

ELEMENTARY SCHOOL STUDENTS' MENTAL REPRESENTATIONS OF FRACTIONS

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Based on psychological approaches that evoke mental representations through verbal and visual cues, this paper investigates the different kinds of mental representations projected by 8 to 11 year old children of identified arithmetical achievement when responding to verbal and visual stimuli associated with fractions. It examines how the visual and verbal cues may affect the kind of mental representations the students' project. The paper traces the way these mental representations may change from an immediate "first" response to a "30 seconds" response. The study reveals that different formats of the stimuli and the elapse of time not only evoke different kinds of mental representation but that these different kinds may be strongly associated with the level of mathematical achievement.

INTRODUCTION

Students' understanding of fractions is an area of research within the field of mathematics education that has received considerable attention in the recent years. This interest has produced a wealth of information associated with students' operations and representations of fractions, the complexity of the concept, student difficulties and advice on the way fractions may be approached in the classroom (Behr, Harel, Post, & Lesh, 1992). In addition, research has also provided a wealth of information associated with the abstraction that denotes the cognitive shift implicit in doing mathematics and knowing mathematics (Beth & Piaget, 1966; Dreyfus, Hershkovitz & Schwartz, 1997).

It is students' difficulties in making these abstractions in the context of fractions that is the rationale for this paper. Both research findings as well as school experience, indicate that students provide a variety of responses when asked "what is a fraction?" (Pitta, 1995). Some students tend to see it as "something very small", "a circle cut into pieces" or "a shape with a lot of lines". Other students may tend to see the more intrinsic qualities of the mathematical symbolism and think of fractions as "the relationship between a numerator and a denominator, quotient and decimals". From such findings we may discern that there are differences in the quality of students' thinking about fractions and the mental representations they associate with the notion.

Our efforts to gain insight into why these differences occur, have consciously taken a route that considers cognitive development and more specifically mental representations. Whilst acknowledging that there is a wide spectrum of inherent, social and educational influences on this development, our interest focuses on

seeking answers to the kind of mental representations students' form. The underlying assumption is that the qualitative differences in thinking may arise because students form different kinds of mental representations.

The paper reports part of a larger study investigating the way in which different kinds of mental representations may be associated with students' achievement in elementary arithmetic. However, when we refer to different kinds of mental representations we do not intend to discuss differences in format associated with visual or non-visual characteristics. Irrespective of these formats, our guiding principle is that there are other aspects of mental representations that require further discussion. We believe these are important in understanding what it is children select to form in their mind and how they may relate to mathematical concept development.

Starting from De Beni's and Pazzaglia's (1995) classification of different kinds of mental representations, we broaden this debate to illustrate how these different kinds of mental representations may be related to different levels of arithmetical achievement. We also investigate whether mental representations are affected by the presentation of visual and verbal forms of stimulus and whether they change from an immediate or "first" response to a lengthier, "30 seconds", response.

THEORETICAL BACKGROUND

A number of mathematics education researchers have considered different kinds of images and mental representations. Pirie and Kieran (1994) indicate that the development of a learner's understanding can be seriously influenced by strong attachments to initial particular images. Thomas, Mulligan and Goldin (2002) have suggested that children's internal systems of representation of numbers go through a series of changes, from a semiotic one, in which meaning is established through previously constructed representations, to an autonomous stage in which a new system of representation functions independently of its precursor. Brown and Presmeg (1993) suggested a distinction between concrete, pictorial, memory and pattern imagery. The first two appeared to be dominant amongst instrumental thinkers and the later amongst relational thinkers. In elementary arithmetic such differences may emerge because those who mainly use procedures display less inclination to filter information. Relational thinkers appear to reject, temporarily ignore or select information which is more relevant to the task (Gray & Pitta, 1999).

Within the field of cognitive psychology De Beni and Pazzaglia (1995) have identified different kinds of mental representations which seem to share the view that mental representations may have different contexts and different levels of abstraction. They identified several kinds of mental representation: general, specific, contextual and autobiographical which were seminal to our work and to this study.

