COLLECTIVE MATHEMATICAL UNDERSTANDING AS AN IMPROVISATIONAL PROCESS

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This paper explores the phenomenon of mathematical understanding, and offers a response to the question raised by Martin (2001) at PME-NA about the possibility for and nature of collective mathematical understanding. In referring to collective mathematical understanding we point to the kinds of learning and understanding we may see occurring when a group of learners, of any size, work together on a piece of mathematics. In employing and extending the theoretical work of Becker (2000), Sawyer (1997; 2000; 2001) and Berliner (1994; 1997), we characterise collective mathematical understanding as a creative and emergent improvisational process and illustrate how it can be observed in action.

THE NATURE OF MATHEMATICAL UNDERSTANDING

Drawing on the work of Pirie and Kieren (1994)[1], we see the growth of mathematical understanding as a dynamical and active process. The Pirie-Kieren theory offers a way of considering mathematical understanding that recognises and emphasises the interdependence of all the participants in an environment. It shares and is intertwined in an ecological view of learning and understanding as an interactive process. This location of understanding in the "realm of interaction rather than subjective interpretation", together with a recognition that "understandings are enacted in our moment-to-moment, setting-to-setting movement" (Davis, 1996, p.200), allows and requires the conceptualisation of understanding not as a state to be achieved but as a dynamic and continuously unfolding phenomenon. Hence, it becomes appropriate to talk not about 'understanding' as such, but about the process of coming to understand, about the ways that mathematical understanding shifts, develops and grows as learners move within the world.

Davis (1996) claims that "a significant strength of the [Pirie-Kieren] model is that it can be used to interpret the mathematical actions of either individuals or groups of learners" and that "the model highlights the manners in which collective understandings do emerge…that cannot be located in any of the participants but which rather are present in their interactions" (p.203). However, as acknowledged by Kieren and Simmt (2002), the Pirie-Kieren theory is still essentially one of dynamical *personal* understanding, although it has been applied in limited ways to groups of learners. In considering the growing understanding of two learners working together, and to acknowledge the interactions and shared activities of the learners, Kieren, Pirie & Gordon Calvert (1999) chose to show the students' growing understandings with one pathway, although in doing so they did not

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[1] The Pirie-Kieren theory has been fully presented and discussed in a number of previous PME meetings and many of its features have been set out there and elsewhere. Hence it is not intended to elaborate on the theory here.

"intend to imply that [the students'] histories or understandings [were] identical" (p.220). We would suggest that in considering the notion of dynamical *collective* understanding it is not that the individual understandings and histories of learners are identical, but that alongside a personal dynamical understanding there also co-exists an understanding at another level, located in the interactions of the learners (see Davis & Simmt, 2002). We see mathematical understanding as an emergent phenomenon requiring simultaneous analysis at multiple levels, not just at the level of the individual. As mathematical understanding emerges at the collective level from the actions of individuals, so the collective also constrains and influences the individual, and thus "a purely reductionist model [one focused on individual learners] will still fail to represent the emergent higher-level entity - the collaboratively created frame" (Sawyer, 2001, p.214). However, we are not claiming that collective mathematical understanding is an automatic or simple occurrence whenever two or more people are working together. Indeed in other papers we have highlighted the way in which collective understanding fails to emerge from the interactions of a group of learners, and that what is observed is merely a set of individual understandings occurring simultaneously (see Martin, Towers, & Pirie, 2000). In contrast, in this paper we will illustrate what we are characterising as collective mathematical understanding and how we observe it to occur in action.

THE NATURE OF THE IMPROVISATIONAL PROCESS

In attempting to characterise collective mathematical understanding as an emergent process, we draw on the work of Becker (2000), Sawyer (1997; 2000; 2001) and Berliner (1994; 1997) on improvisational traditions within jazz and theatre. We extend this theoretical frame to consider collective mathematical understanding as a process with a similar nature and characteristics. Berliner (1997) claims that "the study of one musician's creative process cannot capture the essence of jazz, because more than any other performance genre, a jazz performance is a collective, emergent phenomenon" (p.10). We contend that the same can be true of a group of individuals working together mathematically, and that to simply focus on the understanding actions of one or all individuals does not necessarily fully explain nor characterise the growth of the mathematical understanding as it occurs.

