RESEARCH FORUMS

RF01: THE SIGNIFICANCE OF TASK DESIGN IN MATHEMATICS EDUCATION: EXAMPLES FROM PROPORTIONAL REASONING

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In the context of the overall focus of PME29 on *Learners and Learning Environments*, we have chosen the topic of pedagogical task design for this Research Forum. We see task design as a crucial element of the learning environment, and wish to explore further the role that it plays for learners. The overarching question for this Research Forum is: Why is task design significant?

To make progress on this question, we raise two issues: how does the task design impact on student learning? How does the agenda of the researcher or teacher shape the task design? More specifically we ask: how does the nature of the task influence the activity of students? What is important for mathematics educators in designing a task?

In order to work on these questions, both in the preparations for the Forum, and within the sessions at the conference, we have chosen to take a specific topic within the curriculum, that of *proportional reasoning*, and to invite the contributors to the Forum to work on designing tasks for the learning and teaching of proportion for pupils of around 11-12 years old.

The contributors

There are four groups of researchers contributing to this Forum, all of whom work on aspects of task design from different perspectives.

Dirk De Bock, Wim Van Dooren and Lieven Verschaffel explore features of the use of words problems in a number of mathematical areas, and have focussed on the ability to discriminate proportional and non-proportional situations.

Koeno Gravemeijer, Frans van Galen and Ronald Keijzer use design heuristics from Realistic Mathematics Education (guided reinvention through progressive mathematization, didactical phenomenology, and emergent modeling) in an approach which also draws on design research.

Alex Friedlander and Abraham Arcavi have many years experience within the Compumath project, which is developing a technology-based curriculum and studying the effects on pupils' learning.

Janet Ainley and Dave Pratt have developed an approach to task design based on creating tasks which are purposeful for pupils within the classroom environment.

We hope that our understanding of task design will be enhanced by making explicit reflections on these differing perspectives in the context of specific examples of tasks and their use by pupils.

The design brief for the contributors

Each of the teams of contributors was asked to design a task which focussed on proportional reasoning. The task had be suitable for pupils aged about 11-12 years, and it also had to be a ‘stand alone’ task, which could be tackled within one lesson. This condition was a significant constraint for some of the contributors, who would normally design tasks as part of a sequence. Contributors were asked to prepare their task in a form that could be presented to pupils, and were also asked to provide teachers’ notes.

Each of the tasks has been trialled with pairs of pupils and the papers by each of the contributing teams which follow this introduction draw on this data to illustrate the discussion of the principles which underpinned their task designs.

Dirk, Wim and Lieven’s task

This task focuses on similarities and differences in a set of word problems, some of which require proportional reasoning, while others have a similar format, but are not, in fact, proportional problems.

Yesterday, Mrs. Jones made some word problems to use in the math lessons. But they got all mixed up! Can you help Mrs. Jones to put some order in the word problems? Look at the problems very carefully and try to make groups of problems that belong together.

A Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 rounds. When Ellen has run 30 rounds, how many has Kim run?

B Mama put 3 towels on the clothesline. After 12 hours they were dry. The neighbour put 6 towels on the clothesline. How long did it take them to dry?

C Mama buys 2 trays of apples. She then has 8 apples.Grandma buys 10 trays of apples. How many apples does she have?

D John runs a bakery. He uses 10 kg of flour to make 13 kg of bread. How much bread can he make if he uses 23 kg of flour?

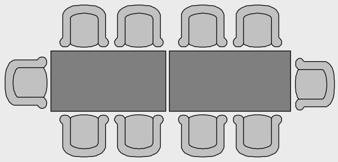
E The locomotive of a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. If there were 8 carriages connected to the locomotive, how long would the train be?

F Today, Bert becomes 2 years old and Lies becomes 6 years old. When Bert is 12 years old, how old will Lies be?

G A group of 5 musicians plays a piece of music in 10 minutes. Another group of 35 musicians will play the same piece. How long will it take this group to play it?

H Yesterday, a boat arrived at the port of Rotterdam, containing 326 “Nissan Patrol” cars. The total weight of these cars was 521 tons. Tomorrow, a new boat will arrive, containing 732 “Nissan Patrol” cars. What will be the total weight of these cars?

I In the hallway of our school, 2 tables stand in a line. 10 chairs fit around them. Now the teacher puts 6 tables in a line. How many chairs fit around these tables?



J In the shop, 4 packs of pencils cost 8 euro. The teacher wants to buy a pack for every pupil. He needs 24 packs. How much must he pay?

Now answer the following questions:

- Write here the different groups of problems. (Use the letters on the sheets)
- Why did you make the groups in that way?
- Can you think of a different way to put the problems in groups? Explain that as well.

Koeno, Frans and Ronald's task

This task is based around the story of Monica and Kim making a cycle trip from Corby to Cambridge. Various resources such as a map of the route, photographs and background information (the reason for the trip, the weather conditions) are provided.

After cycling for 1 hour 30 minutes, the girls reach a village called Catworth where there is a signpost showing 18 miles from Corby and 30 miles to Cambridge. "Okay", Monica says, "this is going well."

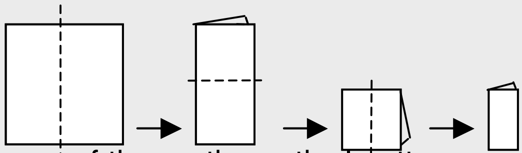
1. Could you tell why she might say this?
2. How much time has it taken them to get to Catworth? And what is the distance they have covered?
   So what can you say about the speed of Monica and Kim? You can use the table to judge their speed.

Cycling at a slow speed:	8 miles per hour
Cycling at a normal speed:	12 miles per hour
Cycling at a fast speed:	18 miles per hour

3. In the table, the speeds of various kinds of cyclist are given. However, if you want to compare the speeds of cyclist who are not riding the same road on the same day, conditions might be different.
   Could you mention the things that have to be taken into account, if we were to measure the speed of a cyclist.
4. After a short stop, Monica and Kim are moving on. They get on the road from Catworth to Cambridge, a distance of 30 miles. At about what time do you think they will arrive in Cambridge?
5. Of course, you cannot be absolutely sure about how long it will take them.
   Could you mention some reasons why you cannot be sure? Still, to make a sensible guess, it might be helpful to know how much time she would need if she were to keep up the same speed.
6. How much time would the ride to Cambridge take if they were to keep up the same average speed as before?

Alex and Abraham's task

This task is based around the practical activity of folding a 32x32 square piece of paper, as shown below. There are then a series of questions to address, some of which use a spreadsheet. In the pupils' materials some guidance for using the spreadsheet is included, which has been omitted here.



2. Describe some of the mathematical patterns you notice as you fold the shapes.
3. ***Predict***: What is the pattern of change in the perimeter, as you fold the shapes?

4.a. Write on the drawings the ***dimensions*** and the ***perimeter*** of the first four shapes in the sequence.

b. ***Collect your data in a spreadsheet table*** that shows the dimensions and the perimeter of the first ten shapes in the sequence.

5. ***Draw a graph*** to show the perimeter of the first ten squares and rectangles in the sequence.
6. Look for ***patterns*** that describe the change in the perimeter, as the square is folded. Explain the connection between your patterns and the folding shapes.

7.a. The teacher asked: *By how many length units does the perimeter get shorter at each folding?* Daniel replied: *At each folding the perimeter gets shortened by the same length.* Do you agree with Daniel?

b. ***Collect data*** that may help you to answer the teacher's question.

c. Do you see any ***patterns*** in the collected data? Explain the connection between your patterns and the folding shapes.

d. Did you change your initial opinion about Daniel's answer? Explain why you did or did not.

8.a. The teacher asked: *By what ration does the perimeter get smaller at each folding?* Daniel answered: *At each folding the perimeter of the new shape is half the perimeter of the previous one.* Do you agree with Daniel?
(b, c and d as for question 7)

9.a. Find pairs of shapes that have a **perimeter ratio of one half**.

b. Give a "rule of thumb" for finding such pairs.

c. Convince a fried why your rule always works.

Janet and Dave's task

For this task pupils have measuring tapes, a spreadsheet. Each group also has a different item of dolls' house furniture.

Children in a primary school want to make a 'dolls' house classroom'. Use the piece of furniture you have been given to work out what size they should make some other objects for their classroom.

DIFFERENT PERSPECTIVES ON TASK DESIGN

The four tasks presented here offer significant differences in the kind of activity that pupils may be engaged in when working on them, but they also arise from different approaches to task design. These are explored and elaborated within the individual papers, but we also draw attention here to one issue which may be discussed within the Forum sessions: the role of the teacher.

Gravemeijer, van Galen and Keijzer emphasise the central role which they see the teacher as playing when a class is working on the task in guiding discussion to focus on mathematical issues and the development of tools to support proportional reasoning. De Bock, Van Dooren and Verschaffel have designed a task which it appears pupils may work on independently, but they also acknowledge the potential role of the teacher in encouraging whole class discussion around the task. Friedlander and Arcavi have constructed a task made up of a sequence of questions, which balances structured questions with more open invitations to make conjectures. Some of the questions are based on hypothetical conversations between the teacher and a pupil, and clearly offer support for pupils to work independently, or for an inexperienced teacher to use the materials. Ainley and Pratt's task is stated very briefly. There is clearly a crucial role for the teacher, who would need an understanding of the approach, in leading discussion to explore and develop the task, but the authors also contrast the activity of pupils who need to rely on continuing support from the teacher, and those for whom the task itself determines the direction of their activity.

NOT EVERYTHING IS PROPORTIONAL: TASK DESIGN AND SMALL-SCALE EXPERIMENT

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INTRODUCTION

Proportional (or linear) reasoning is a major tool for human beings in many cultures to interpret real world phenomena (Post, Behr, & Lesh, 1988; Spinillo & Bryant, 1999), even when the phenomena are not linear 'stricto sensu'. Therefore, not surprisingly, proportional reasoning constitutes one of the major topics in school mathematics from the lower grades of the elementary school to the lower grades of secondary school. From Grades 2 and 3 onwards children learn to multiply and divide and to apply these operations in simple word problems like "1 pineapple costs 2 euro. How much do 4 pineapples cost?", which are predecessors of proportional reasoning tasks. During Grade 4 and afterwards, proportional reasoning skills are further developed. From this age on, students are frequently confronted with proportionality problems, most often stated in a so-called missing-value structure such as: "12 eggs cost 2 euro. What is the price of 60 eggs?", and are trained to set up and solve the corresponding proportion $12/60 = 2/x$ for the unknown value of x. However, in the last decade, mathematics educators formulated two main deficiencies of this current school practice for teaching and learning proportionality.

