A COGNITIVE MODEL OF EXPERTS' ALGEBRAIC SOLVING METHODS

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We studied experts' solving methods and analyzed the nature of mathematical knowledge as well as their efficiency in algebraic calculations. We constructed a model of the experts cognitive functioning (notably teachers) in which the observed automatisms were modeled in terms of schèmes and instruments. Mathematical justification of transformation rules appeared as the basis of teachers' algebraic competencies. Our research led us to consider the management of mathematical justification as a fundamental teachers' task.

The mastery of literal elementary calculations concerns the conceptualization of mathematical objects (equation, inequation, systems…) and efficiency in using transformations rules. We studied experts solving methods (notably teachers' solving methods); the nature of their competencies and efficiency. On the one hand, subjects constructed mathematical knowledge and on the other hand, mental structures (schèmes) which allowed them to rapidly perform algebraic calculations. Mathematical justification of transformation rules appeared as the basis of the efficiency.

The identified invariant task in algebraic calculation and the different types of justifications given by experts and students, led us to take the theoretical concept of the operational invariant further than previous authors. The different types of operational invariants involved in algebraic calculations concerned operational knowledge and the efficiency of solving methods. Experts' solving methods included analysis tasks which allowed choosing a pertinent transformation. Observed automatisms in the subjects' cognitive functioning were modeled by means of the theoretical concept of instrument.

Our theoretical framework is fundamentally based on the research of Vigotsky, Piaget and Vergnaud in the area of psychology; and the research of Rabaldel and Pastré in the field of cognitive ergonomics.

THEORETICAL FRAMEWORK

Transformation rules state a mathematical property on the one hand, and on the other, they propose an action. Most transformation rules, given by teachers or textbooks, state mathematical properties which are considered to be self-evident (self-justified). But later, students adopt transformation rules that are more "economical", closer to the written transformations. For example, the rule *"Add (or subtract) the same number to each member of the equation"* is replaced by another: *"Transfer a term to the other side of the equation by changing its sign"*. The latter states a mathematical property that is not at all self-evident, but it does describe the transformation that students perform on paper. Also, other rules (used in factorization, development, numerical calculation, etc.) are sometimes totally obscure. Using the rule, the student develops a mental structure that enables him to take action in a structured manner.

The utilization schème of a rule is constructed by a subject through using the rule and through writing transformations. This schème (the concept of schème was introduced by Piaget) allows performing a transformation quickly and without focussing attention on the action at hand. Thus, subjects can acquire some distance with the task. Consequently, *the utilization schème of the rule may be described as a controlled automatism.* Schèmes can be observed in the fact that the subject performs transformations quickly and his notations are stable. In general, human beings construct adapted schèmes for any type of task. The utilization schème of the rule organizes the way the transformation is written and thus leads to new rules ("economics rules") which correspond to the transformations as written.

Instrument : the utilization schème of a rule is constructed using the rule, but it is not the rule itself. The given rule (symbolic object) and the utilization schème (subjects' mental construction) form a whole that we call an "instrument". P. Rabardel (1995) introduced the theoretical concept of instrument in the field of cognitive ergonomics.

The solving methods. Experts identify mathematical objects and define relevant goals (goals are not the same in solving equations, inequations, systems…) and a solving strategy. Consequently, it can be said that subjects construct a particular solving method for each type of mathematical object. At the level of the solving method, subjects analyze the particularities of the mathematical object and choose relevant transformations, which are performed by means of instruments. For each transformation, experts use an instrument and the same instruments are used in solving different mathematical objects. Then, instruments are not associated to a particular mathematical object and *they establish continuity between methods.*

Solving methods are "instrumented" schèmes, another type of schème, which involves analyzing task. The efficiency of solving methods depends on the constructed operational invariants. If the method is well constructed, subjects can reconcile speed with efficiency in performing transformations.

THE THEORETICAL CONCEPT OF OPERATIONAL INVARIANT.

The concept of operational invariant was introduced by Piaget. For example, Piaget (1950) considered conservation principles in physics (e.g. the conservation of energy) as operational invariants of physical thought: *"conservation principles constitute both absolutes of the considered reality and operational invariants of the deductive processes used to analyze this reality...".* Later, Vergnaud (1990) identifies two types of operational invariants concepts-in-action (concepts pragmatically constructed by subjects) and theorems-in-action (for example, mathematical properties, true or false, pragmatically constructed by students. In the present research other types of operational invariants were analyzed.

