

## RF02: ALGEBRAIC EQUATIONS AND INEQUALITIES: ISSUES FOR RESEARCH AND TEACHING

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Algebraic equations and inequalities play an important role in various mathematical topics including algebra, trigonometry, linear programming and calculus (e.g., Hardy, Littlewood & Pólya, 1934/1997). Accordingly, various documents, such as the U.S. NCTM Standards, specify that all students in Grades 9-12 should learn to represent situations that involve equations and inequalities, and that they should understand the meaning of equivalent forms of expressions, equations and inequalities and solve them fluently (NCTM, 1989; 2000). To implement these recommendations it is crucial to analyze students' ways of thinking about equations and inequalities when designing instruction and in teaching.

Indeed, in the last decade there has been growing interest in the learning and teaching of algebraic equations and inequalities. Discussions regarding related issues have been conducted, for instance, in PME 22 Discussion Group meetings (1998) and continued during PME 23 Project Group sessions (1999). There was a consensus among 1999 PG participants that the meetings of the group should be temporarily postponed, while calling on researchers to invest more efforts in various facets of algebraic reasoning related to the solution of equations and inequalities. Among the benefits of the 1998-1999 discussions were the selection of key research questions and the initiation of collaborative research teams that since then have been working in this area. The Research Forum at PME 28 provides an opportunity for presenting some fruits of the research that has been conducted since then, for discussing theoretical frameworks for data analysis, and for examining the different educational implications that were suggested and tried. Indeed, in their presentations, all the invited speakers of this RF make due reference to previous related PME publications (e.g., Boero, Bazzini, & Garuti, 2001; Duval, 2000; Garuti, Bazzini, & Boero, 2001; Linchevski & Sfard, 1991; Nunez, 2000; Radford, 2002; Tsamir, & Bazzini, 2001).

The theme of PME28, *Inclusion and Diversity*, is well reflected in this research forum. *Diversity* may be found in the different lines of research, different theories to account for the findings and the educational implications that have been put forward by the researchers. For example, among the theoretical frameworks mentioned here to analyze students' solutions are the Vygotskian model and Nunez's grounding metaphors, in Boero and Bazzini – [BB], Duval's theory on semiotic registers and Frege's theory of denotation, in Sackur – [S], and Fischbein's model, in Tsamir, Tirosh and Tiano – [TTT]. Kieran [K] offers three categories for analyzing algebraic activities: generational, transformational, and global meta-level.

The presentations address a variety of difficulties occurring in students' solutions of equations and inequalities, and suggest different reasons for these difficulties. When analyzing students' performances, [BB] and [TTT] mention students' tendencies to make irrelevant connections between equations and inequalities as a problematic phenomenon. It should be noted, however, that [K] presents connections made between equations and inequalities as an important step in solving algebraic problems by means of non-algebraic methods. [BB] mention traditional, algorithmic teaching approaches as a main reason for students' errors, Dreyfus and Hoch [DH] mention the need to enhance the internal structure of equations that students hold, while [S] carefully analyzes difficulties with reference to the various solving methods and indicates that even the functional approach and the use of graphic calculators do not automatically lead to errorless solutions.

However, beyond their differences, the presentations share common goals. One such goal is to investigate ways to promote performance on algebraic equations and inequalities by seeking means for analyzing students' reactions to various representations of equations and inequalities in different contexts, while considering the way this topic was taught. Thus, this forum will also shed light on the more general issues concerning the interplay between theory, research and instruction.

The two reactors intend to react on all the papers and make concluding statements, but their review is made from different perspectives.

Further discussion will address a number of key questions, like:

- What are students' conceptions of equations / inequalities? What is typical correct and incorrect reasoning? What are common errors?
- What are possible sources of students' incorrect solutions?
- What theoretical frameworks could be used for analyzing students' reasoning about algebraic equations / inequalities?
- What is the role of the teacher, the context, different modes of representation, and technology in promoting students' understanding?
- What are promising ways to teach the topics of equations / inequalities? What curricular innovations can we suggest?
- Is there a global theory that may encompass the local theory of equations and inequalities?

The discussion of such issues could give further support to research and teaching. During the sessions at the conference, each of the presenters is allotted only ten minutes to present the central points of his / her ideas, and each of the reactors is invited to react on all presentations during a fifteen minutes presentation. Most of the two sessions are dedicated to the participants' work in small groups, and to whole RF discussions.

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## INEQUALITIES IN MATHEMATICS EDUCATION: THE NEED FOR COMPLEMENTARY PERSPECTIVES

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### 1. Introduction

This contribution deals with inequalities: an important subject from the mathematical point of view; a difficult subject for students; a subject scarcely considered till now by researchers in mathematics education. Our working hypothesis is that *different tools belonging to different disciplines (cognitive sciences, didactics of mathematics,*

*epistemology of mathematics*) related to mathematics education are needed to interpret difficulties met by students and plan and analyse teaching experiments intended to cast new light on this subject. Data coming from some teaching experiments conceived in this perspective and other experimental investigations will be used to support our working hypothesis. In particular, we will present the guidelines and some results of a research program conceived according to the above perspective and concerning the approach to inequalities in 8<sup>th</sup> - grade. We will show how a functional approach to inequalities (i. e. an approach based on the comparison of functions and suggested by the *didactical, epistemological* and *cognitive analyses* of the subject), when suitably managed by the teacher, can reveal (from the research point of view) and allow to exploit (from the curriculum design point of view) a students' potential which goes far beyond the mathematics content involved (inequalities). We will use the *Vygotskian perspective* and the "*didactical contract*" construct to frame the teacher's role in the classroom and analyse the teacher-students relationships. Amongst the cognitive tools, in particular we will use the "*grounding metaphor*" construct (Nunez, 2000) to analyse some aspects of the students' behaviour and open the problem of how to enhance students' use of those metaphors in this mathematical domain.

## **2. Inequalities: a challenge for teaching.**

In most countries, inequalities are taught in secondary school as a subordinate subject (in relationship with equations), dealt with in a purely algorithmic manner that avoids, in particular, the difficulties inherent in the concept of function. This approach implies a "trivialisation" of the subject, resulting in a sequence of routine procedures, which are not easy for students to understand, interpret and control. As a consequence of this approach, students are unable to manage inequalities which do not fit the learned schemas. For instance, according to different independent studies (Boero et al, 2000; Malara, 2000), at the entrance of the university mathematics courses in Italy most students fail in solving easy inequalities like  $x^2-1/x>0$ . In general, graphic heuristics are not exploited and algebraic transformations are performed without taking care of the constraints deriving from the fact that the  $>$  sign does not behave like the  $=$  sign (Tsamir et al., 1998). Similar phenomena were described in some studies concerning the French situation (Assude, 2000; Sackur and Maurel, 2000).

We may ask ourselves what are the reasons of this situation. In a *didactical-anthropological perspective* (Chevallard, 1987), one reason could be the fact that equations (and inequalities) are considered (in most of European countries, including Italy) as a typical content of school Algebra; this subject matter is distinguished from Analytic Geometry and does not include functions. This might explain why inequalities (and equations) are not dealt with in those countries from a functional point of view. But even in countries where functions (and Analytic Geometry) belong to school Algebra (see NCTM Standards, 1989 and 2000) the procedural, algebraic approach prevails in many curricula and even in innovative proposals (Dobbs and Peterson, 1991). So the *didactical-anthropological analysis* must be refined and

integrated with an *epistemological analysis*: we must consider the big distance between the subject as a school subject, and the mathematicians' professional approach to the subject. Indeed the functional aspect of inequalities plays a crucial role when mathematicians solve equations with approximation methods, deal with the concept of limit or treat applied mathematical problems involving asymptotic stability. We can make the hypothesis that an alternative approach to inequalities based on the concept of function could provide an opportunity to promote the learning process of the difficult concepts involved and the development of the inherent skills (see Harel and Dubinsky, 1992 for a survey). It could also ensure an high level of control of the solution processes of equations and inequalities (Sackur and Maurel, 2000; Yerushalmy and Gilead, 1997).

