# CHARACTERISTICS OF $5^{TH}$ GRADERS' LOGICAL DEVELOPMENT THROUGH LEARNING DIVISION WITH DECIMALS

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*If we consider the gap between mathematics at elementary and secondary levels, and the logical nature of the higher level, it is important that aspects of children's logical development in the latter grades in elementary school be clarified. We focused on $5^{th}$ graders' learning "division with decimals" as it is known to be difficult to understand in its meaning because of the implicit model. We discuss how children may develop logic beyond the implicit model in terms of formational operational thinking. We suggested that children's explanations based on two kinds of reversibility were effective to overcome the model, and that the overcoming processes enabled them to conceive multiplication and division as a system of operations.*

## INTRODUCTION

In learning operations with decimals or fractions, children tend to acquire only the mechanistic procedures like "invert and multiply" in division with fractions. However, if the gaps between primary mathematics and that at secondary levels are considered, mindful of the logical nature of the latter, we think it necessary to encourage children to develop their logical reasoning in upper grades at elementary school.[1]

Previous studies on the operations reported that the implicit model (e.g. Fischbein *et al*., 1985) had a great influence on the child's decision making in solving problems. Recently such phenomena have been examined extensively as the intuitive rule theory (Tirosh and Stavy, 1999). However, the following problems are still existent; De Bock *et al.* (2001) listed similar research tasks. In what stages may the implicit model be overcome in learning operations with decimals or fractions, and how will children develop their logical reasoning in the overcoming process. To clarify these tasks, this study analyzes the characteristics of $5^{th}$ graders' development of logical reasoning through classroom lessons on division with decimals.

## THEORETICAL BACKGROUNDS

### (1) Formal operational thinking

Inhelder and Piaget (1958) used the notion of "formal operation" to characterize adolescent thinking starting from about 11 years of age. They noted that, (a) it can proceed from some hypothesis or possibility, (b) it can be characterized as propositional

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[1] In Japanese educational system, the elementary school continues from $1^{st}$ to $6^{th}$ grades, lower secondary school from $7^{th}$ to $9^{th}$ grades, and upper secondary school from $10^{th}$ to $12^{th}$ grades.

logic by combining the statements p□q, p□q, □p, p□ q (cf. Jannson, 1986), (c) the object for thinking is the generality of the law, the proposition, etc., and (d) it includes two kinds of reversibility. In (d), one reversibility is inversion, which enables one to “return to the starting point by canceling an operation which has already been performed” (Inhelder & Piaget, 1958, p.272), and the other one is reciprocity, which is related to “compensating a difference” (p.273) and is “required for equating operations which are oriented in opposite directions”(p.154). Later this notion is used to analyze the characteristics of $5^{th}$ graders’ reasoning.

**(2) Mathematical meaning and the child’s implicit model in division**

We can assume that in learning division with decimals, the mathematical meaning predominates and the difference from the child’s model causes his/her difficulty. Here, we briefly examine two problems. (A) “If 12 apples are fairly shared among 3 persons, how many apples does one person get?” and (B) “The price of 2.8 m of ribbon is 560 yen. How much does 1 m cost?” The both have a same structure because each answer is the quantity-per-unit and permit proportional reasoning. Mathematically saying, “If (a, b) is any ordered pair of rational numbers and (a, b)~(ma, mb) [m: integer], the relation ‘~’ is equivalent. Then (partitive) division means to transform the element (a, b) into (quotient, 1) of the equivalent class (a: dividend, b: divisor)”.

However, children’s conceptions of problem (A) and problem (B) are very different. Though division with integers permits one to imagine the situation that divides something into equal parts and to have the model that division makes the answer smaller, division with decimals doesn’t permit this thinking. Instead, the latter must be conceived proportionally. For example, 560÷2.8=200 should be reinterpreted that 560 is to 2.8 what 200 is to 1. In the following we will focus on how this reinterpretation might occur and children develop their reasoning in the process.

## METHODOLOGY

Data were collected from a fifth grade classroom in a university-attached school (20 boys and 18 girls). 7 lessons, in which the topic was division with decimals, were recorded by video camera and field notes.

