YOUNG CHILDREN'S UNDERSTANDING OF GEOMETRIC SHAPES: THE ROLE OF GEOMETRIC MODELS

Iliada Elia, Athanasios Gagatsis and Leonidas Kyriakides

Department of Education, University of Cyprus

In this paper, we explore the role of polygonal shapes as geometrical models in teaching mathematics, so as to elicit and interpret children's geometric conceptions and understanding about shapes. Primary pupils were asked to draw a stairway of figures (triangles, squares and rectangles) each one bigger than the preceding one. Pupils use two different strategies to tackle this task: (a) conservation of shape by increasing both dimensions of the figure and (b) increasing mainly one dimension of the figure. Each strategy seems to reflect a different way of reasoning, possibly corresponding to a different level of development in geometric thinking. Implications of findings for teaching geometry are discussed.

INTRODUCTION

Previous research concerning children's geometric conceptions has provided useful foundations affecting the development of mathematics education. In the present study three dominant lines of inquiry, which have been based on the theories of van Hieles and Clements, as well as, the use of geometric models, are taken into account. According to the van Hiele theory, children initially conceive a shape as a whole and not as a sum of its parts (visual level) (Hannibal, 1999). At the descriptive level children are competent in recognizing and expressing in words the components and properties of familiar shapes. At the next level, the informal deduction level, properties are logically ordered; this means that they are deduced from one another (van Hiele, 1999). Van Hiele supports that students' progress through levels of thought in geometry is more dependent on instruction rather than on age or biological maturation, and that types of instructional experiences can foster or impede development (Clements et al., 1999; van Hiele, 1999).

Clements et al. (1999) found that a prerecognitive level exists before van Hiele level 1 ("visual level"). At this level, children perceive shapes, but are not able to identify or distinguish among several shapes (Clements & Sarama, 2000). They propose that children develop stronger imagistic prototypes and gradually gain verbal declarative knowledge. Consequently, the visual level is reconceptualised as syncretic (a synthesis of verbal declarative and imagistic knowledge) (Clements et al., 1999).

Gagatsis and Patronis (1990) approached children's geometric conceptions from a different perspective, associated with geometrical models and their use. They argue that a geometrical model of _ is a collection S of points, lines or other figures in n-dimensional Euclidean space, representing a system _ of objects or a situation or process, if the intrinsic geometric properties of the elements of S are all relevant in this representation, i.e. they correspond to properties of the system _. Therefore a "polygon", according to Gagatsis and Patronis (1990), is a convex polygon, i.e. the convex hull of a finite set of points in the plane. In a more general sense, a "polytope" is the convex hull of any finite set of points in n-dimensional Euclidean space E^n. As for the dimensions of polytopes: 0-

polytopes are points, 1-polytopes are line segments, 2-polytopes are (convex) polygons. Polytopes are, therefore, considered as geometrical models, since we are not interested in a particular polytope as a specific set of points in space, but rather in all polytopes similar to it. Similarity is an equivalence relation in the set of polytopes. A specific example that constitutes a geometrical model of polygonal shapes is the class of equilateral triangles, which is actually a similarity class whose members have the properties of an equivalence relation: reflexive, symmetric and transitive.

However, according to Robertson (1984), in the space of all similarity classes of n-polytopes, any shape of polytopes can be continuously deformed into any other with the exception of 0-polytope. This kind of intuitive thinking, which involves a continuous variation process, is called “dynamic intuition” (Castelnuovo, 1972). Furthermore, Castelnuovo (1972) pointed out that children do not easily observe figures when they are steady, but rather when they move or vary in a continuous manner.

In this context, an investigation concerning a specific family of geometrical models has been conducted in order to identify the specific ideas that young children develop about geometric figures and explore their dynamic intuition. More specifically, the aims of the study were: (a) to investigate the extent to which children would conserve the shape (constant path) of a polygon or to what extent they would implicitly use some model of variation of shape (continuous path), (b) to examine how the above processes would vary with children's age, and (c) to identify implications of findings for theoretical descriptions of children's geometric thinking, for advancing children’s understanding of shapes and thus for teaching geometry in the early childhood years.

METHODS

The sample of the study consisted of 99 children ranging in age from 4.1 to 7.2 decimal years. All subjects were asked to "draw a stairway of triangles, each one bigger than the preceding one" and to repeat the same procedure with squares and with rectangles. The results concerning children’s reactions to the above tasks were codified in three ways:

(a) "T" was used to represent “conservation of shape”, i.e. pupils attempt to increase both dimensions of the figure (complete series of figures).

(b) "O" stands for pupils attempt to increase mainly one special dimension (complete series of figures: possibly producing rectangles in the series of "squares" or a square in the series of rectangles or an isosceles triangle in the series of "equilateral triangles").

