THINKING IN IMAGES AND ITS ROLE IN LEARNING MATHEMATICS

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In this theoretical essay four stages of functioning of thinking in images are distinguished and thoroughly described: 1) creation of a primary image (on the basis of some visual material), 2) creation of a secondary image based on memory, 3) operating with images, 4) creative formation of new images. The role of thinking in images at the solving of geometric problems is revealed and illustrated by examples.

In pedagogical psychology usually three types of thinking of the child, which "sequentially substitute each other, being indicators of growth development" (Yakimanskaya, 1980) are distinguished.

In the early stages of its development the thinking is inseparable from practical operations – the visual-active thinking develops. It means, that at the presence of a problem the child has to immediately investigate real objects. Later such mental operations as the analysis of conditions, statement of the purpose, evaluation of the correspondence of results of transformations to the purpose of study etc. start to be formed. Thus, the basic feature of visual-active thinking is that a real situation serves as the object of direct transformations. This form of thinking is the first step and at the same time the basis for the development of other forms of mental activity. In the process of the development the child encounters with more and more complicated problems. There is a necessity to plan the operations, predicting their results beforehand. For this purpose the child needs the skill "to act mentally". However, the development of such skill is very complicated process. Before the thinking of the child will be able "to come off" the concrete reality, it should pass through the phase of visual-pictorial thinking.

The development of this form of mental activity is very important for the further shaping of thinking of the child. As O.K.Tihomirov (1969) marked, "...The visual-pictorial thinking plays the important role in the formation of child's understanding of processes of the change and development of things and phenomena".

In the opinion of A.V.Zaporozhets (2000), "... The mind of the person in whom in a childhood the visual perception of the reality and the visual-pictorial thinking have not been generated properly, can receive afterwards one-sided development and excessively abstract, separated from the concrete reality, character".

With time the child realizes the presence of interior, hidden connections between various phenomena, and on the basis of visual-pictorial thinking conceptual thinking arises. At this stage the whole system of mental operations is actively formed. The child's mind distinguishes in concepts individual and general attributes. As a result the thinking gets the inductive and deductive character. Furthermore, the reasoning in terms "as if", becomes possible, i.e. the probabilistic-hypothetical judgements can be used. The

generated conceptual thinking assumes the possibility of arbitrary self-regulation of the person's intellectual activity.

The traditional selection of three stages of development of thinking has resulted, in the opinion of I.S.Yakimanskaya (1980), in the underestimation of the independent role of thinking in images in the intellectual development of pupils. In particular, "... It was not taken into account, that the thinking in images develops itself, that it is the equivalent form of intellectual activity, has rather complicated forms of display and various functions".

By many researchers the process of development of thinking is understood not as a sequential change of the forms of mental activity listed above, but rather as the gradual growth of complexity of mechanisms of processing of information. S.L.Rubinshtein (1958) wrote: "...Genetically earlier forms of visual thinking are not replaced but be transformed into to the superior forms of visual thinking". O.K.Tihomirov (1969) also argued that "... These three forms of thinking coexist and function at the solving of various problems by the adult". In the opinion of V.V.Mader (1994), "... The thinking can not be pointless and only abstract — it needs to be supported by concrete images".

Thus, one should not conclude that in the process of the development of logic (conceptual) thinking visual-active and visual-pictorial forms of thinking become rudimentary fragments in the general structure of person's thinking. The efficiency of processes of thinking in many respects depends on the level of development of both logic and visual components, and also on the degree of their integration.

The important research problem arises — to determine a role and place of pupils' thinking in images in the geometry learning in secondary school. For this purpose one has to analyze features of thinking in images, namely stages of its functioning and appropriate for these stages mental operations.

STAGES OF THINKING IN IMAGES.

In the opinion of I.S.Yakimanskaya (1980), "... The thinking in images should be considered as a complex process of transformation of sensual information". Perceiving some real object, the person distinguishes details and attributes in it, and the certain emotional attitude to this object arises. As a result the mental representation of its image is formed: a mental "picture", reflecting the object's most essential features (the term "picture" is rather conditional here, as the images may be not only visual, but also acoustical, motive, emotional etc. and frequently they may combine in themselves some of these qualities. Further this image can vary, being generalized, and finally it will reflect features not only of a concrete object, but also of a whole class of phenomena. Consider in more details, how the process of creation of images and thinking on their basis (i.e. the process of functioning of thinking in images) takes place.

