

PERSISTENCE OF DECIMAL MISCONCEPTIONS AND READINESS TO MOVE TO EXPERTISE

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This paper describes features of a group of misconceptions about decimal notation that lead to students selecting as larger, decimals that look smaller. A longitudinal study identified approximately 900 students from a variety of schools who exhibited these misconceptions and whose subsequent progress could be traced. The data demonstrates that the progress that students makes depends to a certain extent both on the nature of their misconceptions and the grade at which the misconception is held. Such phenomena could be expected to hold for misconceptions in other topics.

AIMS

This paper aims to describe some features of several interesting misconceptions about decimal notation. The results are derived from careful analysis of a longitudinal study of students' misconceptions, with data collected in Melbourne, Australia from 1995 to 1999. Over 3000 students, ranging from Grade 4 to Grade 10 (ages about 9 to 16), completed nearly 10000 tests and the responses of individual students have been tracked, in some cases for up to three years. Students were, in principle, tested twice yearly, although with absentees and some delays in testing, the average time interval between consecutive tests by students was 8 months.

This data has enabled us to track how the students' thinking about decimal notation typically evolves with time. In this paper, we discuss what happens to students who hold a group of misconceptions that generally result in a belief that a decimal that *looks* smaller is in fact larger; precise details are given below. These misconceptions are interesting for several reasons. First, the existence of such thinking is very surprising, and is not commonly recognized by teachers. Second, the misconceptions themselves are interesting and variations have recently been uncovered from interview studies that accompany our quantitative analysis. We believe that some of these misconceptions relate deeply to the ways in which mathematical ideas are understood in terms of metaphors – in this case arising from three different mirror metaphors (Stacey, Helme, & Steinle, 2001). Understanding the evolution of the thinking of such students may therefore have wider significance to theories of embodiment and mathematical thinking which has been a theme of PME included in discussion groups in recent years. Third, our early preliminary analysis of the longitudinal data showed that different groups of misconceptions were likely to evolve differently (Stacey & Steinle, 1999). Students generally “grow out of” some misconceptions; they are common in young students but when students leave them, they do not return. In contrast, the group of misconceptions studied in this paper is attractive to students of all ages; Steinle and Stacey (2003a) found that about 1 in 3

students will use one of these ways of thinking at some time during primary school, as do about 1 in 4 secondary students.

Some of the results reported in this paper are qualitatively the same as those that have been reported earlier from the preliminary analyses of the data. However, this analysis is new; it has overcome many technical difficulties of the preliminary analyses (e.g. relating to the composition of the longitudinal sample). Thus numerical results in this paper should be quoted in preference to our earlier publications.

The paper is specifically concerned with two issues: to what extent misconceptions persist in students (so that later tests show exactly the same ideas) and whether it is 'better' to hold one of these misconceptions in preference to others.

THE TEST AND THE CODES

The data arises from a 30-item Decimal Comparison Test (DCT), which consisted of pair-wise comparison items with the instruction *For each pair of decimal numbers, circle the one which is LARGER*. Two sample items are comparison of 4.8 with 4.63 and comparison of 4.45 with 4.4502. Students' misconceptions are identified by the *pattern* of correct and incorrect choices in the 30 items. Students of interest to this paper will be correct, for example, on the first comparison (63 looks larger than 8, so 4.63 is smaller than 4.8) and incorrect on the second (4502 looks larger than 45, so 4.4502 is smaller than 4.45). More details of the reasoning are given below. Many researchers (e.g. Nesher & Peled, 1986) have used this technique of examining the pattern of correct and incorrect answers to probe students' thinking about decimals and our test builds on this work. Steinle and Stacey (2003a) provided evidence that the majority of errors that students make on the DCT are systematic and predictable, not random. Along with interview evidence (e.g. Swan, 1983), this demonstrates that the technique provides reliable information about common decimal misconceptions.

Four patterns of responses to the DCT are used in this paper and they are coded A2, S1, S3 and *other-S*. The codes are allocated according to strict rules about which items are correct and incorrect. Full details of the rules and the other 9 codes are given in Boneh, Nicholson, Sonenberg, Stacey and Steinle (2003). Whilst the data can only be in terms of the codes, our interest is in the underlying misconceptions, which are inferred from them.

STUDENTS' THINKING

Each of the codes identifies groups of students who hold particular misconceptions. Steinle and Stacey (2003b) give a comprehensive list of various ways of thinking with explanations. Ideally, one response pattern would correspond to one way of thinking, but as our investigations paralleled the data collection of the longitudinal study, several different variations were found for most codes with this version of the DCT.