First, because almost all proportional tasks students encounter at school are formulated in a missing-value format – and at the same time, non-proportional tasks are very rarely stated in this format – students tend to develop a strong association between this problem format on the one hand and proportionality as a mathematical model on the other hand. Recently, De Bock (2002) provided empirical evidence for that claim. In a series of exploratory studies in one specific mathematical domain, namely, problems about the relations between the linear measurements and the area or volume of similarly enlarged or reduced geometrical figures (such as the dolls' house context in Janet and Dave's task), it was shown that 12-16-year old students have an almost irresistible tendency to improperly apply direct proportional reasoning to length-area or length-volume relationships, especially when the problems are stated in a missing-value format. Changing the problem formulation by transforming the problems into a "comparison format" proved to be a substantial help for many students to overcome the trap of inappropriate proportional reasoning in this domain. This study – together with analogous findings by other researchers – suggests that teachers should at least bring more variation in proportionality tasks and especially take care that these tasks are not *always* formulated in a missing-value format.

Second, as reflected in the *Standards 2000* (National Council of Teachers of Mathematics, 2000, p. 217), "facility with proportionality involves much more than setting two ratios equal and solving for the missing term. It involves recognising quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship". In the same respect, Schwartz and Moore (1998, p. 475) explicitly stated that "when proportions are placed in an empirical context, people do not only need to consider at least four distinct quantities and their potential relationships, they also need to decide which quantitative relationships are relevant." The example they gave relates to mixing 1 oz. of orange concentrate and 2 oz. of water, compared to mixing 2 oz. of orange concentrate and 4 oz. of water. If the question is which mixture will taste stronger, the ratios should indeed be compared, but if the question is which mixture will make more, a ratio comparison is of course inappropriate. The claim for the unwarranted application of proportionality was made even stronger by Cramer, Post and Currier (1993, p. 160). They argued that "we cannot define a proportional reasoner simply as one who knows how to set up and solve a proportion".

For the design of a task, we focussed on students' ability to discriminate between proportional and (different types of) non-proportional situations.

DESIGN OF A TASK

Inspiration for the task design was found in a recent study by Van Dooren, De Bock, Hessels, Janssens and Verschaffel (2005). These researchers studied how students' tendency to overgeneralise the proportional model develops in relation to their learning experiences and their emerging reasoning skills. For that purpose, they presented 1062 students from Grade 2 to 8 with a test containing 8 word problems: 2 proportional ones (for which a proportional solution was correct) and 6 non-proportional ones (2 additive, 2 affine and 2 constant). The following are examples of the non-proportional items:

- Additive problem: "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 round. When Ellen has run 30 rounds, how many has Kim run?" (correct answer: 40, proportional answer: 90)
- Affine problem: "The locomotive of a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. How long is the train if there are 8 carriages connected to the locomotive?" (correct answer: 92 m, proportional answer: 104 m)
- Constant problem: "Mama put 3 towels on the clothesline. After 12 hours they were dry. Grandma put 6 towels on the clothesline. How long did it take them to get dry?" (correct answer: 12 hours, proportional answer: 24 hours)

The results showed that many 2nd graders already could solve simple variants of proportional word problems, but the firm skills to conduct proportional calculations (i.e. to solve proportional word problems) were acquired between 3rd and 6th grade. With respect to the non-proportional items, more than one third of all answers contained an erroneous application of the proportional model. The tendency to over rely on proportionality developed in parallel with the ability to solve proportional word problems: it was noticeable already in 2nd grade, but increased considerably in subsequent years, with a peak in 5th grade where more than half of the answers to non-proportional items were proportional errors. After this peak, the number of proportional errors gradually decreased, but they did not disappear completely: in 8th grade still more than one fifth of the answers contained a proportional error. There were some remarkable differences according to the mathematical model underlying the non-proportional problems: One would expect that the word problems with a "constant" model (like the "clothesline" problem mentioned above) were the easiest ones in the test (since there was no need for calculations), but these problems got the highest rate of proportional errors (up to 80% in 5th grade). For some word problems (like the additive "runners" item), the performances even decreased (with 30%) from 2nd to 6th grade. The authors concluded that, throughout primary school, students not only acquire skills to calculate proportions and solve proportional problems. The proportionality scheme becomes so prominent in students' minds that they also begin to transfer it to settings where it is neither relevant nor valid.

For the task that we designed, we worked with the same kind of word problems (4 proportional ones, labelled with the letters C, D, H and J) and 6 non-proportional ones, namely 2 additive, 2 affine and 2 constant, respectively labelled with the letters A and F, E and I, and B and G). The exact formulation of the different problems is given in the introductory section of this research forum. To avoid confusion, we didn't include problems for which the proportional model gives a more or less good approximation, but one can discuss its accuracy on the basis of realistic constraints (such as it is the case in the task of Koeno, Frans and Ronald). Although all ten problems in our task have an exact numerical answer, the task that we gave the students was not to calculate a numerical answer, but to group the problems in at least two different categories and to explain the motivation for their grouping. To allow at least one other way of grouping than the one based on the underlying mathematical model, two of the proportional problems (D and H) were given with a non-integer internal ration, while all other problems were based on easy, natural ratios.

To clearly explain and illustrate the nature of the task (and, at the same time, to show its open-ended character), we first confronted the participants with 13 cardboard figures (stars, triangles and circles) in three different colours (grey, black and white). Two fictitious students, Tommy and Ann, were asked to help their teacher, Mrs. Jones, to classify these figures. Tommy suggested grouping all figures with the same shape (i.e., a grouping based on a "mathematical" criterion), while Ann proposed to bring together the figures with the same colour (i.e. a grouping based on a "non-mathematical" criterion). Then, it was stated that Mrs. Jones made a series of 10 word problems to use in the math lessons (labelled with the letters A to J), but again, they got all mixed up. Students were asked to do as Tommy and Ann had done and to help Mrs. Jones to classify the word problems. More concretely, they were invited to "look very carefully at the problems and to try to make groups of problems that belong together". After that, they had to answer the following questions:

- Why did you make the groups in that way?
- Ann and Tommy did something different when they made groups of the figures. Can you think of a different way to put the problems in groups? Explain that as well.

A SMALL-SCALE EXPERIMENT

The task was given to four students (aged 11 years): Alice, Freya, Hans and Jonas. The researcher first introduced the task and checked pupils' understanding of the instructions. Then, for about 20 minutes, the children were allowed to read the problems and sort them into groups. As each finished, the researcher directed the pupils to record their reasoning, and then to find other groupings.

Alice worked for about 14 minutes to find a first grouping in three categories: group 1 (A and F, the two additive problems) because "they sound similar", group 2 (B and G, the two constant problems) because "it is all like 'how long will it take this person to do this?' and stuff like that", and group 3 with the six remaining problems (the four proportional and the two affine problems). Alice's grouping is based on the underlying mathematical model of the problem, although she was unable to articulate this criterion. In her grouping, she made no distinction between the "pure" proportional problems and the affine problems (which, in fact, ask for a combination of multiplication and addition). After the researcher insisted, Alice came with a second (rather superficial) grouping into two categories (discriminating the problems with "how" and the problems with "what" in the problem statement).

Freya needed about 14 minutes to find a first grouping into three categories: group 1 (H), group 2 (B, C, D, E, F, I and J) and group 3 (A and G). She explained her criterion as follows: "I made the groups due to the operation you have to do to work out the answer. E.g. in group 2, you have to do multiplication to find the answer, and in group 3, you have to divide to find the answer". Clearly, Freya's actual grouping was not based on the criterion she formulated. Being invited by the researcher to find other ways of grouping, Freya proposed a second grouping in three categories: group 1 (A, B, C and F), group 2 (D, E, G, I and J) and group 3 (H) and gave the explanation "I sorted my groups in this way by how easy, moderate or hard the questions were to work out".

Hans who worked for about 19 minutes before coming up with a first grouping also proposed three categories: group 1 (C, D and I), group 2 (A, B and E) and group 3 (F, G, H and I), explaining the motivation for his grouping as follows: "because group 1 is 'times question', group 2 is questions you divide by and group 3 are add and multiply". We cannot see any rationale in Hans' grouping, nor a link between his actual grouping and the explanation he gave for it. After the researcher directed Hans to find a second set of groupings, Hans came with a categorization in four distinct groups: group 1 (A, E and H), group 2 (B and C), group 3 (I and J) and group 4 (D, F and G), but, once more, his justification remained unclear for the researcher.

John, who worked for about 17 minutes, found a classification into two different groups: group 1 (C, E, F, G, H and J) and group 2 (A, B, D and I). He rather superficially explained the motivation for his grouping as follows: "I made these groups because I think it was the most common way and I managed to make them into two groups without any left over". After directed to find a second grouping, John proposed four categories: group 1 (E), group 2 (A, C, I and J), group 3 (B, G and F) and group 4 (D and H). He now explained: "I put them into groups of weight, time and number (respectively groups 2, 3 and 5) and I could not find a group for the 'train and locomotive' one (problem E)" (which is not in line with John's actual grouping).

CONCLUDING REMARKS

The scale of the experiment was very small, so one can hardly infer definite conclusions from it. We observed that the four participating students showed great difficulties in making and motivating classifications of the ten word problems. They mainly looked for linguistic or other superficial differences between the problem formulations and not for an underlying mathematical structure. Possible explanations refer to the nature of the task and the type of problems that we used.

With respect to the task, one can argue that, these students were unfamiliar with classification tasks. Typically, students are expected to "solve" mathematical problems, i.e., to give numerical answers (most often based on the numbers given in the problem formulation), and not to classify problems. Moreover, in retrospect, we think the task was also rather difficult or too "abstract" for 11-year old students. A possible alternative approach meeting more or less the same goals would have been to ask students to combine different problem statements with correct and incorrect (proportional or non-proportional) solution strategies provided by the teacher or experimenter.

With respect to the problems we used, one can argue, in line with Ainley (2000) and several other authors, that the "word-problem" format is inadequate or insufficient to meaningfully contextualise mathematics in the mathematics classroom. Several authors (e.g. Reusser & Stebler, 1997) showed the beneficial effect of meaningful, authentic tasks also for problems where students inappropriately tend to apply linear methods. In this respect, Van Dooren, De Bock, Janssens and Verschaffel (2005) recently showed that students' problem-solving behavior strongly improves when non-linear problems are embedded in a meaningful, authentic context and students are invited to perform an authentic action with concrete materials (i.e. when students are invited to cover a dollhouse floor with "real" tiles instead of calculating this number of tiles in a word-problem context).

