

professional development benefits of research. Boaler's study is a rare example in which there is no intervention from outside in the more successful of two schools, and the commitment of some of the teachers is not particularly high, but the unusual way they teach seems to benefit students more than methods used in the more traditional school.

In this study there is high commitment from teachers and an inevitable professional development effect, but no imposed innovative methods or materials. We are instead learning about the teachers' and pupils' abilities to incorporate thoughtful activity into *every* lesson, of whatever type, while avoiding a skill-focused remedial approach. We are also learning about how high commitment translates into action.

DEFINING MATHEMATICAL THINKING

We did not define 'mathematical thinking' (MT) or 'key ideas', nor provide teachers with descriptions of what they ought to be doing. Rather, we asked project teachers what they felt they could do, within their current practice, to develop MT. Like Ruthven (1999) we value teachers' practices and aim to contribute to a warranting process of teachers' knowledge through 'triangulation of implementation against intention; experience against evidence; internal participants and external standards; continuing analysis and evaluation of model in light of evidence and development' (p.210)

Whereas all the teachers could agree on *some* aspects of MT, such as the importance of generalisation, there was much early discussion about its full meaning throughout school mathematics. For example: is 'ordinary thinking in the domain of mathematics' a more useful activity on which to focus? Can choice of operation in a word problem be described as MT? Discussions around these issues were rich but inconclusive (Pitt, 2002). We chose to take an empirical approach and compare the practices of teachers who had deliberately taken the decision to work with the target students in ways which (a) did not follow the provided materials and methods and (b) included specific attention to development of MT, *whatever meaning a teacher attached to that phrase*.

Thus the meaning of MT is grounded in researched practice and emerges through a process of co-configuration (Engestrom, 2002) in which differently positioned participants in the process create situated knowledge together in response to a crisis in the existent system. In this case, the crisis is the sustained underachievement of a significant number of students within a prescriptive curriculum and assessment system. The teachers are largely self-selected, committed to evaluating their work and they also recognise the value of working within a supportive group; details of their selection and operation of the group are outside the scope of this paper and can be found elsewhere (Watson, De Geest & Prestage, 2002)

METHODS

Data are collected about the practices of the teachers and their students' responses in the classroom. Some teachers are deliberately introducing new (to them) strategies into their work with the target groups, others are working to become more articulate about their existing practice. Data are collected about pupils' written work, test scores, oral responses and interactions in lessons. Further data include teacher diaries, lesson plans and evaluations, interviews, recorded lessons and observation notes of lessons, and recordings of discussions between teachers and other researchers.

Analysis is ongoing using, at first, techniques of grounded theory to identify commonalities and differences between teachers. These are fed back into group discussions to see if more differences emerge, or more commonalities can be articulated. Thus our role as researchers contributes to the co-configuration of what counts as ‘knowledge’ in the group. Our aim in this process is to see what can be said, if anything, *in general* about teachers who are working towards similar aims. In addition, we are developing descriptions of individual teachers’ practices and the mathematical activity of their students in classroom settings. Thus at some time in the future we shall present portraits of secondary classroom micro-cultures in which teachers offer, and pupils take up, opportunities for mathematical thinking.

Data from classroom observation, teacher notes and group discussions led to the creation of descriptions of the types of pupil activity that the teachers in the group identified as evidence of mathematical thought (Watson *et al*, 2002). These are summarised in Table 1 into two types, those which were specifically prompted and those which occurred unprompted in classroom settings. Many of those which occur in both columns as actions teachers can prompt but for which some students eventually take responsibility.

<u>Prompted</u>	Unprompted
Choosing appropriate techniques Contribute examples Describing connections with prior knowledge Finding similarities or differences beyond superficial appearance Generalising structure from diagrams or examples Identifying what can be changed Making something more difficult Making comparisons Posing own questions Predicting problems Working on extended tasks over time Dealing with unfamiliar problems	Choosing appropriate techniques Contribute examples Describing connections with prior knowledge Finding similarities or differences beyond superficial appearance Generalising structure from diagrams or examples Identifying what can be changed Making something more difficult Making extra kinds of comparison Generating own enquiry Predicting problems Changing their mind with new experiences Creating own methods and shortcuts Initiating a mathematical idea Using prior knowledge

Table 1: Observable actions indicating mathematical thought

The list is similar to descriptions of the mathematical behaviour of high achieving mathematical learners yet arises solely from the data of this project with weaker students. Rather than expecting this kind of behaviour spontaneously, as one might from stronger students, the teachers deliberately framed and organised the classroom environment to make it more likely that students would behave in these ways.

CHARACTERISTICS OF TEACHERS’ PRACTICES

Analysis of the project classrooms has enabled us to identify several characteristics which are seen by the teachers to be central to higher achievement of the target students.

