OBSTACLES FOR MENTAL REPRESENTATIONS OF REAL NUMBERS: OBSERVATIONS FROM A CASE STUDY

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To probe beyond declarative knowledge about real numbers in secondary school, the authors interviewed students in Grades 9, 10 and 12. The main question seems to be whether the length of a decimal expansion is "indefinite" or "infinite". It blurs the mental representation of rational numbers as well.

INTRODUCTION

A central goal of mathematics instruction in secondary school is a conceptual understanding of numbers, not only as an important ingredient of the basic numeracy, but also as a foundation of continuous mathematics (e.g., the calculus) - and even as a major step in the history of ideas. This goal is by no means trivial, since the concept of real numbers intertwines numerical, arithmetical, algebraic, and topological strands.

The motivation of this study were confused and contradictory statements about irrational numbers made time and again by university students who had successfully taken at least one analysis course, and were competently studying fairly advanced mathematics. We wanted to find the source of this confusion.

Of course there had been previous studies of the subject, notably the recent ones by Fischbein et al. [FJC 95] and Peled and Hershkovitz [PH 99]. The former found that the irrational muddle was caused neither by the inability to conceive of incommensurable lengths nor by the notion that the rationals took up every bit of space on the number line. The second one, surveying pre-service teachers, found that many had trouble placing certain items like 0.33333... and $\sqrt{5}$ on the number line - although $\sqrt{5}$ occurred as the diagonal of a 2×1 rectangle in a geometry problem they solved - but that their *declarative* knowledge of irrationals was quite satisfactory.

Since quantitative investigations are largely limited to this declarative aspect, we looked for a qualitative approach, and decided to use the form of videotaped interviews (subsequently transcribed) with small groups of students, which would prompt the participants to discussions among themselves and allow the interviewer to follow up with questions formulated on the spot. The conversation in the interviews was guided by the queries and results mentioned in [FJC 95] and [PH 99].

With this aim in mind, we collected evidence on four levels: Grades 9, 10, and 12, as well as prospective teachers in their fourth year of university. The results so obtained point to consistent difficulties, which are still demonstrable in the prospective teachers' understanding: the insufficient internalization of the notion of irrational number (and thus, *the problems with real numbers* mentioned throughout the literature) *are already visible in the inconsistent mental representation of rational numbers*. It is the weak conceptual tie-in between a number like 22/7 and its theoretically equivalent decimal counterpart 3,142857… According to the students on all our levels, the latter representation is flawed by a connotation of inaccuracy. Considering such " infinite" decimal expansions as legitimate and complete mental objects lies outside the naive range of acceptance: to regard such serpents of digits as "rational" goes against the grain of the students.

This sobering realization recalls the distinction found in [Kl 28], where Felix Klein differentiates between *approximation mathematic*s and *precision mathematics*[1]. The numerical evaluation (!) of 22/7 as 3,142857, plus perhaps another few hundred places which interest nobody, is a natural problem of approximation mathematics; but the idealizing step of accepting this as the representation of an infinite, periodic expansion implies a major shift of paradigm toward precision mathematics, a shift which apparently Plato and Aristotle already argued about. It seems inevitable to us, that Klein's statement ([Kl 32], *ibidem*) "that the concept of irrational number belongs certainly only to precision mathematics", must be extended to include the mental objects known as infinite periodic decimal expansions.

As far as the role and treatment of rational numbers in school mathematics is concerned, our observations clearly entail and support certain consequences, which are discussed in the literature again and again (cf. Stowasser [St 79], Groff [Gr 94]) but go beyond the scope of this paper.

HISTORY

Klein's differentiation between *approximation mathematics* (AM for short) and *precision mathematics* (PM for short) appears to us to be the key to understanding the process of concept formation and the nature of concept images (for the terminology, cf. [TV 81]). To avoid misunderstandings, we hasten to add that there is nothing imprecise about AM: it means "approximate but refinable to any desired degree". On the other hand, PM means "totally precise", i.e., zero-tolerance for error.

The opposition of those two styles goes back at least 25 centuries, to about the time when the torch of scientific innovation passed from Babylon to Greece. Toeplitz ([To 63], Ch.I, §4) says that Plato had a strong preference for PM, while his student Aristotle favored AM - another good reason for these acronyms. However, it would be misleading to confuse this distinction with that of Applied versus Pure Mathematics. Courant and Robbins ([CR 41], Ch. I, § 6) point out that PM has brought with it a "tremendously simplified description of physical phenomena", and Klein calls it an "indispensable support" for the development of AM itself. One page later, he finds nevertheless that school is not the place to deal with it, since it "would hardly be adapted either to the interest or the power of comprehension of most of the pupils."

