MODELING OUTCOMES FROM PROBABILITY TASKS: SIXTH GRADERS REASONING TOGETHER

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This report considers the reasoning of sixth grade students as they explore problem tasks concerning the fairness of dice games. The particular focus is the students' interactions, verbal and non-verbal, as they build and justify representations that extend their basic understanding of number combinations in order to model the outcome set of a probabilistic situation. Several models are posed and students discuss and debate their differences and similarities. Analyses of student-to-student and student-to-researcher dialogue reveal an emphasis on sense making and collaboration in the quest for meaning. This resulted in richer and deeper understanding of the conditions of the problem and the varying interpretations developed by the students.

INTRODUCTION

The 15 children in the study, 7 boys and 8 girls, were students in a K-8 elementary school in a small, working-class community in the United States. During the students' elementary and secondary grades, the researchers periodically facilitated problem-solving sessions during extended classroom sessions[1]. The problem-solving episodes described here document the discourse and work of the children as they developed solutions for a series of combinatoric and probability tasks in March of their 6th grade year. Our intent was to study in detail how the children dealt with the complexity of the task. In particular, how did they negotiate differences and become convinced of the reasonableness of ideas that were posed?

THEORETICAL FRAMEWORK

The basic tenet of the theoretical perspective guiding this research is that meaningful and useful mathematical knowledge is built by the learner through active engagement in challenging problematic situations over extended periods with opportunities to reconsider and modify ideas over time (Alston, Davis, Maher & Martino, 1994; Davis & Maher, 1997; Kiczek & Maher, 1998; Davis & Maher, 1990). This development is often cyclical, involving the construction and reconstruction of representations from and with which to develop, compare and justify solutions individually and in discourse with others (Davis & Maher, 1997, Maher, 1998). Considerable experience working with what may initially appear as messy data enables learners to gain rich and deep experience in posing theories, testing them, and carry out ideas. Our view is that engaging with complex tasks offers opportunity to work through the confusion and make strategic choices. Immersion in

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complexity requires the making of judgments about what features are essential in reaching the goal (Maher, 2002). Making strategic choices and decisions about mathematical ideas that have appeared strong, well-grounded, and effective, when tested within novel situations require the learner to fold back, extend and rebuild more efficient and flexible representations in order to deal with the constraints of the new experience (Pirie, Martin & Kieren, 1996).

METHODOLOGY

During two consecutive 80-minute sessions, the children worked on 5 tasks. Each group was videotaped throughout the sessions. These videotapes were analyzed for the identification of critical events in the students' mathematical thinking by pairs of researchers. Justification of each critical event and its transcript were verified by at least one objective observer. These transcripts, the researchers' analyses, and the students' work, are the data for this report (Maher, 1995, 1998).

The first two tasks required the students to generate all possible two digit numbers from the four digits: 1, 2, 3 and 4, first with no constraints and second if there was only one of each digit to use in any given 2-digit number. For each of Tasks 3 and 4, the students were presented a Dice Game. They were asked upon a first reading to predict whether the game appeared fair, and then, after playing the game several times, to decide whether or not it was fair, to justify their conclusions, and to modify the game, if necessary, to make it fair. Following are the game instructions:

> Game One (A Game for two players): Roll one die. If the die lands on 1, 2, 3 or 4, Player A gets a point (and player B gets 0). If the die lands on 5 or 6, Player B gets a point (and Player A gets 0). Continue rolling the die. The first player to get 10 points is the winner.
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> Game Two (Another Game for two players): Roll two dice. If the sum of the two is 2, 3, 4, 10, 11 or 12, Player A gets a point (and player B gets 0). If the sum is 5, 6, 7, 8 or 9, Player B gets a point (and Player A gets 0). Continue rolling the dice. The first player to get 10 points is the winner.

During Session One, the three groups developed solutions to each of the first three tasks sequentially, discussed their ideas informally and shared their conclusions in whole-group discussion. The second Dice Game was presented to the groups shortly before the end of this session. The students made their initial predictions within their small groups and began playing the game. As the session ended, the students were asked to continue thinking about this task at home and to bring back any data and conclusions to share at the next session.□Because of the space limitations for this paper, the present analysis focuses on the development of probabilistic ideas through the specific lens of one of the three groups during the second session. This group, sitting at Table 3, included 5 children, 2 boys, Matt and Mike, and 3 girls, Michelle, Amy and Magda. Interactions with other students in the class, the researchers and references to the first session are included as they influence or provide context for the mathematical activity of the five children.

