

# UNCERTAINTY DURING THE EARLY STAGES OF PROBLEM SOLVING

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*This paper discusses the role of uncertainty during the early stages of problem solving. It is argued that students start the problem solving activity with some degree of uncertainty that may vary from high to low. This degree of uncertainty may affect students' decisions at early stages of the problem solving process. It may be suggested that an awareness of the possible effects of uncertainty may better prepare students to approach problem solving and support them to start building their own problem solving strategies.*

## INTRODUCTION

The idea of looking at uncertainty during problem solving emerged from an attempt at conceptualising what students do as they tackle non-routine mathematical problems. By analysing students' work it was observed that, at the outset, students' problem solving processes took place in a context of uncertainty. In other words, it was observed that problem solving usually started as a situation in which actions had to be taken in spite of insufficient knowledge. It was also observed that uncertainty was present in different degrees and that this led to different consequences.

A general look into the study of dealing with uncertain situations suggests that its importance has long been recognized in other areas. In business management, for instance, the study of uncertainty can be traced back to at least the late 1950's. In this area, the focus on uncertainty has evolved from trying to find ways of reducing it, to tapping into it as a source of creativity and stimulation (Jauch & Kraft, 1986; Ogilvie, 1998). Due to its role in modern organisations, attempts at modelling uncertainty (e.g., Downey & Slocum, 1975) have been, and may continue to be, conducted.

Uncertainty in mathematical situations has been considered in various studies. The most common position that has been adopted towards the study of uncertainty in mathematics education has been to acknowledge it (e.g., Rickard, 1996) or to explore ways of making positive use of it in didactic situations (e.g., Hadas & Hershkowitz, 1999; Hadas, Hershkowitz, & Schwarz, 2000). Few studies have attempted to develop models of uncertainty and to explain its relation to other aspects of learning and doing mathematics. Studies of this sort may provide useful information for those studies in which uncertainty emerges as a result or plays an important role.

This aim of this report is to discuss uncertainty as observed in a study of mathematical problem solving. The model of uncertainty that will be presented here is part of the broader model that emerged from this study. This sub-model aims at

conceptualising the main aspects of uncertainty and the way in which they relate and affect each other.

### **THE CONTEXT: A BROADER STUDY**

The results presented here are part of a broader study which investigated students' problem solving processes. The aim of this study was to develop a model that explains how students deal with non-routine mathematical problems. The study made use of the grounded theory methodology (Glaser & Strauss, 1967) and, particularly, of the methods of data analysis described in Glaser (1978; 1998). In general, grounded theory can be summarised as a methodology that consists of constantly comparing data (see Glaser, 1992). The method of constant comparison allows the researcher to generate categories and to start hypothesising about the way in which they are related. Emergent hypotheses are then compared against further data and thus a theory is developed. Grounded theory is, thus, an inductive methodology in which general patterns are derived from, and grounded in, the data.

This study was conducted within the context of a problem-solving course offered at the University of Warwick. The students that participated in the course were undergraduate students doing mathematics, computer sciences or (four-year) teacher training degrees. The aim of the course was to allow students to reflect on their "mathematical thinking and to identify and develop their own problem-solving strategies".

During the course, students were required to work on a number of non-routine mathematical problems and to document their thinking processes as they occurred. This provided students with a written document of their work and made it possible for them to review and share their ideas with the rest of the group. From the researcher's perspective, this provided rich data about the way students solve problems as well as some insights into their beliefs and abilities in relation to mathematics. Students' written accounts, or 'rubrics' (see Mason, Burton, & Stacey, 1982), were used as the main source of data for this study. Observations were also made during each session and informal interviews were conducted with a number of students.

The problems that students had to tackle may be described as non-routine mathematical problems, i.e., problems for which students knew that they were not expected to use any procedure or knowledge in particular. Also, some problems can be considered 'open ended' in the sense that students had to decide what was it exactly that they wanted to achieve as a solution (see appendix 1)<sup>i</sup>.

### **GENERAL RESULTS: A FRAMEWORK FOR LOOKING AT UNCERTAINTY**

The analysis of students' rubrics during the main study gave rise to a considerable number of concepts about what students do with during problem solving. Finding suitable terms for these emerging concepts was, in many cases, a difficult task.

