TEACHING ANGLES BY ABSTRACTION FROM PHYSICAL ACTIVITIES WITH CONCRETE MATERIALS

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An angles teaching sequence was designed, in which students were guided to abstract a general concept from physical activities with concrete materials. The three design principles used were familiarity, similarity recognition, and reification. The resulting teaching sequence was tested in a field study involving 25 teachers of Grades 3-4. The data collected demonstrate the significance of the three design principles, and the results from pre and post assessment interviews provide evidence for the overall effectiveness of the lesson sequence. Also earlier findings that angle is a multifaceted concept which is difficult to learn are supported.

School students have great difficulty learning the angle concept (Clements & Battista, 1992). Douek (1998), for example, described how difficult it was for students in Grades 3 and 4 to interpret the inclination of the sun in terms of angles. The problem seems to be that angle is such a multifaceted concept. Close (1982) and Krainer (1989) have discussed the variety of angle definitions used in mathematics, and many authors have noted the difference between dynamic (movement) and static (configurational) aspects of the concept (Close, 1982; Kieran, 1986).

Mitchelmore and White (2000a) proposed a theory of learning angle concepts by successive abstraction and generalization. According to this theory, an abstract concept is "the end-product of ... an activity by which we become aware of similarities ... among our experiences" (Skemp, 1986, p. 21) and generalisation is the process which extends the meaning of such a concept to include further experiences. Mitchelmore and White proposed that young children initially recognize superficial similarities between physical situations and abstract everyday concepts such as corner, junction, door, and roof. Gradually, they recognize deeper similarities between these objects and form restricted angle concepts such as corner, turn, and slope. A general abstract angle concept emerges as children recognize the even deeper similarity between these restricted angle concepts. At this point, students can interpret a wide variety of situations in terms of angles because they recognize them as consisting of two linear parts that meet at a point and realize that the relative inclination of these two parts has a crucial significance.

Mitchelmore and White (2000a) showed empirically that the major difficulty in learning to identify a physical angle situation lies in identifying the two linear parts of the angle. They found that children can identify so-called 2-line angles (e.g., corners of a room, road intersections, pairs of scissors), where both arms of the angle are visible, as early as Grade 2. On the other hand, 1-line angles (e.g., doors, windscreen wipers, ramps) are more difficult to understand, and even by Grade 6 many students cannot interpret 0-line angles (e.g., doorknobs, pirouettes, wheels) in terms of angles. For 1- and 0-line angles, one or both linear parts have to be imagined or remembered.

Mitchelmore and White (2000b), incorporating the ideas of Sfard (1991), advanced a method of teaching angles called Teaching for Abstraction, based on the following three principles:

> Familiarity. Students should first become familiar with a variety of angle situations.
>
> Similarity. Teaching should then focus on helping students recognize the similarities between these situations.
>
> Reification. Activities should be undertaken whereby the recognized similarity becomes abstracted to an angle concept that can be operated on in its own right.

Following exploratory research on this model using one-to-one and small-group teaching (White & Mitchelmore, 2001), a sequence of lessons was designed and tested in the field during 2001 and 2002. This paper reports and analyses the results of this study.

METHOD

The authors wrote a sequence of 15 lessons which initially explored 2-line angles (corners, scissors, and body joints) and then moved on to 1-line angles (doors, clock hands, and slopes). The lessons followed the principles of Teaching for Abstraction as follows.

> Familiarity. Students explored the angle situations separately to learn about their crucial, angle-related features.
>
> Similarity. Lessons involved frequent direct matching, indirect matching, and other forms of attention to angular similarities:
>
>> In direct matching, the angle in one situation was physically superimposed upon the angle in the other situation. For example, the corner of a pattern block was fitted into the corner of a window frame.
>>
>> In indirect matching, an intermediate angle-like object was used to indicate how the angles in the two situations were similar to each other. For example, a bent straw was used to show that two angles were "the same".
>>
>> In selective attention, the arms, vertex, and opening of an angle were identified.
>
> Reification. Several activities were aimed towards abstracting the angle concept from the concrete situations:
>
>> Students made abstract drawings of angles with approximately the correct size and orientation.
>>
>> Acute, obtuse, and right angles were defined.
>>
>> Students described the angle concept in their own words.

