AN ANALYSIS OF MENTAL SPACE CONSTRUCTION IN TEACHING LINEAR EQUATION WORD PROBLEMS

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This study is to explore the potential of Fauconnier's mental space theory in understanding the teaching and learning processes of mathematics. Notions from cognitive linguistics have been introduced to research in mathematics education recently, but there have been little studies about what potential the framework of mental space has in understanding mathematics teaching and learning. This paper will present an analysis of classroom data based on that framework. The data was collected at lessons of linear equation word problems of a Japanese seventh grade classroom. It is argued that the mental space theory will provide a useful framework for understanding the process of mathematics instruction, and that the idea of mental space blending could be a powerful tool for analyzing mathematical problems.

Introduction

The framework of cognitive linguistics has been introduced in recent research in mathematics education. Especially, the notions of metaphor and metonymy have been pointed out as essential in teaching and learning mathematics by several researchers (e.g., English, 1997; Presmeg, 1992). Also, "Mathematical Idea Analysis" in the embodied cognition perspective (Lakoff & Núñez, 2000; Núñez, 2000) has showed the power of metaphor in understanding and creating mathematical ideas.

This paper is to explore the potential of Fauconnier's mental space theory (Fauconnier, 1994, 1997; Fauconnier & Turner, 2002) in understanding the teaching and learning processes of mathematics. The mental space theory has provided an interface for understanding the connections between meanings and language expressions, without falling into formalism like previous linguistic theories. It provides a framework for understanding meaning construction *process* in the cognitive linguistics, as well as theorizing the notions of metaphor and metonymy as essential tools of meaning construction. This paper will attempt to show some of its potential for mathematics education by using classroom data.

THEORETICAL FRAMEWORK

Human experiences are always mediated by conceptual structures, and human beings construct realities through conceptual structures, and give meanings to the experiences. Thus, conceptual structures play fundamental roles in human cognition. These are called cognitive models (Lakoff, 1987). The basic components of cognitive models are categories and schemas. Categories are mental constructs resulting from human's practice of grouping experiences. Schemas are mental constructs resulting from experienced patterns of activities, events or phenomenon. Cognitive models are constructed through various projections from these categories and schemas.

To understand the *process* of meaning construction, it is necessary to have tools to represent how these cognitive models are employed in meaning construction. Fauconnier's theory of mental spaces and mappings (Fauconnier, 1997) provides such tools. A mental space is a field of thinking, which is either temporally constructed (e. g., "today's weather"), or relatively stable. Meaning construction is explained as making connections ("mapping") between mental spaces, and producing a network of mental spaces. There are three kinds of mapping (for details, see Fauconnier, 1997):

(1) Schema mapping: This to directly map a schema, frame, or model to a mental space.
(2) Projection mapping (analogy or metaphor): This is to project a structure of a mental space to another mental space.
(3) Pragmatic function mapping (metonymy): This is to connect two spaces based on a pragmatic function.

There is another important operation in mental space construction, "conceptual blending." It is a mental operation to connect more than one mental space and produce an integrating mental space. It is a fundamental and powerful tool for human beings to develop imaginative and creative conceptualization. Fauconnier and Turner (2002) illustrate it by using a solution of the following "Buddhist monk problem":

> A Buddhist Monk begins at dawn one day walking up a mountain, reaches the top at sunset, meditates at the top for several days until one dawn when he begins to walk back to the foot of the mountain, which he reaches at sunset. Make no assumptions about his starting or stopping or about his pace during the trips. Riddle: Is there a place on the path that the monk occupies at the same hour of the day on the two separate journeys? (p. 39)

One way to solve this problem is to imagine the monk walking down from the top at the dawn of the same day when another monk (a double role) walks up the mountain: the monk must meet himself somewhere on the path.

