DESCRIPTIONS AND DEFINITIONS IN THE TEACHING OF ELEMENTARY CALCULUS

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In this paper, we discuss the (potentially positive) pedagogical role of intrinsic limitations of computational descriptions for mathematical concepts, with special focus on the concept of derivative. Our claim is that, in a suitable approach, those limitations can act for the enrichment of learners' concept images. We report a case study with a first year undergraduate student and place this in a broader empirical and theoretical context.

INTRODUCTION

Giraldo (2001) defined a *theoretical-computational conflict* to be any pedagogical situation with apparent contradiction between the mathematical theory and a computational representation of a given concept. We have argued that the approach to the concepts of derivative and limit can be properly designed to prompt a positive conversion of theoretical-computational conflicts to the enrichment of concept images (Giraldo & Carvalho, 2002a, 2002b, Giraldo, Carvalho & Tall, 2002). In addition, we distinguish between a *description* of a concept, which specifies some properties of that concept and the formal concept *definition*. Descriptions commonly employed in mathematical teaching include numeric, graphic and algebraic representations that individually involve limitations that do not fully reflect the mathematical definition. We will argue that suitable use of these limitations can stimulate students to engage in potentially enriching reasoning.

RESEARCH FRAMEWORK

Our theoretical position is grounded in the theory of concept image and concept definition (Tall and Vinner, 1981). The *concept image* is the total cognitive structure associated with a mathematical concept in an individual's mind. It is continually being (re-)constructed as the individual matures and may (or may not) be associated with the *concept definition* (the statement used to specify the concept). Barnard and Tall (1997) introduced the term *cognitive unit* for a chunk of the concept image on which an individual focuses attention at a given time. Cognitive units may be symbols, representations or any other aspects related to the concept. A rich concept image should include, not only the formal definition, but many linkages within and between cognitive units.

In a strictly formal standpoint within a formal system of rules of inference, a mathematical object is perfectly characterized by its definition, so that the definition completely exhausts the object and, in this sense, a mathematical object *is* its definition.

However, the theory of concept image suggests that the teaching of a mathematical concept must include different approaches and representations to enable learners to build up multiple and flexible connections between cognitive units. The three main forms of representation for functions, numeric (tables), algebraic (formulae) and geometric (graphs), each have their own limitations. A table can have only a finite number of entries that does not necessarily determine the whole function, a formula may be presented in a way that does not mention the range or domain and a physical graph can only approximately present the information required for the formal function. Each of these is a *description* that lays stress on certain aspects of the concept, but also casts shadows over others.

The literature reveals examples of the *narrowing effect* (described in Giraldo, Carvalho & Tall, 2002) of the students concept image as a result of focusing only on certain aspects, particularly computational ones. For instance, Monaghan et al (1993) reported that students using Derive to study calculus explained the meaning of the expression $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ by replacing $f(x)$ with a polynomial and referring to the sequence of key strokes to calculate the limit. Research in Brazil (Abrahão, 1998; Belfort & Guimarães, 1998) reveals many instances of students accepting numeric and visual output of technology without query, even when software limitations produce results that clearly conflict with their prior knowledge.

However, we believe that the limitations of the various descriptions need not necessarily lead to a narrowing of the concept image. On the contrary, such limitations have a potentially positive role. Sierpinska (1992), for example, affirms that the awareness of the limitations of each of the forms of representation, when they are all meant to represent the same concept, is essential for the understanding of the concept of function. We believe that the emphasis on theoretical-computational conflicts can lead not to a narrowing, but to the enrichment of learner's concept images.

THE CASE OF THE DERIVATIVE

One of the most widely used descriptions for the derivative concept in elementary calculus courses is the following: The gradient of the function $f(x)$ at x_0 is the slope of the tangent line to the graph of *f* at the point $(x_0, f(x_0))$. However, as Vinner (1983) and Tall (1989) observe, the notion of tangency in students' concept images is often strongly linked to geometry problems about the construction of tangent lines to circles. The approach to those problems focuses on global geometric relationship of the curve and the line, particularly, on the number of points of intersections. Thus, the idea of being tangent—to "touch" in one single point—is featured in opposition to the idea of being secant—to "cut" in two points. This leads to a narrowing of the concept image of a tangent that is not consistent with the notion of tangent in infinitesimal calculus.

