NATHAN'S STRATEGIES FOR SIMPLIFYING AND ADDING FRACTIONS IN THIRD GRADE

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Nathan entered third grade having already constructed a Generalized Number Sequence (GNS) for whole numbers that enabled him to coordinate sequences of multiples to find a common multiple of two whole numbers. Nathan was able to apply the operations of his GNS to his understanding of part-whole relations and his emerging use of fractional language, in order to rename fractions generated through iterations of unit fractions (e.g. 3/8 as another name for six iterations of 1/16). Through structured teaching episodes, Nathan was eventually able to find common partitions for different unit fractions and use the common partition to produce (and name) the sum of the two fractions. The on-screen actions with the computer manipulatives, used in this teaching experiment, were both engendering and constraining for Nathan's construction of these mental strategies.

THE CONSTRUCTIVIST TEACHING EXPERIMENT[1]

The research reported in this paper is part of an on-going retrospective analysis of videotaped data from a three-year constructivist teaching experiment with 12 children (Steffe & Olive, 1990; Steffe, 1998). A team of researchers began working with the children at the beginning of their third-grade and continued through the end of their fifth grade year in a rural elementary school in the southern United States. Pairs of children worked with a teacher/researcher using specially designed computer tools (TIMA) (Olive, 2000). The major hypothesis to be tested was that children could reorganize their whole number knowledge to build schemes for working with fractional quantities and numbers (the rational numbers of arithmetic) in meaningful ways (Olive, 1999; Steffe, 2002).

Previously Reported Results.

At PME 25 and PME 26 I reported how two children (Joe and Patricia) who had constructed an *Explicitly Nested Number Sequence (ENS)* (Steffe and Cobb, 1988) were able to construct iterable unit fractions and commensurate fractions during their second year (fourth grade) in the teaching experiment (Olive, 2001, 2002). In this report, I focus on the strategies that Nathan developed during the first year of the teaching experiment (third grade) that were based on his *Generalized Number Sequence (GNS)* (Steffe and Cobb, 1988). The GNS is a generalization of the operations on units of the ENS to composite units. It marks the transition from an "ones" world to a world of composite units.

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While a detailed report of Nathan's advanced multiplicative operations on fractions, developed during his fourth and fifth grades in school, has been published elsewhere (Olive, 1999) this report will be the first publication of Nathan's strategies developed during the first year of the teaching experiment (his third grade in school). A comprehensive publication that will report the complete retrospective analysis of the key case studies from the Project is currently being assembled (Steffe & Olive, in press).

The Special Case of Nathan

Nathan was a special case in a very important aspect of our teaching experiment: we were attempting (as teacher/researchers) to build models of the children's schemes for operating with fractions. Because I had been engaging in mathematical conversations with Nathan for many years (he is my son), my models of Nathan's mathematical schemes were much more viable than my emerging models of the schemes of the other children in the project. I was able to travel with Nathan along his reasoning paths and could provide meaningful signposts at critical junctions in those paths. I was not able to do this as successfully with the other children in the project, which was sometimes frustrating for them and for me! Nathan was also my "test pilot" for the TIMA software and so was more familiar with the actions available to him when using these tools.

Our goals for working with Nathan in the project were to take advantage of his special relationship with me, with the computer tools and with mathematics in order to map out possible ways in which the children might make use of the computer tools to construct their own strategies for operating with fractions. He was able to show us what was possible and demonstrate the enabling power of a GNS in constructing meaningful operations on fractions.

THE EVIDENCE FOR NATHAN'S GENERALIZED NUMBER SEQUENCE

Early in our work with Nathan he demonstrated evidence of a GNS when working with a problem involving coordination of two composite units (a three and a four). The problem had been posed within the context of our TIMA: Toys computer environment (Olive, 2000). Nathan was asked how many strings of three toys and strings of four toys would be needed to make a string of 24 toys. Nathan reasoned out loud as follows:

> *Three and four is seven; three sevens is 21, so three more to make 24. That's four threes and three fours!*

Nathan created an iterable seven consisting of an iterable three and an iterable four. He used his units-coordinating scheme and unit segmenting scheme (Steffe, 1992) to produce 21 as the result of iterating seven three times and then saw 24 as requiring one more iteration of his three-string. Nathan had constructed 24 as a partitioned unit with two sub-partitions: three fours and four threes. He also used his decomposed seven later in the same episode to work out seven times eight: "That's 32 and 24!" -- strong confirmation that part-whole operations were available operations within his units-coordinating scheme.

