

# ON THE FRAGILE, YET CRUCIAL, RELATIONSHIP BETWEEN MATHEMATICIANS AND RESEARCHERS IN MATHEMATICS EDUCATION

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*The relationship between mathematicians and researchers in mathematics education has often been fragile. Yet it is crucial. We conducted a series of themed Focus Group interviews with mathematicians from six UK universities. Pre-distributed samples of mathematical problems, typical written student responses, observation protocols, interview transcripts and outlines of relevant bibliography were used to trigger an exploration of pedagogical issues. Here we elaborate the theme “the relationship, and its potential, between mathematicians and researchers in mathematics education” that emerged from the data analysis. We do so by presenting the participants’ views on this relationship in terms of: **obstacles**, **desired characteristics** and **potential benefits**.*

## INTRODUCTION

The relationship between theory and practice in mathematics education is often fraught with suspicion (Sierpinska & Kilpatrick 1998), even hostility. This applies across the educational spectrum – for example within the primary and secondary sectors where policy makers often suffer criticism that their decisions are rarely and marginally informed by research in mathematics education [Brown in (Sierpinska & Kilpatrick 1998)]. Nowhere however is this more evident than within the tertiary sector (Ralston 2003). Recent developments in the world of university mathematics, such as the changing enrollment and profile of the student intake (Holton 2001), have resulted in a need for mathematics departments to rethink curricula, e.g. (Kahn and Hoyles 1997), and pedagogical practices, e.g. (McCallum 2003). In doing so a rapprochement between the worlds of mathematics and mathematics education research has become vital [Artigue in (Sierpinska & Kilpatrick 1998)].

Here we elaborate this issue through drawing on the views of university mathematicians participating in a study currently in progress in the UK. For an outline of the methodology of the study see the ENDNOTE. Participants were twenty mathematicians, pure and applied, with teaching experience ranging from a few years to several decades, all but one male and of varying rank. In six out of the eighty *Stories* which formed the analytical units of the study (see ENDNOTE) the participants expressed views on what we present here grouped as: *obstacles* and *desired characteristics* of the relationship between mathematicians and researchers in mathematics education; and, *potential benefits* for mathematicians engaged as educational co-researchers.

which the students conceptualized two main concepts: the *n*-star and the *common denominators* concepts (for further details: Lavy and Leron, in press).

### Methodology

The computerized environment under study consisted of MicroWorlds Project Builder (MWPB) –a Logo-based construction environment.

A group of ten 7<sup>th</sup>-grade students met several times after school hours in the school computer laboratory and explored the effects of the instruction *repeat n [jump k]* on geoboards of varying number of pegs. The instruction *repeat n [jump k]* results in the command *jump k* being executed *n* times in succession. Each choice of specific values for *n* and *k* results in a screen display of a regular polygon or a star with varying number of vertices (figure 1). The students were encouraged to look for mathematical patterns connecting the input numbers (*n* and *k*) and the shapes and the number of vertices of the resulting polygons or stars. These investigations led to the emergence in the students' discourse of concepts such as prime number, divisor and greatest common divisor (gcd). For detailed description of the computerized environment see Lavy and Leron ( in press).

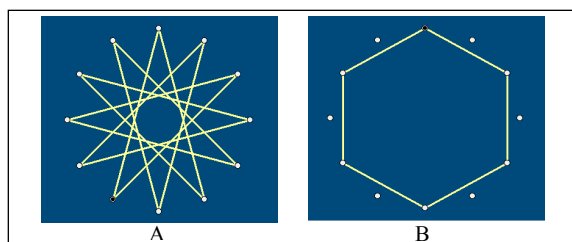


Figure 1: A – 12-star resulting from the command *repeat 12 [jump 5]*  
B – 6-sides regular polygon resulting from the command *repeat 6 [jump 2]*

The students Noam and Jacob, whose discursive, and in particular, screen productions were captured by a video camera are in the focus of this paper. These students were selected because, more than their other colleagues, they tended to “think aloud” during their work. The major part of the research data is the verbalized discourse, which took part between the students during the exploration process. In addition, the research data included the screen pictures at every stage of the inquiry, the students' body language and every piece of written paper they produced.

