EFFECTIVE VS. EFFICIENT: TEACHING METHODS OF SOLVING LINEAR EQUATIONS

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The choice of teaching an effective method—one that most students can master—or an efficient method—one that takes the fewest steps—occurs daily in Algebra I classrooms. This decision may not be made in the abstract, however, but rather in a ready-to-hand mode. This study examines how teachers solve linear equations when the purpose is pedagogical and when it is mathematical. Their answers reveal a definite preference for students to solve linear equations in a standard order even if it is not the most efficient way. The most efficient solutions show much more variation, but these answers still exhibit a clear tendency to follow the standard order in general.

INTRODUCTION

Many discussions of mathematics teaching and learning set up a dichotomy between conceptual and procedural learning based on the work of Hiebert (1986). Star(2000, 2002) examines a third alternative of "intelligent" procedural knowledge or "procedural understanding." This type of understanding, he contends, leads students to more efficient methods for solving routine problems. Procedural understanding also makes students more flexible and innovative with greater understanding of algebraic processes. Star's work looks at a way to enhance or develop this type of understanding in students. However, he takes as unproblematic that teachers recognize and value the most efficient methods for solving linear equations. He also contends that there is a standard way to solve linear equations taught in the United States. I examine these two issues from a teacher perspective. While there may be a general method that teachers present for solving linear equations, their own work may not be constrained by that method. Furthermore, teachers have competing pedagogical demands of effectiveness and efficiency when teaching students how to solve linear equations. In this paper, I theorize that teachers may value effective techniques more than efficient for students' solutions to linear equations.

THEORETICAL FRAMEWORK

Teachers, like all of us, operate in a world with many demands. They must teach particular content—at least in most schools—that will be tested in procedural ways, and at the same time, they must help students become mathematically literate problem solvers. This work occurs within a culture bounded by the nation, the state, the community, the school, and the department within which they teach. Recognizing that teachers must often manage dilemmas, as Lampert (1985) describes, brings pedagogical decisions to the forefront in any discussion of what happens in a classroom. Working from the hermeneutical tradition of Heidegger and Gadamer allows an examination of teacher actions and decisions in particular instances without losing sight of the larger context within which they work (see Williams, Walen, & Ivey, 2000). The pedagogical decisions that teachers make are bounded by what they perceive as necessary and possible—their hermeneutic "horizons" (Gallagher, 1992). What they come to know about teaching is always done in context, never in isolation. Furthermore, many pedagogical decisions are made in a "ready-to-hand" or unreflected upon manner. Thus not all choices are based on what is mathematically best or most efficient. Many decisions of what and how to teach are based on what is most effective for producing procedural knowledge, particularly in basic classes such as Algebra I.

Historically, we have known that a dilemma exists between effective and efficient methods. Tall (1989) points out that a student's ability to conceive of algebraic expressions as objects is an important step in moving from arithmetic to algebraic reasoning. But the ability to recognize algebraic expressions as chunks to be treated as a single variable is closely related to how recently students have worked with this kind of activity (Linchevski and Vinner, 1990). At the same time, Wenger (1987) describes the inability of students to make good choices about what to do next in simplifying expressions. These findings delineate the dilemma for teachers—how to encourage strong algebraic reasoning in students while making sure that they can actually simplify expressions and solve equations.

Teachers' own understanding of methods for solving linear equations may be flexible and innovative in Star's (2002) use of these terms, but their expectations for students may not be. By examining teachers' solution methods under different circumstances, the dilemma can become overt, and their "ready-to-hand" pedagogical choices become "present-at-hand" or objects available for examination.

METHODS AND DATA SOURCES

Twenty pre-service and in-service teachers volunteered to participate in this study. The in-service teachers included middle-grades, high-school, and community-college teachers. Each participant completed a three-page questionnaire anonymously. The various pages of each questionnaire were coded for identification, but participants' names were not recorded. Page 1 contained questions about demographic information including number of years of teaching experience, number of years of beginning algebra teaching experience, mathematical preparation, and level taught. Other questions asked participants i) to "list the steps that a student would need to know how to do to be able to solve any linear equation," ii) if there were a standard order for performing those steps, and iii) if their textbook presented a standard order. Additionally, on pages 2 and 3, participants were asked to solve linear equations "showing all the steps that you would want your students to show" and "in the most efficient way," respectively. Figure 1 contains the questionnaire condensed by removing the space for answers.

