# DIFFICULTIES IN VECTOR SPACE THEORY: A COMPARED ANALYSIS IN TERMS OF CONCEPTIONS AND TACIT MODELS

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*This paper reports on the first stage of a research project still in progress aiming at identifying undergraduate students' difficulties and errors in solving linear algebra problems. Some students' difficulties related to very basic notions of vector space theory are shown and discussed with respect to both Balacheff's theory of conceptions and Fischbein's theory of tacit models. Finally a brief discussion concerning the complementarity of the two frameworks is proposed.*

## INTRODUCTION: THE CURRENT STATE OF RESEARCH IN LINEAR ALGEBRA EDUCATION

As many researchers underline (Dorier, 1998; Harel, 1990; Tucker, 1993), the importance of linear algebra in many fields of mathematics, science and engineering is widely acknowledged by both mathematicians and scientists, who consider linear algebra as an important mathematical prerequisite for undergraduate students in science and technology. Coherently linear algebra courses are basic for a wide variety of disciplines: mathematics, physics, computer science, engineering and so on. Nevertheless up to the 90's linear algebra has not seemingly raised the interest of the community of the researchers in math education. It is only in the last decade that studies on this subject have been carried out and all of them highlighted a number of difficulties in teaching and learning linear algebra.

Two years ago Jean-Luc Dorier collected the most advanced studies in linear algebra education to present the state of research on the subject (Dorier, 2000). A complete overview of these researches is out of the goals of this brief report, we only would like to highlight one aspect shared by many of these studies, that is the effort of describing and interpreting the students' difficulties and errors in terms of very general paradigms. For instance Harel (2000) shows that the teaching of linear algebra usually violates fundamental didactical principles, that is the concreteness and necessity principles. Dorier and other French researchers (2000) point out what they call "obstacle of formalism" as the main source of students' difficulties and claim the need of explicitly taking into account the "formalizing, unifying, generalizing and simplifying nature of the concepts of linear algebra" when teaching linear algebra.

But formalisation, unification and generalisation are characteristics of any mathematical theory, and the above didactical principles are usually violated in any purely axiomatic approach to a mathematical field. Therefore, notwithstanding the undeniable progresses which these studies brought, we think that the adopted approaches could lead not to adequately take into account the specificity of linear algebra and of its fundamental notions. In particular seemingly no researches have been carried out according to paradigms like that of "misconceptions" (in the sense of Confrey, 1990) nor that of "concept images" (Tall&Vinner, 1981) or of "epistemological errors" (Bachelard, 1938), which on the contrary are strictly related to the nature of the specific notions and provided efficient tools for analysing students' difficulties in other mathematical domains (e.g. calculus).

In this report we are going to describe and interpret students' difficulties in terms of *conceptions* and discuss the potentialities of this paradigm.

## THE CK¢ THEORY[1] BY BALACHEFF

> Within the theory of situations (Brousseau, 1997), the ck¢ theory - conception, knowing and concept - is an attempt to model "students' knowing of mathematics under the constraints of acknowledging both their possible lacking of coherency and their local efficiency" (Balacheff&Gaudin, 2002, p.1).
>
> The problem of elaborating such a theory is solved taking into account and formally defining the notion of "conception", which has been widely used as a tool in didactical analysis, but which has been used most often as a common sense notion without any explicit definition (Artigue, 1991).

Starting from the definition of concept given by Vergnaud (1991), a formal definition of conception is proposed. Without adopting the formal language of the theory, one can say that a *conception* is characterised by:

- a set of problems, the sphere of practice of the conception;
- a set of operators, i.e. a set of actions performed to solve the problems;
- a representation system, which allows to represent both problems and operators;
- a control structure, which allows "to decide weather an action is relevant or not, or that a problem is solved" (Balacheff&Gaudin, 2002, p.7 ).

The three first components are those identified by Vergnaud in order to define a concept, to these components the control structure is added. According to this theory a conception may be seen as the instantiation of a knowing which has proved to be efficient with respect to a certain domain (i.e. a certain set of problems). Once conceptions are defined, one can also formally define *knowings* and *concepts*, anyway we won't present here the complete modelisation which can be found in Balacheff, 1995 and Balacheff, 2000.

One main hypothesis is that one can describe and interpret what is called the *state of knowings* of a student and hence her/his difficulties in terms of conceptions.

