NOTATION ISSUES: VISUAL EFFECTS AND ORDERING OPERATIONS

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Abstract: *This paper reports on part of a wider study on the teaching and learning of the conventions of formal algebraic notation. Through the analysis of paper tasks given to teachers I argue that not only does the inherent mathematical structure have an effect on the way in which an equation is manipulated but also the visual impact of notation itself. Students' interpretation of arithmetic equations written in words and then structurally similar equations written in formal notation raise issues about when students perceive order which is not strictly left-to-right and the possible significant role the equals sign has in this.*

Hughes (1990, p132) gave an example of a six year old, Andrew, who used bricks to work on the problem '13 □ 5 = '. Andrew made three piles of bricks containing one, three and five bricks and then proceeded to add up the bricks to make his answer of nine. When confronted with notation, all anyone can do is try to relate that notation to existing knowledge or awareness and this boy did just that. There is nothing within the notation itself which would tell this child he was wrong. Notation is arbitrary (Hewitt, 1999) in the sense that it is a social convention of arrangements of symbols where there is no necessity for the notation having had to be the way it has turned out. Decisions were made a long time ago and certain decisions have been accepted within the community of mathematicians. According to Walkerdine (1990, p2) *Saussure is credited with recognizing the importance of the fact that the relationship between the signifier and the signified is arbitrary; that is to say, conventional rather than necessary*. Meaning has to be brought to the notation. Goodson-Espy (1998) talked of often coming across issues about student interpretation of notation when trying to explain student learning of algebra. Elsewhere (Hewitt, 2001) I have argued that some of the difficulties within algebra might be due to difficulties with notation per se rather than mathematical notions themselves. The fact that Andrew ended doing an addition of $1+3+5$ when confronted with '13 □ 5 = ' does not tell us anything about whether he can subtract five from 13.
With regard to the mathematics, we only learn that he can add up those numbers successfully. Any difficulty he might have had could just be about interpreting the notation, rather than the mathematics of carrying out such subtractions.

Students can create their own notations for arithmetic operations from an early age (Hughes, 1990; Steffe and Olive, 1996). Attaching a label or symbol to a concept is a natural activity with which young children engage. The act of symbolising is not so much the issue, as Sáenz-Ludlow and Walgamuth (1998, p153) pointed out *the inverse process of interpreting symbols to unfold mathematical concepts challenges all learners*. It is this inverse process of interpreting someone else's notational conventions which often causes difficulty. Sfard and Linchevski (1994) pointed out that algebraic thinking appears before notation – both historically and with learning. For the creators of notation, they already have an algebraic awareness which they are choosing to notate. Which notation they choose is a matter of preference, not a matter of right or wrong. However, for inheritors of someone else's notation, all they are given is a collection of symbols and they are left with the much harder task of searching through their own knowledge and awareness to find something with which they can relate to the notation. For learners this is not a matter of preference as it is often judged by others as right or wrong. For a learner the formal notation can be a barrier to engagement in mathematical problems and can obfuscate otherwise accessible mathematics.

Deacon (1997) pointed out that learning any language is not as simple as a one-to-one correspondence between signifier and signified. He talked of moving from associations between symbols and objects, to relationships between symbols. This is certainly significant with formal notation as there are not only associations between a symbol such as '2' with the mathematical notion of two, or '+' with addition, etc, but also a symbol has varying significance dependent upon its position in relation with other symbols. For example, the meaning of '2' in '24', '42' and '4^2' changes due to its position in relation to the '4'. As Mercer (2000, p67) pointed out *words gather meaning from 'the company they keep'*. Kirshner (1989b) argued that many students develop meaning for ordering of operations in an expression such as $3 + 2x$ through the visual impact of the position of symbols rather than through analysing the associated mathematical content of operations. For example, the '2' and the 'x' are closer together than the '3' is to the '2'. He proposed *that attaining and expressing algebraic skill is principally a linguistic rather than an intellectual exercise* (Kirshner, 1989a, p38). Perhaps a greater awareness is required of the pedagogic job to be done to help students develop linguistic skills of reading and writing algebraic notation as well as working on the algebraic notions that end up being expressed in formal notation.

