AESTHETIC VALUES IN MATHEMATICS: A VALUE-ORIENTED EPISTEMOLOGY

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Much has been said on matter of aesthetics and mathematics—perhaps most commonly articulated in relation to properties such as 'beauty' and 'elegance,' which are used to distinguish good from not-so-good mathematical products. Despite its importance in the work of mathematicians, it has been argued that aesthetics cannot be incorporated into school mathematics given students' difficulties with basic problem-solving skills (Dreyfus and Eisenberg, 1986; 1996). In contrast, this paper argues that it is both possible and desirable to incorporate aesthetic concerns into the mathematical activities of students. The argument is based on a re-articulation of both the nature and purposes of the aesthetic in school mathematics that extends beyond the objective, product-oriented interpretations more commonly discussed.

The evaluative role of the aesthetic pervades the world of the professional mathematician where terms such as 'elegant' are regularly used to judge 'good' theorems, solutions and proofs. Some researchers, such as Dreyfus and Eisenberg (1986), propose that students should also develop aesthetic mathematical competencies, in addition to their cognitive ones, in order to appreciate the 'elegance' of certain solutions and proofs. Their proposal follows from the tenet that "one of the major goals of mathematics teaching is to lead students to appreciate the powers and beauty of mathematical thought" (p. 2).

In order to determine the feasibility and sensibility of such a goal, Dreyfus and Eisenberg designed a study in which they investigated whether college students were able to appreciate elegant solutions. They find that the students were unable to find elegant solutions to these problems. Furthermore, the students did not find the solutions which were later presented to them—and more elegant, according to "experts"—any more attractive than the ones they had come up with on their own. Dreyfus and Eisenberg conclude that their students were incapable of aesthetic appreciation. Might the researchers be wrongly equating the lack of agreement between students' and mathematicians' judgements with students' lack of aesthetic sense?

The issue is partly an epistemological one. When mathematicians evaluate entities such as proofs and solutions, they do so for two reasons: one, to establish personal value; and two, to establish collective value. As Alibert and Thomas (1991) note, these two purposes are often absent in school mathematics, where "the subject matter is presented as finished theory, where all is calm… and certain" (p. 215). Students approach mathematics as something to be accepted and learned while mathematicians approach it as something to be evaluated and negotiated. Alibert and Thomas are particularly concerned with the problems this epistemological disconnect produces when it comes to proofs. While mathematicians use proofs to convince (both themselves and others), students see proofs as difficult, formal, and sometimes arbitrary things. Ironically, Alibert and Thomas believe that the perceived need to preserve the precision and the 'beauty' of mathematics—by emphasising the rigour of formal proofs in the classroom—may compromise students' concern for meaning and value as well as their appreciation for the functional role of proof. In other words, why would students develop preferences for one proof over another if proofs only have already-established truth values? That would be an epistemological category mistake.

In their study, Dreyfus and Eisenberg suggest the possibility that proofs can have an aesthetic value, but they take an objectivist view of aesthetics—that a certain solution is elegant in and of itself, independently of human perceivers—and, in considering students' evaluations, look for aesthetic preferences that match those of professional mathematicians. Since they do not find these, they conclude that students do not show aesthetic appreciation. Yet perhaps these students are showing and developing quite different aesthetic preferences, which suit their own current goals and needs.

For example, Brown (1973) describes what might be called a "naturalistic" conception of beauty manifest in the work of his graduate students. He recounts showing them Gauss' apocryphal encounter with the famous arithmetic series: $1+ 2 + 3 + \ldots + 99 + 100$. He asks his students to investigate variations of the general scheme (that the sum of the first n numbers is $(n +1) * n /2$). They come up with many geometric and algebraic approaches, each equivalent but expressed in various ways. Brown asks them to discuss their approaches in terms of aesthetic appeal. Surprisingly, many of his students prefer the rather messy, difficult-to-remember formulations to Gauss' neat and simple one. Brown conjectures that the messy formulations do a good job of encapsulating the students' personal history with the problem as well as its genealogy, and that the students want to remember the struggle more than the neat end product. This is in clear opposition to the way that mathematicians like to present their results: they are almost always devoid of any of the guesses, supporting sketches, and history of the solving process. Brown's observation highlights how the contrasting goals, partly culturally imposed, of the mathematician and the student lead to the use of different aesthetic criteria.