Notwithstanding these limitations and shortcomings and the rather disappointing results of our small-scale experiment, the various reactions of the four participating students also suggest that that this type of task design can be a rich starting point for significant classroom discussions on mathematical modelling: which operation is needed in a given problem situation?

DESIGNING INSTRUCTION ON PROPORTIONAL REASONING WITH AVERAGE SPEED

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Instructional design in Realistic Mathematics Education aims at both fostering student reasoning, and at putting instructional tasks in a perspective of long-term learning processes. We try to illustrate this with a task on reasoning about average speed.

TASK DESIGN

There is a long history of instructional design, within which instructional tasks were designed with a primary focus on behavioral objectives. Central instructional design strategies were task analysis and the construction of learning hierarchies. Lessons would be planned on the basis of well-defined prerequisites and precise lesson goals. Teachers were expected to evaluate each lesson by assessing whether those goals were reached at the end of the lesson.

Today this type of instruction is criticized as being ‘instructionist’ or as reflecting a ‘transmission model’ of teaching. In contrast to teachers *instructing*, the emphasis is now on students *constructing*. Following Cobb (1994) we may argue that constructivism—as an epistemology—does not have direct implications for teaching, as “the constructivist maxim about learning may be taken to imply that students construct their ways of knowing in even the most authoritarian of instructional situations” (Cobb, 1994, 4). Still, constructivism may inspire one to consider how we can influence the construction processes of the students. One of the results of such considerations is a shift in attention from behavioral objectives to the mental activities of the students. In this respect, we may refer to Simon’s (1995) notion of a hypothetical learning trajectory. We may notice the flexibility and the situatedness of this concept. A teacher will design a hypothetical learning trajectory for the students in his or her classroom, given where the students are at this moment, while taking into account goals and teaching practices. Moreover the teacher will adjust the hypothetical learning trajectory on the basis of his or her interpretation of how the students act and reason. This puts the notion of task design in a different perspective. What the task entails is not fixed, as tasks are interactively constituted in the classroom. When we expect teachers to orient themselves on the mental activities of the students, and consider those in relation to the intended end goals, we might argue that teachers should be supported in making these considerations.

In the Netherlands we constructed an instructional design strategy, which is aimed at developing prototypical instructional sequences and local instructional theories that are to offer teachers a framework of reference for constructing their own hypothetical learning trajectories. This strategy is based on what is called design research and on the use of three design heuristics from realistic mathematics education (RME), namely, guided reinvention through progressive mathematization, didactical phenomenology, and emergent modeling. In the following paragraphs we explain this in more detail.

Design research can be thought of as a combination of design and research aimed at developing both a sequence of instructional activities and a local instructional theory. A classroom teaching experiment forms the core element of this type of research (Gravemeijer, 1998). This consists of an interactive and cumulative process of designing and revising instructional activities. To this end, the designer conducts anticipatory thought experiments by envisioning both how proposed instructional activities might be realized in the classroom, and what students might learn as they engage in them. These instructional activities are tried out in the classroom. Then, new instructional activities are designed or redesigned on the basis of analyses of the actual learning processes. At the end of a cumulative process of designing and revising instructional activities, an improved version of the instructional sequence is constructed. After some design experiments, the rationale for the instructional sequence eventually acquires the status of a local instructional theory.

The other core element of our instructional design strategy is the use of the three design heuristics that characterize the domain-specific instruction theory of RME. This educational theory originated in the Netherlands inspired by Freudenthal's idea of mathematics as an activity of organizing or mathematizing. The first heuristic has to do with Freudenthal's (1973) idea that students should be given the opportunity to experience a process similar to the process by which mathematics was invented, and is called guided reinvention through progressive mathematization. According to this heuristic, the designer takes both the history of mathematics and the students' informal solution procedures as sources of inspiration (Streefland, 1990), and tries to formulate a provisional, potentially revisable learning route along which a process of collective reinvention (or progressive mathematization) might be supported.

The second heuristic concerns the phenomenology of mathematics, and asks for a didactical phenomenological analysis. The developer looks at present-day applications in order to find the phenomena and tasks that may create the need for students to develop the mathematical concept or tool we are aiming for. The goal of a phenomenological investigation is, in short, to find problem situations that may give rise to situation-specific solutions that can be taken as the basis for vertical mathematization.

In the instructional design we are reporting in this paper, the focus is on the emergent modeling heuristic (Gravemijer, 1999). Models in RME are related to the activity of modeling. This may involve making drawings, diagrams, or tables, or it can involve developing informal notations or using conventional mathematical notations. It is important that these notations have the context situation of the problem as starting point and are developed by the students as they attempt to come to grips with the problem and find ways to solve it. The conjecture is that the emergence of the model is reflexively related to the construction of some new mathematical reality by the students, which may be labeled as more formal mathematics. Initially, the models refer to concrete or paradigmatic situations, which are experientially real for the students, and are therefore to be understood as context-specific models. On this level, the model should allow for informal strategies that correspond with situated solution strategies. As the student gathers more experience with similar problems, the model gets a more object-like character, becoming gradually more important as a base for mathematical reasoning than as a way of representing a contextual problem. The *model of* informal mathematical activity becomes a *model for* more formal mathematical reasoning.

THE DESIGN TASK: (UN)JUSTIFIED PROPORTIONAL REASONING

In the context of the research forum, we were asked to design a single task on proportional reasoning, while also addressing the issue of unjustified proportional reasoning. We chose a task on speed. Reasoning about speed in everyday-life situations asks students to coordinate pure proportional reasoning with realistic considerations on what may distort the proportionality in actual reality. The task we designed was a problem about two girls who make a bicycle trip. After 1 1/2 hour they pass a signpost telling them that they have already cycled a distance of 30 kilometers, and they still have 45 kilometers to go. In the story one of them comments: ‘This is going well’, and the question the students have to answer is why she would say so. There were five more questions, but, in a sense, the first one covers them all; the other questions discuss the relevant points more explicitly. The remark ‘This is going well’ is expected to raise a discussion about questions like:

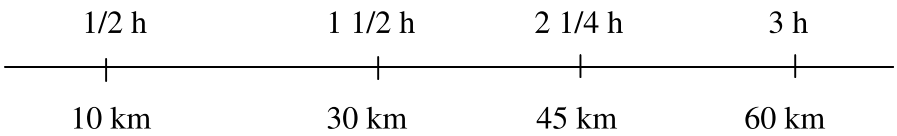
- Is 30 kilometers in one hour and a half an achievement one would be happy with? What would have been their speed, in terms of kilometers per hour, and would that be fast, or slow?
- The girl might be happy because she sees that they have done a big part of their trip already. So what is the relation between the 30 kilometers and the distance the girls still have to cycle? Would it be possible to estimate how much time they need for the rest of their trip?
- Will a calculation lead to an exact prediction, or are there other factors to take into account?

Note that the numbers were chosen carefully as to make easy computations. The task was tested both in the Netherlands and in the UK; the English version was about a trip from Corby to Cambridge, with 18 miles done and 30 miles still to go. Note also that there are various ways to calculate the time needed for the second part of the trip. Students can compare 30 km and 45 km and conclude that the second part will take 1 1/2 time as long, they might see that 30 km in 1 1/2 hour gives 10 km in half an hour and reason from this, or they might calculate the average speed in km per hour.

The student activities that we anticipate are threefold:

- The students will (start to) reason proportionally in the context of speed.
- The students' explanations will allow the teacher to start a discussion about how to record proportional reasoning on paper. This could be a lead in to a discussion about the use of models like the double number line or the ratio table.
- The students will realize that proportional reasoning does not predict the arrival time in a precise manner, but do realize that calculations are a useful tool in making estimations.

Models for proportional reasoning, and therefore also for reasoning with average speed, are the double number line and the ratio table. They both offer a systematic way of writing down the relation between distance and time. On the double number line the position of points is meaningful, whereas the columns of the ratio table can be in any order. Both models can function as a tool, allowing one to break down complicated calculations into intermediate steps.



time	1 1/2 h	1/2 h	2 hs	2 hs 15 min
distance	30 km	10 km	40 km	45 km

In our view students should be stimulated to reinvent these models; they should not be offered as a ready-made products. This does not mean that students are expected to reinvent the exact way numbers are written in rows and columns in the ratio table, but they should be stimulated to think about systematic forms of notations, and thereby learn to appreciate the 'official' ratio table as one of the possible forms.

Following the emergent modeling perspective, the students' activity with double number line and ratio table will be grounded initially in thinking about its contextual meaning. Doubling in the ratio table, for example, will be justified by thinking of traveling twice as long. Later the ratio table may be used for reasoning with linear relations. As we argued elsewhere, students may eventually start to use the ratio table in a semi-algorithmic manner to execute multiplications, without necessarily having to think of possible contextual meanings of the numbers involved (Gravemeijer, Boswinkel, Galen, & Heuvel-Panhuizen, 2004).

SOME FINDINGS

The task was tested twice, once with a small group of four students in England and once in a class with 10 to 12 year old students in the Netherlands. In the experiment in England the teacher introduces the problem by focusing heavily on exploring the situation and the circumstances that influence the time one needs to cycle from Corby to Cambridge. The situation is meaningful enough for the students to bring forward many aspects that could influence the cycling time. They mention that the time to travel the whole distance could be influenced by the weather, the hills alongside the route, the breaks the girls take, etcetera. In this setting the students developed ideas on how much time it takes to cycle the whole tour, but the numbers they bring forward are mostly guesses. They agree that it should take the children at least two hours to ride the 30 miles from the road sign to Cambridge. Only two students replace their guesses about the time needed to cycle from Corby to Cambridge by calculations and schemes.

The Dutch experiment also starts with an exploration of the context. As the students here are more familiar with a bike a means of transport, they easily bring forward what should be done if one undertakes a tour as mentioned in the task. When next the students receive the worksheet with the map and the road sign, they find little problem in interpreting the situation. The teacher here, like her English colleague, discusses one of the girls saying ‘This is going well’, when they arrive at the road sign.

In the Dutch version of the task in took the children one and a half hours to cover the first 30 kilometer. At that point there is still 45 kilometer to go. The students formulate several arguments why 30 kilometer in one and a half hour is quite a distance for such a short time.