An alternative to the traditional approach, based on the notion of local straightness, has been proposed by Tall (e.g. Tall, 2000). This is grounded on the fact that the graph of a differentiable function 'looks straight' when highly magnified on a computer screen. Tall claims that local straightness is a primitive human perception of the visual aspects of a graph, deeply related to the way an individual looks along the graph and apprehends the changes in gradient, that is suitable as a cognitive root for the concept of derivative.

However, the notion of local straightness is also a *description* for the concept of derivative, since it comprises limitations that can trigger theoretical-computational conflicts. For example, floating point approximations made by computer software may cause unexpected results, as the one shown on figure 1. It displays the process of local magnification of the curve $y = x^2$ (in the neighborhood of (1,1)) run by software *Maple*. Until a certain stage of the process, the curve does look like a straight line, but afterwards (for graphic window ranges lower than 10^{-6}) it becomes polygonal.

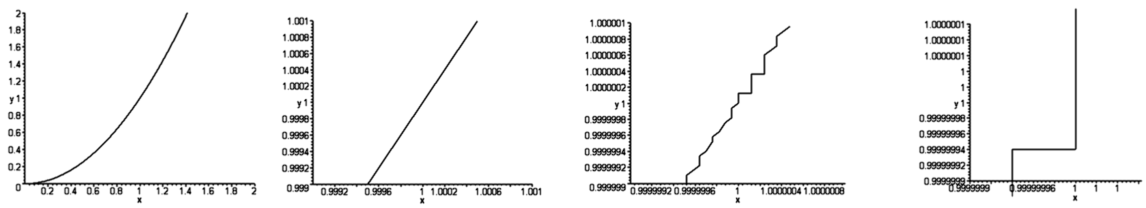

Figure 1. A theoretical-computational conflict observed on the local magnification process.

Theoretical-computational conflicts like this are deeply related to the fact that a finite algorithm is being used to describe an infinite limit process. These intrinsic limitations may lead to narrowed concept images, if computational descriptions are over-used. Nevertheless, our hypothesis is that a suitable approach, where theoretical-computational conflicts are not avoided, but highlighted, can prompt the positive conversion of these same limitations: they can make for the enrichment of concept images, by underlining that *the notion of limit, in the sense of infinitesimal calculus, is beyond computers accuracy, no matter how good it is, or, more generally, any finite accuracy.*

A CASE STUDY

The experiment reported in this section is part of a wider study, in which six first year undergraduate students from a Brazilian university were observed in personal interviews dealing with theoretical-computational conflict situations from different natures. We summarize the responses of one of the participants, Antônio (pseudonym) to four interviews, concerning the concept of derivative (translated from Portuguese).

Interview 1: Participants were given a few general questions concerning their conceptions about functions, continuity and differentiability.

Antônio was asked how could he decide whether a function is differentiable or not, given the algebraic expression. He stated that a function would be differentiable if he could apply known formulae to evaluate derivatives. Afterwards, he was asked how he could decide about the differentiability if the graph of the function on a computer screen was given, instead of the expression. He stated that he would zoom the graph in to have a more careful view, but it would be impossible to be sure, as computers are not flawless.

Interview 2: Participants were asked to gradually zoom in the graph of the function $y = x^2$ around the point (1,1) using the software *Maple*, and simultaneously explain what they were observing. They would obtain screens similar to the ones shown on figure 1.

At the beginning, Antônio declared he would see something similar to the tangent straight line, as he zoomed in on the graph. When the software started to display a polygonal for the curve, he claimed that the computer was wrong, as this was not the expected result. After thinking for a while, he explained the computer's error:

Antônio: It's because the computer hasn't got idea what it's doing. It's kind of messing up the points. […] As the computer sketches the graph by linking the points and these points are results of approximations, so it links without thinking. It links the points, and whatever it gets will be the graph for it, as it doesn't know what goes on.