### **3. The teaching experiments**

Keeping the previous analysis into account we have planned two teaching experiments at the VIII-grade level with rather limited aims: investigating the feasibility of an early functional approach to inequalities; and revealing students' potential and difficulties in dealing with this subject as a special case of comparison of functions. According to a Vygotskian perspective, we choose to guide our VIII-grade students in a cooperative, gradual enrichment of tools and skills inherent in the functional treatment of inequalities. Then we have analysed how (in relatively complex tasks) they had been able to use their knowledge and increase their experience in an autonomous way.

As concerns the content, the concepts of function and variable have been approached through activities involving tables, graphs and formulas. According to existing cognitive and epistemological analyses, at the beginning the function was presented as a machine transforming  $x$ -values into  $y$ -values (machine view in Slavit, 1997), then classroom activities focused on the variation of  $y$  as depending on the variation of  $x$  (covariance view). By this way a dynamic idea of function gradually prevailed on the static consideration of a set of corresponding pairs (correspondence view). As a consequence, a peculiar aspect of the concept of variable was put into evidence (a variable as a "running variable", i.e. a movement on a set of numbers represented on a straight line) (Ursini and Trigueros, 1997). Finally, the approach to inequalities was realised by comparing functions.

The specific didactical contract demanded to compare functions as global, dynamic entities. Students knew that they had to compare functions by making hypotheses based on the analysis of their formulas. The point-by-point construction of graphs was discouraged. As a consequence, the ordinary table of  $x$ ,  $y$  values was sometimes exploited as a tool to analyse how  $y$  changed when  $x$  changed (column-vertical analysis) and not as a tool to read the line-horizontal point-by-point correspondence between  $x$ -values and  $y$ -values. The algebraic and the graphical settings were strictly related (formulas were read in terms of shapes in the  $(x,y)$  plane, while graphs evoked formulas). The teachers promoted classroom discussions about "what do we loose and

what do we earn" when a function is represented through formulas or graphs or tables or common language. Also different ways of describing given functions have been enhanced (see Duval, 1995: coordination of different linguistic registers). They became personal tools exploited and to compare functions. Even the metaphors used by students to describe the role of different pieces of the same formula have been encouraged and discussed.

#### 4. Some remarks and questions.

First of all, some remarks on the role of metaphors are worth noticing. Since the beginning of the eighties metaphors have been reconsidered as crucial components of thinking. Nunez (2000) describes *conceptual metaphors* as follows: "*Conceptual metaphors are in fact fundamental cognitive mechanisms (technically, they are inference-preserving cross-domain mappings) which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference*". Considering *conceptual metaphors*, Lakoff and Nunez (2000) (see also Nunez, 2000) make a distinction between *grounding metaphors* (i. e. conceptual metaphors which "*ground our understanding of mathematical ideas in terms of everyday experience*") and other kinds of conceptual metaphors (*Redefinitional metaphors, Linking metaphors*).

Concerning *grounding metaphors*, our research study aims to show how different kinds of *grounding metaphors* can intervene (as crucial tools of thinking) in novices' approach to inequalities and to discuss possible refinements of the idea of a *grounding metaphor*, deriving from the analysis of students' behaviour and related to the cultural variety of everyday life source domains. Finally we aim to investigate how *grounding metaphors* can become a legitimate tool of thinking for students.

In particular, it would be interesting to discuss the following questions:

- I) what theories and what tools do offer the best opportunities to interpret students' behaviors when they deal with inequalities?
- II) can the study of teaching and learning inequalities be reduced to the study of teaching and learning functions?

Some research findings have been already presented in Boero & al., (2001); and Garuti & al, (2001). Further results related to on-going research will be discussed.

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## **THE EQUATION / INEQUALITY CONNECTION IN CONSTRUCTING MEANING FOR INEQUALITY SITUATIONS**

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Recent algebra learning research has included a focus on students' understanding of and approaches to inequalities. For example, Bazzini & Tsamir (2001) have researched 16- and 17-year-old students' ways of thinking when solving various types of algebraic inequalities. Bazzini, Boero, and Garuti (2001) have studied the feasibility of a functional approach in the teaching of inequalities to eighth grade students. Tsamir, Almog, and Tirosh (1998) have observed high school math majors' methods for solving equations and inequalities and have noted that the most common

were algebraic manipulations, drawing an graph, and using the number line. This body of research has advanced the field with respect to our knowledge of students' conceptions of inequalities in several ways. It has pointed out, for example, the positive role that graphical representations can play in helping students to better conceptualize the symbolic form of inequalities, as well as the pitfalls involved in attempting to apply to the solving of inequalities some of the transformational techniques used with equations. Despite its foray into graphical representations, this same body of research has been quite narrow in emphasis with its almost exclusive focus on the manipulative/symbolic aspects of inequalities.

### **Theoretical Framework**

By means of a model recently developed and presented at the ICME-8 conference in Sevilla (Kieran, 1996), algebra can be viewed according to three main categories of activity: generational, transformational, and global meta-level. For the case of inequalities, meaning for the symbolic form is often derived via the global meta-level activity of contextualized problem solving, which activity tends to then be harnessed to generate the symbolic form of inequalities. However, these two types of activity seem absent from the current research on inequalities. Because the students involved in those studies are often older secondary level students, we presume that they have already constructed meaning for the symbolic form of inequalities; nevertheless, the research remains relatively silent on this issue.

### **Data Source**

The goal of this contribution to the PME Research Forum is to present a brief analysis of a classroom sequence that aimed at introducing inequalities. The data are drawn from the TIMSS-R 1999 video study of 8th grade mathematics teaching in algebra classes around the world (Hiebert et al., 2003). The lesson, which was the first of a set of seven such lessons, involved a Japanese class where the teacher used a specific problem situation to create meaning for mathematical inequalities and for their algebraic form ([www.intel.com/education/math](http://www.intel.com/education/math)). In this analysis, both the global meta-level activity of problem solving and the accompanying activity of generating an inequality are interwoven as we witness the teacher orchestrating both his overall aims for the lesson and particular students' approaches to the solving of the problem situation, which was as follows:

It has been one month since Ichiro's mother entered the hospital. He has decided to give a prayer with his small brother at a local temple every morning so that she will soon be well. There are 18 ten-yen coins in Ichiro's wallet and just 22 five-yen coins in the younger brother's wallet. They decided to place one coin from each of them in the offertory box each morning and continue the payer until either wallet becomes empty. One day after prayer, they looked into their wallets and found the younger brother's amount was bigger than Ichiro's. How many days since they started prayer? (translated version)

## Brief Analysis of Student Work

After the teacher spent a few minutes clarifying the problem situation to the class, students began to work individually on the problem. The teacher circulated, taking note of the various solution methods being worked and encouraging students to try more than one method. After some time had passed, the teacher asked certain students to present their solutions at the blackboard in the following order (see figure below):

**Public Class Work: Students Presenting Solutions to Class**  
 The teacher asks the students to present their solutions in the following order and place a pre-written title over each solution. He asks for each solution is shared, "How many others solved it in the same way?"

