THE EFFECTS OF DIFFERENT MODES OF REPRESENTATION ON MATHEMATICAL PROBLEM SOLVING[1]

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The main objective of this study is to investigate the role of four different modes of representation in mathematical problem solving (MPS), and more specifically to develop a model, which provides information about the effects of these representations in the solution procedures of one-step problems of additive structures. Data were collected from 1447 pupils in Grades 1, 2 and 3 of elementary school in Cyprus. Confirmatory factor analyses (CFA) affirmed the existence of four first-order representation-specific factors indicating the differential effects of each particular type of representation and a second-order factor representing the general ability to solve mathematical problems. Results provided support for the invariance of this structure across the three groups of pupils.

INTRODUCTION

There is strong support in the mathematics education community that students can grasp the meaning of mathematical concepts by experiencing multiple mathematical representations (e.g., Sierpinska, 1992; Lesh, Behr, & Post, 1987). Furthermore, Principles and Standards for School Mathematics (NCTM, 2000) include a standard referring exclusively to representations and stress the importance of the use of multiple representations in mathematics learning. The present study purports to throw some light about the nature and the contribution of three systems of representations, pictures, number line and verbal description (written text) of the problem to MPS.

THEORETICAL FRAMEWORK

A representation is defined as any configuration of characters, images, concrete objects etc., that can symbolize or "represent" something else (Kaput 1985; Goldin, 1998; DeWindt-King, & Goldin, 2003). In elementary mathematics teaching and curriculum design, a representation that plays an important role in the teaching of basic whole number operations, and generally in arithmetic, is the number line (Klein, Beishuisen, & Treffers, 1998). Despite the widespread use of number line diagrams as an aid to whole number addition and subtraction, doubts about the appropriateness of using them have been raised (Hart, 1981). Ernest (1985) supports that there can be a mismatch between students' understanding of whole number addition and their understanding of the number line model of this operation. In fact, number line constitutes a geometrical model which involves a continuous interchange between a geometrical and an arithmetic representation. Based on the geometric

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dimension, the numbers depicted in the line correspond to vectors. According to the arithmetic dimension, the number corresponds to a point on the line. The simultaneous presence of these two conceptualizations may limit the effectiveness of the number line and thus hinder the performance of pupils in arithmetical tasks (Gagatsis, Shiakalli, & Panaoura, 2003).

Although, the mental processes, and particularly the visual-spatial images, used in MPS or mathematics learning have received extensive research in the field of mathematics education (e.g., Presmeg, 1992; Gusev, & Safuanov, 2003), the role of pictorial representations or number line in MPS, has received much less attention. Based on the functions that pictures serve in text processing, as proposed by Carney and Levin (2002) (decorative, representational, organizational, interpretational and transformational), the present study attempted to examine the role of two divergent categories of pictures, decorative and informational pictures, in MPS. Decorative pictures do not provide any information to the pupils for the solution of the problem, but simply decorate the page bearing little or no relationship to the problem context. Informational pictures provide information that is essential for the solution of the problem; the problem is based on the picture.

What is new in this study is that, besides the effect of pictorial representations, it aims to provide information about the effect of the use of number line on problem solving, and compare these effects between each other, and with the effect of the verbal description of the problem. Further, as concerns the use of representations in MPS within the present study, we assume that a major distinction is needed, between auxiliary and autonomous representations. Auxiliary representations are not necessary for the solution of the problem but may assist the process of MPS. Autonomous representations have an essential role in MPS, since, through them, any information related to the problem can be expressed. In terms of this study, we theorize that the number line and the decorative picture are auxiliary representations, while the verbal description (written text) of the problem and the informational picture are autonomous representations.

The present study focuses on one class of problems with additive structures, based on the classification of additive problems, proposed by Vergnaud (1995). In particular, we used one-step change (measure-transformation-measure) problems. We have included problems with positive (join situation) and negative transformation (separate situation) and problems with the placement of the unknown in the starting amount (a) and the transformation (b), that is four situations in total (2x2).

