CURVED SOLIDS NETS

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The transformation of a solid to its net is based on something quite different from simple perceptual impression. It is a mental operation performed by manipulating mental images. The aim of this study was to observe pre-service and in-service teachers' ability to visualize the transformation of a curved solid to its net and vice versa, and to try to classify and to analyze students' mistakes and difficulties related to this kind of visualization. The most significant conclusion is that when lacking any experience, many students are unable to imagine the process of unfolding a curved surface. The findings show poor performance in attempts to produce nets or judge drawings as being possible nets of a cylinder or a cone.

INTRODUCTION

The transformation of a solid to its net is not a copy of the corresponding perception, but rather a mental operation performed by manipulating mental images. It may, therefore, significantly contribute in developing students' visual ability, which is an important goal of mathematical education. The aim of this study was to observe pre-service and in-service teachers' (herein referred to as students) ability to visualize the transformation from a curved solid to its net and vice versa, and to analyze and classify students' mistakes and difficulties related to this kind of visualization.

The net (sometimes called development) of a solid is the 2-D shape obtained from the solid by "unfolding" its whole surface so it lies in one plane. Visualizing solids nets, in general, requires a level of mental imagery students often lack, but can benefit by experience, awareness and analytical analysis. However, there is a difference in mental processes needed for visualizing curved solids nets as opposed to polyhedra nets: in polyhedra nets, every surface of the solid (which is a flat polygon) appears in the exact same shape. In curved solids, such as cones and cylinders, there are curved surfaces that look totally different when rotated to lie on a plane.

Because we are dealing with adults, we have to take in account that although they often know how to draw the standard net of a cylinder or a cone, this does not mean that they are able to visualize the process. They "know" the expected shape only because they have seen it in books or in class. This phenomenon is especially common among teachers. In this study, there is an attempt to overcome this problem by asking students to draw additional nets, or to identify nonstandard nets. When doing so, they function very much like those who have never seen a net of a cone or a cylinder before, and we can obtain a better idea of their actual ability to mentally fold and unfold surfaces.

THEORETICAL BACKGROUND

Folding and unfolding a solid is considered part of visual ability by many researchers: McGee (1979) includes it among "spatial visualization" which in his classification is one of two kinds of spatial abilities. Bishop (1989) differentiates between two abilities in visualization: the *visual processing* (VP), and the *interpretation of figural information* (IFI). The process of folding and unfolding solids belongs to the VP, because it involves the manipulation and extrapolation of visual imagery, and the transformation of one visual image to another. In the organizing framework suggested by Gutierrez (1996), the VP and IFI mentioned by Bishop are seen as processes of visualization, that is, mental or physical actions where mental images are involved.

Piaget (1967), in a chapter dedicated to rotation and development of surfaces, sees visualizing solids developments (i.e. nets) as "something quite different from simple perceptual impression… Between perception and imagination there lies a whole series of increasingly systematized actions, internalized in the form of images" (p.272). He explains that to pass from perception of the intact solid to an image of a drawing of its development (i.e. net), it is necessary to perform a mental action, and at the same time, to mentally co-ordinate the different points of view. This ability is reached, in Piaget's terms, only in sub-stage IIIB, that is, the second sub-stage of the concrete operational stage (age 9-11). Before that stage, although they have all the necessary prerequisites for drawing developed and rotated surfaces, children are limited in mental operations such as internalizing concrete actions in the form of symbolic images. At Piaget's stage IIA (about age 5-6) children are limited to reproducing as the net their drawing of the intact solid, expecting that their drawing could be cut and folded to reproduce the solid with all its sides. At Piaget's stage IIB (age 6-7) there is a first attempt to distinguish between the intact and the developed solid (by descriptions, "hints" in the drawings or gestures), but children still cannot rid themselves of their current perception. At Piaget's stage IIIA (age 7-9), there is a beginning of coordination between actual viewpoints: children represent some phases of the unfolding process, but they are unable to predict the final outcome and to represent the mental unfolding movement to the stage that all surfaces are arranged in one plane. They are unable to distinguish between different viewpoints and co-ordinate them well, and thus we can still find confusion between perspective view and net. Only at stage IIIB are they able to project all the parts of the curved surfaces on one single plane..