Thinking that leads to the code S1

Two ways of thinking lead to students completing tests coded as S1, referred to as *denominator focussed thinking* and *place value number line thinking*. Students with the former will consider 0.73 to be smaller than 0.6 by focussing on the size of the denominators. Knowing that one hundredth is smaller than one tenth is then incorrectly generalized to any number of hundredths is smaller than any number of tenths. These students interpret decimals in terms of place value, but do not reunite. Students with *place value number line thinking* create a false analogy between place value columns and the formal mathematical number line or an informal version of it. Just as all 2-digit whole numbers come after all 1-digit whole numbers and before all 3-digit whole numbers, so they believe that all 2-digit decimals (e.g. 0.43) are smaller than all 1-digit decimals (e.g. 0.4) and larger than all 3-digit decimals (0.432). As a result of the analogy with place value columns, their number lines are often inverted, with larger numbers to the left and smaller numbers (for these students this means longer decimals) to the right. They may have little idea of place value.

Thinking that leads to the code S3

The difference between the codes S1 and S3 shows in responses to equal length decimal comparisons, where S3 students make errors. Two ways of thinking lead to students coded as S3. A student with *reciprocal thinking* makes an analogy with the fact that $1/73 < 1/6$ and concludes 0.73 is smaller than 0.6, while a student with *negative thinking* makes an analogy with $-73 < -6$. These students all effectively treat the decimal portion as a whole number and then choose the smallest whole number as creating the largest decimal. Stacey, Helme and Steinle (2001) explain *reciprocal*, *negative* and *place value number line thinking* in terms of mirror metaphors.

Thinking that leads to the other-S

In this paper, students who generally choose the shorter decimal as the larger, but do not follow the S1 and S3 patterns exactly on items with special features (e.g. zeros) are labeled other-S. Either they do not consistently apply their S1 or S3 thinking, or they have an unidentified misconception, or they use a combination of ideas.

Thinking that leads to the code A2

Students coded as A2 are correct on all types of decimals except one, but they may in fact have very little idea about decimal place value. Students who use *money thinking* believe that 4.4502 is equal to 4.45 (both analogous to \$4.45, maybe with some “round-off error”). These students are effectively rounding or truncating decimals to two decimal places, and treat the 2 digits on the right of the decimal point as a (second) whole number. Steinle and Stacey (2001) explored this thinking by using a variation of a decimal comparison test that allowed students to choose the option of equality and found that, of the students who made errors on these items, about 25% chose this option and about 75% chose $4.4502 < 4.45$.

A second group of A2 students compare decimals with the left-to-right digit comparison algorithm, but reach an impasse when they run out of digits to compare. In the example above, they match 4 with 4, 4 with 4, 5 with 5, but then need to compare 0 with a blank, and do not know what to do. If they complete tests coded as A2, it is because they then selected the shorter decimal as the larger. These A2 students are thus like S students. Features of this code are in Steinle and Stacey (2002).

SAMPLE

For this paper, students who completed tests that were coded as A2, S1 or S3 were tracked to their next test (on average 8 months later). For simplicity, we refer to the unit of analysis as the “student”, but it is actually a pair of tests belonging to a student (many students contribute multiple pairs of consecutive tests over the longitudinal study). Points such as this make good analysis of longitudinal data logically difficult.

Two subsets of the full data are used for this paper. The *full sample* consists of all students (in the codes of interest) who completed another test in the study. The *reduced sample* excludes those whose next test has the same code. For example, 33 students completed A2 tests in Grade 6 and also completed another test in the study. On the next test, 3 students retested as A2 and the other 30 did not. The full sample contains all 33 students and the reduced sample contains the 30 students.

Table 1 shows the number of students in the full and reduced samples in each grade for A2, S1 and S3. Due to small numbers, Grades 4 and 5 will be combined, as will Grades 9 and 10. These students come from 12 different schools and many classes, so their future development is not explained by any particular teaching.

