RESEARCH FORUMS

RF1 PERCEPTUO-MOTOR ACTIVITY AND IMAGINATION IN MATHEMATICS LEARNING

Nemiroskvy, Ricardo (Co-ordinator)

Arzarello, Fernando & Robutti, Ornella

Chazan, Daniel, & Schnepp, Marty

Borba, Marcelo (Co-ordinator) & Scheffer, Nilce

Rasmussen, Chris & Nemirovsky, Ricardo

RF2 EQUITY, MATHEMATICS LEARNING AND TECHNOLOGY

Vale, Colleen; Leder, Gilah; & Forgasz, Helen (Co-ordinators)

Forgasz, Helen

Keitel, Christine

Setati, Mamokgethi

Leder, Gilah

Kaiser, Gabriele

Vale, Colleen

Secada, Walter

RF1: PERCEPTUO-MOTOR ACTIVITY AND IMAGINATION IN MATHEMATICS LEARNING

Ricardo Nemirovsky Marcelo Borba

TERC UNESP

The idea that perceptuo-motor experiences are important in mathematics learning is not new, of course; it is often associated with the use of manipulatives. The use of manipulatives in mathematics education is part of a long tradition enriched by noted educators such as Maria Montessori, Georges Cuisenaire, Caleb Gattegno, and Zoltan Dienes. Like many teachers, these educators have observed that numerous students will become engaged with materials that they can manipulate with their hands and move physically, with an intensity and insight that are not present when they simply observe a visual display on a blackboard, a screen, or a textbook. While researchers justly observe that students' experimentation with manipulatives and devices does not automatically cause them to learn mathematics (*1-5*), there is something valuable that sustains the use of manipulatives even though it is straightforward to simulate most physical manipulatives on a computer. It is a very different experience to watch a movie displaying a geometrical object than it is to touch and walk around a plastic model of the same object. Clearly both experiences can be useful, but even if one would argue that they both reflect the same mathematical principle, they are not mere repetitions. One difference is that the use of appropriate materials and devices facilitates the inclusion of touch, proprioception (perception of our own bodies), and kinesthesia (self-initiated body motion) in mathematics learning.

An emerging body of work, sometimes called "Exploratory Vision," describes vision as fully integrated with all the body senses and actions. Our eyes are constantly moving in irregular ways, momentarily fixing our gaze on a part of the environment and then jumping to another one. It is as if we are constantly posing questions to the visual environment and making bodily adjustments that might answer them. The bodily adjustments enacted in search of those answers constitute a critical aspect of what one calls seeing:

> On this view, no end-product of perception, no inner picture or description is ever created. No *thing* in the brain *is* the percept or image. Rather, perceptual experience consists in the ongoing activity of schema-guided perceptual exploration of the environment. (*6, p. 218, italics in the original*)

A reason often drawn on to set aside touch, kinesthesia, etc. in mathematics learning is that mathematical entities cannot be "materialized", one cannot touch, say, an infinite series or the set of even numbers. While true, the fact that these entities are imaginable with the symbols we use to work with them, is profoundly connected to perception and bodily action (*7*). In fact. it is increasingly evident that there is a major overlap between perception and imagination (*8, 9*). To imagine, for instance, a limit process, one extends perceivable aspects to physically impossible circumstances and conditions. In this regard, touch and kinesthesia can be instrumental to imagining. It is not unusual that to imagine inexistent objects and events one gestures shapes and motions or takes hold of an object, say a cardboard box, to help see them from different sides.

This research forum attempts to advance these themes by addressing the following research questions:

- What are the roles of perceptuo-motor activity, by which we mean bodily actions, gestures, manipulation of materials, acts of drawing, etc., in the learning of mathematics?
- How do classroom experiences, as constituted by the body in interaction with others, tools, technologies, and materials, open up spaces for mathematics learning?
- How does bodily activity become part of imagining the motion and shape of mathematical entities?
- How does language reflect and shape kinesthetic experiences?

The ensuing text encompasses five different papers. The first one outlines conjectures on the relationship between perceptuo motor activity and mathematical understanding. The ensuing four papers describe classroom-based cases, examine the research questions, and elaborate on the initial conjectures.

THREE CONJECTURES CONCERNING THE RELATIONSHIP BETWEEN BODY ACTIVITY AND UNDERSTANDING MATHEMATICS[1]

Ricardo Nemirovsky. TERC. Cambridge, MA. USA

This paper is divided in three sections. Each section starts by reviewing and interpreting a small selection of references. They were chosen to provide an initial background for the ensuing conjecture about the relationship between body activity and mathematical understanding.

I

In 1924 Josef Gertsmann reported the first case of the syndrome that is nowadays called by his name. He published the case of a 52-year old woman who had become unable to recognize or name fingers either of her own or of someone else's hands, a symptom that he termed "finger agnosia". While the patient had difficulties in producing some isolated finger movements there was no noticeable overall motor or sensorial loss; in fact, she had not been aware of her finger agnosia until she was tested (*10 p. 203*). In addition to finger agnosia, she showed other symptoms; three them became later associated to the Gertsmann syndrome: 1. right-left disorientation, especially for her and others' body parts; 2. Spontaneous Agraphia, that is inability to write text that she originated, as opposed to copying text which she was able to do; and 3. Dyscalculia, or inability to understand and operate with numbers.

Much has been discussed over the years on whether the four symptoms that characterize the Gertsmann syndrome are just a set of contingent ones gathered by the idiosyncratic topology of different cases of brain damage or whether they all reflect a common "principle". In any case, the re-appearance of cases in which these four symptoms occur in isolation from other neurological problems, as well as the localization of the damage in a small region (*11*), keep open the possibility that the understanding of hands, right-left orientation, spontaneous writing, and arithmetic might be deeply interwoven.

While the understanding of numbers grows also out of sources that are not rooted in finger counting[2], the fact that the latter has such a prominent role in the development of number in all cultures and historical periods, has prompted many researchers to reflect on the nature of its function (*12-14*). Because numbers can be used to quantify anything whatsoever, they are often viewed as a primary example of what an abstraction is. The

[1] The writing of this paper has been supported in part by the "Math in Motion" project, which is funded by a grant from the National Science Foundation (Grant REC-0087573). All opinions and analysis expressed herein are those of the author and do not necessarily represent the position or policies of the funding agency. The author wishes to thank Tracy Noble and Chris Rasmussen for their feedback based on a previous version.

[2] An outstanding example could be subitizing, that is, recognizing numerosity at a glance or touch without counting. Subitizing appears to be limited to 4 items; it encompasses the ability to perform addition, subtraction, and size comparisons with them. As opposed to counting, numerous researchers, but not all, view subitizing as innate and widely present in animals.

property of, say, three, can be attributed to a collection of three items regardless of whether they are things, ideas, sounds, and so forth. In neuroscience this is sometimes asserted by saying that numbers are "amodal", in the sense that they are not restricted to a particular perceptual modality (vision, touch, etc.). This amodality might have the same roots that the body activity of pointing at things in the surroundings has, whether they are visual, tactual, auditory, and so on. One can use numbers to quantify anything for the same reasons that one can point at anything in one's surroundings. Or for the same reasons that one can trace a shape with one's finger independently of what the traced thing is made of, its color, its size, etc. Even though pointing and tracing have a spatial reference that is not always present in finger counting, in all these cases there is at play a bodily activity that can eventually be felt and enacted by itself, detached from its original object, as it were. We say "eventually" because the transformation of these bodily activities into self-referential ones (e.g. pointing at one's own pointing, counting one's own counting, etc.) that can be extended and refined on their own, demands an immersion into complex cultural practices. In the case of finger counting, cultures offer diverse technologies in the form of sequences of number words, devices to keep track of counted items, or specialized notations, as well as customary ways in which knowledgeable adults guide learners.

First conjecture: mathematical abstractions grow to a large extent out of bodily activities having the potential to refer to things and events as well as to be self-referential. Think of measurement as an example. The primal units of measurement are body parts—feet, arms, thumbs, etc.—that are imaginarily repeated and laid out next to each other. The whole process is a body activity of covering an extension with successive presentations of units leaving material or imaginary traces along the way, which are to be counted. These techniques are the subject of recurring training, linguistic expressions, and procedural fluency. As in the case of finger counting, a wide collection of cultural resources are made available that enable these practices to eventually become self-referential. This self-referential aspect is sometimes alluded to by assertions of the sort "twelve inches make a foot", when they are intended to mean the measuring of a measure.