**1. Student one (Manipulating actual objects)**  
 Actually take one coin from each wallet and put it in affortary box until Ichiro's wallet contains less money than the brother's wallet. Or crossing out one coin from each wallet until the same condition is met. Answer: 15th day

**2. Student two (Solving it by making a table)**

Number of days	1	2	3	.....	14	15	16	17
Amount left	170	160	150	.....	40	30	20	10

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Number of days	1	2	3	.....	14	15	16	17
Amount left	105	100	95	.....	40	35	30	25

Answer: 15th day

**3. Student three (There is a difference of 5 coins per day)**

$$180 - 110 = 70$$

$$70 \div (10 - 5) = 14$$

$$14 + 1 = 15$$

15th day

**4. Student four (If X is the day when the monetary amounts become the same)**

$$180 - 10X = 110 - 5X$$

$$70 = 5X$$

$$X = 14$$

15th day

**5. Student five (If X is the day when the brother's monetary amount exceeds Ichiro's)**

$$180 - 10X < 110 - 5X$$

**Task: Complete the chart and check if the value of x holds true for this inequality. What is the relationship?**

X	1	2	3	.....	14	15	16	17
Ichiro	170	160	150	.....	40	30	20	10
Brother	105	100	95	.....	40	35	30	25

The teacher assigns a task and students work on it individually for six minutes.  
 The teacher asks a student to write her results on the chalkboard while the other students are still working.

The above methods that were used to solve the “inequality” problem situation show that the majority of the students who were invited to the board approached this problem as an equality situation, which enabled them--by means of a slight adaptation of the solution to the equality--to provide the solution for the inequality. They used the following language to express this idea:

**Student three:** “Well, in the beginning, Ichiro had 180 yen, and the smaller brother had 110 yen. And since there is a difference of 70 yen, and since the difference between them becomes smaller by five yen each day, so it’s 70 divided by 10 minus 5. And since by the fourteenth day it becomes exactly the same amount of money, so since on the day after that there will be a difference, so 14 plus one is 15 and it’s the fifteenth day.”

**Student four:** “On the fourteenth day they become the same amount of money. And the next day since Ichiro puts in 10 yen and the smaller brother puts in 5 yen .... the amount of money put in is bigger for Ichiro. So the next day Ichiro’s amount of money left is less so it becomes the fifteenth day.”

The student work (Student five) that involved the symbolic form of an inequality right from the start served as a tool for the teacher to introduce to the rest of the class both the inequality symbol and an expression containing this symbol:

S: (wrote on the blackboard:  $180 - 10x < 110 - 5x$ ). The one labeled  $x$  is that one day after they finished their prayers. On that day since the smaller brother's amount of money was bigger than his brother ... and the person who had 110 yen is the amount of money the smaller brother had in the beginning.

T: So these kinds of expressions ... the fact is these are the ones we're going to use from now on. Equations that use symbols like this ... umm, we'll call them inequalities. ... So today I think we would like to find the value for  $x$  that holds true for this mathematical expression while actually putting in numbers.

The teacher then asked students to complete a table (see figure above). When the table displayed on the blackboard was filled in, the teacher noted: "x holds true for 15, 16, 17, and 18; these are the '<'. The first value of  $x$  [13] was a '>'; the second one [14] was equal -- 'the standard'". The teacher then asked about the 19th day, to which one student responded that Ichiro's wallet was then empty and that the situation was finished.

### **Concluding Remarks**

In this classroom segment, we have witnessed the close relationship between inequality and equality concepts in eighth grade students. In using a problem-solving context involving a situation of inequality, an algebraic activity that we have characterized as being at the global meta-level, the teacher aimed to help students acquire some meaning for the form of algebraic inequalities. The problem provided a backdrop for generating an expression containing an inequality symbol. The solution to this inequality, having already been found by the students by means of non-algebraic methods, was regenerated by substituting values, from the vicinity of the solution, into the two algebraic expressions that formed the algebraic inequality. In this way, the relationship between the solution to the linear equality ( $180-10x=110-5x$ ) and those of its two related inequalities ( $180-10x>110-5x$  and  $180-10x<110-5x$ ) could be drawn out – implicitly appealing to a number-line interpretation of these solutions. It is also noted that, among the students' attempted solving approaches to the given problem, no one used a Cartesian graphical representation.

It has been argued from the research carried out with older students (e.g., Tsamir, Almog, & Tirosh, 1998) that there are clear pitfalls involved in attempting to apply to the solving of inequalities some of the transformational techniques used with equations. Yet, if the Japanese students' thinking about inequalities is at all representative of other students of this age range, then the interweaving of inequalities and equalities would seem to be rather deeply rooted. The didactical challenge is to find ways to help students beware of the traps of the equality/inequality connection in their transformational work with symbols, while they still enjoy its benefits in algebraic activity of the generative and global meta-level types.

## Some Questions

- The global meta-level activity of contextualized problem solving has successfully been used to provide meaning for inequalities and for their symbolic form. This leads to the question of whether, in a similar way, certain aspects of such contextualized activity can be found to be effective in helping students make sense of some of the exceptional transformation rules used in solving inequalities.
- The properties underlying valid equation-solving transformations are not the same as those underlying valid inequality-solving transformations. For example, multiplying both sides by the same number, which produces equivalent equations, can lead to pitfalls for inequalities. As the differences between the two domains are critical, the following question arises: What is the nature of instructional support that can generate in students the kinds of mental representations that will enable them to think about these critical differences when engaging in symbol manipulation activity involving inequalities?
- In which ways, if any, and for which age-ranges of students, can symbol-manipulation technology be harnessed so as to provide viable approaches for developing students' algebraic theorizing with respect to inequalities and their manipulation?

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# PROBLEMS RELATED TO THE USE OF GRAPHS IN SOLVING INEQUALITIES

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## I. Introduction

Graphs of functions are used increasingly to solve algebraic inequalities. This phenomenon is most probably in relation with the increasing use of graphic calculators in schools.

Most teachers seem to see the use of graphs as something that should help students in their solving of inequalities. In relation to some observations in our classrooms (students aged 15 and 17), we came to consider that this is not always the case and that there is a need to study some of the problems that arise when one changes a problem in algebra into a problem on graphs.

Solving an inequality graphically means, at first look, comparing the position of two curves. Starting from an algebraic inequality, it supposes that the student does the following work:

Inequality  $\rightarrow$  create the two functions  $\rightarrow$  emergence of the graphs through the emergence of  $y \rightarrow$  compare the  $y \rightarrow$  come back to  $x$ .

We will first address Duval's theory on semiotic registers to point out some of the difficulties that can arise. Then, as dealing with graphs means dealing with functions we will question some differences between denotation in algebra and denotation in calculus as they appeared in some recent, and still ongoing, work by Maurel & Sackur.

## II. Some Observations

We will first give a quick look to some results coming from the classroom. We asked our students to solve the inequality  $3/x > 2+x$ . As we expected, all the students who used an algebraic method to solve it made the expected error. They multiplied by  $x$  whatever the sign of  $x$  could be, thus giving an incorrect answer:  $x \in ]-3; 1[$ . Quite a few students used a graphical solution, drawing the graphs of the two functions:  $y=3/x$  and  $y= x+2$ . Then we found two types of errors: the first one came from reading the solution of the inequality, the second one from the writing of the solution for  $x$  even if the reading on the graph was correct. Older students (age 17) encountered the same type of difficulties on working with graphs. Our purpose is to give some interpretation of these errors and to show that the use of graphs for solving inequalities should be carefully prepared.

