

# ALGEBRAIC AND GEOMETRIC APPROACH IN FUNCTION PROBLEM SOLVING

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*This study explores students' algebraic and geometric approach in solving tasks in functions and the relation of these approaches with complex geometric problem solving. Data were obtained from 95 sophomore pre-service teachers, enrolled in a basic algebra course. Implicative statistical analysis was performed to evaluate the relation between students' approach and their ability to solve geometric problems. Results provided support for students' intention to use the algebraic approach to solve simple function tasks. Students who were able to use geometric approach had better results in solving complex geometric problems. Implications of findings for teaching functions are discussed.*

## INTRODUCTION AND THEORETICAL FRAMEWORK

There is an increasing recognition that functions are among the most important unifying ideas in mathematics. Functions form the single most important idea in all mathematics, in terms of understanding the subject as well as for using it for exploring other topics in mathematics (Romberg, Carpenter, & Fennema, 1993, p.i). The understanding of functions does not appear to be easy, given the diversity of representations associated with this concept, and the difficulties presented in the processes of articulating the appropriate systems of representation involved in problem solving (Yamada, 2000). Although it is complex and difficult, attaining a deep and rich understanding of the concept of function is crucial for success in mathematics. Yet, despite its importance, numerous studies suggest that many students, including pre-service teachers (Even, 1993), demonstrate an instrumental understanding of the concept of function (Sfard, 1992).

In this study the term representations is interpreted as the tools used for representing mathematical ideas such as tables, equations and graphs. Concern has been growing about the role of representations in teaching and learning mathematics. The NCTM's Principles and Standards for School Mathematics (2000) document includes a new process standard that addresses representations (p. 67), while the status of function representations has been elevated even further. The use of different modes of representations and connections between them represents an initial point in mathematics education at which students use one symbolic system to expand and understand another (Leinhardt, Zaslavsky, & Stein, 1990, p. 2).

Several researchers addressed the importance of connections between the different modes of representations in functions and in solving problems (Moschkovich, Schoenfeld, & Arcavi, 1993; Yamada, 2000). Students' handling of different representations permits ways of constructing mental images of concepts in functions.



Moreover, the connections between the different modes of representations influence mathematical learning and strengthen students' ability in using mathematical concepts of functions (Romberg, Fennema, & Carpenter, 1993, p.iii). Knuth (2000), indicated that students have difficulties in making the connections between different representations of functions (formulas, graphs, diagrams, and word descriptions), in interpreting graphs and manipulating symbols related to functions.

The theoretical perspective used in the present study is related to a dimension of the framework developed by Moschkovich, Schoenfeld, & Arcavi (1993). According to this dimension, there are two fundamentally different perspectives from which a function is viewed. The *process perspective* and the *object perspective* have been described. From the *process perspective*, a function is perceived of as linking  $x$  and  $y$  values: For each value of  $x$ , the function has a corresponding  $y$  value. Students who view functions under this perspective could substitute a value for  $x$  into an equation and calculate the resulting value for  $y$  or could find pairs of values for  $x$  and  $y$  to draw a graph. In contrast, from the *object perspective*, a function or relation and any of its representations are thought of as entities—for example, algebraically as members of parameterized classes, or in the plane as graphs that are thought of as being 'picked up whole' and rotated or translated (Moschkovich et al., 1993). Students who view functions under this perspective could recognize that equations of lines with the form  $y = 3x + b$  are parallel or could draw these lines without calculations if they have already drawn one line or they can fill a table of values for two functions (e.g.,  $f(x) = 2x$ ,  $g(x) = 2x + 2$ ) using the relationship between them (e.g.,  $g(x) = f(x) + 2$ ) (Knuth, 2000). The algebraic approach is relatively more effective in making salient the nature of the function as a process while the geometric approach is relatively more effective in making salient the nature of function as an object (Yerushalmy & Schwartz, 1993).

