GUESS MY RULE REVISITED

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We present classroom research[1] on a variant of the guess-my-rule game, in which nine-year old students make up linear functions and challenge classmates to determine their secret rule. We focus on issues students and their teacher confronted in inferring underlying rules and in deciding whether the conjectured rule matched the rule of the creators. We relate the findings to the tension between semantically and syntactically driven algebraic reasoning.

FROM SEMANTICS TO SYNTAX

There are diverse approaches to algebra depending upon the relative mix of modeling, generalized arithmetic, mathematical structures, functions, and other considerations. Clearly, not all approaches will be equally appropriate for the young learner. It stands to reason, for example, that if algebra is introduced to elementary students as the syntactically-guided manipulation of formalisms (Kaput, 1995), many young learners are going to be left behind. We find compelling the evidence that children's early mathematical learning benefits from reasoning about rich contexts, from thinking about relations between quantities, from trying to solve word problems. This general approach to mathematics is shared by many schools of thought and research traditions (Vergnaud, 1985; Schwartz, 1996; Davydov, 1991; Smith,)[2]. It has led us to highlight modeling and mathematization in developing learning tasks during our three year longitudinal investigation of Early Algebra learning among 70 second to third grade students in greater Boston. The mere names of tasks we developed—the Heights Problem, the Piggy Bank Problem, the Best Deal, Phone Calling Plans—reveal our bias in grounding early algebra activities in rich situations about which students had considerable intuition and prior experience.

But early on we came to realize that certain representational tools—tables, number lines, graphs, and algebraic-symbolic notation, for example—were going to assume increasingly important roles in our students' mathematical lives. And the students clearly were not going to invent these notational systems on their own. This led us to elaborate activities in which special mathematical representations would become the object of direct discussion and reflection over the course of many lessons. It has been encouraging

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[2] Carpenter and Franke (2001) have built a program of Early Algebra instruction and research based on the premise that open sentences can constitute the main point of departure for introducing algebra into the early mathematics curriculum. Although we take a different view, we follow their work with interest, and find it noteworthy that our work in theirs draws inspiration from the groundbreaking work of Davis (1966-67), among others.

to see students incorporate tables, number lines, and graphs into their repertoire of "spontaneous" representations.

The case for algebraic-symbolic notation, however, has been somewhat different. We found that our students generally used algebraic notation to describe functions that they had come to identify through reflecting upon rich situations. However, we saw little evidence that such notation was exerting an influence on the course of their thinking. They seemed to be merely using the notation to register what they had concluded.

We realize that, as many have noted, at some point students will have to be able to reason directly upon and with the written notation. When should this transition occur? Will the prior emphasis upon richly contextualized reasoning militate against this progression towards notation and syntactically driven reasoning? What sorts of situations are likely to promote this new form of reasoning? What issues must students contend with along the way?

The Guess My Rule game suggested itself as a promising context for getting students to focus on written algebraic notation as an object of discussion because students would have to compare expressions written by the rule makers with those posed by the rule guessers and decide whether certain variations express the same or different underlying rules.

The Guess-my-Rule game has often been used in mathematics education at different grade levels as a way to introduce children to linear functions (see, for example, Davis, 1967, 1985; Carraher, Schliemann, & Brizuela, 2001, 2003; Schliemann, Carraher, & Brizuela, 2003). This activity essentially provides students with values from a function's domain (input) and the corresponding value from the range (output); based on the data, students try to infer the function. In order to play the Guess My Rule game, students must accept that: 1) each input must result in a single output; and 2) a function is consistently used for all values of the solution set, that is, the rule cannot change. Children in younger grades enjoy participating in this guesswork, even though they may not be fully aware of the teacher's role in moving the discussion towards the recognition of linear functions. Davis (1967) refers to children's experience in Guess My Rule as "readiness-building for functions" and makes a case for initially allowing students to figure out how to solve functions through their own means.

OVERVIEW OF CLASS AND THE RULES STUDENTS CHOSE

The following data come from a 90 minute classroom of 18 students in third grade; data from three other classes will be analyzed elsewhere. The students were already somewhat familiar with algebraic notation for functions. However, they had partaken in limited discussions about equivalent expressions for functions or about operating on functions. After several examples, students broke into 'groups' of one to four to make up their own rule. They made considerable efforts to ensure that other groups did not eavesdrop and discover their choice of rule as they were choosing it. After they chose their rule, they completed a table with input values of their choosing and the respective output values.

As students formulated their rules in groups, they discussed which combinations of operations and addends would result in a tricky rule yet one they could manage. Further, they openly discussed the importance of choosing a single rule that would account for all the data. All the groups applied their rule to inputs of 1 through 10, and several included higher, landmark numbers, such as 50 or 100.

In the following section, we will summarize the discussions from one of our four classrooms using Table 1 to provide an overview.