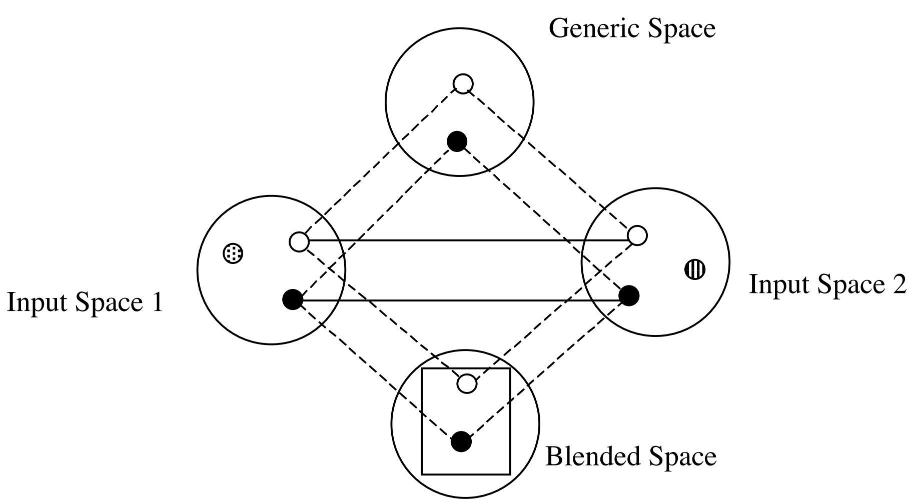

Figure 1. Blended Space (from Fauconnier & Turner, 2002, p. 46).

Fauconnier and Turner (2002) analyzed this solution by making a network of four mental spaces (Figure 1):

> Input Space 1: The journey to the top of the mountain. Input Space 2: The journey back to the foot of the mountain. Generic Space: Journey at the mountain.
> Blended Space: A situation where the monk begins walking down from the top at the dawn of the same day when another monk (a double role) walks up the mountain.

Input Spaces are the sources to be connected and integrated. They have some elements corresponding to each other (e.g., moving individual, mountain, motion): the correspondence forms a "cross-space mapping." Generic Space is a space that has a structure common among Input Spaces. Blended Space ("Blend") is a space onto which the structures of Input Spaces are partially projected. It is not just a mixture of projected structures. The most important feature in the Blend is that a new structure emerges that is not in Input Spaces. In the Buddhist monk problem emerged is that two monks are moving toward opposite directions.

DATA COLLECTION

The data collection was conducted in mathematics lessons of a teacher in a public junior high school in Japan. The teacher had a master's degree in mathematics education and was very interested in improving his teaching. His seventh grade mathematics class was the target of data collection. The class had 19 students. Almost all the mathematics lessons had been observed and audio-and-video recorded from November 2001 to February 2002. Interviews were also conducted with the teacher and students. The lessons contained a unit of linear equations and a unit of proportional, and inversely proportional functions. Those units were chosen because they contained important ideas of mathematics, equation and function, and mathematically rich activities. The present paper uses data from only the lessons about word problems in the linear equation unit.

ANALYSIS

In Japan, the use of letter in mathematical expressions is introduced early at the seventh grade. The linear equation is taught after that. At the beginning of the lessons of linear equation word problems, the teacher prepared large worksheets, called "thinking sheets." Each sheet contained one word problem. Students were asked to write their solutions, their thinking processes, and the solutions discussed in the class on their sheets.

Single-schema-mapping problems

The first word problem the teacher used in the worksheet was as follows:

> "We went shopping carrying one thousand yen. We bought six pencils and one 480-yen pencil case, and 280 yen remained. How much was the price of one pencil?" (Nov. 21, 2001)

This problem could be solved by mapping the following a "shopping payment" model on to the mental space of the problem (Figure 2):

> The amount of money you possessed before shopping subtracting the amount of money you spent results in the amount of money that remained after shopping.

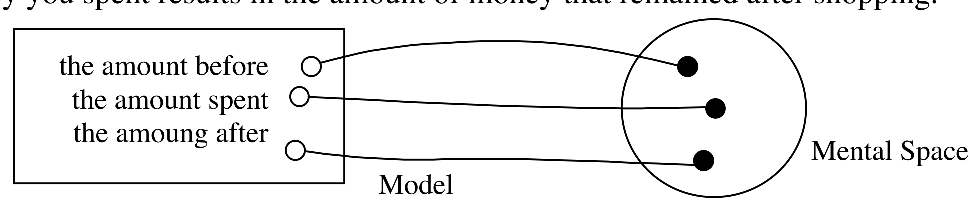

Figure 2. Single-schema mapping.

There are several models equivalent to this, all of which are structured by the part-whole schema. By projecting one of these to the mental space of the problem, we can give an overall structure to the latter. Since the price of one pencil is unknown, we may assign a letter x to it. By using a shopping payment model, an equation "$1000 - (6x + 480) = 280$" is obtained.

At the beginning of the word problem lessons, each problem was solved by mapping single models like the above. Another model appeared was a "total payment" model: The total of the number of each product one bought multiplying its unit price results in the total payment.

Blending problems

After working on several single-schema-mapping problems, the teacher delivered a new worksheet containing a "surplus-shortage problem": "There are children. If we distribute three oranges to each child, then eight oranges remain. If we distribute four oranges to each child, then we are five oranges short. Find the number of children."

