FOURTH GRADERS SOLVING EQUATIONS[1]

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We explore how fourth grade (9 to 10 year olds) students can come to understand and use the syntactic rules of algebra on the basis of their understanding about how quantities are interrelated. Our classroom data comes from a longitudinal study with students who participated in weekly Early Algebra activities from grades 2 through 4. We describe the results of our work with the students during the second semester of their fourth grade academic year, during which equations became the focus of our instruction.

INTRODUCTION

Successful implementation of algebraic activities in elementary school are described by Bodanskii (1991), Brizuela, Carraher, and Schliemann (2000), Brizuela and Lara-Roth (2001), Carpenter and Franke (2001), Carpenter and Levi (2000), Carraher, Brizuela, and Earnest (2001), Carraher, Schliemann, and Brizuela (2000, 2001, 2003), Davis (1985), Kaput and Blanton (2001), Schifter (1999), Schliemann, Carraher, and Brizuela (2001), Schliemann and Carraher (2002), and Schliemann, Goodrow, and Lara-Roth (2001). Our own work has shown that third graders can learn to think of arithmetical operations as functions rather than merely as computations on particular numbers, that they can operate on unknowns, and work with mapping notation, such as $n \rightarrow 2n - 1$. We have also found that graphs of linear functions are within reach of fourth graders.

These demonstrations, however, may not have convinced some mathematics educators that young children can learn algebra. Previous research has highlighted students' *difficulty* in solving equations when unknown quantities appear on both sides of the equality (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Many attributed such findings to developmental constraints and the inherent abstractness of algebra, concluding that even adolescents were not ready to learn algebra (Collis, 1975; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Linchevski, 2001; MacGregor, 2001; Sfard & Linchevski, 1994). Further, some have claimed that students are engaging in algebra only if they can understand and use the syntax of algebra and solve equations with variables on both sides of the equals sign (see Filloy & Rojano, 1989).

It is our belief that, as previously stressed by Booth (1988), Bodanskii (1992), Kaput (1995), and Schliemann and Carraher (2002), among others, the difficulties middle and high school students have with algebra result from their previous experiences with a mathematics curriculum that focuses exclusively on arithmetic procedures and computation rules. With the classroom data we will describe, we will show that, if children are given the opportunity to discuss algebraic relations and to develop algebra notations, even fourth graders will be able to solve algebra equations.

OUR APPROACH TO ALGEBRAIC NOTATION

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Algebraic-symbolic notation is one of several basic representational systems of mathematics. In a *narrow* sense, algebraic reasoning concerns only algebraic-symbolic notation. In the *broad* sense we adopt in our research and in this paper, algebraic reasoning is *associated with* and *embedded in* many different representational systems. Although some educators argue against any and all uses of algebraic-symbolic notation in the early grades, we feel it is better to frame the issue in a *broad* context. By *broad context* we mean to ask more generally how written notations relate to mathematical reasoning and algebraic concepts in particular.

In a previous study, we found that children can use mathematical notations not only to register what they understand, but also to structure their thinking; that is, notations can help further children's thinking (Brizuela, Carraher, & Schliemann, 2000), allowing them to make inferences they might otherwise not have made. Conventional notations help extend thinking (Cobb, 2000; Lerner & Sadovsky, 1994; Vygotsky, 1978), but if they are introduced without understanding, students may display premature formalization (Piaget, 1964). For these reasons, students need to be introduced to mathematical representations in ways that make sense to them. Much of our research has focused on introducing mathematical symbols in meaningful ways. Our approach relies on introducing new notations as variations on students' spontaneous representations (Brizuela & Lara-Roth, 2001; Carraher, Schliemann, & Brizuela, 2000). Our classroom intervention data have shown that young students can meaningfully learn to use *algebraic-symbolic notation* to express generalizations they have reached while exploring problems in open-ended rich contexts. Our next step was to investigate whether elementary school children could also deal with written algebra equations and with the syntactic rules of algebra.

