THE ACTIVITY OF DEFINING

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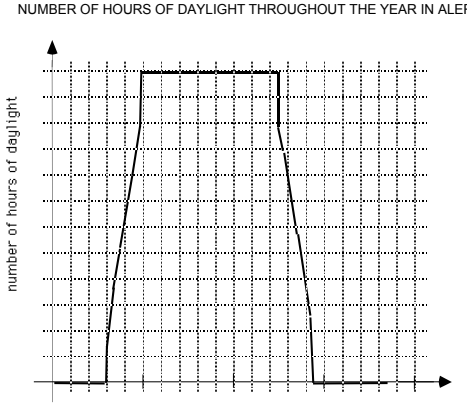
This paper presents a rationale and a conceptual framework for a wider research project dealing with mathematical communication, and in particular with actions performed by interlocutors whenever they wish to clarify their use of a symbol, a word or an expression. The aim of such actions is often to repair a communicational breach resulting from differences in the interlocutors' uses of words. As was found in our study, only some of the defining actions would result in texts known as mathematical definitions. The point of departure of our research project is that the effectiveness of the defining actions is as much a function of the action itself as of its contexts. Our focus in this research is thus broader than in the past studies on definitions, and includes the when and why of defining along with their how.

Today, it is a common belief that learning with peers in small groups has many advantages over frontal learning, where the teacher is often the only speaker. And yet, such face-to-face interactions would sometimes be ineffective and, as such, would be lacking the basic feature that is a necessary condition for successful learning. Let me begin with an example of a situation where the participants fail to communicate[1]. In the episode presented in Fig. 2, two 7 grade students are working together to answer question 3 appearing in Fig. 1.

The number of hours of daylight on any given day is a function of what day it is in the year, and of the latitude of the location. The number of hours of daylight in Alert, NWT (near the North Pole) was recorded every day in 1993. The graph below shows the information.



Describe what happened to the number of hours of daylight over the year by answering the following questions.

1. How many hours of daylight were there on January 1, 1993? ____
2. For how many days did this occur before there was a change in the number of hours?
3. During which period of time did the number of hours of daylight increase most rapidly?
   From day _____ to day _____

Figure 1: Daylight Episode, Activity sheet

At the first glance (see Figure 2 below) it looks like the boys are trying to collaborate in solving the task. At a closer look, we see a communication breach that persists all along the episode. This miscommunication is clearly apparent to both students, as they question

[1] This example appears in Sfard & Kieran (2001) in a different context.

each others solutions and cannot agree about the answer (see, e. g. [14]-[20]). They do try to repair the breach but their attempts are ineffective.

What is done	What is said
	[12] A: 60 to 100. From day 60 to 100
	[13] G: Cause, Oh no, no, no no no. Look, look. Up here. It's day 100 to day --- to day
	[14] Ari: What are you talking about?
	[15] Gur: 55.
	[16] Ari: Where?
[17]"here and here": G. points back and forth several times to the extremes of the upper horizontal line, about (100,24) to (250,24)	[17] Gur: Look, it changed most rapidly in between here and here. You see?
	[18] Ari: Oh? It's exactly the same.
[19]G. traces the "descent" of the line from x=100 to 0.	[19] Gur: No, because see, it moves up (mumble)
	[20] Ari: It goes up most rapidly
	[21] Gur: So it's from day 100
	[22] Ari: To day 100
[23]G. is still pointing at about 250 on x axis.	[23] Gur: No, from day 100 to day ---
	[24] Ari: No, No, No.
	[25] Gur: 2 hundred and sixty,
	[26] Ari: That's not how you're supposed to do it.
	[27] Gur: two hundred and eighty. To day
	[28] Ari: See, during which time. The time, the period of time has to change rapidly.
[30] A. underlines "time" on G's question sheet.	[29] Gur: Oh. No, it says from day to day what?
[31] G. traces horizontal line at 24 hours	[30] Ari: Read the question. During which period of <u>time</u> - time.
	[31] Gur: Up here, time.
[32] "that didn't change: here A. traces horizontal line at 24 hours. "right here": A. puts pencil mark along graph from y=20 to 24.	[32] Ari: No, but that didn't change, it stayed still, which means it has to be right here,
	[33] Gur: No
	[34] Ari: which is about 90
[35] G. traces ascent of graph from 0 to 100	[35] Gur: Right here
[37]G. traces a curve along graph from 0 to about 250.	[36] Ari: No, right here
	[37] Gur: You don't get it, do you? If it was like this
	[38] Ari: Fine, it's from 60 to a hundred, ok?
	[39] Gur: No.
	[40] Ari: Yes. I'm writing that.
	[41] Gur: Why?
	[42] Ari: We can have different answers.
	[43] Gur: Why? I don't care.