### III. Duval's theory of semiotic registers

#### III. 1. The concepts

Mathematics is working with representations of objects. The large variety of semiotics representations for the same mathematical object is stressed as a factor of difficulties for students in learning and understanding mathematics.

#### A. The registers

Duval considers that there are four different types of semiotic registers in mathematics (Duval, 2000). We will not give any exhaustive description of the registers, those that we will be interested in for this presentation will be described in part III. 2. The most interesting point for us is that two representations in two different registers of the same mathematical object do not have the same content, the same meaning (Frege 1985). Change of register makes explicit different aspects and different properties of the same object.

Duval emphasises the fact that comprehension in mathematics assumes the co-ordination of at least two registers.

#### B. The two types of transformation of semiotics registers

- Treatment inside one register corresponds to all transformations that can be made on a representation of one type. For instance all algebraic operations on an expression.
- Conversion between two registers is more interesting for us. Conversion is the origin of many difficulties as it is generally not reversible and can be very easy (Duval says *congruent*) in one direction and difficult (*non-congruent*) in the other.

#### III. 2. Application to our Problem

If we come back to the table in the introduction, we can identify 4 registers involved in the solving of an inequality graphically.

I	II	III	IV
Algebraic	Fonctional	Graphical bi-dimentional	Graphical mono-dimentional
$3/x > x + 2$	$f(x) = 3/x$ $g(x) = 2 + x$	$y = 3/x$ $y = 2 + x$	$x \in [\dots]$

An algebraic resolution of the inequality consists of “treatment” inside register I.

For simplicity, we will consider that students shift directly from register I to register III, and we will now study the two “conversions” I→III and III→IV.

To explore the congruence between two registers we have to look at the different ways students act in both of them.

### Conversion I→III:

1. First of all students must identify two different graphs in place of one inequality. The emergence of  $y$  and its role is a source of difficulty for students (Bazzini & al., 2001).
2. Then the type of transformations that the students have to make in register I have no correspondence in register III. In register I, one writes a sequence of algebraic expressions. The different graphs corresponding to these different expressions do not appear in register III. Another aspect of this conversion is the fact that the transformations in algebra are done for “all”  $x$ , whereas on a graph one can only visualise the graph for limited values of  $x$ .
3. Graphically one has to focus on the  $y$  for different values of  $x$ , which means looking at the intersection of the curves with straight lines whose equations are  $x=a$ . The process of solving depends on the position of the curves, on the number of points where they intersect each other. For simplicity sake we will just observe four different situations as shown in Fig. 1. In the simplest case, “ $f(x)>0$ ”, the conversion is congruent, as this can be translated as “the curve  $C_f$  is above the  $x$ -axis”. Difficulties arise when the slope of the function is steep such as  $y=1/x$  when  $x$  is close to 0 or  $y=1000x$ . The problem is no longer “ $C_f$  being above  $C_g$ ”, but  $C_f$  belonging to one or other of the parts of plane limited by  $C_g$ . One can see, easily, that there is then a difference between solving equations and solving inequalities.
4. Concerning the inequality  $3/x > x+2$ , the situation is interesting in the following way: algebraically, one has to distinguish between  $x>0$  and  $x<0$ . To this separation corresponds the fact that one of the graph ( $y=3/x$ ) has no intersection with the straight line  $x=0$ . Thus the algebraic activity has its correspondence in the graphical register and vice versa. See Fig. 2.

Our conclusion is that, most probably, the conversion is not congruent.

**Conversion III→IV:** The situation here seems simpler. There is a one to one correspondence between the points on the graph of function  $f$  where  $f$  is greater than  $g$  and the abscises of these points. We can then say that the conversion between those two registers is congruent.

The question is to interpret the difficulties that the students encounter to come back to the solution of the inequality with  $x$ . As we will see, the situation is different from the one we can observe with an equation and enforces the link between inequalities and functions.

### III. The Theory of Denotation

The theory of denotation in algebra is well known (Frege, 1985) and has been used in many situations to interpret the difficulties of the students. We have been lately interested in understanding what could be the concept of denotation in calculus and functional analysis (Maurel & al. 2001). The results we mention here are a very first attempt in this direction.

### III. 1. Denotation in Calculus

We studied several situations: primitives, error terms in Taylor formulas. We came to the conclusion that a symbol like  $\int f(x)dx$  doesn't correspond to one function (one mathematical object) as it does in algebra but to a class of functions. Ignoring this can lead to some difficulties such as the demonstration that  $1=0$ . Legrand (Legrand, 1993) has emphasised the fact that very often in calculus one has to abandon some information in order to obtain the result. We think that the difficulties of the students in shifting from the graph to the solution in  $x$  could come from this aspect.

### III. 2. The Case of the Inequality

Very shortly, we can say that, here also, there is not a one to one correspondence between the graph and the set of solutions in  $x$ . Different graphs can lead to the same set of solutions as is shown in Fig 3. There is not one point for one  $x$  but an infinite number of points. The situation looks very similar to the situation of the primitives. One has to abandon information, the precise graph of the functions, to focus only on the abscises of these points.

## IV. Conclusion

Concerning inequalities, the use of graphs induces new difficulties for students, some of them being specific of functions. It should not be taken for granted that when "solving graphically" students learn the same mathematics as when "solving algebraically". Our interest is not so much "how to have students learn to solve inequalities?" but "what do they learn in mathematics when they solve algebraically or graphically?".

Another important question is the apparent similarity between the solving of equations and the solving of inequalities. This issue appears to be a crucial one.

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## Figures (these belongs to the preceding paper)

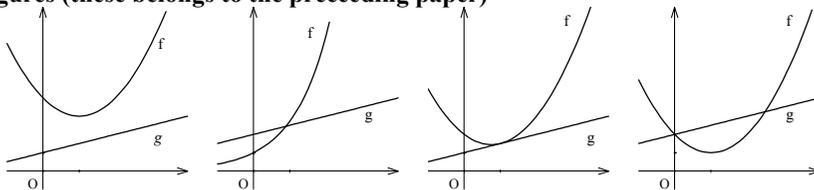


Fig. 1.

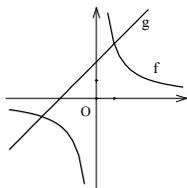


Fig. 2

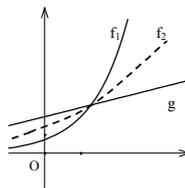


Fig. 3

## EQUATIONS – A STRUCTURAL APPROACH

**Tommy Dreyfus** and **Maureen Hoch**, *Tel Aviv University (Israel)*

### Structure

Reading papers on teaching and learning algebra (and other topics in mathematics, including calculus) one frequently meets the term structure. Some examples of papers in which structure plays a substantial role are Sfard & Linchevski (1994), Dreyfus & Eisenberg (1996), Linchevski & Livneh (1999), Zorn (2002). Structure appears to be a convenient term to describe something many of us may have some vague feeling for but cannot grasp in words. In fact, in few papers is there an attempt at defining, or even circumscribing what the authors mean by structure.

According to Sfard & Linchevski algebra is a hierarchical structure. In algebra what may be considered to be an operation at one level can be acted on as an abstract object at a higher level. Dreyfus & Eisenberg variously describe structure as the result of construction; as involving symmetry; as being composed of definitions, theorems and proofs; as being a method of classification; as relationships. Zorn states that “Understanding basic mathematics profoundly means proficiency at detecting, recognizing and exploiting structure, and at drawing useful connections among different structures”. While giving no definition of structure he hints that it may be connected to pattern. Linchevski & Livneh discuss students’ difficulties with mathematical structures in the number system and in the “algebraic system” but nowhere do they define what these structures are. They also use the term algebraic structure without explanation, and refer to surface structure, hidden structure and structural properties.