In interview studies, Brito Lima and da Rocha Falcão (1997), Schliemann, Brito-Lima, and Santiago (1992), and Schliemann, Carraher, Pendexter, and Brizuela (1998) have shown that seven year-olds can understand the basic logic of equations, and that third graders can develop representations for algebraic problems and, with help from the interviewer, solve linear equation problems using different solution strategies, including the syntactic rules of algebra. Furthermore, Bodanskii (1992) found that fourth graders introduced to algebra notation and equations from grade 1, could solve algebra problems and equations, performing better than sixth and seventh graders who received five years of arithmetic instruction starting algebra in grade six only. Other promising results come from Lins Lessa (1995) who found that, after only one individual teaching session, fifth graders could solve verbal problems or situations presented on a balance scale that involved equations as complex as $x + y + 70 = 2x + y + 20$ or $2x + 2y + 50 = 4x + 2y + 10$. She also shows that, in the post-test, the children's solutions were based on the development of written equations and in more than 60% of the cases they used algebra syntactic rules for solving equations.

In the longitudinal study we partially report here, we introduced children to equations as an extension of their work on functions and on graphs of linear functions. In this paper we will report on the final results from one of the classrooms we worked with.

The Classroom Intervention and its Results

We worked with 70 students in four classrooms, from grade 2 to 4. Students were from a multiethnic community (75% Latino) in Greater Boston. Each semester, from the beginning of their second semester in second grade to the end of their fourth grade, we implemented and documented six to eight Early Algebra activities in their classrooms, each one lasting about 90 minutes. The activities related to arithmetic operations, fractions, ratio, proportion, and negative numbers. Our goal was to examine how, as they participated in the activities, the students would work with variables, functions, positive and negative numbers, algebraic notation, function tables, graphs, and equations.

The last six lessons we taught in fourth grade focused on equations and algebraic notation. Each lesson focused on a problem that had unknown amounts in it and that could be represented with equations, as in the following example:

> Mike and Robin each have some money. Mike has \$8 in his hand and the rest of his money is in his wallet. Robin has altogether exactly three times as much money as Mike has in his wallet.
>
> Which phone plan is better? Plan #1: You pay \$0.10 per minute for all calls. Plan #2: You pay \$0.60 per month plus \$0.05 per minute for calls.

When presented with the problems, children were not asked to find a "right" answer, but to consider all possibilities, to draw the graphs of two functions, and to consider an answer only after they had gone through these steps. During the weeks leading up to the lesson we will focus on in this paper, the children felt fairly comfortable dealing with unknown amounts and some of the children were able to gradually use N to represent the unknown amounts, although some of them still used iconic representations. During the last lesson in fourth grade, the following problem was presented to the class:

> Two students have the same amount of candies. Briana has one box, two tubes, and 7 loose candies. Susan has one box, one tube, and 20 loose candies. If each box has the same amount and each tube has the same amount, can you figure out how much each tube holds? Each box?

A box, two tubes, and 7 candies in a transparent bag are put on Briana's table; a bag, a tube, and 20 candies in a transparent bag are put on Susan's table.

The students start by discussing the problem and Aarielle recalls that it is similar to the "wallet problem" (see above) they had solved six weeks before. Kauthaumy states that Susan has 13 more candies in her bag than Briana does, and Albert observes that Briana has an extra tube of candy. When the teacher of this lesson (David Carraher) asks if they could figure out how many candies there are in a tube or in a box, most of the students answer that they couldn't. However, less than 14 minutes into the class, Albert explains that Briana's tubes have to have 13 candies in them so that the tube plus the 7 loose candies could be equal to Susan's 20 candies. Briana agrees with Albert and Cristian notes that it doesn't matter how many candies are in the boxes.