(c) The symbol "N" was used to show that pupils produced a defective series (i.e. very irregular figures or non-increasing at all in a regular way).

This paper is focused on the first two types of responses. Moreover, implicative analysis (Gras, Peter, Briand & Philippé, 1997) was used in order to identify the relations among the possible responses of students in the test tasks. Therefore, six different variables representing the extent to which students used a specific strategy when they attempted to solve the three different types of test tasks emerged. More specifically, the following symbols were used to represent the six variables involved in this study:

a) Symbols “Trt”, “Sqt” and “Ret” represent the production of a series of similar triangles, squares and rectangles, respectively, of continuously increasing dimensions

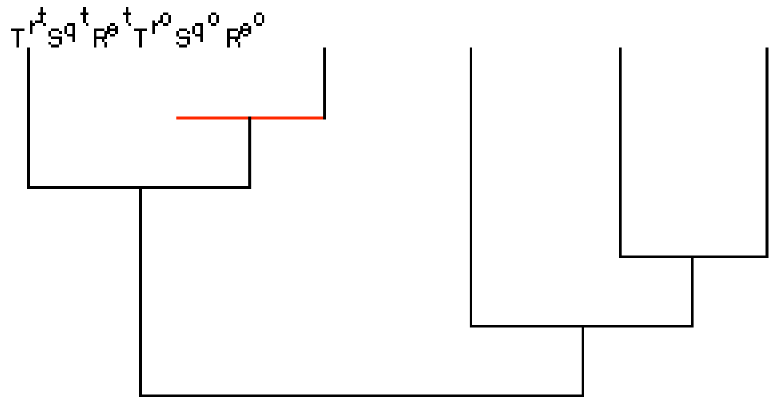
b) Symbols "Tro", "Sqo" and "Reo" represent the production of a series of triangles, squares and rectangles, respectively, by increasing mainly one dimension of the figures.

For the analysis and processing of the data collected, implicative statistical analysis was conducted by using the computer software CHIC (Bodin, Coutourier, & Gras, 2000). A similarity diagram and a hierarchical tree were therefore produced. The notion of 'supplementary variables' was also employed in the particular analysis. Supplementary variables enable us to explain the reason for which particular clusters of variables have been created and indicate which objects are "responsible" for their formation. In our study, pupil's age was set as a 'supplementary variable'. Consequently, we were able to know which age group of children contributed the most to the formation of each cluster.

RESULTS

The similarity diagram is shown in Figure 1. We can observe that pupils' responses to the tasks can be classified according to the strategy they applied. More specifically, two clusters (i.e., groups of variables) can be identified. The first group consists of the variables "Trt", "Sqt" and "Ret" which represent the application of T-strategy (i.e., increasing both dimensions of the figures). The second group consists of the variables "Tro", "Sqo" and "Reo" and refers to the O-strategy (i.e., increasing one dimension of the figures). The emergence of these two clusters are in line with the assumption of our study and reveal that children tend to approach the variations of all three kinds of figures in a similar way; that is, they apply the same strategy for all the paths of figures. The most contributing variable to the establishment of the group of variables representing the use of T-strategy is the group of pre-primary pupils, while the most contributing variable for the establishment of the class of responses concerning O-strategy is the group of primary pupils. In order to clarify further this finding, graphs 1 and 2 illustrate the percentages of pupils who use the two strategies according to their age. Graph 1 shows that the percentages of primary pupils who used T-strategy for the tasks involving triangles and rectangles are lower than the corresponding percentages of pre-primary pupils. On the other hand, the percentages of primary pupils who use the strategy of increasing one dimension of the figures are higher than the percentages of the pre-primary pupils. Chi-square test revealed that the six differences in the percentage of the two age groups of pupils who use each strategy are statistically significant ($p<.01$). This finding seems to be in line with the fact that the variable age was the most contributing variable for the creation of the two clusters emerged through the similarity diagram.

From the similarity diagram, we can also observe the existence of the same pattern of grouping in the test tasks students were asked to deal with. More specifically, students´ responses to tasks dealing with squares and rectangles are closely related. This close connection may indicate the children's intuitive (or taught) conception that rectangles and squares are "similar" geometric shapes and are therefore "completely different" from the shape of triangle. It is, however, important to acknowledge that there is only one statistically significant similarity at level 99% and this refers to the responses of pupils who use the T-strategy and "solve" the tasks concerning rectangles and squares.



Note: Similarities presented with bold lines are important at significant level 99%.

