

TRACKING PRIMARY STUDENTS' UNDERSTANDING OF THE EQUALITY SIGN

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Recent curricular reforms are following the lead of the Standards 2000 and, in diverse ways, integrating some algebraic work into primary school mathematics. Our research aims at producing a tool that will allow researchers to track the development of algebraic thinking in children as they progress through various primary programs. In this paper we will illustrate how we have proceeded and what we have learned in our exploration of one small but widely recognized element of algebraic thinking, a rich understanding of the equal sign. An analysis of questionnaires administered to a kindergarten, a grade three and a grade six class in a Montreal area school allows us to present a wide range of student errors and to suggest some key questions for tracking students' thinking and for comparing them across curricula.

INTRODUCTION

Over a decade of research into students' understanding of introductory algebra (Hart, 1981; Booth, 1984, 1988; Wheeler & Lee, 1987; Kieran, 1992; etc.) confirmed what was already popularly recognized¹: algebra constitutes a major obstacle for a significant number of middle and high school learners. There have been a large number of responses to this problem which have mainly centered on reforming the teaching of high school algebra: problem solving approaches, strengthening arithmetic skills, slowing down the introduction of algebra, various technological approaches. One response—by a growing group of researchers and considered unthinkable two decades ago²—has recently found expression in a large number of curricular reforms in North America. The American *Standards 2000* has enshrined the response of the early introduction of algebra by creating an “algebra strand” running in parallel to the traditional arithmetic and geometry from kindergarten onwards.

Many primary curriculum reforms have followed suit. In Quebec³, where the primary reform is now in its third year of implementation, the development of algebraic thinking is present as an aim though always hidden. The proposed high school curriculum admits that the student has already been introduced to algebra in primary

¹ This opacity of algebra found expression in jokes and phrases such as the French “C'est de l'algèbre” [It's algebra] for the English phrase “It's Greek to me”.

² In 1984, an ICME-5 working group softened its name from “Algebra in Elementary School” to “Algebraic thinking in the early grades” because, as Davis (1985) expressed it, some participants “were opposed to the whole idea” (p. 198).

³ The Quebec reform, which is much more timid in its recommendations concerning algebra, appears to have been inspired by the 1989 Standards and the 1994 Vision of Algebra document.

school but “à son insu” [without his being aware of it]. The Canadian Atlantic provinces are much more explicit about the inclusion of algebra in their K-8 curricular agreement of the year 2000. The document, *Mathematics Learning Results* (Education Foundation of the Atlantic provinces, 2000), mentions algebra in the formulation of all four of its didactic learning principles and puts “patterns and relationships” as one of the four content domains.⁴

The task of creating appropriate didactical tools for the teaching of algebra in the early grades—textbooks, inservice and preservice teacher training programs, evaluation tools—has barely begun. NCTM has suggested a few classroom activities in its *Navigating through algebra* (2001) series. Carpenter, Franke and Levi (2003) have recently produced a textbook for use in teacher training, *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. An analysis of reform textbooks in Quebec indicates that most all but bypass the algebraic goal⁵.

Our research falls within the “evaluation tools” category; it aims at tracking students’ algebraic thinking throughout primary school. This tool, intended chiefly for researchers, will allow us to compare various primary school programs—those that aim at the introduction of algebraic thinking to varying degrees, those that build algebra on arithmetic, those that develop algebra independently or in relation to the entire curriculum and those that begin school with algebra and build arithmetic on a basis of algebra⁶—and to trace the development of algebraic thought in individual students as they move through the grades within these programs.

In this paper, we will look at one small but widely recognized element of algebraic thinking, the concept of equality and the related understanding of the equal sign, as an example of the construction and use of this instrument. Many researchers have concluded that one of the requirements of the passage into algebra is a much richer understanding of the equal sign than that which is provided by traditional arithmetic and most traditional algebra curricula deal with this early on or relegate it to what they call pre-algebra.⁷

⁴ The others are: number and operations, shapes and space, data analysis and probability.

⁵ Indeed, students in the master’s program at the Université du Québec à Montréal, concluded that it would be entirely conceivable that a classroom teacher might not even notice the new algebraic aim, so well is it hidden in the curriculum and textbooks. The one textbook series (Défi Mathématiques) for primary school that went furthest in developing algebraic thinking before the reform, has scaled back the algebraic component, apparently to meet the acceptance requirements of the ministry!