II

It is still not uncommon to find the idea of thinking as encompassing three elements: 1. Perception channels that count as "inputs", 2. High level processing—where the thinking really occurs, and 3. Motor channels conveying the "output". Zeki (*15*) relates how, until 30 years ago or so, this perspective was the dominant one for the study of vision. Until then, the visual cortex was commonly understood as generating, out of retinal information, an internal image of what is visibly outside. This was deemed complicated but still "low level" input processing, which was in turn processed by the "association cortex" to make meaning out of this mental image. As a consequence of understanding generated by the association cortex, action in the form of, say, eye and head movement would follow (a "low level" output task). The science of vision has moved out of this narrow notion. As Zeki (*15*) has pointed out, vision is not currently viewed as split along the lines of low-level perception-action and high-level intelligence. Instead, the strands of seeing are of a different kind that cuts across levels: they specialize in different subjects such as faces, shapes, colors, motion, etc., and they are all active in diverse bodily activities, such as reaching, grasping, avoiding, and so forth. Perception does not happen in an input mode: that which one cannot understand one cannot see, and we see to the extent that we understand. Losing totally or partially the capacity to see color, for example, is also losing totally or partially the capacity to understand color (e.g. to imagine colors, to use color words properly, etc.; see (*16*) as an example)

Within the neuroscience of number, similar issues arise. One of the first models for the processing of numbers (*17, 18*) was based on the typical structure: 1. Input (e.g. perceiving numbers spoken, written in Arabic notation, in words, etc.), 2. Calculation System, and 3. Output (e.g. writing numbers, saying them, etc.). The authors assumed that an abstract amodal number code is used by the Calculation System. A growing body of empirical evidence has made this scheme either much more complicated or just harder to support. For example, Cipolotti at al (*19*) have studied the case of a woman in Northern Italy (named C.G.), who after a stroke showed the Gertsmann Syndrome. C.G. was deeply dyscalculic: she could not say how many days there are in a week, tell her age, subitize, or when asked what the word 'seven' means she responded: "I've never heard this word before". Nevertheless, she was quite knowledgeable about numbers 1, 2, 3 and 4. Within this number range C.G. could recognize their Arabic symbols and names, compare which one is greater, count sets of objects, memorize number sequences, and so forth. If there were a discrete "Calculation System" for general number understanding, why would such understanding stop precisely at '4'? And why would its damage obliterate her perceptual ability—presumed to be separate—to see that '7' is a common symbol whose meaning she forgot, instead of a strange mark on paper?

So far we have discussed the merging of perception and understanding. But this is only part of the issue because understanding is also interwoven with motor action. This is to be expected given the intimate connectedness between the perceptual and motor "sides" of cognition. Several studies indicate that merely seeing a tool, such as a comb, a fork, or a screwdriver, activates areas of the cortex involved by the motor actions enacted in their use (*20, 21*). In addition, there is growing evidence that the close bonds between the perceptual and the motor are also active when we remember or imagine actions and situations (*22*).

We know from functional brain imaging studies and from cases of neuropathology that our understanding of different subjects is distributed across the perceptuo-motor cortex in ways that depend, in part, on the type of experiences we have had with the subject. For example, many cases have been reported of selective impairment in the understanding of, say, tools, body parts, musical instruments, etc. It now appears that because, for non musicians at least, the primary mode of relating to musical instruments is by looking at them, damage to the cortical regions devoted to visual recognition may cause a patient to lose their understanding, including aspects such as their names and use (*23*); whereas because the primary mode of use of tools is motor, an impairment in cortical regions dealing with visuo-motor manipulation can lead to these becoming alien objects (*24*). The partial correspondence between use on the one hand and distribution on the perceptuo-motor cortex on the other does not imply that an uninformed examination of the former can tell the latter in any simple way. For instance, we might expect that understanding numbers represented in Arabic numerals or in words would overlap because we can use them in similar contexts. However, cases of brain damage have been reported in which patients preserve one to the exclusion of the other (*25, 26*). These types of results can be of great relevance to recognize fine distinctions within what at first appear to be manifestations of the same practice.

Second conjecture: While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based in part, on how we have learned and used the subject itself. This conjecture implies that the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities which become more or less significant depending on the circumstances. For instance, seeing a trigonometrical function as a component of circular motion or as an infinite sum of powers may entail distinct and separate perceptuo-motor activities. Learning a different approach for what appears to be the "same" idea, far from being redundant, often calls for recruiting entirely different perceptuo-motor resources.

III

In his *Principles*, William James (*27 p. 1130*) examined the notion of "ideomotor action", that is, the activation of muscular systems inherent in the thought of a bodily movement. He cites an eloquent text by Lotze:

> The spectator accompanies the throwing of a billiard-ball, or the thrust of the swordsman, with slight movements of his arm; the untaught narrator tells his story with many gesticulations; the reader while absorbed in the perusal of a battle-scene feels a slight tension run through his muscular systems, keeping time as it were with the actions he is reading of. These results become the more marked the more we are absorbed in thinking of the movements which suggest them (ibid, p. 1133).

Nowadays it appears reasonable to assume that ideomotor actions span a continuum from those that activate peripheral muscular systems to those that remain circumscribed to areas of the motor cortex. A key question that arises is why *overt* ideomotor action is more or less intense, to the point that sometimes we imagine bodily movements without any evident change in muscular tone. James' explanation was unequivocal: it all depends on the simultaneous thought of "antagonistic" ideas, of thoughts of not-moving: "try to feel as if you were crooking your finger, whilst keeping it straight. In a minute it will fairly tingle with the imaginary change of position; yet it will not sensible move, because *its not really moving* is also part of what you have in mind" (p, 1135. Italics in the original). It takes effort to inhibit ideomotor action. This explains why being *fully absorbed* in something we watch or *talking aloud* about something we imagine, increases overt body motion; because in those circumstances there is less room for the "antagonistic" ideas. After many years of neglect, work on the nature of ideomotor actions is emerging anew (*28, 29*).

Analogous phenomena have been documented in the case of eye motion. It appears that as we imagine something, we move our eyes similarly to how we would move them if we were watching the imagined scene (*30*). Other studies suggest that if one imagines something as being far away, the eyes' crystalline lens adjusts as if one were actually looking far away (*31*). Why does all this happen? Why does the body invest in physically acting as if the imaginary were there, tangibly next to us? One approach to addressing these questions runs contrary to the view that mental models are the actual objects of thought. Rather than, for instance, assuming that seeing is examining a mental image – a mental version of what is "outside" – the point is that the objects of thought are experienced in and from the perceptuo-motor activity involved in thinking about them. This thesis has been articulated by O'Regan and Noe (*32*) who talk about the things we see and touch as a type of "external memory" we explore by enacting numerous "sensory contingencies" (e.g. eye motion, blinking, hand shape, etc.). The object of our perception is not a mental entity but the thing we touch or look at; a thing that we experience woven in the touching and looking themselves, which is inseparable from the individual and cultural history of how it has been used and perceived in the past. It is natural to extend this analysis of perception to the phenomenon of imagination: we move our eyes to imagine a scene because eye movement is an important aspect of the perceptuo-motor activity of seeing it; we need to enact these aspects to experience its imaginary appearance.

Third conjecture: in connection to the previous statement that "while modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities", we add here that that of which we think emerges from and in these activities themselves. It has been noted that to imagine a static object swinging we move slightly our eyes while looking at it, as if we were "pushing" it with our eyes. It is not the full-blown motion of the eyes tracking a swing but just a little thrust. Rather than assuming that we generate a mental image of the swing in motion, this conjecture suggests that we achieve such imagining by enacting bits of perceptuo-motor activities that would be engaged in seeing it. We think of, say, a quadratic function, by enacting "little thrusts" of what writing its equation, drawing its shape, uttering its name, or whatever else the use of a quadratic function in a particular context might entail. The actions one engages in mathematical work, such as writing down an equation, are as perceptuo-motor acts as the ones of kicking a ball or eating a sandwich; elements of, say, an equation-writing act and other perceptuo-motor activities relevant to the context at hand are not merely accompanying the thought, but are the thought itself as well as the experience of what the thought is about. Think of the case of a mental calculation. This conjecture departs from the idea that a mental calculation is a matter of processing information derived from general properties of numbers and operations; instead, it points at a complex perceptuo-motor activity that combines elements of writing those numbers, uttering their names, watching their shape, grouping objects, tracing lines, moving inscriptions, scratching others, counting marks, and so forth, all of that energized by a contextualized focus of attention and an emotional drive to address certain questions.

APPROACHING ALGEBRA THROUGH MOTION EXPERIENCES

Ornella Robutti Ferdinando Arzarello
Dipartimento di Matematica, Università di Torino, Italia

This paper describes didactic situations which can help and support students in a meaningful approach to algebraic rules, symbols and relationships. The focus is on developing the symbol sense, as well as interconnecting the syntactic and semantic aspects. The didactical aim is the construction of the concept of function as a tool for modelling motion. The research aim is the analysis of students' cognitive processes involved in the construction of a meaning for functions and how these meanings get reflected by the ways in which real data are interpreted, represented, and grasped.

The paper analyses (a part of) an ongoing teaching experiment in secondary school (from grade 9 up), where Calculus is early taught within different *experience fields* (*33*). Even if the experiment concerns all the basic topics in Calculus, we shall illustrate only an approach to the function concept within the experience field of pupils' motion. While describing the genesis of such a concept in our students, we shall sketch also to what extent and in which way the conjectures formulated in the "Three Conjectures…" section of this Research Forum are corroborated and refined by our findings (they will be indicated as Conj.1, Conj. 2, Conj. 3).