To sum up, the purpose of the present study is to explore and compare the effects of decorative, informational pictures, number line and verbal description (text) of the problem on MPS by pupils of Grades 1, 2 and 3 and investigate how these effects vary in pupils of different age. The following research questions were formulated: First, which are the effects of the particular forms of representations (decorative picture, informational picture, number line and verbal description) on pupils' MPS performance? Second, what are the differences between pupils of different age in

regard with the effect of representations on their MPS performance? It should be noted that the questions of the present study are beyond the mean differences and concern with the structure of pupils' MPS within different modes of representations.

METHOD

The instrument used in this study, to collect information for pupils' MPS, involving different modes of representations, was a questionnaire. The questionnaire consisted of 16 one-step change problems with additive structures (a+b=c). The problems were accompanied with or represented in different representational modes. The exact classification of the problems included in the questionnaire and the symbolization used for them in terms of the analysis of the data are provided in Table 1.

Type of representation	Join situation (b>0)		Separate situation (b<0)	
	Placement of the unknown		Placement of the unknown	
	a	*b*	*a*	*b*
Verbal	V10	V20	V6	V17
Decorative picture	D16	D3	D8	D13
Informational picture	I4	I14	I19	I9
Number line	L12	L7	L18	L5

Table 1: Specification Table of the problems included in the questionnaire

The written questionnaire was administered to 1447 pupils of Grades 1, 2 and 3, from 26 elementary schools in Cyprus (479 1st graders, 477 2nd graders and 491 3rd graders). Pupils were ranging in age from 6.5 to 8.6 years at the time of testing. It should be noted that mathematics elementary books in Cyprus include many mathematical activities based on representations (pictures and number line). Pupils were instructed to use the representations, if they believed they could help them resolve the problems. Answers were marked as 0, 1 and 2. Each correct solution procedure (equation or description in words) was marked as 2, each correct answer without equation or explanation as 1, and each wrong answer or solution procedure as 0. Cronbach's alpha coefficients for the test were found to be well above commonly accepted levels (0.9) of reliability.

Multiple methods of analysis were performed, using the collected data, including Gras implicative analysis (Gras, Peter, Briand, & Philippé, 1997) and Rasch model analysis. Since, however, the present study firstly aims at the articulation of a model explaining the effect of the different types of representations used in MPS, data analysis will focus on CFA. Specifically, the data was analysed by using CFA for the total sample and multiple-group analysis for the different groups of pupils, to explore the theoretical assumption that both the semantics of mathematical problems

(additive structures, placement of the unknown) and the different modes of representation involved in the problems (verbal description-text, decorative picture, informational picture and number line) affect MPS. One of the most widely used structural equation modelling (SEM) computer programs EQS (Bentler, 1995) was used to test the proposed models. In order to evaluate the extent to which the data fit the models tested, the chi-square to its degree of freedom ratio (x^2/df), the Comparative Fit Index (CFI), and the Root Mean Square Error of Approximation (RMSEA) were examined. It is generally recognized that observed values for x^2/df < 2, for the CFI > .9 and for the RMSEA < .05 are needed to support model fit. Finally, the factor parameter estimates for the model with acceptable fit were examined to help interpret the models and their divergence.

RESULTS

In order to refute the assumption that MPS is influenced only by the semantics of the problem, a first-order model, which presumes that representation is not a factor in MPS, was examined within the SEM framework. This model involved one first-order factor, which associated all of the tasks and could be taken to stand for general ability of MPS. Specifically, the model hypothesized that: (a) responses to the questionnaire could be explained by one first-order factor representing the ability to solve one-step change problems with additive structures; (b) each item would have a nonzero loading on the factor; and (c) measurement errors would be uncorrelated. This model did not have a good fit to the data [x^2 (98) =1210.29; CFI=.86; RMSEA=.09], and therefore, could not be considered appropriate for explaining the ability of MPS.

