COMPARING COMPETENCE IN TRANSFORMATIONAL AND GENERATIONAL ALGEBRAIC ACTIVITIES

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The Purposeful Algebraic Activity Project[1] is a longitudinal study of the development of pupils' algebraic activity in the early years of their secondary schooling. Here, we report on our empirical findings from the initial semi-structured interviews. We analyse the responses of three pairs of 12-year-old pupils to a range of algebra questions. In our analysis, we identify broad similarities in the 'answers' pupils gave to transformational questions and quite significant differences in the pupils' responses to generational questions. We consider the implications for assessment, and discuss the potential of spreadsheets for developing pupils' appreciation of the need for an algebra-like notation.

BACKGROUND

Kieran (1996) describes three kinds of activities within the scope of school algebra:

- *generational activities*: generating expressions and equations that are the objects of algebra, expressions of generality from geometric patterns or numerical sequences, and expressions of the rules governing numerical relationships
- *transformational activities*: rule-based activities including collecting like terms, factoring, expanding, substituting, simplifying expressions and solving equations
- *global, meta-level activities*: such as problem solving, modelling, finding structure, justifying, proving and predicting (p. 272)

In the project Purposeful Algebraic Activity, we use Kieran's classification above, together with a broad notion of *algebraic activity* (Meira, forthcoming).

Generational activity, particularly the translation of a verbal representation of a problem into an algebraic one, has been identified as a major obstacle for pupils (Kieran, 1997). MacGregor and Stacey (1997) interpret pupils' early misrepresentations, not as indicative of low levels of cognitive development, but as 'thoughtful attempts to make sense of a new notation' (p. 15). They identify the sources of these errors as analogies with other symbol systems, intuitive assumptions and pragmatic reasoning about an unfamiliar notation system. Ainley (1999) found that 11-year-old pupils were comfortable with talking about, representing and operating mathematically with unknown quantities. Their written representations reflected their lack of experience of the conventions of notation rather than their difficulties with algebraic activity. Research has also shown that technology can be highly successful in helping students give meaning to algebraic expressions (Thomas and Tall, 2001; Sutherland and Rojano, 1993).

Transformational activity has traditionally been given significant attention both in schools and in research (e.g. Küchemann, 1981). Within the literature, some researchers use the transformational activity of solving equations in order to define a boundary between arithmetic and algebra (Filloy and Rojano, 1989; Herscovics and Linchevski, 1994). Rather than establishing levels of pupils' competence or defining boundaries, we

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focus our attention on the *meaning* of *algebraic activity*. Whilst we acknowledge that detachment of meaning is powerful in transformational activity, we recognise that while pupils may engage in routines of action, they may not appreciate the value of doing so:

> 'Even those students who manage to handle algebraic techniques successfully, often fail to see algebra as a tool for understanding, expressing and communicating generalisations, for revealing structure, and for establishing connections and formulating mathematical arguments' (Arcavi, 1994, p. 24)

Although there appears to be relatively little research comparing *generational* and *transformational activity*, the literature on assessment raises two issues of particular relevance. It is widely acknowledged that school tests tend to assess quantifiable skills, and that where particular aspects of mathematics are assessed, there is an incentive to 'teach to the test'. We suggest that while transformational activities are relatively easy to measure, assessing generational activity and global, meta-level activity is potentially more problematic. As well as the *kind* of activity assessed, Cooper and Dunne (2000) discuss the *context* of the question. They refer to pupils' responses to two algebra questions used to assess whether pupils can express a simple function symbolically (from a U.K. national test taken by 13-14 year olds). One item is *realistic* with the imagined context of planting trees in an orchard; the other is *esoteric*, requiring pupils to work out the perimeter of labelled figures. Cooper and Dunne observe that pupils are significantly more successful at answering the *esoteric* question, and have difficulties constructing the intended goal of the problem as algebraic for the realistic item.

DATA COLLECTION

As part of our longitudinal study we are conducting semi-structured interviews at regular intervals to trace the development of pupils' algebraic activity during the first three years of secondary schooling. The interview questions cover a number of themes, and include *generational*, *transformational* and *global, meta-level activity*. We report here on the initial set of interviews with twelve pairs of pupils at the end of their first year of secondary school (mostly aged 12), during which they have experienced some formal algebra teaching. Their teachers were asked to identify compatible pairs of pupils across the perceived ability range from those who were willing to take part. The first named author conducted the interviews, with each question presented in written form and read aloud. Throughout the interviews, pupils were encouraged to articulate and discuss their responses, all of which were video taped and audio taped. Calculators, paper and pens were available for the interviews, but most pupils chose to say aloud their responses. The transcripts were annotated to include non-verbal behaviour, and any written work.