RESULTS

In the final minutes of Session One, the students were asked to share their predictions as to whether the game was fair. Almost all agreed that the game was not fair with Player A having the advantage for receiving points for 6 sum scores (2, 3, 4, 10, 11 and 12) as compared to 5 scores for Player B. A few students noted that Player B had numbers that were "easier to roll" which might make the game fairer. The following discourse occurred among several of the children during this whole-group discussion and provides a context for the debate in Session Two.

Michelle: *(Table 3, predicting that player A had an advantage in the game): ...because Player A has .... um ...*

Amy: *(Table 3): More numbers ...*

Michelle: *Yeah - They actually have 6.*

Amy: *That's what I said.*

Ankur *(Table 1): but 7 is the most commonly rolled number.*

Amy: *It's 6 to 5 (numbers for the two players).*

Ankur: *7 is the most commonly rolled number. Look at all the combinations it has ....6 and 1, 5 and 2, and 3 and 4 - and they can, they can be reversed because you have two dice. There's like 6 possible ways.*

As the students entered the classroom for Session Two, they immediately began discussing the task. Some of the students shared ideas from what they had done at home, while others began to play the game. As pairs of students began playing, the researchers encouraged them to record their actual scores, to show not only the sums, but also the numbers on each die for each roll. The ideas and conclusions within each of the three groups were quite different. The students at Table1 returned with data from home that supported Ankur's earlier statements. Stephanie and Milin had each played numerous games with their fathers and had developed charts to represent what they had found. The four students at Table 2 and the five students at Table 3 appeared not to have thought as deeply about the game and were still uncertain about which player had the advantage.

The 5 children at Table 3 immediately separated into two smaller groups (the boys and the girls) and spent the first half-hour of the session playing the game, carefully recording not only the sum for each roll, but the number on each of the dice. Michelle first stood quietly as an observer behind Amy and Magda, but was encouraged by one of the researchers to develop her own record, playing "against herself". As they shared their results, the five agreed that their original prediction about Player A’s advantage was not correct and began to develop strategies to modify the game. The researcher provided overhead transparencies and asked students to prepare their ideas to present to the whole group. The following transcripts of critical episodes during the session document the students' continuing debate about the meaning of the number combinations that were rolled as a means of modeling the outcome space for the game.

Episode One, Negotiating Meaning

Stephanie (Table 1) joined the group at Table 3 to compare solutions.

Stephanie: *Matt, explain your theory to me. I have to find out if they're the same. OK ….Show me.*

Matt: *All right. Since B has the most chances. B has 13 chances. A has only 9. So we figure the way to make it at least a little bit fair is to give them 12 and give them 8, or 7, or which ever one you want to give them.*

Stephanie: *But that doesn't ensure a chance of a win.*

Matt: *Well why don't we try and do it that way? Neither does yours.*

Stephanie: *I know. ………… I didn't say it did. Okay. What's with this? (pointing to Matt's overhead transparency) These are the probabilities? Right?*

Matt: *That's what's with that.*

Stephanie: *So you switched the 12 and the 8?*

Matt: *Yeah. Basically. Yeah.*

Stephanie: *Okay, but………B still has a better chance. Because look… what I'm saying is…I'm not saying that this is going to make a difference because for everybody's, no matter what you do, there's never a sure chance of winning. So I'm not saying this is bad. What I'm saying is…let me write on this.*

Stephanie takes a sheet of blank paper and wrote as she was speaking. She copied the two sets of outcomes, drew a line across the page under each set and then wrote the frequency under the first four outcome numbers.

Stephanie: *We have 5, 6, 7, 12 and 9 against 2, 3, 4, 11 and 8. Okay. Now, the probabilities. 5 is 4, 6 is 5, 7 is 6, 12 is 1.*

Matt: *7 is not 6.*

Stephanie: *Yeah. Here let me show you mine. I'll be right back.*

*Matt (*speaking to his group while Stephanie is gone)***:*** *How did she get 5 for 6?*

Magda: *How does she …….* (takes Stephanie's paper and begins to write on it).

Stephanie went back to her seat at Table1 and returned with a matrix chart that she had prepared. The chart headings were the numerals 1 through 6 across the top and down the side. In each cell was the resulting sum of the two digits. Referring to this chart, she continued writing numbers and frequencies on another paper for Matt while she talked (See Figure 1 below.)

