OPEN-ENDED REALISTIC DIVISION PROBLEMS, GENERALISATION AND EARLY ALGEBRA

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This paper explores the role of open-ended realistic division problem in the development of algebraic reasoning. A written test was administered to 672 students. From the results of this test students were selected for semi-structured interviews. The students were interviewed in pairs and were asked to explain to each other how they would solve the problem. The results of the written tests and the semi-structured interviews indicated that both students understanding of division and the real world context of the problem played an important role in abstracting generalisations.

A principle focus of the mathematical reform movement has been to embed mathematics in reality (Heuvel-Panhuizen, 1996). This embedding in reality serves as a source of application and learning, with the links to the real world providing practical knowledge (Treffers, 1993). This enables the students to learn to mathematise their world from the early years using mathematical sign systems interwoven with natural language (Filloy & Sutherland, 1996). The students can draw on understandings that they have constructed in relation to everyday life.

When realistic problems are open ended in nature as well as contextual, they have an added advantage. They draw on the same content but allow the possibility of the students investigating the situation for themselves and so coming to a better appreciation of the concept as a result of their own thinking (Sullivan, Warren, White, & Suwarsona, 1998). They also produce quite different classroom interactions in that the students reporting on their own insights and the variety of solutions they find becomes a part of instruction. This can lead to the development of a classroom culture that supports the processes of justification and deliberate argumentation (Brown & Renshaw, 1999), another principle focus of mathematics reform classrooms.

This paper reports on a study of Grades 7 and 8 students' responses to an open-ended realistic division problem that required them to generalise in a real world context. In this generalisation, the problem reflects those used in the modern practice of introducing algebraic thinking through problems situated in 'real world' contexts (e.g., the handshake problem, making fences and the number of poles required). It underpins the development of algebraic reasoning in its several forms by facilitating processes of justification and argumentation that are believed to accompany acts of generalising and formalising (Kaput & Blanton, 2001). It can be seen to be an example of algebrafication of existing arithmetic problems by transforming them from one-numerical answer arithmetic problems to opportunities for pattern-building, conjecturing, generalising and justifying mathematical facts and relationships (Carpenter & Franke, 2001; Kaput & Blanton, 2001). Thus, the paper explores how students' solutions to an open-ended realistic division problem in order to: (a) identify students' ability to generalise from a 'realistic' open-ended division situation; and (b) delineate students' thinking that supports and interferes with generalising and formalising, that is, thinking algebraically.

The trend to algebrafy the primary curriculum as a result of the recognition of the difficulties many beginning algebra students have with the domain is worldwide (Carpenter, & Levi, 2000; Carraher, Schliemann, & Brizuela, 2001; Warren & Cooper, 2001; Kaput & Blanton, 2001). Such algebrafication is not about introducing formal algebra in the early years but rather about developing arithmetic reasoning in conjunction with algebraic reasoning. Its focus is deeper understanding of mathematics by fostering fundamental skills in expressing and systematically justifying mathematical generalisation not formal manipulations of symbols that is characteristic of secondary algebra (Kaput & Blanton, 2001). It emphasises sense making and understanding, the abstraction of structures from computation, relations and functions, and provides the generalisation and formalisation that, along with the syntactic manipulation of symbols, is seen as the essence of algebra (Lamon, 1998). It is an important part of the transformations within and across representations, the translations between mathematical signs systems and non-mathematical signs systems and the consolidation, simplification, generalisation and reification of actions that are the basis of sign systems such as mathematics (Kaput, 1999).

However, the use of open-ended realistic problems is not without contention. An overemphasis on mathematics in everyday life can (almost inevitably according to Filloy & Sutherland, 1996) result in an under emphasis on algebra. Also, the use by many elementary students of informal methods, while appropriate at a beginning level, can result in avoidance of the later development of formal algebraic methods. Thus, this paper will also explore whether such avoidance is evident in the students' responses to the division problem.

METHODOLOGY

The study used a mixed method approach. A large number of students completed the problem in a written form from which responses a small number of students was chosen for interview. As stated earlier, the problem was open-ended, realistic and directed the students to generalise. It contained two parts to complete:

> *Sarah shares $15.40 among some of her friends. She gives the same amount to each person. a. How many people might there be and how much would each receive? (Give at least 3 answers.) b. Explain in writing how to work out more answers.*

This problem was situated in the "real world" context of sharing money and fits the criteria of a 'good question' as it requires more than remembering basic facts and has several answers (Sullivan & Liburn, 1997). It represented a transformation of a one-numerical answer arithmetic problem. It was chosen for its realism, openness and its support of algebraic reasoning (Kaput & Blanton, 2001) that occurs in the generalising and formalising processes attached to part b, that is, in articulating how multiple answers can be generated by simply dividing the amount of money by the number of friends. Its solution involves translation between language and symbol system and generalisation of the process required to generate solutions.

