

TEACHING ARITHMETIC AND ALGEBRAIC EXPRESSIONS

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A teaching intervention study was conducted with sixth grade students to explore the interconnections between students' growing understanding of arithmetic expressions and beginning algebra. Three groups of students were chosen, with two groups receiving instruction in arithmetic and algebra, and one group in algebra without arithmetic. Students of the groups that learnt arithmetic developed a strong understanding of the concept of term and applied it to reason about equal expressions. They performed better at some questions in algebra, especially those that required a sense of the structure and meaning of the expression.

INTRODUCTION

Many of the difficulties that school students face in learning algebra may have their source in the poor understanding of two important concepts – the variable and the algebraic expression. Sfard (1991) and Tall (1999) have pointed to the difficulty in understanding the process-product duality of algebraic expressions, which encode both operational instructions as well as denote a number that is the product of these operations. The difficulty in understanding the multiple meanings encoded by expressions may underlie the inability of many students to operate with unclosed expressions (Booth 1984).

It has been recognized that students' understanding of arithmetic and algebraic expressions are interconnected. For example, students who make errors in manipulating algebraic expressions repeat some of these errors when dealing with arithmetic expressions (Linchevski and Livneh, 1999). Many students have a poor sense of the structure of arithmetic expressions and are unable to judge the equivalence of expressions like $685-492+947$ and $947-492+685$ without recourse to computation (Chaiklin and Lesgold, 1984). Algebra, as generalized arithmetic, symbolizes and exploits the structural aspects of arithmetic. However, it is not clear whether instruction oriented to developing the structure sense of arithmetic expressions transfers to algebra. It might well be the case that learning algebra paves the way for a better understanding of arithmetic expressions since the algebraic symbolism enhances the structure of the expression. Linchevski and Livneh (1999) have recently raised doubts about whether focusing on teaching structured arithmetic as a preparation for algebra is a good pedagogic strategy.

The present research study is aimed at exploring the interconnections between students' growing understanding of arithmetic expressions and beginning algebra. Sixth grade students were taught the topic of arithmetic expressions using a novel approach that emphasized structural aspects. They were also taught beginning algebra broadly following the approach used in school classrooms. The study was carried out

in two phases. The first phase was exploratory and was aimed at developing suitable instructional material that allowed students to develop a better understanding of the meaning and structure of arithmetic expressions. The second phase of the study had a two-group design, where one group was taught arithmetic expressions in conjunction with algebra, while the other group received instruction in algebra without instruction in arithmetic.

THEORETICAL FRAMEWORK AND ASSUMPTIONS

In the primary mathematics classroom, children encounter arithmetic expressions as ‘questions’, i.e., instructions to carry out certain operations and produce an answer. The transition to algebra requires pupils to understand the three-fold meaning of arithmetic expressions: process, product and relation. The relational meaning of an expression is expressed when we say that $12+7$ denotes a number which is ‘seven more than twelve’ (or alternatively, ‘twelve more than seven’). Thus $12+7$ and $10+9$ denote the same number but express different relations. In algebra, the relational meaning is generalized to the concept of a function. From a pedagogic point of view, the relational aspect is intuitively meaningful for children and can serve as an introduction to the structure of arithmetic expressions.

The equality of expressions is the core notion around which the structure of expressions can be grasped. The instructional units aimed at developing a structure sense in arithmetic were organized around this notion. Pupils were required to compare expressions without computation, by examining the terms in the expression. This provided a meaningful situation for leaning the concept of term and allowed students to appreciate the importance of this concept.

The concept of term and the concept of equal expressions may function as possible loci of the transfer of knowledge from arithmetic to algebra. Thus they may be called ‘bridge concepts’. The emphasis on terms allows students to perceive a number in an expression together with the attached sign. The concept may be extended to include product terms and to bracketed terms, which are again useful concepts in both arithmetic and algebra.