During the first two lessons division problems were solved and discussed in which the divisors were bigger than 1 (e.g., “The price of 2.4 m of ribbon is 108 yen. How much does 1 meter cost?”), and the following ideas were constructed.

(a) There are many situations that are same as what 108 yen is to 2.4 m. (e.g., 216 yen is to 4.8 m; 540 yen is to 12 m; 1080 yen is to 24 m).

(b) If we multiply each number by 5 or 10, we can transform the problem into a division with integers (e.g., 108(yen)÷2.4(m) = 1080(yen)÷24(m) = 45).

(c) We can solve by firstly finding the price of 0.1 m (e.g., 108(yen)÷24_10 = 45).

Though the teacher next presented the problem in which the divisor is less than 1 ("The price of 0.8 liter of juice is 116 yen. How much does 1 liter cost? "), they easily made the expression "116÷0.8" and found the answer utilizing the thinking in (b) or (c) above.

However, when teacher asked them to explain what the expression (116÷0.8=145) should represent in the 3rd lesson, they began to feel uncertain and the cognitive state of disequilibrium became apparent (Piaget, 1985).

C1: 116 divided by 0.8… Why is the expression right? It might not be 145.

CA (Children: affirmative): It must be 145.

C1: It might be 145, but the answer for the problem is not 145 yen.

CA: Why? It must be 145 yen.

T: If 116 yen is to 0.8 liter, then 145 yen is to 1 liter. Is that wrong?

C2: I think the answer is the price of 0.1 liter.

C3: I also think that if we do 116÷0.8, we get the price of 0.1 liter.

CN (Children; negative): I agree!

CA: No, its wrong!

Some children considered the answer 145 as the price of 0.1 liter. We consider this the influence by the implicit model "division makes the answer smaller"; for such phenomena didn't come up in the previous lessons. Analyses that follow are devoted to the stages and characteristics during the time children were overcoming the difficulties.

## PROCESSES OF OVERCOMING THE IMPLICIT MODEL

### Logical explanations and the robustness of the implicit model

The idea "the answer of the expression is the price of 0.1 liter" was soon refuted.

C4: If 145 yen were to 0.1 liter, then 0.1 liter was more expensive than 0.8 liter.

C5: We must do 116÷8 in order to work out the price of 0.1 liter.

C6: (After pointing that both 116÷0.8 and 580÷4 have the same answer) If we divide by 4 liter, of course the answer is the price of 1 liter. The idea of 0.1 liter is strange.

C7: …(Referring to the expression 116÷8_10) If we put this 10 in front, it is the same as 1160÷8. So, I think this (116÷0.8) summarizes these expressions which were made to get the price of 1 liter of juice.

We can find the initial form of deductive reasoning in the above explanations. For example, C6's utterance is interpreted as the reasoning that P3 is deduced from P1 and P2.

P1: If we divide 580 by 4, we get the price of 1 liter. (Agreed)

P2: 580÷4 can be equivalently changed into 116÷0.8. (Agreed)

P3: Therefore, 116÷0.8 is the expression for finding the price of 1 liter.

It should be noted that such syllogism occurred through intersubjective conflict (Cobb, Yackel, Wood, 1993). However when the teacher asked children whether they feel the expression uncertain at the beginning of the 4$^{th}$ lesson, 70 % of them disclosed their feeling of uneasiness.

C8: Though I don't know the reason, even with the division the quotient is bigger… than the dividend.

C9: I can understand that if we divide something by 2, we get half. But I don't know how we get 145 when we divide by 0.8.

T: Do you think that "divided by 0.8" is a problem?

Cs (Some children): Yes. It's unclear and strange.

Most children implicitly experienced the cognitive state of disequilibrium. This episode suggests that even if logical explanations are given, they aren't sufficient to overcome the disequilibrium resulting from the implicit model. Though the expression 116÷0.8 was transformed into the other expression (e.g. 1160÷8), it seems that the implicit model does not vanish without discussing what "divided by 0.8" itself means.

**The process of equilibration based on the reversibility "inversion"**

The equilibration began from a child's utterance based on the inverse operation.

C10: It is not good to consider 116÷0.8. By reversing it, if we think of the problem as "The price of 1 liter of juice is 145 yen. How much does 0.8 liter cost?", it will be 116 (He calculated it)… I got it. So, division means … even if the divisor may be a decimal or an integer, the answer is… 1 liter… to get 1 liter.