In Cortés A & Pffaf N. (2000) mathematical justifications of transformation rules were identified as operational invariants. The most general justifications are the conservation of the truth-value and the conservation of an identity.

The concept of numerical function and the conservation of the identity. In all definitions of numerical function two types of variables can be distinguished. For example, in the equality $f(x) = 50\ (x\text{-}2)$, x is the independent variable and $f(x)$ is the dependent variable. The definition of this function is the algebraic expression given, which allows calculating the corresponding value of f(x) for a given value of x. We analyze the meaning of the equal sign as an "identity": the numerical value of the algebraic expression will be the numerical value of f(x), for all given values of x.

Allowed transformations (factorization, development, numerical term reduction, etc.) conserve all couples $x \Rightarrow f(x)$. For example, f(x) = 50 (x-2)= 50x-100. Both algebraic expressions have the same numerical value for all given values of x: *the transformation conserves the identity*. Such transformations can be analyzed as "rewriting", at least the most elementary among them.

This analysis can be generalized to all the "rewriting transformations" used in algebraic calculations. Transformation made on only one member allow the equation to be rewritten: (e.g. (-6) (4x + 20)= 3x+100$\Leftrightarrow$ -24x-120= 3x+100), an implicit function is conserved (the left-hand member). The fact of focussing attention on a particular term implies constructing an implicit function, which is transformed (e.g. $x^2 - 4 + (x - 2) \geq 0$ $f(x) = x^2 - 4 = (x+2)(x-2)$; $(x+2)(x-2) + (x-2) \geq 0$) The implicitly constructed identity is conserved.

The conservation of the truth-value. The left and the right sides of an equation are different algebraic expressions (e.g. (-6) (4x + 20) = 3x+100). For given numerical values of x, the numerical value of each side varies according to its own law. In general, the equation can be seen as the equality of two functions. The equal sign has a different meaning in comparison to the equal sign of functions. This equality is only true for a particular value of x (the solution of the equation): the equal sign expresses equivalence for the solution of the equation. However, x can take other values and the equality then becomes false. Consequently, the concept of solution leads on the one hand to particular values of x and on the other, to the truth-value of the equation. The truth-value is a fundamental property of the equation.

Transformations which change the direction or the unequal sign of inequations (-6x < -3 ; 6x > 3) can be justified by conservation of the truth value in numerical expressions (-6 < -3 ; then 6 > 3). Allowed transformations always conserve the truth-value of the initial expression. The concept of truth-value is implicitly used by students in calculating solutions to trivial equations, by giving values to the unknown (without making transformations). Likewise, students implicitly use the truth-value of equations when they check the calculations they have made.

EXPERIMENTAL WORK

Our research is based on individual interviews concerning the solving of equations, inequations and systems of equations. We conducted: a) Ten recorded interviews with "experts" in algebraic calculations (engineers and scientists). b) Five recorded interviews with high school mathematics teachers. c) Ten recorded interviews with with10th grade technical high school students. In all cases the number of subjects was small, but was enough to identify the main characteristics of the subjects' solving methods.

The different types of operational invariants constructed by experts for algebraic calculation.

In our research, we take the theoretical concept of operational invariant farther than previous authors. Indeed, we have identified invariant tasks, in other words, tasks that the expert always undertakes (implicitly or explicitly) in algebraic calculations. Each task is undertaken by means some specific piece of knowledge or by means a competence, that we call the operational invariant of the task.

Five invariant tasks were identified: 1) The identification and the analysis of the mathematical object. 2) The respect of the priority of operations. 3) The checking of the validity of the transformation. 4) The checking of transferred terms in a new written expression. 5) Numerical calculations.

1) The identification and the analysis of the mathematical object. For these tasks the operational invariant is a concepts (equation, inequation, function, system) which allow subjects to choose a particular solving method (to define goals and a particular transformation strategy). A relevant transformation is always chosen after the analysis of the particularities of the mathematical object.

2) Operational invariant concerning the respect of the operation to be given priority. The identification (usually implicit) of the operation to be given priority allows choosing a relevant transformation (for example, factorization and development change the priority of operations and allow other transformations). There is a multiplicity of operations (square root, power of a number, additions, etc), and the operation to be given priority depends on the situation. Subjects need knowledge concerning each pair of operations involved in a particular situation; in this sense, this knowledge (operation invariant) is composite. Similar operational invariants were find by Pastré (1997), in the area of cognitive ergonomic.