The angles teaching sequence was tested in a total of 25 Grades 3 and 4 classrooms. All the teachers (20 female and 5 male) attended two one-day workshops, one before and one after teaching about 8 lessons selected from the sequence. Grade 3 teachers taught mainly lessons on 2-line angles, whereas Grade 4 teachers taught both 2-line and 1-line angles. Each teacher identified a target group of 8 students in her or his class and administered them a pre and post assessment interview designed by the researchers.

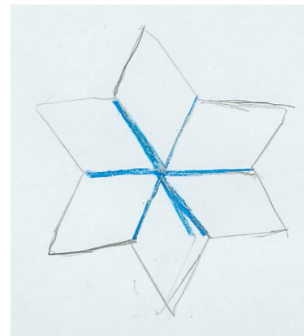
Data were collected from four sources: (1) teachers' written comments and comments made in focus group discussions at the second workshop, (2) the researchers' notes taken during the workshop discussions, (3) work samples from 130 students, and (4) pre and post assessments from about 200 students.

RESULTS AND DISCUSSION

In this section, we present examples illustrating the significance of familiarity, similarity, and reification in the learning of angles. We also present evidence that students learnt to recognize angular similarities and did, to some extent, reify the concept of angle.

Familiarity

There were several instances where the importance of students' familiarity with physical angle situations became clear.



Students were allowed a period of free play with pattern blocks prior to exploring how their angles fitted together (see student's drawing on the right). The pilot testing had shown that, without this familiarization process, students were distracted by making creative patterns and concentrated on the whole shapes instead of the angles at their corners.

A lesson in the sequence dealt with angles made by opening a pair of scissors. Students were clearly unfamiliar with the function of the pivot, but quickly grasped its significance.

Without this first step, it would not have been possible to discuss angles of opening at all. Similar, essential familiarization processes took place when students learnt in other lessons about how the bones of the body are jointed and how doors are hinged.

A negative example occurred in a lesson devoted to angles of slope. Although students easily recognised the angle of slope when a ruler was placed on a table, they had difficulty when this prop was removed and many drew arbitrary second lines (see student's drawing at right). Many students were clearly not sufficiently familiar with the idea of the horizontal, and drew arbitrary second lines.

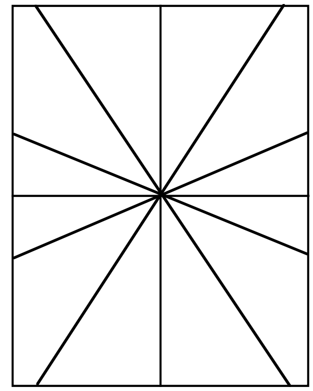
This lesson should have been preceded by lessons devoted to the concepts of vertical and horizontal.

A more positive example occurred when measuring angles. The approach taken by the researchers was to use the 30o corner of one of the pattern blocks as a unit angle. We felt that the introduction of degrees at this point would be unnecessarily complicated, so suggested that teachers invent a name for this unit such as a "Kevin." However, several teachers argued that, since students were familiar with the idea that a right angle is 90o they could easily work out that each unit was 30o. This proved to be correct.

Similarity

Students focused on angular similarities in different situations by matching the angles or identifying the arms and vertex in each situation (selective attention).

Matching. There were several situations where the use of matching assisted development of an abstract angle concept. In one such example, students were challenged to find whether the angles at the centre of a "windmill" (see figure on right) were equal. Most students initially thought that the angles at the top and bottom were larger than those at the sides. When they directly matched the 30o corner of a pattern block with each of the angles at the centre of the windmill, they were surprised to find that all the angles were equal. This exercise focused students' attention on angle as an amount of opening and demonstrated dramatically that the length of the arms is irrelevant.



Another instance occurred with the opening of a door. Students were introduced to this angle situation through examples such as their classroom door, which provided a visible line for the closed position of the door. Students then looked at a "floating door" (one made from a piece of card where the line of the closed position was not visible). A bent straw was used to compare the opening in the two types of door and to help students identify the invisible line of the angle in the second case.

In the assessment interviews, students were given three pairs of angle situations and asked to check that the angles were equal which they did either by direct matching (placing one angle on the other) or indirect matching (placing an intermediary object on both angles). Figure 1 shows the percentage of the sample that could correctly match the pairs of angles before and after the lesson sequence.