The research data were analyzed by three tools: Inductive analysis, interactional analysis and scanning the students' discourse through the lens of Toulmin's terminology (1969). Inductive analysis (Goetz & Lecompte, 1984) is a method which integrates between scanning the data while looking for phenomenological categories, and successive refinement of them when confronted with the new events and interpretations. In keeping with this approach, there were no predetermined criteria or

working such as the mode of environment-dependent and the gradual shift towards concentrating mainly on mathematical concepts and their attributes. In addition, it helped to form the different logical 'blocks' relating to each one of the emerging argument in the discourse.

In order to be able to characterize the ways each one of the student expresses himself during the exploration process, a discursive tool of interactional analysis was applied (Sfard and Kieran, 1997). This analysis enabled the understanding of the contribution of each student to the argument construction.

Scanning the discourse through the lens of Toulmin's terminology enabled the characterization of the different emerged arguments. Toulmin in 'The Uses of Argument' (1969, pp. 85-113) proposed a useful model for analyzing and constructing arguments. The two most important elements of his model are *data*: facts serving as the basis for a claim; and the *claim*: a conclusion or generalization to establish or support. Additional components in the model are: *warrants* ( or reasoning), which are general authorizing statements justifying the logical leap from data to claim and *backing*, which is the information that supports or offers a foundation for the warrant statement.

## Results and Discussion

The exploration process of the relations between the logo instruction: *repeat n [jump k]* and the resulting shapes of stars and polygons for various values of  $n$  and  $k$  was accompanied by arguing, trying to justify and convince each other by the validity of one's claims. In this study, since the students were only in 7<sup>th</sup> grade, the emerging arguments included visual and/or intuitive justifications rather than formal ones.

The distinction between the different kinds of the emerging arguments was made according to the character of the data components and the reasoning used. For example, arguments which included screen images of a certain geoboard were classified as one kind of arguments while arguments which included claims received from previous arguments as data were classified as another kind of arguments. The following sections will include the identification of the four kinds of arguments found in the students' discourse and example to each one of them will be brought.

### 'Basic' argument

After checking few examples of jumps in 11-peg geoboard, Noam said: "There will be here only one polygon". In fact, Noam formulates the claim:" In an 11-peg geoboard there is only one polygon and the rest are stars"[4.52]<sup>1</sup>. Noam's claim was a result of viewing few screen images which were the outcomes of different jumps in 11-peg geoboard. These picture screens were actually the data which was the basis for the arrived claim. The reasoning Noam gave for this claim was not verbal. It was based only on demonstrating additional examples of screen images on the same

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<sup>1</sup> The number 4 stands for the fourth session of the investigation process and the number 52 stands for the line number of the discursive transcript in this session

geoboard. Since the validation of this claim was involved only by checking a finite number of cases, the students' reasoning for this claim was consisted of concrete examples of different jumps on the same geoboard. I termed this kind of arguments as "*basic argument*". The level of generality of the claim in this argument is local. In this kind of argument the regularity relates to a *certain* geoboard and the data and reasoning consists of *additional concrete examples on the same geoboard only* – screen pictures of polygons or stars (figure 2).

Figure 2 includes a schematic description of a basic argument using the graphical description and terminology used by Toulmin (1969). The left rectangle refers to the argument's data, the right rectangle refers to the argument's claim, and the bottom rectangle refers to the reasoning used in this argument.

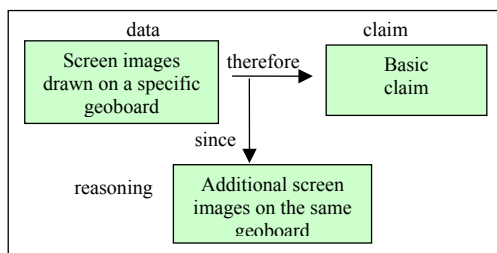


Figure 2: Schematic description of basic argument

Arguments of this kind were found at the first three sessions of the investigation activity. At this stage of the investigation, the students were not able to predict what will be the resulting shape (polygon or star) of a certain input ( $k$ ) in a specific  $n$ -peg geoboard using the instruction *repeat  $n$  [jump  $k$ ]*, and their claims were phrased only *after* testing some inputs of  $k$  in a specific geoboard.

### 'Compound' argument

From the third session of the investigation process, a different kind of arguments was found. Before terming the new argument, the raw data of an example to such an argument is presented:

[4.73] **Noam**: you have already said that in geoboards with prime number of pegs there will be only one polygon and the rest will be stars.

[4.74] **Jacob**: it is like this, but why.

[4.75] **Noam**: because they are primes, 'cause they do not have any divisor. The divisors they have are one and themselves.