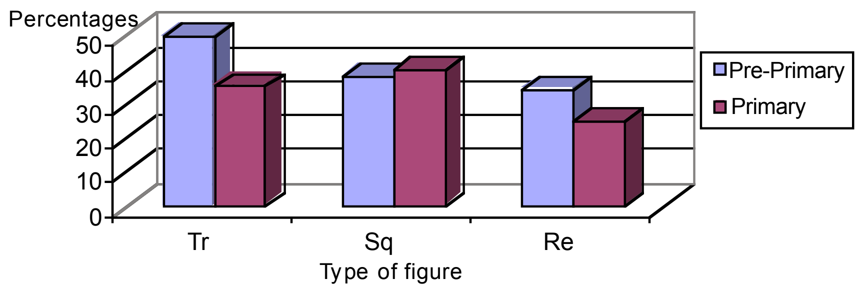


Figure 2. Percentages of pupils who used the T-strategy for solving the three types of tasks according to their age

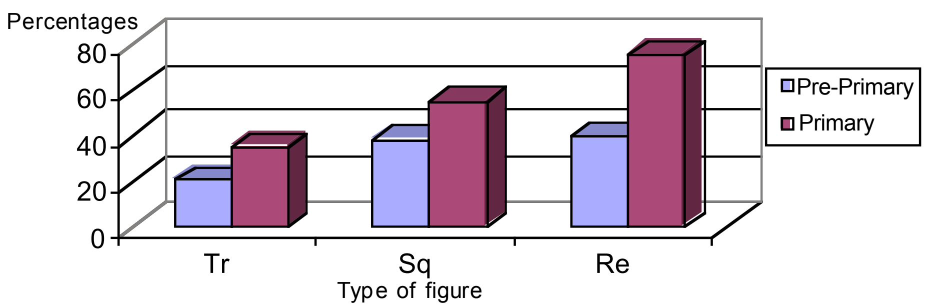
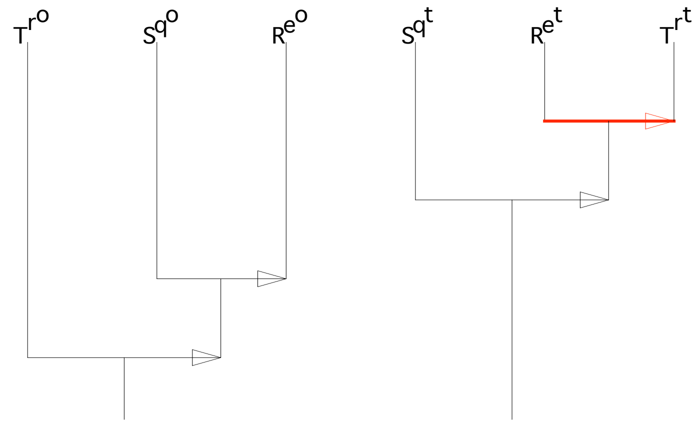


Figure 3. Percentages of pupils who use the O-strategy for solving the three types of tasks according to their age

The hierarchical tree, which shows significant implicative relations between variables of our study, is illustrated in Figure 4. The following observations arise from Figure 4. First, two groups of implicative relationships can be identified. The first group of implicative relations refers to variables concerning the O-strategy, while the second one provides support to the existence of a link among variables concerning the use of T-strategy. This finding is in line with the findings emerged from the similarity diagram. The formation of these groups of links indicates once again the consistency that characterizes children's reactions and strategies towards the tasks for different figures. Second, the implicative relationship (Sqo, Reo) indicates that the construction of a series of squares by increasing mainly one special dimension implies the application of the same strategy for rectangles. An explanation for the certain result is that, in the case of rectangles the dimension that children selected to increase was the longer one, whereas in the case of squares they were not sure which side to increase (since all of them had the same size) and this might caused them some further difficulty in dealing with this task. Eventually, however, they produced rectangles instead of squares.



Note: The implicative relationships in bold colour are significant at a level of 99%.

Figure 4: Hierarchical tree illustrating implicative relations among the six variables

DISCUSSION

The results of our study reveal that children tended to use a specific strategy in a consistent way in their attempt to solve the problems. This kind of behaviour was expected for two reasons: (a) each task was given to them as an open-ended one for all kinds of figures and (b) children were at different levels of thought in geometry. This observation leads us to the rejection of the traditional view of learning, which assumes that identification of a member of a class of shapes from other figures (Clements et al., 1999) or its construction can be apprehended by all pupils in the same instructional approach.

The application of O-strategy may indicate the function of mental operations (corresponding perhaps to the prerecognitive level), which allowed children to construct a path of "polytopes". Such kind of affined transformation may imply that there is a close relationship between the geometrical model of all similarity classes of polytopes and pupils drawings, which represent two alternative strategies. More specifically, a path of continuous deformation of any shape of polytopes into any other corresponds to O-strategy, whereas a constant path in the space of all similarity classes of polytopes, i.e. a path confined into one similarity class is closely associated with the use of T-strategy.

