FUNCTION AND GRAPH IN DGS ENVIRONMENT

Maria Alessandra Mariotti*, Colette Laborde** & Rossana Falcade**

*Dipatimento di Matematica di Pisa – **Laboratoire IMAG – Grenoble

Assuming that dynamic features of Dynamic Geometry Software may provide a basic representation of both variation and functional dependency, and taking Vygotskian perspective of semiotic mediation, a teaching experiment has been designed with the aim of introducing pupils to the idea of function. First data coming from the observations in Italian and French classrooms are presented.

INTRODUCTION

Since a long time, the notion of function has been at the core of a great number of studies, and the rich literature reports on a number of difficulties related to different aspects of the notion of function (Goldenberg, Lewis and O'Keefe 1992, Harel and Dubinsky 1991, Sfard 1991, Sierpinska 1992, Tall 1991, Vinner and Dreyfus 1989, Leinhardt, Zaslavky & Stein 1990). Difficulties of interpreting graphic information in terms of function are widely reported. It seems that for students there is a lack of explicit relationship between function and graph[1], (Vinner & Dreyfus, 1989, Dreyfus & Eisenberg, 1983); students are not able to move from the one to the other identifying domain and image of function or confounding decreasing behavior with negative values (Trigueros, 1996). Rigid and stereotyped ideas are often related to functions and their graphs (Markovits et al. , 1986, 1988 ; Schwarz & Hershkowitz, 1996).

Even with Graphic calculators students show difficulties to relate the graphs they saw on their GC to the algebraic representation of the function (Cavanagh & Mitchelmore, 2000). In summary, students have problems to grasp the idea of function as a relationship between variables (one depending on the other). Pupils have a discrete view of a function relating separate pairs of numbers, where each number may be considered as an input giving another number as result; pupils consider that there is a relationship between numbers, but the relation is conceived separately for each pair. In any case, the relationship of dependency between the two variables is not visible in the graph, that remains a static representation of the couple (x,y) and does not afford the meaning of dependency between the two variables that rather play a symmetrical role.

This paper is based on the starting assumption that the crucial aspect of the idea of function is the idea of variation or more precisely of co-variation, i.e. a relation between two variations (cf. below section Function as change).

Primacy of numerical setting (Goldenberg et al., 1992) and the lack of experience of functional dependency in a qualitative way may be considered as the main sources of the students' difficulties. This is why we assume that it is important to start in an environment providing a qualitative experience of functional dependency, independently

[1] In French the word "graphe" refers to the algebraic notion of ordered pairs of numbers and the curve is called "répresentation graphique" (graphical representation). According to the current use in English, in this paper we will use " graph" to refer to both meanings.

of a numerical setting. DG incorporates functional dependency and allows thinking the geometrical links in terms of functional dependency.

Starting from these assumptions and taking the Vygotskian perspective of semiotic mediation (Vygotsky, 1978, Mariotti, 2002) as well as using the theory of didactic situations (Brousseau, 1997), teaching experiments have been set up. Some data coming from classrooms observations will be presented here.

FUNCTION AS CHANGE

Although not expressed in the classic mathematical definition of function, the idea of variation and co-variation is a crucial component of the notion of function, as Tall clearly states: “One purpose of the function is to present how things change” (Tall, 1996. p. 288). Our assumption is that grasping the idea of function requires grasping the idea of co-variation, i. e. conditional change. As cognitive analysis highlights, motion – space changing over time – can be considered as one of the basic primitive perceptions of “dynamic and continuous” (by using the terms of Malik, 1980) variation.

A deep gap separates early notions of function, based on an implicit sense of motion, and the modern definition of function, that is "algebraic in spirit, appeals to discrete approach and lacks a feel for variable❒ (Malik, 1980).

A connection to the basic metaphor should be preserved in the idea of graph, that is in the spatial representation of a function in a coordinate plane; however, this can be done by considering the graph as the trajectory of a moving point P (Laborde 1999, p.170), representing the dependent variable, according to the variation of a variable point M on the axis of abscissas, representing the independent variable. This complex interpretation requires to reintroduce time and to consider the co-variation of P and M as a relation between two interrelated variations depending on time, what we call a <u>dynamic interpretation of a graph.</u>

Unfortunately, this dynamic interpretation is often neglected in the textbooks. In any case, a dynamic interpretation of a graph cannot be externally experienced and remains a sort of mental experiment, impossible to be shared. Unlike the paper and pencil environment, which cannot afford the representation of change through motion, the DGS environment can. In DGS environments the idea of variation is grounded in motion, so that it is possible to experience variation in the form of motion: points can be dragged on the screen and represent the basic variables. As a consequence, DGS environments incorporate and represent the idea of variation and that of functional dependency.