Sawyer (2000), in considering acts of collaborative emergence, suggests that "in an ensemble improvisation, we can't identify the creativity of the performance with any single performer; the performance is collaboratively created" and that although each individual is contributing something creative, these contributions only make sense "in terms of the way they are heard, absorbed, and elaborated on by the other musicians" (p.182). In applying this to improvised theatre and verbal performance, Sawyer offers a number of key features of collective emergence. Of particular importance is the notion of 'potential', of the unpredictability of pathways of actions. Sawyer notes that "an improvisational transcript indicates many plausible, dramatically coherent utterances that the actors could have performed at each turn. A combinatorial explosion quickly results in hundreds of potential performances" (p.183). However, quite early in a scene Sawyer suggests that a "collectively created structure" has emerged from the interactions, which "now constrains the actors for the rest of the scene" (p.183). To remain coherent, subsequent actions must fit with this structure, yet of course will still add to it. Thus we have a complex interplay between individual and structural actions as these co-emerge together.

In considering improvisational jazz, from a similar perspective as a collective emergent phenomenon, Becker (2000) offers a number of key characteristics of collective improvisation in a jazz performance. Of particular importance is the requirement that everyone pay attention to the other players and be willing to alter what they are doing "in response to tiny cues that suggest a new direction that might be interesting to take" (p.172). Becker notes the subtlety of the etiquette operating here, as every performer understands

> that at every moment everyone (or almost everyone) involved in the improvisation is offering suggestions as to what might be done next. As people listen closely to one another, some of those suggestions begin to converge and others, less congruent with the developing direction, fall by the wayside. The players thus develop a collective direction that characteristically…feels larger than any of them, as though it had a life of its own. It feels as though, instead of them playing the music, the music, Zen-like, is playing them (p.172).

He also notes that unless the performers listen carefully, and where necessary "defer" to the collective mind, the music will "clunk along" with each individual doing nothing more than playing their own "tired clichés" (p.173). In improvisation, when one person does something that is obviously better (in the view of the collective) then "everyone else drops their own ideas and immediately joins in working on that better idea" (p.175). Of course this requires some understanding of what "better" might look like and of how to recognise it. So how does a group recognise what is better? In mathematics, "better" is likely to be defined as an idea that appears to advance the group towards a solution to the problem, the drawing on a concept that seems appropriate and useful in the present situation. Interestingly, Becker also suggests that in collaborative improvisations, as people follow and build on the leads of others, they "may also collectively change their notion of what is good as the work progresses" (p.175) leading to a creative production or performance that could not have been predicted prior to the activity.

COLLECTIVE MATHEMATICAL UNDERSTANDING AS AN IMPROVISATIONAL PROCESS

In this section we discuss how some of the theoretical principles of improvisation offered by Becker, Berliner and Sawyer might apply to children working mathematically. The ideas we are advancing here are based on data from many observational studies we have conducted over a number of years.[2] Here, we illustrate our thinking through considering an extract of video data of three Grade Six students starting work on a problem. The students have been posed the problem of calculating the area of a parallelogram, a figure with which they have not worked before.

Interviewer: Okay. So what I have here…is a little bit of a different shape for you. (*She passes over a piece of paper on which is drawn a parallelogram with no dimensions provided*)

Natalie: Parallelogram, I think?

[2] See Kieren (2001) for a discussion of our collective "research-in-process" (p.225).

I: Mmmmm! It's a parallelogram…well done. I wonder if you could figure out for me how to find out the area of that parallelogram. I have rulers here if you need a ruler.

N: Can try it…measure…sides. (*She measures two adjacent sides*).

Stanley: Ten

N: Mm hm. And that's ten right? (*She points to the opposite side*). They're both ten. And then the top one is longer, and that is…

Thomas: it's eighteen.

N: Yep. So that's ten… and eighteen. (*She writes the numbers on the sides*). Ok, so we could…

T: multiply

N: …the area, yeah ten by eighteen …and then see what we get.

T: It would be this…

N: yep (*pause*) and then if we wanted to do…so that would be the area then?

S: But look at the shape.

N: I know that's what I'm saying that can't be right cause that - that's a little bit…

S: Okay. Wait, I know…. draw a straight line, here and here, you get triangles, and squares. (*He adds lines to the parallelogram, see Figure 1*)

Figure 1

T: Oh I know…

S: Now these triangles…I think…

T: is half?

N These triangles will make up that square though. So then if we just measure that….

S: and times by two.

N: Yeah. Because these two, these triangles make that square, right? .

T: That'll work.

S: I think…

N: So if we…what is…this it's still, it's not eighteen though, anymore. Because we're cutting it… it will be twelve. (*She measures the sides of the rectangle that has been created*).

