REPRESENTATIONAL ABILITY AND UNDERSTANDING OF DERIVATIVE

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The ability of students to interpret from, represent mathematical problems in, and interact with, a number of differing representations appears to be a crucial part of understanding mathematical concepts. Here we consider the role of representation in student understanding of derivative and present an outline framework for determining its influence. Analysis of questionnaires and interview data from case studies of two students on the basis of the framework revealed qualitative differences in their understanding of derivative and its associated concepts, and suggesting that representational fluency may be a key marker for this difference in thinking.

Since mathematical objects are conceived with the support of tools such as language, symbols, graphs, and artefacts, it has been considered (Brown, 2001) that representation cannot be divorced from the process of mathematical understanding. For example, an important indication of emerging understanding is the capability of the learner to represent a problem in a number of different ways, allowing them to approach solutions from different perspectives (Sigel, 1999). Indeed it has been suggested (Cifarelli, 1998) that problem solving success may be due to the ability to build appropriate representations of the problem, using these to construct meaning for the given problem situation. In addition, the ability to perform mathematical operations is contingent on the ability to recognise the syntax and rules of production in a chosen representational system (Hiebert & Lefevre, 1986) and to interact with representations according to these rules (Thomas & Hong, 2001). However, each representational form will support and enhance differing aspects of mathematical thinking and learning, and hence the formation of integrated multiple representations for the same phenomenon, or representational fluency (Lesh, 2000), is most likely to encourage meaningful understanding.

One crucial variation in thinking which interacts with the formation of representational ability is the process/object duality of mathematical conceptions described in the literature. While the ideas vary somewhat depending on the researcher's perspective, it is generally agreed that a dynamic process can become viewed as a static object (Sfard, 1991). This occurs, in the words of Dubinsky (1991) via reflective abstraction, and *encapsulation* of the process as an object. In this present research we have considered a theoretical perspective on possible differences between action (or procedural), process and object views of derivative, and how these might be characterised in terms of their relationship to graphical, algebraic and tabular representations.

A REPRESENTATIONAL FRAMEWORK OF KNOWING DERIVATIVE

Other recent research has tried to categorise student activity associated with derivative. For example, Kendal and Stacey (2000) have developed a *Derivative Competency Framework* that enables the monitoring and comparing of student achievement in differentiation across three representations (symbolic, graphical and numerical).

However, that framework concentrates on the solution processes employed by students and the relationship of these to the representations, analysing these in terms of the activities of formulation, interpretation and translation (and combinations). Our perspective here is that it would be useful to construct a framework allowing student understanding to be related to qualitatively different dimensions of knowing and to representational ability. We describe below five dimensions of knowing arising from a process-object theoretical approach to advanced mathematical thinking, and then present the framework in matrix form, describing possible abilities of students within the different representational modes (symbolic, graphical and numerical).

Five Dimensions of Knowing

We have attempted to describe a classification of knowing (and understanding) in terms of the representational abilities which they mediate. The five dimensions of knowing we have identified are: *procedure-oriented, process-oriented, object-oriented, concept-oriented* and *versatile*. It is important to state that these are not claimed to be mutually exclusive categories; we are trying to describe the dominant perspective of an individual.

- *Procedure-oriented knowing* —can successfully obtain an answer by following a sequential set of rules (a procedure), which may or may not be meaningful for them, to solve a given problem. Procedural knowing is not always meaningless and limited to an application of known procedures (Hiebert & Lefevre, 1986). It includes the ability to interpret and represent a problem in a particular representational system, and to know when and how to use the procedure (Skemp, 1976).

j. *Process-oriented knowing*—the learner has condensed and interiorised a procedure in its totality. Rather than being step-oriented and sequential it is global and holistic. The learner has an idea of what process may be used to solve a given problem and when it is appropriate. It includes the ability to describe and reflect on procedures without necessarily performing them (Cottrill *et al.*, 1996).

k. *Object-oriented knowing* —a process can be operated upon as an object. The learner can reflect on the process, but can also generate a mathematical entity out of it. They can perceive a representation as portraying an operation and the result of the operation.

l. *Concept-oriented knowing*—the level where the learner has created a 'bigger picture,' comprising schemas containing procedures, processes, and objects arranged in a relational manner. The learner with concept-oriented knowing can provide answers to why certain procedures and processes work, is able to create conceptual links across representations and relate process and object tools used in solving problems.

m. *Versatile knowing*—the learner has a sufficiently wide range of the four types of knowing described above to enable choice in problem solving, along with sufficiently developed metacognitive ability to choose an appropriate perspective at any given point in time, and ability to move fluently between the chosen perspectives as required.