Guiding learners into mathematical cultural practices

Teachers see their task in terms of structuring teaching to enable students to make contact with mathematics using their powers of thought. They recognise the learners' entitlement to access mathematics as an established cultural artefact and not to be limited to watered-down, concrete, procedural versions, or a version constrained by the learners' current context. They do not simplify mathematics to make it more accessible. Teachers focus on adapting the habits of the learner and classroom so that the learner may be enculturated into the world of mathematics. For example, they offer situations in which there are dimensions of choice so that students learn how to choose appropriately; they ask for examples in order for exemplification to become a habit; they find playful ways to elicit more mathematically sophisticated responses.

Making connections

All the teachers want students to view mathematics as connected rather than as separate topics, using structural links between topics to design tasks. Some teachers were explicit with students about making links within and across topics. Sara reorganised the imposed scheme of work so that links were obvious; Anthony explicitly encouraged students to express any connections they saw with previous work, with other mathematics or with other contexts.

Preparing to go with the flow

Teachers deliberately plan to 'go with the flow' of student response. This may entail planning a range of approaches to allow flexibility so that practical, spatial and numerical approaches are all possibilities and the teacher can decide which approach is appropriate, or when to move between them. They also respond to students' moods, using them constructively to mould the progress of lessons rather than battling against them.

Allowing thinking time

All teachers find they were giving students a long time to think, including long wait-times with whole-class questions, and in general throughout their work and interactions. Giving more time, creating space rather than imposing pace, is seen in the project as having emotional, behavioural and cognitive effects. Class discussion, with space given for individual thought, was used more and more but not without difficulty (see below).

Varying task-type

All found themselves, either deliberately or incidentally, using fewer worksheets and textbooks and more activities, developments from starter tasks and students' own questions.

Extending duration of tasks

There has been in general a shift towards longer tasks in the project, if for no other reason than the fact that teachers are building more thinking time into their expectations. However, for a few teachers this is a deliberate major move in order to create an atmosphere in which students are embedded in a mathematical situation for several lessons. This goes completely against the normal belief that such students 'cannot

concentrate' and need to be offered task variety. Some project teachers have therefore focused on developing ongoing questions and enquiries. This is particularly well-developed in Becky's case. For example, she gave pupils straws to make and discuss angles and their relationships by intersecting the straws. Students managed to find all the usual angle rules for intersecting lines, triangles and parallel lines over a two week period. This method allowed them to generalise because it offered them unlimited possible angles and the space to make conjectures, experiment and think things through. In contrast some others introduced shorter, high-concentration tasks to enable students to learn that they *could* concentrate, so that they could build on this new behaviour later with more extended tasks.

Creating own examples

All our teachers used 'create your own example' tasks as part of their everyday lesson structure. For example, Andrea deliberately included some blank places in an otherwise teacher-driven activity so that students could create their own examples with which to work. Several teachers use 'if this is the answer, what is the question?' tasks. One student said:

Making my own examples makes me think. I think about half the time in class now.

Respecting learners

Overwhelmingly, the teachers respect students as learners. They do not guess where the students are in their mathematical development, they ask and listen. The teachers provide the scaffolding, the students construct. The curriculum does not dictate progress, the learners do.

Differences.

We have learnt that it is possible for two teachers to make apparently contradictory decisions about classroom norms, but to have and achieve similar aims. What seems to be important here is not the decision that is made, but the *purpose* of the decision. For example, all teachers thought it was important for students to discuss mathematics with their peers, be it in pairs, in groups, or whole class discussions, where the whole-class discussion provides models for how to discuss mathematics. They also believe that everything said in class is valuable and everyone should hear it. **But for one**, this leads to the practice of repeating everything which is said by students (ensuring everyone hears); for another this leads to the practice of repeating nothing and orchestrating discussion around what each student says (ensuring everyone listens).

All the teachers recognise a link between thinking and writing about mathematics. For one teacher, the act of writing is seen as forcing thinking because it has to be expressed in a linear form, using logical connectives like 'and, but, if, then, because, so, ...' The effort required to communicate forces clarity – speech demands transformation of thinking into what makes sense to others. Others believe that writing gives you something of your own to look back at; a way to remind yourself what it is that you know. One, however, sees writing as a serious *distraction* from thought. She sees mental visualisation, and struggling to visualise, as acts which make future access to mathematical facts and methods easier because the memory has been activated by the effortful creation of an image. For all of them the purpose is to promote thought, but different decisions about writing arise as a result.