## OUR RESEARCH

The study presented in this report is the first stage of a research still in progress aiming at *identifying undergraduate students' difficulties and errors in solving linear algebra problems*. More precisely, our study focuses on students' difficulties and errors related to *basic notions of vector space theory*, e.g. linear combination, linear dependence/independence, generators.

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[1] Balacheff speaks of *ck¢ model* rather than *ck¢ theory*, notwithstanding we prefer here to use the term *theory* in order to avoid any confusion with the term *models* as used in Fischbein's theory of tacit models to which we will refer later.

Up to now our study involved five medium-high achieving students in mathematics in their first year undergraduation. Students have been individually interviewed after the end of their first semester in linear algebra, during which they were introduced, via an axiomatic approach, to the basic notions of vector space theory, and whilst they were attending a second more advanced one. Students were presented with three different problems to be solved in individual interviews; no time constraints was imposed over the problem solving sessions, which were recorded.

The analysis of the transcripts of the interviews highlights a number of students' difficulties concerning basic notions of vector space theory. In the following we will present some excerpts from students' protocols showing some of such difficulties.

The problem we will refer to during our discussion is the following:

> ***Problem.*** Let $V$ be a R-linear space and let $u1$, $u2$, $u3$, $u4$ and $u5$ be 5 linearly independent vectors in $V$. Consider the vector $u=\sqrt{2}u1-1/3u2+u3+3u4-\pi u5$.

- Do there exist two 3-dimensional subspaces of $V$, $W1$ and $W2$, such that $W1 \cap W2=Span\ \{u\}$ ?
- Do there exist two 2-dimensional subspaces of $V$, $U1$ and $U2$, which do not contain $u$ and such that $u$ belongs to $U1+U2$ ?

The answer to both the questions is that such subspaces of V exist. In order to successfully approach the problem one might try to describe the conditions which the subspaces must fulfil in terms of their possible generators. For instance Span{u, u1,u2} and Span{u,u3,u4} verify the conditions posed in the former question and Span{u1,u2} and Span{u3+3u4,u5} verify the conditions posed in the latter one; this is just one possible solution, it is not the only one.

None of the interviewed students succeeded in solving this problem. In the following section we will present some excerpts from the transcripts of the interviews and we will propose a first analysis in terms of Balacheff's theory of conceptions.

## 4. Protocols

For the sake of brevity we can not discuss here all the 5 collected interviews, so we present some excerpts from only 3 protocols. The difficulties we are going to discuss can be considered exemplar being shared by the other interviewed students.

<u>Protocol A: Fra.</u>

Fra has just correctly answered the former question of the problem and she is now approaching the latter. Since the very beginning she expresses the feeling that the answer to the second question is negative.

(a) **Fra:** I think that it is not possible because $u$ is linear combination of 5 linearly independent vectors… and if one can write it as… that is it should be an element which can be written as the sum of an element of $U1$ and of an element of $U2$, and then it should be linear combination of at most 4 linearly independent vectors… now let's see

Then Fra seemingly faces the task of proving her assertion or at least of ascertaining its truth ("let's see", item 74). In fact she does not prove it, on the contrary her assertion is never put under discussion and it is used as the conclusive and decisive argument in her solution to the problem.

1 **Fra:** […] I find that anyway $u$ is written as linear combination of 5 linearly independent vectors… then I cannot write $u$ with only 4 linearly independent vectors

1 **Fra:** anyway $u$ is written as linear combination of 5 linearly independent vectors, then I cannot write it as belonging to this sum [$U1+U2$]

<u>Protocol B: Jas.</u>

Jas approaches the problem starting from the last question. One of her first remarks concerns the uniqueness of representation of the vector $u$.

**2** **Jas:** […] a vector $u$ is given, such that $u=\sqrt{2}u1-1/3u2+u3+3u4-\pi u5$. It is written in a unique way

2\. **Jas:** then… let's see, I must… I find $u$ in just one way

The uniqueness of representation of $u$ leads Jas to state that $u$ belongs to $U1+U2$ only if vectors $u1$, $u2$, $u3$, $u4$ and $u5$ belong to $U1+U2$ too.

**3** **Jas:** […] $u$ has to belong to their sum [$U1+U2$], let's see. I don't think that it is possible, because if I take… let's see […] in order to get that $u$ belongs to their sum [$U1+U2$] I have to find in this sum at least both $u1$ and $u2$ and $u3$ and $u4$ and $u5$ […] but if $U1$ has dimension 2 then I get that it does not contain more than 2 linear indipendent vectors which I can suppose to be $u1$ and $u2$ …

Starting from the above argument and from considerations concerning the dimension of $U1$ and $U2$, Jas supposes $U1=Span\{u1,u2\}$ and $U2=Span\{u3,u4\}$ and negatively answers to this question. She similarly approaches the former question of the problem and as a consequence she fails to solve it.