Piaget asserts that imagining the rotation and development of solids depends largely on the actual experience of folding and unfolding solids. "The child who is familiar with the folding and unfolding paper shapes through his work at school is two or three years in advance of children who lack this experience" (p. 276). In our case, it seems that students, who lack this experience, act in many ways on a level corresponding to stages IIB or IIIA. This supports Van Hiele's claim that the level of geometrical performance depends on experience, teaching and learning, and does not necessarily develop spontaneously with age (see Fuys et. al., 1988).

There is some more recent research dealing with solid nets, but most investigate polyhedra nets, and focus mainly on children's performance (cf., Mariotti, 1996, Meissner, 2001). Meissner argues that a solid net can be seen as a *Procept*, as defined by Tall (Tall et. al., 2000), because it has both a static and a procedural aspect, and is both process and product. This study accepts this opinion. As Piaget (1967) demonstrates, the development of a solid is definitely not a perceived object, like most geometrical objects are, but rather a conceived object. As such, it requires a pseudo-empirical abstraction, deriving from the process that the individual performs on the object. Those processes are encapsulated, as Meissner shows, into a static object. The term "net" serves as the symbol, representing both the process and the object. Those objects can be manipulated, as in this study, when students worked on nets of regular cones and cylinders as elementary objects. (A discussion of this matter is outside the purvey of this paper.)

THE STUDY

The study was carried out on 43 pre-service teachers (all learning to be mathematics teachers in elementary or junior high school in a college of education), and 78 in-service mathematics teachers in elementary schools.

The study uses a combination of quantitative and qualitative methods:

1) Two questionnaires were given to the whole population: one open and one closed.
2) The questionnaires were followed by a teaching unit of 2.5-3 hours. The unit began with group discussions and manipulating pre-cut paper shapes. Students were encouraged to describe their discoveries and their mistakes verbally. Then, the teacher conducted class discussions and additional activities dealing with some general aspects of non-polyhedral solids nets (such as investigating how various shapes of flat surfaces look when "folded" to surround a flat disc, what the connection between the net of a solid and the way this solid rolls is, which curved surfaces can be unfolded to flat nets etc.) The discussions dealt with generalizing students' discoveries, analyzing their own beliefs and mistakes and some didactic aspects of the activity. Some of those units were videotaped and analyzed.
3) Ten of the students were interviewed and videotaped, some singly and some in pairs.

This paper focuses only on analyzing the finding from the two questionnaires, and not on the teaching unit.

The open questionnaire:

As a first step, the students answered an open questionnaire in which they had to draw several nets of cylinders and of cones. For each of the solids, they had to draw one net, and then three more, as different as they could from the first one. It was expected that their first drawing would usually show if they were familiar with the standard net (the prototype), whereas the three others would reflect their ability to visualize the "unfolding process". It was made clear to the students that we would accept any kind of net, even if the "cutting" was done in nonstandard ways (through the middle of the circle, along a broken or a curved line etc.), but it had to be "in one piece" (possibly connected by one point). They were told to think of wallpaper on which they could draw their net so that it could be cut and pasted onto the entire solid surface (they had the solids at hand). This requirement of non-conventional nets enabled us to make the students actually try to visualize the process, instead of just producing familiar drawings.

The closed questionnaire:

In the closed questionnaires, students had to identify drawings as being or not being possible nets for a cylinder or for a cone (separately). Almost all the drawings in this questionnaire were drawings produced by students or by school pupils in previous studies (some of them as answers to the open questionnaire in a pilot study). The aim of this stage was to examine students' ability to visualize the process of folding the shapes to create the solids, and to detect typical mistakes, beliefs and difficulties in doing so.