Mariah asks Albert to explain why he thinks there are 13 candies in each tube. He answers that the amount in a tube plus the 7 loose candies would be equal to Susan's 20 candies. Mariah asks about Briana's second tube and Albert assures her that it still works because Susan also has one tube. Carissa further explains that the candies in Susan's bag make up for the extra tube that Briana has. David (the instructor) asks how many candies in the bag make up for the tube and Albert replies 13, which would leave 7. A few minutes later, David asks, "How do we know that the tubes have 13 and that the girls are holding the same amount if we haven't peeked in the tubes yet?" Cristian replies that this is called algebra and Briana and Mariah explain that they used algebra to subtract and make educated guesses. David challenges the children to prove that there are indeed 13 candies in a tube. Cristian explains that we can use N to stand for a tube, and the class as a whole agrees that a different letter should be used to stand for the boxes. When the students sit down to work on ways of representing the problem in writing, Anne, a member of the research team, asks Carissa and Susan to explain the problem for her. Carissa explains that Briana and Susan have the same amount and thus Susan's bag, that had 20 candies, is really like 13 plus 7. So Susan has 13 extra candies, so that has to be the amount in Briana's extra tube.

Each of the students in the class produce their written account of the problem. Although most of the children in this class of 18 students made iconic representations for the problem (78%), one third of them included an equation in their representation and more than one third (39%) included a letter in their representation, to stand for one or more of the unknown values.

Figure 1. Nancy

Nancy's written work (see Figure 1) is an example of an iconic representation. She first works with the amounts given for the loose candies (20 and 7) and correctly uses the difference of 13 between these two amounts as the value for what is inside the tubes, showing one tube on Susan's table and both of Briana's tubes as having 13 in them. Although Nancy acknowledges that Susan starts out with 20 loose candies, on the table she shows her as having 7 and 13—just like Briana. One interesting feature of Nancy's work is the question mark that she places on the two boxes—the amount of candies in the boxes is unknown (hence the question marks) and will and can remain unknown to the very end. In our longitudinal study, this was not the first time that we had observed children using question marks to represent unknowns.Ramón's written work (see Figure 2) is also interesting in the way in which he is able to integrate both an iconic representation and algebraic notation (N+7). He consistently uses the N to show what is in the tubes of candies. In addition, he uses his notation to solve the problem and show his solution to the problem. He represents Susan's (1) and Briana's (2) candies iconically, then matches what they have and crosses out the matching amounts on both sides. He does not assign a value for the boxes and appears to have no problem crossing them out since there is one on each side. Through his matching, he arrives at the conclusion that, in order to have equal amounts of candy, Susan must be left with 7 and 13 (20) and Briana must be left with N+7, making N be 13 candies.

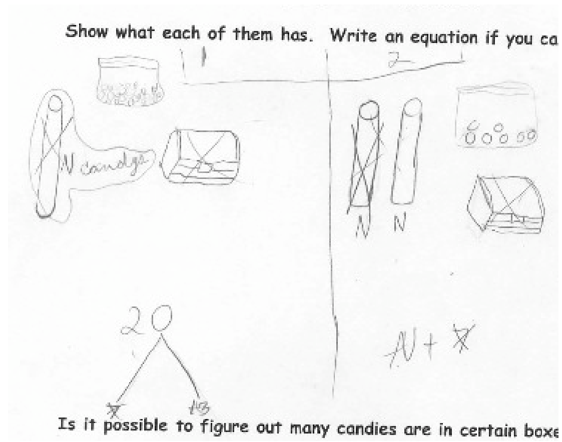

Figure 2. Ramón

Figure 3 shows that Albert uses an equation to represent the problem. He uses both N and Z as the unknowns. He starts by using N to represent the amounts in the tubes and in the boxes but soon uses Z for the amount of candy in the tubes. After matching the equal amounts, he appears to have used the letters interchangeably as he finally reaches the equation 20=N+7.

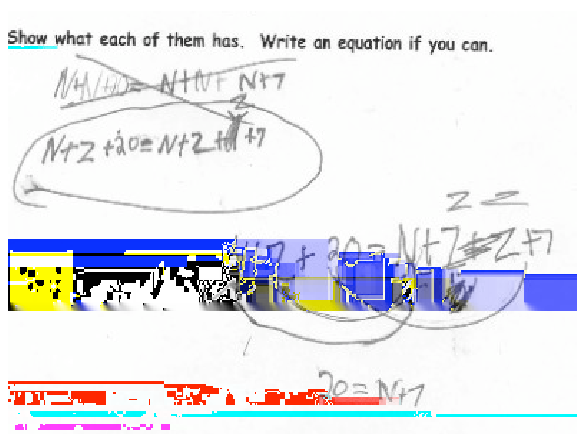

Figure 3. Albert

Figure 4 shows a very sophisticated representation by Cristian. Although similar to Albert's, Cristian's notations are of added relevance given the explanation written out at the side of the equations that he matches up. Cristian has set Susan's and Briana's amounts equal and matched the elements in the two sides of the equation.

