed here. It's no use to put small fractions on the number line, only thing you need is the interval between zero and one – which you can expand as long as you want to. It is too much for me to think of the cardinality of irrational numbers, because it is easier to think the number line as a line of whole numbers. Some numbers I associate to different situations, some are lucky numbers, and sometimes they are jus a grey mass for me. Quite often I associate them only to mathematics and after that to some of my teachers.

<u>The students in the fourth group </u>were boys, who gave high level answers already in the pre test and most of them even higher in the post test. These students were patient with the tests, and gave advanced level of explanations. Their essays referred to spontaneous and intentional pondering of numbers. They demonstrated some unique and elaborative ways to deal with the abstraction of number line, which for them still was somewhat framed with the concrete. For example, John was tackling this problem by writing about the difficulty from uneven divisions:

> I see number line as a long line, but in order to help my thinking there are cross lines on the places of whole numbers like in the set of coordinates. It is not difficult to think of integers, but the difficulty comes from uneven divisions. It's easy to fold a paper in half or in four parts, but 1/3 is not so easy and because of that, thinking of those numbers is more difficult. Thus, with paper it is easily done, but as imagined it is difficult to focus – the numbers smaller than 1/20 are already difficult to think, they finally pile up to a same place.

Tim tackles the problem by abstracting a third dimension for his dynamic model of number line:

> As a whole I see the number line from very far – a vague mass of white numbers on black background. Then when I think the numbers in the beginning, like numbers 1-10, I zoom closer. If I want to, I can browse the integers back and forth, which I could continue forever, but usually it's not needed…. If I want to examine the fractions I zoom deeper between numbers one and two, take the left half of the interval and zoom in to that. It is possible to continue forever, but usually 1/21 is already too much. It is frustrating to think of something that is endlessly continuing. In order to examine the irrational numbers I usually zoom back outside of the number line, take an approximate value of needed number and using that dive into the depths.

In Tim's explanation, there was a little mistake, but his description on finding the needed irrational number with the help of approximate values was profound.

CONCLUSION

Most of these students were still far from a radical conceptual change in their number concept, namely changing their frame of reference of numbers from natural numbers to real numbers with rational numbers, integers and natural numbers as sub sets. However, this experiment, although it was small, gave some important results for future research in finding ways to support the conceptual change in mathematics.

The main effect of this intervention was evident in the groups two and three, where the students totally or partly changed their abstraction of discrete numbers to an operational level of abstraction of density of numbers by adding decimals and moreover, this change seemed to be quite stable. Thus the main results from the intervention were that it is relatively easy to foster a change as long as it stays on the operational level or as an enrichment kind of change.

In order to foster a more radical kind of change in mathematics, metaconceptual awareness referred also by Vosniadou (1999) seems to be mandatory. This means that the students need to be aware of how they think about numbers and consciously pay attention to the differences between different kinds of numbers. The writings of the students in the fourth groups referred to this kind of situation. The elaboration level of the best writings refers to metacognitive skills of the best succeeded students. Thus it suggests also to the need to consider metacognitive, motivational and intentional factors in the endeavours to facilitate conceptual change. Using writing as a tool in teaching mathematics (e.g. Conolly & Vilardi, 1989) an in research of supporting conceptual change has been used earlier (e.g. Mason, 2001) according to our results it might be a useful especially in teaching metacognitive skills.

The writings and answers of the students in the two first groups revealed especially the low level of mathematical thinking of these students. Thus, what are needed are more experience and exercises to awaken and develop the metacognitive skills in mathematics. Low levels of abstraction were also related to deficiencies in basic knowledge of numbers. There were also a clear division between boys and girls, all the girls were in the two first group and almost all the boys in the third and fourth group. It is possible that the possibility of earning extra money during the intervention called especially the attention of the boys who sat in the front part of the class, while all the girls sat in the back. Thus, the boys had a real experience of the potential infinite division. On the other hand in several studies it has been found that in questions pertaining to infinity, girls seem to be more cautious in their conclusions, which was found also in our previous study (Merenluoto 2001).

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