RESEARCH PROJECT

In our project secondary school students use technologies (a calculator connected with a motion sensor) to study models of different motions (e.g. students walking or running with different velocities, or toys moving). The kinaesthetic experiment of body motion is the first step to introduce students to modelling. In this experiment, they are involved with vision, perception, movements and rhythms (when they have to control their invariance or changing of velocity). First, they can "feel" the motion by themselves, in terms of changes of space in time, then they can see its mathematical representation by a graph on the display in real time. Subsequently, they are asked to interpret the graphs and tables containing the data (distance vs. time) related to their motion: they must use first the written (natural) language, then the technical terms of increasing-decreasing functions, and finally the numerical terms, to calculate the slope at certain points. The classroom activities involve working groups, classroom discussions, and final remarks made by the teacher. Each group experiments with motion and collects data with one sensor, then analyses them with one calculator: the restriction to only one instrument forces students to interact with each other and to share the process of knowledge construction.

In this way, everyday experience fields worked out suitably by the teacher are the environments where the students approach the mathematical concepts. In their genesis, language and instruments play a crucial role.

On the one hand, language activities have a genuine embodied nature (as neurological research points out) and support the students in the development of the scientific discourse, whose concrete features are blends, metaphors and gestures. These features accumulate into clusters of perceptual activities which the students experiment, describe by language and represent in different ways. The linguistic description allows them to deepen the different activities, to link each other, in an interactive and reflective attitude: see the discussion on self-referential activities in the "Three conjectures…" section of this Research Forum. Such clusters condense and compress into invariants, which constitute the abstract scientific concepts.

On the other hand, the use of instruments is crucial, because they support and enhance learning abilities, putting forward the different representations of a mathematical object. For example, a symbolic-graphic software can use different representations of data collected through instruments, like graphs and number tables. Such a representation *"can provide a powerful environment for doing mathematics and, with suitable guidance, to gain conceptual insight into mathematical ideas"* (*34*). In fact mathematical symbols can be used as cognitive *"pivots between concepts for thinking about mathematics"* (*34*). The dynamic of such a conceptualisation can be described within a Vygotskian frame: it represents a transition from the immediate intellectual processes to the operations mediated by signs and illustrates the dialectic between *everyday* and *scientific* concepts.

To investigate the specificity of such a dialectic within our teaching experiment, we use three analysis tools: the *embodied cognition* approach by Lakoff & Núñez (*7*); the *instrumental analysis* by Rabardel (*35, 36*) and others (*37, 38*); the definition of *concept* given by G.Vergnaud (*39*), in particular the notion of *operating invariant.*

The embodied approach reveals crucial for describing pupils' cognitive evolution within technological environments and for designing suitable teaching experiments. It shows a basic unity in their cognitive evolution from perceptions, gestures, actions to the theoretical aspects. Embodied cognition is also useful to analyse the dynamics of the social construction of knowledge by the students: specifically the metaphors, introduced by students in a group or classroom discussion, or by the teacher when (s)he wants the students to concentrate on a particular concept or to construct a new one, reveal powerful tools for supporting and sharing new ideas. The instrumental analysis by Rabardel explores the interactions among students, mathematical concepts and technologies at school. It considers the way in which the technological tools act on the mathematical concepts and the way by which such concepts can model the didactic transposition (*process of instrumentation*). Vergnaud's definition of concept (as composed of: (i) a *reference* system, i.e. "l'ensemble des situations qui donnent du sens au concept"; (ii) the *operating invariants*, which allow the subject to rule the relationship between the reality and the practical and theoretical knowledge about that); (iii) the *external representations*, e.g. language, gestures, symbols,…) is useful to give reason of other complex systemic features of the mathematical conceptualisation: e.g. the abstraction as an invariant, the role of symbols, graphs etc. in such a process, and so on.

Analysing our experiment with the three theoretical tools sketched above, we find that the function meaning built by the students is deeply featured by the mediation of the tools they have used. Typically: the variational and co-variational aspects of functions (*40*), the related building of the space-time blending, the compressed position-velocity relationship represented by pupils' gestures, and so on. Such a rich conceptualisation is marked by a rich linguistic activity, plenty of the embodied features described by Lakoff and Núñez

(*7*): metaphors, blends, fictive motions. As well, conceptualisation is supported and possibly produced by a suitable mediation of instruments and of external representations (often a representation is framed within an instrument through its functions: e.g. the data tables and their scrolling on the display). At the end, the way students describe a function shows deep traces of their actions and interactions with instruments and representations. Such traces are not complementary to the concept but are an essential component of its meaning.

Now we shall sketchily describe two fragments of our experiment, which illustrate the points above: of course, only seeing them in the videotape can give the flavour of what we mean, since pupils' gestures are essential ingredients of the story.

EPISODE 1: Perception and concepts.

We illustrate how the evolution of pupils' *conceptualisation grows to a large extent out of bodily activities* (Conj. 1). Our attention is focused on the *ways* the students construct the meaning of the graph they observe on the screen after the motion experiment. The students of the class (grade 9) are divided in small groups of three-four, each with one calculator.

They have walked starting from the sensor up to a red line traced on the floor, approximately at 4 meters in front of the sensor, with a uniform motion, and then they have come back, with a similar motion. A student says: *"Here* [he is pointing at the starting point of the horizontal stretch of the graph]*, here is when I got to the red line"*. Here the red line is a real object, to which one can refer to describe both the motion and the graph. Then, after some interventions, it becomes an abstract reference point, to indicate a distance, both in space and in time: *"Walking always the same distance"*, *"…He tried to keep the same steps always in the same…in the same time"*. These two aspects support students' transition to the concept of velocity: *"…During the entire period, in which I walked, I tried to…, to make, to keep always the same velocity"*.

After having interpreted the graph, the students work with the calculator, to observe the table of data collected by the sensor and represented on the screen, in order to describe suitably the table itself in terms of their motion. In the interpretation of the table, the students pass through the same steps as in the interpretation of the graph: at the beginning, the red line is conceived in its real meaning: "*I arrived to the red line"*. Then, it is progressively transformed into other concepts, by the exploration of the table through scrolling: "*Scroll!*" and observing that the covered distance is around 4 meters: *"I did, so I did 4 meters"*. Then, the observation that the distance (from the sensor to the student) is increasing: "*The space increases up to 4 meters, then it decreases"* and that there is a maximum value, which corresponds to a distance (from the sensor to the red line) in space and to a time interval: "*Yes, but we wanted to know in how much time he arrived"*.

These excerpts show the nature of the cognitive dynamics of students' conceptual genesis: this is rooted in their actions (running in the class), activities with the artefact (scrolling data) and linguistic productions. At a certain point, the pupils realise that two quantities are needed to describe scientifically the movement: time and space. The students have entered into the mutual relationships between these two quantities, exploring the numerical data on the screen through scrolling. Namely they realised conceptualisation, utilising a function of the instrument (instrumental genesis) and looking for invariant through it. As a result of their activities (language, scrolling, etc.) the very nature of objects changes its status. The red line is emblematic: it represents first a tangible object, traced on the floor, but, going on the conceptualisation, it becomes a reference point, a *cognitive pivot*, for the interpretation both of the graph and of the table. In doing so, it acquires new meanings. In fact, it becomes the initial point of a horizontal stretch in the graph or the maximum value of a number table. In this process, it loses its physical features and become *self-referential.* For example, when a student refers to the table of data, saying *"I arrived to the red line"*, he is referring to the distance in meters observed on the table, more than to the red line. The students' *thinking has grown from their perceptual facts in a self-referential way* (see Conj. 1) and it is spelled out through the same words of *their perceptual-motor activity*, whose meaning is changed and has evolved in time (Conj. 3).

EPISODE 2: Gestures and concepts.

This episode, worked out by D. Paola, illustrates the systemic and non-linear nature of pupils' understanding and thinking mathematics, and its deep connection with the *multidimensional variety of their own perceptual-motor activities* (Conj. 2).

In this episode, a student, Mattia, tries to reproduce a graph, sketched by the teacher at the blackboard, through his movement: the graph of his movement is shown in real time by the sensor on the screen. Erik, a school-mate from the group (grade 9), comments Mattia's movement with expressive gestures of his hands: "*That is first slow* [he moves horizontally his right hand towards right], *then fast* [moving up his right hand very fast], *then down fast* [moving down his hand fast towards left], *then slows down* [moving his hand towards left and describing a concave descending curve in the air], then fast again [again his hand up to the right]…*then it stops* [he moves his hand horizontally towards right]".

Erik's gestures show clearly that he has understood both the movement and the graph. His hand gestures incorporate in a *compressed* way the features of the time law. In fact when the speed is increasing, he moves his hand faster, and when the speed is decreasing, he moves his hand slower. In a Cartesian graph the information concerning the function change and its derivative is coded in a unique sign (i.e. the graph) and, as such, it is not accessible to everyone. The movement of Erik's hand condenses two features: the first (namely the trajectory of his hand) expresses how the function varies (the time law shape); the second (his hand's speed) incorporates the velocity of the moving body. This double embodiment of information is not a coding into an unknown language; it is a 'natural' representation of the movement. In fact, his gestures are more direct representations than the blackboard graph (i.e. a static Cartesian plane with different quantities on the two axes): they represent a mediating tool for grasping the situation in a more feasible way (no transcoding is needed, apart the embodied one). Erik's intervention represents an intermediate level between the external movement and the time law, (i.e. through the Cartesian graph), which is useful to start an understanding process of the scientific features of the motion. It represents a stage towards the interiorisation of this scientific meaning for Erik, but it also creates a possible space of communication for the class, which was not evident before. In fact, other students in the class take again Erik's words and gestures: most of them use the same type of gestures than Erik while discussing the problem.