Sfard (1992) has argued that the ability of seeing a function as an object is indispensable for deep understanding of mathematics. Furthermore, developing competency with functions means moving towards the object perspective and graphical representation (Moschkovich et al., 1993). Students being able to profitably employ object perspective can achieve a deep and coherent understanding in functions. We believe that this point to point approach, that is the algebraic approach, gives students only a mere and local image of the concept of function. On the contrary, the geometric approach gives students a global approach of the concept of functions, so that students, who can manipulate and use it, will perform better in solving complex geometric problems.

The purpose in this study is to contribute to the mathematics educational research community's understanding of the algebraic and geometric approach students develop and use in solving function tasks and to examine which approach is more correlated with students' ability in solving complex geometric problems.

## METHOD

The analysis was based on data collected from 95 sophomore pre-service teachers enrolled in a basic algebra course during the spring semester in 2003. The subjects were for the most part students of high academic performance admitted to the University of Cyprus on the basis of competitive examination scores. Nevertheless there are big differences among them concerning their mathematical abilities.

The instruments used in this study were two tests. The first consisted of four tasks, implementing simple tasks with functions. In each task, there were two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical representation. There was always a relation between the two functions (e.g.  $f(x) = x^2$ ,  $g(x) = x^2 + 2$ ). Students were asked to interpret graphically the second function. The second test consisted of two problems. The first problem consisted of textual information about a tank containing an initial amount of petrol and a tank car filling the tank with petrol. Students were asked to use the information to draw the graphs of the two linear functions and to find when the amounts of petrol in the tank and in the car would be equal. The second problem consisted of a function in a general form  $f(x) = ax^2 + bx + c$ . Numbers  $a$ ,  $b$  and  $c$  were real numbers and the  $f(x)$  was equal to 4 when  $x=2$  and  $f(x)$  was equal to -6 when  $x=7$ . Students were asked to find how many real solutions the equation  $ax^2 + bx + c$  had and explain their answer. The tests were administered to students by researchers in a 60 minute session during algebra course.

The results concerning students' answers to the above tasks and problems were codified in three ways: (a) "A" was used to represent "algebraic approach – function as a process" to tasks and problems. (b) "G" stands for students who adopted a "geometric approach – function as an entity". (c) The symbol "W" was used for coding wrong answers. A solution was coded as "algebraic" if students did not use the information provided by the graph of the first function and they proceeded constructing the graph of the second function by finding pairs of values for  $x$  and  $y$ . On the contrary, a solution was coded as "graphical" if students observed and used the relation between the two functions (e.g.  $g(x) = f(x) + 2$ ) in constructing the graph of  $g(x)$ . This paper is focused on the first two types of responses. Moreover, implicative analysis (Gras, Peter, Briand & Philippé, 1997) was used in order to identify the relations among the possible responses of students in the tasks and problems. Therefore, twelve different variables representing the algebraic and geometric approaches emerged. More specifically, the following symbols were used to represent the solutions, involved in the study: (a) Symbols "T1A", "T2A", "T3A" and "T4A" represent a correct algebraic approach to the tasks and "P1A" to the first problem (second problem could not be solved algebraically). (b) Similar, symbols "T1G", "T2G", "T3G" and "T4G" represent a correct geometric approach to the tasks and "P1G" and "P2G", correct graphical solutions to the problems respectively.

For the analysis and processing of the data collected, implicative statistical analysis was conducted using the statistical software CHIC. A similarity diagram and a

hierarchical tree were therefore produced. The similarity diagram represents groups of variables, which are based on the similarity of students' responses to these variables. The hierarchical tree shows the implications ( $A \Rightarrow B$ ) between sets of variables. This means that success in A implies success in B (Gras, et al., 1997).

## RESULTS

The main purpose of the present study was to examine the mode of approach students use in solving simple tasks in functions and to test which approach is more correlated with solving complex geometric problems. Most of the students correctly solved the tasks involved linear functions (T1 and T2). Their achievement radically reduced in tasks involved quadratic functions (T3 and T4) and especially in solving complex geometric problems, only 27.4% and 11.6% of the 95 subjects correctly provided appropriate solutions (Table 1). More than 60% of the students that provided a correct solution chose an algebraic approach, even in situations in which a geometric approach seemed easier and more efficient than the algebraic. Furthermore, in the second problem, most of the students failed to recognize or suggest a graphical solution as an option at all.