3) Operational invariants concerning the checking of the validity of transformations. The mathematical justification of a transformation (for example the conservation of an identity) makes a link between the mathematical properties of the transformed object and the rule used. We consider the mathematical justification of rules as the most important operational invariant because it allows subjects to check the validity of transformations. Different types of justifications were identified:

a) Operational invariant of the type "principle of conservation". All allowed transformations conserve the truth-value of mathematical expressions (invariant property) which allows checking the validity of transformations (operational for the thought). The conservation of the truth-value (or the conservation of the identity) constitutes the most general filiation and justification of transformations; teachers do not explicitly use it. In France, only conservation of solutions is explicitly used.

b) Operational invariant of the type "self-justified or evident mathematical property" justifies one or several transformation rules. For example, transformation rules given in 8^{th} grade school textbooks (e.g. "One adds the same number to each side of the equation") later justify the "economic" rules used by students and teachers (e.g. "transfer a term to the other side of the equation by changing sign"). Another example: the reversibility of factorization justifies development and vice-versa.

c) Operational invariants of the type "theorems-in- action". For many students, transformation rules do not have mathematical justification. We consider these rules (true or false) as theorems in action (mathematical properties without justification). Some times teachers who are sure of using right properties were not able to justify the transformation rules they used; these rules are "theorems in action".

4) The operational invariant concerning the checking of transferred terms in a new written expression is a competence: subjects must exhaustively check the written terms in the new mathematical expression. Executing this task, subjects' attention frequently moves from the previous expression to the new one and vice-versa.

5) ***Numerical calculations*** were made by means of mental calculation instruments (we do not analyze these calculation in this article).

THE EXPERTS' ALGEBRAIC SOLVING METHODS

The teachers were asked to solve different mathematical objects (equations, inequations, systems d equations and inequations…). They always began by identifying the mathematical object and choosing goals (for example, to isolate the unknown in solving equations and inequations, to obtain a single unknown equation in solving systems of equations…). The analysis of the particularities of the expression allows subjects to choose relevant transformations. For each mathematical object, particular goals were defined and a particular solving method was employed (a particular organization of the solving task).

The situation is "evident" For most exercises, experts and teachers immediately chose a solving strategy. They have reached a very high degree of expertise and they do not need to justify the "economics" rules they use. Teachers use "economic" rules daily and these rules become "evident": they have the intimate convection (they believe) that they are true, its allows them to work quickly.

In solving a system of equations, for example $x + y = 24$; $50x + 100y = 1750$, teachers first verified the existence of the solution (analysis of the particularities task): the system was considered as two straight lines and teachers verified that they were not parallel. Then, a goal was stated : to obtain a single unknown equation. After transformation, the expression $y = 24-x$ (which prepares a substitution) was considered as an equivalence; x and y were considered as the coordinates of the solution point. This transition was totally implicit.

In France, teachers do not explicitly use the truth-value property of mathematical expressions. But, only the explicitation of the truth-value of the system allows exploring the double meaning of the system (obscure for most students): the points of a Cartesian representation are the truth-values of each equation and both equations are true only for a couple of values (x and y, called solution). Allowed transformations conserve the system's truth-value, then the solution.

By substitution, a single unknown equation was obtained: $50x + 100\ (24-x)= 1750$ and a goal was (implicitly or explicitly) stated: to isolate the unknown. The choice of a relevant transformation (development) implied the analysis, in general implicit, *of the priority of operations*. The development transformation was made by means of an instrument (evident rule, plus a utilization schème) and numerical calculations were made by means of mental calculation instruments. A first check was made taking into account the rule used: teachers verified multiplied terms as well as the signs of numerical results.

The mathematical justification of development was “it is the same thing”; which is to say, that the conservation of the identity (f(x)= 100 (24-x)= 2400-100x) was implicit. The checking of the transferred terms was made at the level of the method (independently of the used rule), once the equation was written. These checking tasks were made very rapidly and often subjects were not conscious of doing them. Other transformations can be analyzed in the same manner.

The situation is not evident and the property used must be justified.

Some situations are not evident for teachers and experts. For example, for certain teachers, numerical calculations of the type $((n^s)^p)$ were not “evident” problems, which correspond to an “evident” rule. Subjects hesitated between power addition ($s+p$) or power multiplication ($s{\bullet}p$). Then, they constructed a simple numerical model in order to justify the relevant operation. In this type of situation, teachers (or experts, in general) always justified the transformations they used. We consider mathematical justifications as fundamental.