Defining or circumscribing what we mean by structure is not an easy undertaking. Many mathematicians, especially algebraists, tend to give the definition of some category that includes algebraic objects such as groups, rings, fields, ideals etc. But this is not helpful for us if we want to deal with high school algebra and with learning it. What does structure mean if we talk about high school algebra, and more specifically about equations with all their technical aspects?

In this contribution, we will remain guilty of the same sin of talking about structure without saying what we mean by it. Further issues of definition of structure and of examining the meaning of the definition in practice are discussed in Hoch & Dreyfus (2004). Here we will concentrate on why one might want to look at structure.

There are two quite different areas where structure is important for equations, recognizing an equation and dealing with its internal structure.

### **Recognizing an Equation**

One would expect that one of the mathematical objects most easily recognized by students is an equation. (In order to avoid the need to distinguish between equation and identity, we include here identities under the general category of equations – more specifically, an equation with the entire substitution set as solution.) We have asked some Israeli high school students who learn mathematics at above average level to say what they think an equation is. Responses included:

1. An exercise where the aim is to find  $x$ .
2. An exercise that has a solution, that is, an exercise before you've solved it, and in the end you can do something to it and get to the solution. You need to find the variable.
3.  $x$ -s on one side, numbers on the other, an equal sign between them; need to find  $x$ .
4. A statement including two sides, an equal sign, and one or more  $x$ -s.
5. Two sides connected by an equal sign and certain rules for solving.

We see responses 1 and 2 as being purely procedural, referring to what has to be carried out. The others refer to external form. This might qualify as structure and be useful from the formal language point of view but it remains surface structure. Response 3 also mentions procedure, whereas response 4 focuses on external form only. Response 5 comes closest to indicating that there may be some underlying structure by mentioning "certain rules". The responses do not refer to what we might call the deep structure of equation, the mathematical properties of the object "equation". If these responses are typical, our data indicate that structure is not something that is in the realm of awareness of high school students.

### **The Internal Structure of Equations**

Equations also have internal structure – at a finer level than the one needed to say whether something is an equation or not. Recognizing and using this internal structure may make solving the equation easier and increase success. Internal structure may be the actual or potential structure of the equation. By actual structure

we mean the equation as it is given. For example the actual structure of the equation  $\frac{1}{x-2} = 3x + 4$  could be described as a rational equation describing the intersection of a rational function with a linear function. The potential refers to what can be reached by transforming the equation. In the case of this example, the potential structure is quadratic, specifically the quadratic equation  $3x^2 - 2x - 9 = 0$  whose structure is rather different from the original one. There might be an intermediate case where minor operations such as adding or removing brackets lead to a different structure.

The above equation could be written  $\frac{1}{x-2} = 3(x-2) + 10$ .

Wenger (1987) provides a classic example of where recognizing actual structure is helpful in solving the equation. When solving the equation  $v\sqrt{u} = 1 + 2v\sqrt{1+u}$  for  $v$ , recognizing the linear structure yields a relatively easy solution process. Many students, of course, focus on the square root sign which is a signal for them to square both sides of the equation. Another classic example is this parametric equation in

$x: \frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-a)(x-c)}{(b-a)(b-c)} = 1$ . It appears in Movshovitz-Hadar & Webb (1998). Here “brute force” leads to a solution only at the hands of a very determined and very adept solver while examination of the structure provides a much more efficient solution. Examining a structure that is just below the surface, the structure of the individual terms that make up the equation, reveals that  $x=a$  is a solution, and then that  $x=b$  and  $x=c$  are also solutions. The internal deep structure – the properties of a quadratic equation – provides the information for the final solution, that this equation is true for all values of  $x$ .

A typical equation from high school algebra is  $(x^2 - 4x)^2 - x^2 + 4x = 6$ . It can be solved by recognising that a simple substitution transforms it into a quadratic equation. Thus a minor operation reveals structure and gives a handle on solving the

equation. Another example is  $\frac{1}{4} - \frac{x}{x-1} - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$ . An examination of the structure reveals that this is a linear equation masquerading as a rational equation. For many students however the presence of an algebraic fraction is a signal to multiply by a common denominator leading to a long and error prone solution (see Hoch & Dreyfus, 2004). Here recognizing a hidden structure and transforming the equation (by subtracting the same term from both sides), so as to show this hidden structure, is used to solve the equation. We see that recognizing and using structure is likely to increase success in algebra substantially.

## In Conclusion

Our experience is that Israeli students have little difficulty in actually recognizing equations but extreme difficulty in talking about this recognition. They rarely relate to equations in any way apart from the procedural. They usually do not recognize the internal structure of equations. If they do recognize structure they rarely use it (see

for example Hoch, 2003) and in fact they have difficulty solving all but the most standard equations. Also, teachers do not seem to be aware of what recognizing and using structure could do for the student. The emphasis in the algebra classroom is on mechanical methods for solving equations. For example, the method of substitution is taught in tenth grade, but usually on a very technical level, and is soon forgotten (see Hoch & Dreyfus, 2004).

We suggest that the forum address the issue of ways of presenting algebra that will focus students' attention on structure.

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## “NEW ERRORS” AND “OLD ERRORS”: THE CASE OF QUADRATIC INEQUALITIES

**Pessia Tsamir, Dina Tirosh, Sarit Tiano, Tel-Aviv University (Israel)**

A prominent line of research in mathematics education is the study of errors. The early research on mathematics learning viewed students' errors as flaws that interfere with learning and need to be avoided (e.g., Greeno, Collins & Resnick, 1996). From an instructional perspective, students' errors were traditionally perceived either as

signals of the inefficiency of a particular sequence of instruction or as a powerful tool to diagnose learning difficulties and to direct the related remediation (e.g., Ashlock, 1990; Fischbein, 1987). Borasi (1987) argued that errors could and should be used as springboards for problem solving and for motivating inquiry about the nature of mathematics, and Avital (1980) claimed that the best way to address common mistakes is to intentionally introduce them and to encourage a mathematical exploration of the related definitions and theorems.

But what is actually happening in the classrooms with respect to errors? Do teachers intentionally introduce common errors and if so: How? When? To whom? How do teachers address errors made by students? What are the factors that influence their reactions? In this paper we discuss our initial attempts to explore teachers' declarations and practices regarding the role of errors in their classrooms. We describe here the case of Rami, a very experienced mathematics teacher who has a reputation of an excellent teacher. We shall focus here on a variety of ways in which Rami addressed errors when teaching quadratic inequalities. We shall first briefly describe what is known from the literature on students' common errors when solving quadratic inequalities and to their possible sources.

### **Literature on Quadratic Inequalities: *Errors, Sources and Instruction***

In the last decade there is a growing interest in students' performances when solving various types of algebraic inequalities in general and quadratic inequalities in particular (e.g., Linchevski & Sfard, 1991; Tsamir & Bazzini, 2001). Several *common errors* were identified, including the tendencies to: (1) multiply / divide both sides of an inequality by a factor that is not necessarily positive, (2) deal with products in the following manner:  $a \cdot b > 0 \Rightarrow a > 0$  and  $b > 0$ ;  $a \cdot b < 0 \Rightarrow a < 0$  and  $b < 0$ ; (3) make inappropriate decisions regarding logical connectives, and (4) reject  $\{x | x = a\}$ , "R" and " $\phi$ " as solutions. Several publications mentioned *possible sources* for these errors, mainly relating to possible overgeneralizations from equations to inequalities (e.g., Tsamir, Tirosh & Almog, 1998), and to the grasp of transformable inequalities as being equivalent (Linchevski & Sfard, 1991). Some of these errors are intuitive (Fischbein, 1987), and are, thus, likely to evolve in every class. Consequently, we decided to explore how various teachers address errors in their classrooms. Here we focus on a class that dealt with quadratic inequalities.