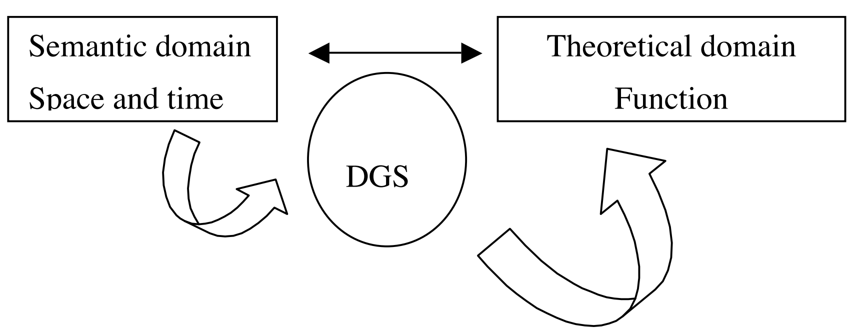

Figure 1

Thus DGS offer a powerful environment incorporating the semantic domain of space and time, where the notion of function can be grounded. We call that particular instance of function "Dynamic Geometrical Function". This general idea can be interpreted in a Vygotskian perspective, according to the notion of semiotic mediation. The following section will develop this idea and explain how it was used in the design of our experiment.

THEORETICAL FRAMEWORK: TOOLS, SIGNS AND MEANING

Vygotsky distinguishes between the function of mediation of *technical tools* and that of *psychological tools* (or *signs* or *tools of semiotic mediation*) (Vygotsky, 1978: 53). The use of the term *psychological tools*, that refers to signs as internally oriented, is based on the analogy between tools and signs, but also on the relationship that links specific tools and their externally oriented (for the mastering of nature) use to their internal counterpart (for the control of oneself) (ibid.: 55).

Through the complex **process of internalization** (Vygotsky, ibid.), a tool becomes a «psychological tool» and will shape new meanings; in this respect a tool may function as a semiotic mediator.

As far as the DGS Cabri is concerned, previous studies (Mariotti & Bartolini, 1998; Arzarello, 2000, Mariotti 2001,Mariotti & Cerulli, 2001, Mariotti, 2002,) focused on the analysis of the specific elements of the microworld (dragging facility, commands available, macro …) as instruments of semiotic mediation that the teacher can use in order to introduce pupils to mathematical ideas. We assume this theoretical hypothesis in the case of a set of particular tools, among those available in the microworld Cabri, and the meaning of function, as discussed above. In particular, assuming motion as a primitive metaphor of variation in the semantic domain of space and time, a crucial element emerges as a privileged representative (metaphor) of the idea of variation: the idea of trajectory.

The notion of trajectory entails the twofold meaning of motion: punctual and global; a trajectory is at the same time a sequence of positions of a moving point with respect to time and the whole object consisting in the set of all such positions.

The Trace tool, activated on a point, provides the track of the motion of that point. Trace tool can be used in the case of both a direct variation by dragging and an indirect variation . Although the final product of the Trace tool is a static image consisting in a set of points, the use of Trace tool involves time: actually, one can feel time running in the action of dragging, in particular when changing the speed of dragging, but also one can feel time running in the variation of the dependent point. As a consequence it is possible to grasp simultaneously the global and the punctual aspect of the product of Trace, which can be related to the global and punctual aspect of a **trajectory**.

In this sense**,** the use of the Trace tool and the mental constructs related to it, may refer to the idea of trajectory, and it can be considered as a potential tool of semiotic mediation for the mathematical meaning of trajectory. That supports our main hypothesis about the use of Trace as a tool of semiotic mediation.

Personal meanings concerning the idea of variation and co-variation as it emerges from pupils' activities in the Cabri environment may evolve into the mathematical meanings of function as a covariation .

THE TEACHING EXPERIMENT

Basic assumptions

Taking into account the main results obtained from the analysis on the correspondence between some Cabri tools and the meanings related to function, a sequence of activities was designed and implemented in two classes, in France and Italy (10^{th} grade). This sequence was based on some general assumptions about the teaching/learning processes:

- Tools are part of the construction process of meanings, then they can be used by the teacher to foster this process according to intended meanings
- Learning is both an individual and a social construction.
- As a consequence the general structure of the activities consisted of two stages:
- firstly students were faced with tasks to be carried out in the Cabri environment,
- the various solutions were then discussed collectively under the guidance of the teacher.