Here, we report on the responses of three pairs of pupils to a selection of *generational* and *transformational* questions from the initial interviews. The three pairs were chosen to illustrate typical responses from each of the ability groups:

Megan and Thomas	High ability	School M
Natasha and Holly	Middle ability	School M
Mollie and Grace	Low ability	School N

In our analysis we discuss the contributions of each pair, but where appropriate (where they disagree, for instance) we focus on the contributions made by specific individuals.

TRANSFORMATIONAL QUESTIONS

During their first year of secondary school, pupils learn to simplify expressions and construct and solve simple linear equations. We wanted to see whether pupils could *do* these things, and how they talked about doing these things. The following table summarises the pupils' spoken responses to the transformational questions. The first four questions involve simplifying expressions (from Küchemann, 1981); the last four involve solving equations (first two of which are from Herscovics and Linchevski, 1994).

Question	Megan and Thomas (High ability)	Natasha and Holly (Middle ability)	Mollie and Grace (Low ability)
2a+5a	**Seven a**	**Seven a**	**Seven a**
2a+5b+a	**Three a plus five b**	**Three a plus five b**	G: **Three a add five b** … the answer would be eight … two a add five b … M: Three a equals five b
3a-b+a	**Four a minus b**	**Four a minus b** Three a minus b Three a plus one b	**Four a take away the b**
(a-b)+b	a minus two b a plus b	H: a minus two b or a plus two b … N: I don't think you can simplify	[Evaluates for particular numbers]
14+n=43	**Twenty-nine**	**Twenty-nine**	**Twenty-nine**
23=37-n	M: Sixty … T: **Fourteen** [Megan agrees]	N: What minus thirty seven equals twenty three? H: **Fourteen** [Natasha agrees]	G: You'd do n take away thirty seven equals twenty three M: **Fourteen** [Grace agrees]
3t=t+3	T: (frowns) t is one M: … Nought or something … T: **One and a half**	H: That just means the same thing … N: Three times t plus t plus three	Four t add three
7+4u=70-3u	[Unable to solve]	[Unable to solve]	[Unable to solve]

Table 1: Extracts from pupils' spoken responses to transformational questions, correct answers highlighted in bold

Using the distinction between *answers* and *strategies*, (van den Heuvel-Panhuizen and Fosnot, 2001) we observe that there are broad similarities between the answers given by the three pairs. With the exception of one equation (3t=t+3), the three pairs were either all correct or all incorrect in their response.

Megan and Thomas (High ability) were noteworthy in terms of the immediacy and confidence of their responses. Thomas was the only pupil of the three pairs who was able to solve the equation 3t=t+3. Clearly, solving an equation had some meaning for Thomas, for when asked what he thought algebra was useful for, he responded 'If you, like, don't know a number and something then, but you know the rest of the thing you can, like, call it x and try and work out what it is.' He was able to use trial and improvement, going beyond the taught method of using inverses.

Natasha and Holly (Middle ability), although giving the same answers, engaged in significantly more discussion, evidenced by the lengthy transcripts. Their talk is an honest account of some of their dilemmas. For instance, when trying to simplify 3a-b+a:

Holly: You can't minus the b. These are the ones what I don't understand really, because it's really confusing, I think, because you don't know whether to plus a or take away a …

Natasha: I think you do, I think it's four a minus b [writes 4a-b] I think that's how you write it, 'cause you can't minus a b from an a because they're different letters

Holly: Yeah, but then, it doesn't exactly say we have to add an a does it? 'Cause it could mean take away an a. No it doesn't.

Importantly here, we observe that they gave the correct *answer* in the end. Natasha and Holly said that they had done a lot of algebra in class, and particularly referred to the transformational activities of simplifying expressions and solving equations.

