

A TEACHER'S MODEL OF STUDENTS' ALGEBRAIC THINKING ABOUT EQUIVALENT EXPRESSIONS

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This research report describes the findings of a study on teachers' ways of interpreting student responses to tasks involving equivalent expressions. The teachers in this study were engaged in model-eliciting activities designed to promote the development of their knowledge and reveal their models (or interpretations) of their students' algebraic thinking about equivalent expressions by creating a library of their students' work. This report focuses on one teacher's model of his algebraic practice. Results showed that this teacher devoted a significant amount of time to the implementation of the algebraic unit. The teacher employed visual strategies for the first time and began to perceive their usefulness in demonstrating the equivalency of two expressions.

INTRODUCTION

Coupled with changes in the past decade advocating a reform-based vision of teaching algebra and the pressing need for effective algebraic instruction in the school curriculum, study about teachers' understanding of algebraic teaching becomes critical. At least a year of high school algebra is now the norm for most students in the United States. Currently, reform efforts advocated by the National Council of Teachers of Mathematics (NCTM) in the United States call for teachers "to analyze what they and their students are doing and consider how those actions are affecting students' learning" (NCTM, 2000, p. 19). Research on the teaching of algebra shows instruction tends to emphasize procedural rather than structural interpretations (Kieran, 1992; Nathan & Koedinger, 2000). Under the reformed vision, instruction should focus on helping students make interpretations between procedural and structural conceptions within algebra. Often the difficulty of students making the cognitive leap from arithmetic to algebra is related to instructional strategies (Kieran, 1992), however, as Kieran noted, "there is a scarcity of research emphasizing the role of the classroom teacher in algebra instruction" (p. 395). A decade or so after Kieran's observation, the research base on teachers' knowledge for teaching algebra is still quite limited (Doerr, in press).

There also exists a need for more effective models of how teachers interpret the learning process (Ball, 1997; Shepard, 2000). The use of student work samples (Chamberlain, 2003; Moskal & Magone, 2000) offers teachers the possibility of detailed information from which to examine students' reasoning processes. However, not all teachers acquire the same information nor interpret it in consistent ways. For example, research on equivalent expressions showed that neither novice nor expert teachers used spatial arrangements to help students see that two expressions might be equivalent (Even, Tirosh, & Robinson, 1993). The primary



goal of this study was to focus on the nature of the teachers' models or interpretations about their students' algebraic thinking as the students completed a series of lessons on equivalent expressions. Specifically, the core research question addressed in this study was: How do teachers interpret their practice when they focus on their students' algebraic thinking about equivalent expressions by building a library of student work?

THEORETICAL FRAMEWORK

The theoretical framework guiding this study is that of a models and modeling perspective on teacher development (Doerr & Lesh, 2003). A modeling perspective of teacher development focuses upon the ways teachers think about and interpret their practice. This perspective is based upon the premise that:

it is not enough to see what a teacher does, we need to understand how and why the teacher was thinking in a given situation, that is, interpreting the salient features of the event, integrating them with past experiences, and anticipating actions, consequences, and subsequent interpretations. (p. 127)

What teachers do is inherently complex. A modeling perspective draws upon the mathematical knowledge teachers currently possess and uses that base to engage teachers in expressing, revising, and refining their knowledge. The intent is to extend that knowledge into increasingly powerful models of classroom teaching. This perspective suggests that teachers be viewed as evolving experts. Doerr and Lesh assert that teachers' models serve as interpretive and explanatory frameworks to make sense of their students' mathematical thinking.

METHODOLOGY AND DATA SOURCES

This study used Lesh and Kelly's (1999) multi-tiered research design, with the following levels:

- *Tier 3: The Researcher Level.* Researchers develop models to make sense of teachers' and/or students' model-eliciting activities.
- *Tier 2: The Teacher Level.* Teachers develop shared tools (such as a library of student work), and then construct and refine models to make sense of students' modeling activities.
- *Tier 1: The Student Level:* Students work on a series of model-eliciting activities that reveal how they are interpreting the situation. (adapted from p. 198)

Involvement with a thought-revealing activity should result in a tool or artifact that can be shared and reused (Schorr & Lesh, 2003). The thought-revealing activity used in this report consisted of asking the teachers to select, analyze, and interpret "exemplary and illuminating" (Doerr & Lesh, 2003, p. 136) samples of student work. The purpose of gathering student work was multifaceted:

As the teachers select, organize and compare student work, they reveal how they are seeing the students' mathematical ideas. This may lead to mismatches

between their expectations of some students... It may lead to seeing students give mathematical interpretations of problem situations that the teacher had not seen. It is the resolution of such mismatches that provided the impetus for the development of teachers' knowledge. (p. 137)

Model-eliciting (or thought-revealing) activities for teachers attempt to help them become reflective of their teaching efforts.