The different aspects of the function concept (e.g.: its variational and co-variational features, see Slavit (*40*); its first derivative) correspond to different *perceptual-motor activities of the students more or less active according to the context* (Conj. 2). Although their experience contains everyday concepts, their gestures already incorporate the scientific aspects. The teacher linguistically helps the students to transcode their conceptualisation into the scientific language. During the discussion, the scientific words suggested by the teacher give a name to the gesture representation used by the students to describe the situation: they repeat the words together with the corresponding gestures. In this blending of representations they conceptualise in a conscious, intentional and willing way, namely they conceptualise a scientific idea according to Vygotsky. The blending of gestures and words they use shows that their conceptualisation embodies their actions: *that of which they think in that moment emerges from and in these activities themselves* (Conj. 3).

CONCLUSION

The theoretical tools we have chosen to describe the phenomenon of mathematical conceptualisation give a good description of it. In fact, Vergnaud's description of concepts, the instrumental analysis of Rabardel and the stress on the metaphorical nature of mathematical language seem unavoidable tools to start with. Our analysis supports the conjectures made in the “Three conjectures…” section of this research forum, stressing the role that language, external representations and instruments play in developing such an embodied conceptualisation. However, the theoretical tools needed for the analysis must be deepened and widened.

On the one side, our idea of *clusters of experiences* seem a flexible and rich tool for describing mathematical conceptualisation which takes into account also the existence of the symbolic language and does not reduce all mathematics to metaphors. A wider approach is perhaps needed to grasp the phenomenon in its complexity, taking into account cognitive as well neurological results. The results of all people who have worked to this Research Forum show that mathematical concepts must be re-thought today in the light of such new results, namely not only from an epistemological and cognitive but also from a biological point of view (see *13, 41, 42*).

On the other side, it is necessary to analyse the connections between such an embodied approach and other theoretical frames, which describe abstraction and concept building in mathematics from other points of view. For example the anthropological approach to the *ostensives* by Chevallard (*43*) is an intriguing point to debate, insofar it seems to present a complementary analysis of the same kind of phenomena.

INCORPORATING EXPERIENCES OF MOTION INTO A CALCULUS CLASSROOM

Marty Schnepp Daniel Chazan
Holt High School University of Maryland

This contribution to the research forum builds on an examination of one session in a Calculus classroom discussion. Focusing on the teachers' role in supporting mathematical activity in classrooms, this contribution explores implications for the mathematics classroom of views that root mathematics in bodily activity.

If mathematical abstractions grow, to a large extent, out of bodily activities and mathematical understanding and thinking are perceptuo-motor activities what sorts of implications are there for the learning of mathematics in classrooms?

Our contribution to the research forum is based on attempts to integrate discussion of shared experiences with motion into a high school Calculus class. This discussion is grounded in the detail of one teacher's Calculus classroom (that of Marty Schnepp, one of the two authors of this paper) and in the use of a Line Becomes Motion (LBM) device that links the motions of cars on linear tracks with analytic functions displayed on a computer screen. Early on in his Calculus course, Marty uses this device (and others) to teach his students to develop their understandings of motion, to learn to associate mathematical calculations with aspects of motion, and to see and understand velocity graphs in a disciplined (mathematical) way. In our view, this is one attempt to develop mathematics instruction that takes bodily activity seriously as a source of mathematical understandings and insights.

Stimulated in large measure by a comment of Rene Thom's (*44*) and Brousseau's (e.g., (*45*) analyses of the mathematics teaching, we focus on the role of the teacher. In the context of concerns about the New Math, Rene Thom (*44*) says: "Whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (p. 204). In other words, what teachers do in the mathematics classroom with their students reflects their ideas of what mathematics is and how it develops. More recently, Brousseau's (e.g., (*45*) analyses of teaching suggest that this is the case because mathematics teachers have the responsibility to teach mathematics; justification of activity in mathematics classrooms must include an explanation for how the activity is mathematical.

Since teachers' views of mathematical activity are imbedded in instruction, instruction based on the conjectures of this forum would represent a departure from standard practice. In particular, in the context of Calculus, it suggests alternative perspectives both on what it means to learn Calculus and what teachers can do to support student learning. We are interested in interrelationships between a teacher's conceptualization of instructional goals (for example, what it means to understand motion and the role of such understandings in a Calculus course) and issues (for example, challenges in learning and teaching the mathematics of motion) and the instructional possibilities the teacher entertains as plausible or reasonable ways to reach these goals, as they seek to exploit the psychological / physiological links between body moment, eye movement, and imagination.

The task of teaching students to read velocity time graphs, for example, is in part helping students learn to imagine the types of motion that particular graphs may describe, consistent with Tracy Noble, Ricardo Nemirovsky, Cara DiMattia, and Tracey Wright's (*46*) argument that "interpreting a graph or a table entails perceiving a range of possibilities distributed across its spatial layout" (p. 2). This is a part of what they view as learning a "disciplined" way of seeing, mathematical vision. In this way of thinking, learning to interpret a velocity over time graph is inextricably linked with understanding what velocity is. One cannot read such a graph simply as the graph of a function. One must read it as a graph of a velocity function and understand the field of possible motions that such a graph might describe. As they argue:

> Such gradual mastering of visual interpretations is not achieved by the performance of isolated and self-contained sequence of steps, but by interpretive efforts that encompass ways of doing things and domains of familiarity. Experimenting with partial interpretations based on familiar contexts leads, not to a 'blind' set of procedures, but instead, to a complex way of seeing that summons explicit and tacit expectations. (*46 p.31*).

This point of view suggests that there is much more to learning to interpret velocity graphs than simply understanding that negative velocities suggest speed in an opposite direction. Such an understanding is important. But, those who read velocity graphs effectively also appreciate the ways in which such negative velocities interact quantitatively with positive velocities. And, they also appreciate that a velocity graph does not imply a particular starting place.

This is a view of the learning of mathematics where mathematics and lived experience are always in contact, and not just at the beginning and the end of problem solving as is suggested by the words “applying mathematics.” In order to incorporate activity predicated on such views into the mathematics classroom, however, Brousseau's theory (e.g., (*45*) suggests that an argument must be made for the mathematical-ness of such activity.

Fortunately, for the adherents of such classroom activity, there is a range of views of mathematics, including ones that conceptualize a role for experienced motion in the development of mathematics. For example, Philip Kitcher (*47*) offers an evolutionary theory of mathematical knowledge. He suggests that the origins of mathematics lie in sensory perception and the world around us. He then suggests that built on this substratum of experience mathematics grows as an idealizing theory of the world.

> Mathematics consists in idealized theories of ways in which we can operate on the world. To produce an idealized theory is to make some stipulations—but they are stipulations which must be appropriately related to the phenomena one is trying to idealize (*47 p.161*).

Such a theory describes the world not as it is, but as it would be if accidental or complicating features were removed. "Thus we can conceive of idealization as a process in which we abandon the attempt to describe our world exactly in favor of describing a close possible world that lends itself to much simpler description" (p. 120). An important aspect of the development of such simpler descriptions is, in Kitcher's view, a desire to make such descriptions internally consistent.

> … It would be futile to deny that observation is one source of scientific change. The burden of the last paragraph is that observation is not the only such source. There are always "internal stresses" in scientific theory, and these provide a spur to modification of the corpus of [scientific] beliefs…. To oversimplify, we can think of mathematical change as a skewed case of scientific change: all the relevant observations are easily collected at the beginning of inquiry; mathematical theories develop in response to these and all the subsequent problems and modifications are theoretical… (p. 153).

From this point of view, rather than being surprising or inexplicable, the effectiveness of mathematics in the natural sciences is support for the idealizing nature of mathematical theory and for its origins in the world of our senses.

Returning to the classroom, such a perspective on mathematics suggests that if Calculus teachers spend time on students' conceptions of motion, by watching or physically experiencing it in other ways, they have not abandoned mathematics for physics. Instead, by doing so, they are allowing students to built an important proto-mathematical (Kitcher's word!) substratum of experience and vocabulary upon which the mathematics of motion can be built. Similarly, when the use of LBM software reverses the arrow of representation, and examines the degree to which the world of motion represents idealized mathematical theories, the idealized theory is being made accountable to the world it is meant to idealize.

In our contribution to this research forum, we grapple with the question with which we began by focusing on the role of the teacher. Our discussion is grounded in examination of two clips from a single classroom session. Stepping back from the particulars of this classroom session, we hope that our contribution to the research forum will stimulate reflection on relationships between teachers' beliefs about mathematics and the nature of the instruction that seems justifiable to them. In particular, we are interested in the question of relationships between teachers' beliefs about mathematics and the introduction of bodily experience and imagining into the mathematics classroom.

Our focus leads us to the following observations. First, and perhaps obviously, the perspective on mathematics outlined in the "Three conjectures…" section of this research forum departs from many views of the nature of mathematical activity. Second, it suggests alternative perspectives both on what it means to learn Calculus and what teachers can do to support student learning. Finally, reflection on the videotaped class session also suggests some themes that are not central in the central conjectures of the forum. In particular, examination of this classroom session and the teacher's intentions underlines the importance of the social nature of the classroom context. Throughout this session, there are many instances of issues related to language as a vehicle for capturing individual intuitions related to a common demonstration; there are conflicts that arise among student usages and the teacher also plans purposefully to raise issues in an effort to build shared understandings of accepted usages.