General structure of the sequence

The activities of the sequence can be grouped in three phases:

1. Introduction of the variation and co-variation through exploring the effect of Macro constructions; a first definition of function, domain and image was given, based on the interpretation of a geometric situation in terms of function. "Dragging", "Trace tool" and "Macro tool" are the key elements on which semiotic mediation is based.
2. Introduction of the idea of graph, working on a text, drawn from the original work of Euler (Euler, 1743). According to this text, a graph of function gives the means to represent geometrically a numeric function.

> "[…] Because, then, a unlimited straight line represents a variable quantity x, let's look for a method equally comfortable (useful) to represent any function of x geometrically.
>
> […]Thus any function of x, geometrically interpreted in this manner, will correspond to a well defined line, straight or curve, the nature of which will depend on the nature of the function.(Translated by the authors)
>
> (Euler, 1743, p. 4 -5)

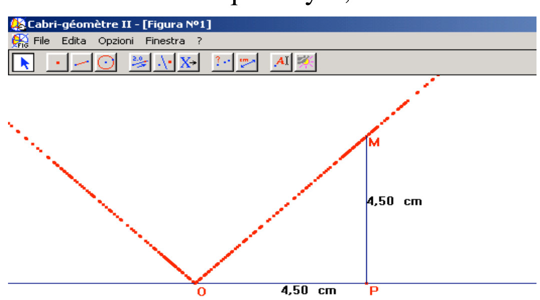

Fig. 2 Graph of f(x) = |x| according to Euler's method

Euler's method consists in generating the **trajectory** of the extremity M of a segment PM. One of the extremities, P is a variable point of the axe of abscissas and has a distance x from the origin O. The other extremity, M, is on the line perpendicular to the axe of abscissa passing through P and such that PM measures f(x). Figure 2 shows how, according to Euler, the graph of the real function f(x) = |x| would appear in Cabri.

3. the use of the graph of a function in solving problems.

In the following section, some results will be discussed, concerning the first phase of the sequence, when the idea of function has to be related to the grounding metaphor of motion and in particular to the idea of trajectory.

THE FIRST ACTIVITIES AND THE NOTION OF TRAJECTORY

The first teaching session is carried out in the computer laboratory. Pupils are grouped in pairs and must produce a common written answer on a worksheet. In the first task the students apply an unknown macro-construction to three given points□A, B and P□they obtain a fourth point H. They have to explore systematically the effect of moving a point and have to fill in a table explaining what moves and what does not move when dragging each point. In a second question, they are suggested to use the Trace tool and asked to observe what happens, then to describe the movement of the different points, using the current language of geometry.

Students easily solved question 1 in both countries. They all perceived the difference between direct move of points A, B and P, which can be dragged directly by taking them with the mouse and indirect move of H, which cannot be grasped with the mouse but moves only when A, B or P is dragged. This became a reference situation on the basis of which the meaning of variable (dependent and independent) was going to be treated.

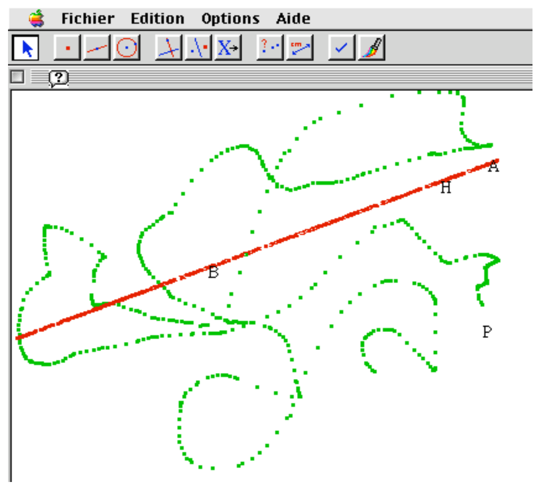

Fig. 3 Traces as they appear on the screen

The analysis of the pupils' answers confirms our main hypothesis about the contribution of the use of Trace in the emergence of the twofold meaning of trajectory. In fact, both the conception of trajectory as an object and as ordered sequence of points can be found in the pupils' formulations, as shown in the following examples drawn from pupils' written reports. Different meanings could be drawn from the interpretation of their words and expressions. ((I) Italian pupils and (F)French pupils).

> Federica (I) drew the trace of both the variable point (independent variable) and the point H (dependent variable) and wrote:
> "Dragging B, H <u>forms a circle</u>, passing through P and A
> Dragging P, H forms a straight line, passing through B and A, which touches the circle in two points .
> Dragging A, H forms a circle, passing through P and B."
> Laurent (F) writes: "Quand on déplace P, le point H fait une droite qui suit [AB].
> Quand on déplace A, le point H <u>forme un cercle</u> par B et P. …"

The expression "forms a circle" refers to the global aspect of the trajectory, the circle is the final product of a completed process.

