

# HOW DO HIGH SCHOOL STUDENTS INTERPRET PARAMETERS IN ALGEBRA?

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## Abstract

*This paper presents an analysis of high school and starting college students' work with algebraic expressions and problems involving parameters. We suggest that parameters should be considered as general numbers that are used to make second order generalizations. The Three Uses of Variable model (3UV model) is then used as the theoretical framework to analyse students' interpretation, symbolisation and manipulation of parameters in different contexts. We found that, in general, students had great difficulties in working with parameters but, when they could assign a clear meaning to them, their difficulties decreased. Our results suggest that parameters are general numbers that acquire a clear algebraic meaning only when a specific referent can be given to them.*

## Introduction

Several studies have investigated students' work with the different uses of variables. The great majority stress students' difficulties and errors when working with unknowns, general numbers or related variables. However, only few studies have paid attention to students' work with parameters (Bloody-Vinner, 1994, 2001; Furinghetti & Paola, 1994). These researchers point out that the difficulties students have when working with parameters are both of semantic and syntactic nature. They stress as well, that in order to work properly with parameters it is necessary to be able to distinguish them from unknowns and variables (meaning by "variables" related variables) and that this distinction depend on the context. Bloody-Vinner (2001) refers to parameters as "another usage of letter" additional to unknowns and related variables; we agree with this affirmation, however it seems to us that parameters should not to be considered only as "another usage of letter" different from an unknown or related variables.

From our perspective parameters are general numbers, but of second order, that is, required when generalising first order general statements. By first order general statements we mean those derived from generalising statements involving only numbers. For example, general numbers of first order are present in the general term of a numeric sequence; the general method for calculating the area of a square; equations with numeric coefficients; etc. Parameters appear when we represent families of first order general statements (families of equations, families of functions, families of open expressions), therefore they can be considered general numbers of second order. When parameters are involved in an algebraic expression their role depends on the context, as it is stressed by Bloody-Vinner (2001) and Furinghetti &

Paola (1994), and this role may change within the same problem, for example, from general number to unknown (e.g. *When does the equation  $3x^2 + px + 7 = 0$  have a unique solution?*); from general number to a variable related to another variable (e.g. *Given the general equation of a straight line  $y = mx + c$  write the equation representing all the straight lines passing through the point (5,6)*).

### **Theoretical framework**

From our perspective parameters are general numbers that assume the role of unknown or of related variable, depending on the context. Therefore, the 3UV model (Trigueros and Ursini, 2003; Ursini and Trigueros, 2001) that considers these three uses of variable and the aspects characterising each one of them, can be extended in scope and be a useful theoretical frame to understand students' work with parameters.

From the 3UV model perspective, in order to understand students' difficulties with parameters it is necessary to focus on their capability to interpret them, to symbolise them and to manipulate them in different contexts. Moreover, when the problem requires shifting between the different roles a parameter can assume (that of general number, unknown or related variable) it is necessary to be able to handle the aspects characterising each one of these specific uses of variable in an appropriate way.

When the parameter is perceived as a general number students need to be able to: relate parameters to patterns' recognition, or to the interpretation of rules and methods; interpret it as representing a general, indeterminate entity that can assume any value; deduce general rules and general methods by distinguishing the invariant aspects from the variable ones in first order general statements; manipulate it; symbolise it for representing second order general statements, rules or methods. When a parameter assumes the role of an unknown it is necessary to: recognise that it represents something unknown that can be determined; interpret it as representing specific values that can be determined by considering the given restrictions; determine its values by performing the required algebraic and/or arithmetic operations; substitute the required values to the parameter in order to make the given condition true. When a parameter assumes the role of a variable related to another one, it is necessary for the student to: recognise the correspondence between the two variables in the analytic expression; determine its values in terms of the value of the related variable or determine the value of the related variable in terms of the value of the parameter; recognise the joint variation of the parameter and the related variable; determine the interval of variation of the parameter or the related variable when the interval of variation of the other one is given. To work successfully with parameters it is therefore necessary to be able to shift between different uses of variable and to master each of them with its own characteristics.

