THEMATIZATION OF THE CALCULUS GRAPHING SCHEMA

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This article is the result of an investigation of students' conceptualizations of calculus graphing techniques after they had completed at least two semesters of calculus. The work and responses of 27 students to a series of questions that solicit information about the graphical implications of the first derivative, second derivative, continuity, the value of limits, and the inter-relationships among these concepts was analyzed from their interviews. A double triad was developed to describe students' schema as a framework for the analysis. The study centered on the way students coordinated the various elements of each question, their strategies and difficulties. It was found that coordinating concepts to solve complex problems in a graphical setting is a difficult process. Only two students were considered to have thematized the schema.

Many research studies have been carried out to find the difficulties students have with specific calculus concepts. There is not that much research about the way students coordinate knowledge to solve complex calculus problems, so this study contributes to this area. This study extends a previous study (Baker, Cooley & Trigueros, 1999, 2000) in examining students' abstractions of calculus graphing techniques and the coordination of the elements of a calculus graphing schema. It enlarges the previous work in probing students' understanding to determine if there is evidence of thematization of this schema.

ANTECEDENTS

Many researchers have used the concept of schema to describe cognitive structures of individuals. Piaget studied the concept of schemes in many of his works. Whatever is repeatable and generalizable in an action is what Piaget refers to as a scheme. Any single scheme in itself does not have a logical component, but schemes are coordinated with each other, and this fact results in the general coordination of actions.

Robert Davis (1984) wrote about "frames", a concept which has some similarities to schemas. He wrote that each person holds in memory a vast collection of knowledge "representation structures" (RS), or "frames". The collection continues to grow as one learns new things. The growth manifests itself as new, more complex frames which are built on previously held frames. He relates these frames to explicit observable behaviors, and includes properties that frames have in common.

Skemp (1987) explains schema as a hierarchy of concepts, each level building from the previous ones. A schema for him is a conceptual structure that includes not only the complexities of mathematics, but also the relatively simple structures which coordinate sensori-motor activity. Doerfler (1989) discusses protocols and these have some similarities with Davis' representation frames and Skemp's schema. The basis for his work is the genetic epistemology as developed by Piaget and, in particular, the interpretation of Piaget's work by von Glasersfeld.

Piaget and Garcia (1983, 1989) wrote in detail about schemas and their development. In their book Psychogenesis and the History of Science (1983/1989), They discuss the development of a schema progressing through the three stages called the triad. These stages are referred to as the intra-, inter- and trans- stages. At the intra- stage of a schema, particular events or objects are analyzed in terms of their properties. Explanations at this level are local and particular. An object in the intra level is not recognized by the learner as necessary, and its form is similar to the form of a simple generalization. The student's use of, comparison of, and reflection upon isolated ideas leads her or him to the construction of relations and transformation in the inter level. In the inter- stage, the student is aware of the relationships present and can deduce from an initial operation, once it is understood, other operations that are implied by it or can coordinate it with similar operations. When a student reflects upon these coordinations and relations, new mathematical structures evolve. Through synthesis of the inter level transformations, the student is reconstructing an awareness of the completeness in the schema and in the trans-level, can perceive new global properties that were inaccessible at the other levels.

The Action Process Object Schema theory (APOS) is based on Piaget's work and has been illustrated and elaborated in a number of papers (see e.g., Asiala et al., 1996; Clark et al., 1997; McDonald, Mathews, and Strobel, 2000). In this theory, a schema is a dynamic collection of action conceptions, process conceptions, object conceptions and other schema and the relationships between them. The most distinguishing characteristic between the three stages is the ability of the person to demonstrate an understanding of the relationships between the actions, processes, objects and other schema and the development of a schema is not necessarily a linear progression through the three levels.

According to Piaget and Garcia, and in APOS theory, at the trans stage, a person may thematize the schema. A schema is considered thematized when it becomes a reality for the individual, has reached a conscious level, and may be treated as a new interesting object. When a schema has been thematized it can be unpacked so that its various parts can be used and repacked as needed. When presented with a new situation, the subject knows which parts of the schema are applicable and which are not. The structures of the schema provide true comprehension of what would be understood before. A new level of intelligibility, where the understanding is now fundamental is reached.