THE STUDY

This paper reports on part of an ongoing study into how some students are interpreting and using formal algebraic notation before they have had many lessons on the topic of algebra within their secondary schooling. A wider aspect of the study, not reported here, involves observation of students working with recently developed software which is aimed at helping them to create and work with formal notation. Additionally, I am interested in how teachers themselves work with notation when solving some algebraic tasks and their opinions about working with students on formal algebraic notation. I used a questionnaire for teachers which included asking them to solve specially chosen linear equations and to answer various questions on notational matters. I also asked students to complete two written tasks. The first involved a series of arithmetic equations written in word form. Students were asked to state whether they felt the statement was mathematically correct and then asked to write the statement out (whether correct or not) as a formal mathematical equation. This was completed before the second task was given out, which involved a series of arithmetic equations, this time written in formal notation. The task was for the students to put a tick or a cross next to each one depending upon whether they felt the equation was correct or not. Each word statement in the first task sheet was mathematically equivalent in form to a statement written in formal notation on the second task sheet. The numbers involved in these equations were deliberated kept small in an attempt to reduce arithmetical difficulties.

The analysis of data was carried out in two ways. First, the design of some of the statements in the tasks and questions on the questionnaires were such as to test pre-study conjectures about some effects notation can have on the way notation is interpreted and worked with. Second, I used a mixture of collecting quantitative information and also coding some of the students' formal notation versions of the word statements. As per Grounded Theory (Glaser & Strauss, 1967) the coding was not determined in advance. However, the development of classifications was based more upon a Discipline of Noticing (Mason, 2002) framework where I used my initial awareness of related issues in order to notice similarities and differences and thus begin classification. This in turn sharpened my sensitivities and helped me to notice other features which led to further classification and re-forming of previous classifications. This is a cycle of using awareness to notice, which in turn sharpens sensitivities to related issues, which in turn informs my awareness to help further noticing, etc.

The focus of this paper is on the way notation is interpreted, created and used by teachers and students in the study. To this end I will focus on preliminary results from 40 teachers and from one particular class of Year 7 (11-12 year old) students.

RELATIONSHIP BETWEEN MATHEMATICS AND NOTATION

Notation not only can have an effect on the interpretation of a formal algebraic statement, but it can also have an effect on the way in which someone works with, and manipulates, algebraic statements. I asked teachers to solve some equations, two of which were:

$$\frac{13}{2kx} = 47 \qquad \text{and} \qquad 13 - (2 + k + x) = 47$$

The first involves multiplication and division, the second involves addition and subtraction. As multiplication is to division what addition is to subtraction, there is an certain isomorphism between these two statements in a structural sense. Although they have similar structure, there is a significant difference in the visual impact of the two equations. I looked at the first line of the teachers working as they solved each of these equations for x. Just over a quarter of the teachers moved the x in the first equation on its own without moving the k or the 2, such as $\frac{13}{2k} = 47x$. When solving the second equation, the x was never moved on its own until later lines of working when the bracket was no longer present. I argue that although the mathematical structures are equivalent, there is a significant difference in the visual structures and that it is the visual structure which affects how someone is likely to manipulate the equation as well as the mathematical structure. In the first equation although x is adjoined to k there is still 'free space' for it to take a journey to the right-hand-side of the equation. On the other hand, the second equation has x held within brackets which visually holds and keeps the x together with the k and 2. It is not until the bracket disappears (usually through either distributing the subtraction across the whole bracket, or by taking the whole of the bracket across to the other side) that the x is 'visually free' to be manipulated on its own. Thus the notation affects the way in which the equation is manipulated.

