# SECONDARY SCHOOL STUDENTS' IMPROPER PROPORTIONAL REASONING: THE ROLE OF DIRECT VERSUS INDIRECT MEASURES

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A systematic series of studies by De Bock et al. revealed a strong, deep-rooted tendency among secondary school students to apply the proportional model in non-proportional problems involving lengths, areas and volumes of similar geometrical figures. In these studies, however, items were used involving direct measures for area and volume as well as indirect measures (e.g. the time to manure a piece of land as an indirect measure for its area), assuming that this would make no difference. The current study confirmed that there were no significant differences in performance related to the presence of direct or indirect measures in the items, but there were some differences in the applied solution strategies. These findings confirm the internal and external validity of the earlier studies on students' illusion of linearity.

## THEORETICAL AND EMPIRICAL BACKGROUND

It is known that many students have a tendency to apply proportional or linear solutions "everywhere", also in situations where they are not applicable. This so-called "illusion of linearity" has been exemplarily described in several mathematical domains, such as elementary arithmetic (Verschaffel, Greer, & De Corte, 2000), algebra (Matz, 1982), and (pre)calculus (Leinhardt, Zaslavsky, & Stein, 1990), and recently also in probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2002).

The best-known case of the overreliance on the linear model, however, is situated in the domain of geometry: many students of different educational levels believe, for example, that when the sides of a figure are doubled, the area and volume will be doubled too (National Council of Teachers of Mathematics, 1989). In the past years, we performed a series of empirical studies to evidence this particular tendency in secondary school students. In these studies (see De Bock, 2002; De Bock, Verschaffel, & Janssens, 1998), large groups of 12–16-year old students were administered (under different experimental conditions) written tests consisting of proportional and non-proportional word problems about the relationship between lengths and perimeters/areas/volumes of different types of similarly enlarged and reduced figures. The following item is an example of a non-proportional problem about the area of a square: "*Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?*" The majority of the students in these studies gave a proportional answer on this type of non-proportional problems, thinking that the time to manure the large piece of land would be tripled too. Even with considerable support (such as the provision of drawings, enhancing metacognitive awareness, and/or embedding the problems in an authentic problem context), only very few students appeared to make the shift to the correct non-proportional solution. A further in-depth investigation using individual interviews (De

Bock, Van Dooren, Janssens, & Verschaffel, 2002) showed that students' unwarranted proportional reasoning was due to a set of closely related factors: an intuitive approach towards mathematical problems, particular shortcomings in geometrical knowledge, inadaptive beliefs and attitudes, and a poor use of heuristics.

In many of the studies by De Bock et al., it was not explicitly stated that the problems were dealing with the perimeter, area or volume. Instead, an indirect measure for these quantities was used. For example in the above "farmer Carl"-item, the problem statement mentions the *time needed to manure* a certain piece of land, *not the area* of the piece of land itself. Of course, it can reasonably be supposed that this time is directly proportional to the area, so that "time to manure" can be considered as an appropriate indirect measure for the area. Analogously, the time to dig a ditch around the piece of land was used as an indirect measure for its perimeter. However, during the in-depth interviews study (De Bock et al., 2002), the suspicion arose that the use of indirect measures in previous studies might have strengthened students' tendency towards unwarranted proportional reasoning. For example, some students had difficulty in immediately recognising the problem offered during their interview as dealing with area. And since the problem in the interview study asked about the millilitres of paint needed to cover a particular figure, some students were even more confused because millilitres reminded them of volume rather than of area. In the present paper, we report an empirical study aimed at investigating whether the use of direct or indirect measures for area in non-proportional problems has an influence on students' solution processes and performances.

## THEORETICAL INDICATIONS FOR THE INFLUENCE OF THE NATURE OF THE MEASURE

Besides our own anecdotal experiences during the above-mentioned interview study (De Bock et al., 2002), the research literature on (mathematical) problem solving yields several indications that the type of measure in the problem referring to the perimeter, area or volume indeed may have an influence on students' solutions, and therefore possibly on the occurrence of unwarranted linear reasoning.

Rogalski (1982) investigated elementary school children's reasoning about lengths and areas. She reports that some students overgeneralised the properties of "unidimensional" lengths in a figure (e.g., the side of a square in metres) to "unidimensional" area measures (e.g. the amount of paint needed to cover that square in litres), while this was not the case for problems with "bidimensional" area measures (e.g., $cm^2$). In other cases, students overgeneralised the properties of direct measures of length (e.g., cm) to direct measures of area (e.g., $cm^2$), by proportionally relating the area of a figure to its length.