Code	Sample	Grade					Total
		4,5	6	7	8	9,10	
A2	Full	24	33	88	79	56	280
	Reduced	23	30	74	60	40	227
S1	Full	50	57	80	35	23	245
	Reduced	42	49	68	32	22	213
S3	Full	70	59	107	116	33	385
	Reduced	53	43	70	69	23	258

Table 1: Numbers of tests in full and reduced samples by grade

PERSISTENCE AND READINESS TO MOVE TO EXPERTISE

We first discuss *persistence*, which is plotted on the X-axis in Figure 1. Persistence, the percentage of students who retest the same on the next test, is calculated using the full sample. For the 33 students who completed A2 tests in Grade 6, three retested as A2 on their next test, hence the Grade 6 persistence of A2 is 3/33 or 9%. The overall persistence of A2 is 19%, more than for S1 (13%) but much less than for S3 (33%). These results can be compared to the persistence of 89% for experts (i.e. correct on

all item types) and an average persistence of 25% for all of the non-expert codes. Figure 1(a) shows the grade-related data for students initially in A2. The persistence of A2 increases directly with age (the markers have higher X-axis values with increasing grade). More than a quarter of older A2 students (those in grades 8 to 10) retest in this code on the next test, but less than a tenth of the primary school A2 students retest as A2. Figures 1(b) and (c) provide the same measure of persistence for the codes S1 and S3, respectively. Code S1 is less persistent than A2 and S3 (the maximum X-value is less than 20%) and S1 shows the opposite trend to A2 (persistence decreases with grade). Code S3 is highly persistent, with at least a quarter of all age groups retesting in S3 at the next test. With the exception of the oldest students (Grades 9 and 10), S3 shows a general trend of increasing persistence with grade, like A2.

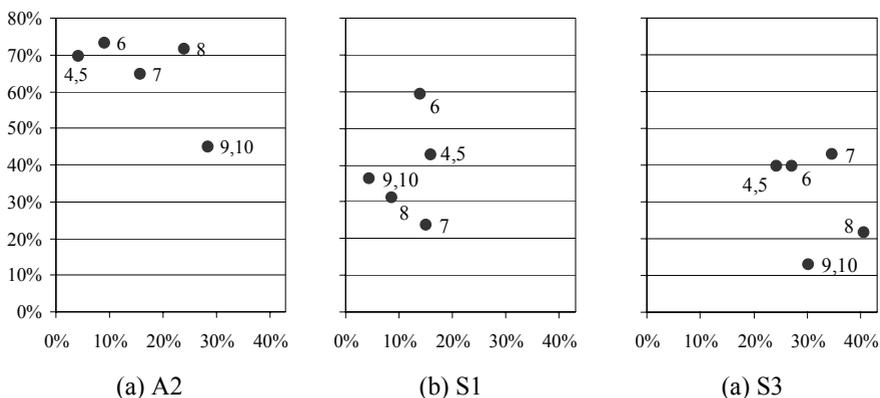


Figure 1: Persistence (on X-axis) against Readiness to move to expertise (on Y-axis) for A2, S1 and S3 by grade.

Next we report the likelihood that a student with a given misconception will become an expert (i.e. complete the test with very few errors). A simple measure shows that in the full sample above, 53% of A2 students are expert at the next test, as are 33% of S1 students and 22% of S3 students. Prospects are better for A2 students than for S1 students, with S3 students least likely to achieve expertise on the next test.

To complete this picture, it is important to examine the likelihood that a student who is looking for new ideas (rather than staying in a previous misconception) becomes an expert. Thus, the proportion of students not retesting in the same code who become experts on the next test is referred to as *readiness to move to expertise*. This is calculated on the reduced sample. Readiness to move to expertise is therefore the conditional probability of a student becoming an expert on the next test, given that they change code. For example, of the 30 Grade 6 A2 students who changed code, 22 moved to A1, hence the readiness to move to expertise is 22/30 (73%). The conditional probability is used for readiness so that it is unaffected by the persistence

of the code; there is no logical relationship between the two variables. Figure 1 displays readiness to move to expertise on the Y-axis for students testing as A2, S1 or S3 at each grade. Hence the two values for A2 students in Grade 6, mentioned above, are graphed in Figure 1(a) as the point (9%, 73%).

Across all grades, the readiness to move to expertise for A2 is a high 65%; while the values for S1 and S3 are much lower (38% and 33%, respectively). Figure 1 shows that students in A2 in Grades 4 to 8 are more likely to become experts (given that they change code) than any other group here, including the oldest A2 students (less than 50% of those who change code). It is important to note that the reason for this decrease with grade level is *not* related to the increase in persistence, as there is no logical relationship between the two measures.

S3 shows a somewhat similar grade pattern to A2, although the readiness to move to expertise is much less in each case. Older S3 students who change code are much less likely to become experts at their next test than younger students (about 20% or less in Grades 8 to 10, compared with about 40% in Grades 4 to 7).