The question that must be addressed is whether the goal of nurturing aesthetic preferences is to align them with those of professional mathematicians. In contrast with Dreyfus and Eisenberg, who want to initiate students into an established system of mathematical aesthetics, I propose that educators nurture students' development of aesthetic preferences according to the animating purposes of aesthetic evaluation in school mathematics. The starting point is not to train students to adopt aesthetic judgements that are in agreement with "experts" but, rather, to provide students with opportunities in which they want to—and can—engage in personal and social negotiation of the worth of a particular idea. A student's aesthetic capacity is not equivalent to her ability to identify formal qualities such as economy, cleverness, brevity, simplicity, structure, clarity or surprise in mathematical products. Rather, her aesthetic capacity is her ability to combine information and imagination when making purposeful decisions regarding meaning and pleasure—this is a use of the term 'aesthetic' drawn from interpretations such as Dewey's (1934).

Thus far I have focused on solutions and proofs—the 'ready made mathematics'—as objects of aesthetic evaluation. However, as Le Lionnais (1948/1986) points out, mathematicians will also judge the aesthetic value of many other mathematical entities including definitions, diagrams, theories, methods and algorithms—entities that students encounter far more frequently than finished solutions and proofs. But as with proofs, students are presented with definitions, methods, and algorithms, as if, echoing Alibert and Thomas, they were finished—all being calm and certain. Students accept the division algorithm and the definition of a quadrilateral without being invited to consider questions such as: Is it good? Is there a better one? Does it have value?

In the following example, I invite a group of four middle school students to consider these questions about a method for constructing a square. Their responses suggest that students do indeed, with little guidance, show a strongly developed aesthetic capacity when considering the value of mathematical entities.

AN ILLUSTRATIVE SKETCH FROM RESEARCH

The following sketch is taken from the observation of four grade eight students in a western American city who were working independently on a geometry course using *The Geometer's Sketchpad*. It is from one site of an ongoing research programme being carried out to study the aesthetic dimension of student mathematical activity. This is the third class of the semester and the students are attempting to re-construct an iterated image involving many squares (for more details, see Sinclair, 2002).

The four students Aleah, Becca, Sara and Zhavain are attempting to construct their first square with Sketchpad. *Constructing* a square in Sketchpad is not a trivial matter; one must first know what defines a square, and then know how to use the appropriate tools. Most students start by using the segment tool to draw four equal sides (the salient property of the square for them) and then attempt, when the time comes, to put the segments at right angles (the more tacit property). I let the students draw squares using only the segment tool, and then show them how to use the circle tool to construct equal segments. Since they have already learned to construct perpendicular and parallel lines, they are then able to construct their squares. Except Aleah. She is stuck on her horizontal segment, insisting on "turning it" up to a vertical position—not wanting perhaps to bother with circles and perpendicular lines. I show her how to turn her segment using the rotate command. Once she has completed her square, she proudly shows the technique to Sara.

I ask Sara which technique she prefers: the rotation method or her "compass and straightedge" one? Sara thinks the rotation method much easier and quicker to perform (given the grammar of Sketchpad's tools at least, where rotation is a one-step action). But, she describes the compass and straightedge method as "more perfect and more mathematical." I ask her what she means by "perfect," and she tells me that the compass and straightedge gives a better construction because she knows that the "points are at the right place." Sara manages to convince Becca and Zhavain of her opinion, but not Aleah.

I then show the students how to create a custom tool that will allow subsequent squares to be created effortlessly. Sketchpad's custom tools are accompanied by scripts, which provide a symbolic representation of the steps involved in the construction associated with the tool. Aleah thinks that the brevity of her script—"look, only six steps!" (compared with Sara's ten)—will help convince her classmates of the rotation method's superiority, but alas, they still prefer their "perfect" method.