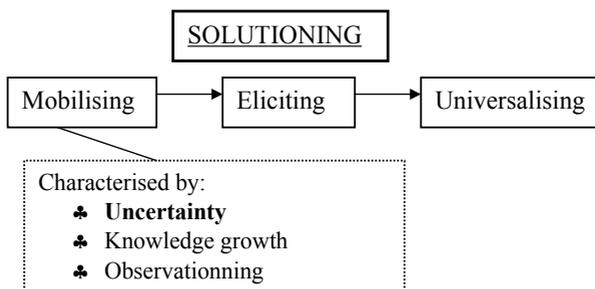
Furthermore, in the cases where terms were readily available, variations in basic aspects of their definitions made it difficult to make use of them. For this reason, the reader will notice that the study introduces of a number terms that, at first sight, may appear strange or even unappealing. These terms were chosen because, at the time, they were the most suitable way of illustrating and naming the concepts that emerged. They will continue to be in use until more suitable ones are found or until new data suggests that they need to be modified.

The following is a brief outline of the general results of the study. Due limitations in space, the concepts that will be introduced cannot be fully discussed. The outline, however, must serve to give a general picture of the framework from which uncertainty is being considered.

### **A model of students problem solving processes**

An analysis of students' rubrics suggested that problem solving is best conceptualised as a cognitive process. This process was called 'solutioning' to highlight the on going nature of the situation. Solutioning consists of three main stages, namely, *mobilising*, *eliciting* and *universalising*. Mobilising is the stage at which information and understanding start to be generated in order to start solving the problem. Eliciting refers to the stage at which students go beyond the data that is given and what is being observed and start developing a solution. Universalising is the last stage whereby elicited solutions are improved and transformed into more general, rigorous and compact results.

For most students, the aim at the initial stages of problem solving is, in general, to generate data, information and understanding. Thus, it may be said that solutioning starts with mobilising. As said, mobilising is the process through which students start dealing with the situation, learning about it and understanding it. Mobilising is characterised by *uncertainty* and has two main consequences, namely, *knowledge growth* and *observationning*. The following figure provides a general representation of the model described above.



## UNCERTAINTY

Students start to ‘solution’ by mobilising. Mobilising may start to be conducted in a context of uncertainty. *A context of uncertainty is one in which students lack the knowledge and understanding necessary to know exactly what to do next in order to deal with the situation.* In spite of this lack of knowledge, students are required to make decisions and take actions if they want to eventually provide a solution.

Uncertainty can be analysed from students’ perspectives as they experience it. Students express uncertainty by raising questions about the situation or by making decisions based on insufficient or inadequate information. In the following example, Hannah started by asking herself some questions about the situation and trying to provide answers to them. In the case of the first question, her queries were easily resolved. As for the second question, the answer that she provided was based on the intuitive information she had at her disposal.

What is the question asking me? I want a rectangular piece of paper, and take away from it the largest possible square. I then want to repeat the process with the left-over rectangle. I want to know what different things can happen, and when they will.

Is there anything else I want to know? Well, I want to know where to stop—do I only repeat the process once or do I keep going until I reach an endpoint. I think I’ll only repeat it once or otherwise the process could be infinite! So, I want to know what the different things are that could happen. (Hannah, Square Take Away, p. 1)

Uncertainty can vary in degree from high to low (and in some cases – where a student believes they know a precise, direct route to a solution – it can be null). Students may meet the problem with little or no knowledge about the situation, i.e., with a high degree of uncertainty. On the other hand, they may be presented with what they perceive as a familiar problem or with a problem that refers to material with which they are already experienced. In this case, there is a lower degree uncertainty. A lower degree of uncertainty means that the student has some ideas about the situation and about how to start tackling it.

The following quotes briefly illustrate cases of high and low uncertainty. The first example corresponds to a case of high uncertainty (as expressed by the students) whereas the last two correspond to cases in which uncertainty seemed to be low.