Another indication is the finding that students sometimes tend to use "key word strategies" (e.g., Verschaffel et al., 2000). Superficial characteristics of the problem text (the presence of certain words) immediately guide the choice for a particular operation. In our case, the presence of the words "perimeter", "area" or "volume" or expressions with direct area or volume units (such as $cm^2$ or $cm^3$) might remind the student to apply another strategy than the most straightforward proportional solution scheme, while the student might not be reminded to do so if an indirect (though proportionally related to the area or the volume) measure is mentioned.

Finally, a rational task analysis indicates that problems formulated with indirect instead of direct measures for perimeter, area or volume essentially involve extra thinking steps: students need to notice that the indirect quantity is related to the perimeter, the area or the volume of the figure under consideration. Moreover, they need to know that the relationship between the indirect and the direct measure is a direct proportional one. Because of these extra steps, more errors can appear when solving problems with indirect measures. Therefore, the use of problems involving indirect measures in our previous research might have strengthened students' tendency towards unwarranted proportional reasoning and have had a negative impact on their performances. An empirical study was conducted to find out whether the measure in the problem has indeed this influence.

## RESEARCH QUESTIONS AND HYPOTHESES

The current study aimed at investigating the influence of the nature of the measures in a word problem on students' tendency to (improperly) apply proportional solution methods. Can we identify differences in students' performances on and solution processes for non-linear word problems, when they are formulated in terms of direct or indirect measures?

Based on the theoretical indications described above, we hypothesized that students would perform better on problems in which the measure of perimeter/area/volume is explicitly mentioned, and especially that they would less often apply a proportional solution method when it is not applicable. Moreover, we expected that the strategies used by students to solve problems involving direct measures would differ from the strategies used to solve problems involving indirect measures. For example, for problems explicitly expressed in area measures, we expected students to apply more often previously learnt knowledge and strategies for calculating areas because they were provoked to do so by key words in the problem statement, while other strategies (such as applying internal ratios or the "rule of three") would appear more often with problems involving indirect area measures (wherein such key words are lacking).

## METHOD

A paper-and-pencil test was administered to 145 secondary school students aged 15–16, attending two different typical schools for general secondary education in Flanders (Belgium). All participants received two problems: a (proportional) item about the perimeter of an enlarged irregular figure (where the perimeter of the smaller version was given), and a (non-proportional) item about the area of that enlarged irregular figure (where the area of the smaller version was given). These problems were presented in random order.

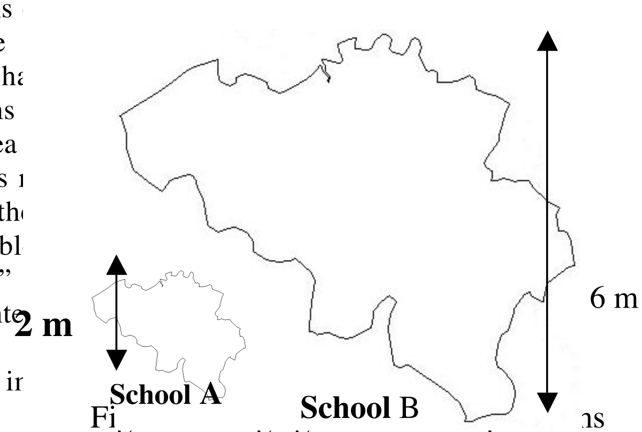
Students' solutions were analysed in two ways. First, they were scored with 1 or 0, depending on whether the response was correct. The interrater agreement of this categorisation was □=□.933. Next, the underlying solution strategy was identified, using

| | ***Direct version*** | ***Indirect version*** |
|---|---|---|
| ***Intro*** | In school A, pupils made a chalk drawing of the map of Belgium. The drawing had a height of 2 m. | In school A, pupils made a chalk drawing of the map of Belgium. The drawing had a height of 2 m. |
| ***Perimeter problem*** ***(propor-tional)*** | In the mathematics lessons, the pupils figured out that the drawing had a perimeter of 11 m.<br>In school B, pupils drew a map of Belgium on a bigger scale: it was 6□m high. What would be the perimeter of this map? | Afterwards, the pupils put coins of 1 euro on the chalk lines of the map. They needed 3 kg of coins to do that.<br>In school B, pupils drew a map of Belgium on a bigger scale: it was 6□m high. How many kg of coins would they need to put on the chalk lines? |
| ***Area problem*** ***(non-pro-portional)*** | In both schools, the students also made an estimation of the area of their map. In school A, pupils estimated that their map had an area of 3 $m^2$.<br>What would be the area of the map in school B? | In both schools, the students also paved the whole map of Belgium with euro coins, turning it into a coin carpet. In school A, pupils needed 40 kg of coins to do that.<br>How many coins would the students in school B need? |