In this stage of the investigation, the students had already checked various geoboards with prime number of pegs. Before [4.73] was argued, they investigated 5-peg, 7-peg and 11-peg geoboards. The claim "In a geoboards with a prime number of pegs there will be one polygon and the rest will be stars" was actually phrased by Noam although it can be understood from the above excerpt as if Jacob have said it before. Jacob's reaction [4.74] also implies that this claim was not phrased by him. First Noam phrases the claim [4.73] and since Jacob was wandering why [4.74], he gives a vague justification: "because they are primes"[4.75]. Although Jacob did not say anything in return, looking at the video reveals that he was not satisfied with Noam's

response. Being aware of Jacob's dissatisfaction of his answer, Noam tries to elaborate his justification by referring to properties of prime numbers (having only two divisors: one and the number itself). This is the first time in the discourse that mathematical considerations were given as justification in addition to relevant screen images. Yet, there were no explicit attempts made by Noam to connect between the mathematical considerations and the relevant screen images.

The above argument is an example of the second kind of the emerging arguments and is termed as "*compound Argument*". In this argument the mathematical claim relates to a group of geoboards sharing common attributes and the reasoning is composed of concrete examples and mathematical considerations related to number properties (figure 3).

The construction of compound argument was based on: basic conclusions constructed so far during the exploration process and controlled selection of screen images. The verbalized discourse of this stage of the investigation, shows that the students picked certain inputs

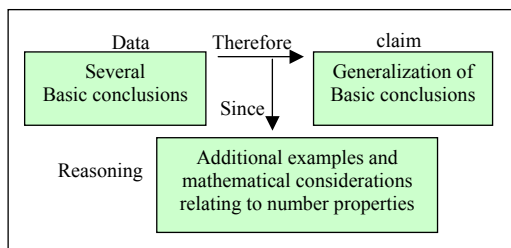


Figure 3: schematic description of a compound argument

for  $n$  (in this case, prime numbers) and checked only few inputs of jumps ( $k$ ) for the instruction: *repeat  $n$  [jump  $k$ ]* to verify their conjecture, while in the formulation of basic arguments they checked *all* the relevant inputs for  $k$  (from 1 to  $n-1$ ). Connecting the mathematical properties of the related numbers to the resulting geometrical shape on the computer screen presumably reduced the need to check all the cases for each geoboard with prime number of pegs.

### 'Elaborated' argument

The third kind of argument was found during the fourth session. Before defining this kind of argument, first given the raw data of an example to this argument:

[4.179] **Noam:** I think I arrived to some regularity in stars of 24, of the... all the vertices. Here *jump 6*... here I got it [goes on typing]

[4.180] **Jacob:** Look, if you do a prime number, you get a star.

[4.181] **Noam:** No, not just a star, a star of 24.....

[4.240] **Jacob** (to the teacher): and you put in the *jump* an odd number, then I get a star. Why? Because it is not divisible....

[4.277] **Noam:** [...] For me a star is only if it is not divisible....

[4.289] **Noam:** but look, in 24, but in a prime which is not divisible, it always comes out a star of 24. [pointing at the screen] look. *Jump 7* is a star of 24.

At this stage of the investigation activity, Noam and Jacob had already discovered that in geoboard with prime number of pegs there is one polygon and the rest of

shapes are stars. In the above excerpt they phrase the claim that when the jump size ( $k$ ) in the command *repeat  $n$  [jump  $k$ ]* is relatively prime to the geoboard's number of pegs ( $n$ ) the resulting shape is an  $n$ -star. By  $n$ -star I mean a star of  $n$  vertices, where  $n$  is the number of pegs on the geoboard.

Until this stage of the inquiry, Noam and Jacob distinguished only between cases in which one can get a polygon and cases in which one can get a star. This is the first place in the discourse that they noticed that there are different "kinds" of stars. This was made possible after getting the screen image of the instruction *repeat 12 [jump 5]* in 12-peg geoboard. When Noam saw the 12-peg star resulting from the latter instruction, he moved back to 24-peg geoboard and only after verifying his conjecture, using Jacob's formulation, he putted it into words: "No, not just a star, a star of 24" [4.181].

The above example belongs to third kind of the emerging arguments which I termed as an "*elaborated argument*".

This kind of argument is a refinement of compound argument. It is based on compound conclusion arrived earlier in the discourse, and additional examples which do not "fit" with the former conclusion. The argument's justification is based on concrete examples and mathematical considerations related to number properties (figure 4).