To verify that apart from the semantics of the problem, the modes of representation within the problem have a major role in MPS, a second order CFA model was designed. Specifically, the model (see Figure 1 for a diagram of this model and Table 1 for information about the tasks for each factor) hypothesized that: (a) responses to the questionnaire could be explained by four first-order factors that would stand for the four types of representational assistance used here, i.e., pupils' abilities in solving problems represented verbally (V), as an informational picture (I), accompanied by a decorative picture (D), and a number line (L) respectively, and one second-order factor, i.e. pupils' general ability to solve one-step change problems of additive structures (MPS); (b) each item would have a nonzero loading on the factor it was designed to measure and zero loadings on all other factors; (c) measurement errors would be uncorrelated (d) covariation among the four first-order factors would be explained fully by their regression on the second order factor. Figure 1 presents the results of the elaborated model, which fits the data reasonably well [x^2 (94) =483.83; CFI=.95; RMSEA=.05], and shows the parameter estimates. By comparing the second order factor model with the first order factor model, a decrease of the RMSEA and an increase of the CFI could be identified (see Table 2). Thus, the second order model is considered more appropriate for interpreting the ability of MPS.

To test for possible differences between the three age groups in the structure described above, multiple-group analysis was applied where the second order model

was fitted separately on each age group. The model was first tested under the assumption that the relations of observed variables to the first-order factors and of the four first-order factors to the second-order factor would be equal across the three age groups. The fit of this model, although acceptable, was not very good [x^2 (32) = 816.72; CFI=.92; RMSEA=.03]. This was due to the fact that some of the equality constraints were found not to hold (especially, those involving the tasks with an informational picture). As a result, the equality constraints were released.
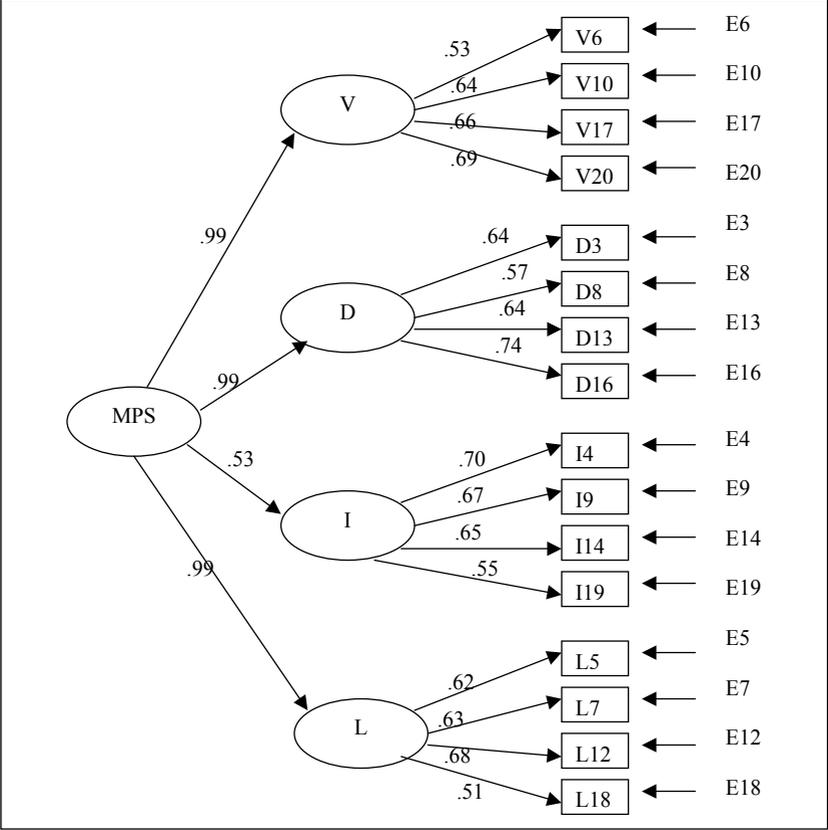

