TEXT TALK, BODY TALK, TABLE TALK: A DESIGN OF RATIO AND PROPORTION AS CLASSROOM PARALLEL EVENTS

Dor Abrahamson

Northwestern University[1]

The paper describes the rationale and 10-day implementation in a 5th-grade classroom (n=19) of an experimental ratio-and-proportion instructional design. In this constructivist-phenomenological design, coming from our theoretical perspective, design research, and domain analysis, students: (1) link 'real-world' and 'mathematical' objects reciprocally through classroom enactment of word-problem situations vis-à-vis guided reading/writing of spatial-numeric inscriptions; (2) interpret and invent rate, ratio, and proportion texts as patterned cells in and from the multiplication table; (3) revisit and consolidate addition and multiplication as conceptual domain foundations. Students of diverse ethnicity, SES, and mathematical competence engaged successfully in discussing and solving complex problems, outperforming older students on comparison items.

INTRODUCTION

Substantial effort has been put into documenting and analyzing students' notoriously low performance in the domain of rational numbers (e.g., Behr, Harel, Post, & Lesh, 1993)—performance which has often been typified as evidencing 'additive reasoning' where 'multiplicative reasoning' may have been more effective. For instance, a student may reason that 3 : 4 is equal to 6 :7 because in both ratios there is a difference of 1 between the two numbers. It may be the case that students do not use multiplicative reasoning because they lack coherent cognitive models for understanding and applying multiplicativity in these cases. Plausibly, instruction should help students build such models on the basis of existing knowledge. Insightful design research has called for sensitive adaptation of instruction according to students' intuitive understandings (Confrey, 1998), and Vergnaud (1994) has suggested a broad view of the instructional domain of multiplicativity as encompassing not only schooled but also everyday knowledge. Some of this intuitive (Fischbein, Deri, Nello, & Marino, 1985), streetwise (Schliemann & Carraher, 1992), and folk (Urton, 1997) knowledge has been described in ways that illuminate prospective didactic solutions.

Our design-research principles have emanated from the perspective that students self-construct knowledge (Piaget, 1952) in response to their goals within social settings (Vygotsky, 1978) such as classrooms (Cobb & Bauersfeld, 1995). We thus strive to engineer an instructional design that affords: (1) opportunities for individual students to draw on their experience and intuitions as well as on their schooled knowledge in self-constructing understandings; and (2) engaging participatory activities that foster a supportive classroom climate (Fuson, De La Cruz, Smith, Lo Cicero, Hudson, Ron, & Steeby, 2000). When collaborating with teachers and students, we iteratively modify our design so as to maximize resonance with their difficulties, strategies, vocabulary, and emergent understandings.

We have found designs to be most effective when reading/writing/building activities allow students to 'connect' (Wilensky, 1993) the material to their previous understanding, where 'understanding' does not comprise static 'concepts' but "ways of acting and thinking" (von Glasersfeld, 1990, p. 37; see also Freudenthal, 1981). We are informed by Phenomenology (Heiddeger, 1962) in assuming that students' previous understandings—both the academic and non-academic components of their domain-specific 'conceptual field'—are implicit in their ways of acting and thinking, and become self-explicated through classroom problematizing that stimulates appropriation of mathematical artifacts. These mathematical-didactic views together with our domain analysis of ratio and proportion (Abrahamson & Fuson, 2003b) have brought us to conceptualize and design the domain in the following way:

