TEACHER AND STUDENTS' JOINT PRODUCTION OF A REVERSIBLE FRACTION CONCEPTION

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Within an elaborated social-constructivist perspective, I conducted a teaching experiment with two fourth graders to study how a teacher and students can jointly produce the content-specific reversible fraction conception. Ongoing and retrospective analysis of the data revealed the non-trivial process by which students can abstract multiplicative reasoning about fractions without using the activity of splitting. The study contributes to the psychology of mathematics education by articulating a culminating advance in a developmental sequence of iteration-based fraction conceptions and the teacher's role in fostering such an advance in students.

In this study I addressed the problem of how teachers and students jointly produce mathematical conceptions (Bauersfeld, 1988). In particular, I generated a grounded, content-specific explanation of students' learning of a *reversible fraction conception* as a transformation in their iteration-based fraction conceptions and of the teacher's role in this process. Content-specific explanations are critical for utilizing general models of growth in mathematical understandings while teaching particular ideas to particular students (Tzur, 2002). Generating content-specific explanations is particularly important in the domain of fractions, because, as Davis, Hunting, and Pearn (1993) asserted, "The teaching and learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure" (p. 63).

CONCEPTUAL FRAMEWORK

The study drew on a social-constructivist approach (Cobb & Bauersfeld, 1995), which coordinates social and psychological perspectives for research. The explanation of concept formation that I used is an elaboration of the psychological perspective and of the possible role for teaching. In this section I briefly present key aspects of the general and the content-specific constructs that, combined, provided a framework for the study's design and data analysis (see Simon et al., 1999; Tzur & Simon, 1999).

General Constructs. In this framework, a mathematical conception is considered a dynamic, mental relationship between an activity and its effects ('conception' and 'activity-effect relationship' are used interchangeably). That is, activity generates and is a constituent of a conception—it is not just a catalyst to the process or a way to motivate learners. Abstracting a new conception is made possible by the mechanism of reflection on activity-effect relationship, which is an elaboration of Piaget's notion of reflective abstraction (note that 'reflection' does not imply learner's awareness). Two qualitatively distinct stages in abstracting a new conception—participatory and anticipatory—are identified on the basis of the nature of the learner's anticipation. At the participatory (first) stage, the learner anticipates effects of an activity and utilizes them in problem situations. However, the learner cannot independently call upon the activity (hence the activity-effect relationship); he or she must somehow be cued for which activity to use (e.g., chance, engaging social interaction). At the anticipatory (second) stage of a new conception, the learner can independently call upon and utilize that conception proper to the problem situation. The content of the anticipated relationship is the same in both stages; they only differ with respect to the availability of the relationship in a given situation. Teaching is viewed as a cycle of 3 principal activities: inferring learners' current conceptions, hypothesizing a learning trajectory (Simon, 1995) from current to intended conceptions, and selecting/using tasks (problem situations) that learners can assimilate and solve as a means to form the intended conceptions. Teacher tasks are geared toward: (a) engaging learners in setting goals and initiating activities toward those goals and (b) orienting learners' noticing of effects of activities and their reflection on the designated activity-effect relationship.

Content-Specific Constructs. Researchers identified two fundamental activities that constitute fractional conceptions, splitting (Confrey, 1994) and iteration (Steffe, 2002). Splitting refers to recursively acting on the results of previous activities (e.g., part of a part of a part, etc.) and was considered to generate multiplicative structures, such as a geometric sequence, better than the iteration of units (e.g., repeated addition). Yet, Kieren, Mason and Pirie (1992) noted that rational numbers are both additive and multiplicative quantities, and splitting activities such as paper folding are insufficient for generating the additive meaning. For example, one cannot expect splitting activities alone to generate the critical understanding of non-unit fractions such as 6/11 (let alone the 'improper' 13/11) as the effects of iterating 6 (or 13) abstract units of size 1/11 each (Behr et al., 1992). Tzur's (1999; 2000) identification of the partitive and iterative conceptions indicated how the activity of iteration, which is available via whole number conceptions, allows students to abstract unit fractions (e.g., 1/11) and non-unit fractions (e.g., 6/11, 13/11). Moreover, he showed that iteration could generate fractions as a multiplicative relation: the child conceives of, say, 13/11 as a unit that is 13 times as much as 1/11, where 1/11 is itself not just part-of-whole but any unit which fits-in-a-given-whole exactly 11 times. The study reported here was set to address the content-specific problem of how students transform (reverse) those iteration-based conceptions of non-unit fractions so they can decompose such fractions. In the analysis section I will show that such a critical conceptual advance—abstraction of a reversible operation in Piaget's sense—is not as trivial as it might appear.