The teacher frequently asks the students to explain their ideas. Therefore the discussion focuses more and more on mathematical arguments. One of the students for example claims that he cycles 3 kilometers in a quarter of an hour. He argues that in that speed it takes one and a half hours to cover 18 kilometers. 30 kilometer in one and a half hour therefore is fast cycling.

Unlike her English colleague, the Dutch teacher at certain points redirected the discussion to the use of mathematical arguments. The Dutch students therefore all reasoned in terms of ratios to calculate the arrival time. Moreover, the arrival time is next discussed in terms of the context, where the students decide to add about an hour for breaks, flat tires and weather conditions.

We were in the fortunate position to thus find two settings where the teachers both choose a different manner to guide the students. This enabled us to analyze the teacher’s role and to test (in this specific context) our ideas on this. We noticed that the Dutch students did not have any problem with putting their calculations into perspective. They could easily compute how much time would be needed for the next 45 km, but it was also obvious to them that such calculations only give you a first approximation. In the English experiment the students were aware that one could only estimate the arrival time, but the setting did not stimulate them to further mathematize the problem.

CONCLUSION

In Realistic Mathematics Education instructional design concerns series of tasks, embedded in a local instruction theory. This local instruction theory enables the teacher to adapt the task to the abilities and interests of the students, while maintaining the original end goals. The task we designed should be viewed from this perspective. In an educational setting it would not be an isolated task, but part of a longer learning route. Goals of such a learning route would be:

- Students learn to reason proportionally.
- They develop tools for proportional reasoning, tools that can also be used for calculations, like the double number line and the ratio table.
- At the same time, however, they learn to see the relativity of their calculations; when making predictions other factors in the context may have to be taken into consideration.

When our task was tested, the emphasis was on the third goal. Within a longer learning route, however, the challenge would be more to help students develop the right tools for proportional reasoning. Among other things, these tools would help children to discriminate between situations where proportional reasoning is, and is not justified. RME describes this process of developing mathematical tools as emergent modeling.

In the test situations there was no discussion, or only a limited discussion about tools like the double number line and the ratio table. Within design cycles of testing and revising this could lead to the decision to make certain changes, in this case, for example, to change the numbers in such a way that students would not be able to do the calculations in their heads. But even when an activity, after some revisions, has found its definite form, success cannot be guaranteed, of course. This underscores the central role of the teacher in supporting the learning process. The teacher should be capable to make changes, like asking certain questions, focusing the discussion on certain topics, and so on. An essential condition to establish this is, that the teacher knows and understands the local instruction theory behind the activities.

FOLDING PERIMETERS: DESIGNER CONCERNS AND STUDENT SOLUTIONS

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In this paper we first describe some of the concerns and approaches that have influenced the process of designing the *Folding Perimeters* activity. Then, we will present several selected episodes from the actual solutions produced by two pairs of 12-year-old, higher ability students, in view of the design concerns that were encountered in the development of this activity.

TASK CHARACTERISTICS

Folding Perimeters was designed as the last and most advanced activity in a learning series on ratio and proportion. This section describes the main characteristics of the activity, and some considerations that led to its present design.

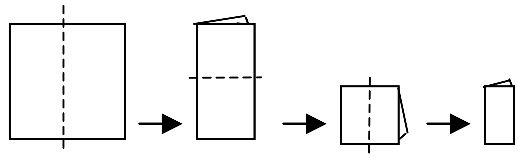

Figure 1. Context of *Folding Perimeters*

Context. In this activity, students investigate the perimeters of an alternating sequence of squares and rectangles, during a process of repeated folding-in-two (Fig. 1). The use of context enables a constructivist path of learning (Hershkowitz et al., 2002). When students start with a problem situation such as the above, they can rely on their acquaintance with its non-mathematical components and on their ability to observe, to experiment and to act on the situation itself. As indicated by Ainley and Pratt in this collection of papers, the characteristics of a task may also contribute to provide a sense of purpose and ownership. Moreover, a problem situation can also contribute to students' understanding of the need for constructing appropriate tools and concepts, first investigating the problem at an intuitive level and later on, analysing the newly formed tools and concepts in a more extended and mathematically formal manner. Tourniaire and Pulos (1985), in reviewing the research on proportional reasoning, concluded that context plays a crucial role in student performance and that use of a wide variety of contexts is needed in the teaching of this domain. In our case, we considered the context of paper folding to be simple and familiar, on the one hand, and to be rich in mathematical opportunities on the other hand.

Mathematical content. The activity integrates various mathematical domains - for example, geometry (squares, rectangles, perimeters, opposite sides, measurement), arithmetic (numerical tables, operations, difference, ratio), and algebra (*Excel* formulas and pattern generalizations). The mathematical content is stated clearly throughout the activity, and is one of the factors that determine the sequence of tasks. The first three tasks in the activity require a more geometrical and visual investigation, there is a task that relates to the differences between the perimeters of two adjacent shapes, and the last two tasks focus respectively on the perimeter ratios of two adjacent, and of every other shape. However, some other tasks in the activity are less directive with regard to content or solution strategy open. More specifically, these tasks require students to find any patterns of perimeter change and justify them. Similarly to Dirk, Wim and Lieven’s task, the patterns of change in our activity do not constitute a classical and straightforward application of the idea of proportionality, common in many textbooks.

Multiple representations. The presentation of mathematical concepts and operations in various representations is central in investigative activities (Friedlander & Tabach, 2001a). One of our reasons for using spreadsheets as a mathematical tool is their ability to simultaneously support work on various representations, and to present the algebraic representation as an efficient and meaningful means of constructing data. In our activity, students are specifically required to present perimeters and perimeter changes in actual paper, in drawing, in numerical tables, as algebraic formulas, in bar diagrams, and in verbal descriptions. Some of the tasks focus on the construction and use of a specific representation, whereas others leave this issue open to the students. Figure 2 presents a numerical and graphical representation of the data and some of the results obtained by the observed students, regarding the alternating sequence of shapes in the activity. Some of the algebraic formulas used by the observed students will be discussed in the next section.

	A	B	C	D	E	F
1	fold #	side	side	perim.	differ.	ratio
2	0	32	32	128	32	3/4
3	1	32	16	96	32	2/3
4	2	16	16	64	16	3/4
5	3	16	8	48	16	2/3
6	4	8	8	32	8	3/4
7	5	8	4	24	8	2/3
8	6	4	4	16	4	3/4
9	7	4	2	12	4	2/3
10	8	2	2	8	2	3/4
11	9	2	1	6	2	2/3
12	10	1	1	4		

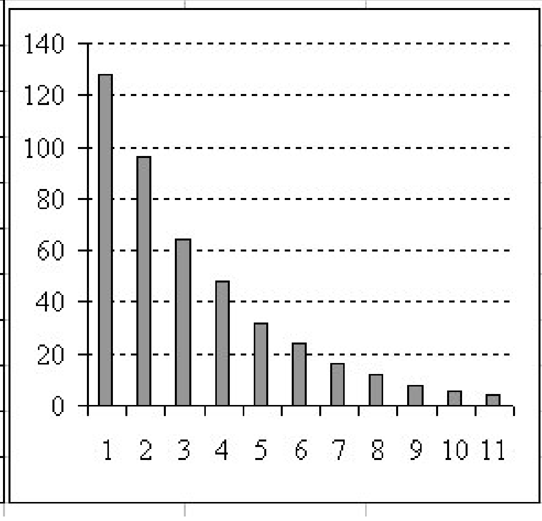

Figure 2. Spreadsheet representation of data and results in *Folding Perimeters*.

Task sequencing. Investigative activities (including *Folding Perimeters*) frequently follow a flow pattern that is in many ways similar to the PCAIC investigative cycle (*pose*, *collect*, *analyze*, *interpret, and communicate*) proposed by Kader & Perry (1994). This cycle is adapted from the domains of data investigation and scientific research, and is inductive in nature. First, specific cases are collected, organized, and analyzed, and then general patterns are formed and conclusions are drawn, interpreted and applied.

Generalization of patterns. Many activities associated with generalization – including ours, assume that the process of pattern generalization is inductive and based on a limited number of cases. In the next step, the discovered pattern is explained and justified (Friedlander et al., 1989). This flow pattern is frequently used in the design of generalization tasks. In our activity, this sequence of tasks is applied in several cycles, with regard to any patterns of perimeter change, then regarding the difference, and finally regarding the ratio of perimeters of two consecutive shapes.

Level of task openness. The process of task design is based on a constant state of tension that exists between the design of unstructured open tasks that do not require that the problem posed be solved by a specific method, a certain representation or an implicitly given sequence of steps, as opposed to a structured approach that poses specific requests with regard to the variables mentioned above. The open approach reflects the designers' striving to develop problem solving skills, to develop creative mathematical thinking, to provide opportunities for students to actually experience investigation, and to achieve a meaningful construction of knowledge. The structured approach enables students to pursue a more predictable and planned agenda in the domains of mathematical content and the processes of problem solving. The activity discussed here addresses this issue by presenting a sequence of tasks of both kinds. Open tasks require students to identify any properties of the presented sequence of shapes, make predictions, and then look for patterns that describe the change in perimeter. Tasks that are more directive require the student to collect data for the first ten shapes in the sequence, organize it in a spreadsheet table, present it as a diagram, investigate patterns of perimeter change by considering first the difference and then the ratio between pairs of adjacent shapes, and of shapes placed in the sequence at a distance of two steps. One may argue that leading students through a sequence of tasks, rather than presenting only a problem situation and a "big question", decreases in itself the extent of freedom in student work. We suggest, "walking a fine line" between opening and closing a task by directing students to some extent through a sequence of leading questions, within an open problem situation. This approach to task design supports a convergence towards a meaningful progress in the students' solution, without curtailing their sense of ownership of the task (in the sense of Ainley and Pratt in this collection of papers). Such a sense of ownership stems from the opportunity to observe, experiment and act on a "realistic" situation, and not necessarily from the task's degrees of freedom.

Verbalization. Requests for descriptions of patterns, explanations, discussions of another (fictitious) student's solutions and reports of results are included in this, as well as many other activities. These requests are the result of designers' desire to develop communication and documentation skills, to make students consider verbal descriptions as mathematical representation, and to change the stereotypic view of mathematics as the exclusive domain of numerical and algebraic symbols only.