Having established what exactly the question is asking me, I feel this problem is going to be incredibly difficult. I could literally place 2002 pieces of paper in the hat, however, there are some many possibilities of what number can be drawn out that I don’t know how this would help. (Gina, Hat Numbers, p. 1)

I can immediately see that this problem is very simple if I use squares, which are just regular rectangles. So I will first look at the squares briefly... (Jasmine, Diagonals of a Rectangle, p. 1)

From first looking at the problem I can see that the answer will have something to do with common factors. (Jasmine, Visible Points, p. 8)

The degree of uncertainty is related to the type of decisions that students make during mobilising. Mobilising requires students to make decisions about what ideas to explore, about what to focus on. These decisions may vary from arbitrary to well informed. Arbitrary decisions are decisions for which there is insufficient information to make a choice and where students end up deciding on the basis of less relevant factors (e.g., choosing a course of action over another because it seems easier). High uncertainty means that decisions will be made arbitrarily, almost at random. As uncertainty decreases, however, decisions are likely to become more informed.

From the two examples below, the first one belongs to a student that seemed to be considerably uncertain about the situation. His decision on what to focus on at that time seemed to be more arbitrary than well informed. As for the second example, it belongs to a situation of low uncertainty. As it can be observed, the decision that the student made about how to proceed seemed a more informed one.

Well, the question of the problem is perhaps ambiguous. I could make out of this question that I will aim to find what sizes work or what particular configurations work. For now I will concentrate more on the sizes (dimension) problem. (Marcus, Faulty Rectangles, p. 1)

We know  $(a+b)$ ,  $(b+c)$ ,  $(c+a)$ , but we want to know  $a$ ,  $b$ ,  $c$  explicitly. A solution to this problem would be to express  $a$ ,  $b$ ,  $c$  in terms of  $(a+b)$ ,  $(b+c)$ ,  $(c+a)$ . (Alan, Arithmagons, p. 1)

The decisions that students make during mobilising may or may not lead to sustainable courses of action. A sustainable course of action or idea is one that turns out to be manageable for the student and that leads to generating results. High uncertainty means that students will be unsure about whether a particular course of action will be sustainable or not; they will need to engage in it in order to find out. Furthermore, the less uncertain the situation is, the more likely it is that the student will choose a sustainable course of action. The consequence of choosing a course of action that is not sustainable is that it will be eventually abandoned and that a new one will need to be found.

The following quotation illustrates how a decision taken in a context of uncertainty led to an unsustainable course of action. Realising that the chosen course of action was not going to be very productive provided the student with some experience that might have helped towards decreasing her uncertainty. Nonetheless, the student had to abandon the present course of action and face the need of finding other possible avenues to pursue.

I will start by writing a grid of numbers, and using it to specialise systematically to find sums of lines with different gradients and different starting points...

Stuck! I don't think this will work (i.e., I don't think I will be able to find any patterns) using this method, because the amount of numbers in the sum does not necessarily increase as you move down the table (i.e., as you start with a higher number) because the line does not necessarily pass through a number in the top row. (Hillary, Sums of Diagonals, pp. 1–2)

Arbitrary decisions made in contexts of uncertainty are not as valued as informed decisions based on logical deductions. However, making an arbitrary decision may be the best option for a student dealing with a highly uncertain situation. On the one hand arbitrary decisions may lead to an unsustainable course of action. On the other, arbitrary decisions may also bring unexpected knowledge and understanding and allow the student to open promising avenues. This suggests that, in some situations, making arbitrary decisions can be a good action after all.

Reducing complexity<sup>ii</sup> is an important aspect of mobilising that seems to help students deal with uncertainty. By reducing complexity, students focus on specific and relatively simple aspects of the situation. In students' words, they "start at the beginning", looking at simple cases or examples first. Students' aim in reducing complexity is to gather information and to gain understanding in order to be able to eventually move on to more sophisticated cases. Reducing complexity helps students deal with uncertainty by allowing them to focus on manageable aspects of the situation and helping them to start gaining knowledge and understanding.

In the example quoted below, the student was relatively uncertain about the situation. Reducing complexity helped her to deal with some of this uncertainty by allowing her to start learning about the situation.

As  $m, n$  increase, what percentage of points is visible from  $(0, 0)$ ?

Stuck! I have no idea what the percentage should be as it would vary when  $m$  and  $n$  are varied.

Let me try to draw some planes with different sizes...