Figure 2: Daylight Episode

The extensive use of the indexical *it* in [17] and [18] is one of the reasons. The boys do not employ the word in the same way. In [17], Gur says 'it changed most rapidly' while pointing to a part of the graph that represents a constant function. It is thus plausible that

while using the word *it* Gur refers to the graph itself rather then to a mathematical object (function) which the graph is supposed to represent. Consequently, he tries to interpret the terms appearing in question 3 as referring to properties of the graph. However, the graph is a stable object and the terms *rapid* and *increase* both refer to processes. It seems that Gur helps himself out of the dilemma by interpreting the words 'increase most rapidly' as 'there is the most *extreme* change in the shape' (of the graph). To sum up, in [17] Gur uses the word *it* as a substitute for *the graph*.
In contrast, Ari seems to be using the words 'increase most rapidly' in the way intended by the authors of the worksheet, and thus in his case, the word it in [18] comes to replace the number of daylight hours (as represented by the graph). In [19] and [20] each of the boys continues to use the word it in his own way. In [28] Ari performs a defining action: he repeatedly stresses that it is 'the time, the period of time [that] has to change rapidly', thus clarifying retroactively that the word it in [18] had to do with time, or period of time, rather than with the graph as such. This clarification does not seem to work for Gur, so Ari takes a more direct attempt to focus Gur's attention on the use of the word time ([30]). He does this by pointing to the written word time in the worksheet. In [32] the boy tries yet again to repair the miscommunication, but he uses the indexical that instead of explicitly saying that the number of daylight hours is what he has in mind. As a result, also this attempt at straightening things out proves futile.2

To sum up, throughout the episode Gur refers to the graph itself while Ari speaks about what is represented by the graph. The boys do not seem to be aware of this significant difference. Because of cases like this, the cases in which the participants use the same words in different ways, I undertook the present study. My focus is at what will be called here activity of defining - the activity that aims at keeping communication effective by clarifying uses of words. Defining actions are a natural answer to the type of communication breaches we saw in the Daylight episode. Yet, as the example shows, such actions do not have to be effective. This observation is a point of departure for the questions I ask in my study: What types of defining actions people use to perform spontaneously? Why is it that these attempts are often ineffective? Can the skills of defining be taught? Let me stress that the focus of the study is on the actions of defining and not just on their products. In particular, I will be looking at defining actions performed in mathematics classrooms, but not only on those that lead to mathematical definitions. Of course, mathematical definitions will be dealt with as one of many possible products of such an action. The unique role of such definition in keeping the mathematical discourse effective will also be discussed. And yet, the scope of defining actions to be considered in the study is much broader than that. The research project

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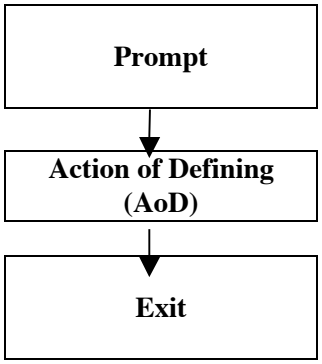
[2] Another possible cause of confusion might be the fact that Ari uses the word *time* ([28] and [30]) in two different ways. In [28] he states: *The time, the period of time has to change rapidly*. The reference to a change in time suggests that he may, in fact, use the word time as a shortcut for *the number of hours of daylight per day*. Yet, in [30], Ari uses this same word as it is asked for by question 3 - *the time in year during which the number of daylight hours increase most rapidly*. His confusion is probably the result of the fact that the word *time* has a double meaning here and may be measured in days along the x-axis, and in hours along the y-axis.

devoted to the activity of defining deals above all with the question of when defining actions are undertaken by interlocutors and of what makes them effective or ineffective. In this paper, I will present only a very small part of this larger study.

COMMUNICATIONAL BREACHES AND EVENTS OF DEFINING

As was shown, the reason for the salient difficulty observed in the Daylight episode was the fact that Ari and Gur used the same words (increase most rapidly, time) in different ways. This type of communicational breach is not limited to mathematics. Indeed, this is a very common type of miscommunication that occurs in other types of discourse as well. One way to deal with situations like this is to engage in the activity of defining: The interlocutors have to make a transition to meta-discourse, that is, to the talk about their talk, in order to explicitly discuss and coordinate their uses of words. Two conditions must be fulfilled if such action is to happen and to be effective: First the participants must realize that the reason for the breach is their different uses of the same words, and second, they must arrive at an explicit agreement about the words' use. If so, in a study like this two questions must be considered: First, what prompts people to undertake defining actions? And second, how do they do this? The first question deals with the when of defining, whereas the second one with the how.