SENSORS, BODY, TECHNOLOGY AND MULTIPLE REPRESENTATIONS

Marcelo C. Borba Nilce Scheffer
UNESP URI

This paper presents a case in which collectives of students, a teacher and graphing calculators linked to sensors struggled to coordinate body motion with one of the standard mathematical representations: graphs. The case presented is then linked to the discussion regarding an extended idea of multiple representations in the learning of functions, and discusses how different interfaces change the nature of what is known and how it is known. Epistemological issues regarding the body, humans and non-humans in the production of knowledge are discussed.

As emphasized in the introduction of this research forum section, new theories have been developed regarding the connection between eyes and body. In particular, two questions have been spelled out regarding the connections between kinesthectic activity, body movements and the learning of mathematics: What are the roles of kinesthetic activity, by which we mean bodily actions, gestures, manipulation of materials, acts of drawing, sensory-motor coordination, etc., in the learning of mathematics? How does bodily sensory-motor activity become part of imagining the motion and shape of mathematical entities?

One possible answer to these questions is based on an epistemological view of the role of technology and on a revision of the notion of multiple representations. The case which will be presented illustrates how technologies of information can create links between body activity and representations which are officially recognized by the mathematics academy. We want to claim that open-ended tasks with the use of sensors connected to calculators and mini cars can add new dimensions to the discussion regarding multiple representations which was popular up to the mid 90's. In this way, coordination of multiple representations would encompass more than just representations of mathematical objects which are accepted by academy such as tables, algebra and graphs. Such representations would also have to be coordinated with body actions allowing for the expression of the being (*48, 49*). We claim that this new aspect of coordination expands the epistemology of multiple representations proposed by Confrey and Smith (*50*). In our theoretical framework, knowledge is constructed by collectives that include humans and technologies of intelligence, such as orality, writing and computer technology. In such a view (*51-57*), knowledge is always produced by collectives of humans-with-media, and it is transformed as different media or humans join a given collective.

The analysis presented is about the theme of body movement articulated with the representations attributed to them, i.e., the graphs on the Cartesian plane represented by the software and the calculator, taking into consideration the significant contributions of the gestures, the oral communication, and the interpretation of the students' narratives regarding their experience with that activity.

The fieldwork involved teaching experiments composed of sessions carried out with six students, with the researcher interacting with one pair of students at a time, in a combination of interviews and teaching-learning situations based on several authors (*57-61*). The teaching experiments were conducted in a computer laboratory at UNESP - a university in Rio Claro, São Paulo, Brazil - in at least six sessions per pair during the year 1999. The sessions, lasting 60 minutes each, involved ten different activities related to the theme of movement which were carried out with the use of CBR and LBM. Computer based Ranger (CBR) and Line Became Motion (LBM) are devices which connect standard mathematics representations with movements developed by humans or things. The sessions were video-taped by a technician. The research subjects were eighth-grade students, between the ages of 13 and 15, from a low income public school in the city of Rio Claro, São Paulo, Brazil. Prior to the teaching experiments, they had participated in classroom activities involving calculators, computers and sensors.

The six students were divided into groups of two. The decision to work with pairs of students was due to the fact that, when working with pairs, more discussion occurs between the students, with each showing their reasoning in a more detailed manner, explaining, clarifying answers, promoting their ideas, and mutually supporting each other. Regarding this interaction, Fontana and Frey (*62*) point out that, in addition to the personal revelation of feelings and emotions, difficulties may arise because the group may be dominated by one person. Attentive to this possibility, the interviewer (*63*) sought to take care to maintain balanced participation between the members of the pairs throughout the development of the activities.

Each session was structured in accordance with the activities previously elaborated, in such a way as to promote discussion between the students as well as with the interviewer. Paper, pencil, chalk and a chalkboard were always available to the students to use whenever necessary, in addition to the graphing calculator, sensors, and a computer when the LBM was being used. The video-tapes were viewed following each session to look for situations that stood out relative to the research questions, as well as problems, the students' understandings, and the effort and interventions of the interviewer; these, in turn, contributed to the re-organization of subsequent sessions in such a way as to give more attention to certain situations. In the episode we chose to present, the LBM was not involved. Only the CBR linked to a graphing calculator was available. It was made clear to the students that the sensor (the TI-CBR) would measure the distance from itself to an obstacle in front of it. The sensor has an internal clock that runs for 15 seconds. Once the students started the device, the graphing calculator would graph the distance to an obstacle for 15 seconds.

The main actors in the episode that will be described are André (A) and Naíta (Na), interviewed by one of the authors of this paper, Nilce Scheffer, who is identified as "Ni", in the transcription. The episode was extracted from the third meeting with these eighth graders. The episode took place in a small classroom (4.5 meters x 6.5 meters) which is normally used for research seminars. André and Naíta were already familiar with the calculator and the CBR, since all the students had used the calculator in regular class prior to the experiment. As André was asked to make any movement he wanted with the sensor, he chose to position himself in the middle of the room and turned his own body around with the CBR pointing against the walls as he moved. As he did this, the following dialogue took place:

Ni: Look, he's making the movement rotating the . . .

Na: No, but if you stay in one place and just keep rotating the hand like that?

Ni: That could be, too.

A: Draw it on the blackboard?

Ni: Yeh. . . . .

[André draws a circle on the board, Fig. 1].

Ni: Why?

A: Because I was turning, right?

Ni: Where were you?

A: In the center.

Ni: So you think that the movement would turn out like that?

A: Right.

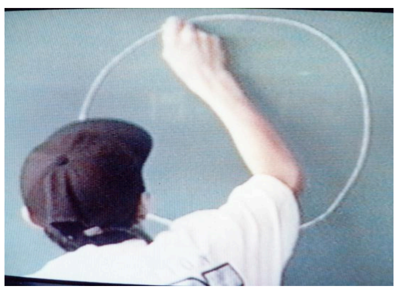

Figure 1

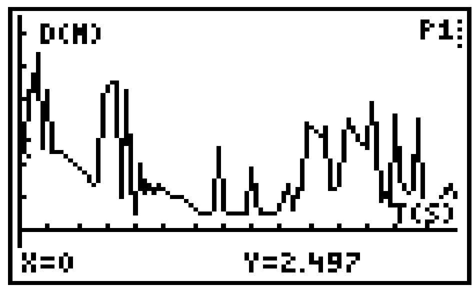

Figure 2

Before André and Naíta could see what kind of graph was generated by the graphing calculator as he moved his own body around, André remembered that the researcher had asked him to draw a graph on the blackboard representing what he expected to see on the calculator screen. André drew a graph that resembled the movement of his body. If we are correct in our interpretation, the graph he drew represents the circle made by his hand as he turned himself around. Probes were made by the researcher which strengthened our interpretation. As the episode continued, the graph of the calculator was shown to them. They were both very surprised, both because of the fuzziness of the graph, and because they could not make sense of the peaks of the graph (see figure 2). The former is linked to limitations in the accuracy of the sensor and to some shaking in Andre's hand, and the latter is due to the corners of a rectangular room. It took quite a while to make sense of it, and it was Naíta who said "And so it catches the distance from the point where he was to the board, or the window, or the door [pointing to those locations in the room]". Naíta, in our interpretation, came up with an explanation for the bumps in the graph displayed.

The discussion was richer then we can describe in this paper, but it became even more interesting after Naíta convinced André that the graph could not be a circle and the latter tried to defend his conjecture by claiming that, if the room had a circular shape, then the graph he drew on the blackboard would be correct. His attempt to coordinate his body movement with the graph led to the following dialogue:

> A: There, in that circular motion [pointing to the figure he drew on the board], being in a round room, right, and exactly in the center [repeats the movement with his body], then the distance would be only one.
>
> Ni: Then how would the graph turn out?
>
> A: It would turn out circular.
>
> Ni: Would it turn out circular?
>
> Na: I think that, since the calculator represents it using straight lines, then it would be just a straight line. [she makes a gesture suggesting a constant function]
>
> Ni: Hmm. Interesting. Then if we were in a circular room like André is saying, how would this graph turn out? [pointing to the graph on the calculator (Fig.2)]
>
> Na: With it straight- it would turn out just a straight line . . . If he kept his arm, for example, stretched out, and if he didn't pull his arm, it would be just a straight line, because it would be the same distance.

This last part is very rich for our analysis. The first sentence André said reinforces our conjecture that he was thinking of a circle-like movement. Next, he attempted "to save" his solution as he argued that, if he were in a circle-shaped room, his graph would be correct. Naíta takes the lead at this point, arguing that the graph would be a straight line, and using regular mathematical terminology, it would be a constant function, $y(t)=c$, where "t" is time and "y" is distance. She uses her hand and arm to describe it, and uses the expression "one distance", which we interpreted as being "the same distance from the wall as he rotates in a circle-shaped room".