Each participant completed the questionnaire sequentially with no knowledge of what the next page would ask, nor with an opportunity to revise any answers. The resulting work was coded for demographic responses and for operations performed in each step of the various problems. A step was construed to be what was completed between one line of written symbols and the next. All coding was completed by the author. There were no ambiguous cases of what constituted a step. Comparisons of the methods for solving each equation were made across participants on each question, and methods for solving equations were compared across questions for each participant. These comparisons resulted in several themes emerging from the data. The primary results are considered in the next section.

Page 1

1. I teach at ____middle school ____high school ____community college ____ college or university
2. I have taught for ____0 years ____<3 years ____3 - 10 years ____>10 years
3. I ____regularly teach ____have taught ____have never taught Algebra I .
4. My highest earned degree in mathematics or mathematics education is: ____none ____undergraduate minor ____undergraduate major ____Master's degree ____Ph.D.
5. List all the steps that a student would need to know how to do to be able to solve any linear equation. (For example: add the same thing to both sides.)
6. Is there a standard order that you expect students to perform the steps you listed above?
7. Does your algebra textbook present a standard order of steps to solving linear equations?

Page 2

Solve each problem showing all the steps that you would want your students to show.

1. 2x + 4 = 10
2. 3(x+2) = 21
3. 3(x+1) =15
4. 4(x+1) + 32 = 5(x+1)
5. 4(x+1) + 3(x+2) = 20
6. 3(x+1) + 6(x+1) + 6x + 9 = 6x + 9
7. 4(x+1) + 2(x+1) = 3(x+4)
8. 4(x+1) + 2(x+2) = 3(x+4)
9. 4(x+1) + 2(x+1) = 3(x+1)
10. 4(x+2) + 6x + 10 = 2(x+2) + 8(x+2) + 6x + 4x + 8

Page 3

Solve each problem in the most efficient way.

a. 3(x+1) = 15
b. 4(x+1) + 2(x+2) = 3(x+4)
c. 4(x+2) + 2(x+2) = 3(x+6)
d. 4(x+3) + 2(x+3) = 3(x+3)
e. 3(x+1) + 6(x+1) + 6x + 9 = 6x + 9
f. 4(x+3) + 32 = 5(x+3)

Figure 1. Solving Linear Equations Questionnaire

RESULTS

Standard Order

From their answers to the question about a standard order for solving linear equations, five participants do not believe there is a standard order for solving linear problems. The other 15 participants name from two to seven steps that are more or less specific, but generally name the same order—to simplify (distribute, get rid of fractions, combine like terms), add or subtract from both sides, then multiply or divide both sides. In solving the problems on page 2, most participants want students to show essentially the same work in a relatively standard order—distribute, combine like terms, subtract like terms, and divide. The main difference is whether participants want to see subtraction of variable terms and constant terms in separate steps or in one step. The majority of responses, approximately two to one on each question, show subtraction of variable terms and constant terms separately. Table 1 shows how many participants (N=20) perform the standard order on each problem 1 through 10. Also in Table 1 are the numbers of participants who use a chunk, such as (x + 1), as a unit in solving these problems. These results indicate that participants do expect a standard order for solving linear equations.

Problem number	Completed Standard Order	Worked with Chunks
1	20	NA
2	17	0
3	17	0
4	20	2
5	19	NA
6	20	2
7	20	3
8	20	NA
9	19	3
10	19	2

Table 1. Standard order completion and use of chunks on problems 1-10. (N=20)

Problem number	Completed Standard Order	Worked with Chunks
a	9	NA
b	20	NA
c	19	11
d	19	10
e	20	10
f	20	8

Table 2. Standard order completion and use of chunks on problems a-f. (N=20)

Using Chunks

In contrast to the answers on page 2, the questions on page 3 show substantially different responses. Table 2 shows the number of standard order completions and the use of chunks on problems a through f. One of the main differences obvious from the results given in Table 2 is the use of chunks. While few participants use chunks in answering the first ten problems on page 2, a substantial increase in use of chunks is seen in the last four problems on page 3. The standard order is generally followed but without distributing first when chunks are used. This use of chunks indicates that participants have a flexible and innovative understanding of algebraic variables when solving problems themselves.

Individual Responses

In the first ten problems, all uses of chunks are by the same three individuals. One person is a pre-service teacher, and the other two people have been teaching for three to ten years and regularly teach Algebra I. All three individuals consistently use chunks in the last four problems on page 3.