Our purpose in this study was to identify students' interpretations and difficulties when working with parameters. In order to do so, we have used the 3UV model as

theoretical framework to analyse students' interpretation, symbolisation and manipulation of parameters in different contexts.

## **Methodology**

The 3UV model guided the methodology we used in the design of the items of a questionnaire and in their analysis. This methodology was already used in previous work where attention was centred in the analysis of the way students work with the concept of variable in elementary algebra (Trigueros and Ursini, 2003; Ursini and Trigueros, 2001). The way questions are designed using this methodology has shown to be an efficient way to probe students' reasoning on variable. In fact, in previous studies it was found that the results obtained from the analysis of the responses given to a questionnaire and those obtained from the analysis of interviews gave the same results in terms of the analysis of students' reasoning. This methodology allows us to classify the questions into categories related to the interpretation, manipulation and symbolizations of the different uses of variable. It makes also possible to isolate and identify with some precision students' strengths and weaknesses when they work with algebraic expressions.

A 16 items questionnaire was designed (Table 1) which includes items requiring interpretation, symbolisation and manipulation of parameters. The questions can also be distinguished between those in which the parameter has a geometric referent and those in which it has a strictly algebraic one, in order to make it possible to observe if different contexts influenced students' responses.

The questionnaire was piloted and the final version was answered by 112 students: 50 of them were still attending the last year of high school, and 62 were initiating their first mathematics course at college level at a small Mexican university. Students' answers were analysed independently by two researchers and, when differences appeared, they were negotiated for validity.

## **Results**

### *Results related to interpretation of parameters*

Most of the students interpreted a parameter as a general number and, in most cases, they had difficulties in differentiating it from the other variables involved in an expression. This happened both, when questions were posed in a geometric or in a strictly algebraic context. There were students, however, who had less difficulties in differentiating the role of a parameter when they could attribute a geometric referent to it. This happened, for example, when a parameter was involved in an equation of a line (question 2). Students' responses to this question demonstrated their capability to interpret the parameter as the slope of a line. The responses of two students exemplify typical answers to this question: "*a is the value of the slope of the line*"; "*a represents the slope of the line represented by  $y=ax+3$* ". We can conclude, however, that students were using a memorized fact when assigning meaning to the symbol *a* since, when asked to calculate the value of the parameter in a particular case (*which is the line through the point (2,3)*), almost all the students showed

confusion, and they were not able to do it. In the previous example the same two quoted students were not able to calculate the slope using the given information. One of them wrote: “ $y=mx + 3$ ,  $y=-2x+3$ ,  $(3/2)x+y-3=0$  so,  $m=-(3/2)x + 3$ ”.