⁶ The Hawaiian “Measure Up” program is currently adapting and testing the Russian Elkonin-Davydov curriculum in two of its schools. Children begin by comparing and operating on quantities before number is introduced. Thus algebraic thinking precedes arithmetic.

⁷ In Quebec, for example, the equal sign has been studied in the first year of high school in what is called a “preparation for algebra”.

STUDIES OF EQUALITY

The NCTM *Standards* (2000) consider “equality is an important algebraic concept that students must encounter and begin to understand in the lower grades” (p. 94). This recognition of the importance of the equality concept is based on decades of research from Ginsburg (1977) to Falkner, Levi & Carpenter (1999) and, even more recently, Theis (2003) who did his doctoral research on the learning and teaching of equality in the early grades. One of the findings that have stood the test of time and constituted the basis for the reform programs are that children have a “do something” or operational view of the equal sign rather than a relational one (Behr, Erlwanger & Nichols (1980), Kieran (1981)). A second, and very important finding, is that this unhelpful view of the equal sign does not sort itself out over time or with mathematical maturation. In a study of primary school children, “performance did not improve with age” and “in fact, in this sample, results for the sixth grade students were actually slightly worse than the results for students in the earlier grades” (Carpenter et al., 2003, p. 9). Kieran (1981), who looked at understanding across primary, secondary and university students, suggested that understanding of the equal sign did not necessarily improve beyond primary and may be responsible for a number of errors in secondary and post-secondary mathematics.

A third finding concerning the equal sign—less discussed in the literature but of interest to us for reasons that will soon become evident—was noticed and remarked upon by Ginsburg (1977). He saw how children link the equal sign to the operation and cannot envisage the one without the other.⁸ Behr et al. (1980) made a similar observation in their research: “There is a strong tendency among all of the children to view the = symbol as being acceptable in a sentence only when one (or more) operation signs (+, -, etc.) precede it” (p. 15). Rather than interpreting the equal sign as a “do something” signal, it is possible that children perceive $+ =$ as a single operator symbol where the + indicates the type of operation to be performed.

OUR RESEARCH

Context

In the spring of 2003, we began exploring a number of “early algebra” themes in kindergarten (35 students, ages 5-6), two grade three classes (20 and 11 students, ages 8-9) and a grade six class (23 students, ages 11-12) in a French Montreal area private primary school with a math enrichment program for all students. Our study of children’s understanding of the equality sign involved from two to three hours of video-taped classroom time with each group. We began each class with a full group discussion eliciting reactions to a number of statements involving the number 8 such as $8 = 8$, $8 = 3 + 5$, $3 + 5 = 4 + 4$. Students were then asked to complete a two-page questionnaire that involved filling in the missing numbers in statements of the form

⁸ For example, children find $3 + 5$ and $3 = 3$ unacceptable statements. They will insert an equal sign in the first ($3 + 5 =$) and an addition sign in the latter (such as $0 + 3 = 3$).

$a = a$, $c = a + b$, $a = b + c$, $a + b = c + d$ where one of a , b , c or d was missing (a box or a blank) and the numbers involved in the statements were grade appropriate.⁹ In the kindergarten class, a second one-hour session a few weeks later allowed us to repeat the activity using concrete material. The full group discussion involved numbers of candies in small opaque boxes rather than numbers on the board. The questionnaire made the link with the boxes of candy through representations of the various shaped boxes sometimes with small circles to represent the candies (Smarties) and sometimes with numbers.

Results

Few errors occurred in problem types $a = a$ and $a + b = c$. The tables below summarize the errors which occurred in problems of the form $c = a + b$ (Table 1a) and problems of the form $a + b = c + d$ (Tables 1b and 1c).

Grade	Problem	Reflect number	Sum(not difference)	Difference
K	$_ = 4 + 3$	$\underline{4} = 4 + 3$		$\underline{1} = 4 + 3$
Gr. 3	$_ = 45 + 50$	$\underline{45} = 45 + 50$		$\underline{5} = 45 + 50$
Gr. 6	$_ = 6500 + 500$			$\underline{6000} = 6500 + 500$
K	$7 = _ + 1$		$7 = \underline{8} + 1$	
Gr. 3	$67 = _ + 67$ $67 = _ + 50$	$67 = \underline{67} + 67$ $67 = \underline{67} + 50$		
Gr. 6	$67000 = _ + 50$		$67000 = \underline{67050} + 50$	
K	$7 = 7 + _$ $7 = 5 + _$	$7 = 7 + \underline{7}$ $7 = 5 + \underline{7}$	$7 = 7 + \underline{14}$ $7 = 5 + \underline{12}$	
Gr. 3	$67 = 5 + _$ $67 = 67 + _$		$67 = 5 + \underline{72}$ $67 = 67 + \underline{134}$	
Gr. 6	$67000 = 5 + _$ $67000 = 67000 + _$		$67000 = 5 + \underline{67005}$ $67000 = 67000 + \underline{134000}$	