THE EQUALS SIGN

The imbalance of the '=' sign is well known (for example, Boulton-Lewis et al, 2000). Through the practice of teachers and textbooks of presenting 'the answer' on the right-hand-side of an equation, the '=' sign has become to take on a meaning different to that of indicating that two sides are equivalent. Electronic calculators have reinforced this situation due to the temporal order in which buttons are pressed and the fact that an 'answer' is produced following the pressing of the '=' sign. This is one reason which can account for students feeling that equations with the right-hand-side involving an operation are somehow unfinished (Lack of Closure - Collis, 1975) whereas they can accept an operation on the left-hand-side. This is supported by the fact that on average over 20% of students transposed the word statements where a single number was on the left of *equals*, into notation statements with the single number on the right of the '=' sign. This is not something that just students do. There were two linear equations teachers were asked to solve for x where it is perhaps 'natural' for the x to finish up on its own on the right-hand-side of the equation, for example $\frac{13}{2kx} = 47$. However, in 49 out of 80 cases the teachers changed the sides of the equation in their working so that the x ended up on its own on the left-hand-side (i.e. $x = \frac{13}{94k}$ rather than $\frac{13}{94k} = x$). This appears to support the conjecture that having what is perceived to be 'the answer' on the right-hand-side (in this case the answer to the question *what does x equal?*) is something which stays with teachers as well as students. This helps fuel a self-perpetuating cycle with teachers offering examples with the 'answer' on the right-hand-side of the equals sign and so students abstract from such examples that there is a rule that *equals* is always followed by the 'answer'. Some of these students then becomes teachers and the cycle continues.

ORDER WITHIN STATEMENTS

Students were presented with arithmetical equations written in words. Generally students ordered the operations as they read them - left to right. For example, out of 29 school students, 20 read the statement *Two plus one times three equals nine* as correct and 24 said the statement *Four times two add three equals twenty* was wrong (see Table 1). The final statement of 17 presented to students was *Three plus two times four equals eleven* and I expected there to be a clear majority of students who felt that this statement was wrong, which would support the left-to-right ordering of the operations. However, the results showed that students were divided between whether they felt this was correct or not: 15 said correct; 14 said wrong. So what might account for the difference between this statement and the two earlier ones? Why did so many students think that this statement was correct? One possibility was that if the word statement was literally translated into a symbol statement without any brackets involved, it would be $3 + 2 \times 4 = 11$ and with the convention that multiplication is carried out before addition, then this would make it a statement not ordered left-to-right and the statement would be correct. However, the same would have been true with the first statement and by far the majority of students placed a left-to-right ordering on the operations.

Word statement and notation statement	Left-to-right ordering	Non left-to-right ordering	Unsure
Two plus one times three equals nine $1 + 3 \square 2 = 8$	20 14 (correct)	8 16 (wrong)	1 0
Four times two add three equals twenty $2 \square 3 + 4 = 14$	24 26 (wrong)	5 4 (correct)	0 0
Three plus two times four equals eleven $4 + 2 \square 3 = 10$	14 11 (wrong)	15 19 (correct)	0 0

Key: ct' conventional answer (if word statements were written in symbols without brackets).

Table 1: Different readings of three word statements and structurally equivalent notation statements

I could see little difference in the structure of the two statements which makes this last one stand out as being more ambiguous than the first. So I looked at the statements which appeared prior to this final one. There were four statements each of the form:

NUMBER operation NUMBER *equals* NUMBER operation NUMBER

In fact these were the only occurrences of statements where there was not a single number on one or the other side of the word *equals*. In such situations, the word *equals* can break up the flow of left to right by creating a new beginning after the word *equals*. These questions were invariably answered with success and so the students were able to read the statements as having a break in the flow and a new beginning part way through. This is exactly what is required in order to interpret *three plus two times four equals eleven* as correct, by having a new beginning with *two times four*, rather than carrying out the initial *three plus two*. Thus I conjecture that a number of students had become practised at making such a mental break in statements and could see a way of creating a break in the final statement to make that statement correct.

In the sheet of statements written in formal notation, each statement was structurally equivalent to the statements on the word sheet. Table 1 also shows the statements which appeared on the notation sheet underneath their equivalent word statements. There are several possible interpretations of the figures for these notation statements. One interpretation is that since the sheet with statements written in formal notation was completed immediately after the word statement sheet, any awareness of the possibility of breaking the flow from left-to-right might be carried into these statements, hence the increased number of students opting for a non left-to-right ordering. Another interpretation is that students were more aware of conventions of order when reading formal notation than when reading a word statement, and so generally more students agreed with the conventionally correct interpretation. Either way, it is still the case that more students decided upon a non left-to-right ordering in the last statement. Again, the four statements before this last one on the notation task sheet were of the format $a \oplus b = c \quad d$ (where $\oplus$ and stand for some operation) which students almost universally answered correctly and which 'enforced' a break in a strict left-to-right ordering.