Although within the fields of mathematics education and cognitive psychology there are various classifications of mental representations, research seems to converge at least on one basic principle: individuals construct different kinds of mental representations from any learning (or daily life) activity. This observation has crucial

implications for the development of the fraction concept, the initial development of which usually involves activities with concrete objects and geometric shapes. From these activities learners are expected to abstract the concept of fraction and, eventually, the associated subconstructs of part-whole, number, quotient, operator, and ratio. The issue for this paper is the relationship between abstraction and mental representations.

The abstraction of mathematical constructs from concrete situations is considered to be an important outcome of mathematical learning activity. However, the individual needs to identify the same concepts, structures and relationships from many different but structurally similar tasks (Dreyfus, HersHKovitz & Schwartz, 1997; Charles & Nason, 2003). The absence of such identification may lead to the mere completion of the task and superficial memorization of the procedure or the activity (Bereiter, 1994).

Mental representations may be seen to be the product of a suitable form of abstraction. Beth and Piaget (1966) have identified three kinds of abstraction, each of which contributed to qualitative different levels of thought. Whilst “empirical abstraction” derives its knowledge from the properties of objects “pseudo-empirical abstraction” teases out properties from the actions. A third kind, “reflective abstraction”, teases out properties that the actions of the subjects introduce into the objects. These three kinds of abstraction have special connotations for this study since they are concerned with the focus for the abstraction: the object, the action or the common properties that the actions have introduced into the objects, and the level of specificity or generality of these abstractions.

AIMS OF THE STUDY

The aim of this study is threefold. First, to identify what it is that 8 to 11 year old students at different levels of achievement have selected to abstract and keep as mental representations of fractions. Second, to investigate what is the “first”, immediate mental representation that comes to mind when students are presented with different fraction items and examine how this mental representation may change and expand during a 30 seconds period of clarification. Through the two part questioning process, the “first” response and a “30 seconds” response, it was believed that respondents would have an opportunity to create a “first” mental representation that had the potential to be enriched with detail resulted from a network of other relationships (Drake, 1996). The third aim of the study was to investigate whether visual and verbal stimuli caused different kinds of responses.

METHOD

One cannot of course observe students’ mental representations. Spoken words and written representations are used to make inferences about these mental representations. Thus the data gathering technique was the semi-structural clinical interview (Ginsburg, Kossan, Schwartz & Swanson ,1983). Students were

interviewed over two separate occasions approximately 8 weeks apart. All interviews were video-recorded, linked to field notes and transcribed.

The research was conducted in a “typical” primary school in the English Midlands. The participants were sixteen children aged 8 to 11 years old, representing the extremes of numerical achievement in each of the four years of schooling. Thus there were two children at the each extreme of achievement within each of four years. The students’ arithmetical ability was measured by criterion based test results available in the school and a numerical component which formed part of the larger study of which this paper is part. The underlying assumption was that the analysis of responses provided by students at the extremes of arithmetical achievement would demonstrate a clear distinction between the different kinds of mental representation they project and these in turn would demonstrate a relationship with arithmetical achievement.

A modified version of the defining feature approach (see for example, Roth & Bruce, 1995) was used to gain a sense of what students feel is important to communicate when faced with visual and verbal stimuli. Within this paper we report on the outcomes of children’s responses to three verbal cues: fraction, half, three quarters, and three visual cues: the symbols $1/2$ and $3/4$ and the representation .

Each verbal and visual cue was presented with the following instructions:

- 1st: What is the first thing that comes to mind when you hear the word (fraction, half, three quarters) (or see $1/2$, $3/4$, )?
- 30 seconds: Talk for 30 seconds about what comes in your mind when you hear the word... (fraction, half, three quarters)
- 30 seconds: Look at this, ($1/2$, $3/4$, ) when I tell you close your eyes and put this in your mind. Talk to me for 30 seconds. Do it now.”

RESULTS

Classifying Responses

The classification of responses is mainly based on those identified by De Beni and Pazzaglia (1995). However, after a first analysis of the data collected we felt that some modification of these classifications was necessary in order to make them more appropriate to mathematics education. One of the most important modifications made was the replacement of the classification “contextual” with three separate components, “episodic”, “generic” and “proceptual” (Gray & Pitta, 1999). In order to provide a more comprehensive description of the way in which students’ responses were classified for this paper, examples of the different kinds of mental representations projected by the students for the item “fraction” are presented below.

