USING RESEARCH TO INFORM PRACTICE: CHILDREN MAKE SENSE OF DIVISION OF FRACTIONS

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The purpose of this paper is to share the strategies that children invented to solve problems involving division of fractions in the absence of algorithmic instruction. The data come from two sources, the first being a teaching experiment and the second being the regular classroom practice of this teacher/researcher. It was found that when the conditions of the teaching experiment were replicated, the three methods that students constructed to solve the identical problems involving division of fractions were the same as those seen in the original group. The three methods involved reasoning concerning natural numbers, measurement and fraction knowledge, all of which relate to counting.

INTRODUCTION AND THEORETICAL FRAMEWORK

In the 1993 – 1994 school year, a research study was conducted in the form of a teaching intervention into a regular classroom, in a 4th grade suburban elementary school in NJ.[1] The topics addressed included several related to fractions. The premise of the study was that students could be provided with opportunities to develop an understanding of fraction concepts before the formal introduction of algorithms. The selection of a 4th grade is significant because it is the year prior to the one in which students are formally introduced to most algorithms involving fractions.[2] Several research papers (including Maher, Martino, R. B. Davis, 1994; Alston, R. B. Davis, Maher, Martino, 1994; Steencken, Maher, 1998; Martino, Maher, 1999; c.f. Steencken, 2001; c.f. Bulgar, 2002; Bulgar, Schorr, Maher, 2002) have traced and documented the emergence and development of powerful mathematical ideas about fractions, and the conditions that were in place in an attempt to better understand how children construct knowledge about fractions in the absence of algorithmic instruction. Other researchers have also investigated the development mathematical ideas prior to the introduction of more formalized procedures. Kamii and Dominick (1997) compared students who were taught algorithmically with a group that was not. They found that the non-algorithmic group was more successful and that even when they made errors, the errors resulted in more reasonable responses. They concluded that teaching algorithmically may impede number

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2 New Jersey's Core Curriculum Mathematics Content Standards have been designed to be philosophically consistent with the NCTM Standards. The current version represents revisions made in 2002.

sense as it adversely affects what was taught about place value and forces children to give up their own thinking.

> When we try to teach children to make relationships between numbers (logico-mathematical knowledge) by teaching them algorithms (social-conventional knowledge), we redirect their attention from trying to make sense of numbers to remembering procedures. (Kamii & Dominick, 1997 p. 59).

Detailed studies of several of the sessions in the above-mentioned teaching intervention have been undertaken (Steencken, 2001; Bulgar, 2002). During the four sessions dealing with division of fractions, it was observed that children used one of three successful means, all related to counting, to solve the problems.

In an attempt to achieve similar outcomes in a regular classroom setting, the conditions of the teaching experiment were identified, studied and then replicated by this teacher / researcher, as part of regular classroom practice. This teacher / researcher had spent years studying the development of children's mathematical ideas and thought deeply about what was observed. Three important elements were noted and replicated. First was the establishment of a classroom community, described below in greater detail, that is consistent with recommendations set forth by the National Council of Teachers of Mathematics (NCTM 2000) and conducive to the development of an inquiry-based classroom. The other significant elements are the selection of appropriate tasks and teacher interventions designed not to interfere with the natural trajectories of children's thinking.

This particular paper seeks to resolve the following issues. How can we use what has been gleaned from the teaching experiment to make a positive impact on classroom teaching? And perhaps more importantly: Are the strategies that children constructed so robust that under similar conditions comparable favorable outcomes can be achieved?

METHODS AND PROCEDURES

This study involved replication for a multiple-case research methodology. Further, the design is considered to be a literal replication since each case was expected to yield similar outcomes as a result of similar conditions being in place (Yin 1994).

BACKGROUND, SETTING AND SUBJECTS

The teaching experiment took place in a small suburban NJ district, over the course of a year. The focus of this intervention was to investigate the development of children's mathematical ideas about fractions. This particular 4th grade class consisted of 25 students. The four sessions involving division of fractions took place in December 1993.