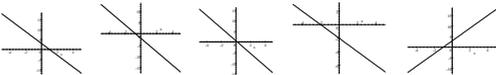
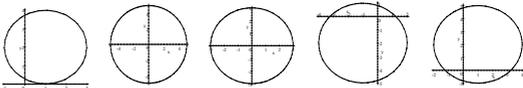
<p>1. Given the family of lines <math>ax + by = c</math> in the plane <math>xy</math>. Where does each line cross the axe <math>X</math>? Where does each line cross the axe <math>Y</math>? What does <math>a</math>, <math>b</math>, <math>c</math>, <math>x</math> and <math>y</math> represent?</p>	<p>9. Give a particular example of the general equation <math>(m + 2)x^2 + 3mx + 6 = m + 2</math>. If <math>x = 0</math>, the expression can still be considered an equation? Explain your answer. If <math>m = -2</math>, the expression can still be considered an equation? Explain your answer.</p>
<p>2. Given the family of lines <math>y = ax + 3</math>, which is the line through the point <math>(2,3)</math>. What does <math>a</math> represent?</p>	<p>10. Use the formula <math>x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}</math> to solve the equation <math>2x^2 - 5ax + 8 = 0</math>. What does <math>a</math> represent in the equation? For which values of <math>a</math> the equation has no solution in real numbers?</p>
<p>3. Given the expression <math>x^2 + y^2 = m^2</math> explain what is constant and what varies. What do <math>x</math>, <math>y</math> and <math>m</math> represent?</p>	<p>11. Which of the following lines are represented by <math>y = -3x + C</math>? What does <math>C</math> represent? Which is the value of <math>C</math> in the graphic you chose?</p> 
<p>4. Simplify and solve the equation <math>a^2x - 3a^2 - bx + 3b^2 = 0</math>. What do <math>a</math>, <math>b</math> and <math>x</math> represent?</p>	<p>12. Which of the following circumferences are represented by <math>x^2 + y^2 = K</math>? What does <math>K</math> represent? Which is the value of <math>K</math> in the chosen circumference?</p> 
<p>5. Develop the expression <math>(x^2 - a)(x^2 + b)</math>. What do <math>x</math>, <math>a</math> and <math>b</math> represent?</p>	<p>13. Use symbols to express: The whole numbers that multiplied by 3 produce a multiple of 7. Explain the meaning of the symbols used.</p>
<p>6. Given the equation <math>m(x - 5) = m + 2x</math>, explain the role of <math>m</math> and <math>x</math> in the equation. For which values of <math>x</math> the equation has no solution?</p>	<p>14. Use symbols to express: The real numbers that after being multiplied by a constant are added to 4 and give a multiple of 5. Explain the meaning of the symbols used.</p>
<p>7. Given the equation <math>3x^2 + px + 7 = 0</math>, for which values of <math>p</math> the equation has only one solution? Which are the roles of <math>p</math> and <math>x</math> in this equation?</p>	<p>15. Use symbols to express: The lines with constant slope equal to <math>-2</math>. Explain the meaning of the symbols used.</p>
<p>8. All these equations belong to the same family: <math>3x + 2y = 5</math>; <math>23x + 7y = 5</math>; <math>-8x + 19y = 5</math>. Which is this family? Describe it in your own words and write an algebraic expression to represent it.</p>	<p>16. Write the symbolic expression you would use to solve the following problem but do not solve it: A rectangular land should be circled. There is a fixed longitude for the fence. Which is the area of the circled land in terms of the length of one side? Explain the meaning of the symbols used.</p>

TABLE 1

Usually the letter used for the slope in the general equation is  $m$  so, in order to help himself calculate it, this last student changed the original letter  $a$  to  $m$  and tried to calculate its value. The other one wrote: " $3=2a+3$ " substituting the given values in the equation, but it was observed from his work, that he was not able to continue because he could not accept  $a=0$  as the solution for the slope and  $y=3$  as the equation he was looking for.

Another evidence of students' tendency to use memorized facts to make sense of the role of the parameter is shown in their responses to question 3. The equation does not represent a line, but students who tried to distinguish between the roles of  $x$ ,  $y$  and  $m$  replied " $x$  and  $y$  are variables and  $m$  is constant because  $m$  is the slope". Most of them could not find any referent and they considered that the parameter was just a general number. They replied to this question writing " $x$ ,  $y$  and  $m$  are the variables" or "*all of them,  $x$ ,  $y$  and  $m$  can change*". Only one student referred to the equation of the circle and interpreted  $m$  as the radius.

When the geometrical referent was familiar and was supported with graphic representations (question 11), we observed that college students found a firm support to the interpretation of the parameter as a generalization of the Y intercept of a line in the graphical representation. This, however, was not the case for the majority of high school students, who did not even answer this item. When the geometrical referent was a circle (question 12) students were able, in general, to interpret the parameter as a generalization of something, but were not able to attribute a meaning to it, even when they were able to identify the circle related to the given equation. Only few students recognized the parameter as the radius of the circle.