SUMMARY

Learning to use formal notation involves not only developing meaning for symbols but also developing meaning for the positioning of those symbols in relation to other symbols. The symbols within an algebraic equation and their relative positions produces a visual impact which affects the way those equations are manipulated when re-arranging the equation. In particular, brackets have a visual effect of holding symbols together and these are unlikely to be separated until other manipulations have first been carried out effectively to make the brackets disappear.

Equations where operations are carried out on both sides of the equals sign provide a source of examples whereby a natural left-to-right reading is broken up by the equals sign and a new beginning is established. Early data gained raises the issue of whether students experiencing such enforced breaks might carry over that experience to consider such breaks and new beginnings in equations where the convention is not a strict left-to-right ordering of the operations involved. An additional pedagogic benefit of students meeting equations with operations on both sides of the equals sign is that such examples can help to prevent the otherwise self-perpetuating practice of both students and teachers always following *equals* with an 'answer'.

References:

Boulton-Lewis, G. M., Cooper, T. J., Atweh, H., Pillay, H. & Wilss, L. (2000), 'Readiness for algebra', in T. Nakahara & M. Koyama (eds), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* Volume 2, pp89-96, Hiroshima University, Japan.

Collis, K. F. (1975), *The development of formal reasoning*, Report of a Social science research Council sponsored project (HR 2434/1), Newcastle, NSW, Australia: University of Newcastle.

Deacon, T. W. (1997), *The Symbolic Species: the co-evolution of language and the brain*, New York: W.W. Norton & Company.

Glaser, B. G. & Strauss, A. L. (1967), *The discovery of grounded theory: strategies for qualitative research*, Chicago: Aldine.

Goodson-Espy, T. (1998), 'The Roles of Reification and Reflective Abstraction in the Development of Abstract Thought: Transitions from Arithmetic to Algebra', in *Educational Studies in Mathematics* 36, pp219-245.

Hewitt, D. (1999), 'Arbitrary and Necessary: Part 1 a Way of Viewing the Mathematics Curriculum', in *For the Learning of Mathematics* 19(3), pp2-9.

Hewitt, D. (2001), 'On learning to adopt formal algebraic notation', in Chick, Helen; Stacey, Kaye; Vincent, Jill & Vincent, John (eds), *Proceedings of the 12th Study Conference of the International Commission on Mathematical Instruction: The Future of the Teaching and Learning of Algebra*, Volume 1, pp305-312, The University of Melbourne, Australia.

Hughes, M. (1990), *Children and Number. Difficulties in Learning Mathematics*, Oxford: Basil Blackwell.

Kieran, C. (1996), ‘The changing face of school algebra’, in Claudi Alsina, Jose Ma Alvarez, Bernard Hodgson, Colette Laborde & Perez, Antonio (eds) *Eighth International Congress on Mathematics Education Selected Lectures*, Seville, Spain: S.A.E.M. ‘THALES’, pp271-286.

Kirshner, D. (1989a), 'Abstract impressionist algebra', in C. A. Maher, G. Goldin & R. B. Davis (eds), *Proceedings of the Eleventh Annual Meeting, North America Chapter of the International Group for the Psychology of Mathematics Education* Volume 1, New Brunswick, New Jersey.

Kirshner, D. (1989b), 'The visual syntax of algebra' in *Journal for Research in Mathematics Education* 20(3), 274-287.

Mercer, N. (2000), *Words and Minds*, London: Routledge.

Sáenz-Ludlow, A. & Walgamuth, C. (1998), ‘Third graders’ interpretations or equality and the equal symbol’, in *Educational Studies in Mathematics* 35, 153-187.

Sfard, A. & Linchevski, L. (1994), 'The gains and pitfalls of reification - the case of algebra', in *Educational Studies in Mathematics* 26, 191-228.

Steffe, L.P. & Olive, J. (1996), ‘Symbolizing as a constructive activity in a computer microworld’, *Journal of Educational Computing Research* 14(2), 113-138.

Walkerdine, V. (1990), *The Mastery of Reason: cognitive development and the production of rationality*, London: Routledge.