Two individuals—both high school teachers, one with fewer than 3 years experience and the other with 3 to 10 years experience—show a check step on the problems on page 2. One shows it on every problem, the other on the first four problems only. Neither person shows a check step on any problems on page 3.

$$4(x+1) + 3(x+2) = 20$$

$$4x + 4 + 3x + 6 = 20$$

$$7x + 10 = 20$$

$$(20\text{-}10) \div 7 = \frac{10}{7}$$

Figure 2. An unusual solution

A different individual—also a high school teacher with 3 to 10 years of experience—shows an unusual solution technique on all problems on page 2. (See Figure 2.) Standard algebraic notation is used for the distributive step and for the combine like terms step. At that point, the "arithmetic" steps—subtraction and division on both sides—are performed as a unit with no variable showing.

DISCUSSION

Standard Order

As noted by Thompson (1984), what teachers say they believe and what they do are not always the same. In this study participants are much more uniform in their expectations of a standard order for solving linear equations when demonstrating how they want students to solve problems than when discussing the idea in the abstract. Essentially all of the participants show a standard order when solving problems, even though a third of the participants say there is no standard order for solving linear equations. Roughly half of the participants say that their textbooks do not show a standard order, or they do not know if their textbooks show a standard order. One participant provides her students with a template to follow in solving linear equations that outlines the standard order because the book does not show it. Thus there is a mixed response on the existence of and in some cases the need for a standard order when participants discuss the issue. When the actual expected work is shown, however, the standard order is almost universally followed.

The instructions given on page 2 and page 3 produce different responses to essentially the same questions. All page 3 questions have an exact or very nearly exact analog on page 2. Two interesting variations in solutions on page 3 are worth noting. On problem a, 11 out of 20 participants change the standard order and divide first, resulting in a two step solution, the most efficient possible. This change is particularly noteworthy in that on problem 3 (the same problem as problem a) only three participants expect students to divide first. This difference in responses supports participants' views that a standard order is not necessary for solving the problems, but also indicates their preference for students to follow a standard order. The second variation is in how the subtraction steps are performed. On page 2, most participants want to see students subtract variables and constants in separate steps. On page 3, however, almost all participants subtract variables and constants in a single step. Again we see a difference in how participants want students to demonstrate solutions and how they demonstrate solutions themselves.

Use of Chunks

Participants are also unlikely to treat chunks as single variables when showing what they expect students to do, but instead begin each problem by using the distributive property to remove parentheses. When working problems for efficiency, however, participants use chunks freely, but generally still adhere to the same standard order for solutions. The use of chunks, in most of these problems, does not lead to more efficient solutions in terms of number of steps, but it does show flexibility. In one case the use of chunks does create a much more efficient solution. On problem d, one participant notes that the "variable [x+3] is the same on both sides and constants are not equal so x+3 must equal 0." This observation reduces the problem to one step of subtract 3 from both sides.

Individual Problems

There is much more uniformity between participants' answers on page 2 than within a single participant's answers on pages 2 and 3. In general, the responses on page 2 are interchangeable between participants. Responses to page 3 problems show much more innovation in order of steps and flexibility in using chunks between individuals. Interestingly enough, there are no apparent differences between individuals based on any of the demographic categories.

The inclusion of a check step on page 2 problems also points out the variation within one person's answers. She saw the check step as important for students to show, as evidenced by her willingness to include it on all ten problems. The check step is abandoned, however, when an efficient solution is the goal. This implies that the check step has a pedagogical use but not an efficient use.

CONCLUSIONS

Teachers, both pre-service and in-service, in this study expect students to solve linear equations in a standard order—distribute (and other simplifications), combine like terms, add and subtract, multiply and divide. This expectation is clearer from worked examples than from descriptions of a standard order. Furthermore, differences in level of teaching, years of experience, frequency of teaching Algebra I, and level of mathematics preparation do not affect teachers' expectations.

The purpose for solving linear equations appears to affect what teachers do in solving linear equations. When the purpose was clearly pedagogical—show what students should show—teachers' expectations were uniform. When the purpose was efficiency, teachers' solutions varied more, showing both innovation and flexibility.

The dilemma that teachers face, to teach effective methods or efficient methods, is thus brought out in a way that can be examined. A standard order provides an effective way for students to routinely solve linear equations correctly. That order, however, is not necessarily the most efficient way and is certainly not the only possible way. Within the context that teachers work, the solution to the dilemma appears to be the recognition that there are multiple orders possible and a variety of innovations possible in solving linear equations, however, the importance of this skill makes a standard order the expectation for students' work.

This study supports Star's (2002) contention that there is a standard order for solving linear equations in the United States. It also demonstrates that teachers expect that standard order from students but are able to vary from it themselves when trying to solve problems efficiently. This portion of the study brings out another pedagogical decision made every day by teachers, but one that is made perhaps in a ready-to-hand way without conscious reflection.

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