METHODOLOGY

The study was part of a larger constructivist teaching experiment in which, together with 5 researcher-teachers, I taught fractions to two fourth graders, Linda and Jordan. During that year, we conducted 29 videotaped teaching episodes once or twice a week, about 30 minutes each, and collaborated in the ongoing analysis/planning sessions between every two consecutive episodes. Retrospective analysis of students' conceptions and the teacher's role in fostering them consisted of rigorous, line-by-line interpretations of participants' language and actions: asking questions about the data, making grounded hypotheses about possible explanations for these questions, and systematically searching for confirming/disconfirming evidence.

To support activities needed for students' learning of fractions the study utilized a computer microworld called Sticks, which was developed along with and informed by the teaching experiment. In Sticks (see Figure 1), using Draw, one could produce on the computer screen a linear figure (a 'stick') of various lengths and replicate any such stick as many times as desired using Copy. One could mark a stick vertically using Marks and erase or move those marks, cut any stick at a desired point using Cut, and join one stick to any other stick on the screen using Join. One could also partition any stick into a desired number of equal pieces (2-99) using Parts, break a marked or partitioned stick into pieces using Break, and iterate a stick (plain or marked) using Repeat—a coordination of Copy and Join. One could pull pieces of a marked or partitioned stick using Pull-Parts and measure any stick by using Measure. One could also fill a stick or any part of it with 10 different colors using Fill, use Label to attach a fractional symbol to any stick, and cover or uncover sticks or parts of them.

ANALYSIS

In this section I explain how Jordan, Linda, and I (the researcher-teacher) jointly produced the reversible fraction conception. I focus on how their work oriented my teaching, how my teaching occasioned their abstraction of the reversible conception, and how their advances occasioned my articulation of the abstraction process.

Prior to our work on the reversible fraction conception, Linda and Jordan had established two major iteration-based conceptions—partitive and then iterative—as reorganization in their whole number conceptions (Tzur, 1999; 2000). Using these conceptions, they were able to generate and operate on symbolized unit and non-unit fractions (i.e., anticipatory stage). For example, in a playful context where we considered ourselves workers at a pizza stand, they could independently and mindfully explain the solution to problems such as, "How much of a pizza would you sell if four customers bought 3/10 of a pizza each?" Therefore, I decided to present a task that required decomposition of a non-unit fraction in the pizza-stand context.

Initial Assessment of Reversibility. I began episode 26 (May 3) by drawing an unmarked stick, told the children that it represented 5/8 of a pizza but the original pizza was lost, and asked if they could rebuild the original pizza. Immediately, both Jordan and Linda activated Parts to solve the problem by partitioning the unmarked 5/8 into 8 rather than 5 parts. This was a clear indication of calling upon their prior, anticipatory iterative conception, so I decided not to pursue the problem further and moved to working on tasks of selling and buying pieces of a 10-slice (10/10) pizza.