From the first trial, I can see that the percentage that we want to know is:

$(\text{visible points on a plane} / \text{visible} + \text{invisible points on a plane}) \times 100$ . (Karina, Visible Points, p. 1)

Another important aspect of uncertainty is the way it affects *observationning*. As a result of mobilising, students start noticing salient facts and pointing them out as observations. Observationning is about noticing and making a note of these facts (thus the term *observation-ing*). As a result of uncertainty, observationning, especially at early stages, can be exhaustive, meaning that students will try to make a note most of what is being observed. In other words, students may engage in noting most (if not all) seemingly salient facts or ideas "just in case" they are useful or relevant at a later time. The less the student knows about the situation, i.e., the higher the degree of uncertainty, the more likely it is that observationning will be conducted in this way.

The following example illustrates how uncertainty affects observationning. It may be suggested that it was due to uncertainty that, at the start of her process, Carolyn chose to reflect on the first observation made. It can be speculated that, not knowing too much about the situation led her to closely consider a variable that would later be considered irrelevant.

I have noticed that there is a line of symmetry running through the grid from the top left through to the bottom right [see appendix]. Does this mean that, for example, 9 to 5 will give the same result as 5 to 9? Will diagonals that go to and from the same number (e.g., 4 to 4) need a different formula than those that go to a different number (e.g., 9 to 5)? ... Hopefully I will be able to answer these questions by the end of this investigation. (Carolyn, Sums of Diagonals, p. 1)

Finally, uncertainty decreases as the student's knowledge and understanding of the situation increase. In other words, uncertainty decreases with *knowledge growth*. Knowledge growth is the change in what students know about the situation and in the way they deal with it. Knowledge growth is usually set off as a consequence of mobilising, however, it is not limited to it. Knowledge growth can also be the result of any other action that brings understanding and generates information. Since knowledge growth increases what students know and improves their understanding of the situation, it may help to reduce uncertainty.

### **IMPLICATIONS OF THE STUDY**

The model of uncertainty proposed in this report suggests that the more uncertain the situation experienced by students during problem solving, the more likely it is that they will have to make arbitrary decisions. Moreover, it suggests that arbitrary decisions are less likely to lead to sustainable courses of actions than informed decisions. The model also suggests that students' uncertainty decreases with knowledge growth. This knowledge growth is a consequence of mobilising but it may also be said that uncertainty can serve to stimulate it. For instance, the exhaustive way in which students conduct observationing at early stages seems to be the result of working in an uncertain environment.

Recognising the fact that uncertainty may be present in various degrees can help students prepare for best dealing with it. By this, it is neither suggested that a context of uncertainty is a negative situation nor that it should be avoided. In fact, it seems that that students could benefit from learning to tolerate some uncertainty and even from actively creating certain levels of it. As suggested by other areas, a context of uncertainty can be a stimulating environment in which interesting questions can be raised and novel perspectives can emerge (see, e.g., Schoemaker, 2002).

Initial uncertainty may be a characteristic not only of problem solving but of the mathematical activity in general. Further studies can help to explain the role of uncertainty in other areas of mathematics as well. Such studies would help teachers and students to better understand the role of uncertainty and would support them in developing strategies for effectively tapping into uncertain situation.

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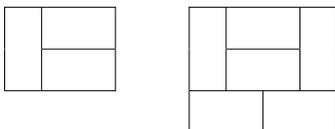
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<sup>i</sup> The following are some examples of the problems students tackled during the problem solving course:

Square Take Away: Take a rectangular piece of paper and remove from it the largest possible square. Repeat the process with the left-over rectangle. What different things can happen? Can you predict when they will happen?

Hat Numbers: A hat contains 1992 pieces of paper numbered 1 through 1992. A person draws two pieces of paper at random from the hat. The smaller of the two numbers drawn is subtracted from the larger. That difference is written on a new piece of paper which is placed in the hat. The process is repeated until one piece of paper remains. What can you tell about the last piece of paper left?

Faulty Rectangles: These rectangles are made from ‘dominoes’ (2 by 1 rectangles). Each of these large rectangles has a ‘fault line’ (a straight line joining opposite sides).



What fault free rectangles can be made?

<sup>ii</sup> *Reducing complexity* is a term that seems to increasingly point towards *specializing* (see Mason, Burton, & Stacey, 1982). For the purposes of this study, the former term was considered more appropriate to convey students’ main concerns as they focus on specific or simplified aspects of the situation.