In the episode described above, the student draws a graph on the blackboard which resembles the trajectory of the CBR (connected to his body) in the air. There are numerous examples in the literature, since at least the 1980's, of students who draw graphs which resemble movements. What is interesting in the situation under scrutiny here is that the students are dealing with their own body, their very first notion of space according to Bicudo (*49*), and connecting them to graphs, which are possibly the most familiar official representation of function they have. As André performed the action, they (André, Naíta, Nilce and the different technologies of intelligence available) had to coordinate body experience and graphic representations. It should be emphasized that this problem can be seen in different degrees of complexity. Generating a graph of distance (versus time) of the CBR to a target in a rectangular room, with doors, windows and blackboards, is a difficult task. If the speed at which his body was turning is brought into account, the problem is even more complicated. If we consider the room as a square and remove the blackboards and other objects which can generate non-smooth lines on the calculator, the problem becomes easier; and if we consider that the movement is performed at a constant speed, it is still not easy to arrive at the kind of prototypic function that would model the movement, much less come up with a specific trigonometric function either in the graphical or algebraic representation.

Of course, we would typically not expect 8th graders to take this path to coordinate algebraic expressions with graphs and body movement. But they did, in their discussion, consider the corners and other “points” which could cause problems; they seemed to consider that the room was a square, and one of the students found a solution for the case of a circular room. It is relevant to observe that open-ended, “simple” tasks with the different technological interfaces available were able to generate the difficult tasks that this collective of humans-with-media (*51, 52*) had to face. The analysis suggests that there is potential to explore mathematical concepts at different levels (middle school, high school and introductory mathematics at the university level). Coordinating the interfaces of the body with the technological interfaces can be seen as an overall goal of a collective when it is producing knowledge in situations like the one presented in this paper.

We believe that this episode illustrates how different interfaces change the possibilities of our thinking. In this case, the CBR and the imagination of the student were able to transform an open-ended task into quite an interesting, defined problem. Moreover, it makes room for the transformation of the notion of multiple representations, bringing a new twist to the theoretical discussion regarding multiple representations issues started in the late 80’s with the popularization of micro-computers and the availability of softwares which enabled students to deal with graphs and tables in ways not possible before. Again, we believe that now technologies like LBM and CBRs, which became more popular and available in the late 90’s, help to shed new light on this discussion. Sensors and interfaces linked to software were able to closely connect the movement of students and of objects to graphs in the examples developed in this teaching experiment. Body movements and graphs can be linked to tables and algebraic representations, and new research has been developed since then to show how this is possible. Many authors have emphasized the role of the body in an attempt to overcome the body-mind dichotomy. Such a discussion emerged naturally as researchers saw students and teachers refering to their bodies and using them as they used interfaces which connected motion to graphs and other representations. Of course, such a discussion was not completely absent two decades ago. For example, several years ago, Borba (*58*) referred to pointing and gesturing as students dealt with Function Probe (*64*), a multiple representation software which had tables, graphs and algebraic “facilities”. But the body has been only peripheral to the multiple representation discussion (*65, 66*); similarly, the discussion about functions, and even different representations, has not been emphasized enough in the discussion about body motion, although the relation to graphic representation has been stressed (see, for example, the special issue of ESM (*67*).

The availability of interfaces such as sensors makes it possible to expand the notion of multiple representations beyond the coordination of standard mathematical representations to include even the notion of body. Our example supports this idea and adds evidence to the third conjecture presented in the “three conjectures...” section of this research forum. This conjecture states that thinking is not an internal process; even mental calculation, for example, often involves sketching lines and moving inscriptions. Conjecture three is in line with this extended notion of multiple representations, as thinking is seen as a complex activity involving elements which are “inside” and “outside” humans. If such a conjecture were to become more widely accepted, there would be consequences regarding the elaboration of didactical material (written and manipulatives) and the design of new computer technology, as it will matter how to scratch, what to group and so on. There is another point in which conjecture three gives support to the case presented in this paper and to the theoretical construct we have developed: the notion of humans-with-media. Since this conjecture emphasizes that the complex activity involved is not an internal process - but involves grouping objects and sketching lines, in the case of calculating for instance - the notion of "inside" and "outside" is not so clear anymore. The one who knows is not only the "lonely knower" nor a collective formed only of humans. The basic unit of knowing always involves non-humans, and in particular, non-human actors such as media, that Levy (*55*) called technologies of intelligence and the knowledge produced changes as new interfaces are added to collectives of knowing. Likewise, the coordination of multiple representations can gain new dimensions, as we hoped to illustrate with our case. This is a possible way to answer the research questions outlined at the beginning of this Research Forum.

BECOMING FRIENDS WITH ACCELERATION: THE ROLE OF TOOLS AND BODILY ACTIVITY IN MATHEMATICAL LEARNING[1]

Chris Rasmussen
Purdue University Calumet

Ricardo Nemirovsky
TERC

As part of a larger study investigating students' learning of central ideas in dynamical systems, this case study traces the evolution of one university student's mathematical learning as she worked with a physical tool called the "water wheel." In particular, we characterize how bodily activity and tool use combine in mathematical learning and how this combination suggests alternative characteristics of knowing. We then relate this analysis to the three conjectures outlined at the beginning of this paper.

In this paper we elaborate on a particular type of knowing that Broudy (*68*) called "knowing with." He proposed that this form of knowing is different from two other types of knowing that are usually proposed: "knowing that" and "knowing how." Knowing-that is declarative knowledge, the type that is typically expressed in verbal assertions and theory-like elaborations. Knowing-how is performative and gets expressed in actual performances. A typical example used to make the distinction is, say, that of the tennis player who masters certain types of serves by being able to do them (knowing-how) and the tennis analyst who describes what bodily abilities enables certain players to be good at those servings (knowing-that). The phrase "knowing with" means that there is something else of great significance: the sensitivities and perspectives that we come to hold as we become familiar with a tool, technique, lexicon, and so forth. Using a different language, Polanyi (*69*) made similar distinctions.

For example, take the case of knowing a foreign language. Becoming a fluent speaker in a foreign language entails, in addition to knowing how (e.g., utter correct expressions appropriate to the circumstances) and knowing that (e.g., stating grammatical rules), developing certain views and sensitivities regarding the things talked about. These views and sensitivities enable us to grasp humor, poetry, word games, and many other phenomena that are difficult or impossible to translate—they constitute what we would call knowing-with the foreign language. In reality all three forms of knowing play out together with more or less relative prevalence.

We propose that knowing-with is an essential and not always recognized aspect of developing fluencies with tools and techniques in mathematics education. As an elaboration and refinement of what Nemirovsky, Tierney, and Wright (*41*) refer to as "tool perspective," knowing-with suggests an important alternative characterization to knowing.

[1] This paper is based upon work supported by the National Science Foundation under Grant No. 9875388. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

The study. We conducted a total of eight, 90- to 120-minute open-ended individual interviews with three students. During each interview, students engaged in a number of different tasks involving a physical tool called the water wheel. Our goal was to use these tasks as a springboard for student exploration of mathematical ideas that were of interest to them, rather than as a strict progression of problems to complete. As described next, the water wheel includes an optional computer hook-up, which enables one to generate real-time graphs of angular velocity and angular acceleration. Reflection on such graphs became a major theme in the interviews.

As shown in Figure 1, the water wheel consists of a clear, circular acrylic disc holding 32 plastic tubes around its perimeter. The disc is mounted on an axle and is free to rotate a full 360° and tilt between 0 and approximately 45°. In the middle of the disc are two concentric clear plastic cylinders that contain a variable amount of oil that acts as damping for the system. The amount of oil can be adjusted by raising or lowering the oil reservoir.

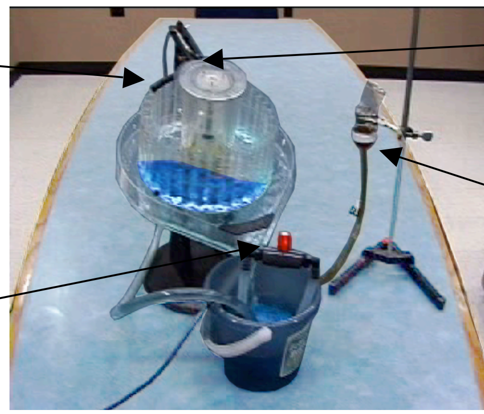

Figure 1: Water wheel

Colored water from a bucket with a submersible pump with adjustable flow rate, showers into several contiguous tubes. Each tube has a small hole that allows water to drain out, which then collects in a drip pan and is directed back into the bucket containing the submersible pump for a continual flow of water into and out of the tubes. When the wheel is tilted and there is even a slight variation in the way water is distributed, gravity causes the wheel to rotate. The optical sensor collects data for real-time displays of angular velocity and angular acceleration versus (*70*).

Each student we interviewed had completed three semesters of calculus and had taken or was taking differential equations. All interviews were videotaped and transcribed. Summaries of each interview were developed and compared across all interviews. The analysis of these interviews focused on how tool use and bodily activity combine in mathematical learning and resulted in the elaboration of the knowing-with construct (*71*). In this report we focus on the learning of one student, Monica, because it was most helpful in our understanding the role of tools and bodily activity and for framing the mathematical learning of the other two students.