Accidentally, but fortunately for the goal of the study, Jordan ruined and erased the original 10-slice pizza. My on the spot reaction was to utilize this as an opportunity to further examine the children's thinking about rebuilding the pizza from parts of that pizza (1/10, 2/10, 9/10) that were still available on the screen. I asked a question of the type "Think and tell me what would you do" in order to both enable each child time to think of a solution and examine their anticipation of activity-effect relationship. After a few seconds Linda said that she would copy and repeat the 2/10 five times and Jordan said he would copy and join the 9/10 and 1/10. Then, each child implemented his or her

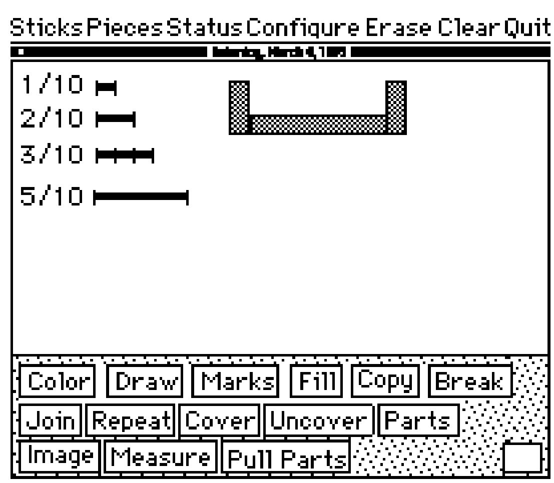

Figure 1. The initial screen of Episode 27

plan.

Each child's response indicated that they anticipated the effects of both activities: iterating 2/10 five times or adding 9/10+1/10 <u>must</u> reproduce a whole like the original. While reflecting on their anticipations and my task I realized that they have just "taught" me how they might decompose a non-unit fraction (i.e., iterating non-unit fractions). My critical role, then, was to devise a task that would transform their independent use of that anticipatory conception into the reverse activity. Eventually, I designed a task for them to set the same goal (rebuild the whole) in the same context (pizza-stand). I chose units (1/10, 2/10, 5/10) that would lead them to bring forth iteration and also a marked unit (3/10) that would require further activity of partitioning (see Figure 1). The hypothesis was that even with the 3/10 being marked, the children would begin with 5/10 (doubling), then use the 1/10 (iterating a unit fraction), then 2/10 (anticipated, non-problematic iteration of a non-unit fraction), and finally use the 3/10. Using the 3/10 was likely to bring forth its decomposition, at least once, which I would then use to orient their reflection on and foster further utilization of such activity.

<u>Reversing an Available Anticipatory Conception</u>. I posed the task two days later, in episode 27 (May 5). Their goal was to rebuild the pizza so it would fit in the 'oven,' using one of the pieces at a time. I asked them to think and tell what they planned to do before actually doing it. As hypothesized, Jordan said he would use the 5/10 twice; Linda said she would use the 1/10 ten times. Both children seemed confident that their solutions must work. Then I asked which of the other two pieces they could use next. Linda immediately and excitedly said she would use the 2/10, then implemented her anticipated action while saying: "Cop-I … (she copies the 2/10 and repeats it 4 more times while counting the non-unit fractions) 2, 3, 4, 5." Linda's work indicated that she anticipated the effects of iterating the 2/10 five times by using multiplication in conjunction with her iterative fraction conception. To convince me that Linda solved the problem properly Jordan said, "two plus, umm, 2-times-5 is 10," and Linda said that she thought of the same reason. The groundwork was laid for the designated transformation.

<u>Excerpt 1 (Episode 27, May 5)</u>

Res.: That's great. You see, what I like very much is that you have different ways of doing things. N-O-W (intonation emphasizes "here's the real challenge"), we have only one piece left.

Linda: (Echoing Res.) One – Piece - Left. (Then, enthusiastically) I know how.

Jordan: (Immediately following Linda) I know how. (Excited, to Linda) It's my turn. (Utilizes the microworld efficiently in a seemingly well-planned solution. Using Copy he makes 4 replicates of the 3/10, breaks the last one into three pieces of 1/10 each, joins the three pieces of 3/10 first, then also the 1/10, and trashes the remaining two pieces of 1/10.)

Res.: (To Jordan) Very nice. (To Linda) Do you have another way to make the original pizza?