This short episode shows a few different aesthetic preferences emerging from the students' negotiation. Sara seems to have a 'classical' orientation, preferring the Euclidean approach to constructing a square. But perhaps she had been enculturated into believing that things which are more technical, more complicated, are in turn more mathematical. In fact, I was initially surprised at Sara's answer, convinced that she would prefer Aleah's method. But Sara also seems convinced that the compass and straightedge construction is somehow more precise. This may be due to the sense of determinacy that points of intersection provide; after all, she constructs each vertex by finding the point where a circle and a perpendicular line intersect.

The rotation method does not have the same sense of determinacy though, of course, it also provides a precise location for each vertex. Since Aleah never actually followed Sara's method, she may not have experienced that sense of determinacy that comes from finding a point of intersection.

Aleah's penchant for the rotation method has several sources. First, the rotation method is hers; she is the one who discovered it. Second, the rotation method grew out of her unique way of seeing a square. Even when I showed the students how to use the circle as a compass, Aleah had a specific idea about how the square should evolve that did not involve circles and perpendicular lines. She knew she wanted to rotate; all I did was to show her that Sketchpad could help her accomplish her goal. So not only was Aleah's method her own, but it answered the particular question *she* had about making a square. In contrast, since the other students hadn't seen the square as Aleah did—as rotated segments—, Aleah's method answered a question they did not ask.

Lastly, Aleah seems to adopt a familiar mathematical aesthetic for simplicity and economy; she uses both as criteria for evaluating the two solutions. Her rotation method is simpler because it does not require using the circle as a compass tool, and it essentially repeats the same step over and over, instead of requiring several constructions that are not transparently related to the square (what does a circle have to do with a square?). Her solution has more economy because it literally takes fewer steps, and saves her from having to hide extra geometric objects (in Sara's method, the circles and line segments used to determine the vertices need to be hidden and replaced with segments).

Possibly, Aleah prefers her method because it is hers and merely appeals to the criteria of simplicity and economy as less subjective-looking reasons. Professional mathematicians might be accused of doing the same thing—ultimately preferring their own discoveries and solutions. In fact Wells (1986) raises this issue, suggesting that they might indeed be "aesthetically biased, as many artists seem to be, towards their own fields and their own works" (p. 39). As with Aleah, professional mathematicians might also invoke aesthetic criteria such as simplicity and economy when trying to convince colleagues of the significance of their work.

The first and second sources of Aleah's preference for the rotation method match Brown's findings that students prefer their own solutions. However, where Brown emphasises the students' solution process, and the attachment they feel to their own solution paths, I believe Aleah's preference is not so much about the process as it is about the relationship between the problem and the solution. Aleah's method *fit* her problem, which happened to be slightly different than her classmates' problem. Here the aesthetic plays a slightly more experiential role since the aesthetic 'sense of fit' straddles the process of inquiry instead of operating only at the final evaluative phase.

CONTEXTS OF NEGOTIATION

The four students do not come to a consensus, however, each student develops a sense of the value of their different square-constructing methods, thus establishing a personal connection with some of the mathematical ideas they will continue using. This episode shows quite clearly that middle students can draw on some of the aesthetic criteria used by professional mathematicians. More provocatively, it suggests that students, if allowed to, naturally behave in ways that are just as aesthetic as they are cognitive or affective—a suggestion consistent with the theories of a wide range of scholars (see Dissanayake, 1992; Dewey, 1934; Johnson, 1993). The accessibility of students' aesthetic sense may seem surprising given Dreyfus and Eisenberg's research. However, the apparently conflicting conclusions reveal different assumptions and goals for aesthetic appreciation.

In Dreyfus and Eisenberg's study, students who had already found their own solutions for a problem were presented with an outside, "expert" one; this set up a very different context of negotiation than the one in the episode related above. The "more elegant" solutions were essentially presented as the *right* solutions, eliciting responses from the students such as "Oh, that's how you do it." The students even became defensive "My way works too." Furthermore it is not clear that the students really understood the "aesthetically superior" solutions; in fact, Dreyfus and Eisenberg report that the students wanted to "pick up the pencil and start working—without first reflecting upon different solutions paths" (p. 7). This slight condemnation reflects Dreyfus and Eisenberg's belief that aesthetic judgements can be made based on objectively accessible features that determine the aesthetic merit of a solution, which would be agreed upon by the "experts."