<u>Protocol C: Ele.</u>

Since the very beginning of the session Ele seems a bit confused: at item 4 she wonders whether the space generated by a single vector [$\sqrt{2}u1 -1/3u2 +u3 +3u4 -\pi u5$] might have dimension greater than 3. Some minutes later (item 23.) something similar occurs when Ele represents the space generated by $u1$, $u2$ and $u3$ as $<\sqrt{2}u1-1/3u2+u3>$ instead of $<u1,u2,u3>$.

**4** **Ele:** I have to establish whether this one [$Span\{\sqrt{2}u1-1/3u2+u3+3u4 -\pi u5\}$]… I mean, whether the subspaces generated by this vector [$\sqrt{2}u1-1/3u2+u3+3u4 -\pi u5$]… if its dimension is greater than 3, then it does not exist

**5** **Ele:** […] I'm looking at… [she writes $W1 = <\sqrt{2}u1-1/3u2+u3 >$] … I want $W1$ to be a subspace generated by $u1$… by $u1$, $u2$ and $u3$ [she writes $W1 = <u1, u2, u3 >$]

In the meanwhile, Ele states that the basis of the searched subspaces ($W1$ and $W2$) have to be subsets of the set of the given vectors $u1$, $u2$, $u3$, $u4$ and $u5$.

- **Ele:** well… then… I do not know… the intersection between these two [subspaces $W1$ and $W2$]… then, the intersection between these two subspaces has to be generated by $u$ and then it has to be a linear combination, then… a multiple, because we are dealing with only one vector [$u$], it [$W1 \cap W2$] has to be a multiple of this linear combination [$\sqrt{2}u1-1/3u2+u3+3u4 -\pi u5$]

- **Ele:** then […], if the dimension of these two subspaces has to be 3, then I have to take, in order to have 3-dimensional subspaces, I should have… a basis with three linearly independent vectors… now, I can choose such vectors only among these five ones [*u1*, *u2*, *u3*, *u4* and *u5*] …

Coherently with her assertion Ele only investigates the case in which *W1=Span{ui,uj,uz}* and *W2=Span{up,uq,ut}* (*i*, *j*, *z*, *p*, *q* and *t* are integers less than 5) and thus she negatively answers the question. The same assumption and the same behaviour are at the core of the (incorrect) solution which Ele gives to the latter part of the problem.

## A FIRST ANALYSIS OF THE THREE PROTOCOLS

As we underlined above, all the students failed in solving the problem (Jas and Ele failed to answer both the questions while Fra failed to answer only the latter). In this section we analyse the previously highlighted students' difficulties and errors with respect to the theoretical framework of conceptions.

In each protocol we may identify the source of students' failure in incorrect assumptions which underlie and affect students' search for the subspaces *W1* and *W2*, and *U1* and *U2*, that is:

- **Fra:** *u* is a linear combination of 5 linearly independent vectors then it can not be written as a linear combination of only 4 linearly independent vectors (see item 86.)
- **Jas:** in order to get that *u* belongs to *U1+U2* one needs that *u1*, *u2*, *u3*, *u4* and *u5* belong to *U1+U2* too (see item 13.)
- **Ele:** the generators of the searched subspaces must be taken out from vectors *u1*, *u2*, *u3*, *u4* and *u5* (see item 21.)

These *assumptions* are never questioned during the problem solving activity, on the contrary as we remarked above they guide students' activity and provide them with the ultimate criteria to answer the posed problem. So students' behaviours, their actions and strategies are strictly related to these assumptions. According to the ck¢ theory these assumptions and students' related actions may be interpreted in terms of *controls* and *operators* (students' schemas). We could go further in our analysis and detect a more complete list of *operators* and *controls* related to the highlighted assertions in order to model the students' behaviours as conceptions. Moreover we could eventually use this list as a diagnostic tool both to describe these students' behaviours in solving more problems and to describe other students' behaviours in solving this same problem.