CLASSIFICATION OF TYPICAL MISTAKES

The theory and classification suggested here is based on both the quantitative findings and an analysis of the interviews and of the discussions. It is not an exclusive classification. Each type of mistake deals with one aspect, so that the same mistaken drawing can fit several types. Most types are relevant for both cylinders and cones. All the examples are taken from students' answers to the open questionnaire.

Type1: Confusion between the perspective view of the solid and its net

1-A) "***Ellipses***": drawing ellipses instead of discs as bases. (cyl.**1**, cyl.**2** , cyl. **3** in Fig.1)

1-B) "***Proportion as seen***": drawing the lateral surface[1] the same width as the disc. Here we can detect a mix-up between the perspective view of the solid intact exterior and the imaginary net of its surface. (cyl.**4**, cyl.**5**, and cone**6** in Fig1)

1-C) "***Front and back***": although taking into account that the solid has a back side, students sometimes draw the width of the back and the front "as seen". This means that they are able to control some points of view, but not all of them. (**7** and **8** in Fig1)

Fig. 1

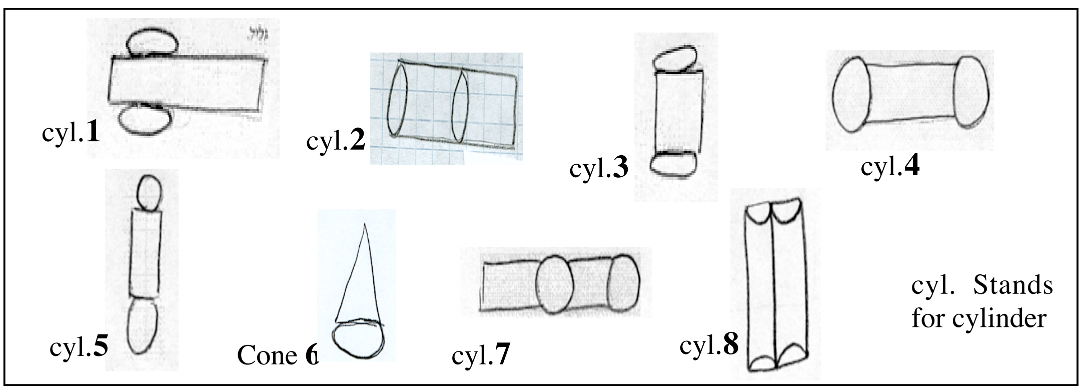


Figure 1.

Type 2. Joining the disc and the lateral surface along a line.

Students have a clear tendency to join the disc to the developed lateral surface along a line, and not only at one point. Here only clear cases were considered mistakes. In many cases, although considered here correct, we can see a slight distortion or overlapping of the disc that might result either from the above or from inaccuracy.

2-A) "***Overlapping disc***": drawing the disc partly overlapping the lateral surface (**9** and **10** in Fig.2). In many cases, students drew the lateral surface, then the disc (or discs), and finally erased the part of the drawing "under the disc". The *overlapping disc* creates a curved joining line. In the teaching unit, students discover why this is not possible.

2-B) "***Peeling without detaching***": drawing pieces of the lateral surface around the disc as if it was peeled, but without detaching the pieces from the disc. This mistake, like 2A, creates a

[1] When speaking of the drawing, the meaning of 'lateral surface" will be the unfolded lateral surface.

curved joining line. (In fact, it can be seen as a variation of 2A, but students often judged it as different.) (**11**, **12** and **13** in Fig.2)

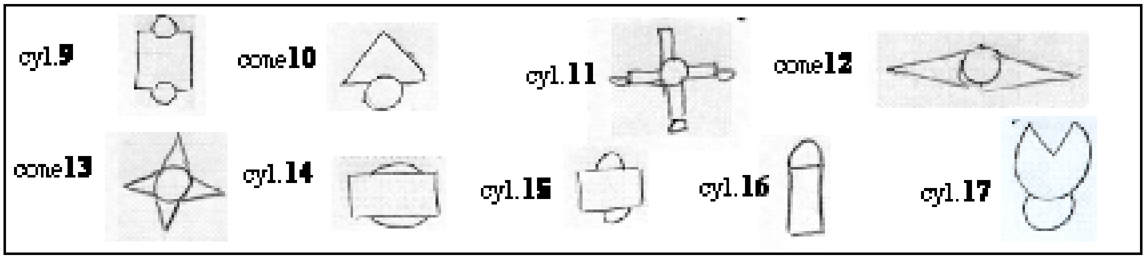


Figure 2.