DEVELOPING A LANGUAGE OF FRACTIONS

When we first started working with Nathan and his partner, Drew, on simple sharing tasks using the TIMA: Bars software (November of third grade), both children demonstrated a naïve use of fraction language to describe the shares they created. For both children, their fraction words were associated with the number of visible parts in a share, not to a part-whole relation (e.g. when sharing 3 bars among 4 people, the share of one person consisted of 3 of 4 strips of one bar, that they named as one third "because there are three rows"). They also appeared to only have words for unit fractions (a fourth and a third). Nathan eventually related the unit fraction to the number of parts in the whole rather than the number of parts in one share.

During the next two teaching episodes (two weeks later) Nathan and Drew began to use a language of parts that made sense to them. They had set themselves the problem of sharing two bars among three mats. They had partitioned each bar into 36 parts (six rows of six parts each). They eventually found that each mat would get four rows each and that would make 24 parts each. Drew at first named the share of one mat "one fourth" (because the share consisted of four rows) but Nathan disagreed because there were *six* rows in each bar, so it couldn't be "fourths." They eventually came up with "four parts out of six" but did not have a fraction name for this quantity.

After working with Nathan and Drew for two months (through December) with TIMA: Bars, Nathan had constructed a meaningful language of fractions that included both mixed numbers and improper fractions. He could also create a whole bar given an unpartitioned part of the bar (e.g. make a whole bar given 2/7 of the bar). The language was *internalized* in that it represented mental images of subdivided and subdivisible regions. It may not, at this point in time, have been *interiorized* in that the representations were still figurative rather than abstract quantities.

DEVELOPING OPERATIONS ON FRACTIONS

Nathan was asked in the last session of January to find out how much of a bar he would have if he joined a half of one bar with a third of another, congruent bar. His first response was "a whole bar!" He saw that he was wrong after carrying out the actions with TIMA: Bars and eventually reasoned that a whole bar would be a half plus a half. His strategy for finding a fraction to describe the half plus a third was, again, figurative. He tried different partitions and made visual comparisons. Three fourths was not quite right so he then tried sixths. Four sixths wasn't big enough so he tried five sixths and found that it was the same size as the half plus a third. He described the amount, however, as "one sixth less than a whole bar!" The next series of episodes were designed to help Nathan construct a scheme for making commensurate fractions (Olive, 2002).

Renaming Fractions

In the first two sessions in February, Nathan began to reason multiplicatively when finding new names for fractions. In the first session he was asked to shade 1/4 of a 12-part bar and reasoned that it would be three parts because "four threes in 12, so three pieces make one fourth." Such reasoning was in stark contrast to Nathan's reasoning in the first sharing activities the previous November. He no longer associated the fraction name with the number of parts in a share (3). Instead, the fraction (1/4) was firmly associated with the *multiplicative* relation between the share (number of parts) and the whole. Nathan had been able to make this important shift in a very short time span because he could reason with composite units in the same way that Joe (Olive & Steffe, 2002) could reason with singleton units.

In the second session he was set the task of making as many different fractions as possible from a 12-part bar. He was asked to copy one of the 12 parts and asked what fraction this was; he responded with "one twelfth." [Note: all of the fraction names were given verbally; there was no written symbolism at this point. For purposes of brevity I shall use the normal ratio notation to refer to the verbal fraction name.] He was asked to make 1/2 and 1/3 with this 1/12 part. He had no problem with these fractions, making 6/12 and 4/12 respectively. He also named them in terms of the twelfths. He was next asked how many twelfths in 2/3? This question confused him initially. Eventually he said: "Oh! Now I've got it -- 1/3 is 4/12 so 2/3 would be 8/12!" He went on after this to work with fourths: "1/4 is 3/12, 9/12 is 3/4, 6/12 is a half which is 2/4."

Even though Nathan appeared to be reasoning with progressive numerical integrations and using both his whole number units-coordinating and unit-segmenting schemes (Steffe, 1992) the next question from the teacher indicated that these schemes were undergoing functional modifications and were in a fragile state of flux. The teacher asked Nathan if he could do the same thing for fifths. Nathan responded: "That would be 5/12, 10/12." The teacher did not respond immediately and Nathan eventually asked: "Is it possible with fifths?" The teacher asked Nathan what he had done. Nathan indicated the 5/12-piece that he had created and said "5/12 in 1/5." The teacher suggested he check that. Nathan copied the original bar, wiped it clean and partitioned it into 5 parts. He made visual comparisons between the twelfths and the fifths and eventually said: "You can't make 1/5 out of 1/12 because it's not even."