From a consideration of these theoretical levels the *Representational Framework of Knowing* matrix in Table 1 was constructed. It describes general criteria, along with brief examples of the kind of representational abilities that might be associated with thinking about derivative and differentiation in that domain.

Dimensions	Representations		
	Symbolic	Graphical	Numeric/Tabular
Procedure-oriented	Manipulate symbols according to rules e.g. $\frac{d(x^n)}{dx} = nx^{n-1}$	Calculate from graphical forms. e.g. the gradient of a tangent at a point or a chord from two given points	Use procedures to obtain numerical results from tables. e.g., interpolation for approximating an average rate of change
Process-oriented	Interpret the meaning of symbols $\frac{dy}{dx}$, $f'(x)$, y' as a differentiation process	Have a pointwise approach derivative of graphs e.g. can draw the graph of a derived function given the graph of a function. Can understand second derivative as rate of change of gradient.	Understand and apply rate of change, differentiation and gradient processes in a tabular setting. e.g., use $\frac{f(x+h)-f(x)}{h}$ for small h from a table.
Object-oriented	Operate on the symbols $\frac{dy}{dx}$, $f'(x)$, y' as objects, e.g., assign meaning to $\frac{d\left(\frac{dy}{dx}\right)}{dx}$ Interpret nth derivatives as functions	Interpret derivative graphs as representing functions Operate on graphs as an entity e.g., relate the derivatives of $f(x)$ at $x = m$ and $f(x - a)$ at $x = m + a$ $\forall x$	Interpret a table of values as representing a discrete approximation to a continuous function e.g., transform a table of values for $y=f(x)$ to $y=f(x-a)$, for all x.
Concept-oriented	Relate the differentiation procedures, and processes applicable in one representation to each other and to relevant concepts. Make procedural and conceptual connections between corresponding differentiation procedures, processes and objects in different representations. Identify and operate on conceptual objects such as derivative and function presented in different representational forms.		
Versatile	Have sufficiently well-formed differentiation procedure-, process-, object- and concept-oriented knowledge to be able to identify and use appropriate objects, processes and procedures in their various representational manifestations. Choose appropriate representational system perspectives to solve a differentiation problem. Move seamlessly and fluently between the chosen perspectives as required.		

Table 1. The Matrix for a *Representational Framework of Knowing Derivative.*

METHOD

The case studies reported on here describe two male Form 7 students (aged 18 years), James and Steven, from a private and a public school, respectively. Both schools are situated in high-level socio-economic catchment areas in Auckland, New Zealand. The two students are part of a larger study of understanding of derivative currently taking place with students from four schools where the teachers agreed to 'integrate' graphic calculators in their calculus teaching. The students were selected on the basis of a test given to their respective classes, with James selected from among the higher achievers, and Steven from the lower achievers. Each of the two subjects completed a pre-test, a pre-intervention interview, a module of work on derivative using the TI-89, a post-test, and a post-intervention interview. The pre-intervention interviews were video-taped, while the post-intervention interviews were audio-taped. Both interviews were transcribed and the data analysed together with the results of the tests.

RESULTS

On the pre-test James scored 12.5 (2nd highest) and Steven 3 out of 31. However, it was clear by the second interview that Steven and James not only differed in test scores, but more importantly had qualitatively different types of thinking about derivative. For example, Steven, though appearing to be confident with his rules of differentiation has difficulty understanding what derivative and differentiation are. When asked if he understood what a derivative is he stated “Hmm..I know how to do that. I don't know how to describe it. Let's say it's related to graphs." Asked what the derivative gives you he replied "I don't know really." He recognises that he has had problems with his understanding and recall of the subject, "I did differentiation. I understood it. I haven’t done it since a little of last year…I’m not sure how much I remember. I hardly remember doing it.” It seems that his understanding of derivative is iprocedural and he does not see derivative as a process, or as an object resulting from a differentiation process.

One major disadvantage of a procedural approach occurs when procedures are incorrectly remembered, or used in an inappropriate context. This was apparent when Steven was asked to differentiate the function presented algebraically as $y = \sqrt[3]{(x^2+4x)^2}$ in the second interview. His responses given aloud included “First rule: rearrange the cube root, so you have [writes an answer]. Yeah, that’s right. And then I’ll expand that out first. So…[continues writing]…then I expanded the bracket out…so then I’ll use the chain rule.” While it may appear that Steven was confident that he had the correct procedures to employ to solve the problem (namely expansion and the chain rule), we can see from his written work (Figure 1) that he has errors in both the bracket expansion and in the differentiation procedure.

$$\left[\left(x^2+4x\right)^2\right]^{1/3} = \left(x^4+4x^2\right)^{1/3}$$

$$y' = \frac{1}{3}\left(x^4+4x^2\right)\left(4x^3+8x\right)$$

Figure 1. An example of Steven's procedural errors.