In our design (see Figure 1): (1) '*Rate*' is an iteration of constant increments, e.g., "Every day Robin puts \$3 in her kitty bank" ('+3 per day') or "Every day Tim puts \$5 in his doggy bank" ('+5 per day') that produce successive cumulative totals, e.g., 3, 6, 9, 12, etc. or 5, 10, 15, 20, etc., which appear in this order in multiplication-table columns; (2) '*ratio*' is two linked rates occurring as a succession of parallel (contemporaneous paired) events, e.g., "Every day Robin puts \$3 in her kitty bank and Tim puts \$5 is his doggy bank," so '+3 & +5,' that produce successive rows in linked multiplication-table columns, e.g., 3 & 5, 6 & 10, 9 & 15, 12 & 20, etc.; (3) a '*proportion*' is two rows out of a ratio progression, e.g., 6 & 10 and 21 & 35 (forming a 'proportion quartet'); (4) the multiplicative relation between values in a proportion—whether 'scalar' (within column) or 'functional' (within row, Vergnaud, 1994)—emerges as a strategically useful property that shortcuts and can substitute for repeated addition in the solution of unknown-value proportion problems, e.g., 6 & 10 and 21 & '?' (see also Abrahamson & Cigan, in press).

METHOD

A class of 19 5th-grade students of mixed ethnicity and SES (34% free lunch) in an urban/suburban school participated in a 10-period experimental design that spanned two weeks. The class was co-taught by the author-designer and the teacher, who collaborated on preparing and debriefing each lesson. The author also held daily tutoring sessions with two students whom the teacher selected as having difficulty in mathematics. These sessions, conducted as semi-clinical interviews, served to help the students and probe for difficulties they were experiencing with the design. All lessons were audio- and video-taped and micro-analyzed daily towards modifying the design. The class had a large laminated multiplication table on the wall, and each student had two small multiplication tables, from one of which they cut columns. These columns could be put together to form rate tables or ratio tables, thus relating these forms to the multiplication table.

We pre- and post-tested the students on 13 items used in previous studies and gave the same pre-test in 5th-grade classes in two additional schools in the same district to check for the representativeness of the experimental class. These other students had the advantage that the test was conducted 2 months later in the school year.

Solution to:
Every day Robin and Tim each save at a constant rate. If on a certain day Robin had $6 and Tim had $10, then how much did Tim have when Robin had saved $21?

PQ as mini MT

Robin | Tim
3 | 5

2	6	10	2
7	21	35	7
	3	5	

Proportion Quartet (PQ)
factored outside

PQ as selected RT rows (here, Rows 2 & 7)

Multiplication Table (MT)

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Ratio Table (RT)
Coming from a situation

	3 : 5	
0	0	0
1	3	5
2	6	10
3	9	15
4	12	20
5	15	25
6	18	30
7	21	35
8	24	40
9	27	45
10	30	50

Every day, Robin puts $3 into her kitty bank, and Tim puts $5 into his doggy bank