T: and then ten

N: and this side is…no it wouldn't be ten. (*She is referring to the width of the rectangle, i.e. the perpendicular height of the parallelogram*).

T: Like, eight?

N: this side is eight, yep. So…

S: and then 12 times 8…is…

T: …is…ninety…
S: Yeah, ninety-six.
T: …six
N: So ninety-six times…
S: So ninety-six times two, so uh ninety-s….
T: a hundred…I mean no, not a hundred…yeah a hundred – eighty-uh… three?
S: *ninety*- three…
N: ninety-six. One ninety-six.
S: Okay.
N: How does that work? Two times six is twelve. One ninety…two.
T: So it's one ninety-two.
S: Yeah, one ninety-two…
N: is the area of that. So it's the area of the whole…

We recognise that in this short piece the students do not find a correct answer to the problem. However, they continued to work for several more minutes after this, as they realised (after a question from the interviewer) that their strategy of simply doubling the area of the rectangle did not quite work. Our focus here though is not the correctness of their mathematics, but the improvisational character of their mathematical interactions.

There are many interesting aspects to this transcript, and to the ways in which the students work collectively together here. Firstly, the discourse appears as though one person (rather than three) was speaking. The students complete one another's sentences, but more than that they seem to be speaking with one voice. Indeed, it is almost as though you could remove the names of the speakers in the transcript and read it as a monologue. This is also true of the emerging mathematical understanding, which like the conversation, cannot be separated into three pathways of growth, and indeed only makes sense when considered collectively. The students are engaged in making a collective image for the concept of area of a parallelogram, and although this is emerging from the contributions of individuals, we see a single image being made - that to find the area of a parallelogram one doubles the area of its internal rectangle. It is not possible to discern individual pathways of growth of mathematical understanding, and we would contend that in this case these do not really exist.[3] The growth in mathematical understanding is occurring at the collective level. This claim echoes Sawyer's (2000) suggestion, which we cited earlier, that "in an ensemble improvisation, we can't identify the creativity of the performance with any single performer; the performance is collaboratively created" (p.182).

It is interesting too to note that although these students were working in an interview scenario, they pay little attention to the presence of the interviewer. Instead, like Becker (2000) noted of jazz performers "the improvisers are trying to solve a problem or perform a feat for its own sake or their own sake, because it is there to do and they have agreed to

[3] Interestingly, when the researchers first worked with this data they attempted to produce individual mappings for each student's growth of understanding and found this be to be both frustrating and impossible.

devote themselves collectively to doing it" (p. 174). The students here are committed to solving the problem that is posed, for its own sake. There is a powerful sense of collective purpose throughout the whole session, beyond what is transcribed here, with the priority always being meaningful and useful engagement with the mathematics. The devotion displayed by these three students to the problem is striking and it is worth noting that the students were so engrossed in the collective problem-solving that they failed to notice the bell ringing to signal the end of the session. Their focus is uncommon, and has been remarked upon by other teachers and researchers who have viewed the episode, including the students' own Grade 6 teacher, who had not often asked these three students to collaborate as a group throughout the year, preferring, as do many teachers, to "spread the wealth" of these mathematically-able students by distributing them around various other groups in the classroom. It should be noted that our focus here is not group work, per se, but rather collective mathematical understanding, so we will refrain from further comment on group organisation, other than to note that our data suggest that, not to deny the importance of mixed-ability groupings in the classroom, there is also much to be gained by having students of comparable mathematical ability work together.

When offered the parallelogram, a shape that the students had not worked with in the context of area before, they initially began to cast around for strategies to find the area. At this stage there is no way to predict how the mathematics will unfold, nor even what mathematics will emerge from the interactions. Like the early stages of any improvisational performance, there is both 'unpredictability' and 'potential', with the group listening to the suggestions of each other as they work on making their image. Initially, no one idea is collectively taken up, nor discarded. Natalie takes the lead in measuring all the sides. Stanley and Thomas follow along. Thomas suggests multiplying, which is briefly picked up by Natalie, but no-one seems satisfied that that will produce an appropriate solution. Suddenly, when Stanley adds two lines to the diagram (see Figure 1) saying "draw a straight line here and here, you get triangles and squares" the energy of the group lifts and all three students appear to recognise the potential of the strategy. At this stage we see a collectively created structure starting to emerge, as the group have now collectively chosen which pathway to pursue, effectively rejecting all other previously offered ideas in favour of something which, for the moment, appears to be 'better'. That is, as suggested earlier, it is a lead that seems likely to further their image making and continued growth of understanding. Here, there is a high level of attentiveness between the three students, they listen to each other, and to the mathematics as it emerges from their engagement. We see a deferring to a group mind, including a willingness to abandon personal motivations, which allows a collective image to emerge.