The systematization of research in the area allows to distinguish four stages of functioning of thinking in images: 1) creation of a primary image (on the basis of some visual material), 2) creation of a secondary image based on memory, 3) operating with images, 4) creative formation of new images. Consider each of these stages in more detail.

The creation of a primary image at the level of sensual perception is not simply "mental photographing of a real object and "absolute" reflection of its properties. In an image those attributes of object are fixed which the perceiving subject considers (consciously or unconsciously) as the most important. As a result any image reflects in itself properties and attributes of objects selectively, depending on conditions of a task and personal preferences, on beliefs of a person. And this set of the chosen attributes and properties may freely vary during the further analysis. Therefore, the image becomes a dynamical and multidimensional reflection of a real situation.

In the learning of mathematics (especially geometry) this stage of functioning of thinking in images appears to be very important. Many typical errors of the pupils are born just at the stage of creation of primary images, appropriate to investigated geometric concepts

At the following stage of thinking in images the creation of a secondary image happens. As a rule, it is based on memory (at the absence of a real object of perception or in conditions of the conscious refusal of its use as a visual support). Thus the secondary image will be more "general" than primary one. Really, some properties of an object reflected in a primary image "are lost", and only the general, essential attributes are reproduced. As a result the secondary image reflects attributes of a whole class of objects, that is, essentially, comes nearer to concept. For example, in the opinion of N.S.Podhodova (1997), "...The concept "grows" from the "preconcept which, in turn, is based on images".

This stage is also very important for the process of learning mathematics, in a particular, geometry. However it is much more difficult to connect this stage with concrete activities of the pupils. What the pupil should do? How can the pupils' activity be controlled? How the teacher can rule the process of creation of a generalized image? — All these are open methodological problems.

At the third stage — operating with images — the active transformation of the images, created or reproduced in memory, happens. The direction and the modes of these transformations are determined by a problem situation (requirements of a task) and personal beliefs of the perceiving subject. Nevertheless, all transformations of an image save its basic attributes ("freedom of operation" is limited, for example, by conditions of a task). It means that essentially one cannot say at this stage about creation of new images — the images are the same; only combinations of their components may vary.

All the teaching of geometry at the secondary school takes place in conditions of operating with images. If two previous stages had not received due attention, there may be serious difficulties in the management of the process of learning at this stage.

At the stage of the creation of new images mental operations sometimes become the main purpose. The images will be transformed under the influence of some associations, analogies etc. As a result the new images possessing frequently completely unexpected qualities are born. In this case it is possible to speak about creative imagination. In the opinion of A.I.Gibsh (1995), just "... Possession of spatial representations and the presence of spatial imagination... is one of the basic criteria of educatedness in the field of mathematics".

It is clear, that this stage of functioning of thinking in images is the most significant in the learning of geometry at school. However, one should not think that such concepts as spatial representations and spatial imagination are connected only with this stage. All four listed above stages are necessary steps on the path of their formation, and each stage is connected with certain skills, methods and exercises.

OPERATIONS OF THINKING IN IMAGES

One can distinguish mental operation specific for each stage of thinking in images. At the stage of creation of primary images basic mental operation is the analysis (structuring) of visually perceived object, including: distinction of its separate elements, correlation of the elements with the background etc. Note that L.S.Vygotsky (1934) considered the skill to arbitrarily choose a figure and hum noise as the basic indicator of sensibility and arbitrariness of attention: “If I am able to see a thing only in a way dictated by its structure, my attention is extremely involuntarily. If I am able to see a thing so that I can make any element of this thing a center or a figure and make all the rest the background, my attention becomes extremely arbitrary”.