A similar problem emerged when he was asked to differentiate the function $q = 3m^2 \square \frac{7}{m^3}$, where his answer was $\frac{dm}{dq} = 6m \square \frac{7}{3m^2}$. He seemed to be confused by the unfamiliar symbols and so used the symbol $\frac{dm}{dq}$ instead of $\frac{dq}{dm}$. In addition he exhibited a poor knowledge of the procedure for differentiating powers. When his mistakes were pointed out, he responded: “It’s not actually mistakes. So, just like that…” His confidence seems to be based on his level of “knowledge of the procedure”, since he knows this has worked well in previous problems.

In contrast when James was asked how he would explain derivative to himself he gave a conceptual answer related to a graph, saying that:

> Well, derivative is sort of like…it is the graph of the gradient of this other graph …I think it's more a fact that it's the gradient at a given point on the graph. That makes more sense when I think about it. So, yeah, but the graph gets you just all of these with the corresponding x-y, why makes it a lot easier to read.

However, he was also able to see it as a function in its own right generated by a process, and as a rate of change, commenting that "They're all functions in their own respect…It's … rate of change." This provides evidence of both process- and object-oriented knowing. One of the areas where characteristics distinguishing procedural (action) or process thinking about derivative from object thinking can occur is regarding appreciation of $f'(x)$ and $\frac{dy}{dx}$ as objects. Seeing $\frac{dy}{dx}$, with its two apparent parts, as a single entity can prove difficult. Indeed it has previously been noted that some students find difficulty in the transition from the representation, $f'(x)$ to $\frac{dy}{dx}$. Chinnappan and Thomas (2001, p. 158), for example, record the words of a student who said "I couldn't do $\frac{dy}{dx}$, and as soon as I hit university, it changed… $f'(x)$ was, I mean, that's what I worked with. I liked that… I got completely stumped… and it took me a long time to figure it out, but once I did it was great, like a revelation."

Previously (Delos Santos & Thomas, 2001) we found that only 45% of 22 students could make any interpretation of $\frac{dy}{dx}$ in $z = \frac{d(\frac{dy}{dx})}{dx}$, and only 1 thought that it had anything to do with rate of change or gradient of a tangent. In agreement with Thomas (2002) we believe that this is because students perceive $\frac{d^2y}{dx^2}$ as a repeated application of the differentiation process, but are unable to see $\frac{d(\frac{dy}{dx})}{dx}$ this way, thus they meet a cognitive obstacle. This latter form appears to require one to see the application of the differentiation process to the derivative object $\frac{dy}{dx}$, and this gave us a starting point for an investigation of a difference in thinking. In the context of the framework there is a distinction between the former process-oriented view and the latter object-oriented perception in the algebraic domain.

In the second interview, at the end of the study, when asked to describe his understanding of a number of symbolic forms, James spoke of $\frac{d(\frac{dy}{dx})}{dx}$ as "The differential of…I don't know. Hold on, hold on…or derivative of first derivative, derivative with respect to x, second derivative. So it's a rate of change, derivative of y with respect to x…aha, aha.. rate of change…'cause the 'd' is 'delta'…interesting stuff", while $\frac{d^2y}{dx^2}$ was "the second derivative of the function y … the derived function of the first derived function which gives the nature of the turning point". They were clearly not the same to him. In contrast Steven found the former too difficult, saying "I'm not sure… I got confused by that, over this side. Sort of keeps me…" This was not probed any further since he appeared irritated by his lack of insight, lacking a relevant contribution from his concept image of derivative.

For $\frac{dy}{dx}$ James referred to it as the "gradient function…the first derivative of the function y with respect to x, and that it is used to find points, stationary points on the graph and turning points." Asked about $f'(x)$ he replied that it is "the first derivative, which is what we use in gradient, same as derivative of y with respect to x…kinda similar [to $\frac{dy}{dx}$], sort of interchangeable." When asked why he thought that there are two symbols used to represent derivative he said $\frac{dy}{dx}$ "could be graphical because y is involved…the function y on the vertical axis and that is defined by what x is doing", while $f'(x)$ "could not be as much like graphical". However, he still is a little confused, as seen in his remark that "$f'(x)$ is not implying the gradient of the graph…[it is] implying the derivative of the function $f(x)$…even though they both imply the gradient." Clearly there is a representational aspect here to James' thinking. He associates $\frac{dy}{dx}$ more with the graphical representation and $f'(x)$ more with the algebraic, even though it can be used to find the gradient of a function. This reinforces our view that the context in which symbols are first met has a strong influence on the way we think about them. In spite of this James has a concept-oriented perspective across the representations. He talks about concepts such as derivative, stationary points, turning points, gradient, axes, graphs, etc in a manner which portrays a richer conceptual structure than Steven.