The theory of a triad of stages in schema development has since been utilized in several studies of student understanding in mathematics. In their study of student understanding of the chain rule, Clark et al. (1997) found that the APOS theory involving actions, processes, and objects was not adequate for analyzing their data on student understanding but that the triad of Piaget and Garcia (1983, 1989) was useful in interpreting the levels of understanding. McDonald et al. (2001), also utilizing APOS theory and the triad, studied the development of the sequence schema.

In both the calculus study by Baker, Cooley & Trigueros (1999, 2000) and in the study on systems of differential equations by Trigueros (2001), the notion of schema was explored and expanded. The idea of schema interaction was introduced in the first of these papers, and a framework consisting of the intermingling of two other schemas was described. In this work it was shown that the triad proved to be a useful tool in describing the interaction of two specific schema, referred to as the property schema and the interval schema, that the researchers observed the students using in trying to solve the non-routine calculus problem that was posed. The study indicated that students retain and use certain calculus concepts while disregarding others. In particular, the two predominant areas of concern were coordinating overlapping information across contiguous intervals on the domain and coordinating properties that were explicitly stated rather than derived from a formula. Based on these observations, a model of schema interaction was described for the first time. As a consequence of this study several further research questions were raised. One of them was whether other students, who addressed the same graphing problem, could also be described by the same model. Another issue was if the behavior of students changes when presented with multiple questions of a similar nature instead of only one task. Finally, an important research question was to study thematization of the schema and its characteristics. This current paper addresses these questions.

THEORETICAL FRAMEWORK

The theoretical framework used to design and analyze this research project is a two-dimensional schema referred to as a Calculus Graphing Schema, which was presented in a previous paper. This model was developed on the basis of a genetic decomposition of the processes and objects contained in the schema as well as their relationships.The main objects in the schema are the first derivative, second derivative and continuity. These were then also identified in relationship to the graph of the function with its intervals over the real numbers and the possible relationships between all of these aspects.

The property schema has two important aspects: understanding the graphical property associated with analytical conditions concerning the main objects of the schema and the coordination of all the properties pertaining to a given interval in the domain of the function. At the intra-property level, a student can interpret only one analytical condition at a time in terms of its graphical feature. A student at this level typically utilizes solely the first derivative condition and is often aware of other properties but cannot coordinate them to produce a graph. If two properties overlap, the student describes the behavior of the graph using only one property. If he or she tries to use more than one property, the student cannot complete her or his description and resorts to using only one property.

At the level of inter-property, the student begins to coordinate two or more conditions simultaneously. This coordination, however, is not applied throughout all overlapping graphical properties given for an interval. The student is considered to be at the trans-property level if he or she can demonstrate coordination of all the analytic conditions to the graphical properties of the function on an interval. At this point, the student expresses or demonstrates a coherence of the schema. That is, the student clearly recognizes what behaviors of a function can be included in the graph and what cannot.

The main aspects of the interval schema are understanding the interval notation, connecting contiguous intervals, and coordinating the overlap of the intervals. The interval schema involves coordinating conditions across contiguous intervals as well as different conditions on the same interval. Distinguishing different sections of the domain is a typical problem in introductory calculus. At the intra level of the interval schema, a student works on isolated intervals and is not able to coordinate information across intervals. The overlap of intervals or connection of contiguous intervals causes confusion. At the inter-interval level, the student begins to coordinate two or more contiguous intervals simultaneously. This coordination, however, was not applied throughout all connected intervals or across the entire domain. The student is considered to be at the trans-interval level if he or she is able to describe the coordination of the intervals across the whole domain. He or she is able to overlap intervals and connect contiguous intervals. The student also demonstrates coherence for the schema by describing which manifestations in the graph are allowed by the overlap and connection of the intervals and which are not.

Students who were determined to be reconstructing their knowledge at the trans-property, trans-interval level were considered candidates for thematization. A student is considered to have thematized the schema if he or she is able to continue to coordinate all of the given properties across all intervals as singular properties are withdrawn. If this is demonstrated, then the student is able to unpack the schema and repack it without the removed property, coordinating the remaining property. The student is demonstrating conservation in how the remaining conditions interact, which behaviors of the graph now remain and which have been removed. The mathematical structures of the Calculus Graphing Schema have become a fundamental part of their understanding and may be used in total as an object. Another important aspect of thematization is consistency, in this case the student consistently is able to pack and unpack the properties of the function throughout the questions, which can be solved by means of the same coordinations.