At the stage of the creation of a secondary image the basic operation is the generalization. On the first sight there can be an impression, that an image, in the contrast to a concept, is a reflection of an individual object, i.e. that the possibilities of an image from the point of view of generalization are minimal. However, actually advanced “thinking in images allows to reach rather high levels of generalizations of phenomena, but in the specific pictorial form” (Yakimanskaya, 1980). Really, only a primary image is a reflection of an individual real object, on the basis of perceptions of which it is created. And even to this initial stage the term “a reflection” can be applied only conditionally. As already it was noticed above, the primary image created on the level of sensual perception reflects properties of an object very selectively, conceptually: usually only the most essential properties are distinguished. In the opinion S.L.Rubinshtein (1934), the image created with the help of memory, in the absence of a visual basis, “is released from “the attachment to an individual object and can be the generalized image of the whole class or category of similar subjects ".

The third stage — operating with images — is analysed in the psychological literature basically from the point of view of transformation of visual (spatial) images.

In the work of I.S.Yakimanskaya (1980) on this occasion the following is said: “Mental operations of thinking in images correspond to basic geometric transformations”. In the opinion of I.A.Kaplunovich (1996), “the structure of spatial thinking... is determined by the operations corresponding to... appropriate basic mathematical transformations... Namely: reflection of two- or three-dimensional Euclidean space into itself and affine transformations of the graphs of functions”. In particular, I.A.Kaplunovich (1996) selects the following “... Basic operations fulfilled in imagination with images of spatial figures: “Parallel translation, rotation, central symmetry, axial symmetry, symmetry with respect to a plane, homothety, parallel projection, orthogonal projection, graphic interpretation of operation of addition of functions, graphic interpretation of operation of multiplication of a function to a number, compression and dilation of the graphs of functions”.

Except for separate operations at this stage of functioning of thinking in images I.S.Yakimanskaya (1980) revealed three types of operating with images (in order of increasing of their complexity): “transformations resulting in the change of a spatial position of an image (I-st type); transformations changing the structure of an image (_-nd type); long and repeated performance of transformations of first two types (_-d type)”. Thus, operations listed above form a multilevel system.

Probably, the description of a system of mental operations, “homomorphic to the basic mathematical structures” really represents an effective model of thinking in images (this approach is correlated with the well-known conception of J.Piaget, according to which the structure of mental operations performed by a person corresponds to basic mathematical structures). However, explanation of one abstraction (psychological) through another (mathematical) not always can clarify the essence of mental processes.

At the stage of creative formation of new images the active transformation of initial data happens. It means that all operations of the previous stage, though in specific conditions, are used. Firstly, basically operating on the II-nd or III-d type occurs, and secondly, the direction of operating is set not by any exterior requirements, but by analogies and associations, arising on the basis of available images, and as a result new images emerge.

THE ROLE OF THINKING IN IMAGES AT THE SOLVING OF GEOMETRIC PROBLEMS

The thinking in images of the pupils plays the important role in learning mathematics in the secondary school, in particular, in learning geometry. In the opinion of A.Ya.Tsukar’ (1998), without thinking in images “... The successful study of a geo-metric material is impossible... because there the skill to read images of figures, to mentally imagine required objects, to keep in sight several objects simultaneously and to operate with them is continually required...”.

Further we will analyze displays of features of thinking in images during the solving of geometric problems.

There is a lot of works in which in some form the process of the solving of problems including geometric ones is described.

Since the long time V.A.Gusev and his disciples were engaged in researching so-called skills of the solving of geometric problems and paths of the realization of these skills in practice.

Concerning a role of thinking in images in a realization of the generated skills of solving of geometric problems, its special role is exhibited at the realization of skills in the distinction of figures appropriate for the given element of a problem. More precisely, these are the following skills:

- Construction of a drawing appropriate to the text of a problem:

- Distinction of figures appropriate for the given element of a problem, i.e. those figures available on a drawing, which will participate in the solution of a problem (the additional constructions are also possible).

At the solving of problems the pupils should mentally imagine a spatial figure described in the condition of a problem. In the book of G.Polya (1965) in this occasion the following is said: "If we deal with a geometric problem, we should consider some geometric figure. This figure we can either present in our imagination or to represent on the paper as a drawing. In some cases it may appear to be better to imagine a figure, but not to draw it".