In addition to the global conception of the trajectory (H forms a circle, a line…) the idea that point H is moving on that object was expressed.

> Andrea (I):"When the position of P varies, H leaves a trace which always stays on the line passing through A and B"
>
> Tiziano (I): "If one moves (drags) point P, H moves on the line containing (comprende) the segment AB"
>
> Catarina (F): "Quand on utilise A, H fait un cercle passant par B et P. … quand on utilise P, H se déplace en ligne droite en passant par A et B."
>
> Sarah- Julia (F): "Quand A bouge, H décrit un cercle autour de diamètre BP"
>
> Sonia (F): " Si on déplace A, A forme une trajectoire quelconque et H se déplace sur un cercle de diamètre BP."

Expressions like "describes", "se déplace en ligne" incorporate both the components of the conception of trajectory (global, as an object, and pointwise, as the sequence of positions taken along the time).

The students' answers show the use of dragging to identify on the one hand the nature of variables and on the other hand the domain and the image of function as trajectories.

The dragging tool moved in the following tasks from an external use to an internal one for several students. For example, Chrystelle and Cécile hesitated when identifying between points P and Q the dependent and the independent variables. Finally they evoked the dragging test which led them to write the correct relation F(P) = Q.

> Chrystelle and Cecile wrote:"We associated point P to point Q because when we move P Q is moving on the trajectory (NN'). Therefore Q depends on P (F(P) = Q).

This latter example gives evidence of the internalization of the dragging tool; the following episode gives evidence of the fact that an internalization process was achieved and that the Trace tool became a 'psychological tool'.

THE CASE OF SONIA AND JULIE

Faced with the task of conceiving their own function Sonia & Julie correctly described a construction relating point M' to an independent point M and commented:

> "M' varies in the plane and it's the symmetrical of M with respect to the angle bisector of angle MRM'. M' is dependent of R (M?) through a geometrical transformation."

According to the notes of observation, they used the Trace tool to check that M' depends on M: they activated the Trace Tool on both M and M', then by dragging M, verified that M' is moving because M' is leaving a trace. That is the use of Trace that was initially introduced in relation to dependency between variables. However, evidence of the internalization of this tool is given by the fact that it was also used to solve a new problem. In fact, in order to determine the nature of the function relating M and M', the two girls decided to move M along a small curve, deliberately regular (a knot), looking for an analogy between the trajectories of M and M'. They realized that the two traces looked identical, after this observation they identified the function as a reflection. This may be interpreted in terms of semiotic mediation.

The image and domain of function as reified by Trace are intentionally utilized in checking a conjecture based on their previous geometrical knowledge about reflection. That means that the Trace tool is used not only in exploration (externally oriented), but also in the reasoning which leads to the solution of the problem: it has become an intellectual tool (internally oriented).

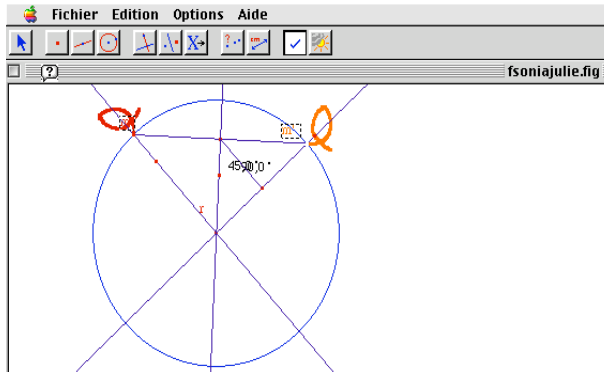

Fig. 3 What appears on the screen of Sonia and Julie

CONCLUSIONS

As shown in the previous example, the expected results occurred in both countries. Pupils seem to have grasped variability as motion. From the combination of observation and action emerges the idea of co-variation, which is experienced through the coordination between eyes and hands and is incorporated in the conditional movement of points on the screen.

Similarly the twofold conception of trajectory, global and punctual, clearly emerged, in relation to dragging and Trace tools. We can say that the internalization of the dragging and Trace tools has contributed to construct both the global and the punctual aspects of the idea of trajectory, and this in connection to the introduction of the notions of domain and image of function. According to our hypothesis, the twofold conception of trajectory as object and sequence of points has been reinvested both in the interpretation of Euler's text and in its appropriation as a method to conceive dynamically a graph of function both as a set of points and as a curve. The size of this report does not allow to go further (see Falcade 2002 for further analysis).