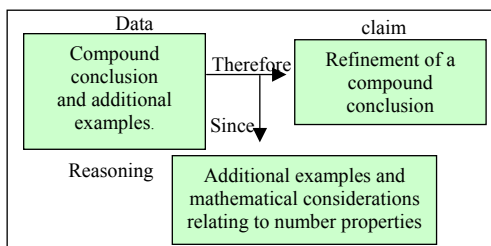


Figure 4: Schematic description of an elaborated argument

### General argument masked as specific

By the end of the fourth and during the fifth session, the students were investigating for which inputs of  $n$  and  $k$  the instruction *repeat  $n$  [jump  $k$ ]*, in an  $n$ -peg geoboard, will result in a star with less than  $n$  vertices. The regularity the pair was trying to uncover was that the number of vertices of the resulting shape equals  $n / \gcd(n, k)$ . In fact, they did not formulate the above regularity; rather they used several local arguments verify it. First the raw data of two local arguments is given:

[5.5] **Noam** (with frustration): no only if you know [re-types the command] it will come out an 8-star because... [Counts aloud 8 vertices on the screen] look it comes out 9 [meaning: a jump of 9 in a 24-peg geoboard will result in an 8-star].

[5.6] **Jacob**: why?

[5.7] **Noam**: because actually 9... what is its common divisor with 24? .....

[5.45] **Noam**: ah, because 8, its common divisor with 20 is 4 and the common denominator is 5 [by this last part he apparently means that the "common denominator" of 4 and 5 is 20].....

[5.55] **Noam**: ah, ok, ok, let's do 14 [changes to 14 peg-board, still thinking of a jump of 8] it's got a 7 star.

[5.56] **Jacob**: why? Because you divide by 2?

[5.57] **Noam**: no. because 2... because 8 is 2 times 4, and 7 is... what you need to multiply to get 14 is 2, right? [Noam and Jacob are looking at the 14 board on the screen].

[5.5-5.7] refers to the local claim: "a jump of 9 in a 24-peg geoboard results in a 8-star". This is the first time in the investigation that the students raise a claim without checking it first in the computer. This fact might imply that Noam had already a generalized assumption which came out in words as a concrete one.

[5.55- 5.57] concerns the example of *repeat 14 [jump 8]* (resulting in a 7-star), in which Noam finally gives a relatively clear procedural reasoning of how to calculate the number of vertices of the star. In fact, what he have said amounts to saying that  $7 = 14/\gcd(8,14)$ . It can be said that the argument related to the case in which one can get a star with less than  $n$  vertices, was build from various concrete examples which implies on the general claim.

The above example belongs to fourth kind of emerged arguments which I termed as "*generalized argument masked as specific*". This kind of argument is based on elaborated and compound conclusions arrived earlier in the discourse, and its justification is based on arguments which are specific examples of the generalized claim and mathematical considerations related to number properties (figure 5).

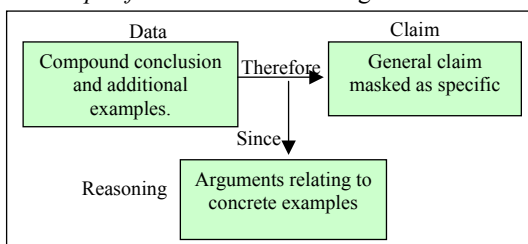


Figure 5: Schematic description of generalized argument masked as specific

## Concluding remarks

The characteristics of the emerging arguments described above, imply on a tight connection to the computerized environment of the geoboard, namely data and reasoning rely on concrete examples of geometrical shapes (stars and regular polygons). However, from the second kind of argument and on, the reasoning used in these arguments include in addition to concrete examples of geometrical shapes, mathematical considerations relating to numbers' properties. The various interactions, namely student-student and students-computerized environment, enabled them to gradually develop the ability to see the connections between number properties and the related geometrical shapes. Each one of the students contributed to the mutual effort of building the mathematical argumentation.

The students' ability to argue during the investigation process was improved since there was no observed 'withdrawal' regarding the kind of the emerging arguments, namely there were not found in the students' discourse basic arguments after the students developed the ability to phrase compound or elaborated arguments. Finally, there is no doubt that this process of the students' collaborative effort to phrase

arguments must be followed by a class discussion in which formal proofs will be built on the basis of them.

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