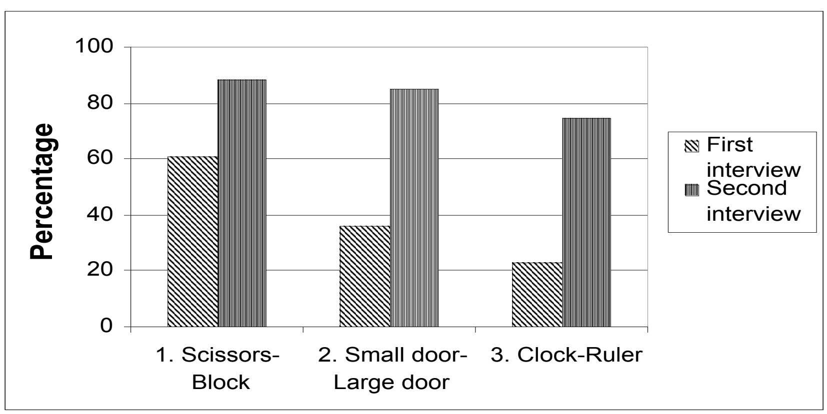
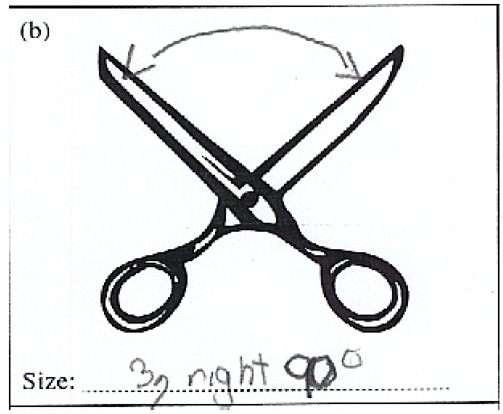


Figure 3: Percentage of sample correctly matching angles.

As expected, the first task, which dealt with 2-line angles, was rather easier than the other two, which dealt with 1-line angles. But the data clearly show that, in all cases, students' ability to match the angles improved substantially over the teaching period.

Selective attention. Many students initially had difficulty identifying the parts of the angle (even in 2-line situations). For example, in the example shown on the right, the student clearly had some idea about the angle of opening, but could not identify the arms or the vertex. This common problem was addressed in the teaching sequence by having students draw lines from the pivot along the blades of the scissors. A similar difficulty occurred with angles made by limbs of the body. Some teachers marked the joint with a dot and the arms with felt pen to make the angles clearer.



Others built models out of pipe cleaners to help students identify the angles better.

In the assessment interviews, students were asked to identify the arms and vertex of various angles. Figure 2 shows the results. Again, students recognized 2-line angles more easily than 1-line angles but in all cases recognition increased dramatically as a result of the teaching. Another task not reported here showed that it was substantially easier for students to identify the arms of an angle than its vertex.

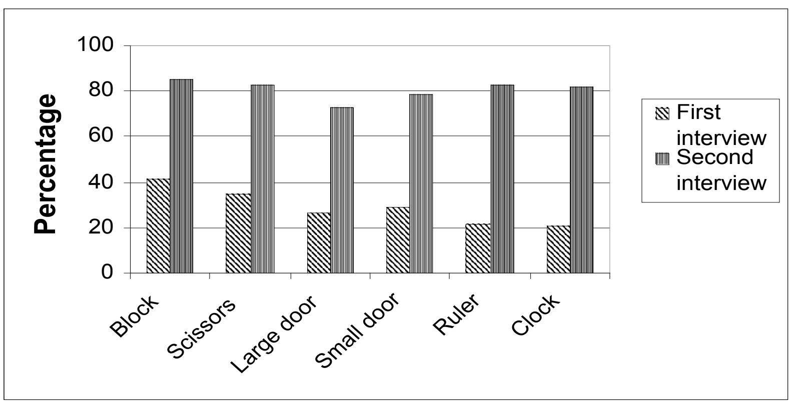


Figure 2: Percentage of sample correctly identifying arms and vertex of each angle.

Reification

In one interview task, students were asked to match an angle in a physical situation with a given abstract angle size. Sample tasks were to move one hand of a clock through a right angle from 2 o'clock and through half a right angle from 7 o'clock, to open a door (represented by a straw on a diagram of a door opening) through an obtuse angle, and to draw an acute angle.. Correct identification increased from 45% to 81% for right angles, from 29% to 82% for obtuse angles, and from 42% to 75% for half a right angle. The large percentage knowing about right angles is not surprising and the large percentage able to draw an acute angle could have been artificial because most angles encountered were acute and angle diagrams are usually drawn acute. However, the dramatic increase in half right and obtuse (in particular) indicates that after the teaching, most students could relate abstract angle sizes with physical counterparts and the technical terms right and obtuse had for most students become quite general.