Starting from these results which show clear evidence of the power of mediation provided by Cabri tools, new teaching experiments have been designed and are in progress, aimed to investigate the mediation process. They allow to refine our assumptions but confirm the main one concerning the idea of covariation. At the end of the second teaching experiments, both in France and in Italy (2002), the students were asked to express the conditions for two functions to be equal. It is possible to observe the emergence of the idea of coincidence point by point, as it is originated by the coincidence trajectory by trajectory. In the same time, most of the students, asked to describe the similarities and differences between geometrical and numerical functions, explained that the similarity consists in the dependence between two kinds of variables.

A final remark: previous descussion does not take into account the role of the teacher in the process of evolution of meanings. The role of the teacher develops at the meta level, when guiding the evolution of meanings: it becomes determinant in a process of semiotic mediation. Further investigations into the delicate role played be the teacher have been designed and are currently underway.

References

Arzarello, F. (2000) Inside and outside: spaces, times and language in proof production. , In *PME Proc.* Hiroshima, II, pp I, 1-23

Brousseau G. (1997) Theory of didactical situations in mathematics. Kluwer Ac. Pub.

Cavanagh,M. & Mitchelmore, M. (2000) Student misconceptions in interpreting basic graphic calculator displays, In *PME Proc.* Hiroshima, II, pp. 161-168,

Dreyfus, T. & Eisenberg, T. (1983) The function concept in college students *Focus on the Learning Problems in Mathematics*, 5, 119-132.

Euler, L.(1743) Introductio in analysin Infinitorum Tomus secundus Theoriam Linearum curvarum , Lousannae.

Falcade R. (2002) Cabri-géomètre outil de médiation sémiotique pour la notion de graphe de fonction, *Petit x*, 58, 47-81.

Goldenberg E. P., Lewis P., and O'Keefe J. (1992) Dynamic representation and the development of an understanding of functions, In Dubinsky E., and G. Harel (1991.)

Harel G. and Dubinsky E. (1991) *The concept of function Aspects of epistemology and pedagogy,* Washington, DC Mathematical Association of America .

Laborde, C. (1999) Core geometrical knowledge for using the modelling power of Geometry with Cabri-geometry, *Teaching Math. and its Application*, Vol. 18 n°4.

Leinhardt, G., Zaslavky, O. & Stein, M.K. (1990) Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research*, v60 n1, 1-64 .

Malik, M.A. (1980) Historical and pedagogical aspects of the definition of function. *Int. Journal of mathematics education in science and technology* , 11 (4), 489 –92

Mariotti, M.A. & Bartolini Bussi M.G. (1998) From drawing to construction: teachers mediation within the Cabri environment, in *PME Proc.* , Stellenbosh, I- 180-95.

Mariotti, M.A. (2001) Introduction to proof: the mediation of a dynamic software environment, (Special issue) *Educational Studies in Mathematics* 44, (1&2), Dordrecht: Kluwer, 25 – 53.

Mariotti , M.A. & Cerulli, M. (2001) Semiotic mediation for algebra teaching and learning, *PME Proc.* The Nederland .

Mariotti M. A. (2002) Influence of technologies advances on students' math learning, in English, L. et al.(eds) *Handbook of International Research in Mathematics Education*, LEA, 695-723.

Markovits Z., Eylon B. & Bruckheimer , M. (1988) Difficulties Students have with the Function Concept, *The ideas of Algebra, K-12*, 43-60, NCTM Yearbook

Schwarz, B.B. & Hershkowitz, R. (1996) Effects of computerized tools on prototypes of the function concept, *PME Proc.*, Valencia, IV –267- 274.

Sfard A. (1991) On the dual nature of mathematical conceptions reflections on processes and objects as different sides of the same coin, *Educational Studiesin Mathematics,* 22 (1), 3-22

Sierpinska A. (1992) On understanding the notion of function, Harel & Dubinsky (1991), 25-58,

Tall D. (1991) The psychology of advanced mathematical thinking, D. Tall (ed.), *Advanced Mathematical thinking,* 3-21, Dordrecht Kluwer

Tall, D. (1996) Function and Calculus , In: A.J. Bishop et al. (eds.) *International Handbook of Mathematics Education* (Vol. 1, 289 – 325) Dordrecht: Kluwer

Trigueros M., et al. (1996) College Students' conceptions of variable PME *Proc.* Valencia, IV - 315-322.

Vinner S. and Dreyfus T. (1989) Images and definitions for the concept of function, *JRME,* 20, 356-366

Vygotskij L. S.:1978, *Mind in Society.* Harvard University Press.University Press.