Rate as single MT-column

Every day, Robin puts $3 into her kitty bank

	3
0	0
1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24
9	27
10	30

Rate
Coming from a situation
(The 3-column from MT)
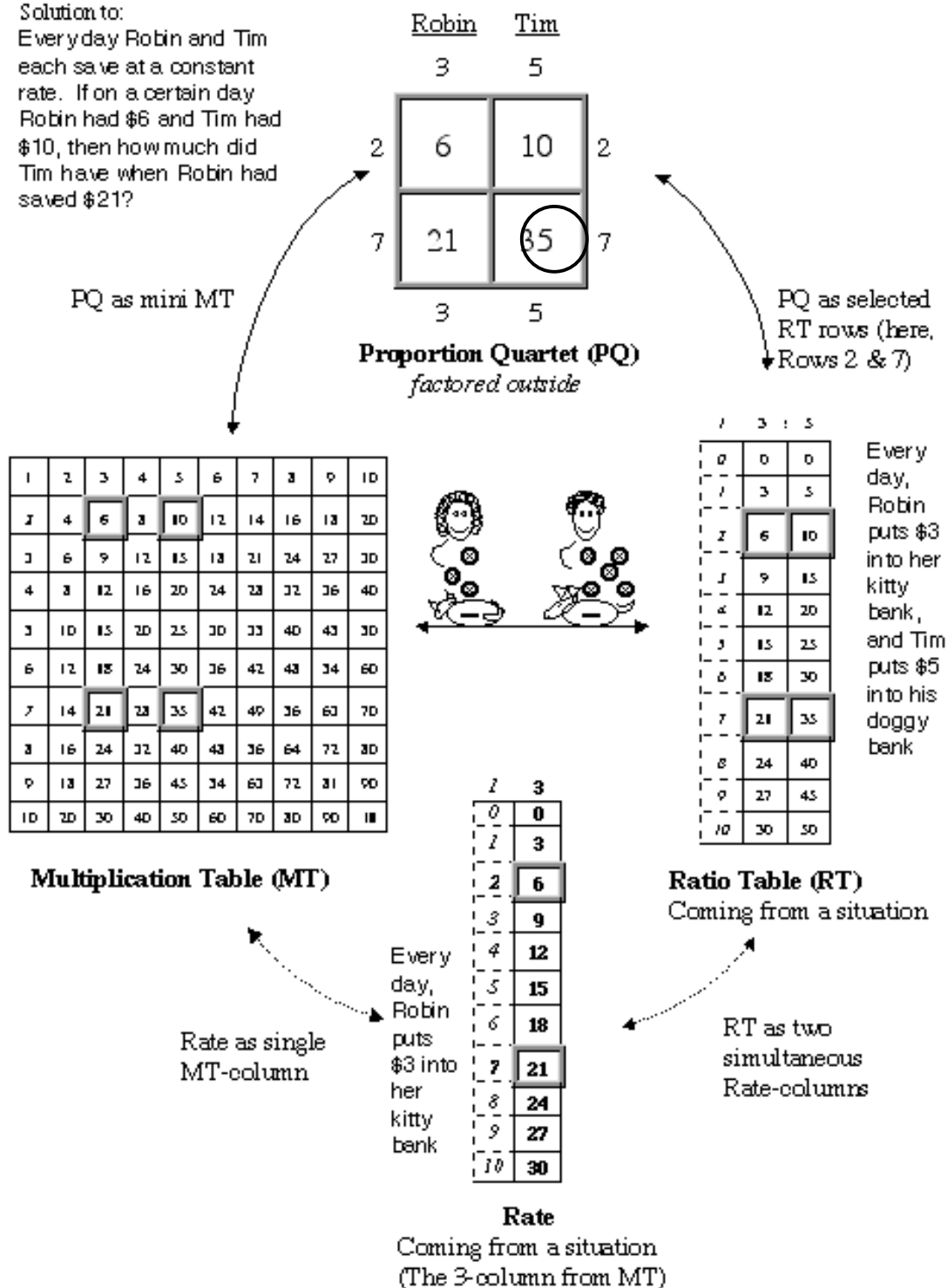
RT as two simultaneous Rate-columns

Figure 1. Design framework for conceptualizing ratio and proportion. From left and anticlockwise: (a) The multiplication table (MT); (b) rate table; (c) ratio table (RT); and (d) proportion quartet (PQ). Products and cells of a specific example problem (top left) are enhanced here for demonstration.

2—3

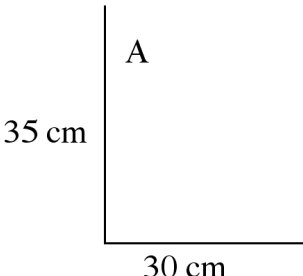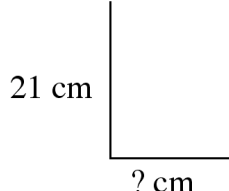**RESULTS**