The division problem was administered as part of a larger written test comprising of 6 tasks to 672 students aged from 11 years to 14 years, with 82% of the sample aged 12 or

13 years, attending six different co-educational schools in Brisbane (2 primary and 4 secondary). Each school was located in a low to medium socio-economic area. The sample was spread across two different grade levels, Grade 7 (n=169) and Grade 8 (n = 503), which straddle the primary and secondary school years (in Queensland, students complete 7 years of primary school before commencing secondary school). A follow up semi-structured interview was administered to 24 students from the 2 schools that made up the Grade 7 cohort. These students were chosen from the school end of year results. Twelve were in the high range, 8 in the medium range and 4 in the low range of achievement. The focus of the interview was to gain insights into the written responses and probe for reasons why students appeared to be experiencing difficulties. The students were interviewed in pairs and the interviews were videotaped. Each problem was presented on a card. They were asked to read the question out aloud and explain to each other how to solve the question. Students of similar ability were interviewed together (e.g., a high achieving student was interviewed with a high achieving student).

RESULTS

The test responses were marked and coded in terms of the number of correct responses (part a) and in terms of the form of the explanation (part b), whether it was attempted, trivial, incomplete and valid. The videotapes of the interviews were transcribed and the students' discussions categorised in terms of the central issue being considered. The results are given in terms of the written and interview responses.

Written responses

The problem involved recognizing that the operation required was "division" and that the application of this operation would generate more than one answer. Students were asked to provide more than 3 answers to the problem (part a) and to explain a process used to generate answers in general (part b). The students' responses to part a, in terms of the number of answers provided, are provided in Table 1.

Table 1. Percentage of responses to part a, *"how many people are there and how much might each get?"*

No of correct responses	No of students (percentage)
0	341 (50%)
1-3	339 (48%)
4-5	11 (2%)

Only 50% of the students responded to part a of the problem, the lowest response rate for all the tasks in the written test. This could reflect student's adversity to answering 'word problems' (this was the only word problem on the test).

The students' written responses to part b, "Explain in writing how to work out more answers", were classified into four categories: i) no response, ii) trivial, iii) incomplete, and iv) valid. Trivial responses simply reflected a single statement. Incomplete responses gave a direction but did not complete how the answers would be found. Examples of incomplete explanations were: *"Try dividing more numbers into \$15.40"*, *"I just divided the number by 4. I guessed I got it right (write)"*, and *"Half it…Quarter it…Third it …"*.

Valid responses provided were complete, providing sufficient to explain how many different answers could be found. Examples of valid responses were as follows: *"Divide $15.40 by how many friends you want and then split the money"*, *"You divide the number of friends to $15 and then afterwards you can divide the 40 between them"*, and *"If you have 4 friends and you divide $15.40 between them so each of your four friends get money ... If you have 5 ..."*. The students' responses to part b, in terms of the categories, are provided in Table 2

Table 2
Percentage responses to part b, "explain in writing how to work out more answers."

Explanation	No of students (percentage)
No explanation	426 (63%)
Trivial explanation	28 (4%)
Incomplete explanation	137 (20%)
Valid explanation	90 (13%)

Two hundred and ninety two students (43%) did not answer both parts, indicating that some students only answered the first part and some only answered the second part. Only 90 students (13%) offered a valid explanation with 11 of these giving 4 or 5 examples of possible solutions.

Responses to the interview

The transcripts of the conversations in the interviews were classified into five broad categories. Each of these categories is described with an example of a conversation from an interview

1. *Not enough information to solve the problem*
(Both low students - **L= Laura, J = Jane, I = Interviewer**)

L Wouldn't be able to work it out. Not enough information to work it out.

I Jamie?

J You can't -You need the number of people.

In this situation, the openness of the problem interfered with these students' ability to reach answers. For them, the problem needed the amount of money and a number of friends, that is, two numbers. The meaning of 'sharing' also caused some concern.

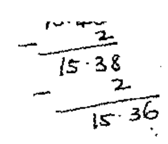
2. *Sharing means continually dividing by 2*
(Both medium students - C = Cheryl , D = Danni)

C I reckon you take 15.40 you could divide it by 2 and keep dividing by 2 and how many times it divides is how many friends. And when you get down to as low as you can, that's how much money each person gets.

I Ok

D Divide 15.40 by two and keep dividing it till you get to zero.

C writes it down on paper to show her reasoning.



C continues to talk through the answers as she works out the solutions $15.38 but then she shares it to her other friends, but then the other person gets 15.36 so you keep doing that.

I So how much does each person get?

C 2 cents.

I Ok it just goes on forever, what do you think about that Danni?

D I think that you find half of $15.40 and share it with a friend. Like she gives one half to her friend and the other half to herself. $7.70 or something.

I So what do you do from then on?

C You get 7.70 and share it among two friends.

D Then you subtract 7.70 again and keep taking it off until zero.