Linda: (Excited) YES. (Using Copy she replicates the 3/10 once, then using Repeat she iterates it three more times while counting in "threes") 3, 6, 9, 12. (She searches for Cut in order to decompose the 12/10 into 10/10 and 2/10 without breaking the entire 12/10. Because Cut is not available, she breaks the 12/10 into 1/10 parts, then joins only 10 pieces into a 10/10 whole.

Res. and Jordan: (Watch Linda's work silently)

Linda: (Trashes her 10/10-stick, copies the 3/10 again, repeats it only 3 times into 9/10, clicks on the trash to bring out the 1/10 she trashed earlier, and joins that last piece to the 9/10.)

The children's work supported our hypothesis: the previous iteration of a unit fraction (1/10) and of non-unit fractions (5/10, 2/10) engaged them in utilizing their iterative fraction conception and thus oriented their reflection on the activity (iterating 1/10) that generated the non-unit fractions. In this sense, the social interaction prompted them for the possibility to decompose the non-unit fraction by reversing that activity. Yet, the decomposition was made only at the end of the children's activity sequences. To foster the transformation further, I continued to challenge them.

Excerpt 2 (Episode 27, May 5)

Res.: Can you solve it in a different way?

Jordan and Linda: (Each takes turns to utilize a different sequence of 'buttons' in the microworld, always decomposing the 3/10 last.)

Res.: Okay. You always started, when you take the 3/10, by making more [of those] pieces. Can you think of making it smaller instead of bigger? Start by making it smaller, then make the 10?

Jordan: (Holds the mouse, then asks) What do you mean, like ...

Linda: Copy!

Jordan: (Copies the 3/10 and says) Like this?

Res.: (To Jordan) Now you have the 3/10, right?

Jordan: (Nodding yes) Mm-hmm ...

Res.: Can you think of a way that, using it, making it smaller?

Linda: (Excited) I do, I do.

Jordan: (Hesitantly) Smaller?

Res.: Smaller than the 3/10 and then reproduce the 10/10.

Jordan: (Pulls the first 1/10 out of the 3/10 and trashes the 3/10 and repeats the 1/10 ten times.)

Linda: That was my idea.

One might say that my researcher's cue ("make it smaller") funneled the children to the intended action. However, Jordan's difficulty to interpret the cue indicates that this was not trivial. Rather, I suggest the following inference on the basis of both children's work. My cue oriented their reflection on the sequence of mental actions that they used in their previous solutions, when anticipating reproduction of a 10/10 whole with decomposition of the 3/10. Through reprocessing the decomposition of 3/10 last they noticed two effects anew: this decomposition created a desired unit fraction (1/10) and, being last in the activity sequence, the decomposition could match the goal ("make smaller first") set by my cue. In this sense, Excerpts 1 and 2 demonstrate the joint production of the initial transition to a reversible conception. Via reflection on their actions in the previous episode, I contributed a task and follow-up prompts that (a) occasioned their utilization of available conceptions and (b) oriented their reflection onto the designated (reversed) activity-effect relationship.

To foster further reflection on that relationship I posed the 5/8 task again. I cleared the screen, made a new 'oven' and an unmarked 5/8 of a pizza, and told them: "My dear workers! You lost your pizza, which comes in 8 slices. But you still have one piece left that is 5/8 of the original pizza." Linda immediately and eagerly said that she knew how to do this even before hearing a question. Jordan, who always grabbed the mouse and began working if he knew the solution, hesitated and asked several times about the task. For him, even understanding the task was not a trivial repetition of the previous one. He did not yet construct even the provisional, activity-dependent anticipation of a new participatory conception, in which the compositions of a non-unit fraction by means of iteration is to be reversed. As Excerpt 3 indicates, Linda also needed to initiate and reflect on some actions before she could clearly abstract the new relationship.

Excerpt 3 (Episode 27, May 5)

Linda: (Activates Parts, dials down from 8 and says to herself) Why not ... eight ... and Parts ... oh, five ... (Mistakenly, she dialed down to 3 so she says) OOPS. (She dials back to **8** and almost clicks on the 5/8-stick, but then changes her mind, says to herself) No (and dials to 5 while saying to herself) I want 5. (She mistakenly turns Parts off so she turns it on, purposefully dials to 5, partitions the 5/8-stick into 5 parts, pulls 3 parts from the partitioned stick while counting them) One, two, three. (She joins the 5/8 with the 3/8 while counting the pieces with the mouse and says) Two, three, four, five; Six, seven, eight.