Figure 1: The Elaborated Model for problem solving with different modes of representations, with Factor Loadings and Factor Correlations for all the pupils

Releasing the constraints, a large improvement of the model fit emerged [x^2 (360) = 708.09; CFI = .95; RMSEA = .03] in comparison with the model for the whole sample, as shown in Table 2. In the multi-group model, the parameter estimates for 2nd graders were higher than the estimates for 1st and 3rd graders for almost all tasks.

Moreover, most of the parameter estimates of the model for 1st graders were lower than the estimates of the model for 3rd graders. This finding indicates that the modes of representations involved in the problems did not influence the solution procedures of 3rd graders and 1st graders in the same way as the solution processes of 2nd graders.

Model examined	x^2	df	CFI	RMSEA
General Model with one first order factor	1210.29 p=.0000	98	.86	.09
General model with four first order factors and a second order factor	483.83 p=.0000	94	.95	.05
Multi-group model for 1st, 2nd and 3rd graders	708.09 p=.0000	360	.93	.03

Table 2: Goodness of Fit Indices

Based on the second order models, it is asserted that the first order factor concerning the informational picture is not as closely related to the second order factor as the first order factors involving the other types of representation. This finding suggests that the tasks involving informational picture require extra mental processes relative to the other tasks. The low percentage of pupils (18%) who accomplished a correct solution of the problems involving informational pictures also supports the above result. The high and similar loadings (.99) of the other first order factors on the second order factor reveal that pupils dealt with the problems in verbal form or accompanied with decorative pictures or number line, in a similar and consistent way.

Implicative statistical analysis of the particular data provides support to these findings. In particular, the similarity diagram derived from the implicative analysis, which allowed for the arrangement of tasks into groups according to their homogeneity, contained two basic groups of problems. The group with the greatest similarity consisted of the problems involving informational pictures and the other group contained a combination of the problems with number line, decorative picture and in verbal form. The commonality between the results of both statistical analyses is justified by the distinction between auxiliary and autonomous representations, proposed in our study. Each of the two groups of tasks formed in the analyses corresponds to an autonomous representation, informational picture and text, respectively. The auxiliary representations, which just accompany the problems, function as an adjunct to the verbal description of the problem in the analyses.

DISCUSSION

The main purpose of this study was twofold, to test whether different forms of representation have an effect on MPS and to investigate its factorial structure within the framework of a CFA, across pupils of three different grades. The results provided a strong case for the role of the use of different forms of representations in combination with the semantics of the problems in MPS. The size of the factor coefficients of the proposed model indicates that the ability of pupils to solve one-

step change problems of additive structures is highly associated with the abilities of solving problems in verbal form, with decorative pictures and number lines. The coherence and similarity in the ways pupils handled these representations within their solution processes implies that pupils overlooked the presence of the line or the picture and gave attention only to the text of the problem. This kind of behaviour towards number line can be attributed to the difficulties caused by the mismatch of the conception of number within the context of the problem, as a quantity of items, and the dual conception of number within the framework of the number line, as a vector and as a point (Gagatsis et al., 2003). As regards the decorative picture, the particular finding is in line with Carney and Levin' s (2002) view that decorative pictures do not enhance any understanding or application to the text.

As concerns the function of informational pictures in MPS, from the results of this study, it is evidenced that it differs significantly from the use of other forms of representation in MPS. It was clear that pupils dealt less flexibly with problems involving informational pictures. This indicates that the cognitive demands of the informational picture in the context of mathematical problems are different from the other forms of representations (examined in the present study). These results support Miller's (2000) conclusions that each representational system has its own regularities. Therefore, the results concerning informational pictures and number lines suggest that both kinds of representations need special attention within the context of MPS during instruction. The above findings are in accord with Stylianou's (2001) conclusions that students do not have adequate training associated with the use of visual representations and lack the particular skill.