Of course, and as Becker (2000) commented, the collective notion of what is 'good' and 'better' may change as an improvisational episode progresses. In our example, Stanley's move was clearly the critical moment in the episode - but only because the other students were prepared to 'take the cue', to adapt their developing thinking to follow a new direction. However, as already noted, the group later recognised that 'doubling the area of the interior rectangle' was an inadequate image and collectively worked to make their image into one that was 'better' - leading to a correct solution that satisfied the group. Their final image (one that is consistent with the particular parallelogram offered in the question, but not generalisable) centres on the idea of the area of the interior rectangle being equivalent to four of the end triangles (see Figure 1). This image was "a result that could not have been foretold from anything they knew and were used to doing before they started" (Becker, 2000, p.175) but instead emerged through the continual and complex interactions of the three students.

CONCLUDING REMARKS

In this paper we have advanced the basis of a theoretical framework that provides a way to observe, consider and characterise the growth of collective mathematical understanding. To view the process of doing mathematics as a form of improvisation provides a powerful mechanism for going beyond an analysis of individual actions and for recognising the power and potential of collaboration for enabling the growth of understanding. A focus on the improvisational character of collective mathematical understanding re-orients our attention to the significance of the level of activity in which the understanding emerges. As Sawyer (1997) notes, "the central level of analysis for performance study is not the individual performer, but rather the event, the collective activity, and the group" (p. 4). Indeed, the science of complexity has already prompted us to acknowledge that "complex unities must be studied at the level of their emergence" (Davis & Simmt, 2002, p. 833) and we believe this to be true of mathematical understanding. Whilst we do not wish in any way to devalue the place of dynamical personal understanding, we would also argue, that as a complex system, mathematical understanding must also be considered at the other levels at which it is seen to emerge, in particular that of the collective. Our work suggests the need to pay close attention to the collaborative work of students and to focus on their 'improvisational performances' in mathematics, rather than just the resulting product of such engagements. We would contend that this attention can provide a rich insight into both how mathematics should be seen as something more than merely an individual activity and also how mathematical understanding is a phenomenon that emerges and exists in collective action and interaction.

References

Becker, H. (2000). The etiquette of improvisation. *Mind, Culture and Activity, 7*(3), 171-176.

Berliner, P. (1994). *Thinking in jazz: The infinite art of improvisation*. Chicago: University of Chicago Press.

Berliner, P. (1997). Give and take: The collective conversation of jazz performance. In R.K. Sawyer (Ed.), *Creativity in Performance* (pp.9-41). Greenwich, CT: Ablex Publishing.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York: Garland.

Davis, B., & Simmt, E. (2002). The complexity sciences and the teaching of mathematics. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. II, (pp. 831-840). Athens, GA.

Kieren, T. (2001). Set forming and set ordering: Useful similies in observing the growth of mathematical understanding? In R. Speiser, C. Maher, & C. Walter (Eds.), *Proceedings of the twenty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. I, (pp. 223-233). Snowbird, UT.

Kieren, T., Pirie, S., & Gordon Calvert, L. (1999). Growing minds, growing mathematical understanding: Mathematical understanding, abstraction and interaction. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp. 209-231). London: Falmer Press.

Martin, L.C. (2001). Growing mathematical understanding: teaching and learning as listening and sharing. In R. Speiser, C. Maher, & C. Walter (Eds.), *Proceedings of the twenty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. I, (pp. 245-253). Snowbird, UT.

Martin, L.C., Towers, J. & Pirie, S. (2000). Growing mathematical understanding: Three layered observations. In M.L. Fernandez (Ed.), *Proceedings of the twenty-second annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Vol. 1 (pp. 225-230). Tucson, AZ.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, *26* (2-3), 165-190.

Sawyer, R.K. (1997). *Creativity in Performance*. Greenwich, CT: Ablex Publishing.

Sawyer, R.K. (2000). Improvisational cultures: Collaborative emergence and creativity in improvisation. *Mind, Culture and Activity, 7*(3), 180-185.

Sawyer, R.K. (2001). Creating conversations: Improvisation in everyday discourse. Cresskill, NJ: Hampton Press.