The last question of the assessment interviews asked students to define an angle. As expected, there was a wide range of responses and it was often difficult to decide exactly what a student was trying to communicate. Eventually, the following categories were determined:

Two arms, vertex, and opening: responses indicating that an angle consisted of two lines meeting at a point and making some attempt to express an angular relation between the lines (using words such as opening, turn, space, area, gap, distance, size and measurement. Typical responses were:

> Two lines that meet at a point. Size between the two arms near the point, not the length of the arms.
>
> Two straight lines that come together at one point vertex. The angle is the opening between the two arms
>
> It measures the turn from one line to another.
>
> An angle is like two sides with apex at the top [uses hands to show] at different degreasers [sic].

Two arms and a vertex: responses indicating that an angle consisted simply of two lines meeting at a point. Typical responses were:

> It's a thing that has a vortex [sic] and two arms.
>
> A pivot which has two arms coming out of it
>
> Two arms: responses that merely mention the presence of two lines, but suggest a general situation. Typical responses were:
>
> Two diagonal lines like a mountain
>
> Two bits of thing which come together

Vague responses named a particular type of angle, referred to degrees or to a part of a specific object, or were vague to the point of being incomprehensible. "Don't know" responses were also included in this category. Typical responses were:

> Something that is 90o, or more or less than 90o
>
> An angle is the amount of degrees between two points.
>
> It's like a triangle. It's used to measure the space.
>
> A thing that bends in a certain way

Figure 3 shows the results for each of the four categories in this item before and after the teaching.

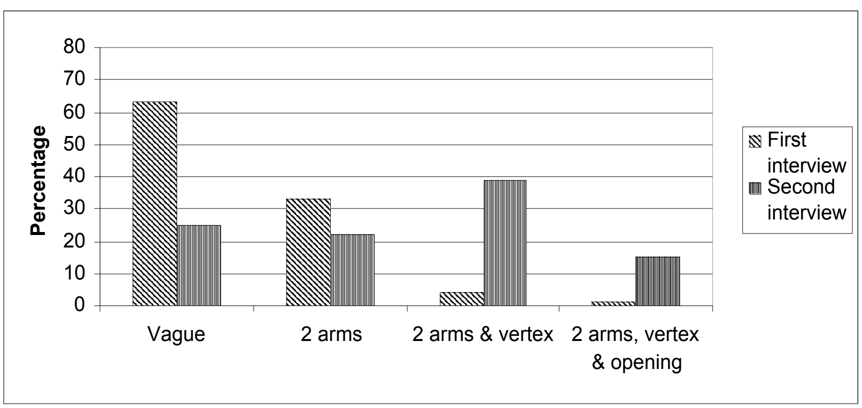


Figure 3: Percentage of sample giving various definitions of an angle.

As a result of the teaching the percentage of responses indicating awareness of at least the two arms and vertex of an angle increased from 5% to 54%, the percentage of vague responses falling off dramatically. The least salient feature of the angle concept was clearly the opening aspect.

CONCLUSIONS

The results of this field study confirm the significance of familiarity, similarity, and reification in learning the angle concept. The importance of familiarity in establishing essential pre-requisite knowledge or necessary understanding of angle situations has been shown to be fundamental to the learning process. The data also show that similarity recognition can be taught and that the three types of matching (direct, indirect, and selective attention) are powerful tools in building an abstract angle concept. There is ample evidence that the emphasis on similarity recognition and reification in the teaching model resulted in many students developing a quite sophisticated concept of angle as early as age 10.

However, our data also emphasize earlier findings that angle is a multifaceted concept which is difficult to learn. The opening aspect of an angle and the significance of the vertex are particularly difficult and would seem to need more emphasis in the teaching. Many of the students in this study were still some way from reifying angle as “an amount of turning between two lines joined at a point.”

Finally, the fact that the Teaching for Abstraction model was so successful in designing an angle teaching sequence suggests that the method could be applied to other mathematical concepts. The processes of familiarity, similarity, and reification are clearly worth further study.

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