The ability to deal with brackets is a critical component of the understanding of symbolic arithmetic and algebra and underlies the capacity to manipulate unclosed expressions. From the experience of the first, exploratory phase of the study, it was felt that this topic could be more effectively taught as a set of clearly specified and connected rules for removing brackets and rewriting an expression. Rules must be connected to concepts in order to enhance their learning and retention. Since concepts occur as referents in the statement of rules, conceptual misunderstanding may lead to incorrect learning of rules. Pupils need to be flexible in their application of rules and conceptual understanding mediates such flexibility. The evidence that we have gathered so far is not sufficient to throw light on the relation between concepts and rules. Further analysis and study is being planned to explore this aspect.

METHOD

The first phase of the study was carried out in the summer of 2003 over 13 one-hour instructional sessions. It was an exploratory study carried out with a small group of about 15 students about to enter grade 6 (about 11 years of age). The focus of the first phase was to prepare an instructional module aimed at developing a sense of the structure of arithmetic expressions among the students. The students belonged to an urban school and were from a low socio-economic background. Although the medium of instruction was English and the students could follow instructions in English, they were not fluent at communicating in English. Students volunteered to enrol in the program and were short-listed on the basis of a random draw.

The pre-test showed that most of the students could not correctly use the rules of order of operations, or find the value of the unknown in simple linear equations or write expressions for sentences presented verbally. These topics are cursorily presented in the grade 5 textbook, but are usually not emphasized during teaching in grade 5. So the instructional module included exercises where the students learnt to express the relational meaning of arithmetic expressions, translate a sentence into an expression, evaluate expressions, and learnt the meaning of the '=' sign as a relationship between two expressions having the same value. This was essential before the students could judge the equality of arithmetic expressions.

Some students could on their own explain why two given arithmetic expressions were equal. For example, the reason given by a student in order to justify that $27+32 = 29+30$ was "take 2 from 32 and give it to 27". In the more difficult examples requiring reasoning about expressions, students needed to be able to deal with brackets. It was evident that some repository of rules is essential to symbolic manipulation and reasoning about expressions. Accordingly, instructional units on rules of order of operations and removing brackets were included in the module.

The second phase of the study was conducted in October-November 2003 to explore the connection between students' knowledge of arithmetic expressions and of beginning algebra. This study had a two-group design. The students were selected in a manner similar to the first phase (by a random draw from applicants) from two nearby English medium schools. These students were studying in grade 6 and had recently been introduced to integers but had not done any algebra. One of the schools was the same as the one that participated in the summer course. The other school caters to students coming from varied socio-economic backgrounds. The students from the second school came from the middle socio-economic group and were relatively more comfortable with the English language although many were not fluent in conversing in English.

A pre-test was given to all the selected students and two equivalent groups were formed on the basis of their performance in the pre-test. Group A had 27 students and group B had 26 students to start with. Some students dropped out and the final analysis was done on 25 students of group A and 21 students of group B. An

additional group, group C of 35 students from school using the local language (Marathi) as a medium of instruction, was also included in this study. These students were from the low and middle income group.

The programme was conducted during a mid-term school break and had 11 instructional sessions of one and a half hours each spread over three weeks. Group A and C were taught arithmetic expressions and algebra whereas group B was taught only algebra. Group B had a few sessions on an unrelated topic: activities in geometry.

The arithmetic expressions module taught to groups A and C included the following topics: meaning of arithmetic expressions, meaning of '=' sign, evaluating expressions using the rules of order of operations, the concept of term, comparing arithmetic expressions without calculation and rules for removing brackets. The algebra module taught to all the groups included the following topics: letters and algebraic expressions as standing for numbers, concepts of term, like and unlike terms, rules for removing brackets and rewriting, simplifying expressions and using algebraic manipulation to justify the outcome of guess-the-number games. In the guess-the-number game, students did a series of operations with a chosen number and recorded the final result. The operations were chosen so that all the students obtained the same answer. The students then had to use algebraic manipulation to justify why the answer was the same.