Mollie and Grace (Low ability). In terms of the *answers* they gave, Mollie and Grace's achievements were comparable with those pairs identified by their teachers as high and middle ability. They correctly simplified three out of the four expressions. However, when asked to explain their method of simplifying 2a+5a, it became clear that Mollie and Grace were using what has become known as *fruit salad algebra*: 'anything can be an a, so you can put two apples add five apples, so then you can, like, add the five and the two … seven, and then you can put seven a equals seven apples.' Whilst they showed some competence in transformational activity, their skills were limited to simple transformations. This relative 'success' may be attributed to previous experience of such tasks. Their language indicates awareness of the *rules* of algebra: 'put the one a on the two a' and 'carry it over.' But with the more difficult equations, unlike the other pairs, Mollie and Grace could not make sense of the *meaning* of the equations; instead focusing on trying to identify which part was the 'answer.'

Analysis of the pupils' responses to transformational questions has shown that there is not a great deal of difference between the three pairs. Whilst all pupils could participate in the discourse of simple transformational algebra, the interview situation revealed some differences in the *meaning* that they constructed for what they were doing.

GENERATIONAL QUESTIONS

One aspect of our research involves tracing the longitudinal development of pupils' generational activity and their attempts to use notation. In the interviews, we wanted to identify whether pupils can generate expressions and equations, and whether they used particular conventions for notation. We report here on the pupils' responses to five such questions.

4. Jack is three years older than Chloe. What can you write for Jack's age?
5. George's big brother gets twice as much pocket money as George.
   What can you write for how much pocket money George's big brother gets?
6. David is 10 cm taller than Con. Con is h cm tall.
   What can you write for David's height?
8. Explain what you think the rule is in the table.
   Write the rule in terms of the letters in the top row.

a	b
1	2
2	3
3	4
4	5

Write the rule in any other way that you can.

11. Can you solve 5x99 in your head?

Can you write down how you could multiply *any* number by 99?

Figure 1: Generational questions called *Jack's age, Pocket money, David's height* (from MacGregor and Stacey, 1997), *Rule in table* and *Multiply 99*

Unlike the transformational questions, analysis of pupils' responses to the generational questions shows quite significant *differences* between how the pairs responded:

Question	Megan and Thomas (High ability)	Natasha and Holly (Middle ability)	Mollie and Grace (Low ability)
4. Jack's age	**c plus three equals j**	H: Eight and Chloe would be five … N: **Three plus c** … H: Yeah, it would be three c … N: Three minus c	G: Jack could be nine and Chloe could be M: Six … they could be any age but as long as they're three years, Jack's three years older than Chloe
5. Pocket money	**Two g equals b**	**g times two** **g two**	G: George could get two pounds and George's big brother could get six pounds … M: However much George gets, then his brother has to get two pounds more
6. David's height	**h plus ten equals d**	**h plus ten**	G: Con could be one centimetre and David could be eleven centimetres … M: The h can be any number (gestures, palms upwards) and, um, as long as David is ten centimetres taller
8. Rule in table	**b equals a plus one**	**a plus one** a plus one equals b [with prompting]	Add the one [with prompting] [Unable to express using letters]
11.Multiply 99	Round ninety-nine to a hundred then times it by the number and take away the number	5 ×100 (− 5) 495	[Unable to solve 5x99; unable to understand method]

Table 2: Extracts from pupils' spoken responses to generational questions, correct answers highlighted in bold

Megan and Thomas (High ability) generated expressions and equations fluently. Their answers indicate their familiarity and confidence with using conventional notation. For example 'two g equals b' in Pocket money shows awareness of the convention of putting the letter term before the number, and 'b equals a plus one' in Rule in table, shows awareness of the convention of putting the dependent variable first. Although in Multiply 99, Megan and Thomas were unable to write anything down when asked, the question does not make it explicit that algebra should be used, and Megan's explanation 'Round ninety-nine to a hundred then times it by the number and take away the number' is entirely appropriate. Over a range of questions, a pattern emerges of pupils who are confident in generating their own expressions and equations, and for whom algebra has some meaning.

Natasha and Holly (Middle ability) Natasha and Holly successfully answered almost all of the questions. They discussed each question carefully but were not always secure in their responses. For instance in Jack's age, they first suggested ages for Jack and Chloe, then generated a number of different expressions: 'three plus c', 'three c' and 'three minus c'. Interestingly, they tended to generate expressions rather than equations, which was problematic for Rule in table, since the rule needed to include both a and b. Natasha initially suggested 'a plus one,' but needed prompting to include 'equals b' in her response. The expressions that they generated do not show the same awareness of conventional notation as in Megan and Thomas' responses. In Multiply 99, Natasha and Holly used the number 5 to illustrate the method; they did not use a letter to represent how to multiply any number by 99.