Items	Our Grade 5 (n = 19)	Comparative Studies
a. In a field trip, 12 people eat 16 boxes of food. How many boxes of food would 15 people eat?	72	45 Grade 6
b. To bake donuts, Jerome needs exactly 8 cups of flour to make 14 donuts. How many donuts can he make with 12 cups of flour?	83	19 Grade 6
c. The Boston Park Committee is building parks. They found out that 15 maple trees can shade 21 picnic tables when they built the Raymond Street Park. On Charles Street, they will make a bigger park and can afford to buy 50 maple trees. How many picnic tables can be shaded at the new park?	89	19 Grade 6
d. The two sides of Figure A are 35 cm high and 30 cm long. Figure B is the same shape but smaller. If one side of Figure B is 21 cm high, how long is the other side? A B 35 cm 21 cm 30 cm ? cm	78	05 Grade 6
e. If the ratio 7 to 13 is the same as the ratio x to 52, what is the value of x? Multiple choice: 7; 13; 28; 364	78	69 Grade 8
f. Fill in the missing number: 3:10 = __ : 100	94	03 Grade 5
g. After a long diet, Fat Cat weighed only 54 lbs. That was 90% of his normal weight. How much did Fat Cat weigh before the diet?	81	na

Table 1: Sample scores in % correct. Comparison items: a. Vanhille & Baroody (2002); b., c., & d. Kaput & West (1994); e. TIMSS (1995); f. Stigler, Lee, & Stevenson (1990)
Students' achievement on the posttest was higher than on the pretest (72% > 36%; p < .0001). In particular, in the 3 critical items, in which the multiplicative relation between the two ratios was non-integer (e.g., 6:10 & 21: 35; see items a, b, & c in Table 1),

students' pre-and-post-test mean scores progressed from 10% to 81%, (p < .005). Also, students progressed on a fraction item even though fractions were not taught in the unit.

Pre-test scores from the 2 control classes in the same school district (36% and 29% correct) did not differ statistically from that of our study class (36%), indicating that this was not a class with special initial knowledge.

Students enjoyed solving numeric missing-value proportion problems given in the 'proportion quartet' format. In particular, students found these problems interesting in that different strategies—choice of factors and order of operations—led to the same solutions. Advanced students reported in feedback questionnaires and interviews that they had figured out how the 'proportion-quartet' solution format relates to their earlier understanding and strategies for addressing proportion situations—some of these students professed that they prefer the proportion-quartet system because its labeling reduces errors of misassigning values to variables, expedites their work, and facilitates interpretation of solution values back in terms of the situation variables and because they can confirm their solutions with the multiplication table.

Some unanticipated outcomes were that: (1) although many of the students were initially quite adept at reciting count-by sequences from multiplication-table columns and by heart, they had great difficulty in articulating these sequences in terms of addends and running totals or as multiplier-multiplicand relations; their use of the multiplication table—locating the cross product of two factors—appeared as acts of referencing a nominal chart that were devoid of any meaning of the multiplication operation; yet (2) using rate, ratio, and multiplication tables, students learned to alternatively interpret the tables' product structure as either additive or multiplicative and thus formed an additive-multiplicative model as a foundation of ratio and proportion; (3) the repeated-addition model of multiplication, e.g., '5 * 3 as +3, +3, +3, +3, +3' is more conducive to effective learning when expressed (uttered, indicated, inscribed) as a rate running total, e.g., as *0 + 3 = **3**, 3 + 3 = **6**, 6 + 3 = **9**, 9 + 3 = **12**, 12 + 3 = **15***, etc., wherein students keep a coordinated <u>parallel</u> double count—tracking the number of addends (running total of *3*'s, e.g., 1, 2, 3,4, 5) and the running cumulative total (e.g., 3, 6, 9, 12, 15); finally, (4) students became gradually quicker at multiplication "basic-facts" which they needed to solve the missing-value proportion quartets (using both multiplication and division).

In the weeks that followed our intervention, the teacher reported that students initiated using our multiplication-table poster and cutouts to find fraction equivalences, e.g., successive columns in the 2- and 5-row form the proportional progression 2/5, 4/10, etc.