One of the goals of this teaching experiment was to create a classroom community in which student inquiry and discovery were of paramount importance. Students were not told that their work was correct or incorrect. Instead, they were questioned and encouraged to justify their solutions, taking personal responsibility for the accuracy and completeness of their work. The overarching perspective was that if students were invited to work together and conduct thoughtful investigations with appropriate materials, they would be able to build mathematical ideas relating to fractions (Maher, Martino, Davis, 1994). Throughout this experiment, the teacher/researchers worked to promote a classroom culture that supported children as they explained, explored, and reflected upon mathematical ideas. Children were always invited to talk about their thinking, and they were challenged to defend and justify their ideas. The children were encouraged to build models of their solutions and share them. In discussing their solutions, children listened to each other and developed convincing arguments to support their ideas. Discourse among students was encouraged and considered a significant component of the classroom community. Cobb, Boufi, McClain & Whitenack (1997) indicated that when students engage in discourse with peers, there is growth and development of mathematical ideas. In this study, the researchers used responsive questioning to elicit explanations, to help students develop appropriate justifications and to redirect them when they were engaged in faulty reasoning. Justification of ideas was considered to be an integral component of the discourse. Cobb, Wood, Yackel & McNeal (1993) clarify the difference between justification and explanation. They say that we are expected to justify our reasoning when our thinking is understood and challenged. In contrast, we are expected to explain when our reasoning is not understood and a clarification is requested.

The students being studied in their regular classroom environment attended a small parochial school that attracts children from a several surrounding communities. For this part of the study, the work of a 5th grade class during the school year 2000-2001 was examined. This academically heterogeneous class consisted of 13 girls. In May 2001, they were assigned the identical task as the students in the teaching experiment.

These students had experienced a very traditional[3] classroom environment prior to the 5th grade. They were used to being told whether or not their answers were correct and being shown procedures for doing mathematics. In contrast, upon entering 5th grade, they were encouraged to take responsibility for convincing others that their solutions were correct and were expected to write about what they were thinking on a regular basis. Discourse was of paramount importance. Responsive questioning took place to encourage mathematical thinking by attempting to elicit verbalization of mathematical thought. The classroom community was one in which students' ideas were always respected. Alternate strategies were encouraged, shared and discussed. They were invited to discuss their thinking and to submit ideas in writing. Students were not taught algorithms. When they recognized patterns and could justify that these patterns were valid, they created generalizations, which they could apply to future problems. In general, the classroom community was modeled on that of the teaching experiment described above.

DATA AND CODING

All of the research sessions were videotaped using two or three cameras. Students' original written work and teacher/researchers' notes were also carefully collected. Transcriptions and detailed narratives of the data were recorded and a coding scheme was designed to flag elements for study. The four classifications of codes used were intended to record teacher interventions, ideas expressed, representations used by students, and justification and reasoning by students.

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[3] Traditional, in this case refers to a more didactic environment of the type described in Cuban, 1993.

Just as the students in the research group had done, the 5th grade students wrote about their mathematical experiences. The original written work that they completed while engaged in the identical activity as the 4th graders form a second set of data, which were examined in this study. In addition, field notes from

two graduate students visiting from Rutgers University and the observations of this researcher, the regular mathematics teacher of the class, provide a triangulation of sources for these data. Audio recordings also exist, but these were made primarily to monitor the nature of teacher interventions.

THE TASK

Both populations worked on the task[4] entitled "Holiday Bows" [5] which was designed to provide students with a meaningful context for understanding division of a natural number by a fraction. The task involved finding out how many bows of several fractional lengths could be made from various sizes of ribbon. For example, one of the questions was how many bows, each one-third meter in length, could be made from a piece of ribbon that is six meters in length. Students in both groups had access to actual ribbons, pre-cut to the specified sizes, meter sticks, string and scissors**.**

RESULTS AND DISCUSSION

What emerged from study of the research intervention was that the students who were part of the 4th grade class used one of three specific means as their basis for solving problems involving division of a natural number by a fraction, and that all three of these solution strategies involve counting. These solution strategies have been identified as:

Reasoning involving natural numbers

Reasoning involving measurement

Reasoning involving fractions.