The tested students have had experience with the equation of a line since the beginning of high school. However, they seem not to have understood it as expected but have only memorized some clue aspects of it. In the case of the circle, all students have had experience with its equation because a course in Analytic Geometry was compulsory for all of them, however, the equation of the circle was something less familiar still to them than the equation of a line. Students' responses show that only a very familiar geometrical context, can give support to the interpretation of the parameter.

When there was not a geometric referent, the structure of the algebraic expression was supposed to provide the referent allowing to differentiate the parameter from the other variables involved. To question 6 most of the students responded " $m$  and  $x$  are variables", " $m$  is an independent variable and  $x$  a dependent variable" or " $x$  is an unknown,  $m$  is a variable". But looking perhaps for a most clear referent, almost 2/3 of the students associated the letter  $m$  to the slope of a line. Nobody could manipulate the expression in order to determine the value of  $x$  for which the equation has no solution. Doing this implies to look at  $x$  and  $m$  as related variables, to express  $m$  as a function of  $x$  ( $m = 2x/(x-6)$ ) and to realise that the value of  $x$  should be other than 6. Students looked for the values that nullified the terms involving  $x$  by direct

inspection, ignoring completely its relation with the parameter, and they considered that the given equation had no solution when  $x = 5$  or  $x = 0$ .

In the case of quadratic equations, less than half of the students could work appropriately with the parameter involved in the expression. In question 7 where the parameter was part of the coefficient of the linear term, students were expected to recognise the algebraic structure of the equation and to recognise the role of the parameter in it. Moreover, they should look at  $p$  as an unknown and find the value of  $p$  that nullified the discriminant of the equation. The great majority of students found it difficult to make sense of this kind of situations. Many of them solved the equation ignoring the parameter. Others responded “*there is no value for  $p$ , since the equation is quadratic, it has to have two different results*”. Some students, however, recognised  $p$  as the unknown but they tried to find its value directly from the given expression and they wrote  $p = (-3x^2 - 7)/x$  and they could not continue.

#### *Results related to manipulation of parameters*

When facing an expression involving parameters, it is necessary to take a decision about the role that each symbol plays in the expression, and then proceed to manipulate the expression according to the requirements of the problem. The great majority of students could not attribute a specific meaning to the parameter and, in consequence, they were not able to manipulate it. Our data suggest that this difficulty is related to the fact that, even when students can differentiate in some way the role that the different letters play in a problem, they are not able to establish clearly their specific role. For example, in question 4, most of the students manipulated the symbols in an appropriate way considering all the letters involved as variables, but they were not able to solve the equation because they were not told explicitly which of the symbols was the unknown. In contrast, when it was clear that  $x$  was the unknown, they could manipulate and solve correctly. When told that the unknown of the equation was another letter than  $x$ , the rate of success decreased a lot.

When facing open expressions (question 5) where all the symbols play the role of general numbers, students were able to think of them that way. Most of them were able to manipulate these kind of expressions. For example, they could develop them when explicitly required and most of them wrote: “ $(x^2-a)(x^2+b) = x^4 + x^2b - ax^2 - ab$ ” and “ $(x^2-a)(x^2+b) = x^2(x^2+b) - a(x^2+b) = x^4 + x^2b - ax^2 - ab$ ”.

#### *Results related to symbolization of parameters*

To symbolise second order generalisations, where parameters should be included, is a very difficult task for students. Most of them ignored the parameter even in the cases where some of them could recognise the pattern leading the first order generalisations involved in the problem. When trying to use parameters, we found that they tended to over-generalise. This clearly appeared in the answers given to question 8. Some of them wrote “ $ax+by=c$ ” over-generalising. A lot of students answered “ $x+y=5$ ”, and many of them showed the contrary tendency, that is, they

gave a particular example such as " $18x+28y=5$ " showing in this way that they had recognised the family but that they were not able to represent it using parameters.