Nathan continued with his task, making all the possible sixths and also skipping sevenths "because it's not even!" He was not sure about eighths. The teacher asked him: "What would 8 pieces be?" Nathan replied "8/12." When asked to think of another fraction name for 8/12 Nathan reasoned that: "4 pieces is 1/3, 2/3 is 8 pieces, so 2/3 is 8/12." He then reasoned that 9/12 would be 3/4 because "3 pieces is 1/4 and 3+3+3 is 9/12." When asked what 10/12 would be he responded that "2 pieces would be 1/6 so would it be 5/6?" He was later asked what 3/6 and 1/3 would be together. He reasoned as follows: "3/6 would be 6 and 1/3 would be 4, so 6 and 4 is 10 -- 10/12." The teacher asked what it would be in sixths? Nathan replied: "Oh! 2, 4, 6, 8, 10 -- so it's 5/6!" Nathan had used the equivalence of 2/12 in 1/6 to count by two's up to 10 (a unit-segmenting activity) and produced the commensurate fraction of 5/6 (a units-coordinating activity between units of 1/12 and 1/6).

The above illustrates the enabling power of Nathan's GNS to both unitize and unite composite unit items that were now iterable unit fractions rather than iterable ones. Nathan was able to reason with three levels of units (the 12-part whole, any one part of that whole and any sub-composite unit of parts), maintaining the multiplicative relations among all three levels. He was able to do so because he had abstract composite units available to him prior to action. The teacher had helped Nathan make the renaming of fractions explicit through this activity. This episode marked the beginning of the *interiorization* of Nathan's fractional language and the *internalization* of his operations on fractions.

Finding Common Partitions for Different Fractions

In the above episode, Nathan was given a partitioned bar (a 12-part bar) and asked to make other fractions using the partitioned bar. Our goal in the next sequence of episodes was for Nathan to construct a scheme for finding a common partition of a bar so that he could make two different fractions from the same bar. The first task that we used proved to be very confusing for Nathan. He was given two bars of the same size: one partitioned into 5 parts and the other into 3 parts. He was asked to cut the two bars into equal sized parts so that children could have the same amount from each bar.

Nathan: How can they get equal pieces without using the same number!?

Teacher: That's right! -- You've got to find out how to cut them into equal pieces.

Nathan: Give me a sheet of paper.

(The teacher did not supply a sheet of paper so Nathan proceeded to use the mouse to draw numerals on the computer screen! He made a 10 and a 6.)

Nathan: I would go 6, 12, --- 15. Ah! I have found it!

(Nathan wiped both bars clean, made two mats and partitioned each bar into 15 parts.)

Teacher: (Pointing to the first bar) How many fives are here?

Nathan: Three fives.

Teacher: (Pointing to the second bar) How many threes are here?

Nathan: Five threes. Oh! So there's 3 fives and 5 threes!

Once Nathan had realized that he had to use the original partitions of the bars to make further partitions he came up with the strategy of comparing multiples of each partition until he arrived at a common multiple (15). The act of forming the numerals for the second multiple of each partition (10 and 6) appeared to have been a critical act for Nathan. Was it simply a matter of providing an anchor of clarity in what had become a very confusing situation? Or did it indicate that the strategy Nathan was forming was based on the numerical relations among his interiorized composite units and as such he needed to focus on the numbers rather than on the partitioned bars?

The following episode (5 days later) indicated that Nathan's strategy for finding a common partition was well established. Given a 5-part bar and a 7-part bar he compared multiples of 5 and 7, talking out loud, until he arrived at 35. He repartitioned each bar into 35 parts and (at the teachers suggestion) filled each of the sevenths (consecutive groups of 5 parts) in one bar a different color and each of the fifths (consecutive groups of 7 parts) in the other bar a different color.

The next episode (two weeks later) confirmed that Nathan's strategy for finding a common partition for two congruent bars was indeed based on the interiorized abstract composite units of his generalized number sequence (Steffe, 1992). Given a 9-part bar and a 12-part bar, Nathan chose 36 as a common partition. He explained his choice *before acting* by saying "9 goes into 36 four times and 12 goes into 36 three times!" He then repartitioned the 9 vertical parts in the first bar with 4 horizontal parts and the 12 vertical parts in the second bar with 3 horizontal parts. His teacher then posed the problem of a 24-part bar and a 6-part bar. Nathan immediately said “That’s easy!” and made a horizontal partition of four parts over the six vertical parts of the second bar. He explained that “6 goes into 24 four times.”

Nathan’s strategy for finding a common partition (or lowest common multiple) indicated a coordination of both partitioning and segmenting operations, both of which could be reversed and used recursively. Nathan was able to posit an unknown partition of both bars that contained the 9 parts and the 12 parts (for instance). In order to find that unknown he created multiples of the existing partitions. But this operation represented, at each stage, a repartitioning of the unit whole NOT an iteration of the existing parts. He implicitly knew that by *splitting* each of the nine parts in two he doubled the number of total parts, but this splitting operation had been curtailed -- he went directly to doubling (and then tripling etc.) the number of parts. He used his segmenting operations with the composite units 9 and 12 to find the number that both would “go into.” He also used his units-coordinating scheme to keep track of how many times each of the existing partitions was used to find the common partition. The two results of these units-coordinations (four 9’s and three 12’s) were then used to repartition the unit bars, but in doing this the segmenting operation was reversed (or inverted) resulting in a further partitioning of each of the existing nine parts into four, and each of the existing 12 parts into three. Thus, “four nines” were transformed into “nine fours” and “three 12’s” into “12 threes”!