Use of spreadsheets. Our experience of students working in a spreadsheet environment shows that spreadsheets can serve as a powerful tool, and allow for some of the design heuristics proposed by Gravemeijer and his colleagues in this collection of papers. They support students' processes of creating emergent models and their "vertical mathematization" of the problem situation. The use of this technological tool to support and promote processes of generalization and algebraic thinking has been amply discussed in terms of theory and investigated empirically (for design considerations in spreadsheet activities, see for example, Hershkowitz et al., 2002; Friedlander & Tabach, 2001b). Because of space limitations, we will only briefly list the following considerations that led the designers to use spreadsheets in this particular activity:

- they serve as a powerful tool for data collection, organization and representation,

- they provide continuous and non-judgmental feedback throughout the solution process,
- they present the concept of proportion dynamically, as a sequence of constant ratios obtained by applying the same rule to numerous pairs of numbers or quantities,
- they enable the analysis of an extended collection of data,
- they emphasize the meta-cognitive skills of monitoring and interpreting results,
- they promote algebraic thinking and present algebraic formulas as a useful and meaningful tool.

STUDENT SOLUTIONS

As previously mentioned, two pairs of students (referred here by the initials of their first names as MS and MG) were observed by one of the authors as they worked on the *Folding Perimeter* activity, during a period of about 80 minutes for each pair. For the purpose of this paper, we will not distinguish between the two members of a pair, and will refer to each pair as an entity. The students had previous experience in using *Excel* in mathematical investigations, but had not pursued the learning sequence of ratio and proportion that included our activity. The interviewer's interventions were minimal and limited to occasional requests to clarify answers or to start working on the next item. The latter case included dealing with "unproductive" paths of solution – defined by Sutherland et al. (2004) as cases of "construction of idiosyncratic knowledge that is at odds with intended learning", and require the teacher's intervention in regular classroom situations. A systematic analysis of student work, according to the eight designer concerns described in the previous section is not possible, because of the space limitation.

In general, the students followed the prescribed sequence of tasks and solved them in a mathematically rich and resourceful manner. However, we will focus here on some differences between the observed students' solution processes and the designers' plans and predictions.

Contrary to our expectations (see the comments on task sequencing and generalization of patterns in the previous section), both pairs reached, at the initial stage of predictions, generalizations that were "scheduled" by the designers to be reached only later on, and on the basis of the collected data. By examining their folded paper square and the drawing of the folding process (Fig. 1), the students considered visual and global aspects regarding the sides that were "lost" through folding, and made the following predictions:

MG: It [the perimeter] gets smaller by the length of the side that gets halved.

MS: In my opinion it [the perimeter] will be 3/4. The vertical lines will stay and the horizontal lines lose one half and one half – and that's a whole side. [After Interviewer asks "And what happens from the second to the third shape?"] It comes out 4/6 because we are left with 4 out of 6 halves [of the longer sides of the rectangle].

Both pairs produced general patterns at a very early stage of the activity - MG is reasoning additively, by looking at differences, whereas MS is thinking proportionally, by considering ratios. The issue of interest for designers and/or researchers is that the processes of pattern generalization can follow two routes:

- inductive generalization based on the collection and analysis of data (as followed by the sequence of tasks in this activity),
- deductive generalization based on a global analysis of the problem situation, and on general reasoning (as followed by the two pairs of students).

We assume that both the students' mathematical ability and task design (e.g., the representation used in the initial description of the problem situation) affect the choice of the route.

The use of spreadsheets was also a source of unexpected developments. The observed students did not encounter any technical difficulties with regard to the handling of the tool. They read, understood, and performed the computer-related instructions, and were familiar with the *Excel* syntax for writing formulas. However, the following three episodes observed during the students' work indicate that the spreadsheet's intrinsic properties can provide opportunities for higher-level thinking, and help both the student and the teacher detect and relate to conceptual difficulties.

a) MS: They construct the spreadsheet table for the first ten shapes (see Fig. 2). They write in the first line of the perimeter column (for the perimeter of the original square) the formula =4*B2 and in the next line (for the perimeter of the rectangle produced by the first folding) =2*B3+2*C3.

"But we can't drag down [two formulas]…Then let's change this [the first formula] into this [the second]". They rewrite the formula for the square as =2*B2+2*C2 and drag it down.

b) MG: They write for the length of sides (see Fig. 2) a formula (pattern) indicating the halving of the above-situated cell, and drag it down cell by cell – one cell at a time, hoping that this method would produce the desired sequence of pairs of identical numbers.

c) MG: They construct the column for the difference of adjacent perimeters (see Fig. 2) by writing in the first line the formula =D2-D3 and dragging it down to the last line. As a result, the last number shows an uncharacteristic increment in the difference sequence (….8, 8, 4, 4, 2, 2, 4) – a result of the difference of the last perimeter (4) and the next empty cell that is interpreted by Excel as zero. They notice the outcome, retype the same pattern (=D3-D4) in the second line and again drag it down to the last line –obtaining of course the same results as before.

In episode (a), work on *Excel* provided an opportunity to perform a higher-level analysis for students without any background in formal algebra: they compared two algebraic expressions and identified one (4B) as a particular case of the other (2B+2C, when B=C). However, episodes (b) and (c) showed that the observed pair of students thought that changing the place or the physical handling of a pattern expressed as an algebraic formula will change its essence.

CONCLUSIONS

The considerations related to the design of the *Folding Perimeter* activity are closely connected to a wide variety of theories and research findings on student cognition, and on the use of technological tools for teaching mathematics. Our experience in implementing many similarly structured investigative activities indicates that they provide opportunities for meaningful learning of mathematical concepts.

We also described here several episodes of student work on a particular activity to show that differences between a designer's planned actions and student work should be expected. Whether, and if so how, these differences should influence the design of this particular activity or the principles of task design remains an open question.

THE DOLLS' HOUSE CLASSROOM

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The design of our task uses the framework of *purpose and utility* (Ainley & Pratt, 2002, Ainley et al., forthcoming). *Purpose* reflects our concern to create tasks which are meaningful for pupils. One strand of research on which we draw is that of mathematics in out-of-school contexts (e.g., Nunes et al., 1993) which has highlighted the contrast between the levels of engagement of learners in mathematical activities in and out of school. In a PME plenary, Schliemann (1995) claimed '*we need school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers. And such situations may be very different from everyday situations.*' (p. 57). We argue that setting school tasks in the context of 'real world' situations, for example through the use of word problems, is not sufficient to make them meaningful for pupils. Indeed, there is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide a purpose that makes sense to pupils (see for example Ainley, 2000; Cooper & Dunne, 2000).

We see the *purposeful* nature of the activity as a key feature of out-of-school contexts which can be brought into the classroom through the creation of well designed tasks. Drawing partly on constructionism (Harel & Papert, 1991), we define a purposeful task as one which has a meaningful outcome for the learner in terms of an actual or virtual product, the solution of an engaging problem, or an argument or justification for a point of view (Ainley & Pratt, 2002; Ainley et al., forthcoming). This feature of purpose for the learner, *within the classroom environment*, is a key principle informing our pedagogic task design.

The purpose of a task, as perceived by the learner, may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a ‘real world’ context. The purpose of a task is not the ‘target knowledge’ within a didactical situation in Brousseau’s (1997) sense. Indeed it may be completely unconnected with the target knowledge. However, the purpose creates the necessity for the learner to use the target knowledge in order to complete the task, whether this involves using existing knowledge in a particular way, or constructing new meanings through working on the task. Movement towards satisfactory completion of the task provides feedback about the learner’s progress, rather than this being judged solely by the teacher (Ainley et al., forthcoming). Harel (1998) proposes the ‘necessity principle’, which addresses the issue of creating the need to learn particular things in a different way. In Harel’s terms an ‘intellectual need’ for a mathematical concept should be created before embarking on the teaching of the concept. However, intellectual need and purpose clearly differ, since intellectual need is related specifically to a mathematical concept, while the purpose of a task is not explicitly mathematical, but relates to the outcome of the specific task. The necessity principle perhaps relates more closely to the second construct within our framework: utility.

UTILITY

Understanding the *utility* of a mathematical idea is defined as knowing how, when and why that idea is useful. A purposeful task creates the need to use a particular mathematical idea in order reach the conclusion of the task. Because the mathematics is being used in a purposeful way, pupils have the opportunity not just to understand concepts and procedures, but also to appreciate how and why the mathematics is useful. This parallels closely the way in which mathematical ideas are learnt in out-of-school settings. In contrast, within school mathematics ideas are frequently learnt in contexts where they are divorced from aspects of utility, which may lead to significantly impoverished learning. Utility thus has some similarity to Harel’s ‘intellectual need’. However, Harel sees intellectual need as providing the motivation for learning a concept, whereas utility, why and how the concept is useful, is seen as an intrinsic, but frequently unacknowledged, facet of the concept itself.

THE DOLLS’ HOUSE CLASSROOM TASK

The Dolls’ House Classroom task focuses on scaling, which is a key idea in proportional reasoning. The outcome of the task is a set of instructions for another group of children to make items for the dolls’ house classroom. The purposeful nature of the task would, of course, be increased if the pupils were involved in the actual manufacture of the product. We developed the idea for this task from the work of a primary school class who used a similar approach to building scenery for a play based on the Nutcracker ballet. There was a need to make the scenery large enough for the people to appear the size of rats.

At the beginning of the task, each group of pupils is given an item from a dolls’ house which corresponds to something they will have in their own classroom (e.g., a chair, a table, a door, a computer). The activity of comparing this with its full-size equivalent will involve measuring and discussion, as pupils decide on which are the most important measurements to use. For example, although the particular design of chairs may vary, the height of the seat above the ground remains fairly constant.

Once they have arrived at a pair of measurements for the full-size and dolls' house items, they enter the most crucial part of the task: deciding how the use these in order to scale other measurements. The role of the spreadsheet is important here in allowing pupils to experiment with different ways of using the measurements, and applying them to other items which they decide to include. It is important that there is an opportunity here for the pupils to make decisions about which other classroom items they will use, as this adds to their ownership of the task. We note here a close affinity with Friedlander and Arcavi, who set out in this collection of papers some of the reasons why they also adopted spreadsheets.