General: the representation of a concept without any reference to a particular example: “*Part of*”.

Specific: De Beni’s and Pazzaglia’s reference to one well-defined example of the concept was extended to allow for representations that included multiple examples that were qualitatively similar: “*Lots of different fractions for example $1/8$, $1/7$* ”.

Episodic: representations associated with a scene or sequence of scenes that was most often narrated: *"It's like doing maths and being taught how to do fractions"*.

Autobiographic-episodic: representations, which allowed for the "occurrence of a single episode in the subject's life connected to the concept: *"My friend wasn't good at fractions and last week she had to take extra work home"*.

Generic: these representations originated from the same general concept that served as the basis for explicit relational connections. They were collections of statements that seemed to have the potential to produce new ideas. Though these representations had a 'general' quality, the statements diverged to produce different ideas: *"Shapes, part of shapes, cutting, cut shapes, sharing"*.

Proceptual: these representations were identified from those that possessed a proceptual nature (procedural and conceptual): *"Half, shaded shape, decimal, percentage, part whole... it is all of these together"*.

Analysis of Results

Tables 1 and 2 display the results related to students' "first" and "30 seconds" responses when presented with the verbal and visual cues. During their "first" response, students provided one kind of mental representation whereas during the 30 seconds it was possible for a child to start from one kind of mental representation and expand to other kinds. This is why whereas in Tables 1 and 2, only 16 responses are recorded for each item in the "first" response, in the "30 seconds" there are more responses and not of an equal number between the two groups of students. It was important to record all the responses, that were provided during the 30 seconds, in order to illustrate the way in which access to one part of the representation prompted the retrieval of other kinds of representations contained in the students' mind.

From Table 1, one can detect that high achievers' "first" response tended to be mainly general (33%) and in a lower extent specific (25%) and generic (21%), while low achievers projected mostly specific (25%) "first" responses. During the "30 seconds" both high and low achievers' general responses dropped dramatically; for the high achievers from 33% to 12% and for the low achievers from 17% to 0%. High achievers appeared to shift mainly towards proceptual (30%) or generic responses (24%). Low achievers, on the other hand, changed to more specific (41%) and episodic (19%) ones. It can be argued that high achievers' "first" responses, whether general or specific, acted as a trigger which was used to search and retrieve actions and objects related to the items in question. This resulted in generic and proceptual mental representations. Indicative of this behavior are the responses provided by a Year 4 high achiever for the word "half":

First response: *"Fraction"*

30 sec. response.: *"2/4, 1/2 of a whole, 1/4 less than 3/4, a part of something"*

After their "first" specific mental representation, low achievers continued by offering more specific examples of similar quality or by including the item in an episode. A

Year 4 low achiever gave the following responses for “half”:

First response: *“Half an apple”*

30 sec. response: *“Cut things in half. You can have half of a broken heart. Cut play dough in half, cut with scissors in half, cut a paper.”*

		Not know		General		Specific		Episodic		Aut. Epis.		Generic		Procept	
		H	L	H	L	H	L	H	L	H	L	H	L	H	L
1 st	Fraction	0	5	4	2	3	1	0	0	0	0	1	0	0	0
	Half	0	0	3	1	3	4	0	0	0	1	1	1	1	1
	3/4	1	2	1	1	0	1	0	2	0	0	3	1	3	1
		1%	29%	13%	17%	25%	25%	1%	8%	1%	4%	11%	8%	17%	8%
30 sec.	Fraction	1	4	1	0	2	2	2	2	0	0	2	1	3	1
	Half	1	0	1	0	0	5	0	1	1	1	1	0	4	2
	3/4	0	2	2	0	2	4	0	2	2	0	5	0	3	0
		5%	22%	2%	0%	12%	41%	5%	19%	1%	4%	14%	4%	30%	11%

Table 1: Students’ “first” and “30 seconds” responses for the verbal cues

Table 2 displays students’ responses to the visual stimuli. The results obtained show that more than half (54%) of the “first” responses provided by high achievers were general, whereas the same percent (54%) of “first” responses provided by low achievers were specific. This is important since it illustrates that once presented with the visual cue high achievers were more inclined to give a general mental representation, which often was the name of the item. For example, seven high achievers when looking at 1/2 said:

“Half” (Year 3, high achiever)

On the contrary low achievers tended to look at the specific example and concentrate on its surface characteristics:

“Black writing. Number 1 with a line underneath and a black 2. It’s on a green card” (Year 5, low achiever).