### **The Study**

#### *Setting and Methodology*

At the time the study was carried out, Rami was the head mathematics teacher in a secondary school. He was a very energetic and highly motivated teacher who invested much effort in his instruction and in establishing open, friendly relationship with his students and colleagues. For the purpose of our study, one of the researchers (ST) observed and videotaped Rami's three lessons on quadratic inequalities in an average, 13<sup>th</sup> grade class (learning for a certificate of electronic technicians). The videotapes were transcribed and all the "error-episodes" were defined (an error

episode consists of an error made in class and the subsequent, related event). Several reflective interviews were then conducted with Rami. In these interviews he was first asked to list students' common errors when solving quadratic inequalities. Then, he was presented with the transcriptions of several "error-episodes" that occurred during his instruction. He was asked to identify the error, to specify its possible sources, to explain the way he addressed it in class, to comment on it and to relate to other, suggested ways of handling this error. Later on, Rami was presented with a list of quadratic inequalities that are known to elicit specific errors. He was asked to list the errors that students are likely to make in each case. Finally, Rami was presented with the typical errors that students commonly make when solving the same quadratic inequalities and asked: "How would you react, in class, to such errors". The interviews lasted about 90 minutes. They were audio-taped and transcribed.

Due to space limitations, we shall focus here only on one main observation regarding Rami's didactical ways of addressing errors in his lessons on quadratic inequalities.

#### *New Errors and Old Errors: A Critical Dichotomy*

Our analysis of the "error episodes" revealed that Rami addressed the errors that occurred in his class in two distinguished, clusters of reactions: The *economic cluster* and the *elaborated cluster*. In the *economic cluster* we included his following, typical reactions: 1) ignore the error and go on teaching, 2) state the correct solution, and 3) when having a mix of erroneous and correct suggestions, address only the correct ones. The following, three reactions are representative of the *elaborated cluster* : 1) ask the student to repeat his erroneous solution and to explain his reasoning to the entire class 2) try to find out if other students in the class hold the same opinions, and 3) try to lead the student (e.g., by counter examples) to realize that she erred.

All in all, it was noticeable that the *economic* and the *elaborated* cluster were distinguished in terms of the time allotted for and the effort invested in discussing the errors. An *economic reaction* is a short, local reaction that highlights only the correct solution, with no reference to the incorrect solutions. An *elaborated reaction* unlike the economic ones, is more time consuming and didactically more demanding. Here, Rami explicitly addressed the incorrect ideas, asking the student for further explanations and trying to trace the source of the error.

A question that naturally arose is: What directed Rami's didactical conduct? Under what circumstances was he acting in an economic manner? In what occasions did he prefer the elaborated reaction? Rami's behaviors could be attributed to various factors, some of which are student oriented (e.g., capabilities, gender) while others are timing oriented (in what part of the lesson the error occurred). Our analysis ruled out the "student" option, since Rami reacted to the same student in different ways on different occasions. At first it seemed that the timing was a major factor that guided Rami's reactions: Economic reactions were more frequent at the beginning of the first lesson while elaborated reactions were more evident by the end of this lesson. But this split was not apparent in the other two lessons. A detailed examination of the

mathematical content of the episodes that were included in each of the two clusters led us to conclude that the episodes in the economic cluster addressed errors that were embedded in mathematical topics that were studied prior to the lessons on quadratic inequalities (e.g., quadratic equation, parabolas). Elaborated reactions were provided by Rami to issues that were part of the topic at hand (e.g., logical connectives). This observation was confirmed by Rami during the subsequent, reflective interview. Indeed, when Rami was asked to relate to various error episodes that occurred during his lessons, he clearly stated that the nature of the error, in terms of being “new” or “old” is a main factor that influenced his reaction to the error.

### **Summing Up and Looking Ahead**

Our results indicate a phenomenon that at first glance seems obvious, i.e., allotting more time and didactical energies to errors in the new topic, and less time and efforts to those that relate to mathematical topics that were studied previously. This observation raises many issues for further explorations, three of which are: (1) Is this conduct a general characteristic of Rami's instruction or is it typical only to his teaching of quadratic inequalities? (2) Is the "new errors"/"old errors" split typical only to Rami or to other expert teachers / novice teachers? (3) What are the pros and cons of this approach? We shall deal with these issues in our presentation.

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## **REFLECTIONS ON RESEARCH AND TEACHING OF EQUATIONS AND INEQUALITIES**

**David Tall**, *Mathematics Education Research Centre, University of Warwick, (UK)*  
In reacting to this forum on ‘Algebraic Equalities and Inequalities’, I take a problem-solving approach, first, asking ‘what is the problem?’ then looking at the five

presentations to see what can be synthesized from their various positions (acknowledging that they are here limited to very short summaries).

The ‘problem’, as initially formulated, focuses on the *algebraic* manipulation of equations and inequalities. Tsamir et al [TTT] focus mainly to this aspect by considering how a teacher might cope with errors that arise from the inappropriate use of earlier experiences in equations that produce errors with inequalities. This focus is broadened in the list of ‘Key Questions’, to encourage the consideration of different theoretical frameworks—and the use of technology—to see how research can improve teaching and learning. The other papers take the key questions in different directions. Boero & Bazzini [BB] and Sackur [S] consider broader issues, with a particular focus on the switch from algebraic to visual representations where an inequality  $f(x) > g(x)$  is visualised by seeing where the graph of  $f$  is above the graph of  $g$ . Underlying both approaches are relationships between different representations (or semiotic registers, as described in the subtle theory of Duval).

Kieran [K] presents a different overall framework (‘generational’, ‘transform-ational’ and ‘global meta-level’) that may be described as a ‘vertical’ theory of development rather than a ‘horizontal’ theory of relationships between representations. Finally, Dreyfus & Hoch [DH] broaden the context to the increasingly sophisticated structure of equations, from a procedure to undo an arithmetic calculation, to solving equations with  $x$ s on both sides, to more subtle cases of equations containing substructures and equations solved using specified rules.)

This brings me back to ‘the problem’. What is it that this forum is really attempting to address? There seems to be an implicit understanding that we need to help students to understand and operate with equations and inequalities. *But for what purpose?* If the purpose is to solve a given equation or inequality, then a graphical picture may be appropriate. For instance, to ‘see’ what happens to the inequality  $x^2 > x + c$  as  $c$  varies, a powerful visual representation is given by the quadratic  $f(x) = x^2$  and a straight line  $g(x) = x + c$  that moves up and down as  $c$  changes. However, if the problem is to enable the student to become fluent in meaningful manipulation of symbolism, then the activities with the graph may involve no symbolic manipulation whatever (particularly if the graph is drawn by computer). [S] considers the strengths and weaknesses of moving between different registers. These focus on different aspects, highlighting some, neglecting others. If an aspect is absent, then its variation does not figure in the link between representations. An example is the *evaluation* of a function by carrying out a procedure:  $2(x+1)$  and  $2x+2$  are different procedures in the symbolic register but are represented by precisely the same graph.

The focus of [BB] on graphs of functions as global dynamic entities uses the idea of ‘grounding metaphors’ of Lakoff & Nunez in a way that ‘could also ensure a high level of the control of the solution process’. But *what* solution process? The visual enactive activity can give a powerful embodied sense of global relationships between functions as entities, but how does it relate to the meaningful manipulation of symbols? It emphasizes the strength of grounded metaphors but not the ‘incidental

properties’ of Lakoff’s theory, which may be usefully employed in a particular context but have the potential to be the sources of errors in new contexts.