When a geometric referent was provided, for example a family of lines, only few of the students were able to recognize the family, to use a parameter to represent a specific feature of the family, and to show some examples of it, but they were not able to write a general expression to describe it. This is clear in the answers given to question 15. Typical responses to this problem were " $m=-2$ " or something such as " $Ax+By=C$ ", and some students gave an example such as " $y=-2x+3$ ".

When faced to verbal second order generalisations and asked to express it symbolically the great majority of students wrote expressions that did not reflect the meaning of the statement. Those who successfully symbolized the expression, showed lots of difficulty in interpreting the meaning of the different symbols they used to symbolize. This was the case of questions 13 and 14. The following answer shows that this student was able to symbolize the expression, but her understanding of symbols was limited: " $x.3=b$ , with  $b=7a$ , and  $x$  is the integer number we are looking for and  $a$  is the result of  $b$  divided by 7". She identified the symbol  $x$  with an unknown, could interpret  $a$  only in terms of the arithmetical operations required to obtain it, and did not explain the meaning of  $b$  in her own terms. Another student wrote " $x.3=7n$ " and then explained " $x$  is  $x$  and  $n$  is a number".

None of the students of this sample was able to symbolize a very simple problem involving parameters (question 16). They showed a tendency to use a symbol as a label for a given data, but not to use it to symbolize a whole expression. Typical responses to this question were: " $2h \times 2l$ ", " $la \times lb$ ", " $A=l^2$ " or " $A=b \times h$ : formula for the area", but none of the students could symbolize it correctly.

## Conclusions

The interpretation, manipulation and symbolisation of parameters was, in general, difficult for students, even for students who were taking college courses and were registered in programs where advanced mathematics are required. Our data suggest that to be able to differentiate parameters from other variables and to give meaning to them it is necessary for the students to have a clear referent or statement that gives meaning to the second order generalization, otherwise they can only perceive in it its character as general number. The referent can come from a familiar situation, in geometric or algebraic context, but students found it very difficult to assign it by themselves when needed. These results suggest that parameters are general numbers that acquire their algebraic meaning of general numbers of second order only when a specific referent can be given to them. We found that students relied strongly on memorized facts. Although, in general they tried to answer all the questions, they did not have enough resources to solve them and they used only what they could recall immediately. When they were not able to remember something useful in the solution of a problem they did not respond or they made up a quick answer out of context. In general they did not try to make sense of the problem in their own terms, even if they

had studied techniques that could be helpful in making sense of a particular situation. For example, when they solved the question related to the equation of the circle, none of them substituted coordinates in the equation in order to determine the role of the parameter in it. We found as well that manipulation of parameters depends on the possibility of the student to interpret them. When they could assign an appropriate meaning for them in the given expression, the manipulation was less of a problem for them. The use of parameters to symbolise a second order generalization was extremely difficult for students. Even in cases where they were able to write an expression they were not able to explain the different roles the symbols played in it.

Synthesising, our results confirm what Bloedy-Vinner and Furinghetti had signalled, that is, that the role of the parameters is very context dependent and that it is impossible to get away of the consideration of great logical complexity involved in the work with them. Our work shows, however, that the most important consideration about parameters is their role in second order generalizations. When teaching the use of parameters students' conceptions of them as general numbers should be taken as the starting point in order to help them approach second order generalisations and give meaning to parameters. Activities related to the different roles parameters can assume, according to the 3UV model, can help students discriminate between the roles they assume in specific problem situations. The results obtained make us wonder if presently students are really learning algebra at school. Gascón, Bosch and Bolea (2001) had already pointed out this problem. They underlined the fact that in school algebra letters usually play the role of unknowns and that parameters are very frequently absent, thus the formulae do not appear in these courses as the result of an algebraic generalisation and they do not play the role of algebraic models, but only of "rules" to be used in order to perform specific calculations. We consider our results totally agree with this assertion.

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