DISCUSSION

Middle-and-high school students' well-documented additive errors in solving multiplicative problems are symptoms and not the sickness itself. Once students are no longer diagnosed as performing these symptomatic errors, one is still faced with the question of whether or not they are operating with understanding. Written explanations students offered in their daily worksheets, classroom participation, post-tests, and interviews to problems they had solved in the *absence* of the multiplication table focused on describing proportion as a spatial-numerical configuration in the multiplication table. That is, the multiplication table first shaped and then served as a criterion of

proportionality. For instance, students used the multiplication table to reject the following problem as expressing proportion: "Bob and Joe are brothers who share the same birthday; When Bob was 3 years old, Joe was 5; Now Bob is 6. How old is Joe?" We view such a preliminary conceptualization of ratio and proportion as robust and not as narrow in that it builds directly on students' understanding of addition, of multiplication as repeated addition, and of ratio and proportion as additive-multiplicative (Abrahamson, 2002a). Moreover, we suggest that our design addresses and fills two lacunae: (1) Generally, students' execution of multiplication computation only ostensibly manifests an understanding of the operation—the running-total model connects addition and multiplication with understanding; (2) designs that do not incorporate any classroom enactment of ratio and proportion as a time-space phenomenon miss out on a crucial conceptual bridge between the <u>parallel</u> mathematical event-elements of text word problems and spatial-numerical inscriptions

Abrahamson & Fuson (2003a) interpret students' developing cognition of ratio and proportion as following: Students who enact ratio-and-proportion word-problem situations embody verbally-kinesthetically and mobilize this text in the classroom time-space. These rhythmic gestures and utterances coincide with students successively tabulating the cumulative narrative-total, e.g., 3, 6, 9, 12, etc., in spatial-numerical structures such as rate tables that they build and fill in. Reciprocally, by interpreting inventively multiplication-table columns as rate and ratio stories, students return the spatial-numerical text to the verbal-kinesthetic classroom time-space and link inscribed forms with their enacted narratives. Thus, mathematizing narratives and narrativizing mathematics are reciprocal acts that, we believe, foster the development of interpretive schemas that are critical for sense making in the domain of ratio and proportion.

Our emphasis on the classroom temporal-spatial phenomenology of ratio-and-proportion commits us to a student-centered conceptualization of the domain: Whereas we appreciate the *mathematical* validity of referring to ratio as an 'intensive' quantity (Shwartz, 1988) and of qualifying the proportional relation as modeled on 'splitting' (Confrey, 1998), we locate the developing *cognition* of ratio in classroom acts of repeated linked adding of two quantities (a ratio) that plays out over classroom time-space. That is, values coming either from texts (word-problems) or spatial-numeric structures (tables) indicate particular cases of proportional relations and the computation of these values is facilitated by various inscriptions that students learn to execute, but the meaning of the proportional relation itself is individually mediated by verbal-kinesthetic embodiment.

To help students see anew the core spatial-numeric structure of our design, the very first task was an open-ended exploration that required students to find "interesting things" in the multiplication table. The surprisingly insightful patterns students discerned served as a "kickoff" for investigating the additive-multiplicative structure of the multiplication table, formed the basis of the classroom's common mathematical vocabulary, and set the tone of this constructivist design. Many activities through which students embodied ratio and proportion involved moving back and forth between lexical and mathematical texts. For example, the teacher responded to a student's additive error in the context of a discussion of the difference in the heights of two plants that grow at different rates. The teacher enacted a "slow-motion running competition" which the class followed gleefully

as the distance between the two "runners"—the very slow runner (the teacher) and the slow runner (a student)—increased at every second. In a related activity, students placed their hands palms down on their desks, lifted their hands at different speeds (the right hand rising faster than the left hand), and watched how the gap between the hands grew.

Following the study reported in this paper, a modified version of the design was carried out in another 5th-grade classroom in the same school (Abrahamson & Fuson, 2003a), and later a further modified version was enacted in a different school. Also, we did extensions of this unit that included the study of percentage, geometrical similitude (Abrahamson, 2002b), and elementary coordinate graphing. The unit has been well received by mathematics teachers and supervisors in our district. Currently, we are working with several teachers in revising the design as a curricular unit of the Children's Math Worlds project. This unit reflects our current thinking on the imperative of initially grounding ratio and proportion in an explicit multiplicative spatial-numeric structure, the multiplication table.

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