In contrast with James' view Steven has a noticeably different perspective on the symbols. When asked about $\frac{dy}{dx}$ he immediately responds in a process-oriented way, saying "I must differentiate." Pressed for what it stands for he responds "Differential of y over differential of x" and appears unable to proceed further. His comments also stress the importance of context to meaning, since when asked why he thinks both $\frac{dy}{dx}$ and $f'(x)$ are used he replies "if there's no y or x in the equation then you can't put one there, so you'll have f'. It is a purely instrumental, context-dependent decision for him.

We decided to examine the students thinking further by taking them into uncharted waters, presenting them with unfamiliar algebraic constructions. While both of the students had used composite functions previously and were familiar with the form $f(g(x))$, we employed the unusual notations $f(f'(x))$ and $f'(f'(x))$ to access their thinking. For the first of these Steven responded "the original function times the differential of the original function", and proceeded to illustrate this by multiplying the function $f(x) = 2x^2 + 1$ by its derivative $4x$. He followed through consistently for the second describing it as "the differential times the differential" Thus when faced with an unfamiliar representational form Steven's recourse is to interpret the contiguous arrangement of f and f' (and later of f' and f') as a known operation or process, namely multiplication. James on the other hand displays the ability to think of the symbol $f'(x)$ as an object, what he calls the derivative function. He states that "this might be the first function here of the derivative function" and in order to elaborate further, he tried to interpret the symbol using specific functions, of the form $f(x) = x^n$, showing that he is able to take $f'(x)$ and treat it as an object, applying the function *f* to it. He wrote

$$f(x) = x^2 \qquad f'(x) = 2x$$

$$f(f'(x)) = (2x)^2 = 4x^2$$

and

$$f(x) = x^3 \qquad f'(x) = 3x^2$$

$$f(f'(x)) = (3x^2)^3 = 27x^3$$

and using the graphic calculator sought to generalise this, getting $n^n x^{(n-1)n}$, and saying "this is always gonna happen…variable *x* to a power, an even power". Moreover, he provided a graphical interpretation saying that "It's going away…it's always gonna be steeper than this original function…it's gonna be steeper…it's also gonna be concave up". Interestingly, when asked to describe $f'(f'(x))$ he said "that does imply second derivative". Hence instead of applying the same composite function thinking he had only seconds previously he now saw this as the second derivative $f''(x)$. It seems possible that this could be the result of a strictly linguistic interpretation of the symbolism. Reading $f'(x)$ as *f*-dashed of *x*, may cause one to read $f'(f'(x))$ as *f*-dashed of *f*-dashed of *x*. This in turn leads to James statement that "It's the derived function of the first derived function.", and hence the second derivative. Whatever the case it again stresses the importance of context in student thinking about symbolic forms. In this case even in the same algebraic representation the student treats expressions with *f( )* and *f'( )* in quite different ways.

CONCLUSION

We recognise that we have only just begun the process of providing evidence for the cells in our matrix of understanding of derivative and more work will come later. However we believe that, based on the data obtained from the two students we can summarise some of the characteristic differences between them that relate to the framework:

- Steven displays primarily procedure-oriented thinking in contrast to James' often concept-oriented approach. e.g., James sees $\frac{dy}{dx}$ and $f'(x)$ in terms of concepts; Steven as an instruction to carry out a differentiation procedure.
- Notation is often linked to representations, e.g., $\frac{dy}{dx}$ is seen as graphical and $f'(x)$ as algebraic.
- Interpretation of a representation appears to be anchored in the context of first exposure and this influences current interpretation of new representational forms. The introduction of new representations should be integrated with previous forms. This is shown in the need to make explicit links between $\frac{dy}{dx}$ and $f'(x)$.
- $\frac{d(\frac{dy}{dx})}{dx}$ and $\frac{d^2y}{dx^2}$ are not perceived as the same by a procedure-oriented thinker. The former requires an object-oriented perspective on derivative.
- Procedure-oriented thinking is more easily forgotten, and it is more difficult to monitor solutions, as Steven's example shows. Concept-oriented thinking is more flexible, allowing different perspectives from other representations to be related to solutions and to inform conjectures, as we see from James' work.

- Concept-oriented thinking is more extensible into new areas. e.g., James coped well with $f(f\square(x))$ by applying the composite function concept, whereas Steven invented a procedure.

Since the ability of a student to understand conceptually may be anchored in the ability to represent an object in different forms, and to relate representations to the object and to other representations of the object, the student who has more representations may have more strategies that can be applied to the problem, and thus, more solution paths. This research suggests that developing richer representational ability could provide a useful means to move from procedure-oriented towards concept-oriented understanding.

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