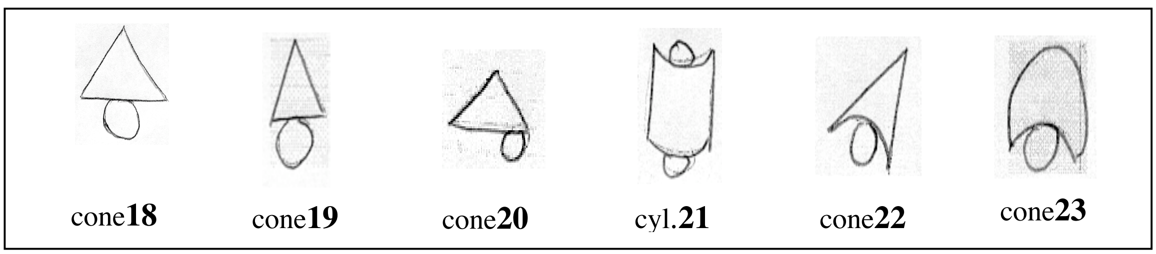
2-C) "***Partial disc***": drawing less than a whole disc (so that the lateral surface's drawing remains intact). This creates a straight joining line. (**14**,**15**,**16** and **17** in Fig.2)

Type 3. Wrong form of the edge to be joined

3-A) "***Triangles***" (relevant only for cones): drawing the lateral surface as one or more triangles (**18**, **19** and **20** in Fig. 3).

3-B) "***Embracing curve***": drawing the edge as a "concave curve", as if is embracing the disc (**21**, **22** and **23** in Fig. 3). Students explain that it feels right that the curve bend towards the disc, because when folded, it must surround it. This mistake can also be connected to type A –

Fig. 3



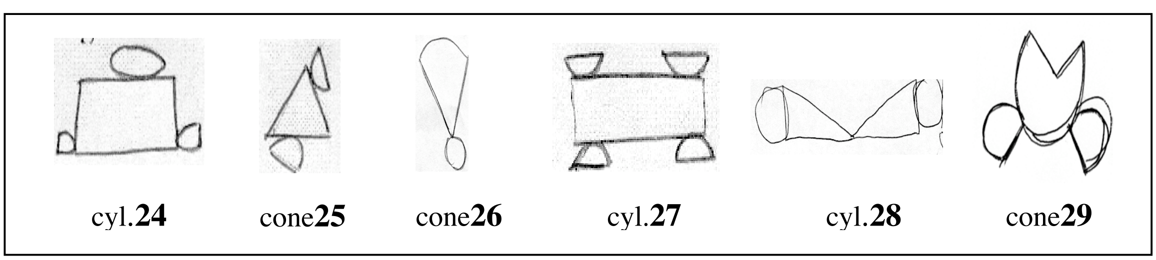
confusion with the perspective view.

Type 4. Wrong placement of the parts

Students sometimes fail to imagine how the net's parts should be connected to one another. They are not able to control the whole process simultaneously.

4-A) "***Wrongly placed bases***": placing the bases or the half bases on wrong sides of the developed lateral surface (**24**, **25**, and **26** in Fig. 4). This phenomenon sometimes occurs when trying to draw an interesting "other net" by splitting the base.