Res.: Now isn't that nice?!

Linda: (Puts the 8/8-pizza into the "oven" and humming to herself, proudly) Doo-doo-dee-doo. (Releases the mouse as if saying "I'm done and I know that it is the right solution.")

Linda's "OOPS" experience is a critical point in her abstraction of the participatory reversible conception. Her initial excitement about the new task was rooted in an anticipation of the need to partition the unmarked non-unit fraction but not yet a clear anticipation of the number of parts needed. I suggest that by reflecting on her nearly executed activity of partitioning the 5/8 into 8 parts and on her mental run of this action to its effect, she noticed that such effect would not match her goal. This reflection led Linda to change her plan and to the desired transformation. She related, in reverse, the activity of partitioning the non-unit fraction with the number of iterations (5) used to produce it from an imagined unit fraction (1/8), as indicated by her purposeful actions and private speech when Parts was accidentally turned off.

Before the end of episode 27, after observing Linda's solution, we only had enough time for Jordan to reproduce another original pizza by partitioning the unmarked 5/8-stick first and then doubling it and joining 3 more pieces (3/8). However, it was unclear to what extent he abstracted that rudimentary, provisional anticipation himself. To test this, a week later (episode 28, May 12), I took advantage of Linda's late arrival and posed the following task: presented with an 'oven' and an unmarked stick labeled 7/10, I asked him if he could reproduce the whole pizza from that piece. Again, for a few minutes, Jordan struggled to make sense of the task itself. Then, he took action.

Excerpt 4 (Episode 28, May 12)

Jordan: (Copies the 7/10-stick once, activates Parts, dials to **10**, and actually partitions the 7/10. The researcher indicates nothing, but Jordan himself seems at unease. He thinks for 10 seconds then says "OOPS" and trashes that stick. He makes one more copy of the 7/10-stick and stops to think. After 20 seconds of silence, he says to himself "Oh," stops for about 2 more seconds, activates Parts, dials immediately to 7, and partitions the unmarked 7/10. At this point he seems to work purposefully and smiles, while pulling 3 more parts from the 7/10 and joining them with the 7/10. Finally, he drags the 10/10 stick into the 'oven' as if to check that it's right, then drags it out of the oven and leaves the mouse to indicate "I'm done.")

Jordan's recurring difficulties to make sense of the task and his first actual partition of the 7/10 into 10 parts highlight that following Linda's solution in the previous week was not sufficient for his abstraction of the new conception. His partitioning of 7/10 into 10 parts is not surprising. It is an example of a predictable 'interference,' implied by the participatory/anticipatory distinction. If a conception has only been established at the participatory stage, the learner cannot independently call upon it so he or she calls upon conceptions established previously at or beyond the anticipatory stage.

Most importantly, Jordan's work provides a significant 'window' to the invisible functioning of reflection on activity-effect relationship. Jordan's uneasiness indicated that his actions created effects he did not anticipate. I suggest that then, during the first 10-second silence, he reflected on mental runs of his action for plausibly producing the desired effect (whole). This reflection led to his realization, indicated by his "Oops" utterance: partition into 10 parts is inappropriate. This realization is compatible with Linda's, when she almost partitioned the 5/8 into 8 parts. Underlying both realizations is noticing of their coordinated actions in the anticipatory production of a non-unit fraction: Partitioning the whole and then iterating the unit fraction. This last inference is supported by the 20-second period of silence: Jordan independently anticipated the need to partition the 7/10, but he still had to figure out the number of parts. The long pause and the resulting confident manner in which he used Parts to select 7 indicated that he did not merely use trial-and-error. I infer that to figure out how many parts he further reflected on and differentiated the process he **had available** (producing 7/10) into separate actions. This reflection led to **his** realization of the need to reverse the sequence: begin by partitioning the 7/10 into the 7 parts of 1/10 each that were previously used (last in the sequence) to compose the 7/10. In turn, Jordan's difficulty and elaborated process brought forth my reflection on and distinction of the non-trivial abstraction of the content-specific, reversible fraction conception. The following episode (29) was last in that year and I devoted most of it to converse with Linda and Jordan about their overall experience during the year. Thus, I can only claim that both had abstracted the reversible conception at least at the participatory stage.