“A one and a two”. (Year 6, low achiever)

The results also illustrate that 21% of high achievers’ “first” responses were proceptual whereas 25% of the low achievers’ “first” responses were episodic.

Given the 30 seconds high achievers’ general responses dropped radically (54% to 23%). This drastic change was not observed in the low achievers’ specific responses (54% to 46%). This suggests that some low achievers started off by projecting a specific example or defined surface characteristics of the object and when more time was given they continued to do the same. The small shift however that seems to have occurred was mainly from specific to episodic. For example:

First response: *“It is 2 numbers like 1 and 2.”*

30 sec. response: *“I don’t know why that line is there but it just is. If you put the 2*

*and 1 together is 12, if you put the 2 there and the 1 there it is 21”
(Half, Year 4, low achiever)*

The item that caused the highest proportion (16%) of specific responses to high achievers was  , since they often attempted to describe it. Still, four of them projected the general response “half”. Low achievers mainly produced specific and episodic mental representations since they attempted to describe it or place it in a real life context. What was very interesting was the fact that five of the low achievers suggested that it was “a window” or a “tennis court”. It appears that these students had embellished it with more surface characteristics and turned it into an item of real life. The remaining three, simply talked about its surface characteristics.

		Not know		General		Specific		Episodic		Aut. Epis.		Generic		Procept	
		H	L	H	L	H	L	H	L	H	L	H	L	H	L
1 st		0	1	0	0	1	3	3	4	0	0	0	0	4	0
	1/2	0	0	7	4	0	4	0	0	0	0	0	0	1	0
	3/4	1	0	6	0	0	6	1	2	0	0	0	0	0	0
		4%	4%	14%	17%	4%	54%	17%	25%	0%	0%	0%	0%	11%	0%
30 sec.		0	0	0	0	6	4	3	5	1	0	0	0	3	0
	1/2	1	1	4	2	3	6	0	4	0	0	1	0	3	0
	3/4	0	0	5	3	4	6	2	3	0	0	2	1	2	0
		3%	3%	13%	14%	13%	46%	13%	34%	3%	0%	8%	3%	10%	0%

Table 2: Students’ “first” and “30 seconds” responses for the visual cues

By comparing the results in Tables 1 and 2, it appears that for both visual and verbal stimuli, high achievers’ “first” responses tend to be general and low achievers’ specific. This phenomenon seems to be exaggerated for the visual cues. For both groups, visual stimuli tend to cause an increase in specific and episodic responses and a decrease in generic responses. It can be argued that the visual stimuli facilitate the generation of specific and episodic representations while the verbal stimuli facilitate the search and retrieval of other objects, action and relationships.

DISCUSSION

This study supports the belief that different kinds of mental representations may be identified amongst high and low achievers. Both groups of children provide specific and episodic mental representations. However, while it is a tendency for low achievers to dominantly provide these, high achievers have the tendency to give generic and proceptual responses. What is intrinsic in the results is the diversity in the way these representations appear to evolve. The “first” response of high achievers tends to be general but when allowed to expand their comments, they can provide a network of relationships with other objects. It can be argued that the general comment needs to occur before the search and retrieval of generic and proceptual qualities is carried out. In contrast, low achievers seem to start from a specific mental representation and given more time they provide more qualitatively similar examples,

concentrate or embellish the items with surface characteristics or place them in an episode. It is also clear that the different format of the stimuli may cause different kinds of mental representations. This may have some serious consequences in teaching practice and especially in relation to the use of teaching aids.

The results appear to have important implications for our understanding of students' abstraction of the fraction concept. It appears that different groups of students concentrate on different qualities of the items and filter out information in a variety of ways. High achievers tend to synthesize aspects of pseudo-empirical and reflective abstraction – they identify the qualities that actions bring to objects and are able to disregard superficial external and contextual characteristics. Low achievers seem to concentrate more on empirical abstractions which appear to cause a disposition towards more specific and episodic representations.

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