S1 has medium readiness values, but does not show a relationship with grade. Of the students who test as S1 in Grade 6 and change code, 60% move to expertise on their next test, which is relatively high. There is a large drop to less than 30%, however, for the corresponding Grade 7 students. What else do these S1 students do, given that they move from S1, but not to expertise? This will be discussed in the next section.

MOVEMENT TO S-CODES ON THE NEXT TEST

The previous section established that the readiness to move to expertise was very low for S1 students in Grade 7 and also for S3 students in Grades 8 to 10. These students do not retest in the same code (by definition), and yet do not become experts; so what do they do? There are many ways in which the data can be queried to answer such a question. Here we look only at the extent of movement around the S-codes. Figure 2 contains the proportion of students (based on the full sample) whose next test is one of the S-codes, i.e. S1, S3 or other-S. The earlier measure of persistence can be observed in (b) for S1 (black) and in (c) for S3 (white). Looking at the three graphs, we see that S ideas are particularly attractive to students around Grades 7 and 8.

DISCUSSION

Since the results above are based on a large amount of data and the students come from many different schools and classes, we can conclude that the nature of the mathematical thinking of students to some extent determines how their thinking progresses. Both the simple rate and the conditional readiness to move to expertise measure show that A2 is the “best” of these misconceptions to have, followed by S1, and then S3. Students testing as A2 on one test have a strong probability of moving to expertise, overall as well as if they change code. This new expertise, however, may be not because of improved understanding but because they resolve the impasse of their failed algorithm differently, in accordance with the bug-migration ideas of

Brown and VanLehn (1982). (Space precludes a full discussion of this point.) The relatively low rates to expertise for S3 can be explained as both ways of thinking behind S3 treat the decimal portion as a whole number, which is more naïve than S1.

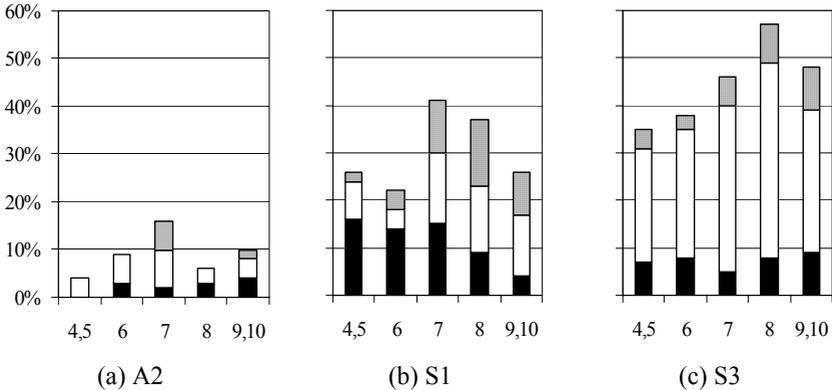


Figure 2: Proportion of A2, S1 and S3 tests, followed by an S test, by grade
Key: black (S1), white (S3), dotted (other-S)

Persistence data shows that S1 is not “sticky”; students in S1 tend not to stay. Code A2 is “sticky” only for older students as younger students have a high probability of moving out, while S3 is “sticky” for students of all grades. The very high persistence of S3 is unexpected as these students cannot order decimals of equal length (e.g. they think that 0.3 is larger than 0.4) a commonly-met situation in classrooms and hence students would be expected to receive regular feedback on errors.

For both A2 and S3, older students have higher persistence, and when they do change code, they are less likely to move to expertise. We explain these trends by noting that these older students are likely to be low-achievers and have *missed the boat*; the main years for decimal instruction have passed. We are unsure why this does not apply to S1. It may be because most S1 students have reasonable place value knowledge.

There are strong tendencies for students in Grades 7 and 8 to move to S-codes on their next test and we explain this as interference from other learning. Negative numbers are being taught then, which may lead to *negative thinking* and generally support *reciprocal thinking* through the mirror metaphors. Scientific notation is also introduced. As 0.00007 can be written as 7×10^{-5} this might also reinforce the association of decimals and negative numbers, as well as long decimals being small.

In summary, this paper has presented a careful analysis of the progress of students who hold an interesting group of misconceptions. Ways of overcoming some of the difficulties of analyzing longitudinal data meaningfully have been indicated. At the most basic level, the results demonstrate that there are significant qualitative and quantitative differences between the progress of students with different misconceptions, as well as grade-related differences. An approach that merges the

groups (often because of insufficient data) will not identify these complexities. This work is in the context of decimal misconceptions, where a well-developed diagnostic test can track the thinking of large numbers of students sufficiently well; it is likely that the same general phenomena will apply to misconceptions in other areas.

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