Finding the Sum of Two Fractions with Different Denominators

Although the above episode might suggest that Nathan’s scheme for finding common partitions of two different fractions was well established, he did not apply it automatically in subsequent episodes. When the context was changed slightly he used the computer manipulatives to make visual estimates for an appropriate common partition rather than reason numerically as he did above. He was able, however, to use the results of his common partitioning scheme to find the sum of the original two fractions as illustrated by the following episode.

The task situation was to form a union of two different fractions and find a fraction name to describe the sum. I set the problem of using a 3-part bar and a congruent 13-part bar. Nathan’s first action was to align the two bars as if checking for an immediate match between the partitions. He then started counting by threes. When he got to 39 he had made a record of 13 threes with his fingers. (He had run through the fingers of one hand twice for ten and had three fingers on the table to complete the 13.) He then partitioned the third, blank bar into 39 parts. He copied his 1/3 piece and moved the copy over the 39-part bar to visually determine how many 39ths were needed to produce 1/3. He did the same for the 1/13 piece. When he had done this Nathan made the following comment:

> <u>Nathan</u>: 39 and the 3 took 13 times to get to 39, so there are 13 pieces in it [the 1/3], and the 13 took three times to get to 39 so there are 3 pieces in it [the 1/13]!
>
> <u>Teacher</u>: How much altogether?
>
> <u>Nathan</u>: There are 13 plus 3 equals 16/39 altogether!

Nathan appeared to have established an inverse relation between the multiples of 3 in 39 and the number of 39ths in 1/3 (and similarly for the number of 39ths in 1/13). This relation, however, may have only been a perceived regularity in his numbers rather than a logical deduction. To reinforce the logical connection I asked Nathan to make the 13 parts in the 1/3 piece and the three parts in the 1/13 piece (he did so). Nathan exclaimed: "The small pieces are the same size!" and then named them as 39ths.

I finally asked Nathan to find 1/4 plus 1/7. He immediately made a 28-part bar, a four-part bar and a seven-part bar (of equal size). He copied one part of the 4-part bar and dragged it up to the 28-part bar in order to determine how many 1/28's in a fourth! He did the same with 1/7. He copied the appropriate number of 1/28's to make first 1/4 then 1/7. He joined these together and counted the parts to arrive at the solution of 11/28! He had an extra 1/28 part so I asked him if he could use that to make 1/4 plus 1/7. He immediately repeated it 11 times!

It appeared from this last activity that, although the numerical relations were becoming explicit for Nathan, the manipulations of objects in TIMA: Bars was a strong attraction that allowed him to use (lower level) visual strategies for finding his solutions. To effect a selection away from the visual solution to the addition of fractions problem, I asked Nathan (in the following episode) to tell me what he would do to find the sum of 1/4 and 2/5 *before* he did it. At first he said he did not know but I asked him to just tell me what he was going to do with the three candybars he had on the screen. [One was blank, one was in four parts and one in five parts. He had copied one fourth and two fifths.] He said he would make 20 parts in the top bar. I asked him why and he replied that both 4 and 5 "went into 20." He then said that he would take five of the twenty parts for the 1/4 and eight for the 2/5 and join them together. I asked him what that fraction would be and he replied "13/20!" I then asked him what 1/4 plus 2/5 was and he replied "13/20."

This episode confirmed for me that Nathan's strategy for adding fractions with unlike denominators was now (at least) a figurative strategy that he could represent to himself without actually carrying out the operations using TIMA: Bars. It was now an *internalized* scheme for adding fractions.

DISCUSSION

Nathan's schemes for simplifying fractions, finding common partitions for fractions with unlike denominators and for adding fractions were all developed through operations on fractional quantities. He was not taught any numerical procedures for these computations. Consequently, he was able to productively apply the computational operations he had constructed with whole numbers to these situations involving fractional quantities. His schemes for whole number multiplication and division, we conclude, were in the process of being reorganized to take into account the inverse relation between the number of parts and the size of a fractional part in relation to a unit whole. The iterable, abstract composite units of his GNS now included unit fractions and composites of unit fractions as items. Such a reorganization of whole number knowledge to incorporate fractional quantities is in contrast with the prevailing assumption that whole number knowledge is a "barrier" or "interferes" with rational number knowledge (Behr et al., 1984; Streefland, 1993), and points to ways in which we can help children avoid the common mistakes they make when trying to apply computational procedures for adding fractions (e.g. add numerators and add denominators).

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