The same instructor taught algebra to students of groups A and B. The arithmetic lessons for Group A were taught by a different instructor. Group C had a separate instructor who taught both arithmetic and algebra in the Marathi language. The lesson plans were carefully drawn up for the three groups, with groups A and C following the same plans. The overall instruction time for the three groups was equal. Group B received more instruction time in algebra than the other two groups. The additional time was spent mainly on simplification of algebraic expressions and applying it to explain the outcome of guess-the-number games. Students of all the groups were regularly given practice exercises and feedback on their learning was collected through tests and worksheets. The lessons were video-recorded for further analysis of the teaching-learning sequence and students' responses. In order to assess the gains in students' understanding in arithmetic and in algebra and to obtain information about the possible interconnections between these, a post-test was administered to all the students. The post-test had components similar to the pre-test, as well as some additional questions testing their ability to simplify algebraic expressions, compare arithmetic expressions without calculations and justify the outcome of a guess-the-number game using algebra.

PRELIMINARY ANALYSIS OF RESULTS

The preliminary analysis reported here is of the performance data from the pre- and post-tests for the three groups. In addition, the justifications that students wrote for their answers to questions on comparing arithmetic expressions have been analyzed.

These have been taken from the post-test and from worksheets that were filled by the students during the instructional sessions.

Arithmetic expressions

The pre-test contained questions mostly on arithmetic. Groups A and B were formed as equivalent groups on the basis of their overall performance in the pre-test. The performance of Group C was slightly lower than the other two groups in most questions in the pre-test. Students from all the groups did well on comparing simple arithmetic expressions using calculation. In these questions students had to fill in the correct sign from ‘=’, ‘>’ or ‘<’, for example, $15 - 5 \underline{\quad} 5 \times 3$. In the questions on evaluating expressions using the order of operations rule, many students from the three groups did poorly. In the question asking them to rewrite expressions by removing brackets, students appeared to be unaware of the rule for doing so, and answered such questions by evaluating the expression and doing the bracketed operations first.

In the post-test, groups A and C, which received instruction in arithmetic, improved their performance considerably on the arithmetic questions. As might be expected, students in Group B did not significantly improve their performance on the arithmetic questions, except in the questions requiring them to rewrite the expression by removing the brackets, where there was substantial improvement. In their algebra lessons during the program, the students had learnt the rule for removing brackets. Group B students were able to transfer their new knowledge to arithmetic expressions because the form of the question items in algebra and arithmetic were similar. This transfer may hence have been enabled by superficial features. Notably, group B did not improve their performance on the questions requiring students to equalize expressions by filling in the blanks, on which 80% of the students from the other two groups answered correctly. None of the groups had directly worked with such questions during the instructional sessions.

	Group A		Group B		Group C	
	Pre	Post	Pre	Post	Pre	Post
Comparing expressions by calculation	97	94	83	85	83	93
Equalizing expressions by filling blanks	82	86	69	65	63	80
Evaluating expressions	51	74	58	60	44	70
Removing brackets and rewriting	22	85	28	71	37	91

Table 1: Percentage of correct responses in arithmetic

Structure of expressions and the concept of term

The lessons done with groups A and C on comparing expressions without calculation proved interesting to the students. Such exercises were new to them. The students had to justify their answers in their own words. Their spontaneous reasoning produced responses that we had not anticipated. As described earlier, one student in the first phase of the study spontaneously justified that $27+32 = 29+30$ because we can “take 2 from 32 and give it to 27”. In the second phase, one of the students produced the following reasoning to justify that $37 - 17 > 37 - 18$: ‘37 is the same and -17 is greater than -18’. He was making use of a property of negative numbers, a topic that the students had briefly encountered in school. This argument was easily assimilated by many students and was the most frequent form of reasoning produced by them in their written responses to this type of problem.

Such exercises made the concept of term very salient for the students. They tried to apply the concept to other comparison problems, but not always successfully. In some problems the increase in one term was compensated by a decrease in the other, for example, $28 + 15 = 27 + 16$ or $36 - 19 = 35 - 18$. Such problems elicited a variety of reasons from the students. When the increase in one term was not compensated by a decrease in the other, or when the expressions had three terms each, students had more difficulty in writing reasons, although many were successful in judging which expression was greater.

In writing reasons, especially for expressions that were related in a simple manner, students often abbreviated their justification by referring to the terms not specifically, but positionally. For example, ‘the first term is one less, the second term is one more’, or ‘one term is the same, the other is one less’. When combined with the high rate of correct judgements, such responses may reflect, at least in the case of some students, an increasing ease with such problems and a trend towards a stable perception of a structural pattern.