Table 1a: Errors in problems of the form $c = a + b$

Here we see that children's errors involved 1) reflecting a given number to the other side of the equal sign as in $\underline{a} = a + b$, $c = \underline{c} + b$ or $c = a + \underline{c}$, 2) inserting the sum of the two given numbers in the blank independently of the position of the unknown, and 3)

⁹ Single digit numbers were used in kindergarten, double digit in grade 3 and up to 5 digits in grade 6.

inserting the difference of the two given numbers in the blank. When three numbers are given and a fourth to be found as in $a + b = c + d$, behavior was a little more complex. In Table 1b we see that when the blank was in the c position, for example, some children behaved as if d were not there and simply entered the sum of a and b. Others appeared to ignore the a and “complete the sum” $b = \underline{c} + d$ while others simply repeated one of the terms a, b, or d in the blank.

Grade	Problem	Direct sum	Complete the sum	Repeat one of terms
K	$2 + 4 = _ + 5$ $4 + 8 = _ + 5$ $3 + 5 = _ + 8$	$2 + 4 = \underline{6} + 5$ $4 + 8 = \underline{12} + 5$ $3 + 5 = \underline{8} + 8$	$4 + 8 = \underline{3} + 5$	$2 + 4 = \underline{4} + 5$ $3 + 5 = \underline{3} + 8$
Gr. 3	$4 + 8 = _ + 5$	$4 + 8 = \underline{12} + 5$	$4 + 8 = \underline{3} + 5$	
Gr. 6	$4 + 8 = _ + 5$		$4 + 8 = \underline{3} + 5$	
K	$3 + 5 = 2 + _$			$3 + 5 = 2 + \underline{3}$ $3 + 5 = 2 + \underline{5}$
Gr. 3	$36 + 54 = 52 + _$			$36 + 54 = 52 + \underline{36}$ $36 + 54 = 52 + \underline{54}$
Gr. 6	$36000 + 54000 = 52000 + _$			$36000 + 54000 = 52000 + \underline{36000}$
K	$2 + _ = 2 + 5$ $2 + _ = 5 + 2$	$2 + \underline{7} = 2 + 5$ $2 + \underline{7} = 5 + 2$	$2 + \underline{0} = 2 + 5$ $2 + \underline{3} = 5 + 2$	$2 + \underline{2} = 2 + 5$
Gr. 3	$25 + _ = 25 + 45$ $25 + _ = 45 + 25$		$25 + \underline{0} = 25 + 45$	

Table 1b: Errors in problems of the form $a + b = c + d$

Three additional errors were also revealed: filling the blank with the sum of all the given terms (K) and filling the blank with either the sum (Gr. 3) or the difference (Gr. 6) of two of the terms (see Table 1c).

Grade	Sum of all terms	Partial sum	Difference of some terms
K	$2 + 4 = \underline{11} + 5$ $4 + 8 = \underline{17} + 5$ $3 + 5 = 2 + \underline{10}$ $2 + \underline{9} = 2 + 5$		
Gr. 3		$36 + 54 = 52 + \underline{88}$	$25 + \underline{20} = 45 + 25$
Gr. 6			$36000 + 54000 = 52000 + \underline{16000}$ $36000 + 54000 = 52000 + \underline{18000}$

Table 1c: Errors in problems of the form $a + b = c + d$ (particular cases)

Analysis of results

The aim of our study is to create an instrument to track algebraic thinking across many different primary curricula and a wide range of students. We are therefore interested in finding the best ways of eliciting this thinking. Using this particular tool of a written questionnaire to look at children's understanding of the equal sign, the focus of our attention is on finding which of the given problems are the most revealing of children's thinking and of its evolution over time. Before discussing that particular question, we will briefly discuss two unexpected but ultimately fruitful results that confirm the need for the instrument we are creating.