Students were asked to solve problems involving equivalent expressions. The tasks were drawn from the Connected Mathematics Project (CMP) book *Say It With Symbols* (Lappan et al., 1998) from Problem 2.2:

Given a square pool as shown, draw a picture to illustrate the border of a square pool in four different ways:

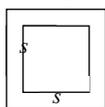
a. $4(s + 1)$

b. $s + s + s + s + 4$

c. $2s + 2(s + 2)$

d. $4(s + 2) - 4$

e. Explain why each expression in parts a-d is equivalent to $4s + 4$. (adapted from p. 22)



□ → 1 ft. by 1 ft border tile

Teachers were directed to save examples of student work that might be helpful to show a pre-service teacher how students in their own classrooms actually solved these problems. This library also served as the focus of the individual interviews. The primary data source used in this research report was the library of student work created by the teacher. Other data sources included fieldnotes from classroom observations, and transcripts from teacher interviews conducted before the series of lessons was taught, immediately after the series of lessons was completed, and after the algebraic unit on *Say It With Symbols* was taught. The teacher being reported on was implementing CMP for the first time. Bruce had 18 years of experience teaching middle school mathematics in an urban setting. He was one of five eighth-grade teachers in this study.

A grounded theory approach (Strauss & Corbin, 1998) was used to analyze the data. I refined my questions and observations based upon emerging themes and patterns of Bruce's actions as the study progressed. First, Bruce selected student work samples for his library. Then, we discussed why he chose each sample, and the similarities and differences amongst the samples in his library. I refined my interview questions based upon Bruce's observations. After Bruce finished teaching the entire *Say It With Symbols* algebraic unit, we discussed his interpretations again. From this process, I developed a profile of Bruce.

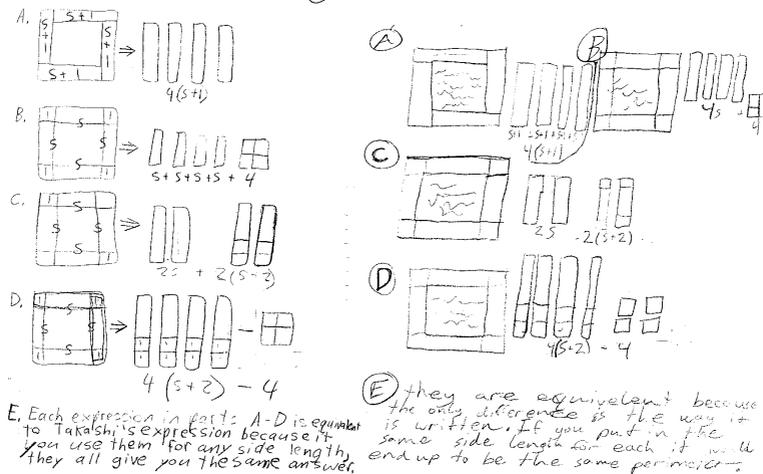
RESULTS

Two results emanated from the study. First, Bruce gave a significant amount of time to the implementation of the unit. Second, Bruce was beginning to develop

significant insight into student work. These results are intimately connected with one another. Bruce's support of the implementation led to his acceptance of the new ideas about how to teach, which in turn led to his new insights on helping his students learn algebra in meaningful ways. I continue with excerpts of Bruce's responses to interview questions that show support for the implementation of the curriculum, and then excerpts from his library of student work that demonstrate his insight into his students' algebraic thinking on equivalent expressions.

First, Bruce was committed to teaching the sequence in its entirety at the pace dictated by the students' progress. Bruce took about one and one-half times as long to teach the series of lessons under study than recommended by the curriculum developers. He spent more time than any other teacher in this study. This appeared to be due to his willingness to implement the curriculum as intended and to give his students ample time to investigate the problems. This action appeared to enhance the extent of the information that he acquired about his students' thinking and was reflected in his extensive library of student work. But, Bruce commented that the length of time "bothered" him because of his concerns about teaching the entire eighth grade curriculum. Nonetheless, Bruce explained that he was willing to extend the time for this series of lessons because he deemed it important enough: "*Time is always one of the biggest factors, you know. I am always fighting with that...*" At the beginning of the study, Bruce decided not to let time constrain his teaching because he wanted to give his students every opportunity for success with the series of lessons. During the teaching sequence, Bruce passionately commented on how his students worked with more interest, "*The kids are involved... and this leads to a better understanding.*" In a final interview, Bruce reflected "*The transition from arithmetic to algebra [is] something to be taken more seriously... I know that I will be doing this [the lesson] again. I will never do it the way I did it in the past. I think this is better.*" Bruce came to embrace the curriculum during the course of the study, leading to the next result.