The concept of the product term was introduced to better understand the equivalence of expressions such as $43 + 68 + 32 \times 35$ and $35 \times 32 + 43 + 68$. Group C was able to apply this concept more successfully than the other groups. In the posttest for this question, students from Group C obtained 81% correct responses as against Group A – 48% and Group B – 40%. However, most students across all the groups were unable to judge the equality of $423-236+423+236+102$ and $423+102+423$, or the inequality of $423+236-423+236$ and $423+236-(423+236)$. Groups A and C performed slightly better (40% and 24% respectively) on these questions than Group B (14%).

Symbolization

On the questions where students had to write algebraic expressions corresponding to given phrases, students from all the groups improved markedly. All the groups had received instruction on such questions and groups A and C learnt to symbolize both in arithmetic and in algebraic contexts. For the simple phrases such as ‘three more

than a ' or 'six times a number t ', all students of groups A and C wrote correct expressions, and group B had over 80% correct answers. For the phrase 'seven less than x ', many students from groups A and B incorrectly wrote $7-x$. However, all students of group C wrote the correct expression $x-7$. It turns out that while the word order in the English phrase follows the order in $7-x$, the word order in the Marathi phrase follows the order in the correct expression.

Two questions taken from the SESM study were included in the pre- and post-tests (Booth 1984). In the first question, the students had to find the perimeter and area of a rectangle with sides $m+3$ and c . In the second, they had to find the perimeter of a star shaped figure with k sides each of length 5. These questions elicited near zero correct responses from all the groups. A simpler version of the second question, finding the perimeter for a star with 10 sides, each of length 4 elicited over 60% correct responses in all the groups. Clearly, knowledge of the concept of perimeter was not the obstacle. Further, in the guess-the-number game, most students chose a letter to represent the unknown and to write an expression, although only a few students could successfully manipulate the expression. Hence we find students able to symbolize using letters in the context of the guess-the-number game, but not in the perimeter and area questions.

Algebraic expressions

In the questions where students had to identify the terms in algebraic expression, groups A and C obtained over 90% correct responses, while group B had about 50% correct responses. In the questions on simplifying algebraic expressions, groups A and C (27% and 25% respectively) performed well below group B (46%). The total time spent by Group B on the simplification topic and on guess the number exercises which required simplification was about twice the time spent on these topics by Group A and C. However, on the questions where students had to identify expressions equivalent to a given algebraic expression, group A (63% correct) performed better than groups B (45%) and C (39%).

SUMMARY

The qualitative analysis of students' reasoning in comparing arithmetic expressions threw up some interesting results. There is positive evidence of the fact that students are able to perceive the structure of expressions from the widespread and ready use of the concept of term in justifying their comparisons and the near absence of justification based on computation. Some students referred to terms by their position rather than specifically, indicating a growing stability in the perception of a structural pattern.

We had hypothesized that learning the bridge concepts of term and equal expressions in meaningful arithmetic contexts prepares a base for a strong understanding of these concepts in algebra. The results from the teaching intervention are suggestive but not definitive. Groups A and C, which received instruction on arithmetic and algebra, did significantly better than Group B in writing an expression for a verbal phrase, and in

identifying terms in algebraic expressions. Group A did significantly better than the other two groups in identifying equivalent expressions.

Students learnt to open brackets by using rules in the context of arithmetic or algebra or in both contexts. On these questions, all the groups improved their performance over the pre-test performance. Since group B did not receive any instruction in arithmetic, this is an instance where the students applied a rule learnt in the context of algebraic expressions to arithmetic expressions. It is not clear if any conceptual transfer is involved here since the rules could be applied purely on the basis of similarity in the form of the questions. Concepts are used in the formulation of rules and hence better conceptual understanding may lead to better learning and retention of the rules. The connection between concepts and rules in the context of manipulation of algebraic expressions needs further exploration.

Although the groups receiving instruction in arithmetic improved their performance on questions requiring the opening of brackets, we did not find direct evidence of its improving performance in algebraic manipulation. The most likely explanation for this is that the time spent on instruction on the latter topic was too short.

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