"IT IS POSSIBLE TO DIE BEFORE BEING BORN". NEGATIVE INTEGERS SUBTRACTION: A CASE STUDY

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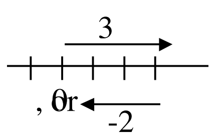
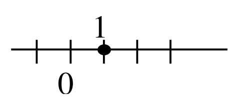
A Case Study is presented in this article, where there is a contradiction between pre-algebraic language semantic and syntax used to solve word problems through a negative integers subtraction.

GENERAL BACKGROUND

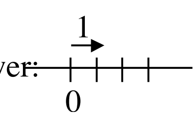
The Case Study reported on this article is based on Vergnaud's research (1982) appearing on the arithmetic to algebra transition which has turned out to be far prolific for the comprehension of negative numbers difficulties students face. Vergnaud sorts additive relations in order to interpret the procedures students use in solving addition and subtraction problems. This classification developed from the fact that knowledge is organized in "conceptual fields" which cover set of problems, situations, concepts, relationships, structures, contents and operations of thought, connected to each other and likely to be interwoven during the process of acquisition. For example, the concepts of state, measure, addition, subtraction, time transformation, comparison relationships, displacement and abscissa on an axis and natural and relative numbers belong to the field of "additive structures". In order to interpret students' behavior when dealing with elementary arithmetic problems, Vergnaud finds it essential to distinguish two sorts of calculus: "numerical calculus" and "relational calculus". Numerical calculus refers to ordinary operations of addition, subtraction, multiplication and division. Relational calculus involves the operations of thought that are necessary to handle the relationships involved in the situation. However, it is highly important to point out that such operations of thought are not expressed or explained by students. As a matter of fact, they can only be hypothesized by observing students' actions. On the other hand, Janvier (1983) showed that signed numbers can be interpreted as two semantically differents sets of objects in the number line model. In this well-known model, numbers are either positions or displacements on this line. Ambiguities are introduced at the beginning. Addition then becomes a combination of different elements. Example:

3 + (-2) is

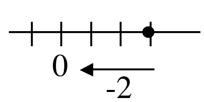




Answer:



3 + (-2) is



Answer as above.

The reader has certainly noted the ambiguity between states and transformations. The difficulty then arises from the fact that adding may be the composition of two transformations or the transformation from a state to a state. Although students can learn a rule and follow it, this basic confusion creeps in when subtraction operation is introduced. On the theoretical basis of Vergnaud relational calculus and Janvier analysis, the following question is risen in the reported study reviewed in this article: How do students do, out of a word problem statement, a negative integers subtraction? This question is approached through the so-called Socrates problem, which statement is: “Socrates, the Greek philosopher was born in 469 B.C. and died in 399 B.C. How old was he when he died?” This problem can be solved by the following negative integers subtraction:

$-399 - (-469) = -399 + 469 = 469 - 399 = 70$ … (1)

Answer: 70 years old.

The two first members of the operation (1) do not belong to arithmetic language, since this language is not proper to represent signed numbers (Vergnaud, 1982). Besides, the horizontal representation of (1) belongs to pre-algebraic language. Thus, -399 – (-469) is a subtraction in the integers domain, while 469 – 399 is a natural numbers subtraction. Likewise, -399 – (-469) equals 469 – 399; that is, there is a syntactic equivalence between both expressions. However, these expressions are semantically different. -399 – (-469) means subtracting Socrates birth date from his death date. Many students solve the problem through the operation 469 – 399 = 70, which is meaningless.

A CASE STUDY

The results of a Study carried out with 41 eight grade students of 12-13 years old, who answered an exploratory questionnaire on fractions and integers, are reported in Gallardo & Novoa (2000). These students were classified in profiles or classes according to their performance in the questionnaire. Nine students from different profiles were chosen to carry out individual clinic interviews. The interviews protocol consisted of the same issues in the questionnaire, that is:

1. Operations with integers and fractions at the syntactic level and their representation on the number line.
2. Solving of word problems.

One of the most important results from this Study, regarding integers, consisted of identifying a “persistent attitude” by most of the students, who always insisted in using the signs multiplication rule, which allowed them to solve subtractions without considering them so. Thus, when using the rule (-) (-) = (+), the subtraction a – (-b) became the addition a + b. The extreme difficulty in the negative integers subtraction in the case of C, the student who best performed in the questionnaire is herein presented, and her actions during the process of solving “Socrates Problem” are analyzed. This problem turned out to be the most difficult item for students. C solved correctly operations in the form of $a \pm b = c$, where a, b, and c are integers both in the syntactic level and on the number line. She also solved the word problems described in the Appendix herein. However, C could not manage to transfer the operating capacity to Socrates Problem. Next, C and the interviewer’s (I) dialogues are presented with the student’s actions analysis. The analysis to each of C’s answers is described in brackets [].