Table 1: Examples of "direct" and "indirect" versions of the word problems administered to the students

a qualitative categorisation scheme based on previous research findings. (For the categories in this scheme, see the results section.) The interrater agreement for this part of the analysis was □ = 0.802.

Two mathematically equivalent versions were developed, and students were assigned to one of two conditions: ha students received a test with problems mentioning the terms perimeter and area the direct measures. The other students r test with problems containing only th measures for perimeter and area. Tabl examples of the "direct" and "indirect" were given to the students. To guarante participants interpreted the word correctly, the test also contained an ir Figure 1).

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## RESULTS

Table 2 presents an overview of the performances of the students on the proportional and non-proportional items in the two conditions. These performances were analysed by means of a $2 \times 2$ repeated measures ANOVA, with "type of problem" (proportional or non-proportional) and "condition" (direct or indirect measures) as independent variables and the number of correct answers as the dependent variable.

| | ***Direct*** | | ***Indirect*** | | ***Total*** | |
|---|---|---|---|---|---|---|
| | **Mean** | **SD** | **Mean** | **SD** | **Mean** | **SD** |
| *Proportional* | 0.792 | 0.410 | 0.876 | 0.331 | 0.848 | 0.360 |
| ***Non-proportional*** | 0.208 | 0.410 | 0.237 | 0.427 | 0.228 | 0.421 |
| ***Total*** | 0.500 | 0.503 | 0.557 | 0.498 | 0.535 | 0.499 |

Table 2: Mean performances (and standard deviations) of the students on the proportional and non-proportional problem in the "direct" and "indirect" condition

First of all, the ANOVA revealed a significant main effect of "type of problem", $F(1, 43) = 55.83$, $p = 0.0001$. It appeared that students performed much better on the proportional item than on the non-proportional item: for the two conditions together, about 85% of the students answered the proportional item correctly, whereas only 23% gave a correct answer to the non-proportional item. This result is in line with our previous research findings (De Bock, 2002; De Bock et al., 1998, 2002), confirming students' overwhelming tendency to improperly apply proportional solutions on non-proportional word problems.

Second, the ANOVA revealed no additional significant effects, neither the main effect of "condition", $F(1, 43) = 1.35$, $p = 0.2473$, nor the "type of problem" $\times$ "condition" interaction effect, $F(1, 43) = 0.33$, $p = 0.5694$. This indicates that there was no difference in performance depending on whether problems involved direct or indirect measures, neither for the proportional item (involving the perimeter), nor for the non-proportional item (involving the area). Therefore, our hypothesis that students would perform better on non-proportional problems if they were expressed with direct measures for the area was rejected.

Despite the absence of an effect of "condition" on students' performance, the question remains whether students in the "direct" condition applied other solution strategies than the students in the "indirect" condition. As mentioned above, a qualitative analysis was performed on the notes of the students on the test. Each correct and incorrect solution[i] was categorised using the scheme in the left column of Table 3, to determine the strategy that was applied to obtain the answer. We will illustrate the categories using the direct non-proportional problem given in Table 1.

| | Direct | | | Indirect | | |
|---|---|---|---|---|---|---|
| *Type of solution strategy* | **Proportional** | **Non-proportional** | **Total** | **Proportional** | **Non-proportional** | **Total** |
| **Proportionality** | **54.2** | **66.7** | **60.4** | **80.4** | **64.9** | **72.7** |
| Internal ratio | 45.8 | 60.4 | 53.1 | 66.0 | 51.5 | 58.8 |
| External ratio | 6.3 | 6.3 | 6.3 | 5.2 | 5.2 | 5.2 |
| Rule of three | 2.1 | 0.0 | 1.0 | 9.3 | 8.2 | 8.8 |
| **Reducing figure** | **22.9** | **16.7** | **19.8** | **4.1** | **12.4** | **8.2** |
| **General principle** | **0.0** | **4.2** | **2.1** | **0.0** | **12.4** | **6.2** |
| **Other** | **22.9** | **12.5** | **17.7** | **16.5** | **10.3** | **12.9** |
| **Total** | **100.0** | **100.0** | **100.0** | **100.0** | **100.0** | **100.0** |