Stephanie: *Here, look, Matt. . 2 has 1. There's one way to roll 2. ... 3, There's 2 ways to roll 3. You get a 2 on one die and a 1 on the other. Or you can get a 1 on one die and a 2 on the other die. 4 has 3. 2 and 2,….3 and 1,* *Matt:* *So you're doubling everything? .....*

Mike: *Will you explain this, I don't understand? (Figure 1 below.)*

Stephanie: *I will in a minute, Mike. All right. 2 has 1 probability. 3 has 2.*

Magda: *No, it has one!*

Stephanie: *No it has 2 ... because 3 can be rolled with a 2 on one die and 1 on another die or with a 1 on one die and a 2 on the other die.*

Amy: *But if you don't look at the graph, then it's the same thing. (referring to Stephanie's matrix shown in Figure 1 below.)*

Stephanie: *It's the same thing but it's used on different dice.*

Mike: *What is the difference if it says 2 on this side? ... What's the difference if it's this or this?* (manipulating the dice as he speaks) *They're both the same thing!*

Amy: *But if you look on the graph.... It shows different ... 1 and 2 and ...whatever.*

Magda: *Can you see the difference? See? (holding the dice) Can you see the difference? That's the ......*

Stephanie: *That's not the point. See? I can't tell the difference either.*

Mike: *Exactly. You don't need that. Okay. What is 1 and 2 and 2 and 1? They're the same answer. You're trying to see the difference – the answer.*

Stephanie: *No. You're trying to show the probability……*

Matt: *Then why do you have 6 probabilities if there's only 3?*

Amy: *Because she's changing them. She's changing.*

Michelle: *She's changing the order of the numbers.*

Amy: *She is going, 1 and 2, 2 and 1 (writing the two combination for Matt as he manipulates the dice) Stephanie? Are you looking at the total? What are you trying to go at, - other than the probability?"*

At this point, one of the researchers joined the group, listened to the discussion that was going on, and provided dice of two colors, one red and one white. She encouraged the children to investigate further and think about whether the two outcomes were indeed the same. Stephanie returned to her group to complete the preparation of the transparencies.