C Half is divided by 2. I think I muddled it up.

E Can you have more than 2 friends.

C Don't know.

D I reckon she'd have just her and a friend. Maybe it could be like 3. 35 and share it with another 2 friends.

Cheryl and Danni seemed to view division, initially, as repeated subtraction of 2 and then as repeated division by 2. Sharing seemed to evoke a "twoness" in their solutions to the problem.

3. *Must go evenly into $15.40*

(Both high students **-** M = Michelle, S = Sarah)

M Have to figure out how many friends she has and how much they will get. They have to get the same amount of money each and it has to equal 15.40.

I Ok so give me a solution.

M Ok you start with the number say 5 and you do 15.40 and you divide it by 5 and if 15.40 can be divided by 5 then she would have 5 friends and you would have to work out how much each gets.

M Yeah probably …..would it be $3.08 each.

I What about if she had 3 friends?

M No you can't do it, so she can't have 3 friends.

I She can't have 3 friends?

M&S No

M Will have to be divided into 40.

I How many answers do you think there are?

M So I think there is only 1- 5 cause that is the only number that can go into 15 <u>and</u> 40.

I Only one, is it what do you think Sarah?

S Ten can but not sure.

M Not ten and ten doesn't go into 15 evenly. Cause its 1 and a half. It says that she gives the same amount to each person.

I So what do you think Sia?

S I guess she is right.

Michelle separated the dollars from cents and shared each separately. Sarah did express some doubts about Michelle's reasoning, but it seemed that Michelle's confidence in her responses acted against any challenges by Sarah. This is not uncommon when students engage in open discussions (Goos, 2000).

4. *Context of the problem - must share evenly and can't have 1c or 2c coins* (Both high students - R = Robin, A = Allan)

R Basically I would keep dividing it, like 3 divided by that, you could try that.
A Wouldn't work - you'd have a couple of cents left over. You can't divide by 3 without having some cents left over.
I Oh ok -What are some numbers?
R No, oh yeah 5 it would be $3.08
A Can't get 8c in money.
R Don't have 1 or 2 cent coins
I Ok so what else works?
R 7
A $2.20.
I What else?
A The highest you can get, because you can't divide by 8.
A $ 2.20 I'd say she'd have 6 friends and herself.
I Can't think of any other solutions?
R No

5. *General solution ignoring the context*
(Both high students - J= John, D = David)

J She had 154 and you could divide 15.4 like 10 cents each.
I Oh ok, among 15 people at 10 cents each.
J and if there were two people it would be $7.70.
I How many people could you have?
J 1,540 and you could get 1 c each.
D Or you can get a fraction of a cent.
I And can you get a fraction of a cent do you think?
J Wouldn't pay them much.
I Wouldn't pay them much. So what other numbers could you have?
J 1,540
I Yeah, go smaller.
J You could have 5.
D You could have lots of numbers. Just divide

John and David ignored the context of the problem and focussed on the general process of division in order to reach an "infinite" number of solutions. In this, they differed from the students described in categories 3 and 4 who situated their solution in a real world context versus students who.

DISCUSSION AND CONCLUSIONS

With respect to division, the interviews provided many insights into how students' experience and understanding with division, particularly for partitive division (or "sharing") problems (Greer, 1992), gives rise to so much difficulty in the written test. First, the difficulties with absence of information seemed to be a consequence of word problems commonly not being open-ended and usually having two numbers. Second, difficulties with "twoness" and remainders appeared to reflect limited intuitive models

(Mulligan & Mitchelmore's, 1997) and restricted cultural conceptions of sharing. Third, difficulties in finding many solutions appeared to be a consequence of embedding the division problem in a real worlds context of money. In realistic contexts, many students seemed to assume that money is discrete and those sharing cannot receive fractions of coins in their solutions (a belief that prevents students going beyond the context of the problem and identifying the structural aspects of "to share you simply divide the among you have amongst the number you are sharing with"). Overall, students' simple early understandings of division and their assumptions about the realistic context of the division problem appeared to act as a cognitive obstacle to later mathematics learning, a conjecture that deserves further research.

With respect to generalising, both the intuitive understandings of division and the real context of the problem impacted on the generalisation process. While the students' discourse supported the notion that the division problem does stimulate argumentation and justification, its impact on algebraic thinking (identifying the underlying structure of arithmetic) is of concern. The context of the problem appeared to evoke certain conceptions of division that restrict algebraic thinking (an interaction that deserves further investigation). Students' attentions seemed to be on the solution itself rather than the method used to solve the problem. While this assisted them in reaching realistic solutions to realistic problems, its-long term effect on thinking algebraically needs further investigation. Is algebraic thinking easier in context-free situations? And does the continual use of open-ended realistic problems result in the avoidance of the difficult transition from arithmetic to algebra (Filloy and Sutherland, 1996). Both these questions need further research.

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