	Tasks and Problems					
	T1**	T2	T3	T4	P1	P2
Geometric approach with correct answer	24.2*	23.2	19	21	16	11.6
Algebraic approach with correct answer	59	56.8	22.1	24.3	11.4	0
Incorrect Answer	16.8	20	58.9	54.7	72.6	88.4

Table 1: Students' responses to tasks and problems.

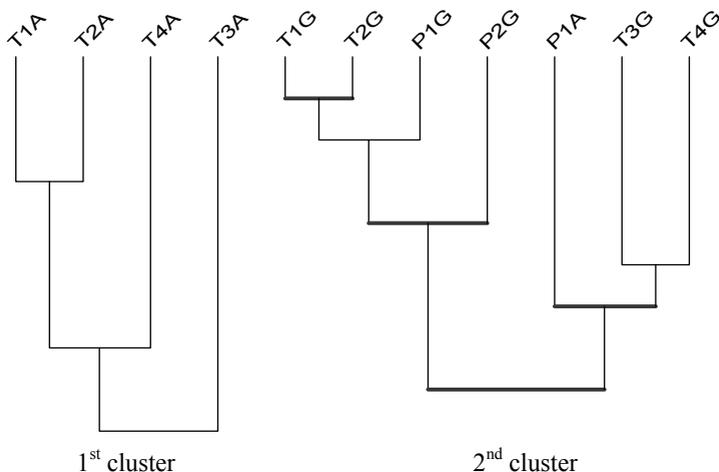
\* Numbers represent percentages.

\* T1 – T4: Tasks in functions. P1 – P2: Geometric problems.

Students' correct responses to the tasks and problems can be classified according to the approach they used and they are presented in the similarity diagram in Figure 1. More specifically, two clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables "T1A", "T2A", "T4A" and "T3A" which represent the use of the algebraic approach (process perspective). The second cluster consists of the variables "T1G", "T2G", "P1G", "P2G", "P1A", "T3G" and "T4G" and refers to the use of the geometric approach (object perspective) and solving geometric problems. The emergence of the two clusters is in line with the assumption of the study and reveals that students tend to solve tasks and

problems in functions using the same approach, even though in tasks that a different approach is more suitable.

It can also be observed from the similarity diagram that the second cluster includes the variables correspond to the solution of the complex geometric problems with the variables representing the geometric approach.



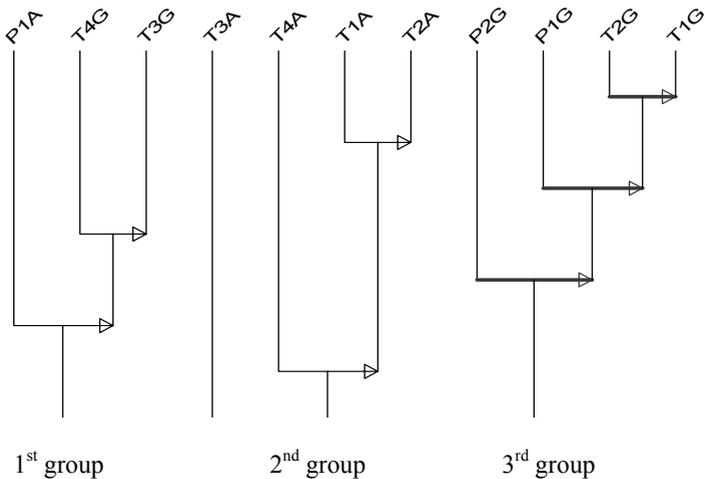
**Figure 1: Similarity diagram of the variables**

Note: Similarities presented with bold lines are important at significant level 99%

More specifically, students’ geometric approach to simple tasks in functions is closely related with their effectiveness in solving complex geometric problems. This close connection may indicate that students, who can use effectively graphical representations, are able to observe the connections and relations in geometric problems, and are more capable in problem solving. It is also important to acknowledge that almost all of the similarities in the second cluster are statistically significant at level 99% and this refers to the geometric approach and the complex geometric problem solving.