The difference between the two styles can be most clearly explained by an example: the result of dividing 144 by 233 is, in decimal notation, 0.61802575... Being the quotient of integers, it must of course be periodic, but its period happens to be of length 232 - too

[1] He later devoted a whole book to it: Elementarmathematik vom höheren Standpunkt aus. Band III: Approximations- und Präzisionsmathematik. Berlin: Springer.

long for practical purposes. AM will stand ready to work out some more places: 0.618025751073..., even more; but PM will insist on treating it as 144/233, not losing a crumb. To PM the Golden Mean is 1/2 of $\sqrt{5}-1$, while AM is content with 0.61803398875..., ready to work out some more places when required. To call one of them "rational" and the other one not, would not even occur to AM. Even PM does not *depend* on that distinction - it only makes it possible.

PM seems to have been a Greek invention, while the Babylonians always worked in "AM-mode". The glories of Greek mathematics are known well enough not to need any further praise, but it is interesting to see ([To 63], p.17) how its greatest master, Archimedes, struggled to get π stuck between the rationals 223/71 and 22/7, while Ptolemy, still working in Babylonian mode 4 centuries later, produced an entire table of sines - all of which (except one) are irrational.

Of course, the Babylonians worked in base 60 (as we still do with minutes and seconds) and that tradition was amazingly durable - even in cultures whose spoken numbers were thoroughly decimal. As late as 1250, Fibonacci displays the solution of a certain cubic equation as 1 plus 22 minutes, 7 seconds, 42 thirds, 33 fourths, 4 fifths, and 40 sixths [To 63, p. 15], which differs from the today's answer by only 0.00000000003 in decimal notation.

This method was translated into the decimal system in 1585 by the Flemish engineer and mathematician Simon Stevin. In the preface to his slender booklet *De Thiende* (The Tenth), he says that he did not invent but only found it, and urges all people having to measure and calculate, be they astronomers or merchants, to use it. He makes no reference to the Babylonians, but the point of this whole excursion is that, with Stevin's modification, their system might be heading for a revival by the present "calculator generation" - to whom the distinction between rational and irrational is an anachronism, and only finite decimal numbers are really "real".

RESEARCH QUESTION AND METHODOLOGICAL CONSIDERATIONS

The over-all research question, then, was: *what is the mental image of rational and irrational numbers in the present generation of students?*

Our study was carried out in the early autumn of 2002, in Northwestern Germany. The first participants were two prospective teachers (Stan and Sue) in their 7^{th} semester at the university, whose ideas and conceptions will play a role in our conclusions, and about a dozen students of Grade 9, who by their vivacious reactions mainly helped articulate our procedure. Our principal sources were three groups of senior secondary students: 4 females (pseudonyms begin with F), later 4 males from Grade 10 (pseudonyms with G), and finally 4 males from an enriched mathematics course in Grade 12 (pseudonyms with M). The "field work" was conceived as a series of open interviews, which were videotaped and are now available in transcribed form. The conversations varied in length between 40 and 70 minutes. Their general themes were known to the students, but there was no sign of any systematic preparation.

As a warm-up, they were asked to make crosses wherever appropriate in the empty fields of an 8×4 table, which was basically copied from [FJC 95]: the columns were labeled by the properties “number, rational, irrational, real” and the rows by the seven symbols in [FJC 95], to which we had added -5 as an eighth. Our intentional questions - as against those, which came up spontaneously - were as follows.

1. When did you encounter these notions, and how did they strike you?
2. Can you give some examples?
3. Are these properties inherent, or are they aspects of the representation?
4. What is the relative abundance of rationals versus irrationals?
5. How can you recognize a periodic decimal?

Most of the time, however, was spent on the unexpected question of whether fractions, by being represented decimally, could give rise to irrational numbers. It is important to note that the interviewer did not somehow insinuate this. The reader can ascertain this at www.pims.math.ca/~hoek/interviews/ where all transcriptions will be posted. The consistency of the high school students' responses might also raise the suspicion that they all got the wrong message at school. In fact, they were from two different schools, whose common curriculum introduces irrationals in Grade 9, with a textbook of exemplary clarity. We know their teachers as very competent, both mathematically and pedagogically.

OBSERVATIONS, RESULTS AND INTERPRETATION

Because of the volume of our material, we must reduce our exposition to very few aspects of our exploration. In particular, we leave aside the students’ ways of wrestling with Question (4) - which led to reasonable conclusions more often than not. The furthest off the mark was Stan, who did *not* wrestle with it, but simply voiced his opinion that irrationals were “stop-gaps” and relatively rare. Nor shall we have enough space to discuss how the “finiteness” of the string of digits for a number is seen to depend on the base (e.g., 2, 10, 60) of the place value system used - cf. Question (3).

On the epistemology of understanding the concept of number.