To address these two questions I will examine closely discursive events in which people try to communicate their use of words. These events will be calles *events of defining* or *EoD*s, for short. An EoD consists of those three components:



A discursive action that causes interlocutors to undertake a defining action aimed at a given signifier (a symbol, a word or expression).

All the discursive actions that aim at clarifying the use of a symbol, a word, or an expression.

A discursive action that is regarded by interlocutors (and by the observer) as showing that the event of defining has terminated.

A very simple example of EoD appears in Figure 33.

Utterances	Activities	Analysis
Ron: What is a cone?		**Prompt**
Iris: It's this clown's hat over here.	Iris picks a cone from a set of solids.	**AoD**
Ron: Ah, ok.		**Exit**

Figure 3: An EoD - a cone

In this example, Ron's request for a clarification regarding the use of the word *cone* prompts Iris to perform the *action of defining* that involves a *metaphor*[4] (Iris names the

[3] EoD may be much more complex, as discursive events of defining have a recursive structure, that is, in any such events other EoDs may be nested.

[4] A metaphor is using words from one discourse in a different type of discourse.

cone a clown's hat) and an *ostensive definition*[5] (she picks up an example for a cone).
Ron *closes* this event by accepting the AoD performed by Iris and thus signaling that an action of defining is no longer needed.
Referring to the three components of the EoD enables us to deal not only with the *how* but also with the *when* of definitions: by studying the AoD we learn about *how* one defines, and by scrutinizing the prompt and exit we get to know *when* one turns to an AoD, how she chooses the specific defining action, and how she decides that the process of defining can be regarded as completed.
AoD is a meta-discursive action supposed to inform others about the way one uses a word. Making the rules of the word use explicit is supposed to allow the interlocutors to mutually adjust their discursive ways and in particular, to coordinate their use the words. Defining actions performed in real-life conversations are many and diverse, and they include pointing (*ostensive defining*), explaining by examples or metaphors, enumerating all the elements of the category, presenting necessary and sufficient conditions for an element to be included in the concept, etc. In mathematics, only the last type on our list counts as the proper type of defining action. In order to get to know all this diversity, there is a need for a very rich and diverse collection of data. In our study, in addition to extensive analyses of whole-class and small-group classroom interactions, we are looking at conversations between teen-agers at home and at youth movement meetings, at exchanges typical of TV and radio talk-shows, at discussions held in different committees in the Knesset (Israeli Parliament), and at problem solving activities the participants of which would not be able to proceed without making explicit decisions about their use of words. Numerous samples of all these types of discursive activity have been video-recorded and transcribed and they constitute the data base of this study.

STUDYING DEFINING IN CONTEXT

Much attention has already been given by researchers to the products of AoD known as definitions and to the ways in which these products are used (or not!) in school students' discursive activities. In this research project the focus is at the action of defining itself, and much attention is given not only to the how of this action, but also its when and why. In this way I hope to be able to answer some of questions that my own former research left open (Nachlieli, 1997). One of these questions concerns the ineffectiveness of mathematical definitions. More specifically, the problem under study is "What is it that often makes mathematical definitions ineffective?" Let me present this question in more detail.

The mathematical definition, which is one of the products of AoDs, is supposed to be the perfect answer for keeping communication effective. It seems that using this type of definition, as is the case in mathematics, increases the chance that all participants will start using the defined word in the same way. Mathematics is an attempt of mathematicians to create a perfect discourse where there are no ambiguities and where, in particular, no place is left for differing uses of the same words. And yet, as has been eloquently argued by philosophers of science on the one hand (see e.g. Lakatos, 1976), and by mathematics education researchers on the other hand, this hope is rather naïve,

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[5] The term *ostensive* means: by pointing, by showing physically.

since it does not into take into account the complexity of human discourse and its essential dependence on contextual factors.

Indeed, past research on definitions and on their role in constructing concepts shows that formal mathematical definitions fail to determine students' use of words even when the students know these definitions (Hershkowitz & Vinner, 1984). Some of the studies simply document the existing gap (Hershkowitz & Vinner, 1984; Wilson, 1990), whereas others attempt to explain the phenomenon (Fischbein, 1993; Vinner, 1990; Fischbein, 1996). Various theories of learning imply that explicit definitions should be useful in constructing concepts, but different studies show that in many cases the definition is not helpful at all and that the students tend to ignore it even when they know it by heart (Hershkowitz & Vinner, 1983). What we found so far confirms all this and more: It not only shows that mathematical definitions are not always listened to, but it also makes us aware that the effectiveness of defining actions cannot be foreseen just by examining their final products. Let me show some evidence.