It is my belief that the phenomenon of ‘cognitive obstacles’ arises precisely because the individual’s subconscious links to incidental properties in earlier experiences are no longer appropriate in a new context. Rather than use the high sounding language of ‘metaphor’ for the recall of earlier experiences, I use the prosaic term ‘met-before’. I hypothesise that it is precisely the met-befores in solving linear equations that causes problems in inequalities researched by [TTT]. Students taught to manipulate symbols in equations, will build personal constructions that work in their (possibly procedural) solutions of linear equations but operate as sub-conscious met-befores that cause misconceptions when applied to inequalities.

In a given context there are often several different approaches possible. [K] reveals a spectrum of responses to a problem that may be formulated as an inequality, including a physical representation, the use of tables, equations and inequalities. [DH] presents a compatible spectrum, with different emphases, numerical procedures to ‘undo’ equations, more subtle manipulation of expressions as mental entities, and seeing sub-structures of equations as mental entities in themselves. Some of these approaches may be more amenable to future development than others; in particular, theories of cognitive compression from process to manipulable mental entities (which are entirely absent from all the presentations) address the possibility that the construction of mentally manipulable entities is likely to be more productive for long-term development.

Later developments in the use of inequalities include the formal notion of limit, where the epsilon-delta method will certainly benefit from meaningful grounding of inequalities, but will also need to focus on the manipulation of symbols and the development of formal proof. Inequalities at a formal level involve axioms for order in a field  $F$ , for example, by specifying a subset  $P$  of  $F$  that has simple properties (if  $a \in P$ , then one and only one of these holds:  $a \in P$ ,  $-a \in P$  or  $a=0$ ; if  $a, b \in P$  then  $a+b, ab \in P$ .) In this case  $a > b$  is *defined* to be true when  $a-b \in P$ . This use of ‘rules’ is not a meaningless procedural activity but a meaningful formal approach that has the potential of giving new meanings. For instance, a structure theorem may be proved to show that every ordered field ‘contains’ the rational numbers and may also contain ‘infinitesimals’ that are elements in  $F$  which are smaller than any rational number. In this way intuitive concepts at one stage (infinitesimals as ‘arbitrarily small’ variable quantities) can be given a formal mathematical meaning.

An organization such as PME needs to aim not only for local solutions to problems, but also for global views of long-term development. The papers in this forum present essential ingredients to contribute such a wider scheme.

When the ‘problem’ of equations and inequalities is seen in this way, a wider picture emerges. There are unspoken belief systems that get in the way of our deliberations. For instance, while several of the papers give examples of different individuals using

different methods to solve the same problem, no one attempts to say whether one solution is potentially better or worse for long-term development. Differences are apparent in the success and failure in all the examples given. Do we need to look at different solutions for different kinds of needs? Rich embodiments have strengths that may be appropriate in some contexts (perhaps to solve an inequality in a specific problem) and misleading in others (where concepts of constructed that, if unresolved, become met-befores causing obstacles in later learning). Do all students follow through the same kind of Piagetian development or, does their journey through mathematics find them using methods that are more or less suited to long-term development that gives different kinds of possibilities for future development?

In addition to the horizontal framework of registers and the vertical framework of [K], I offer a third that relates to the algebraic spectrum of [DH]. A study of long-term development of symbolism in arithmetic and algebra (Tall *et al.*, 2001) led to a categorization of algebra (Thomas & Tall, 2001) in three levels, which we termed 'evaluation algebra', 'manipulation algebra' and 'axiomatic algebra'. The first encompasses the idea of an expression, say  $3+2x$  being used simply for evaluation, say in a spreadsheet or in a graph-drawing program. The second encompasses the idea of an expression as a thinkable entity to be manipulated. The third concentrates on the properties of the manipulation and leads to an axiomatic approach to algebra in terms of groups, rings, fields, ordered fields, vector spaces, etc. In what ways do the papers presented in this forum address problems both at a local level and also in producing a helpful global theory? Much of the discussion could involve evaluation algebra, [TTT] considers manipulation and [DH] looks from manipulation to axiomatic. Do we need one kind of algebra for some students and other kinds for others? Richard Skemp once said to me, 'there is nothing as practical as a good theory'. In our forum it would be practical to look for a global theory encompassing the local theory of equations and inequalities.

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## SYNTAX AND MEANING

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A forum is certainly a multi-voiced dialogue, an example of what Bakhtin used to call heteroglossia, or the encounter of multiple perspectives in cultural interaction. With

their own intonation and from their own perspective, the papers of the research forum engage in dialogue with each other about pedagogical, psychological and epistemological questions concerning two key concepts of school algebra, namely, equations and inequalities. They offer us valuable reflections on the search for new contexts to introduce students to inequalities (e.g. functional covariance) and a critical understanding of the limits and possibilities of these contexts. They also provide us with fine enquiries about urgent learning problems along the lines of key theoretical constructs that have played a central role since the 1980s in mathematics education (such as structure and the cognitive status of students' errors).

The papers tackle a general problématique against the background of the present context of discussions about cognition. In the past few years, there have indeed been important changes in conceptions of cognition in general, as witnessed by e.g., a recent interest in phenomenology, semiotics, and embodiment. We have become aware of the decisive role of artefacts in the genesis and development of mathematical thinking and we have become sensitive to theoretical claims from sociology and anthropology that emphasize the intrinsic social dimension of the mind. With their own intonation and from their own perspective, the papers of the research forum have engaged each other in a dialogue on the problem of algebraic thinking as set by the general stage of our current understanding of cognition. Since one of the key common themes of the papers is that of syntax and meaning, let me delve into it and comment on what the papers intimate in this respect.

## 1. Meaning

In the introduction to their paper, Boero and Bazzini find fault with the classical approach to inequalities and claim that the “purely algorithmic manner” that reduces the solving of inequalities to “routine procedures” limits students' understanding. This complaint is not new. In the seminal book edited by Wagner and Kieran (1989) the same reasons led Lesley Booth to object to the considerable attention paid to the syntactic aspects of algebra in the classroom. There is nevertheless a subtle but important difference in how solutions are envisaged on the one hand, by Booth and the structural perspective, and by Boero and Bazzini, on the other.

Booth claimed that difficulties in learning syntax were the result of a poor understanding of the mathematical structures underpinning algebraic representations: “our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations”, she argued, and added that “[t]hese structural properties constitute the *semantic* aspects of algebra.” (Booth, 1989, pp. 57-58). I do not think that Boero and Bazzini disagree with the important role played by structural properties in the constitution of the semantics of algebra. Nevertheless, they seem to disagree with the idea that, ontogenetically speaking, the understanding of structural properties comes *first* as well as with the claim that these structural properties alone constitute the semantics of algebra. Indeed, in their approach (see also Garuti et al., 2001), the study of the production of meaning is located in an activity that transcends mathematical

structures. Their analysis traces elements of students' linguistic activity and body language in an attempt to detect metaphors, gestures and bodily actions that can prove crucial in students' understanding and use of algebraic symbolism. In their analysis of the way in which students make sense of a quadratic inequality, they emphasize the students' allusion to artefacts and to their understanding of symbols in terms of cultural linguistic embodied categories such as "going up" and "going down". As I see it, the covariational functional context that they propose is conceived of as a means for students to produce meaning and understand signs.