To begin with, the observer is surprised that the conceptual equivalence of common fractions and periodic decimal expansions is by no means evident for these students. For one thing, their primary contact with numbers is through calculator displays. For another, methods for moving between periodic decimal and fractional forms (possibly calculator-assisted) are no longer widely known: since the New Math, they are mentioned only in passing. This experiential deficit has the immediate epistemological consequence that fractional and decimal representations are not on the same ontological level.

Even for Stan (4th year university), 1/3 is more accurate than 0.33333… (“*For me, 1/3 is more precise. When I compute with 1/3, I think back to the world of the Greeks, and I calculate more accurately*”). His classmate Sue tries to repair the perceived imperfection of the latter by writing it as $\square_{i=1} 3/10^i$, in other words, by *completing* the transition to infinity.

The four young men of Grade 10, after asserting that irrationals are “*infinite behind the decimal point”* and although coaxed to recognize the equally infinite 0.33333… as rational, flatly declare that 1/3 is irrational, too, and stand by that opinion. The young women in the same grade are less certain and less unanimous. When asked, whether periodic decimals could be irrational, Fanny says: "*yes, I'd say that*", while Flora says about 0.99999…: "*we have learned that it is also 1*", thus avoiding a direct assertion. In the end, three of the four women vote that 1/3 is irrational, while Flora remains doubtful. In both of these Grade 10 groups, the lead-in question about irrational numbers had produced the same reaction "*not imaginable*" (Fanny, Gilbert). More specifically, they have "*many*" (Frieda), even "*infinitely many*" (Guido), places after the period, and cannot be "*determined exactly*" (Fiona) because they are "*apparently unending*" (Flora). But this feature of irrational numbers taints periodic decimals as well.

In the Grade 12 group, similar ideas come to the surface: *"… an irrational number is … not determined, goes on and on, … is not anchored to a certain point. The point can be encircled, infinitely close, but cannot be grasped."* (Marco). "*Put simply: an irrational number has infinitely many places after the period. I think that some fractions, under certain circumstances, can also be irrational numbers …* (Moshe). It is striking that Moshe brings up the word "fraction" which had not been mentioned previously. Manfred summarizes part of the ensuing discussion as follows: "*in the case of a fraction, you still have a computation to do, a fraction is not yet a finished number. When you have a decimal number with 0. and a great many digits behind it, you know that you have got a number, no further computation is necessary*." It is clear from the rest of the conversation that his "many" means "*finitely many*". Michael agrees that 2/7 is "*only the symbol*" for a number.

Let us emphasize that these students have no problem in interpreting a (finite) decimal fraction and in locating it on the number line, at least in principle. They argue (in their own words) that such a number contains an explicit and understandable locating algorithm: after finitely many steps, you arrive at the correct point on the number line. This perspective shows that they look at these entities as generalized natural numbers. In their view, there is no essential difference between integers, decimal fractions, or even common fractions and roots (!), as long as the latter two are regarded as "placeholders" for numbers yet to be computed. It is amazing to see how closely this understanding correlates with Stevin's notion of number from more than four centuries ago. [Ge 90]

To us, the conclusion seems inescapable that these high school students stand firmly on the ground of what Klein calls approximation mathematics. Precision mathematics is far from their minds: 2/7 is not seen as a number, but is relegated to the world of symbols. The two university students do know the difference between rationals and irrationals, but this knowledge is still more declarative than fully realized.

Verbal description and conceptual content

It is well known that everyday language impinges on mathematical notions even if these have been carefully defined (cf. continuity of a function). This phenomenon appears to affect the notion of "rational" number more strongly than we had expected. However, our students are in good company: according to Klein, astronomers dealing with planetary orbits would consider 2/7 as rational but 2021/7053 as irrational ([Kl 32], p. 36).

In our Grade 12 group, the literal meaning of the word "rational" formed an additional obstacle to understanding what is meant. Early on Marco declares: "*Ratio is that which is given by the structure of thought. Ratio has to do with reason. And for me, a reasonable* *number is one that can be nailed down, that can be defined clearly*." The conversation turns quite philosophical. Much later, Moshe says: "*... Generally speaking, I think that a number is rational, when I can represent it in its entirety, i.e., when I can work with it. This I can do only with numbers that can be converted into natural ones. They need not be natural or whole; they can also be decimal numbers, which have an end. That is, when I have a number with an end, I can work with it, then it is rationally understandable …*"

These linguistically induced conceptions make it difficult to accept infinite (periodic) decimal fractions as rational numbers. Official teachings notwithstanding, prompted only by their daily experience with calculators, these students do, in fact, exactly what Simon Stevin had recommended.