Fig. 4



4-B) "***Uncontrolled parts***": drawing parts (half discs or half lateral surfaces) without taking in account how they should take their place when folded. (For instance: ignoring that those half discs may turn over and overlap instead of integrating to a whole disc.) (**27**, **28**, **29** in Fig. 4)

Type 5. Other mistakes

5-A) "***flexible surface***": imagining the unfolding of the lateral surface as if it was made of a sheet of rubber (**30**, **31**, **32** and **33** in Fig. 5).

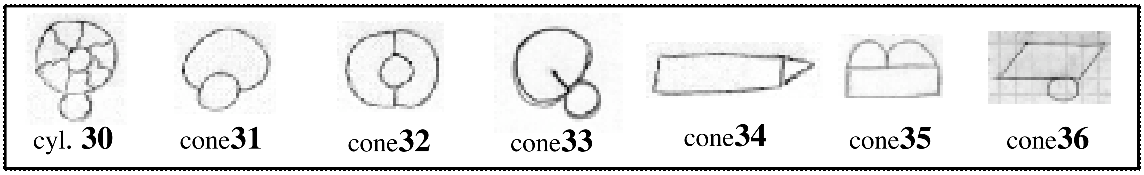


Figure 5.

5-B) "***No idea***": some mistakes attest to a total inability to visualize the outcome of the development process. In this study, it happened only in the cone case, and was accompanied with reflections like "I have no idea how to do it". (Of course, we can detect in those drawings elements of other types of mistakes)(**34**, **35** and **36** in Fig. 5)

ANALYZED RESULTS AND EXAMPLES

Examples from the opened questionnaire

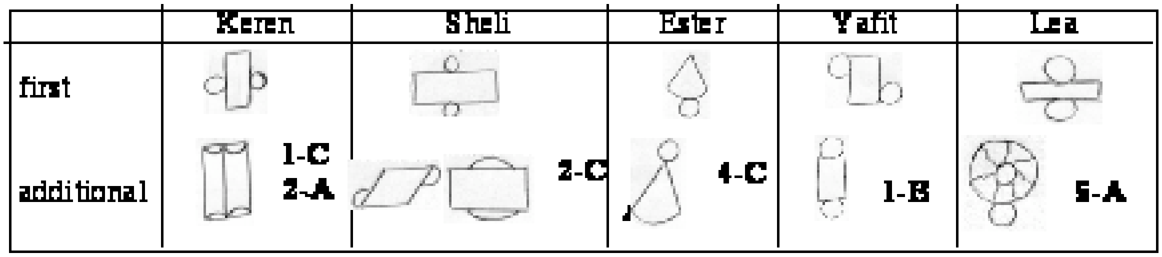
The examples given in the last section give an idea of the variety of drawings students made in the open questionnaire. Of course, there were many creative correct drawings, but in this paper, we focus only on the mistaken ones. Drawing, according to Piaget, can act as an intermediary between perception and imagination for it is "an imitative representation which remains external and material in character, whilst at the same time laying a basis for internalized images" (Piaget, 1967, p. 272). Through their drawings, we have a clue to students' mental processes of visualization.

As mentioned before, students had to draw one net, and then three additional nets for each solid. In the analysis, we have to distinguish between the first drawing and the additional ones. The first drawing is very significant only for those who never saw such nets. The additional drawings may bring to light difficulties of those who gave a correct first drawing "just because they knew" the answer, without actually trying to visualize the process. Hence, the interesting cases in those additional drawings are those in which there is at least one wrong drawing, after a correct first drawing ("wrong after correct").

Table 1: Percentage of mistaken answers in the opened task:

	Pre-service teachers (n=43)	In-service teachers (n=78)	Total (n=121)
First cylinder	21	21	21
Wrong after correct cylinder	58	30	37
First cone	54	34	40
Wrong after correct cone	57	20	28

Let us examine some examples. As can be seen from the table, cones are less well known than cylinders (less mistakes in the first drawing).