Significant implicative relations between the variables can be observed in the hierarchical tree, which is illustrated in Figure 2. First, three groups of implicative relationships can be identified. The first group and the third group of implicative relationships refer to variables concerning the use of the geometric approach – object perspective and variables concerning solution to the geometric problems. The second group provides support to the existence of a link among variables concerning the use of algebraic solution-process perspective. This finding is in line with the findings emerged from the similarity diagram. The formation of these groups of links

indicates once again the consistency that characterizes students' provided solutions towards the function tasks and problems. Second, the implicative relationship (P2G, P1G, T2G, and T1G) indicates that students, who solved the second problem by applying the correct graphical solution, have implied the application of the object perspective – graphical representation for the other problem and the four tasks. An explanation is that, students who have a solid and coherent understanding of functions can identify relations and links in complex geometric problems and thus can make the necessary connections between pairs of equations and their graphs, and easily apply geometric approach in solving simple tasks in functions.



**Figure 2: Hierarchical tree illustrating implicative relations among the variables**

*Note: The implicative relationships in bold colour are significant at a level of 99%*

**DISCUSSION**

It is important to know whether pre-service teachers are flexible in using algebraic and geometric representations in function problems. Although problems used in this study are some of those taught at school, subjects had difficulties, especially when they needed to implement geometric approach. Many students have not mastered even the fundamentals of the geometric approach in the domain of functions. Students' understanding is limited to the use of algebraic representations and approach, while the use of graphical representations is fundamental in solving geometric problems.

The most important finding of the present study is that for the group of pre-service teachers two distinct sub-groups are formatted with consistency: the algebraic and the geometric approach group. The majority of students' work with functions is restricted to the domain of algebraic approach and this process is followed with consistency in all tasks. As a consequence, few students develop ability to flexibly employ and select graphical representations, thus geometric approach. The present study is in line with the results of previous studies indicating that students can not use effectively the geometric approach, which engenders within the object perspective (Knuth, 2000). The fact that most of the students chose an algebraic approach (process perspective) and also demonstrated consistency in their selection of the algebraic approach even in tasks and problems in which the geometric approach (object perspective) seemed more efficient or that they failed to suggest a graphical approach at all, is particularly distressful considering that the students participated in the study thought to be representative of our best students.

Moreover, an important finding is the relation between the graphical approach and geometric problem solving. This finding is consistent with the results of previous studies (Knuth, 2000; Moschkovich et al., 1993), indicating that geometric approach enables students to manipulate functions as an entity, and thus students are capable to find the connections and relations between the different representations involved in problems. The data presented here suggest that students who have a coherent understanding of the concept of functions (geometric approach) can easily understand the relationships between symbolic and graphic representations in problems and are able to provide successful solutions. Moreover, data provided support that there is a close relationship between the use of a geometric approach in functions and better understanding of equations, graphs and functions in general.

Researchers have suggested that the difficulties students have on tasks and problems in functions that require the implementation and use of graphical representations may be due to the fact that students can not focus in the information a graph provides. Thus, students may be unaware that the graphical representation offers a means for determining a solution (Moschkovich et al., 1993). Moreover, in some cases the graphical representations create cognitive difficulties that limit students' ability to make connections between the algebraic and the graphical representations (Gagatsis, Elia, & Kyriakides, 2003). One factor that has a significant influence on students' preference in the algebraic solution is probably the curricular and instructional emphasis dominated by a focus on algebraic representations and their manipulation (Dugdale, 1993). In their textbooks, students are usually asked to construct graphs from given equations using pairs of values, while solving geometric problems follows the same procedure (Leinhardt, et al., 1990). As a result, graphical representations are seemed as unconnected to the corresponding algebraic representations and students fail to make the necessary connections between them and furthermore to effectively employ geometric approach and graphical representations (Yerushalmy & Schwartz, 1993).

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