Second, Bruce began to develop significant insight into his students' algebraic thinking. He employed visual strategies for the first time and began to perceive their usefulness in demonstrating the equivalency of two expressions. Bruce reported that in the past he had provided students with procedural examples of the distributive property. Before this study, Bruce had never used spatial arrangements within the context of teaching equivalent expressions. During the study, Bruce was impressed by the subtle differences between the students' papers and wanted each different way the students answered this problem correctly to be part of his library. His library was larger and showed more details within the various student responses than the other teachers in this study. Bruce selected a large number of papers showing a wide variety of student responses. I now provide excerpts from his library. Bruce selected the work of Student A and Student B as exemplary and indicated that he received a great many papers similar to those represented in Figure 1 below:



Student A

Student B

Figure 1. Bruce’s Library of Student Work--Student A and Student B

Bruce thought that each drawing was clear and the breakdown of the expressions was well laid out. He was quite proud to show me these papers. He selected a number of papers similar to these samples whenever a student drew the representation differently. Bruce was pleased with many of his students’ use of mathematical vocabulary in part e. He noted that his students understood the role of the variable s to represent the length of the side of the pool, and could distinguish the difference between when to solve for a variable in an equation and when to allow an algebraic expression to represent the solution. At the researcher level, I saw additional subtle differences that Bruce was not aware of. This was Bruce’s first time engaged in building a library of student work, and it is likely that his perceptions would grow over time. First, Bruce did not comment about Student A, part c & d, where the student represented both corners of the square at the bottom. A representation in the spirit of the given geometry of the problem might place one square at the top and one at the bottom. I later pointed out these representations to Bruce, but he did not recall the specifics of the instruction. He surmised that perhaps the students were drawing the problem in sequential order, the “ $s + 2$.” Second, Bruce did not point out that these students did not mark out a section for the “1” representing the corner in part a. And last, Bruce did not comment on the fact that the representations drawn by Students A and B were not necessarily the same size as the original squares. Collectively, these details of analysis indicated that he noticed a variety of solutions, but he did not carefully attend to *all* the details of the solutions.

Another part of Bruce’s library included work containing mistakes. (see Figure 2).

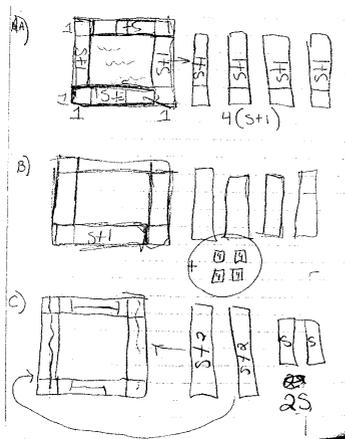


Figure 2. Bruce's Library of Student Work--Student C

Student C drew fairly accurate representations. In part a, the student labeled the individual sections “ $s + 1$ ” and accurately represented the corners. In part b, Bruce circled the use of the 4’s in the squares that should each represent a “1.” In part c, Bruce noted that the student drew the parts of the expression but did not indicate the position of the corners. This student did not attempt part d or e. These mistakes indicated to Bruce that this student had an incomplete understanding of what the corners represented, and hence, the relationship of the expression to the structural aspect of the problem.

The curricular activities guided Bruce and his students to using variables in contexts different from those that Bruce used in the past. In prior years, Bruce lectured about properties like the distributive property. At the end of the study, he reflected, “*They are using pictures and diagrams and they are labeling the different parts of the diagrams with variable terms, and then expressing those areas in different ways.*” Bruce saw that the visual strategies helped his students make sense of the algebraic expressions.

DISCUSSION AND CONCLUSION

Since the quality of algebraic instruction is salient to students’ success in algebra, it is important to understand the models teachers develop of their practice. Results of this study suggested that Bruce both embraced the curriculum and developed new insight into the usefulness of the visual strategies while teaching about equivalent expressions. He was able to attend to many details (but not all details) and see the utility of connecting the visual representation with the algebraic expression. In this context, the use of the visual representation generally drove the understanding of the algebraic expression. Contrary to Even, Tirosh and Robinson (1993), Bruce identified visual arrangements as a highly useful instructional tool. This was in part due to the influence of curriculum, and also his personal willingness to implement it. The use of diagrams to represent a variable, such as the side of a

square was new to him. That Bruce embraced this particular concept showed a shift in his instructional thinking. The use of visual representations in this manner has the potential to enhance the instructional process by better connecting the procedural and structural relationships within algebraic instruction as advocated by Kieran (1992).

In addition, the study demonstrates the potential effectiveness of the models and modeling framework, and the library of student work, as a tool to study teachers' developing knowledge. Consistent with prior research, Bruce did not attend to all the details of his students' responses. Creating a library of exemplary and illuminating student work may also provide teachers in other settings with examples that are useful for algebraic instruction and assessing the learning process.

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