This opinion was very powerful and most children began to regard 116÷0.8 valid as an expression for finding the price of 1 liter. Here it should also be noted that this explanation included the meaning of division (the quantity-per-unit). But the student's opinion was soon rejected because it had the character of checking after solving the division. The next child made the point more explicit. "If it is 116 times 0.8[2], I can regard it to take 0.8 piece of 116. But, please tell me how do we do 116÷0.8".

It seems that most children wanted to conceive the division as a concrete operation. The explanation based on the inverse operation was strong, but it still remained a concrete world, and they needed further explanations to attain a state of equilibrium.

**The process of equilibration based on the reversibility "reciprocity"**

In the second half of the 4th lesson, the teacher reflected on the previous activities on the number line, and proposed to rewrite it as a schema of proportion (fig.1); asking the children to consider the meaning of "divided by 0.8" on the abbreviated schema (fig.2).

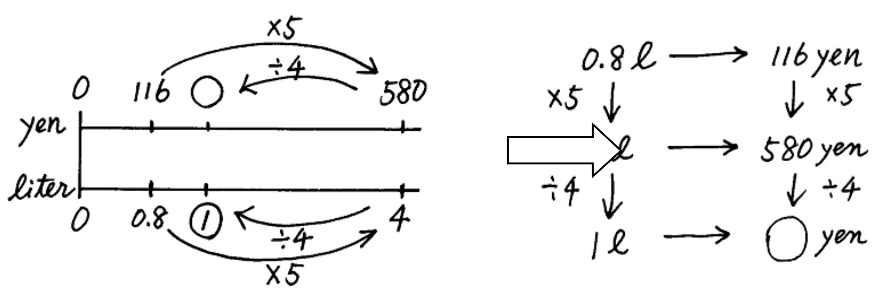


Fig.1 the activity on the number line and the translation to the schema of proportion

[2] In the Japanese notational system, we write 300_5 as the expression for the problem "The price of one apple is 300 yen. How much do 5 pieces of apples cost?" which is different from the English system.

T: Let us discuss by using these two parts.

C11: (He wrote on the blackboard “1.25” beside the left blank)

T: Really?

C11: 0.8 liter is 116 yen and 1 liter is 145 yen. I think some multiple of 0.8 liter is 1 liter. I calculated “1 divided by 0.8”. I found 1.25.

T: Wait. 1 divided by 0.8? Oh, it’s 1.25.

C11: If we multiply 0.8 by 1.25, then of course we must also multiply the price by 1.25. So 116 times 1.25 is 145.

Cs: Yes. It’s the same.

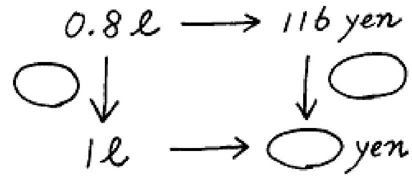

fig. 2.

Though the teacher had expected them to put “÷0.8” into the blank, actually “×1.25” was natural for them. Next they reinterpreted their familiar expressions, e.g. 116×10÷8 as 116×1.25 and had still more confidence in the idea “×1.25”. Here, teacher tried to direct their focus to the relation between “×1.25” and “÷0.8”, though his orientation was suggestive.

T: This is “times 1.25”. Can you represent it by using division? By what do you divide 0.8 liter in order to get 1 liter?

C12: We divide it by 0.8.

T: If you divide 0.8 by 0.8, you get 1. Then by what do you multiply 1 liter in order to get 0.8?

C13: We multiply it by 0.8.

T: If we multiply 145 by some number, we get 116. What is the number?

C14: Oh, it’s 0.8.

T: Is there anything you notice?

C15: “×1.25” and “÷0.8” are same.

T: Everyone, check whether “÷0.8” is same to “×1.25”.

Cs: Oh, they are same.

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T: Really? …. Can you say that this (116÷0.8) is same as “116_1.25”?

C16: Yes. The same. It’s natural that the answer is bigger than the dividend.