METHODOLOGY

The purpose of this study was two-fold: to determine if the double triad would work as a tool in another similar situation and to extend the interview to include questions that would allow a student to demonstrate the thematization of the schema. Therefore, the researchers decided to interview those students found to be successful enough in calculus that there could be a possibility of thematization. These students represented a broad mix. Fifteen students were from a private university in Iowa, while the remaining twelve were from a private university in Mexico City. The majors of these students included mathematics, engineering, mathematics education, economics, and natural sciences.

The reliability of the model derived from the observations of an interaction between the property schema and the interval schema was tested with a series of related questions, as well as the non-routine calculus question used in the previous study by Baker, Cooley & Trigueros (2000). The series of questions was developed by the researchers with the goal of having more in-depth data of student understanding, examining student responses and techniques from various perspectives, in order to provide evidence of conservation in their understanding and if thematization of the schema had occurred. The students were asked to respond to these questions in extensive, detailed, audio-taped interviews. The interviewers were mathematics professors with a background in mathematics education research. The students answered the questions, explained their thought processes, asked questions, and explained the methods they used to assemble their graphs and other work. Their written work was also kept as part of the interview data. The researchers reviewed all the written and oral work. Each student interview was analyzed independently by at least two researchers. After the analysis, the two researchers discussed their observations and conclusions. There were no discrepancies in the final categorization of the students. Those students who were determined to have reached the trans-property, trans-interval level were reviewed again by all three researchers to determine if there was evidence of thematization. The three researchers discussed this analysis and agreed on the students who had thematized the schema.

The questions asked of the students required an understanding of what it means for the first derivative to exist, its representation as the slope of the tangent line and as rate of change, and the relationship between derivative, the limit, continuity, and shape of the curve in different intervals of the domain of the function. They also required an understanding of the significance of the second derivative, of its existence and representation as the concavity of the curve, its graphical meaning and the significance of inflection points. The relationship between first derivative, second derivative, continuity and limit needed to be used in all the problems and must be coordinated on different intervals of the domain. Students needed to decipher which conditions and properties are dependent upon continuity. The questions were posed in different representational forms and were more demanding as the interview progressed in terms of asking for coordination of more objects ending with the same problem used in the previously mentioned paper.

In order to examine possible thematization of the Calculus Graphing Schema, several of the interview questions remove certain conditions (continuity, first derivative and second derivative) while the student is asked to explain what characteristics the graph of the function retains. The ability to describe the relationships among the remaining characteristics, to synthesize the properties and separate them, packing and unpacking them as needed, was considered evidence of thematization.

DISCUSSION

Our analysis of the student interviews showed that students were demonstrating similar behaviors and responses as the students in the previous study. We observed students having the same difficulties at the same points in the final question. Students were found to rely heavily or exclusively on the first derivative and struggle with the limit condition on the first derivative at a point ($x = 0$ in this case). Coordination of differentiability and continuity in cusp points proved to be very difficult for students. The series of questions helped students stay focused on the issues under study and to demonstrate their knowledge in different contexts, but still the students exhibited the difficulties mentioned above consistently throughout all of the problems.

It was interesting to observe students who were able to construct the graph in the last problem, but who demonstrated an unstable conceptualization when trying to answer other questions. We also found that removing the continuity, first derivative and second derivative in the last question provided a good tool to observe the strength and coherence of the students' schema, as well as the ability to unpack it and repack it, thus demonstrating thematization. Indeed, these last questions were very revealing in what the students were able to consistently coordinate and what was a weaker conceptualization.

There were only two students who were consistently able to discuss the relationships between the different processes and objects in the schema, and who were also able to unpack the schema in the final questions related to removing certain conditions while holding others constant and give consistent answers. This was a surprising result considering that all of the students in the sample had successfully completed their calculus courses and were considered above average by their teachers. In contrast with the other students, those with thematized calculus graphing schemas use their knowledge in a flexible way. We show here an example of one of these students when solving the removing conditions parts of the last problem.