Interview 3: Participants were asked to zoom in the graph of the blancmange function around a fixed point using the software *Maple*, and explain what they were observing. The blancmange function is defined in the interval [0,1] as the sum of an infinite series of modulus functions and is continuous but nowhere differentiable (figure 2). However, a finite truncation of the series was being used to draw the graph so that the function displayed was non differentiable at a finite set of points, rather than everywhere. The students were familiar with the functions and its properties, as they had studied it previously on calculus lessons.

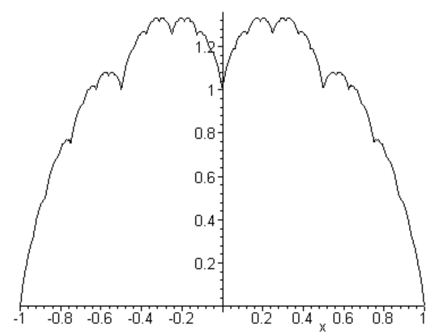

Figure 2. The blancmange.

Antônio started by explaining the construction of the blancmange function. He showed good comprehension of the process:

Antônio: […] You are taking a number and multiplying it by $1/2$, taking that one and multiplying by $1/2$, by $1/2$. So, it's a geometric progression with rate $1/2$. […] Then, it's the sum of a geometric progression. The sum of a geometric progression is a limit, then it converges to a point. […] Then each point there is a geometric progression, it's the limit of a convergent geometric progression. It's there. So you might say the curve is a sum of sums of geometric progressions. [he means the union of sums]. It's well defined.

He then started the process of local magnification and explained that, as the curve was not differentiable, the graph would become more wrinkled as he zoomed in. As the algorithm used a finite truncation of the series, it did not looked more wrinkled, as he expected, but quickly acquired a straight aspect. Antônio showed great surprise, and asked the reason for the unexpected result. After listening to our explanation, he commented:

Antônio: Oh, I see. You could sum a few more steps, but not until infinity.

After thinking for a few minutes, he proceeded, with increasing excitement:

Antônio: But it [the computer] can't make infinity. […] Hey! I think that nothing could make! […] It can't add until the infinite! There will be always an infinity missing. And nothing can represent the infinity, as a whole, but we can show that it goes to that place, that it tends to that. That's the infinite. […] It's impossible to represent it, not on the computer, not on a sheet of paper, and not in anything else! The computer only represents things that a human being knows.

Interview 4: Participants were asked to investigate the differentiability of the functions:

$$v_1(x) = \begin{cases} x\sin(1/x), & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$$ and

$$v_2(x) = \begin{cases} x^2\sin(1/x), & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$$

For that purpose, they were given the graphs of the curves $y = x\sin(1/x)$ and $y = x^2\sin(1/x)$ sketched by Maple in a neighborhood of the point (0,0) (figure 3).

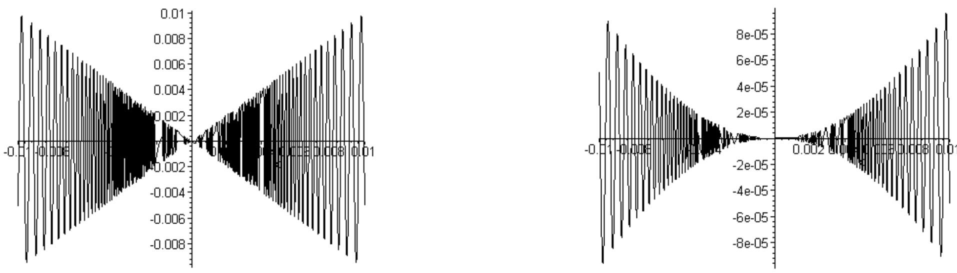

Figure 3. The curves $y = x\sin(1/x)$ and $y = x^2\sin(1/x)$.

Antônio said at first that both the functions should be differentiable, as the formulae he knew applied to the algebraic expression. He then started to zoom in the first graph around the origin, and the curve progressively looked more smudged. Antônio argued that again it should be due to an interpolation error, but the function v_1 should have a derivative. Afterwards, he repeated the process for the second graph. He commented:

Antônio: Look, when it gets closer to 0 it kind of tends to an area. But it's not. We can't see it, but it's the joining of two curves with […] the oscillation tends to zero, that's why we cannot distinguish.