Students who solved the problem using reasoning involving natural numbers converted the meter lengths of ribbon to centimeters so that the division made use of whole numbers. In many cases, these students also used reasoning involving fractions and measurement, but because they performed the division using natural numbers, this is seen to be the primary strategy. For example, when finding out how many bows, each 1/3 meter in length, would be made from one meter, they took the 100 centimeters and divided it into three equal parts, showing an understanding that one-third of a meter involves dividing by three. Many were not able to come up with the exact number, thirty-three and 1/3, but they were able to estimate what 1/3 of a meter would be and then

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4 Students worked on many tasks, however this report focuses on one that provides the context for analysis.

5 This task was developed by Alice Alston to be used with a 5th grade class and was then modified to be more "open-ended" for the 4th grade class. Carol Bellisio (1999) reported on the 5th grade version.

mark off approximately where the 33 centimeters would fall. Then, they counted their markings to find that there would be three such bows.

Students who solved the problem using reasoning involving measurement as their primary strategy created a unit of measurement equal in length to the desired bow, placed it along the given amount of ribbon, and then counted how many times it fit. For example, to find how many bows, each 1/3 meter in length, could be made from two meters of ribbon, they cut a piece of string equal to 1/3 meter and then counted how many times this measuring tool could be placed along the two meters of ribbon. In many cases the measuring tool was constructed by taking a one-meter piece of string and dividing it equally into the required number of parts. That is, if a 1/3 meter measuring tool were needed, a one-meter piece of string would be folded into thirds and then cut.

When students used their knowledge of fractions as their primary strategy to solve the problem, they recognized that each meter contains an equal number of fractional pieces and then they multiplied this number by the number of meters. For example, to find out how many bows, each 1/3 meter in length, could be made from two meters of ribbon, a student would recognize that in each meter there would be three one-thirds, so there would be three bows. They would then take that number of bows, three, and multiply it by two (or use repeated addition) since there were two meters. The counting is evident in the multiplication, which involves the iterated number of thirds.

A significant portion of the activity consisted of division of a natural number by a unit fraction. Students had more difficulty with division involving a non-unit fraction divisor. Those who reasoned using natural numbers, had to work to recalculate what two-thirds of a meter would be in terms of centimeters. Those who used measurement as the primary strategy, had to create a larger measurement tool and did so based upon the measuring tool they used for the unit fractions. For example, they had to construct a one-third-meter piece of string before they could construct a two-third meter measuring tool. Those who reasoned using fractions had great difficulty because they struggled to give meaning to the piece that was "left over". That is, it was not clear how many two-thirds there were in one. In spite of these obstacles, students were able to solve the problems involving the division of a natural number, by a non-unit fraction divisor. In some cases, they had to make adjustments to the schemes they had built to solve the other problems.

When the same task was implemented under replicated conditions, as part of regular classroom practice, the identical three solution strategies were observed.

Nicole used reasoning involving fractions to solve the problems. She writes the following about her solutions.

> What I did for the 1 meter ribbons are: If you have 1 meter and the length of a bow is 1/2 a meter then obviously it's 2. I did the same thing for 1/3 = 3, 1/4 = 4, 1/5 =5.
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> What I did for 2 meters is: Since 1 is 1/2 of 2 I add the #'s for 1. For example: for the 1/2 meter [bow] and 1 meter [ribbon] I got 2. So I added 2 + 2 = 4. So 4 is my answer.
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> What I did for 3m is: I added what I got for 3m. 1m + 2m. For example 1/2 of 1 meter was 2 1/2 of 2 meters is 4 so I added the [sic] together 2 + 4 = 6.

Sarah used reasoning involving measurement. She states the following.

I figured all the answers out by putting the string next to the ruler and finding the "Ribbon Length of Bow" and seeing how many strings I could get to fit to that length.

Being accustomed to justifying her solutions, she continues by attempting to substantiate her work by writing the following.