The above considerations reflect our practical research and teaching experience as well as our theoretical perspective. In order to illustrate some of the characteristic features of such a design approach in action, we gave the dolls' house task to two pairs of eleven year old students (one pairs of boys and one of girls). It turned out that the girls needed considerably more support than the boys from the teacher/researcher. Interestingly, this had the effect of closing down the task for the girls, who followed a much more one-dimensional route through the problem, staying close to the suggestions of the teacher. In contrast the boys were more adventurous in their approach and were able to exploit the opportunities that the task offered. This contrast acts as a useful reminder that the notions of purpose and utility are design imperatives, which act as potentials for the students but how those potentials are realised will vary according to a range of personal attributes (knowledge, confidence and so on) brought to the situation by the children and the structuring resources of the setting, including *inter alia* the approach of the teacher. (Indeed, we note that all authors in this collection of papers found to a greater or smaller extent that there were discrepancies between the learning trajectory that they had envisaged and that which ensued in practice. We make further comment on this at the end of this section.) As a result of this contrast between the boys and the girls, we focus below more on the activity of the boys, which better illustrates the implications of designing for purpose and utility.

PURPOSE AND UTILITY IN ACTION

We were struck by the relationship between the boys' construction of purpose and utility and how the interplay between the two evolved during the 40 minute session. Initially the boys tried to relate the task to their own experiences. One boy told the teacher about how his grandfather used to make dolls' furniture. The other talked about scaling in maps in response to the teacher's mentioning of the term *scale factor*. From an early stage, the boys questioned the nature of the task that they had been set. (Figures in brackets indicate time elapsed in minutes.)

> [6:06] Is this real? Are a Year 6 class really going to do this?

The researcher admitted that this was not actually going to happen.

[6:35] Why can't they just buy the dolls' house?

What do we make of these questions? Are they challenges that suggest the boys are resisting the invitation of the teacher to engage with the problem? If so, it would be hard to explain the subsequent activity, which was marked by the boys' considerable intent and persistence. Rather, we believe that these questions indicate a process in which the boys were beginning to take ownership of the task, They were, in our opinion, delimiting the task, asking where are its boundaries with reality, recognising that is was important to appreciate the true nature of the task as this would later inform their strategies for its solution.

The task itself continued to act as the arbitrator of the activity (in contrast, the girls required the teacher to direct their activity throughout the session). At one point one of the boys encouraged his partner to move on.

[17:14] You can't just keep doing the table; we've got to do something else.

The boys recognised that there was an implication in the task to build a range of artefacts. It was not necessary to ask the teacher what they should do next.

At times, the boys were even prepared to follow the path indicated to them by the task rather than that suggested by the teacher. Thus, at one point, the teacher asked how the boys would find the height of the little shelf for the dolls' house.

[13:40] Before we do that, won't we have to do the width of this table first?

When students take ownership of a task, the levels of engagement can be very high; it is our belief that the opportunity to make choices is influential in helping students to make a problem their own. Furthermore, a well-designed task will also enable students to follow up their own personal conjectures when they try to make sense of the task. Such personal conjectures might be seen by other researchers as misconceptions but our stance recognises the need, from the design point of view, for students to be given the opportunity to test out for explanatory power their own meanings, in this case for proportion. Thus the boys' spreadsheet shows several different attempts at ratio. In one set of cells, they divided the height of the real table by that of the supplied dolls' table (68.5 / 4.3 = 15.93). But when it came to the width of the table, they divided the dolls' table by the real table (5.5 / 134.2 = 0.040983607). In another part of the spreadsheet, they divided the real shelf width by the real table width (75.5 / 134.2 = 0.562593). Each of these calculations has possible utility for their task but whether any particular approach has explanatory power depends on exactly how the boys wanted to use the result and what sense they could make of the feedback. The nature of the task allowed them to explore all three routes, rather than following a route defined prescriptively by the teacher.

Such explorations enabled the boys to construct meanings for the divisions being carried out on the spreadsheet. The spreadsheet handled the calculations, allowing the boys to focus on whether the ratio was actually useful to them in their task. Even so, the technical demands of deciding what to divide by what could become so absorbing that the context could be temporarily forgotten.

[13:20] So, this table [pause] the height of this table divided by the height of that table [pause] I've forgotten how this is going to help!

Nevertheless, the boys recognised that there was a purpose to this technical effort and they were eventually able to reconstruct the reason behind that work. We see this statement and the subsequent activity as evidence that the boys were indeed linking the purpose of the task to a utility for comparing dimensions. The measurements enabled them to derive a scale factor, which could be used to calculate the dimensions of imaginary objects. The utility emphasises how the scale factor might be useful, admittedly in a situated narrative, rather than the technical aspects of how to calculate a scale factor.

This utility was planned. However, when we design for purpose and utility, there is a strong likelihood of other utilities emerging in unpredictable ways. In well-designed tasks there should be a richness of possibilities. When we listened to the recording of the boys working on this task, we were able to identify unplanned opportunities to focus on a utility for rounding. Thus, consider again the occasion when the boys divided the width of the dolls' table by the real table to obtain 0.040983607.

[17:40] How do you shorten that down?

The boys intuitively knew that it would be useful to reduce the length of the decimal. However, they did not know the technicalities of how to do this. Had the teacher been available at that point, there may have been an opportunity to focus on rounding in the context of making numbers more manageable. In the event the boys moved away from this calculation and considered an alternative approach. Nearly ten minutes later [26:50], another rounding opportunity appeared. On this occasion the numbers were easier and so the boys were able to round manually 8.0665 to 8.1.

Another illustration of the richness of such tasks occurred when the boys were considering the area of the tables.

[13:50] We have to find the area of that *(referring to the dolls' table)* and then the area of one of these tables and then combine the area of…

One of the most difficult ideas in secondary level work on proportion is the notion of an area scale factor and how it relates to a linear scale factor. There was potential here for the students to explore the utility of area scale factors.

FINAL COMMENTS

We advocate stressing in task design how mathematical concepts might be useful in particular situations. Such utility does not imply real world relevance. The dolls' house task is somewhat contrived if judged against such a criterion. Nevertheless, the boys took ownership of the task, partly because they were able to make choices of their own and partly because they were able to construct their own narrative for the task. As the activity evolved, the emphasis on making sense of the task itself by relating it to personal experiences and testing its boundaries transformed into creating solution strategies, guided by the purpose of task. In their efforts to construct meanings for the feedback from the spreadsheet, the boys constructed a utility for scale factor. At the same time, there was a richness in the task that is typical in our experience of tasks designed according to the constructs of purpose and utility. This richness manifested itself in the way that the boys followed numerous paths and stumbled into situations that offered potential for engagement with other mathematical utilities.

We note with interest that all the authors in this collection of papers appear to have attempted to include some aspect of purpose or utility in their task designs, without of course seeing what they did in precisely those terms. Word problems in themselves can appear dry, even hackneyed, but in Dirk, Wim and Lieven's task, the problem was transformed. The children had to work on the word problems at a meta level, deciding which problems were like which others. As De Bock, Van Dooren and Verschaffel subsequently observed, the task proved to be rather challenging but we too have seen in the past that this type of transformation can imbue a sense of purpose to the task for many children. In Koeno, Frans and Ronald's task, there was an attempt to connect children's thinking to their experiences of journeys. The approach seemed to offer the children the opportunity to construct a utility for proportion in relation to planning such journeys. In Alex and Abraham's task, we saw the potential for practical activity, which might even have been opened up further by considering other aspects of paper folding that can lead to other interesting proportions.

Finally, and almost as a cautionary tale, we remind you (and ourselves) that the girls working on our own task went down a much narrower predictable pathway than did the boys. One level of response to this result is simply to argue that no task can offer rich pathways for all children. On the other hand, perhaps there are lessons to be learned, not just from the boys' work, but also from that of the girls. Gravemeijer, van Galen and Keijzer have explained how they see the demands of this research forum as at variance to some extent with their normal activity. The principle of progressive mathematization, utilised by designers in the Realistic Mathematics Education school, is not one that sits easily with designing a single task in one shot. We too see task design in terms of design research and, in that spirit, would interpret all these efforts at task design as "bootstrapping" or first exploratory attempts.

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RF02: GESTURE AND THE CONSTRUCTION OF MATHEMATICAL MEANING

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The role of gestures in mathematical thinking and learning is examined from the perspectives of cognitive science, psychology, semiotics and linguistics. Data from situations involving both children and adults, addressing mathematical topics including graphing, geometry, and fractions, are presented in the context of new theoretical frameworks and proposals for the analysis of gesture, language, signs and artefacts.

INTRODUCTION

Recent research in mathematics education has highlighted the significance of the body and, specifically, perceptuo-motor activities in the process of mathematics teaching and learning (Lakoff & Núñez, 2000; Nemirovsky *et al*., 1998). The analysis of the role of the body in cognition takes place within a wide multi-disciplinary effort, involving neuroscience, cognitive science, experimental psychology, linguistics, semiotics and philosophy. These disciplines offer complementary tools and constructs to those who wish to investigate the complex interactions among language, gesture, bodily action, signs and symbols in the learning and teaching of mathematics. The goal of the Research Forum is to examine the role that gesture plays in the construction of mathematical meanings. More specifically, we are concerned with the following questions:

- How can we describe the phenomenology of gestures in mathematics learning (e.g.: What kind of gestures are there? Is the classification created by McNeill (1992) adequate for mathematical gestures?)
- How does gesture function in the processes of learning mathematical concepts?
- Can gesture provide evidence about how mathematical ideas are conceptualized?
- Are gestures context-dependent? In particular, how do they change when students interact with artifacts?
- Which theoretical frameworks are suitable for analysing gestures in mathematics learning taking into account work on gesture carried out within disciplines outside of mathematics education?
- What consequences of the research on gesture can be drawn for mathematics students, teachers, and prospective teachers?

The analysis of gesture, both within and outside of mathematics education, takes place within the broader framework of recent work in embodied cognition and

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cognitive linguistics. As applied by Lakoff and Núñez (2000), this framework holds that human bodily experience, as well as unconscious mechanisms like conceptual metaphors and blends, are essential in the genesis of mathematical thought. In this view, mathematics is a specific powerful and stable product of human imagination, with its origins in human bodily experience. As noted by Seitz (2000, emphasis in the original), "In effect it appears that we *think* kinesically too [….] and has been postulated [….] that the body is central to mathematical understanding (Lakoff & Nunez, 1997), that speech and gesture form parallel computational system (Mc Neill, 1985, 1989, 1992)." In a similar vein, R. Nemirovsky (2003) has emphasized the role of perceptuo-motor action in the processes of knowing:

> While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based on how we have learned and used the subject itself". [As a consequence,] "the understanding of a mathematical concepts rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context. (p. 108)

Furthermore, attention is now being paid to the ways in which multivariate registers are involved in how mathematical knowing is built up in the classroom. This point is illustrated by Roth (2001) as follows:

> Humans make use not just of one communicative medium, language, but also of three mediums concurrently: language, gesture, and the semiotic resources in the perceptual environment. (p. 9)

This attention to the body does not negate the fact that mathematics and other forms of human knowledge are "inseparable from symbolic tools" and that it is "impossible to put cognition apart from social, cultural, and historical factors": in fact cognition becomes a "culturally shaped phenomenon" (Sfard & McClain, 2002, p. 156).