Group 1's rule: N□7 - 3

Table 1 (above) summarizes[3] the results of the activity during which each group of students generated a number of examples of input and output (with the input often coming from the guessing students) and challenged their classmates to guess their rule. For example, group 1 decided to use as their secret function, n□7 –3. When given 4 as an input number, they correctly told the class that 25 would be the output. One of the classmates conjectured that the rule was n□5 +5. This was a reasonable candidate; it is a *local solution*, that is, it matches the data for that particular instance. When the next input-output pair is given, a student among the guessers suggests that the rule is n□5 +7. This is a local solution that does not accord with the first data pair, (4, 25). Likewise a local solution, n□6, is proposed for the ordered pair (3, 18). The final guess, n+7, at first perplexed us. After David wrote the input of 3 above the input 4, the list of inputs was ordered as counting numbers. Joey noticed that each successive output increased by 7. N+7 may not correctly describe the function; however, it captures something about the pattern of outputs: 18, 25, 32, 39.

The class does not solve the function, which is revealed by the rule makers before passing the floor to Group 2.

Group 2: What Counts As a Solution?

Group 2 secretly chose as their rule "N □ 5 □ 4 + 1". One of the guessing students suggests one million as input; the rule makers correctly reveal that the output will be twenty million and one. Cristian immediately raises his hand and says "I know it! I know it!" His conjecture is “N times twenty plus one”. A dialog ensues regarding whether Cristian has discovered the rule used. One of the creators of the rule, Joseph, sounds out a rasping buzzer noise[4] that signals the student has given a wrong answer.

The teacher notes that Cristian's answer happens to be consistent with the data pair. At this point Joseph maintains that Cristian got it wrong but "got it right in a different way".

Group	Functions	Examples	Guesses	Local solution

[3] For simplicity we represent here students’ conjectures through standard mathematical notation even though they were spoken; many of the spoken conjectures were annotated by the teacher on large paper; rule makers also wrote down their rules.

[4] The allusion is to a television game-show contestant being informed that his answer is incorrect.

1 Melissa, Alanna, Nancy, & Maria	N□7-3	4→ 25 5→32 3→18 6→39	n□5 +5 n□5 +7 n□6 n+7	✓ ✓ ✓ ✗
2 Joey, Joseph, & Adam	N□5□4+1	1,000,000→ 20,000,001 1→21 2→41 …→61 …→81 …→101	 n□20+1 n□20+1	 ✓* ✓*
3 Omar, Kevin, & Anthony	K□2-2	5→8 100→198 0→-2 2→2 3→4 4→6 6→10 7→12 15→28	[n+] 3 [n]+3 KK+2 [K] □2-2 K□2-2 K+K-2	✓ ✗ ✓ ✓ ✓*
4 Matthew	[N] +50-20	92→122 0→30 1→31	 N + 8 + 22 N+30	 ✓* ✓*
5 Cristian	C□3+2-4+5	-1000→-2997 0→3 1→6 2→9 3→12 4→15	N□2+997 N□3 N□4+1 Y□3+3	✗ ✗ ✗ ✓*

*expression is not identical to the expression of the rule makers.

Table 1. Summary of secret functions, examples presented in class, and conjectures made by students.

After David leads the students through an additional data pair, Joseph continues to insist that Cristian was wrong. Joey now concedes that Cristian has solved the problem. By the end of the discussion, Joey and Joseph seem to believe that there are two ways to look at the issue. And indeed there are: Cristian's expression is not identical to theirs; nonetheless, it seems to work.

David writes the rule-as-created, N □5□ 4 + 1, on chart paper. To encourage them to see the rules as equivalent, he asks the students to find another way to express the part, □5□4;

they respond correctly, “times twenty”, and David writes ×20 under the factors. In the end, Joseph and Joey, rule makers, agree that Cristian’s rule works but take different stances on the issue of correctness. (Adam, the third rule-maker has not expressed his view.)

> Joseph: He got it wrong, but he got it right in a different way.
>
> Joey: He solved it right, but he did it in a different way.

Group 3: Is K + K the same as K × 2?

Group 3 choose as their secret rule K × 2 –2. The first input-output pair, (5, 8), is written in mapping notation as $5 \rightarrow 8$ on the chart paper. Joseph, now in the role of a conjecturer, says emphatically “three” while Joey says “plus three”. Briana raises her hand and answers that the rule is “plus three”. We transcribed their answers in Table 1 as “[n+] 3” and “[n]+3”, employing brackets to indicate which parts were editorially inserted by us. David realizes that Briana’s rule makes no explicit reference to the variable. So he introduces the letter B (students often prefer to work with the initial letters from their own name) as a means of completing the expression of the rule:

> David: Who thinks they know the rule? Briana…
>
> Briana (hand raised): Plus three?
>
> David: So if you start out with B (writes “B→” on chart paper), B becomes what? What’s the rule, that you think it is? What should we do to the B? [Another student quietly says, “plus three] …According to your rule, Briana,
>
> Briana: Three
>
> David: No…Is three always the answer? So…we’ve got to do something with the B. What do we do to the B? You said it’s plus three, right? [Briana nods in agreement.] So actually, your rule, Briana, is *B plus three* (as he completes writing B → B +3). So that’s Briana’s rule. Let’s see if this works.