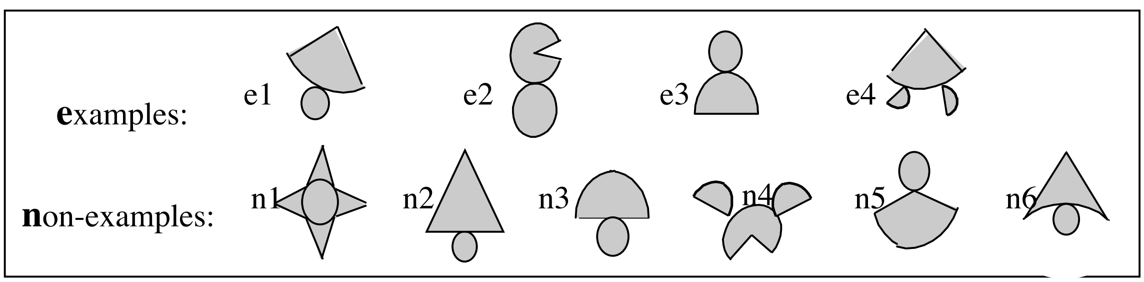


One might think that visualizing cylinders nets is an easy task for adults, but the picture changes dramatically when we come to the “Wrong after correct”: it seems that many of those who automatically drew a rectangle with two discs, do not really have the ability to visualize the process (among pre-service teachers - 57% of the “correct answerers”).

In all cases illustrated, the first drawing would lead us to believe that those students are able to unfold those solids correctly. However, their additional drawings reveal that this is not so! When they produce a drawing after really trying to visualize the process, we can see their difficulties and confusion. The most striking case was of Leah, who explained that she “peels” the lateral surface all around the bottom base of the cylinder (the central disk). It seems that, like young children, she maintains topological properties and neglects Euclidean properties (proportions and distance).

Examples from the closed questionnaire

The following examples are taken only from the cone questionnaire, in which 26



drawings had to be judged as examples or non-examples of possible nets of a cone.

Table 1: Percentage of mistaken answers in the closed questionnaire:

	classification	Pre-service teachers (n=43)	In-service teachers (n=78)	Total (n=121)
e1	Prototype A	14	4	7
e2	Prototype B	40	17	25
e3		74	62	67
e4	**4-B**	75	60	64
n1	**2-B**	72	58	63
n2	**3-A**	65	29	42
n3	**5-A**? **4-A**?	27	15	20
n4	**4-B**	45	60	56
n5	**4-A**	20	15	17
n6	**3-B**	49	31	37

As can be seen from the table, there is a substantial difference between pre-service and in-service teachers. This difference is probably caused by the fact that in-service teachers are familiar with cones' and cylinders' standard nets. Many of them had even taught this recently in their classes. But the difference is impressive only in the classical cases: the two kinds of prototypes, (especially prototype B), and the "popular mistake" – the triangle – again because many of them encountered this mistake when teaching the subject). In unconventional cases, their knowledge did not help them, and they had to try visualizing the whole process, which was, as we can see, far from being easy for them. The case of e3 and a3 is very interesting. While most students do not believe that a half disc could be folded to a cone's lateral surface (e3), most of the remaining students visualized it the wrong way (n3). Some of them reported thinking that the vertex should be at the center of the arc (attesting to mistake 5-A).

CONCLUSION

The most significant conclusion from this study is that when lacking any experience, students sometimes behave as young children do, and are unable to imagine the whole process of unfolding a surface simultaneously. They show some static, incomplete aspect of the action, and often confuse the perspective drawing of the solid with its net. When comparing students' drawings with children's drawings in Piaget's work, we can find correspondence to stages IIB (age 6-7) or IIIA (age 7-9). Some of the drawings produced by the students seem exactly the same as some children's drawing illustrated in Piaget's book. However, it is not enough to know that there is a problem. We must make sure that teacher training includes activities that develop this aspect of visual imagery.

The study, including its teaching unit, also has pedagogical aspects: analyzing and reflecting on their own thought patterns, may help teachers to reach a better understanding of their pupil's possible learning processes.

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