Analysis. We present our analysis of Monica's engagement with the water wheel in two parts. In part one, which occurred during the first interview, Monica synchronized the rotation of the water wheel with given graphs of angular velocity versus time. In the excerpt described, Monica personified the wheel and imaginatively experienced when the wheel will achieve its maximum and minimum velocity. In the process, she accounted for why these maximum and minimum velocities occurred. Our analysis highlights how being the water wheel can engage both knowing-how to be the wheel and knowing-with the water wheel. In part two, which occurred during the second interview, Monica predicted the angular acceleration versus time graph for the same angular velocity versus time graph discussed in part one. Our analysis illustrates how Monica developed a powerful way to think about where and why acceleration is zero and how she coordinated qualities of a graph with animation and personal traits that contributed to her *knowing* angular acceleration *with* the water wheel. In so doing, we trace how Monica became friends with acceleration and highlight the centrality of bodily activity in this process.

Part 1. After having become familiar with the water wheel's various parameters (tilt, water flow rate, and amount of oil), Monica predicted the rotational movement of the wheel given different angular velocity versus time graphs like the one shown in Figure 2

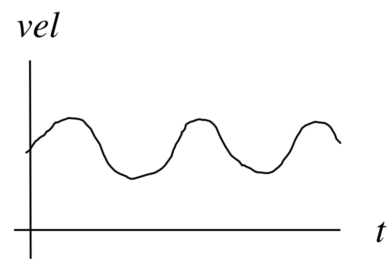

Figure 2: Angular velocity graph

While moving her hand in a counterclockwise direction on top of a motionless wheel, Monica imaginatively engaged in how the wheel is "speeding up speeding up speeding up" and then "slowing down slowing down slowing down." As her hand traced the rotation of the water wheel, her description of the motion did not always follow in real time but rather stretched the temporal space. For example, at one point when she gestured the fastest moment for the wheel her hand actually lingered the longest, as did her words "speeding uuuuup". We see Monica's efforts as a form of being the tool for the purpose of telling the story of the wheel. In knowing-how to be the tool with her hand, a spirit of playfulness is not subjugated to precise coordination of gestures and utterances to the physical constraints of rotational movement.

As Monica continued in her exploration, she gave the wheel a voice. Similar to the way that a puppeteer gives voice to a puppet complete with wants, dislikes and emotions, Monica knows-how to make the wheel talk. Speaking in the voice of the water wheel she pointed at the part of the wheel where most of the weight would be located and said, "Man, I'm no longer, [pointing at bottom of the wheel] being pushed down. Now you want me to go back up. I don't want to go up. So, I'm unhappy. But, I'm going to go up anyway because you're pushing me." Giving the wheel a voice is significant because this puppeteering, which engages linguistic and kinesthetic resources, lends itself to developing certain sensitivities to when and why velocity is maximum and illustrates both knowing-how to be the wheel and *knowing* angular velocity *with* the wheel. As the episode continued, Monica explained that what makes the wheel happy or satisfied is being able to go with the flow not against, being able to go with life and not having to go against it, being able to go with gravity. She goes on to explain that the happiest moment for the wheel is “right before you hit the bottom.”

> *Monica*: Soon as you get, the moment before you hit the very bottom [points to the bottom of the wheel]. That mo[ment], that instant moment, right before you hit the bottom, you are just like, ah, loving it. Because gravity, because you’re going with, with gravity [gestures on the left side of the wheel]. But, as soon as we have to fight against something [gestures on the right side of the wheel], we're not happy. And, it’s that time in between that’s critical. And it’s that time in the, in between [points to the bottom of the wheel] where we reach our peak points in our graph. Maximums and minimums. This is fun!

As the excerpt ends, Monica made explicit the connection between the “peak points” on the graph, the happiest moment of the wheel, and the maximum velocity. We see in her utterances traces of knowing-that, due to Monica’s theory-like elaboration regarding the role of gravity in relation to the wheel’s changing velocity. However, what was a lifeless graph in Figure 2 is now imbued with human traits, wants, dislikes, regions of satisfaction, and regions of dissatisfaction. Important mathematical understandings and interpretations of the angular velocity versus time graph grew in Monica as she acted out and animated the rotation of the water wheel. In part two, Monica continues her animation of the water wheel as she becomes “friends with acceleration.”

Part 2. In the second interview Monica began by physically moving the wheel with her hand to generate angular velocity versus time graphs compatible with those presented in first interview (the water pump was turned off and the computer was connected). She then switched the computer setting to display angular acceleration versus time and gently rotated the wheel to obtain a real time graph of angular acceleration versus time and questioned, “What happens when my acceleration goes across the horizontal?” She explained that she was “trying to think out where these graphs are coming from.” She went on to say, “at the peaks [of the velocity vs. time graph in Figure 2)], well I remember derivatives [laughs], the peaks is where acceleration is zero. But, now that I know that, I want to think about why it is true. I mean it’s cool that I know that, but now I need to know why I know that, that to be true.” Monica knows-that acceleration is zero at the maximum and minimum velocity by recalling the fact that acceleration is the derivative of velocity. On the one hand, Monica knows-that the angular acceleration versus time graph crosses the t-axis whenever there is a “peak” in the angular velocity versus time graph. On the other hand, she reports a certain level of dissatisfaction with this knowledge, indicating that she wants to know why she knows-that.

Monica then simulated the rotation of wheel with her hand for fast slow fast slow rotation in one direction in order to think about what an angular acceleration versus time graph should look like, commenting that, “the thing of it is, is that acceleration and me are not that good of friends.” As Monica continued to simulate the rotation of wheel with her hand for fast slow fast slow rotation in one direction she commented that “everything just left my brain” and that she is “afraid of acceleration.” Nevertheless, she continued to animate the wheel, consistently using the pronoun "it" while she described the rotational movement of the wheel. For example, she starts off by saying, "It starts increasing here [points to the right side of the wheel]. And it's at its highest point when, right before you get to the bottom. And then it starts decreasing." What is the "it" for Monica? We interpret "it" to primarily be the wheel's change in velocity, or what Monica earlier referred to as the incremental velocity. In support of this interpretation, Monica asked herself more than once about the change in speed. For example, she asked herself, "How fast is it decreasing?" In addition, as the episode continued she tended to follow each of her statements that contain "it" with a conclusion about acceleration. Next, we illustrate how Monica comes to *know* where and why acceleration is zero *with* the water wheel.

> *Monica*: Because here [on the right-hand side of the wheel] I'm constantly increasing. So, my acceleration is constantly positive? Because, at first, it's a small increment of time. And it increases in small increments, and it increases kind of slow. So, my acceleration would be on the lower side. But it, it'd be increasing. Here [near the bottom of the wheel on the right-hand side], it'd be increasing even more so. And then across the axis [points to the very bottom of the wheel], because now I'm no longer going faster [gestures on the right-hand side of the wheel]. I'm going slower [gestures on the left-hand side of the wheel]. So, my acceleration, [holds her hand up like an imaginary graph] my zero point, is the difference between when I'm going faster and when I'm going slower.

In this excerpt we see an important shift in the pronouns Monica used. Specifically, Monica shifts from "it" to "I", indicating new ownership of the idea of acceleration. This shift is significant because it signals a new level of being the tool – a level in which Monica begins to *know* acceleration *with* the tool. For example, Monica said, "I'm constantly increasing," and shortly thereafter she said, "I'm no longer going faster. I'm going slower. So, my acceleration …" In the process of telling the story of the wheel, Monica became the wheel and took ownership of acceleration as a quality of the wheel that she can experience through her hand simulation of the wheel's rotation.

By becoming the wheel, Monica becomes friends with acceleration. She experiences where acceleration is zero and why acceleration is zero at the bottom of the wheel's rotation. She herself recognized the significance of her activity, commenting that becoming aware of the location of zero acceleration, what this means, and why it is zero "was kind of nice." The growth of Monica's emerging views and sensitivities to acceleration is not aptly characterized by either knowing-how or knowing-that. Rather, Monica's becoming friends with acceleration is more appropriately characterized as *knowing* acceleration *with* the water wheel.

Discussion. We next reflect on the previous analysis in light of the second of the three conjectures about the relationship between bodily activity and mathematical learning. The point we want to pick up on in this conjecture is the implication that understanding a mathematical concept, like acceleration, spans diverse perceptuo-motor activities rather than having a definitional essence. In support of this implication, the case of Monica illustrates how understanding, in particular a form of knowing we characterized as *knowing-with,* is more like organic growth that occurs over time and engages linguistic, perceptual, and bodily activities. This characterization offers an alternative to the immediate transfer or the direct application of knowledge. From a traditional theory of transfer perspective, we would expect the fact that Monica knows-that the acceleration is the derivative of velocity and the fact that she knows-that the peaks of the angular velocity versus time graph correspond to zeros of the angular acceleration to be the core that would direct her efforts to elaborate on where and why acceleration is zero. As evidenced by our analysis in part two, however, this was not the case. Instead, her emerging understandings are better characterized as *knowing* acceleration *with* the wheel, a process that occurs over time and involves the whole person, not just direct application of some definitional essence like the relationship between acceleration and velocity.