However directly following this direction could cause us to miss one important point of the whole framework. In fact the ck¢ theory shares Brousseau's view that *errors are not only the effect of ignorance, of uncertainty, of chance, but also the effect of a previous knowing which was interesting and successful* (Brousseau,1997, p.82). Going on in a so analytic and detailed analysis of the protocols could lead to excessively fragment students' behaviours (problems, operators, language, control); as a consequence we could not be able to answer the question:

- Which is the *interesting and successful knowing* related to the stressed errors?

In order to answer this question, most researches following the ck¢ theory start from a re-examination of previous works carried out according to paradigms such as that of misconceptions, of concept-images or the like which despite the differences with the ck¢ paradigm offer a suitable ground for an *a-priori* analysis in terms of conceptions. The lack of such researches in vector space theory education opens the problem of answering this question directly on the basis of the collected data. In order to solve this problem we will change perspective and try to use Fischbein's approach in terms of tacit models.

## THE THEORY OF TACIT MODELS BY FISCHBEIN

Starting from the assumption that mathematical concepts and operations are essentially formal and abstract constructs whose meaning and coherence are not directly accessible to us and which we can not "spontaneously" manipulate, Fischbein states that we need to produce mental models providing us with a directly accessible, unifying meaning to concepts and symbols (Fischbein, 1989).

"Generally speaking, a system B represents a model of system A if, on the basis of a certain isomorphism, a description or a solution produced in terms of A may be reflected consistently in terms of B and vice versa" (Fischbein, 1983, p. 121). Among the different kinds of models which Fischbein distinguishes – e.g. intuitive or abstract models, tacit or explicit, analogic or paradigmatic - we are mainly interested in tacit models. Being beyond direct conscious control they very often tacitly influence solution strategies during problem solving. "A person may be convinced that the object of his solving attempts is a certain phenomenon – the object of his interest – while his mental endeavors deal in fact with a model of it" (Fischbein, 1983, p. 203). Students' systematic errors might be due to the limits of a tacit model.

Among the characteristics which according to Fischbein tacit intuitive models share, let us quote: being a structural entity rather than an isolated rule, having a "concrete, practical and behavioural" nature, autonomy, strength.

## ANALYSIS IN TERMS OF TACIT MODELS

In this section we analyse students' reported behaviours in the light of Fischbein's theory in order to investigate the possible sources of students' assumptions and to eventually find out any common aspects among them.

As far as Jas is concerned, we can notice that she initially insists on the uniqueness of representation of $u$ (items 4. and 8.). In fact the representation of $u$ with respect to $u1$, $u2$, $u3$, $u4$ and $u5$ is unique, being these vectors linearly independent, but Jas seemingly extends this property deducing that $u$ can be represented only in the given form. If $u$ can really be represented only in the given form then vectors $u1$, $u2$, $u3$, $u4$ and $u5$ are indispensable for describing $u$ and as a consequence Jas's key-assumption appears well founded.

Differently from Jas, Fra does not give any explicit hint suggesting the origin of her assumption, notwithstanding let us underline how what we identified as key assumption in Fra's protocol may be interpreted in the same terms as before: if the representation of $u$ is unique then $u$ can not be written using less then 5 linearly independent vectors.

Can we interpret Ele's error in a similar way? At a first glance it seems that Ele's assumption has not the same source as those of Fra and Jas. In fact she seems to assume that the only subspaces of $V$ are those generated by a subset of the set constituted by $u1$, *u2*, *u3*, *u4* and *u5*[2]. Is such assumption so different from Fra's and Jas's ones? We think that it is not the case. As a matter of fact the set of generators of a subspace of V has to be such that any vector of this subspace must be represented as a linear combination of the generators themselves. But if any vector of our space is represented in a unique way as linear combination of *u1*, *u2*, *u3*, *u4* and *u5* and *if this is the only way it can be represented*, then the generators of a subspace of *V* can be taken only out from vectors *u1*, *u2*, *u3*, *u4* and *u5*. Moreover note that Ele's initial confusion (see items 4. and 23.) seems coherent with this interpretation.

Then we might interpret Fra's, Jas's and Ele's errors as due to the same tacit model, that is *a vector can be represented as linear combination of other vectors in a unique way*. So, according to this interpretation the tacit model is a distortion of a basic widely used (both to prove theorems and to solve problems) and widely known property[3], i.e.: *the representation of a vector as linear combination of a given set of linearly independent vectors is unique*, but the same vector can be represented in different ways when changing the set of linearly independent vectors.