(a) Sketch the graph of a function that satisfies the following conditions:

h is continuous

$h(0) = 2,\ h'(-2) = h'(3) = 0,$ and $\lim_{x \to 0} h'(x) =$

$h'(x) > 0$ when $-4 < x < -2$ and when $-2 < x < 0$ and when $0 < x < 3$

$h'(x) < 0$ when $x < -4$ and when $x > 3$

$h''(x) < 0$ when $x < -4$, when $-4 < x < -2$, and when $0 < x < 5$

$h''(x) > 0$ when $-2 < x < 0$ and when $x > 5$

$\lim_{x \to -} h(x) =$ and $\lim_{x \to} h(x) = -2$

(b) Do there exist other graphs besides the one you drew that satisfy the same conditions? Justify your response.

(c) If we remove the continuity condition, and the other conditions remain, does the graph change? In what way? Do other possible graphs exist? If other graphs exist, could you sketch one example?

(d) If we remove all of the first derivative conditions, and the other conditions remain, does the graph change? In what way? Do other possible graphs exist? If other graphs exist, could you sketch one example?

(e) If we remove all of the second derivative conditions, and the other conditions remain, does the graph change? In what way? Do other possible graphs exist? If other graphs exist, could you sketch one example?

S: Now b, what did it say? Can any other graphs, apart from the one you drew, satisfy the same conditions? No, like before, since the conditions that are established aren't too specific. They only say that the point is (0,2). So it isn't unique. You can move it to a different place, make it taller or shorter and lower the peak or the maximum. Other than the (0,2) and the asymptote, the rest can be in any other place, and, no, it's not unique.

I: The last part, what happens if you leave all the conditions the same but take away the condition of continuity? Can other graphs exist?

S: If the graph changes, for example at negative 4, it could be broken because there are open intervals and they don't say anything about the negative 4 so that it could break there. Also at 0 there could be a little hole. It could be something like this example in the drawing.

I: Are those the only points that could change?

S: I think so because those are the points where the derivative isn't defined. At the others it is defined. So even if they don't tell you that the function is continuous, you know that it is because if it is derivable, it's continuous, like in the other question. It's the same thing. So only at the 0 and the negative 4.

I: Does the fact that they tell you that it goes through the point (0,2) have any affect?

S: Oh yes, I forgot about that. Then there can't be a hole. There has to be a point. There could be an asymptote and the point there in the axis. No, because it grows and grows, but then it could come out from underneath and it would grow from the left and grow by the right…and the point, but no because then the limit wouldn't exist. What a mess! And then, no, only the negative 4 would change, I think. Yes, only the negative four.

In contrast, another student, also at the trans-property, trans-interval level, was not able to succeed with the removal of the conditions:

I: And if you take off the condition of continuity?

S: If the condition of continuity is removed the graph would change, because now it can be broken in any of the points I marked on the x-axis. It can become something like this, like the one I am drawing here.

I: Are you sure that it can break in all those points?

S: Aha, yes, in all of them there is a change in the properties and if it doesn't have to be continuous well, you can draw little holes there.

CONCLUSIONS

The results of this study show that the double triad model presented above is a good tool to study the coordination of different concepts used to solve calculus graphing problems in different representational contexts. It also shows that the integration of knowledge is a slow process and that even good students finish their courses without having a strong understanding of the relationship between them. Even if the students in this study struggled less with issues that were shown to be difficult before, they still demonstrated the same kind of difficulties as the students had in the earlier study. Coordination of properties on different intervals remained a difficult task for most of the students. This study shows as well that students' knowledge is not stable. What can be considered a slight change in the conditions of a problem causes some students to reconsider their answers and to exhibit difficulties that they didn't show while solving the original problem.

This is, as far as we know, the first study that focuses on thematization of schema. It shows that it is a difficult process and that it takes a lot of time. Even very good students seem to finish their calculus courses with what we can consider weak and unstable schemas.

Further research focused in widening our understanding of thematization and schema development is needed to find ways in which students can be helped through this difficult process and to design didactical strategies that can foster schema development.

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