One of the pointers from the transfer literature that speaks to this issue comes from the work of Judd (*72*), who reacted against Thorndike's split of learning into countless narrow abilities that remain isolated, self-contained, and grouped by "identical elements." Through the analysis of experimental work as well as observations from everyday life, Judd strove to foreground the issue of *how* experiences connect to each other. He stressed that one cannot infer from a description of a task, or from a generic characterization of the subject, what is going to be generalized or "transferred." Indeed, we did not know ahead of time that where and why acceleration is zero would be a source of inspiration for Monica. Our analysis resonates with Judd's view because rather than viewing learning as a mechanistic process that can be planned out in advance once the identical elements are identified, *knowing* acceleration *with* a tool like the water wheel is something that grows and emerges in students and engages linguistic, tactile, and perceptual modalities.

References

1. S. D. Teasley, & Roschelle, J., in *Computer as Cognitive Tools* S. P. Lajoie, & Derry, S. J., Ed. (Lawrence Erlbaum Associates, Hillsdale, NJ, 1993) pp. 229-258.
2. D. L. Ball, *American Educator* **16** (1992).
3. P. Cobb, E. Yackel, T. Wood, *Journal for Research in Mathematics Education* **23** (1992).
4. R. Lesh, T. Post, M. Behr, in *Developments in school mathematics education around the world* R. Streit, Ed. (National Council of Teachers of Mathematics, Reston, VA, 1987).
5. L. Meira, *Journal for Research in Mathematics Education* **29**, 121-142 (1998).
6. N. J. T. Thomas, *Cognitive Science* **23**, 207-45 (1999).
7. G. Lakoff, R. Núñez, *Where Mathematics comes from: how the embodied mind brings mathematics into being* (Basic Books, New York, 2000).
8. J. Decety, *Behavioral and brain research* **77**, 45-52 (1996).
9. J. Decety, *Cognitive Brain Research* **3**, 75-86 (1996).
10. M. Critchley, *The parietal lobes* (Edward Arnold, London, 1953).
11. E. Mayer *et al.*, *Brain* **122**, 1107-1120 (1999).
12. D. Neuman, University of Göteborg (1987).
13. B. Butterworth, *The mathematical brain* (Macmillan, London, 1999).
14. R. Gelman, C. R. Gallistel, *The Child's Understanding of Number* (Harvard University Press, Cambridge, MA, 1990).
15. S. Zeki, *A vision for the brain* (Blackwell Scientific Publications, Oxford, 1993).

16. O. Sacks, R. Wasserman, *The New York Review of Books* **34**, 25-33 (1987).

17. D. I. McCloskey, S. M. Sokol, R. A. Goodman, *Journal of experimental psychology: General* **115**, 307-330 (1986).

18. M. McCloskey, A. Caramazza, A. Basili, *Brain and Cognition* **4** (1985).

19. L. Cipolotti, B. Butterworth, G. Denes, *Brain* **114**, 2619-37 (1991).

20. A. Martin, C. L. Wiggs, L. G. Ungerleider, J. V. Haxby, *Nature* **379**, 649 –652 (1996).

21. S. T. Grafton, L. Fadiga, M. A. Arbib, G. Rizzolatti, *NeuroImage* **6**, 231-6 (1997).

22. J. Decety, M. Jeannerod, *Behavioral Brain Research* **72**, 127-134. (1996).

23. M.-N. Magnie, C. T. Ferreira, B. Giusiano, M. Poncet, *Neuropsychologia* **37**, 67-74 (1999).

24. E. K. Warrington, R. A. McCarthy, *Brain* **100**, 1273-1296 (1987).

25. L. Cipolotti, E. K. Warrington, B. Butterworth, *Cortex* **31**, 73-86 (1995).

26. S. W. Anderson, A. Damasio, H. Damasio, *Brain* **113**, 749-66 (1990).

27. W. James, *The principles of psychology*. (Harvard University Press, Cambridge, MA, 1983).

28. J. Haueisen, T. R. Knosche, *Journal of cognitive neuroscience* **13**, 786-92 (2001).

29. L. Knuf, G. Aschersleben, W. Prinz, *Journal of experimental psychology: General* **130**, 779-798 (2001).

30. B. Laeng, D. S. Teodorescu, *Cognitive Science* **26**, 207-231 (2002).

31. V. Ruggieri, G. Alfieri, *Perceptual and Motor Skills* **75**, 287-290 (1992).

32. J. O'Regan, A. Noe, *Behavioral and brain sciences* **24** (2001).

33. P. Boero, R. Garuti, E. Lemut, T. Gazzolo, C. Lladò, paper presented at the 19th Annual Meeting of the International Group for the PSychology of Mathematics Education 1995.

34. D. Tall, paper presented at the Plenary presentation for ATCM conference, Chang Mai, Thailand 2000.

35. P. Rabardel, "Eléments pour une approche instrumentale en didactique des mathématiques" (IUFM Caen, 1999).

36. P. Rabardel, *Les hommes et les technologies* (Armand Colin, Paris, 1995).

37. M. Artigue, paper presented at the 2° CAME Symposium, Utrecht, The Netherlands 2001.

38. J. B. Lagrange, *Educational Studies in Mathematics* **43**, 1-30 (2000).

39. G. Vergnaud, *Recherches en Didactique des Mathématiques* **10**, 133-170 (1990).

40. D. Slavit, *Educational Studies in Mathematics* **33**, 259-281 (1997).

41. R. Nemirovsky, C. Tierney, T. Wright, *Cognition and Instruction* **16**, 119-172 (1998).

42. S. Dehaene, *The Number Sense: How the Mind Creates Mathematics* (Oxford University Press, New York, 1997).

43. M. Bosch, Y. Chevallard, *Recherches en Didactique des Mathématiques* **19**, 77-124 (1999).

44. R. Thom, in *Developments in mathematics education* A. G. Howson, Ed. (Cambridge University Press, Cambridge, 1973) pp. 194-209.

45. G. Brousseau, *Theory of didactical situations in mathematics*. (Kluwer, Dordrecht, 1997).

46. T. Noble, R. Nemirovsky, C. DiMattia, T. Wright, *International Journal of Computers for Mathematical Learning* (Under review).

47. P. Kitcher, *The Nature of Mathematical Knowledge* (Oxford University Press, Oxford, UK, 1983).

48. M. Merleau-Ponty, *Fenomenologia da Percepção* (Editora Martins Fontes, São Paulo, 1994).

49. M. A. V. Bicudo, *Fenomenologia: confrontos e avanços* (Editora Cortez, São Paulo, SP., 2000).

50. J. Confrey, E. Smith, in *Mathematical Thinking and Problem Solving* A. H. Shoenfeld, Ed. (Lawrence Erlbaum Associates, Mahwah, NJ, 1994).

51. M. C. Borba, M. G. Penteado, *Informática e Educação Matemática*, Coleção Tendências em Educação Matemática (Editora Autêntica, Belo Horizonte, MG., 2001).

52. M. C. Borba, in *Pesquisa em Educação Matemática: Concepções e Perspectivas* M. A. V. Bicudo, Ed. (Editora UNESP, São Paulo, SP, 1999) pp. 285-295.

53. M. C. Borba, M. E. Villareal, paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics, Stellenbosch, South Africa 1998.

54. D. Kerchove, *A pele da Cultura (Uma Investigação sobre a Nova Realidade Electrónica)* (Relógio D´Água Editores, 1997).

55. P. Levy, *As Tecnologias da Inteligência: o Futuro do Pensamento na era da Informática* (Editora 34, Rio de Janeiro, RJ, 1993).

56. P. Levy, *A Inteligência Coletiva: por uma Antropologia do Ciberespaço* (Edições Loyola, São Paulo, SP, 1998).

57. M. E. Villareal, Tese de doutorado, IGCE, UNESP (1999).

58. M. C. Borba, Doctoral Dissertation, Cornell University (1993).

59. P. Cobb, L. Steffe, *Journal of Research in Mathematics Education* **14**, 83-94 (1983).

60. J. Confrey, in *Epistemological Foundations of Mathematical Experience* L. Steffe, Ed. (1991).

61. L. Steffe, P. Thompson, in *Research Design in mathematics and science education* R. Lesh, A. E. Kelly, Eds. (Erlbaum, Hillsdale, N.J:, 2000) pp. 267-307.

62. A. Fontana, J. H. Frey, in *Handbook of Qualitative Research, Sage Publications* N. K. Denzin, Y. S. Lincoln, Eds. (2000) pp. 645 - 672.

63. N. F. Sheffer, Tese de doutorado, IGCE, UNESP (2001).

64. J. Confrey. (Santa Barbara, CA, 1991).

65. M. C. Borba, paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics 1994.

66. M. C. Borba, J. Confrey, *Educational Studies in Mathematics* **31**, 319-337 (1996).

67. J. Ainley, D. Pratt, *Educational Studies in Mathematics* **45** (2001).

68. H. S. Broudy, in *Schooling and the acquisition of knowledge* R. C. Anderson, R. J. Spiro, Eds. (Lawrence Erlbaum Associates., Hillsdale, NJ:, 1977) pp. 1-17.

69. M. Polanyi, *Personal Knowledge. Towards a Post-Critical Philosophy*. (The University of Chicago Press, Chicago, IL, 1962).

70. R. Nemirovsky, *Interactive Learning Environments* **3**, 151-175 (1993).

71. C. Rasmussen, R. Nemirovsky, J. Olszewski, K. Dost, J. Johnson, *Educational Studies in Mathematics* (in review).

72. C. H. Judd, *Educational Review*, 28-42 (1910).