## CONCLUSIONS

We carried out our analysis with respect to both the ck¢ theory and that of tacit models, which respectively allow us to *identify the fundamental controls and operators mobilised by students* in order to solve the problem and to give a unitary interpretation of students' errors *finding out a common tacit model* beyond students' behaviours.

Moreover the provided interpretation, in terms of tacit models, suggests that the highlighted operators and controls may be related to a same conception, concerning the representation of vectors as linear combination of a given set of independent vectors. We could roughly label this conception under the expression *a vector has a unique representation as linear combination of independent vectors*. As already remarked, coherently with Balacheff's view that errors might be interpreted as due to the functional inertia of a knowing which has proved itself by its efficiency in enough situations (Balacheff, 2000), the found out tacit model is a distortion of one of the most powerful and efficient tool in vector space theory, i.e. the uniqueness of representation of vectors with respect to a given basis. *The students' highlighted controls and operators may be seen as consequences of this tacit model.*

The combination of the two paradigms shows its power if didactical implication are considered, in fact the identification of the unifying conception based on tacit models focuses on the potential source of the highlighted difficulties which may be interpreted as a distortion of a fundamental vector spaces property. On the other hand a more detailed description of this model in terms of conceptions (problems, operators, representation system, controls) would facilitate the diagnosis of students' errors related to such a tacit model.

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[2] Ele correctly observes that one can suppose that *V* is generated by *u1*, *u2*, *u3*, *u4* and *u5*.

[3] We are referring to the linear algebra lectures attended by the students involved in our study.

## References


Artigue M., 1991; Epistémologie et didactique, *Rech. Did. Math.*, v.10, n.2/3, pp. 241-285.

Bachelard G., 1938; *La formation de l'éprit scientifique*. Paris: Vrin.

Balacheff N. 1995; Conception, connaissance et concept, in Grenier D. (ed.) *Didactique et technologies cognitives en mathématiques, séminaires 1994-1995*, pp. 219-244, Grenoble: Université Joseph Fourier.

Balacheff N., 2000; Les connaissance, pluralité de conceptions – le cas des mathématiques, in Tcherinikine P. (ed.) *Actes de la conférence Ingénierie de la connaissance (IC 2000)*, Toulouse, pp. 83-90.

Balacheff N., Gaudin N., 2002; Students conceptions: an introduction to a formal characterization, in *Les cahiers du laboratoire Leibniz*, *n. 65, Dec. 2002*,Grenoble, France.

Brousseau G., 1997; *The theory of didactical situations in mathematics*, Kluwer Acad. Publ.

Confrey J. (1990) A review of the research on students conceptions in mathematics, science, and programming, in Courtney C. (ed.) *Review of research in education.* A.E.R.A.16, pp. 3-56.

Dorier J.-L., 1998; Etat de l'art de la recherche en didactique à propos de l'enseignement de l'algèbre linéaire, in *Recherches en didactique des mathématiques*, v.18, n.2, pp. 191-230.

Dorier J.-L., (ed.) 2000; *On the teaching of linear algebra*, Kluwer Acad. Publ., The Netherlands.

Dorier, J.-L., Robert, A., Robinet, J. and Rogalski, M., 2000; The obstacle of formalism, in Dorier, J.-L.(ed.) *On the teaching of linear algebra*, Kluwer Acad. Publ., The Netherlands.

Fischbein E., 1983; *Intuition in science and mathematics*, Reidel Publishing Company.

Fischbein E., 1989; Modelli taciti e ragionamento matematico, in D'Amore B. (ed.) *Matematica a scuola: teorie ed esperienze*, Pitagora Editrice Bologna, pp. 25-38.

Harel G., 1990; Using geometric models and vector arithmetics to teach high-school students basic notions in linear algebra, in *International journal of mathematics education in science and technology*, v.21, n.3, pp. 387-392.

Harel G., 2000; Three principles of learning and teaching linear algebra, in Dorier, J.-L.(ed.) *On the teaching of linear algebra*, Kluwer Academic Publishers, The Netherlands.

Tall D., Vinner, S., 1981; Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity, in *Ed. St. in Math.*; v.12 n.2, pp.151-69.

Tucker A., 1993; The Growing Importance of Linear Algebra in Undergraduate mahematics, in *The college mathematics journal*, v.24, n.1, pp. 3-9.

Vergnaud G., 1991; La théorie des champs conceptuels, in *Rech. Did. Math.*,v.10, n.2/3, pp.133-69.