Formulation: Socrates, the Greek philosopher was born in 469 B.C. and died in 399 B.C. How old was he when he died? 1C. [She writes as in the figure below then she subtracts the birth date from the death date, even though it is wrongly done. She does not get surprised of the size of the number obtained (930) too big for an age, besides it is negative].

$$\begin{array}{r} \square 399 \\ \square 469 \\ \hline \square 930 \end{array}$$

Figure 1. C's wrtten work on the problem of Socrates age at death

1I. WHAT HAPPENED?

2C. I subtracted, added, that is to say I added the death date minus the birth date and then I added. [*She first says subtract and then she says add because she knows a school rule that she puts into words in 3C. The rule expressed by the student is the following: "as both are minus signs, then it is like an addition but with negative sign". Nevertheless, if she had followed the rule she would have gotten: (-399) + (-469) = -868 and not –930 as in 1C*].

2I. Did you add?

3C. Well, I am subtracting but as both are minus signs, then it is like an addition but with a negative sign. [*Contradiction! She makes a wrong subtraction and at the same time she uses an addition rule. Notice that in the vertical representation of 1C, there is no operation sign, because it is an arithmetic representation that is not adequate for operating signed numbers. First she chooses to carry out a subtraction according to the semantics of the formulation (death date minus birth date) and then she uses a sign rule for the numerical expression that appears in1C*].

3I.What is the difference between this formulation and the previous? [See Appendix, problem 1].

4C. That the one before dealt with positive quantities, while this one deals with negative quantities. [*Notice that she refers to quantities and not numbers*.]

4I Why are these negative?

5C. Because it says before the Death of Christ, then the dates before the Death of Christ are represented with a negative sign. [*Again notice that she does not talk about negative numbers but signs*].

5I. Then, how old was this philosopher when he died?

6C. Mmm, 930 years old. [*The answer she gives is a positive number even though she got a negative one*].

6I. Where did it come from?

7C. Because I added the quantities, I correct, I subtracted them but then it would be minus 930 years old. [*She repeats the complete school rule (3C), but this time she adds to 930 the words "years old"*].

7I. Let's see, can you live a negative quantity of years old?

8C. No.

8I. There is another difficulty here, is it possible for anybody to live 930 years?

9C. No.

9I. Is there anything that could help you to clear out the formulation?

10C. But the dates before the death of Christ are upside down that is to say first the big numbers and at the end the smaller ones, because they are negative quantities.

10I. How would you express that?

11C. Well, all the quantities before the Death of Christ, since they are with negative numbers and they have to be placed opposite, I mean, for example, not as year one, year two, but first bigger years and at the end the small ones. [*As saying year one, year two, there is no doubt that she refers to the number line*].

11I. What does your description look like?

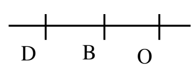
12C. When the numbers on a line are represented or located.

12I. Very good. Could you draw the number line for this?

13C. Yes.

13I. Indicate more or less, he was born here, he died here.

14C. [She writes]



[*Besides the explanation given in 10C and 11C, first she represents he dies (D) and then he was born (B)”*].

14I. Where was he born?

15C. Here [B]

15I. Where did he die?

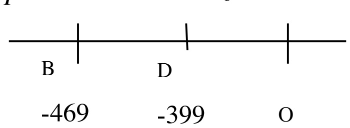
16C. Here [D]

16I. According to these facts, What is first, his birth or his death?

17C. His birth [*She takes the zero in the number line as origin*].

17I. Give it numbers, then.

18C. So it would be all the way around [*permutes B with D and ascribes –469 and –399 respectively*] first it would be his death and then his birth…so I would make a subtraction. [*Contradiction shows up with 17C where she states that his birth was first. However in 18C she states that “his death would be first”. She exchanges B and D in 14C so, new D is now placed closer to zero on the number line:*



18I. Forward.

19C. [*Write*] $\begin{array}{r} 469 \\ -399 \\ \hline 70 \end{array}$. Then, he would be 70 years old. [*At 18C, the student says at the end of the discussion "… so I would make a subtraction". When moving from 1C (a vertical subtraction of signed numbers) to the representation on the number line, she does not give signs to quantities, obtaining a positive result: 70 years old, which describes a real fact].*

19I. Finally, he was born in 469 B.C. and die in 399, he was?

20C. 70 years old.

20I. How would you prove that the answer is 70 years old?