The idea that the production of meaning goes beyond mathematical structures and the claim that meaning is produced in the crossroad of diverse semiotic (mathematical and non-mathematical) systems is certainly one of the cornerstones of non-structural approaches to mathematical thinking. And yet, many difficult problems remain. Algebraic symbolism is undoubtedly a powerful tool. Even if some calculators and computer software are able to perform symbolic manipulations, algebraic symbolism is not likely to be abandoned in schools –at least not in the short term. Kieran's reflections on what happens to meaning when students translate a word-problem into symbolism, Sackur's interest in understanding the outcome of meaning in conversion between, and treatments within, registers and Dreyfus and Hoch's concerns about students recognizing the underpinning structures in equations thus appear to be more than justified. Certainly, one of the crucial problems in the development of algebraic thinking is to move from an understanding of signs having been endowed with a contextual and embodied meaning, to an understanding of signs that can be subjected to formal transformations. The meaning that results from noticing that a graph "goes up" or "goes down" supposes an *origo*, that is, an observer's viewpoint. This *origo* (Radford 2002a) is the reference point of students' spatial-temporal mathematical experience, the spatial-temporal point from where an embodied meaning is bestowed on signs. Algebraic transformations, such as those mentioned by Dreyfus and Hoch, require the *evanescence* of the *origo*. Does this amount to saying that symbolic manipulations of signs are performed in the absence of meaning? To comment on this question, let us now turn to the idea of syntax.

## 2. Syntax

One of the tenets of structuralism is the clear-cut distinction between syntax and semantics. From a structural perspective, the real nature of things is seen not in the world of appearances, but in their true meanings –something governed by the intangible but objective laws that Freud placed in the unconscious, and that structural anthropology, psychology and linguistics, after Saussure and Lévi-Strauss, thematized as "deep structures". Syntax was conceived of as lying on "surface structures", it was merely dead matter, the shadows of deep, structurally governed, mental activity. It is understandable that, in this context, in 1989 Kaput argued that instead of teaching syntax (which would produce "student alienation") we should be teaching semantics (Kaput, 1989, p. 168). Nevertheless, as I have already stated, we have become more sensitive to the claim that every experience, even the more

abstract one found in mathematics, is always accompanied of some particular sensory experience, or –as Kant put it in the *Critique of Pure Reason*– that every cognition always involves a concept and a sensation.

How, then, within this context, can we address e.g. Dreyfus and Hoch’s legitimate concerns? Recognizing equivalent equations is one of the fundamental steps in learning algebra. The formal transformation of symbols in fact requires an awareness of a new mode of signification –a mode of signification that is proper to symbolic thinking (Radford, 2002b) and whose emergence only became possible in the Renaissance. As Bochner (1966) noted, despite the originality and reputation of Greek mathematics, symbolization did not advance beyond a first stage of iconic idealization where calculations on signs of signs were not accomplished. It is not surprising then that the problem of explaining the formal manipulation of symbols puzzled logicians and mathematicians such as Frege, Russell, and Husserl. While for Russell (1976, p. 218) formal manipulations of signs are empty descriptions of reality, for Frege and Husserl formal manipulations do not amount to manipulations devoid of meaning. In fact, for Frege, equivalent algebraic expressions correspond to a single mathematical object seen from different perspectives: they have the same referent but they have a different *Sinn* (meaning). Adopting an intentional, phenomenological stance, Husserl contended that manipulations of signs require a shift in attention: the focus should become the signs themselves, but not as signs *per se*. Husserl insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the “rules of a game” (Husserl 1961, p. 79).

These remarks do not solve the crucial problem raised by Dreyfus and Hoch, also present in the other papers of this forum. It would certainly be of little help to tell students that a seemingly rational equation is, after transformations, equivalent to a linear equation because they are both designations of the same mathematical object.

Perhaps Husserl’s insight intimates that the change in the way we attend the object of attention (e.g. the modeled situation or the equation itself) leading to an *awareness* of the “rules of the game” rests on a process of *perceptual semiosis*, or a dialectical movement between perceived sign-forms, interpretation, and action. Hence, it may be worthwhile to consider the ontogenesis of new modes of signification required by algebraic symbolism as a back and forth movement between interpreting the symbolic expression in its diagrammatic form (Peirce) and the (mathematically structured) hypothetical generation of new diagram-equations.

It might be very well the case that the greatest difficulty in dealing with equations and inequalities resides in: (1) the understanding of the *apophantic* nature of equations and inequalities and (2) the *apodeictic* nature of their transformations.

Number (1) refers to the fact that, in contrast to a symbolic expression like  $x+1$ , an equation or an inequality makes an *apophansis* or *predicative judgment* (in Husserl’s sense; Husserl, 1973): it *asserts* e.g. that  $P(x) = 0$ . Number (2) refers to the necessary truth-preserving transformations of equations and inequalities –if, for a certain  $x$ , it is

true that  $P(x) = 0$ , then  $Q(x) = 0$ , etc., something that Vieta expressed by saying that algebra is an *analytic* art. What I want to suggest is that the predicative judgments  $P(x) = 0$  or  $P(x) \leq 0$ , etc. that rest at the core of solving an equation or an inequality should not be confined to the written register containing an alphanumeric string of signs. We need an ampler concept of predication (and of mathematical text) less committed to the written tradition in which Vieta was writing not many years after the invention of printing. We also need a better concept of predication capable of integrating into itself the plurality of semiotic systems that students and teachers use, such as speech, gestures, graphs, bodily action, etc., as shown clearly in the Grade 8 lesson mentioned by Kieran. Predicative judgments would be made up of a complex string of gestures, written signs, segments of speech and artefact-mediated body actions. Their transformations would not be confined to the realm of logic and formal symbol manipulation, for the passage from one step to the next in a semiotic process is not something predetermined in advance by the logic of deduction alone: what seems to be a formal manipulation is in fact continually open to interpretation. There is, in the end, no opposition between syntax and meaning. Every sign has a meaning. Otherwise, it cannot be a sign. Conversely, every meaning is an abstract entity – “a general” (Otte, 2003) – which finds instantiation in signs only.

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## SOME FINAL COMMENTS

One main aim of this research forum is to have a rich discussion and enable the participants to address the issues presented in the five presentations. For this purpose we decided to have the following structure of meetings: *In Session One*: each of the presenters will briefly [10 minutes] present their studies together with educational implications, and conclude his / her presentation by posing a number of questions for further discussion. Then, all the participants will be asked to discuss these questions, raise additional questions, dilemmas, doubts and comments that will be addressed by the presenters, reactors and all the others during the second meeting. *In Session Two*: each of the reactors will present his analysis of the approaches presented in sessions one [15 minutes], referring both to the presentations and to participants' remarks made during the first sessions. There will be ample time for the audience to add their own thoughts and analyses to those of the reactors.

Another aim of this RF is to discuss issues of *inclusion and diversity*. This will be done by refining the questions posed by the participants so as to meet the needs, abilities and beliefs of different students, teachers, and classes. For example, when discussing students' erroneous solutions to inequalities, we will address the following questions: What are the difficulties of low achievers vs. high achievers? Boys vs. girls? Those who studied the topic in different ways (e.g., graphical vs. algebraic approaches)? When discussing the teaching of equations and inequalities, we may for instance address the following questions: How do different teachers make their related didactical decisions? What is the impact of different teaching approaches on different students?

Finally, this RF aims to create a wide international network to investigate the teaching and learning of algebraic equations and inequalities by deepening existent collaborations and encouraging researchers from additional countries to enter this endeavor. A selection of contributions discussed during the Research Forum could also yield specific publications on the theme.