When going through the next input, 100, David asks what Briana’s rule would predict the output to be. Several students answer, “103, ” and David agrees and Briana confirms by nodding her head. When the rule-makers reveal that their answer is 198, Briana, turns with perplexed surprise to the student sitting beside her. David himself is surprised and he asks to peek at the rule written on the makers’ sheet; he confirms that they have correctly given the output. When Briana acknowledges that her rule did not work, David clarifies that “it only worked for the first one.” In this way he calls attention to the fact that a conjecture might work locally (case 1) without working globally.

> David (summarizing): So we can actually say that this was a good guess but it is wrong, because it doesn’t work for both of them.

The rule-makers proceed to supply the class with additional information, namely the outputs for inputs of 0, 2, 3, 4, 6, and 7. Joey notes with interest, the pattern of the output: “It’s going in a pattern: 2, 4, 6, 8.”

Someone says aloud, “How do you count by twos?”, apparently meaning to say “what rule would yield a pattern that increases by twos?”

Maria conjectures that the rule is “K K plus two”, which David transcribes as “K K +2” and asks, “What is KK? K times K or K plus K?” While Maria is thinking, Cristian suggests “times two, minus two”, a correct answer, although it leaves the variable implicit. We represented his answer in Table 1 as: [K] □2-2. In the classroom, David writes K → on the board while asking “K becomes….?”. A couple students [possibly Joseph and Joey, once again] respond: “…K times 2 minus 2”.

Going back to Maria’s answer “K K plus 2”, David pursues the issue of the identity of “K + K” and “K □ 2”. David now realizes that it may not be clear to the students that K□2 and K+K are interchangeable. He pursues the issue a bit, using N as a variable, but there is no convincing evidence that the students truly accept the identity, n□2 = n+n, in written or spoken form.

Group 4 (Mathew): [N] + 50 - 20

On the basis of the input output data (see Table 1) Cristian conjectures that Mathew’s rule is N + 8 + 22. Others take the rule to be N +30. Matthew states his rule as “+50 –20” yet seems comfortable with the mapping, formulation, “N → N +50 –20”, encouraged by David. Once again, there is a discrepancy between the maker’s and conjecturers’ expressions. David tries to argue that the various formulations express the same underlying rule because they can all be simplified to N +30. Students may not be fully convinced by his points, but in a sense they are being encouraged, through this and other examples, to accept the general notion that expressions that look different may be interchangeable.

Group 5 (Cristian): C□3+2-4+5

Cristian, wishing as he typically does, to provide the class with a very challenging problem, suggests using –1000 as the initial input, yielding –2997 as output. Only a couple of students are following the discussion at this point; it is late in the class and a negative input is a bit strange for them. After going through a number of input-output pairs, students suggest N□3, N□4+1, and Y□3+3 as possible answers. The final conjecture is consistent with Cristian’s rule, although, once again, it is expressed in a different form. Since this discussion is a bit rushed (class is ending and David wants two remaining groups to at least state their rules), it is not fully clear whether some students believe that discovering the rule requires using the same letter adopted by the rule-makers.

The precise letter chosen *was* an issue in other classrooms, as the following dialogue from another class shows, after a discussion in which a student conjectured that the rule was A → A□5 –3.

(Ruler-makers come up to the front and write on the chart paper: K□5-3)

Teacher: Okay. They’re saying that their rule is K□5-3. Is that the same thing [as A□5-3]?

Student: That’s what I said!

[Erica points at the letter “A” in Albert’s iteration of the rule]

Erica: Yeah, but the letter!

Student: The letter’s different!

Teacher: Does it matter if you start with a K?

Assorted students: No! No!

Teacher: And the K becomes K times 5 minus 3? Is that the same as doing this? (pointing to Albert’s rule)

Students: Yes!

Teacher: So they’re really the same rule. (To Erica) So I would say that Albert solved it, right?

Erica: Yeah. Paul got it right too, he said the same thing.

By the end of this interaction, Erica accepts the teacher's statement that Albert got it right, and adds that Paul, another student who concurred with Albert, is right as well. At issue was the idea that letters in algebraic expressions are arbitrary placeholders.

DISCUSSION

There was some evidence that third grade students from an urban public school with a prior background in early algebra activities based on functions and modeling could begin the transition from semantically driven to syntactically driven algebraic reasoning. We would hope to see students taking part in more prolonged and in depth debates about equivalent functions and identities. Furthermore, although students may initially find persuasive the fact that two rules produce the same output from a set of input values, eventually they need to abandon this approach and move towards *proving* the functions are equivalent. The discussion about □4 □5 simplifying to □20 exemplifies this sort of shift. But graphing the data may prove useful in lending meaning to the rules and (dis)proving their equivalence. Ultimately, we want students to be able to operate on equations in ways that preserve the solution set without having to resort to thinking about the original situations that gave rise to the equations. This does not mean that students should abandon, once and for all, semantically driven reasoning, for it may prove useful in other contexts, even some that entail the use of advanced mathematical reasoning. We look forward to encountering additional research that explores the transition and tension between semantically driven and syntactically driven mathematical reasoning.

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