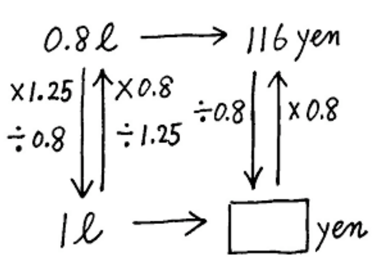

fig. 3.

They made sense of “÷0.8” in terms of the “×1.25” that they had confidence in. In the 5th lesson the teacher and children again discussed those relations, and summarized as in fig.3. It then seemed that they were clearly conscious of the reciprocal relations and fully understood why we should divide by 0.8 and why the answer would then be bigger than the dividend.

## DISCUSSION: THREE STAGES OF LOGICAL DEVELOPMENT

We found that there were three stages in children's logical development as they made sense of division with decimals.

Firstly, they conceived division by a decimal by drawing pictures or manipulating concrete objects. For example, they replaced the situation "108 yen per 2.4 m" with "540 yen per 12 m" by connecting 5 pieces of strip that represented 0.8 m of ribbon and solved it as the division by an integer.

Secondly, they began to reason at the hypothetical-deductive level, detached from the concrete level. Also, their object for reasoning was changed from the answer to the mechanism of the expression. The change occurred through their trials of refuting the idea that $116 \div 0.8 = 145$ was representing the price per 0.1 liter, which was influenced by the implicit model. In this justification process, they developed the syllogistic reasoning by combining some given facts, and sometimes operated on the expression itself; like C7's utterance above. Though these explanations show some characteristics of a formal operation, they didn't attain the cognitive state of equilibrium because of the obstinacy of the implicit model.

Thirdly, they constructed two explanations; each corresponded to two kinds of reversibility. One explanation was based on the inverse operation. It was when C10 inversed the division into the form of multiplication that they firstly realized the correctness of the expression. However, more explanations were needed because the multiplication had the character of checking after solving the division problem. Next they made sense of the expression by using multiplication in another way. It was to consider "÷0.8" as equivalent to "_1.25" which was the flipside of the same coin. It was more natural for them to consider the operation changing 0.8 into 1 as "_1.25" than as "÷0.8" because they had appreciated that _1.25 makes the answer bigger. We can deduce that this eventually led them to conceive multiplication and division as a system of operations, in other words to acquire formal operational thinking.

Here it should be noted that the above stages emerged not linearly, but as equilibration processes in which temporal regressions (disequilibriums) were often involved and more coherent ideas were constructed by coordinating some ideas with each other every time a temporary state of equilibrium was achieved.

Finally, we discuss mathematical characters of children's explanations, which were more or less logical even when the implicit model was not overcome, i.e. at the second stage. There are such properties in division as:

P1: $a \div b = (a \times m) \div (b \times m)$; P2: $(a \times m) \div b = (a \div b) \times m$; P3: $a \div (b \times m) = (a \div b) \div m$;

P4: $a \div b = (a \div m) \div (b \div m)$; P5: $(a \div m) \div b = (a \div b) \div m$; P6: $a \div (b \div m) = (a \div b) \times m$.

Property 1 and 6 emerged frequently (P1: $116 \div 0.8 = 1160 \div 8$; P6: $116 \div 0.8 = 116 \div 8 \times 10$) and the others were also used, though maybe implicitly. For example, in C7's utterance "If we put this 10 in front, it ($116 \div 8 \times 10$) becomes the same as $1160 \div 8$. So, I think this ($116 \div 0.8$) summarizes these expressions", we can find P2 and P4. However, further research will be needed to clarify how these cognitive states develop into secondary mathematics.

## FINAL REMARKS

Findings from this study are:

1. Logical parts, and parts sustained from the implicit model coexisted in 5th graders' reasoning.
2. It was not sufficient for overcoming the model to conceive division proportionally, since the model was more realistic and the expression of division was replaced with another expression.
3. Reasoning based on two kinds of reversibility contributed to overcoming the difficulties, and formal operational thinking was attained in the process. In particular, recognizing the reciprocal relations of operations made their adherence vanish, for the previous image (p) and the constructed image (_p) were combined (p__p), so they no problem deciding whether the answer was smaller or not.

We think it important to study how we can help children to develop logical thought under conditions that their implicit models are made explicit in order not to detach newly learned knowledge from children's minds.

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