Consider the situations when it appears to be useful "to imagine a figure" before the direct performance of the drawing. In each of these situations the condition of a geometric problem suggests the existence of several essentially different cases of the disposition of figures, each of which requires the separate approach to a solution.

Problem 1. On the straight line a points *A, B* and *C* so that $m(\overline{AB})$= 5cm and $m(\overline{BC})$= 7cm mare marked. Find length of the segment *AC*.

The disposition of points *A, B* and *C* on the straight line is not indicated in the condition, and it must be imagined.

Two alternatives are possible: the point *B* lays between points *A* and *C* (fig. 1a) or the point *A* lays between points *B* and *C* (fig. 1b).

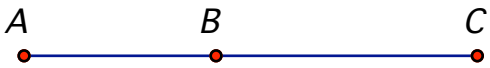

Figure 1a.

Figure 1b

It means that the given problem has two different solutions: in the first case $m(\overline{AC})$= 12 cm, and in the second case $m(\overline{AC})$ = 2 cm.

Problem 2. Two angles have a common side and their degree measures are 65˚and 35˚, respectively. What angle their incoincident sides can form?

In this problem the process of "the imagination of the disposition of angles" is broader, as is connected with the possibilities of disposition of angles not only on a plane but also in space. Considering this problem in a similar way as previous one, we see that there are two possibilities of the disposition of angles: the side of the smaller one is located 1) inside the greater angle (fig. 2a) or 2) outside it (fig. 2b). Accordingly, the angle between the incoincident sides will 30o or 100o.

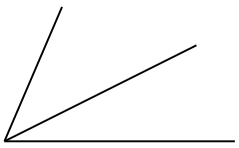

Figure 2a.

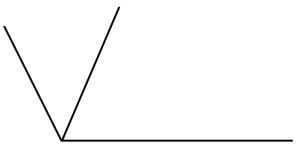

Figure 2b.

However, in the condition it is not indicated that all three rays (sides of the given angles) lay in one plane (fig. 2b). It means that the problem has infinitely many solutions, more precisely, the required angle can have any magnitude from 30˚ up to 100˚.

Problem 3. Bisectors of angles A and D of the parallelogram ABCD divide its side __ into segments of lengths 5cm, 2cm and 5cm, respectively. Find the perimeter of a parallelogram.

In this problem the initial figure is the given parallelogram, and the solution depends on its form and size. There are two possibilities of the disposition of bisectors of angles at the base: these bisectors may be intersected inside the parallelogram and outside it (fig. 3 a, b).

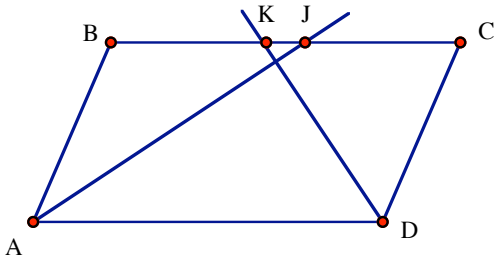

Figure 3a.

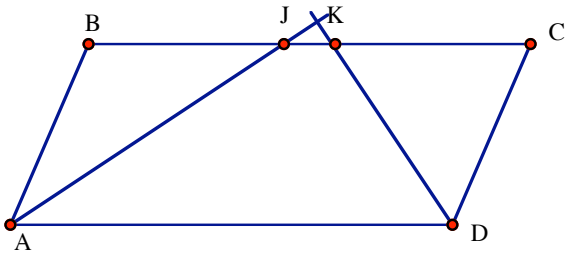

Figure 3b.

The idea of the solution of this problem is the same for both cases (it is based on the fact that triangles ABJ and DCK are isosceles). However, the answers will be different: the perimeter is either 34cm or 38cm. Pupils frequently restrict themselves to considering only one situation fixed by a drawing that they have constructed. As a result the solution appears to be incomplete. That is why the appeal "to imagine a figure before the performance of the drawing" is useful, and the methodology of solving geometric problems should separately elaborate this question.

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