The students worked on it individually. They were then allowed to consult with other students. After that, the teacher asked a student Kawata [pseudonym] to write his solution on the chalkboard and explain it before the class. When he finished the explanation, the teacher went to the board and made a detailed explanation. The teacher did not treat this problem as single-schema-mapping problem.

Before discussing the teacher's explanation, let us see how a process of solving this word problem could be explained using the framework of blending. First, the four mental spaces are constructed:

> Input Space 1: Giving three oranges to each child, leaving eight oranges.
> Input Space 2: Giving four oranges to each child, with five oranges short.
> Generic Space: Distributing each person an equal amount of things
> Blended Space: A situation where oranges are given to children, allowing of two ways of distribution: (1) Giving three oranges to each child, leaving eight oranges, (2) Giving four oranges to each child, with five oranges short.

In this linear equation word problem, the blended space is a space that satisfies the both scenarios projected from the two Input Spaces. An emergent structure in the blended space is that there is only one solution, whereas in either Input Space, solutions are indefinite. In the Buddhist Monk problem mentioned above, it is inferred from common sense (but, mathematically, from the continuity of the path) that two persons walking the same path from the opposite ends meet at one point on the path. In linear equation word problems, it is a mathematical principle that there is only one solution that satisfies two (non-equivalent) linear conditions simultaneously (cf. Fauconnier & Turner, 2002, p. 53).

To make an equation, at least the following processes would be necessary:

(1) To identify two quantities common in both Input Spaces: the number of children, and oranges.
(2) To assign letter x to either of them, say the number of children.
(3) To express the other quantity (the number of oranges) by using the x. Going back to the two Input Spaces, two different expressions are obtained.
(4) To connect the above two expressions because they both express the same quantity.

Thus, an equation $3x + 8 = 4x - 5$ is obtained. (If we assign x to the number of oranges, then another equation $(x - 8)/3 = (x + 5)/4$ is obtained.) The idea of using common quantities in Input Spaces is essential in this equation making. These common quantities are each considered identical in the cross-space mapping.

Let us return to the teacher's explanation. His discussion of this word problem (Dec. 4, 2001) fits the above analysis very closely. He drew diagrams during explanation. While talking about the condition "If we distribute three oranges to each child, then eight oranges remain," he drew the upper half of Figure 3. Saying "On the other hand," he began to talk about the second condition "If we distribute four oranges to each child, then we are five oranges short," while drawing the lower half of Figure 3.

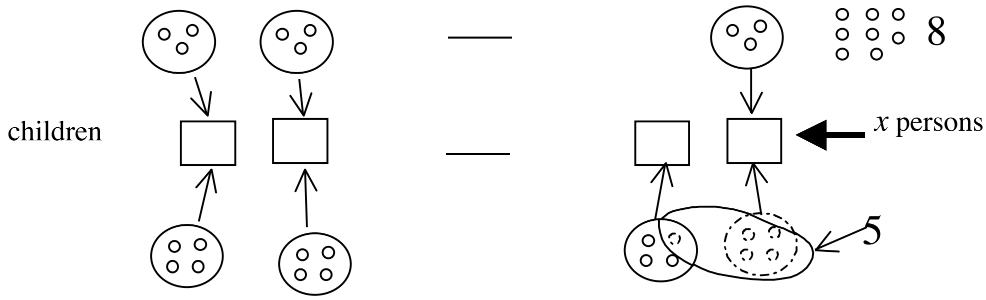

Figure 3. The teacher's diagram.

From his drawing, it is clear that the teacher was introducing the two Input Spaces during his explanation. Also, his contrasting phrase "On the other hand" indicates that he considered both conditions as sharing the "distribution schema."

His attempt to integrate two Input Spaces is apparent in Figure 3. His diagram clearly indicates that the number of children is common in both spaces. After assigning letter x to it, he pointed out that using the letter x, the number of oranges could be expressed. He returned to the Input Spaces, and discussed how to express the number of oranges in each space. He then wrote expressions in words on the chalkboard (Figure 4).

3 x the number of children + the amount of surplus
8

the number of oranges

4 x the number of children - the amount of shortage
5

Figure 4. The teacher's writing on the chalkboard.

He pointed out that the number of oranges did not change in both conditions, and connected the two expressions by an equal sign. Thus, he obtained a linear equation.