Affective Components

Just like Stan (cf. the quote in 4.1 above), Fiona finds 1/3 more dependable than 0.333…: *"I find that 1/3 is more reliable, because you can depend on it. You know in the end, what the result will be. With 0.333…, that is 3-period, I always feel: can I really depend on that …?"* Fanny finds 1/3 *"friendlier"* than its decimal expansion. Marco expresses his incredulity that anybody could really imagine an infinite expression by saying: "*If anyone can imagine* that*, I'd be really impressed."* More importantly, all the students interviewed have a positive, open attitude toward mathematics, and do some honest, serious thinking. Nobody tries to fake it. At the end of the interview, the M-group even agrees it had been "fun".

Criteria of rationality

To explore our "intended" question (5), we handed out a sheet with 5 decimal fractions between 0 and 1, each with 288 places printed out, and asked which ones were rational. Initially, we were working under the assumption that only short periods would be recognized as rational, and therefore tried to test the students' understanding by confronting them with long ones. Even when dealing with the first two, which had moderate periods, it dawned on every group of students, that their rationality was *not* decidable: a viscious teacher could always derail it at the 300th place. In their Platonic existence as fractions or algebraic numbers, the third one had a period of 256, the fifth one of 294, and the fourth one had none. Instead of clarifying the difference between rational and irrational numbers in decimal form, these observations only deepened the students' distrust of infinite (!) decimal representations.

The serpent of nines

The question about the relation between 0.9999… and 1 comes up again and again, as mentioned by many commentators. Tall[2] says that "the primitive brain notices *movement*", and Zeno of Elea might have thought similarly. In our survey, the keyword "asymptote" is brought into the open by Marco: "*… I come infinitely close to it as to an asymptote, but I can never say:* that *is the number*." Since asymptotes play a considerable role in analytic geometry, these Grade 12 students would have been taught that the function $f(x)=1-10^{-x}$ never quite reaches the constant $g(x)=1$. However, this very function produces 0.9, 0.99, 0.999, and so on, for x=1, 2, 3, etc. This remarkable *inconsistency* of

[2] www.warwick.ac.uk/staff/David.Tall/themes/limits-infinity.html

language and imagery (!), between functions, which never quite make it and numbers which are eternally there, erects an additional obstacle.

CONCLUSIONS AND OUTLOOK

We must admit that our expectations were modest when we began to probe the understanding of irrational numbers on these various grade levels. After all, we were aware of Felix Klein's opinion (quoted in Section 2) about the "pupils" appetite for irrationals, and had diverse other reasons (in part also quoted above) to be skeptical.[3]

Nevertheless, we found our observations surprising. We had believed that an adequate treatment of the real numbers was made difficult mainly by the massive influx of mysterious irrationals into the orderly system of good, clean rational numbers. This was clearly not the view of the students we questioned: their horizon was that of Klein's AM, and their working environment was that of Stevin's decimal fractions. Thus, what we had believed to be a *conceptual problem* concerning irrational numbers turned out to be a *notational* one which covered must of the rationals as well.

It is well known that mathematically equivalent statements are not always didactically equivalent. The same is true for representations, for instance common fractions versus decimal ones. For some one equipped with a calculator, the *workability* of the decimals further tips the balance in their favor (remember Moshe's "… *when I can work with it*.") In some sense AM is the mode of action, PM that of contemplation. Conceptual problems are abundant in the latter, while the former is more easily affected by problems of notation and language. In it, the fact that infinite decimal expansions cannot be written down produces a major cognitive obstacle, turned into mockery by the common meaning of the word "rational" – as shown by the students' affective utterances.

In retrospect, our surprise has given way to the realization that a *shift in the mental image* of "number" was to be expected, when the actual contact with numbers had shifted from relatively sparse markings on paper or blackboard to very explicit displays on calculators and computers.

Sparse as it was, the older mode of communication probably left more room for the imagination (of the lucky few) to delve into the immaterial realm of PM, while the modern explicitness holds it back in work-a-day world of AM - which might be just as well, according to Klein.

With hindsight, it all makes sense - but much more research would be required to corroborate our conclusions on a large scale. If this should happen, it would clearly imply that curricula stay with Stevin and in AM as long as possible. This does not mean avoiding all references to common fractions (especially those with finite decimal

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[3] Toeplitz hides his dim view in the folds of the following convoluted comment : *If a high school graduate were asked, what exactly was a number according to the mathematics he had been taught, he would no doubt be able to consent to the suggestion that he say, what he had so far understood to be a number was an infinite decimal fraction*. Unfortunately, this sentence was not included in the English translation [To 63] of the German edition of 1949, where it appears near the top of page 15.

expansions, or at least small denominators) or shirking all irrationals. But, as Stowasser observes [St 79]: "*The measurement of continuous magnitudes provides no sensible motivation for the calculus of fractions*." So, we might make it at least as far as first Year College without it.

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