Children, who applied T-strategy, seem to have the characteristics of the "syncretic" level (Clements et al., 1999), since they conserved the components and the properties of the figure they transformed. Consequently, the application of T-strategy indicates that they didn't judge a figure merely by its appearance, but they were able to recognize and make use of all the components and properties of the figures giving equal importance to each one of them.

Children's behaviour toward the tasks was expected to correspond to their developmental level. For example, younger children were expected to increase mainly one dimension of a figure more frequently and to increase both dimensions of it less frequently than older children. However, the results of our study do not provide support to this assumption. Although further research is needed in order to identify the extent to which these findings can be generalized, it can be argued that our results reveal that good opportunities to learn are probably more important than the developmental level when it comes to children’s learning about shapes (Van Hiele, 1999).

Some further implications for teaching geometry can be also drawn. Various studies seem to agree at one important point in relation to geometry instruction: teaching geometry needs to begin early, since young children's concepts remain constant after six years of age, without necessarily being accurate (Clements & Saramas, 2000). The consistent behaviour of older children towards the tasks of our study seems to provide further supports to the above argument.

Moreover, our findings concur with van Hiele's (1999) notion that in order to develop accurate conceptions teachers must provide teaching that is appropriate to the level of children's geometric thinking. To achieve that, teachers need to elicit, uncover and use the initial knowledge of shapes that children entering primary school already have and should build on children’s ideas. Var de Sandt (2001), also, found that a learner-centred approach leads pupils to better outcomes and higher order thinking in geometry than a teacher-centred one.

The fact that pupils were found to deal in a similar way with rectangles and squares may be an indication of children considering rectangles and squares as closely related classes of shapes (Kyriakides, 1999). Thus, the particular finding support that, geometry instruction for young children, should emphasize shape properties and characteristics, as well as, the interconnective and hierarchical commonalities and differences among shapes {i.e., introducing the general case (e.g., rectangles) and then moving to the specific case (e.g., squares)}. Moreover, teachers should use accurate terminology, label shapes correctly and explain relative properties (Oberdorf & Taylor-Cox, 1999).

An example of a practical teaching intervention, which may contribute significantly to enhance children's conception, that a square is a special form of rectangle, makes use of children's "dynamic intuition". In fact it is based on the non-classical continuous path of varied polygonal shapes (O-strategy), rather than the classical (from mathematical and didactical point of view) constant path in the space of shapes (T-strategy) (Gagatsis & Patronis, 1990). This kind of learning can be achieved, if children following the former strategy are encouraged to realize that a square may occur among the rectangles in a very natural way, as an instance in a continuous variation of a rectangle.

References

Bodin, A., Coutourier, R., & Gras, R. (2000). *CHIC□Classification Hierarchique Implicative et Cohesitive-Version sous Windows – CHIC 1.2*. Rennes: Association pour le Recherche en Didactique des Mathematiques.

Castelnuovo, E. (1972). *Documenti di un' esposizione de Matematica*, Torino□Boringhieri.

Clements, D., & Sarama, J. (2000). Young Children's Ideas about Geometric Shapes. *Teaching Children Mathematics, 6*(8), 482-491.

Clements, D., Swaminathan, M., Hannibal, M.A., & Sarama, J. (1999). Young Children's Concepts of Shape. *Journal for Research in Mathematics education, 30*(2), 192-212.

Gagatsis, A., & Patronis, T. (1990). Using Geometric Models in a Process of Reflective Thinking in Learning and Teaching Mathematics. *Educational Studies in Mathematics, 21*, 29-54.

Gras, R., Peter, P., Briand, H., & Philippé, J. (1997). Implicative Statistical Analysis. In C. Hayashi, N. Ohsumi, N. Yajima, Y. Tanaka, H. Bock, Y. Baba (Eds.). *Proceedings of the* 5^{th} *Conference of the International Federation of Classification Societies* (Volume 2, pp. 412-419). Tokyo, Berlin, Heidelberg, New York: Springer-Verlag.

Hannibal, M. A. (1999). Young Children's Developing Understanding of Geometric Shapes. *Teaching Children Mathematics, 5*(6), 353-357.

Kyriakides, L. (1999). Research on Baseline Assessment in Mathematics at school entry. *Assessment in Education: Principles, Policy and Practice, 6* (3), 357-375.

Oberdorf, C., & Taylor-Cox, J. (1999). Shape up! *Teaching Children Mathematics, 5*(6), 340-345.

van Hiele, P. (1999). Developing Geometric Thinking through Activities that Begin with Play. *Teaching Children Mathematics, 5*(6), 310-316.

Van der Sandt, S. (2001). A case study of four grade 7 geometry classes. In M.van den Heuvel-Panhizen (Ed.), *Proceedings of the* 25^{th} *International Conference on the Psychology of Mathematics Education* (Volume 4, pp. 343-350). Utrecht, The Netherlands.