Mollie and Grace (Low ability). While we observed subtle differences between the high and middle ability pairs, we observed quite significant differences between both of these pairs and Mollie and Grace. In essence, their answers to the generational questions do not include letters or symbols. A pattern emerges whereby Grace sought to evaluate the unknowns, choosing suitable (or unsuitable) values for Jack's age, George's pocket money and David's height. Her responses suggest that she did not recognise the questions as algebra problems, but as arithmetic problems, to which a numerical answer was required. Mollie's language and gestures seemed to indicate that she was comfortable with unknowns and most of the relationships. When asked David's height, Mollie concluded:

> 'Well, h, yeah, again, the h can be any number (pointing to the h and when she says *any number* she gestures with both hands, palms facing upwards) and, um, as long as David is ten centimetres taller than any of that, any number that h is'

Drawing upon Cooper and Dunne's (2000) interpretation of pupils' responses to *realistic* algebra questions, we acknowledge that questions such as *David's height* are problematic in that they are presented in the everyday language of arithmetic and they do not explicitly ask pupils to write an expression. Mollie's response clearly demonstrates an understanding of the *relationship* between the heights, although she did not use notation. We recognise that the use of notation is an implicit expectation, but equally recognise that other pairs of pupils were aware of this convention. Mollie and Grace's difficulties extended over a broad range of generational questions including the *esoteric* questions (*Rule in table, Multiply 99*) where they were unable to generate equations.

Analysis of the pupils' responses to *generational questions* has shown important differences between the pairs. The pairs identified as high or middle ability successfully used algebraic notation to generate expressions and equations. However, those identified as low ability did not use algebraic notation to generate expressions and equations.

DISCUSSION

Analysis of data from the three interviews leads us to tell three different stories about the pairs of pupils. Megan and Thomas, identified by their teachers as high ability, had some sense of what letters are used for in algebraic notation. They solved generational and transformational questions with a confident grasp of notation conventions. Natasha and Holly, identified as middle ability, were also successful over a broad range of questions, particularly when prompted in an interview situation. However, they were less confident with the syntax of algebraic notation. Mollie and Grace, identified as low ability pupils, also engaged in algebraic activity. They could clearly solve simple transformational questions, but had difficulties with generational algebra. In summary, we found broad *similarities* in the pupils' answers to *transformational* questions, and *differences* in their responses to *generational* questions. The interview data gives only a snapshot of pupils' development, and our focus has been on three pairs of pupils, but this finding is representative of what we have found across pairs in the set of interviews. We conjecture that the reason for this finding many lie in the teaching emphasis on transformational activities (which may be particularly prominent in School N), and the nature of pupils' learning experiences of generational activities.

The picture that emerges from our analysis has important implications for assessment and for teaching. If school tests focus on *transformational* activities, they may be giving a misleading impression of how much pupils understand. Competent performance on transformational activities may disguise pupils' difficulties with generational activities, and with constructing meaning for transformational algebra. Hence, whilst we recognise the importance of the assessment of transformational activity, we feel that greater consideration should be given to assessing generational activity (and global-meta level activity). Our initial analysis of pupils' responses to school test items on algebra suggests that our concern about assessment implications is valid. We observe on one test used in School M, transformational items outnumber the generational, and that there are broad similarities between pupils' responses to transformational items, and differences in their responses to generational items, reflecting the findings from our interview data.

We believe that pupils need more opportunities to engage in generational activities, and particularly in activities where they can appreciate why generating expressions is a useful thing to do. In the Purposeful Algebraic Activity project, we have designed a teaching programme using spreadsheet-based tasks (Ainley, Bills and Wilson, forthcoming). The tasks are designed to engage pupils in solving purposeful problems, and incorporate *generational* and *global, meta-level activity*. We feel that generating expressions and attending to the process of denoting is an important foundation for understanding. The spreadsheet provides a context for pupils to construct formulae *themselves*. Importantly, there is a *purpose* in doing so, for example to generate more data in order to solve a problem. We have also built in to the teaching programme reasons to move away from the spreadsheet and engage in *transformational* activity so that 'the non-letter-symbolic representations and their transformations can be used to make contact with or give meaning to the letter-symbolic representations that are traditionally involved in algebraic activities' (Kieran, 1996, p. 275). We see spreadsheet algebra as important in developing pupils' confidence *with* and appreciation of the need *for* an algebra-like notation.

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