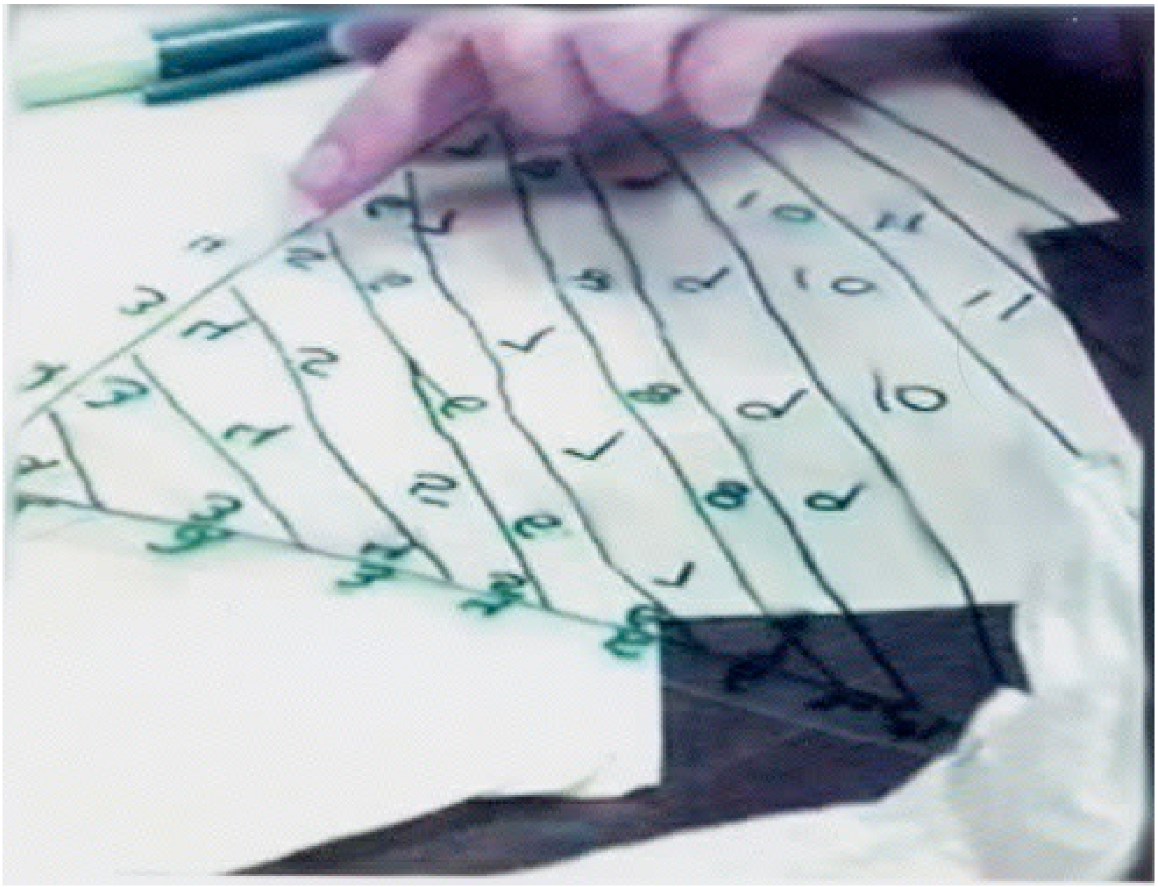

Figure 1: Stephanie's matrix representation of the outcome space

Episode Two: Continued debate

The children at Table 3 continued experimenting with the different dice, discussing the game and challenging each other to predict the sum of a particular roll for some 20 minutes. The following vignette documents that the debate was still unresolved.

Matt:	*What do you want? 6? (rolling the dice).*
Mike:	*There's your 6! (manipulating the dice to form a sum of 6).*
Amy:	*This is the same as ……(reaching for the dice to make a different sum).*
Matt:	*Say you get a 5 and a 1 here, right? It's the same.*
Amy:	*All you have to do is add them up and then …. This is 6 and that's 6. 6 equals 6. It's exactly the same. (writing as she speaks) 6 = 6, 7 = 7, 8 = 8.*
Matt:	*Which one is the one you had before? See, you can't tell .... My point is over!*

Episode Three: Confronting the differences

At this point, 60 minutes into the session, the researcher called the groups together. All the students agreed that Player B had the advantage, despite their earlier predictions. When asked to describe the advantage, Amy asserted that there is one way to roll a 3, with a 2 and a 1. Ankur disagreed.

Ankur: *For the 3, there's 2 and 1 and 1 and 2. 2 is on one die and 2 is on the other die. and 1 is on one die and 1 is also on the other die. It's a different combination.*

The researcher asked for someone to clarify what appeared to be the disagreement.

Shelly (Table 2): He's saying that if you have 1 on one die and 2 on the other, but you could also have 2 on one die and 1 on the other. But it's the same thing. We're working with what it equals up to, not the numbers that are on the dice. We're working with what it equals.

Jeff (Table 2): *Unfortunately, he makes so much sense – because actually you do have to do that to get them. See – look – because if you roll – this die might show a 1 and this die might show a 2 – but the next time you roll it might be the other way around. ……and that makes it 2 chances to get that….even though it's the same number, there are 2 different ways to get it …..*

Stephanie: *Therefore there are 2 different ways to get 3!*

Matt: *For some reason that makes sense. I don't know why, but it does.*

Stephanie then shared her matrix chart, now prepared on a transparency, with the class.

Episode Four: Some resolution

After the discussion and presentations, the researcher presented a final task to the whole group. Their responses indicated considerable resolution about the initial debate, but raised a final issue.

Researcher: *Okay. I have another problem. Roll two dice. If the sum of the two is 7, Player A wins the game. If the sum is 8, Player B wins. Continue rolling until you have a winner. Suppose you have a choice of being Player A or Player B, which would you choose and why?*

Chorus: *Player A.*

Angela (Table 1): Because 7 has the most possibilities…….

Matt: *7 has the same amount as 8 does. . 7 has the same amount as 8 does….*

Several Other Students: No.

Matt: *It does, it does.*

Stephanie: *It doesn't. In this chart, 7 has 6 different ways , 8 has 5. Even though there's really no difference and again, it's a game of luck… I'd rather be Player A because Player A has -maybe - 1 more chance than Player B.*

Matt: *Oh, wait! 7 does! ... She's right because you can't switch the 4 and 4 around. It's the same two numbers. It's like impossible. You can't switch the 4 and 4 around because it's the same two numbers, no matter which way you put it. It's a 4 on one die and a 4 on the other."*

Michelle: *That's why she (referring to Stephanie) said there's only 5."*

Matt: *I know.*

CONCLUSIONS

The children, on their own initiative extended the focus of the debate, moving from group to group as they compared and negotiated shared meaning among themselves, displaying powerful evidence of both individual and collaborative ownership of the ideas. They concurred that 1,2 and 2,1 represented different outcomes. They used the concrete representation provided by manipulating the dice to understand and articulate that a pair of "doubles", even when the dice are moved around, still models one outcome in the set.

Early strong understanding about numbers, multiple representations involving composing and decomposing: "6 is the same as 5 + 1, 4 + 2, etc." is basic to children's understanding of arithmetic. Extending this understanding so that the same number combinations can model more complex situations takes time for exploration and conversation. Through sharing, questioning, and debate, children put forth hypotheses, and explore their reasonableness. In so doing, they cycle back to earlier ideas, test these ideas, and develop a deeper and more flexible understanding.

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