The stability of the models for the different age groups (1st, 2nd and 3rd graders) provides support for the existence of the same structure of pupils' MPS involving the representational modes explored in this study. However, some differences emerged between the models, as regards the strength of the relationships between the tasks and the factors. An explanation for the 1st graders' lower estimates is that they have not yet developed the mental abilities needed for dealing consistently with the particular forms of representation in problem solving. The decrease of the strength of the relations between the general factors and some of the specialized factors for 3rd graders in comparison with 2nd graders indicates that with development, the functioning of the specialized factors becomes less dependent on general abilities. Thus, 3rd graders used the particular forms of representation in MPS in a more autonomous and flexible manner than 2nd graders. This finding reveals the existence of a possible developmental trend in pupils' abilities in MPS based on different forms of representation. Further research is needed to examine and verify this finding.

References

Bentler, M. P. (1995). *EQS structural equations program manual*. Encino, CA: Multivariate Software Inc.

Carney, R. N., & Levin, J. R. (2002). Pictorial illustrations still improve students' learning from text. *Educational Psychology Review, 14* (1), 101-120.

DeWindt-King, A.M., & Goldin, G. A. (2003). Children's visual imagery: Aspects of cognitive representation in solving problems with fractions. *Mediterranean Journal for Research in Mathematics Education, 2* (1), 1-42.

Ernest, P. (1985). The number line as a teaching aid. *Educational Studies in Mathematics, 16*(4), 411-424.

Gagatsis, A., Shiakalli, M., & Panaoura, A. (2003). La droite arithmétique comme modèle géométrique de l'addition et de la soustraction des nombres entiers. *Annales de didactique et de sciences cognitives, 8*, 95-112.

Goldin, G.A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior*, 17(2), 137-165.

Gras, R., Peter, P., Briand, H., & Philippé, J. (1997). Implicative Statistical Analysis. In C. Hayashi, N. Ohsumi, N. Yajima, Y. Tanaka, H. Bock, Y. Baba (Eds.), *Proceedings of the 5th Conference of the International Federation of Classification Societies* (Vol. 2, pp. 412-419). Tokyo, Berlin, Heidelberg, New York: Springer-Verlag.

Gusev, V., & Safuanov, I. (2003). Thinking in images and its role in learning mathematics. In N. Pateman, B. Dougherty & J. Ziliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 4, pp. 87-94). Honolulu: University of Hawaii.

Hart, K. M. (Ed.), (1981). *Children's understanding of mathematics*. London: John Murray.

Kaput, J.J. (1985). Representation and problem solving: methodological issues related to modelling. In E.A. Silver (Ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* (pp. 381-398). Hillsdale, NJ: Erlbaum.

Klein, A. S., Beishuisen, M., & Treffers, A. (1998). The empty number line in Dutch second grade: Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29* (4), 443-464.

Lesh, R., Behr, M., Post, T. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*, (pp. 33-40). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Miller, F. K. (2000). Representational tools and conceptual change: The young scientist's tool kit. *Journal of Applied Developmental Psychology, 21*(1), 21-25.

NCTM, (2000). *Principles and standards for school mathematics*. Reston, Va: NCTM.

Presmeg, N.C. (1992). Prototypes, metaphors, metonymies, and imaginative rationality in high school mathematics. *Educational Studies in Mathematics, 23*, 595-610.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky, & G. Harel (Ed.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25-58). USA: Mathematical Association of America.

Stylianou, D. (2001). On the reluctance to visualize in mathematics: Is the picture changing? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp.225-232). Utrecht, The Netherlands: Freudenthal Institute.

Vergnaud, G. (1995). Problems of additive structures. In A. Gagatsis (Ed.), *Didactics of Mathematics: Theory and Research.* Thessaloniki: Art of Text (in Greek).