DISCUSSION

This study addressed the problem of how a teacher and students jointly produce the iteration-based reversible fraction conception. Analyzing the process of abstracting such a conception and the teacher's role (including useful tasks) contribute to the ongoing effort to identify content-specific understandings and the sequence in which they might develop. A transformation of the partitive and iterative conceptions (Tzur, 1999; 2000), the reversible fraction conception culminates the abstraction of multiplicative reasoning with fractions without/before using the activity of splitting. This helps to further explain two critical issues in the psychology of mathematics education: (a) why is learning of fractions such an obstacle for many students (Behr et al., 1992; Davydov & Tsvetkovich, 1991; Kieren, 1988; Streefland, 1991) and (b) how might teachers support meaningful learning in this domain.

The study also highlights an important aspect of the teaching-learning process. Small group and whole class discussions, alongside teacher tasks and prompts, greatly contribute to learners' abstraction, via orienting learners to their available conceptions—goals they can set, activities they can call upon, effects they can notice, relationships (anticipations) they can form. However, as the example of Jordan's learning with/from Linda demonstrated, these four basics of conceptual progress and the mechanism of reflection that ties them together reside within the learner. Thus, to understand a particular idea in mathematics each learner must abstract the constitutive activity-effect relationship for himself or herself.

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education, *Effective Mathematics Teaching* (pp. 27-46). Hillsdale, NJ: Lawrence Erlbaum.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 296-333). New York: Macmillan.

Cobb, P., & Bauersfeld, H. (1995). *The Emergence of Mathematical Meaning*. Hillsdale, NJ: Lawrence Erlbaum.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 291-330). Albany, NY: State University of New York Press.

Davis, G., Hunting, R. P., & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *Journal of Mathematical Behavior, 12*, 63-76.

Davydov, V. V., & Tsvetkovich, Z. H. (1991). The object sources of the concept of fractions. In L. P. Steffe (Ed.), *Psychological Abilities of Primary School Children in Lerning Mathematics* (Vol. 6, pp. 86-147). Reston, VA: National Council of Teachers of Mathematics.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 162-181). Hillsdale, NJ: Lawrence Erlbaum.

Kieren, T. E. , Mason, R., and Pirie, S. E. B. (1992). *The growth of mathematical understanding in a constructivist environment*. A conference paper at Canadian Society for the Study of Education. Charlottetown, Canada.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. A., Tzur, R., Heinz, K., Kinzel, M., & Smith, M. S. (1999). On formulating the teacher's role in promoting mathematics learning. In O. Zaslavsky (Ed.), *Proceedings of PME-23* (Vol. 4, pp. 201-208). Haifa, Israel.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 102*, 1-41.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research* (1 ed. Vol. 8). Dordrecht, The Netherlands: Kluwer

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

Tzur, R. (2000). An integrated research on children's construction of meaningful, symbolic, partitioning-related conceptions, and the teacher's role in fostering that learning. *Journal of Mathematical Behavior, 18*(2), 123-147.

Tzur, R. (2002). From theory to practice: Explaining successful and unsuccessful teaching activities (case of fractions). In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of PME-26* (Vol. 4, pp. 297-304). Norwich, UK.

Tzur, R., & Simon, M. A. (1999). Postulating relationships between levels of knowing and types of tasks in mathematics teaching: A constructivist perspective. In F. Hitt & M. Santos (Eds.), *Proceedings of PME-NA-21*(Vol. 2, pp. 805-810). Cuernavaca, Mexico.