If, as the literature suggests, children all have difficulty with the equal sign and appear to regress in their understanding over primary school, then there would be no point in conducting this study or in modifying curricula to deal with this problem. The hope is that an early introduction of equality will change this portrait. Although we did not intend to compare our students with others described in the research, we did include, in all four questionnaires, one very fruitful question that was used by Falkner et al. (1999) in their study across the grades: $8 + 4 = \underline{\quad} + 5$. The table below shows the percentage of students getting the correct answer of 7 in the two studies.

Falkner & al. study		Freiman & Lee study	
Grades	% getting answer 7	% getting answer 7	Grades
1 and 2	5	3	K
3 and 4	9	77	3
5 and 6	2	86	6

Table 2: Comparison of results of students on the question $8 + 4 = \underline{\quad} + 5$

Although our kindergarten students were comparable with those in the Falkner study, by grades three and six our students were greatly outperforming their American

counterparts and were drastically improving with grade level.¹⁰ Neither study involved children who were exposed to primary curricula aimed at developing algebraic thinking. Our students, though they represented a full spectrum of abilities, did have one major curricular advantage: they were exposed, for several hours a week, to a math enrichment program. Thus it seems that a written questionnaire could be helpful in comparing understanding of the equal sign in different curricula as well as among students following the same curriculum.

This brings us to the question as to what are the most fruitful questions on the equal sign to be included in a questionnaire on algebraic thinking.¹¹ Among the less fruitful questions are those for which the majority of students were successful. Although statements in the form $a + b = c$ did produce some incorrect answers, from 91 to 100% of students per class were successful. The statement $a = a$ only produced errors in the kindergarten class where nevertheless 25 of the 33 students responded correctly. A third category of less useful questions has nothing to do with question form and more to do with the choice of appropriate numbers for the grade six class. Our use of multi-digit numbers in some of these questions elicited student errors that had little to do with their understanding of the equality symbol and a lot to do with their inability to operate on large numbers. For example, the question $67000 = 5 + \underline{\quad}$ elicited nine different erroneous responses ranging from 65995 to 68995. Indeed the majority of errors in grade 6 could be attributed to errors in addition and subtraction.

The questions that discriminated the most among our students are candidates for a future more global questionnaire. Two problem types—both involving a blank in the last position—caused consistent difficulties across the grades: $c = a + \underline{\quad}$ and $a + b = c + \underline{\quad}$. Three problem types caused difficulties across two grade levels: $\underline{\quad} = a + b$ and $a + b = \underline{\quad} + d$ (K and Gr. 3), $a = \underline{\quad} + b$ (Gr. 3 and Gr. 6 where math errors in $67000 = \underline{\quad} + 50$ accounted for most of the difficulty). Thus, by order of importance in a short list of problems to include in a written test instrument across the grades, we would suggest: $a + b = c + \underline{\quad}$, $a + b = \underline{\quad} + d$, $c = a + \underline{\quad}$, $a + b = \underline{\quad} + d$.

We experimented in grades 3 and 6 with problems having two or more blanks such as: $\underline{\quad} + a = \underline{\quad} + b$, $a + \underline{\quad} = e = \underline{\quad} + d$, $\underline{\quad} + \underline{\quad} + b = \underline{\quad} + a$, and $a + \underline{\quad} + a = b + \underline{\quad}$. Of these problems, $a + \underline{\quad} + a = b + \underline{\quad}$ caused some difficulties at both grade levels. More testing is needed of these forms before we can make a decision as to their usefulness.

Future research

We have not discussed here the video taped whole class sessions where the children's comments were quite revealing of their thinking. Nor have we had time to discuss the

¹⁰ Two of Lee's students (Ouellet & Tremblay) took this question into grades 7 and 8 of a French public school in a different area of the city and, though the results were inferior (ranging from 60 to 66% in the three grade 7 classes and from 74 to 82% in the two grade 8 classes), they were still significantly superior to the Falkner results and did improve from grade 7 to 8.

¹¹ We are assuming here that a questionnaire on algebraic thinking will cover many other concepts besides equality and will necessarily be of limited length.

kindergarten children's work using the concrete materials and compare that with the results reported here. We are also planning some interviews to further explore some aspects of children's thinking about equality and the equal sign.

A number of other elements of algebraic thinking have also been explored and these will be the subjects of future papers. We have learned a number of lessons in our exploration of algebraic thinking so far and these will be applied to our study of other relevant themes until the research tool for tracking algebraic thinking across primary school is ready for testing in a variety of contexts.

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