When teachers were confronted with a choice of transformations (among several), they explicitly checked the validity of the transformations*: they used an “evident” of self-justified mathematical property, which they believed was true (intimate conviction)*. This aspect of the expert’s functioning constitutes a very interesting model for teaching processes, because students are often confronted with the choice of the relevant rule.

WEAK STUDENTS’ ALGEBRAIC SOLVING METHODS.

Most 10th grade students used “economic” transformation rules without mathematical justification (Cotés and Pffaf (2000)). Students’ solving methods resemble algorithms. For example in the solving of equations: first, they processed additive transformations according to the rule *“transfer to the other side by changing signs”*. When they had obtained an $ax=b$ type of equation, they then applied a multiplicative transformation: *“the coefficient transfers by dividing”*.

In general, students did not analyze the particularities of the expression to be transformed. Thus, an equation containing a product of factors (e.g. 50x + 100 (24-x)= 1750) became an insurmountable difficulty for some students. They always began with the same transformation, *“transfer to the other side by changing signs”* (e.g. 50x + 24-x= 1750-100). Students could verify that the number 100 had changed signs, but they were not able to check the validity of the transformation. Furthermore, this error suggested that the absence of a justification for the rule, which is to say, the absence of a link between the property stated by the rule and the properties of the equation prevented students from taking the priority of operations into account.

Nevertheless, most of the students were able to solve the previous equation. They used a prescription, *“first develop the parenthesis”* (teachers often make this prescription). Which is to say that the “prescription” replaced the tasks of analysis which makes it possible to choose a relevant transformation. In general, most of these students had improved their performance by adopting prescriptions or new rules without justification.

Thus, in most classes, success in solving inequations could be attributed to the adoption of a new transformation rule without justification: *“when one multiplies or divides by a* *negative number the unequal sign changes direction"*. We observed in Cortés and Pffaf (2000) that the rule was rapidly forgotten by most students and that they treated inequations and equations in the same manner.

Likewise, systems of equations were solved, in most classes, by adopting news rules without justification. In Cortés A. (1995) we observed that many students were only able to treat a particular form of system (e.g. *ax + by= c; áx + b'y= c'*) and that they failed in treating systems containing other types of expressions (e.g. *y=ax*).

In general, student progressed by matching new transformation rules to new mathematical objects; and often, the rules used were redundant. Which is to say, students often used a particular transformation exclusively for treating a particular mathematical object. Students were successful for a short time only.

CONCLUSION□

Experts' solving methods, main characteristics. Subjects identified the mathematical object and defined goals, which conditioned the solving strategy. The analysis of particularities and the analysis of the priority of operations were always rapidly and often implicitly made in order to choose a relevant transformation. Thus, the change of the operation to be given priority (introduced by several rewritten transformations) and the conservation of an identity remain implicit.

Experts' thought often consists in applying an "evident" transformation (held true by intimate conviction). Consequently, teachers did not need justify used transformations. These competencies can not be directly transferred.

For each transformation, experts used an instrument and the same instruments were used in solving different mathematical objects. Then, instruments are not associated to a particular mathematical object. Solving methods are "instrumented" schèmes.

Conclusion concerning teaching processes. Some situations are not evident for teachers: they must justify their transformations. The checking of the validity of transformations stops when subjects consider the mathematical property used to be "evident".

Our analysis of experts' solving methods led us to consider that the management of mathematical justifications is the basis of theirs methods. Indeed, justifications allow checking transformations and the checking processes involve identifying the operation to be given priority as well as the checking of transferred terms. But, which justifications should be proposed to students? We observed that certain justifications proposed by teachers (at the beginning of the learning processes) were later forgotten by most students.

In Cortés and Pffaf (2000), we analyzed a course on solving inequations; we used general justifications (conservation of the truth-value and the conservation of an identity). All students progressed in learning a new subject and weak students also progressed in a previous subject (equations): general justifications give a filiation to transformations. Otherwise, conservation of the truth-value allows justifying the transformation of inequations, systems of equations, and also allows establishing a link with numerical expressions (used as justifications).

In France, the best students acquire their skills through doing many hours of exercises at home. How can algebraic calculations be taught without asking students to make an extraordinary effort? Daily use of the most general justifications can improve the efficiency of teaching courses, because:

a) At the beginning of learning algebra, the justification must be different from the "evident" transformation rule statement. It is the only way to preserve the justification in the student's memory, because rules change.
b) General justifications can be used in all chapters and allow establishing filiations between chapters: it is possible to teach the solving of a new mathematical object and remediate previous ones.
c) General justifications may constitute the basis and the starting point of the justification process of a particular mathematical property involved in transformations.

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