The embodied approach to mathematical knowing, the multivariate registers according to which it is built up, and the intertwining of symbolic tools and cognition within a cultural perspective are the basis of our frame for analysing gestures, signs and artefacts. The existing research on those specific components finds a natural integration in such a frame.

GESTURES VIEWED WITHIN PSYCHOLOGY

Within a psychological perspective, we begin with the seminal work of McNeill (1992), who stated that, "gestures, together with language, help constitute thought" (p. 245). McNeill (1992) classified gestures in different categories: *deictic* gestures (pointing to existing or virtual objects); *metaphoric* gestures (the content represents an abstract idea without physical form); *iconic* gestures (bearing a relation of resemblance to the semantic content of speech); *beat* gestures (simple repeated gestures used for emphasis). Since his study, much research has analysed how gesture and language work together and influence each other. Alibali, Kita and Young (2000) develop McNeill's view that gesture plays a role in cognition, not just in communication, in the Information Packaging Hypothesis (IPH):

> Gesture is involved in the conceptual planning of the messages, helps speakers to "package" spatial information into verbalisable units, by exploring alternative ways of encoding and organising spatial and perceptual information…gesture plays a role in speech production because it plays a role in the process of conceptualisation (p. 594-5)

According to the IPH, the production of representational gestures helps speakers organise spatio-motoric information into packages suitable for speaking. Spatio-motoric thinking (constitutive of representational gestures) provides an alternative informational organisation that is not readily accessible to analytic thinking (constitutive of speaking organisation). Analytic thinking is normally employed when people have to organise information for speech production, since, as McNeill points out, speech is linear and segmented (composed of smaller units). On the other hand, spatio-motoric thinking is instantaneous, global and synthetic (not analyzable into smaller meaningful units). This kind of thinking, and the gestures that arise from it, is normally employed when people interact with the physical environment, using the body (interactions with an object, locomotion, imitating somebody else's action, etc.). It is also found when people refer to virtual objects and locations (for instance, pointing to the left when speaking of an absent friend mentioned earlier in the conversation) and in visual imagery.

Within this framework, gesture is not simply an epiphenomenon of speech or thought; gesture can contribute to creating ideas:

> According to McNeill, thought begins as an image that is idiosyncratic. When we speak, this image is transformed into a linguistic and gestural form. ... The speaker realizes his or her meaning only at the final moment of synthesis, when the linear-segmented and analyzed representations characteristic of speech are joined with the global-synthetic and holistic representations characteristic of gesture. The synthesis does not exist as a single mental representation for the speaker until the two types of representations are joined. The communicative act is consequently itself an act of thought. ... It is in this sense that gesture shapes thought. (Goldin-Meadow, 2003, p. 178)

Another important aspect of the analysis of gesture concerns the relationship between the content of the speech and the gesture. We can speak of a *gesture-speech match* (M) if the entire information expressed in gesture is also conveyed by speech. If not, that is, if different information is conveyed in speech and gesture, we have a *gesture-speech mismatch* (Goldin-Meadow, 2003). This information is not necessarily conflicting but possibly complementary, and may signal a readiness to learn or reach a new stage of development (Alibali, Kita & Young, 2000; Goldin-Meadow, 2003). According to Goldin-Meadow, mismatch is "associated with a propensity to learn" (p. 49), "appears to be a stepping-stone on the way toward mastery of a task" (p. 51); and may place "two different strategies [for solving a problem] side by side within a single utterance" highlighting "the fact that different approaches to the problem are possible" (p. 126). In general gesture-speech mismatch reflects "the simultaneous activation of two ideas" (p. 176).

GESTURES VIEWED WITHIN SEMIOTICS

The fact that gestures are signs was pointed out many years ago by Vygotsky, who wrote:

> A gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture. (Vygotsky, 1997, p. 133)

Semiotics is a useful tool to analyse gestures, provided that a wider frame, which takes into account their cultural and embodied aspects as well, is considered. An analysis of this kind has been carefully developed by Radford, who introduces the notion of *semiotic means of objectification* (Radford, 2003a):

> The point is that processes of knowledge production are embedded in systems of activity that include other physical and sensual means of objectification than writing (like tools and speech) and that give a corporeal and tangible form to knowledge as well....These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*. (p. 41)

Gestures can be important components of semiotic means of objectifications, whether used when communicating directly with others, or to highlight aspects of artefacts and symbolic representations of mathematical concepts.

Psychologists now distinguish between *linguistic* and *extralinguistic* modes of expression, describing the former as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs (Bara & Tirassa, 1999). When students are learning the signs of mathematics, they often use both their linguistic and extralinguistic competence to understand them; e.g. they use gestures and other signs as semiotic means of objectification. Of course, in all these means of objectification both modalities (linguistic and extralinguistic) are present, with different strengths and in different ways depending on the dynamics of the situation.

SUMMARY OF THE RESEARCH FORUM

The papers of the Research Forum address the main questions and themes summarized above. F. Arzarello *et al.* present an example involving geometric visualization to illustrate a new theoretical framework for analysing gesture and speech in mathematics learning environments. M.G. Bartolini Bussi analyses the genetic links between artefacts and gestures in pupils (9 years old) who use real and virtual artefacts. L. Edwards utilizes data from adult students discussing fractions to argue that the original narrative-based classification of gestures should be adjusted and modified for analysing gestures in mathematical discourse. R. Nemirovsky and F. Ferrara approach gestures from the point of view of perceptuo-motor thinking, showing the connections between parallel strands of bodily activities, in a microanalysis of gestures and eye motions during a graphing activity. L. Radford explains the role of semiotics in analysing gestures as means of semiotic objectification, illustrating his framework with data from modeling activities.

SHAPING A MULTI-DIMENSIONAL ANALYSIS OF SIGNS

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INTRODUCTION AND BACKGROUND

Recently the analysis of gestures and their role in the construction of meanings has become relevant not only in psychology, but also in mathematics education. Gestures are considered in relation with speech, and with the whole environment where mathematical meanings grow: context, artefacts, social interaction, discussion, etc. Mathematics, as an abstract matter, has to come to terms with our need for seeing, touching, and manipulating. It requires perceivable signs and so the environment is crucial in the teaching-learning process.

In this paper, we elaborate on two different ways to look at the cognitive processes of students when they communicate and reason during a mathematical activity. We propose a theoretical frame shaped by the encounter of certain perspectives, developed in the disciplines of mathematics education, psychology, neuroscience, and semiotics. In particular, the theoretical notions we use here are the following: from psychology, the *Information Packaging Hypothesis* (Alibali, Kita & Young, 2000); from semiotics, the idea of *semiotic means of objectification* (Radford, 2003a) and that concerning the different functions of signs, i.e. *iconic, indexical* and *symbolic* (Peirce, 1955; Radford, 2003a), and from psycho-linguistics, the distinction between *linguistic* and *extra-linguistic* modes of expression (Bara & Tirassa, 1999). Let us sketch them here for our purpose; a more detailed account is given in the introduction of the present research forum.

In psychological research, the Information Packaging Hypothesis (IPH) describes the way that gesture may be involved in the conceptual planning of the messages, by considering alternative “packagings” of spatial and visual information, so that this information can be verbalized in speech (Alibali, Kita and Young, 2000). Within the similar perspective that gestures play an active role not only in speaking, but also in thinking, *gesture-speech matches* and *mismatches* are defined (Goldin-Meadow, 2003). A match occurs when all the information conveyed by a gesture is also expressed in the uttered speech; a mismatch happens in all the other cases. Mismatches are the most interesting since they indicate a readiness for learning, conceptual change or incipient mastery of a task. But gestures are also significant from the side of semiotics if seen as *signs*. Vygotsky (1997) already pointed out that "a gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture" (p. 133). Nevertheless, semiotics is useful to analyse gestures only if does not forget their cultural and embodied aspects. Such a direction has been followed in mathematics education by Radford (2003a) with the introduction of the so-called semiotic means of objectification. These semiotic means are constituted by different types of signs, e.g. gestures, words, drawings, and so on. They have been introduced to give an account of the way students come to generalise numeric-geometric patterns in algebra. Different kinds of generalisation have been detected. Among them is the so-called *contextual generalisation*, which still refers heavily to the subject's actions in time and space, within a precise context, even if he/she is using signs that could have a generalising meaning. In contextual generalisation, signs have a two-fold semiotic nature: they are becoming symbols but are still indexes. These terms come from Pierce (1955) and Radford (2003a). An *index* gives an indication or a hint of the object: e.g. an image of the Golden Gate, which makes you think of the city of San Francisco. A *symbol* is a sign that contains a rule in an abstract way: e.g. an algebraic formula. As relevant in communication (in thinking as well) gestures can be considered with respect to linguistic and extra-linguistic modes of expression. The former is characterised as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs: "linguistic communication is the communicative use of a symbol system. Language is compositional, that is, it is made up of constituents rather than parts... Extra-linguistic communication is the communicative use of an open set of symbols. That is, it is not compositional: it is made up of parts, not of constituents. This brings to crucial differences from language..." (Bara & Tirassa, 1999; p. 5). In communicative acts the two modes co-exist. Students who learn the signs of mathematics, often rely on both their linguistic and extra-linguistic competences to understand them: for example, they use gestures and words as semiotic means of objectification. Typically, gestures are extra-linguistic modes of communication, whereas speech is on the linguistic side.

A NEW FRAMEWORK: THE PARALLEL AND SERIAL ANALYSIS

We show a brief example from the activity of some 8th grade students involved in approaching a geometrical problem. They have been asked to describe the geometric solid formed when two square pyramids are placed side by side (with one pair of base sides touching). The solution, which must be visualized by the students, is a tetrahedron seen from an unusual point of view.