The teacher's explanation thus fits the mental space blending very well. What about students? As mentioned, before the teacher explained a solution of the above problem, a student Kawata had explained his solution in front of the class. Immediately after Kawata described how he obtained the expressions "$3x + 8$" and "$4x - 5$," the teacher stopped him and asked questions:

T (the teacher): Wait, wait. What does $3x + 8$ represent in the end?
Kawata: [Showing no hesitation]The number of oranges.
T: Ah-huh. What about $4x - 5$?

> Kawata: [Showing no hesitation]The number of oranges.
> T: The number of oranges, okay. So, what're equal, what're equal?
> Kawata: Huh? The number of oranges.

Kawata was clearly aware that each expression represented the same quantity, the number of oranges. Though he did not draw a diagram like the teacher, his explanation was consistent with the teacher's.

Some of the other students failed to use the common quantities in the Input Spaces in making equations, however. They relied on a pragmatic function mapping between mental spaces of mathematical operations and ordinary phrases. It connected division, addition, and subtraction to "distribute," "remain," and "short," respectively. Using this mapping, they replaced phrases in the problem with mathematical operations, and obtained two (wrong) expressions, $x÷3 + 8$ and $x÷4 - 5$, and connected them by an equal sign to make an equation. This is the well-known "direct translation strategy" (Chaiklin, 1989).

The above connections were motivated from learning of arithmetic word problems. Students seem to have developed close associations between phrases in a word problem and mathematical operations since elementary schools. Mathematical symbols functions as shorthand of phrases. The resulted expressions did not represent quantities, but sequences of events. In single-schema-mapping problems, the schemas used consisted of sequences of events. The teacher's initial use of these problems might have encouraged students' use of direct translation strategy.

The students were assigned a variant of "surplus-shortage problems" later in the class: "We are going to distribute cookies to children. If we give 3 cookies to each child, 12 cookies remain. If we give 4 cookies to each child, 3 cookies remain. Find how many cookies there are." (Dec. 13, 2001). Also, I asked several students to solve similar "surplus-shortage problems" in interview sessions. The above two types of solving processes appeared again among the students.

In the third worksheet, the teacher discussed another well-known type of problems "compound motion" problems:

> A boy left home to a railroad station which was located 2*km* from home. 12 minutes later, his elder brother left home after the younger brother by bicycle. The younger brother was walking 70 *m* per minute, the elder brother pedaled 280 *m* per minute. How long did it take for the elder brother to catch up with the younger brother? (Dec. 5, 2002)

In the discussion of this problem, the teacher's explanation again followed the blending. He advised the students to "think *two boys separately*," drawing a diagram, constructing two separate mental spaces for each boy. Pointing out that the distances two boys advanced were the same when the elder brother caught up with the younger brother, the teacher blended the two spaces and made an equation.

DISCUSSION

Meaning construction is the heart of understanding of mathematics. This study tried to show how useful the framework of mental space theory would be in understanding the meaning construction process in mathematics classrooms. The paper analyzed actual classroom data of teaching linear equation word problems.

It is not possible to assess the full potential of cognitive linguistic analysis in just one small study on linear equation word problem instruction. The analysis presented in this paper was to provide one demonstration of the use of framework and tools of the mental space theory in understanding mathematics teaching and learning. Lakoff and Núñez (2000)'s Mathematical Idea Analysis focused on mathematical concepts. The current paper showed that the cognitive linguistics would provide useful tools to analyze word problems in mathematics, too, though some of the analysis would be familiar to researchers. Especially, the idea of blending mental spaces seems to be very powerful in understanding word problems. The teacher's explanations of several typical word problems were found to be very consistent with the blending process of mental spaces. His ways of explanations are not exceptional. Actually, many of explanations of linear equation word problems in published textbooks fit well with the explanation by mental space blending except single-schema-mapping problems. The mental space theory could be used for analyzing not only "correct" mathematical thinking, but also students' "wrong" thinking. The well-known strategy of solving word problems, "direct translation," could be considered one instance of the use of a pragmatic function mapping.

Comprehension of algebra word problems has often been approached by decomposing problem texts into atomistic formal propositions (e. g., Nathan, Kintsch, & Young, 1992). However, such approach seems to be a long way from explaining meaning construction processes in the classroom because its decomposition is often artificial and loses the sense of reality of classroom discourse. Analysis by the mental space framework starts from the situation where one constructed for understanding, thinking, or explanation: What mental spaces are introduced? How are they introduced? What connections are made between them? and so on. Therefore, the mental space approach would be more compatible with the situated cognition.

There are many other theoretical issues the paper did not address, however. For example, how mental space construction and formal mathematical expressions were related to each other in general. More detailed analysis and theoretical elaboration are necessary to put the theory into practice.

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