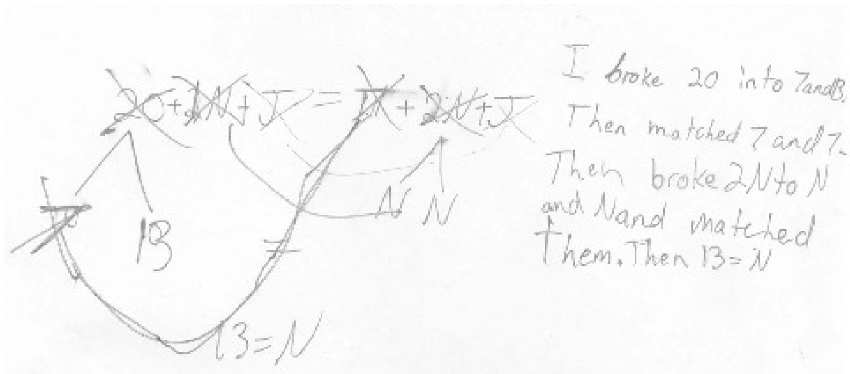

Figure 4. Cristian

Once the group as a whole meets to discuss the problem and the students' written representations, David writes on the board that T is the amount in each tube and B is the amount in each box. He asks the class, "How much did Briana have?" The class calls out "2T + B + 7". He then asks, "How much did Susan have?" and the class calls out, "T + B + 20". When David asks whether these expressions can be simplified, Albert suggests matching up the Bs. David does so and crosses them out: "Now we have 2T + 7 and on the other side we have T + 20. How could we simplify them further?" Carissa suggests putting 7 in Susan's bag and leaving 13 out in a pretend tube. David does so and Aarielle writes on the board, breaking up the 20 into 13 and 7 and matching up the two sevens. David erases the 7 from the board to leave 2T = T + 13. Cristian suggests matching two tubes to leave T = 13. David does this with the actual tubes and records it on the board: "So T has to be 13. Let's count the tube candies." Subsequently, Kauthaumy counts the candies and finds 13. The children shout out, "Hooray", expressing their excitement at their accurate calculations.

INTERVIEW RESULTS

At the end of the school year, 1 to 4 weeks after the last class, we individually interviewed the children on a series of problems. In the last part of the interview, children were asked to represent in writing and to solve the following problem: "Harold has some money. Sally has four times as much money as Harold. Harold earns $18.00 more dollars. Now he has the same amount as Sally. Can you figure out how much money Harold has altogether? What about Sally?" Of the 18 children from this class who were interviewed, 10 represented Harold's initial amount as N, X, or H and Sally's amount as Nx4. For Harold's amount after earning 18 more dollars, eight children wrote N + 18. Four children wrote the full equation N + 18 = N x 4 and eight children correctly solved the problem. However, only one systematically used the algebra method to simplify the equation. Another child, when prompted, correctly explained the algebra method. Apparently, as the children worked in their written representations, they easily inferred that Harold's starting amount was 6. As Albert stated, "I thought about six because it just popped in my head."

DISCUSSION

The kinds of activities we developed over the last six weeks of our longitudinal study were not simple or easy for the students. Nevertheless, they were able to deal with the challenges we proposed and, at the end of only six meetings on equations, many were able to represent and meaningfully discuss and analyze problems involving unknown amounts on both sides of an equality. In the classroom, at least a third of the students in this class could represent the problem as an equation, solve the equation, and meaningfully explain why they could manipulate the elements in the equation. In the interviews, more than half of the children correctly represented the amounts in the problem using letters to stand for unknown amounts. Our results suggest that dealing with equations is not beyond fourth graders' mathematical understanding and that much more can be achieved if the same kind of activities become part of the daily mathematics classes offered to elementary school children.

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