21C. Well, I would add 70 to his birth date and if the answer is 399, then the answer is correct. [Write] $\begin{array}{r} -469 \\ 70 \\ \hline 399 \end{array}$. I've got it! [*She said "add" , however she subtracted because she pictured it* $\begin{array}{r} -469 \\ 70 \\ \hline 399 \end{array}$ *as* $\begin{array}{r} 469 \\ -70 \\ \hline 399 \end{array}$. *This shows that she considered the addition of integers (-469 + 70 = 399) as a subtraction of natural numbers (469 – 70 = 399)*].

21I. Did verification work?

22C Yes!.

22I. What did you say you were going to do to prove your answer?

23C. I was going to add 469 plus 70. [*She is still engrossed in natural numbers*].

23I. 469 plus 70, would be equal to more than 500.

24C. Oh! Yes! [*Here, the student realized she had used up her knowledge in this regard. She is in a vicious circle, in a dead end. At the beginning she had got 930 years old as an answer and now she would get 539 years if following the addition procedure as set out in 23C].*

CONCLUSIONS

C's actions are observed in the interview protocol: In 1C, she writes signed numbers in an arithmetic vertical representation, and she gets a wrong result: -930. In 2C, she does an operation with quantities in context (B.C. chronology). In 2C and 3C she is confused when choosing between subtraction and addition. In 4C, she refers to positive and negative quantities. In 5C, she relates the B.C. chronology to minus sign, instead of a negative number. In 6C, owing to the fact that the interviewer said the word "age", she makes her answer "positive", that is, she leaves out the minus sign and adds instead the word "years". She answers: "930 years". In 7C, she is again confused between addition and subtraction, and she makes her answer "negative". In 10C, 11C and 12C, the student uses "numbers on the number line". She had so far referred only to relative quantities and signs instead of numbers. In 14C, when transferring B.C. chronology to the number line, she represents on it dying (D) before being born (B). In fact, she does not interpret it so, since she reads it from right to left, starting at zero, reaching first B and then D on the number line. Thanks to the interviewer's participation C manages to represent in the right way the problem data (18C). However, 19C shows she does not do operations on the number line, but reads "from left to right", and then she writes a natural numbers subtraction in a vertical line. Both in 19C and 20C, she states: "70 years". In 21C, she says she will add 70 to the birth date in order to check her answer. However, she interprets the integers addition vertical representation as a natural numbers subtraction. In 22C, she is satisfied by her result (70 years old), since she thinks she managed to prove it in 21C. She says: "I've got it". In 23C, she sets out a natural numbers addition which in 24C makes her excitingly say: "Oh, yes!" She is stuck in arithmetic. Regarding Socrates Problem, as the student did not recognize the subtraction of signed numbers she subtracted the death date from the birth date.

To sum up, we can conclude from the dialogue in the interview that the student mostly used natural language throughout the protocol process. In 1C, the student writes a wrong subtraction of signed numbers and it is not until 19C that she writes once more a natural numbers subtraction. Even though she does not know it, it is the syntactic equivalence $-399 - (-469) = 469 - 399$ what helps her get "a possible and actually right answer" of the problem: 70 years old. In 4C, relative quantities (positive and negative) are mentioned. In 10C, when spontaneously using the number line model, she mentions the word "numbers" for the first time. In Gallardo (2002a) four levels of acceptance of negative numbers: subtrahend, signed numbers (plus or minus sign is associated with the number) relative number (idea of opposite quantities) and isolated number (result of an operation or solution of an equation) were abstracted from an empirical study with 35 pupils of 12-13 years-old. In the Case Study reported on this article, it was observed that C showed these levels as well. C's actions analysis suggests the need of re-defining integers and their operations beyond basic arithmetic. In Gallardo (2002a) we read "it is in the transitional process from arithmetic to algebra that the analysis of students' construction of negative numbers becomes meaningful. During this stage the students are faced with equations and problems having negative numbers as coefficients, constants or solutions".

The research approach in this article is in process. The results of a recent empirical analysis with 12-13 year-old students about difficulties due to ambiguity between states and transformations by using the number line model and the word problems solving context are reported in Gallardo (2002b). This empirical analysis shows the difficulties of telling the difference between transformations and states and therefore it does not allow to recognize the dynamic nature of the integers. Most of the students interpreted transformations as static states in themselves. This fact hid the dialectic relation between states (static relation) versus transformations (dynamic relation) of integers.

APPENDIX

Problems C solved correctly:

1. Benito Juarez was born in 1806. How old is he in 1857?

2. A watcher is 135m under the sea level. Where will he get after going up 15m?

3. A watcher is standing in a specific place. He goes up 100m to reach 50m over the sea level, what was exactly his position at the beginning?

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