Explanation: I think my method works because when you measure the string to the right length and see how many strings you can measure it [the pre-cut ribbon length] to, you get an answer.

An interesting difference in the outcomes observed in the two populations is that the students in the 5th grade group rarely used the solution method of reasoning involving natural numbers, while it was a very commonly used method in the 4th grade group. Perhaps this is a function of the difference in grades. In 4th grade, there is a great deal of emphasis on division of natural numbers, while in 5th grade, the curriculum contains more introductory fraction study. This would seem to be consistent with Vygotsky's Zones of Proximal Development (Vygotsky, 1978). Olivia was the only 5th grader to refer to reasoning involving natural numbers and she did not come up with this type of solution initially. At the conclusion of her written work she says the following.

I figured out a shorter way to explain this & it makes more sense. It works as follows: 1 meter = 100 centimeters. You could change the amount of meters you have into centimeters. Thus, let's say you have to make bows each 1/2 of a meter. Figure out how many centimeters = 1/2 of a m. 50 centimeters = 1/2 of a m because half of 100 is 50. Then see how many times 50 goes into 100. However many times 50 goes into 100 is how many bows you can make with each bow 1/2 of a m. & with 1 m. You can also do this with 1/3 of a m. or 1/4… as long as you change 1/3 or 1/4 of a meter into a # amount of centimeters. You can also do this with 2 or 3m… of string as long as you change 2 or 3m… into centimeters. I think this works because you have to figure out how many 1/3rds or 1/4th s of a m. go into 1 m. That is saying the same thing as a certain # of centimeters go into 100 or 200 or 300. Or you could do $1 \div 1/4$ and you would get 4. That is the same thing as $100 \div 25 = 4$. They both = the same thing which proves they both work.

As these students moved from the unit fractions to the non-unit fractions, they also had to adjust their strategies. Linda solved the problems that required division by a unit fraction using reasoning involving fractions. She assumed this method was no longer valid when faced with a non-unit fraction divisor and therefore employed the strategy of reasoning involving measurement. The following appeared in the field notes of one researcher who was present.

When she got to the question of 6m ribbon and 2/3m bow, she started measuring. I asked her why she didn't just use her multiplying method, she replied, "cause there's a 2 there not a 1, so you can't do it, you can only do it when there's a 1, so I have to measure it if there's another number there." It's ironic how she understands that the 2 in the numerator makes her method invalid, but she doesn't understand why. (C. Hayworth, unpublished notes, May 24, 2001)

CONCLUSIONS

The initial research study was undertaken to learn more about how children constructed knowledge about fractions. While there is much research showing that many children have experienced great difficulty in solving problems that involve fractions (for example: Tzur, 1999; Davis, Hunting & Pearn, 1993; Davis, Alston & Maher, 1991; Steffe, von Glasersfeld, Richards & Cobb, 1983; Steffe, Cobb and von Glasersfeld, 1988), these students demonstrated mathematical understanding of division of fractions, thought to be the most complex aspect of the elementary mathematics curriculum (Ma, 1999).

The ability of the students in the teaching experiment to successfully construct means by which to solve problems involving division of fractions prompted an examination of conditions in place during the teaching experiment. The three primary issues which stood out in the investigation are development of a classroom community, choice of appropriate problems and teacher interventions.

The teaching experiment described here became a model for classroom instruction of division of fractions. Specific strategies used by the students in the research group to construct methods of solving problems involving division of fractions were observed. These robust methods were also used by students completing the identical task in their regular classroom setting. These students were also given the time and the opportunity to explore mathematical ideas deeply, in a supportive environment where their ideas are respected, and they became empowered to think like mathematicians. They collaborated, experimented, hypothesized, tested their hypotheses, built concepts and took great pride in their accomplishments.

In this situation, research did inform classroom practice. Teachers need to be provided with more opportunities to study and learn from research. In addition, more research experiments need to be undertaken to provide information to classroom teachers that will lead to a situation wherein all children will have an equitable opportunity to build powerful mathematical ideas and to think like mathematicians.

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