We asked Antônio to conclude about the differentiability. He said:

Antônio: If it were $\sin(1/x)$, without anything else, they wouldn't be. They wouldn't be differentiable at 0, because $\sin(1/x)$ wouldn't be defined. But, for these functions the point (0,0) exists, so it's the joining of two curves there. […] Hey, wait a minute! I think v_1 is not [differentiable], do you know why? Because at 0, it's shaped by the joining of the two straight lines, $y = x$ and $y = -x$. […] When it gets closer to that point the parts approach each other within those lines! They will meet each other at that point, right? But it's a rough joining, it's kind of a corner. […] The other one [v_2] is different, it's a smooth joining. Here, the parabolas shape the curve, not the lines, that's the difference. For that reason, I think that one has a derivative and the other hasn't, v_1 has and v_2 has not. […] But I can't be doubtless sure just looking at the graph. Let me think.

Antônio concludes that the only way to be sure would be using the definition of derivative. He has a little difficulty in evaluating the limits, but reassures himself that it would be the only safe way, even if he could not do it.

DISCUSSION

Since the first interview, Antônio clearly expressed his preference for algebraic description. He states that the criteria for deciding about the differentiability of a function must be based on formulae. Moreover, he appears to be aware of the limitations of computational algorithms. Such mental attitude gave him means to quickly grasp the cause of the unexpected result on interview 2. In this sense, the theoretical-computational conflict involved (represented in figure 4) did not operate as an actual conflict, since it was almost immediately solved by the student.

THEORY	COMPUTATIONAL DESCRIPTION:
The curve is differentiable therefore it can be approximated by straight lines.	The curve does not look like a straight line when magnified

Figure 4. The theoretical-computational conflict in interview 2.

On the other hand, in interview 3 a theoretical-computational conflict (represented in figure 4), played a central role on Antônio's reasoning. In fact, Antônio's enthusiasm suggests the conflict actually triggered a new idea for him: *it is not possible to represent the concept of infinite by any physical means*. Moreover, he points out the reason for the impossibility: *infinity can never be attained*. The theoretical-computational conflict leads Antônio to grasp not only the limitations of the computational description, but of other forms as well; and to figure out a conceptual distinction between finite and infinite.

THEORY	COMPUTATIONAL DESCRIPTION:
The curve is not differentiable, therefore it cannot be approximated by straight lines.	The curve looks like a straight line when magnified

Figure 5. The theoretical-computational conflict on interview 3.

The theoretical-computational conflict involved in interview 4 was slightly more intricate than the ones observed previously, as figure 6 illustrates. In addition to that, the differentiability of the function could not be established by a careless use of the differentiation algebraic formulae, against Antônio's former dominant criteria. However, the confrontation of computational and algebraic descriptions—suggesting different conclusions—impelled him to follow another strategy: he states that *the differentiability of the function could only be doubtless concluded by means of the formal definition*.

THEORY	COMPUTATIONAL DESCRIPTION:
One of the curves is differentiable and the other is not.	Both of the curves seem to be differentiable

Figure 6. The theoretical-computational conflict on interview 4.

Antônio's mental attitude towards conflict situations contributed to the results reported in this paper. The outcomes of the interviews summarized above suggest that the conflict have acted as positive factor for the enrichment of Antônio's concept image of derivative and related notions. Nevertheless, other participants show quite different behaviors. In some cases, the conflicts do prompt students to engage into a rich reasoning. In others, the conflicts are barely noticed by students, as they are quickly solved (like Antônio did on interview 2). But some students very often cannot cope with theoretical-computational conflict situations at all. This obstacle can be due to a more general attitude towards technological devices, transcendent to their use as learning environments. The global results of the investigation in which this experiment is comprised are currently being analyzed. One of our aims is to understand more clearly in which situations conflicts do have a positive role for the enrichment of learners' concept images, in particular, in which sense and in which extent learners' previous attitudes and background determine that role.

The main goal of this work is to put forward an alternative model of approach, not purely grounded on formalism nor purely on imprecise representation forms. This propose does not mean to undervalue of the formalism, in relation to the imprecise. On the contrary, through the emphasis of limitations and differences, we intend to prompt the development of rich concept images, as well to stress the central role of the formal conceptualization on the construction of a mathematical theory.

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