Table 3: Overview of the solution strategies (in %) used by the students to solve the proportional and non-proportional problem in the 'direct' and 'indirect' condition

A first category captures three solution strategies relying on proportionality. These strategies are correct for the proportional perimeter item but incorrect for the non-proportional area item. In the example, students using the "internal ratio" reason that the ratios of the heights of the maps (2 m / 6 m) should apply to the areas of these maps too. Relying on the "external ratio" strategy means reasoning that the ratio between the height and area in the first map (2 m / 3 $m^2$) should be the same in the second map. The "rule of three" strategy first reduces one of the quantities to its unit, e.g. 3 $m^2$ for 2 m, thus 1.5 $m^2$ for 1 m, thus $6 \times 1.5\ m^2 = 9\ m^2$ for 6 m. A second category ("reducing the figure") refers to strategies in which the irregular figure under consideration (the outline of a map of Belgium) is reduced to a more regular figure, such as a rectangle or a right-angled triangle. A third category ("general principle") comprises those solutions that explicitly refer to the general principle governing the similar enlargement of geometrical figures: if a figure is enlarged with factor $k$, its perimeter enlarges with factor $k$, and the area with factor $k^2$. A fourth category contains all remaining solutions, such as unanswered problems and (sometimes correct, but mostly incorrect) solution processes that were difficult to understand or to categorise in one of the other three categories.

Table 3 shows that most of the problems in both conditions were solved with a strategy relying on proportionality, mostly the "internal ratio" strategy. This explains why most students performed well on the proportional item (they correctly reasoned that the perimeter was tripled because the height was tripled), but failed on the non-proportional item (since the area was not tripled). Only a minority of the students thought of applying an approach whereby the irregular figure was reduced to a regular one, and even less students applied the general principle. A comparison of the "direct" and "indirect" condition shows that there were some small but interesting differences between the strategies used by the students. As expected, proportional strategies such as the "internal ratio" or the "rule of three" were more often applied to solve the word problems in the "indirect" condition than in the "direct" condition. When students in the "indirect" condition recognized the non-proportional character of the area problem, this happened sometimes because they knew and activated the general principle, sometimes because they reduced the irregular figure to a regular one. In the "direct" condition, however, considerably more students applied a "reducing the figure" strategy. These findings are in line with our expectations. The presence of direct measures for perimeter and area in the problem statement seems to trigger other strategies in some students: it reminds them to apply previously acquired knowledge about areas of rectangles or triangles, to work on the drawing, etc., whereas problems with indirect measures elicit more often more "general" approaches for solving word problems, such as the application of "internal ratios" or the "rule of three".

## CONCLUSION AND DISCUSSION

Earlier studies have shown that secondary school students have a strong tendency to apply proportional solution strategies even in situations where they are not applicable. More specifically, a systematic line of research by De Bock et al. (De Bock, 2002; De Bock et al., 1998, 2002) has shown that many students believe that if a figure enlarges with factor *k*, not only the perimeter but also the area and volume of that figure increase with that factor *k*. In many of these studies, however, problems were used wherein the quantity under consideration was an indirect – though proportionally related –measure for the perimeter, area or volume, e.g. the weight of an object as an indirect unit for measuring its volume. Implicitly, it was assumed that this would have no significant influence on students' solutions. Recently, however, suspicions arose on this assumption. The use of indirect measures in many earlier studies might have strengthened students' illusion of linearity, and influenced the research findings on the factors influencing this misconception.

The current study explicitly addressed this issue by experimentally manipulating the measures in the problem statement: half of the students solved two items involving direct measures for perimeter and area, while the others solved isomorphic items with indirect measures. A comparison was made of students' performances as well as their solution strategies. We found no significant differences in the performances on the two types of problems. Apparently, the type of measure used in the problem statement has no significant influence on students' performance in general, and on the occurrence of improper proportional reasoning in particular. This seems to confirm the internal and external validity of the findings of the earlier studies on students' illusion of linearity. A qualitative analysis of the underlying solution strategies, however, provided some interesting differences. For items with indirect measures, more students applied a strategy based on the general application of linearity, whereas items with direct units for perimeter and area elicited more content-specific strategies such as working on the graphical representation and the application of formulas for perimeter and area.

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