Difficulties

Of course persuading students to adapt to these ways of working is not easy. Many had developed powerful habits of rejecting the curriculum. Project teachers have not given up attempting to turn rejection into engagement but response can be slow and temporary. Sara reported that in one term the proportion of responsive students had changed from one third to two thirds, but her feelings about the class were dominated by the intransigence of the remaining ones. Andrea gave students the task of learning to take a piece of paper to and fro from home, which can be seen as a behaviourist approach to developing work habits and does not relate easily to her beliefs about how students learn mathematics, and how rewards might be intrinsic. Sian created a board-room arrangement of desks so that students could discuss more easily, but they had been used to being seated separately, all facing front, and it took time for them to cope with new expectations. Another teacher excluded students who quietly refused to work and kept those who were noisy and disruptive but took part in the mathematics. An important part of the study is the recognition of the realities of working with segregated groups of 'failing' students in secondary school, and that the processes involved are of re-enculturation.

STRUCTURE OF MATHEMATICAL TASKS

Apart from some of the techniques used to re-enculturate learners, much of the above is typical of the intentions behind reformed curricula. What is exceptional about these teachers is that they have reconstructed these characteristics for themselves by considering what is best for their most vulnerable students (for an example of an individual teacher's account of similar work see Tierney, 1997). Yet while we can describe the characteristics of teaching which contributes to classroom cultures within which thinking can flourish, we recognised that teachers were doing more than this.

We wanted to identify the nature and effects of the tasks and scaffolds which were initiating and framing thinking. Boaler and Greeno (2000) hypothesise 'a form of connected knowledge that emphasises the knower's being connected with the contents of a subject-matter domain.' (p.191). Our view of mathematics is that 'the contents of the subject-matter domain' are deeply connected within themselves through mathematical structure, and that enculturation into mathematical thinking involves becoming fluent with constructing, creating and navigating similar or isomorphic structures, that is, being intimately attuned to the ways in which mathematics is internally connected. We are influenced in this enterprise by Gibson's (1977) concepts of affordance (what sort of responses are possible to the sensory impacts of the mathematics lesson?) and constraint (how can a teacher usefully reduce that freedom so that the learner focuses on mathematical change and invariance?). We are also influenced by Marton's identification of dimensions of variation in learning (Runesson, 1997). To interrogate our data further we are using a set of analytical questions which arise from co-configured concepts of tasks and scaffolds which promote mathematical thinking.

The initial task: What is the task? How does it open/close possible responses? What is the purpose? Do students know the purpose? How do students work on it? Why this starting point? What representations are offered? What are the dimensions of variation? What is it possible to learn from this task?

Sustaining, motivating, extending: What is valued/praised, and how? How are the students encouraged while working to think more, or to make the work harder? How are incorrect answers dealt with? What are the constraints/freedoms? How are connections made? What is the generality with which they are working? What is it possible to learn with these interventions and emergent features?

Learning: Do the students self-correct? What choices do they make? What do they contribute to the lesson? What is evident in their written work? How much information, variation, elaboration do they give? What range of representations, mathematics, complexity? What evidence is there of awareness of generality? What dimensions have they chosen to vary and how are they varied?

These questions address what is provided by the teacher, what responses are likely, and what learners make of these mathematical affordances and constraints.

Affordances of mathematical tasks

Sara used a task from a published resource to ask students to create patterns using two colours and then express them using letters, for example: a,b,a,b...; a,a,b,a,a,b... By using our questions above we were able to identify why the task failed to stimulate MT: students' attention had *first* been focused on choosing colours and making patterns, the easiest dimension of variation offered to them, rather than on expressing these as generalities about the resultant sequences. The way the task was presented failed to constrain learners to focus on mathematics, and structures in particular. Restructuring so that students had to make initial choices about sequence, rather than colour and pattern, was more successful, the dimension of variation being the letter sequences themselves.

Linda is enthusiastic about what students can do when given an 'answer' and asked to provide a question. She started using this strategy by imposing constraints designed to move them away from obvious decisions. For example, she excluded the use of '+' for numerical questions. She also introduced suggestions such as 'are there any shape questions which give an answer of 4?' Later she would ask 'can anyone make up something which is really tricky?' She structured the idea-sharing formally into her lessons. This helped them learn from each other what might be possible. She then 'faded' her support by asking 'what sort of questions could we ask?', thus beginning to hand over the responsibility for asking questions to the students. Within 45 minutes she had offered freedom to create questions, constraints to guide them away from easier questions and further constraints which could help them to explore further possibilities.

We are in the early stages of applying these analytical questions to *all* our types of data and are finding we can use them to construct a coherent story about tasks and students' mathematics. Even where we have only teacher-reports and written work we can identify affordances and constraints of tasks.

It is too early to report the full implications of this study, but early analysis shows that there are common features in how committed teachers work when they reject a skills-based approach for 'weak' students and adopt a 'thinking' approach instead, and that task analysis as we have described it adds an important dimension to analysis of practice.

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