However, Wells (1986) has shown that the "experts" do not in fact always agree on their aesthetic judgements, and that furthermore, mathematicians take into account their personal experiences with solutions when making aesthetic judgements—sometimes having to 'live through' the solution or proof again. By wanting to pick up their pencils, Dreyfus and Eisenberg's students were showing that they needed to better familiarise themselves with the different solution path before being able to compare it with their own: they could not make spontaneous value judgements.

Instead of a right/wrong context of negotiation, the students in the episode above were invited into a value-oriented context of negotiation. Through the process of negotiation, they were given the opportunity to familiarise themselves with each other's methods, instead of having to make immediate judgements. I have claimed that these students appealed to aesthetic criteria that are similar to ones used by mathematicians but, for two reasons, I have not focused on whether "experts" would have agreed with any of the students' preferences. Firstly, I question whether there is in fact an agreed-upon "expert" opinion and whether there is an aesthetic metric—some hierarchical combination of aesthetic criteria—that could produce an "expert" opinion. Secondly, students usually work with mathematical ideas that are so familiar and evident to mathematicians that they fail to elicit aesthetic responses for them. But even if a professional mathematician is not surprised by or drawn to the 'magical' properties of the number 9, that does not mean the idea is unworthy of aesthetic consideration. Surely, educators cannot hope to help students appreciate the 'elegance' of the Pythagorean proof of the infinitude of primes without first helping them make value judgements on their own mathematics.

I have argued that a primary goal of inviting aesthetic evaluation into the classroom is to encourage students to develop a value-oriented sense of mathematics. In addition to presenting students with a more genuine image of mathematics as professional mathematicians practice it, a value-oriented sense of mathematics can help engage students at a more personal, humanistic level, thus making their experiences in the classroom more memorable and meaningful. After all, as Johnson (1993) writes, the aesthetic provides the very "means by which we are able to have coherent experience that we can make some sense of" (p. 208). A value-oriented sense of mathematics should also provoke meta-cognitive activity since aesthetic evaluation draws on reflections of one's feelings and beliefs about mathematical ideas.

Wells (1986) offers yet another reason for inviting aesthetic evaluation into the mathematics classroom; he points out that teachers might have much to gain in probing students' aesthetic judgements by helping them adapt classroom teaching towards their students' perceptions. For instance, based on Aleah's perception of squares, a teacher might invite Aleah to construct other shapes using transformational geometry. In probing students' aesthetic judgements, teachers might also gain insight into students' ways of thinking and feeling, which can help them adapt the conditions of classroom learning.

SUMMARY

It is tempting, for one who takes pleasure in and values the 'beauty' of certain mathematical entities, to view aesthetic appreciation as a goal in and of itself. Taken into an educational context, the temptation can turn into a conviction that students should be able to take pleasure in and value the 'beauty' and of school mathematics. By showing that students often fail to perceive supposed beauty, Dreyfus and Eisenberg's study suggests that students may not be able to engage in aesthetic evaluation for the same purposes that motivate many professional mathematicians. In contrast, the example above shows that students can and do behave aesthetically in the mathematics classroom, but that their aesthetic behaviours have very functional, yet pedagogically desirable, purposes: establishing personal and social value.

The challenge for educators will be to find ways of productively evoking and nurturing the aesthetic capabilities of their students. But since aesthetic considerations are appropriate in a much wider range of mathematical situations than previously thought, the opportunities will extend beyond problem solving activities. The challenge for researchers will be to determine how students' aesthetic capabilities can also contribute to the *process* of mathematical thinking, particularly in comparison to the way in which aesthetic capabilities contribute to the process of mathematical inquiry for professional mathematicians (Papert, 1978; Poincaré, 1908/1956).

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