First, let us look at the case where mathematical definition turns ineffective. In the following episode Noam, an 11th grade student, is asked whether the shape presented Figure 4 is a kite. The following definition is written in front of him: A kite is a quadrilateral in which there are two pairs of equal adjacent sides6.

WHAT IS DONE	**What is said**
Points to that figure: 	[1] Teacher: Is this figure a kite?
	[2] Noam: No
	[3] Teacher: Why not?
	[4] Noam: Because it doesn't look like one.
	[5] Teacher: What does a kite look like?
Points to that figure: 	[6] Noam: like this
	[7] Teacher: What is the definition of a kite?
	[8] Noam: Reads the definition. According to the definition this is a kite but I know it is not.

Figure 4: the Kite Episode

Noam is familiar with the definition and confirms that the discussed figure fulfills the demands dictated by the definition ([8]). Yet, he refuses to accept the definition as the ultimate touchstone for determining whether the given case belongs to the category of kites. It seems that Noam identifies a figure to be a kite in a direct way, that is, his decision about naming is not mediated by any definition. His tendency for the direct identification is so strong that it seems as if the naming act was not a matter of a mere

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[6] In Hebrew there are two different words for the geometric figure that fulfills the given definition (Dalton) and for the flying object (afifon).

recognition, but rather of discovering the "real nature" of the figure. In situation like this, when the name seems to be a part of the thing itself, the explicit definition has little influence on the way in which the word is used.

In contrast, there are defining actions which, while theoretically insufficient and with little explanatory potential, would nevertheless fulfill their communicational goal. For example, in the episode in figure 3 Iris uses a metaphor and pointing (ostensive definition) to define a cone. These two defining actions, if analyzed independently of their context, seem like having a very small chance for doing their job properly. The fact that this AoD actually works makes it clear that the effectiveness of defining actions can by no means be seen as a straightforward result of the quality of the final definition. In other words, there is no point in trying to evaluate defining actions just by looking only at their textual products (the final definitions) and without considering such contextual factors as the history of the conversation, the common discursive habits of the interlocutors, and more.

All this shows that if the study of defining is to bring any useful results, it has to be conducted in as natural circumstances as possible. In the present context, the expression "natural circumstances" refers to situations in which people undertake defining actions spontaneously, in order to overcome naturally occurring communication breaches. As it turns out, this kind of study is quite difficult to perform because of the fact that people do not engage in defining actions as frequently as could be expected. Indeed, our data so far have shown that interlocutors do not seem too eager to reflect on their uses of words even when their communication limps and becomes obviously ineffective. The discursive activity of defining seems to be pushed aside by our strong tendency to use words in a direct, unmediated manner, without accounting for this use and without monitoring its appropriateness. This inclination for unmediated, spontaneous use of words is the basic characteristic of human communication. And no wonder: after all, the directness is the condition for the very possibility of communication. Indeed, just imagine ourselves deliberating on words' definitions before actually using them. Our fate would be very much like that of the famous centipede who, while asked to think about the way it moved its one hundred legs lost the ability to move. Thus, reflecting on the choice of words before actually putting them into our sentences seems opposed to our most deeply rooted discursive habits. This would be enough to explain why explicit negotiations of words use may be a difficult task for most interlocutors. The additional obstacle stems from the fact that the directness of our choices of words comes together with the deep sense of their uses being extra-discursively determined. Like in the anecdote on a child who was able to understand how astronomers discovered new stars but still wondered how they discover these stars' names, we sometimes have the feeling that things simply come with their names, and that no human definition can change it.

The above observations bring to mind Vygotski's (1987) famous distinction between spontaneous and scientific concepts, made according to the way in which these concepts are learned. Similarly, we can distinguish between spontaneous and scientific uses of

words[7]. The spontaneous use develops through interactions with others, when we pick up discursive ways of our interlocutors. This learning by mimicking happens as if by itself, imperceptibly to ourselves. Scientific use, in contrast, must be deliberately taught. Its learning occurs not just by practicing word use, but also by reflecting on this practice. Explicit defining is a necessary part of this learning. In schools, one's spontaneous uses of words are supposed to be translated into scientific. For this modification to happen, the students will have to learn to suspend their spontaneous discursive decisions for the sake of reflective, meta-discursively mediated choices of words. This, as was already observed, is a difficult thing to learn. The overall aim of my study is to understand the mechanisms of words use and their relation to the activity of defining deeply enough to be able to propose ways for improving students' communication in large, and their mathematical communication in particular.

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[7] According to Vygotsky (1987), *concept* is a word together with its meaning. If combined with Wittgenstein's (1953) interpretation of the notion of *meaning*, the term concept becomes tantamount to the *word together with its discursive use.*