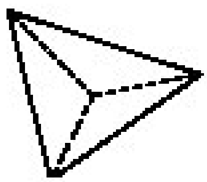

Figure 1

Consider the following utterances by Gustavo, and one of his concomitant gestures:

Gustavo: Yeah, it is a solid, made of two triangles placed with the bases below, which are those starting in this way and going up, and two triangles with the bases above that are those going in this way [see Fig. 2].

We can analyse data like these in a double way, using what we call *parallel* and *serial* analysis. Both analyses take into consideration the dynamics of what we think of as the major components of processes of objectification: not only speech and gestures (respectively ***s*** and ***g*** in Fig. 3), but also written words and mathematical signs (respectively, ***w*** and ***x*** in Fig. 3). The latter, even if not directly part of the communication acts, are a product of them, and often arise from gestures and words used by the involved subjects (Gallese, 2003; Sfard & McClain, 2002).

Figure 2

The components of objectification processes may develop according to two types of dynamics. We call the first dynamics *Parallel Process of Objectification (**PPO**)*; it results when (some of) the different components are seen as a group of processes synchronically developing (e.g. when one talks and gestures simultaneously). They can match or mismatch with each other in the way they are encoding information.

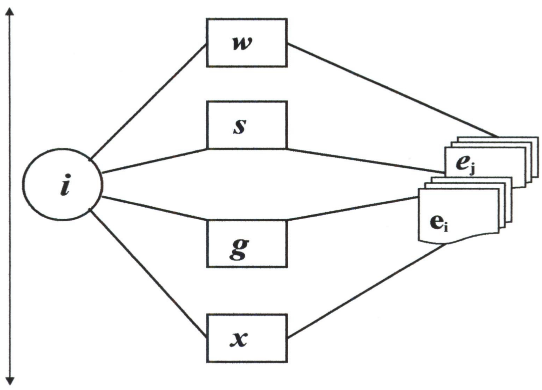

Figure 3: The *PPO*

In such a case, we are interested in a parallel analysis of the components (see the vertical arrow in Fig. 3), which focuses on the mutual relationships among them, where all components refer to the same source ***i*** and possibly to different encoding e_i's. The main elements of a parallel process of objectification are: (i) the idea of semiotic means of objectification; (ii) the Information Packaging Hypothesis; (iii) Match and Mismatch (Goldin-Meadow, 2003).

We call *Serial Process of Objectification (**SPO**)* a second type of dynamics, which results when two different components are spread over time and happen in different moments, as steps of a unique process. An example is given by a sign produced as a frozen gesture (Vygotsky, 1997), or by a gesture embodying some features of a previous sign. In this case, we are interested in a serial analysis (see the horizontal arrow in Fig. 4) focusing on the subsequent transitions from different sources ***i*** to different encoding e_i's.

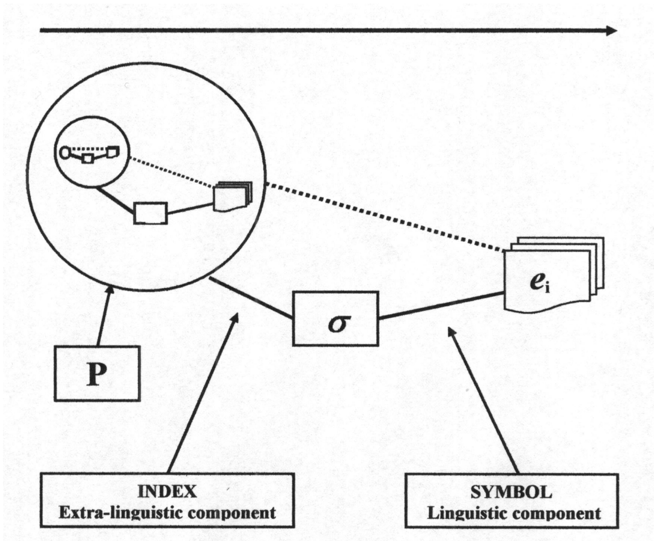

Figure 4: The *SPO*

The Serial Process of Objectification is shown in Fig. 4. Its main elements are again: (i) the semiotic means of objectification; and (ii) the Information Packaging Hypothesis. But there are also two other elements: (iv) the indexical-symbolic functions of signs; and (v) the linguistic and extra-linguistic modes of communicative acts. A serial process of objectification happens when one (or more) serial (or parallel) process(es) **P**, represented in the circle of Fig. 4, is (are) the grounding for the genesis of a new sign (indicated by $\boldsymbol{\sigma}$).

For technical reasons, just one component appears in the circle, but there could be more. The sign $\boldsymbol{\sigma}$ is the pivot of the process; it can be any kind of sign: a drawing, a word, a gesture, a mathematical sign, etc. It is generated by the previous process(es) P and produces an encoding of **P**. The relationships between $\boldsymbol{\sigma}$ and **P** are mainly extra-linguistic, whereas the relationships between $\boldsymbol{\sigma}$ and $\boldsymbol{e_i}$ ***are*** mostly linguistic. In other terms, the sign $\boldsymbol{\sigma}$ has an indexical function with respect to P, but it has also a fresh symbolic function with respect to the encoding $\boldsymbol{e_i}$. Thus, the ***SPO*** could be the basis for a new serial process, and so on, in an ongoing series of nested generalisations. Examples of ***SPOs*** are given by the learning of speech in kids or by that of reading written texts in young pupils. Mathematical examples are the processes undertaken by students who are learning Algebra or some other chunks of mathematical ideographic language, from Arithmetic to Calculus.

Generally both types of dynamics, ***PPO*** and ***SPO***, can support the genesis of signs. As a consequence, each process of objectification may be analysed from both points of view, that is, as a parallel process and as a serial process. We call *parallel* and *serial* the two resulting types of *analysis*. Let us go back to the initial example that we can now interpret through the two analytical lenses. The parallel analysis points out the conflict between the two pieces of Gustavo's theoretical knowledge concerning the 2D and 3D figures. The serial analysis shows that Gustavo's gestures are mediating the transition from the 2D features of the triangles to the 3D ones of the solid. After this episode, the experiment goes on and culminates with the acknowledgement by students of the tetrahedron as a "triangular pyramid". Parallel and serial analysis allow us to focus properly on the dynamics of what is happening. As such they are useful tools of investigation. In fact, parallel analysis reveals itself as a tool suitable for identifying conflicts, even before they appear to block or slow students' activities. On the other hand, the serial analysis represents a tool suitable for focusing on the dynamics through which the subjects try to overcome obstacles met in their activities.

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WORKING WITH ARTEFACTS: THE POTENTIAL OF GESTURES AS GENERALIZATION DEVICES

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INTRODUCTION

We shall summarize some findings of two studies (Bartolini et al., 1999; Bartolini et al. in press) concerning primary school. In the former we have studied the genesis of a germ theory of the functioning of gears. In the latter we have studied the construction of the meaning of painting as the intersection between the picture plane and the visual pyramid. The studies have been carried out in a Vygotskian framework that has been gradually enriched with contributions of other authors. As a result, classroom activity has been designed and orchestrated by the teacher in order to foster the parallel development of different semiotic means (language, gestures, drawing), which form a dynamic system (Stetsenko, 1995, p. 150).

In both studies, concrete artefacts came into play. Wartofsky's distinction between primary, secondary and tertiary artefacts proved to be useful (1979). *Primary artefacts* are "those directly used" and *secondary artefacts* are "those used in the preservation and transmission of the acquired skills or modes of action". Technical tools correspond to primary artefact whereas psychological tools are the individual counterparts of secondary artefacts. *Tertiary artefacts* are objects described by rules and conventions and not strictly connected to practice (e. g. mathematical theories, within which the models constructed as secondary artefacts are organised).

WHEN THE ARTEFACT IS A GEAR.

The role of gestures when concrete tools are into play is obviously very large. Wartofsky himself emphasizes mimicry, among the different representations used to preserve and transmit the modes of action. Gestures are essential to use the artefact, as 'a machine is a device that incorporates not only a tool but also one or more gestures' (Leroi-Gourham, 1943). We found that, from 2nd grade on, when the teacher designs suitable activities aiming at constructing a germ theory of the functioning of

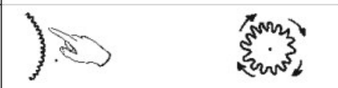
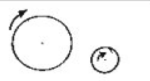
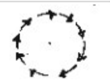
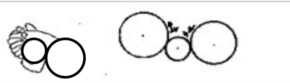
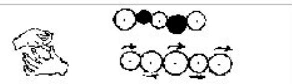
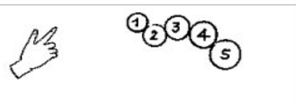
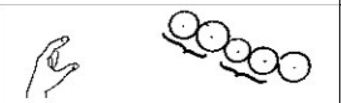
[1] Abridged version of a study (in preparation) carried out together with Maria Alessandra Mariotti, and Franca Ferri, within the National project Problems about the teaching and learning of mathematics: meanings models, theories (PRIN_COFIN 03 2003011072).

gears and supports pupils' work, there is a parallel and intertwined development of three different semiotic means: gesture – drawing - speech (in oral and written forms): the development is towards the appropriation of the meaning of motion direction, represented by a sign ('arrow') with an appropriate syntax, that also allows students to solve difficult problems concerning trains of any number of gears.

Our findings are summarized in Table 1, adapted from (Bartolini et al., 1999, p. 79) which relates the findings of that study to issues discussed in this forum.

The *primary artefacts* are given, in this case, by tools with gears and toothed wheels inside. In the figure, a pair of toothed wheels is represented (courtesy of R. Nemirovsky, TERC). To start the gear a gesture is needed: it creates an action scheme that 'enables students to tackle virtually any particular case successfully' (*factual generalization,* Radford, 2003a, p. 47).

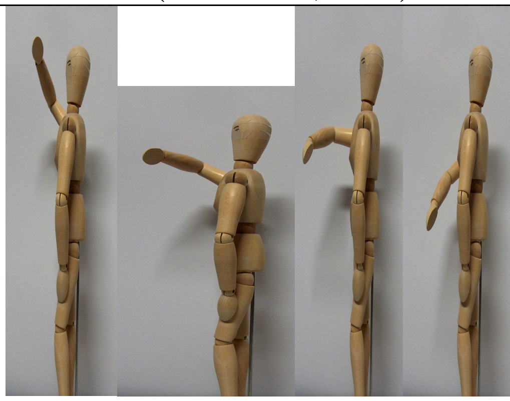
Table 1. *From gesturing to signs (Bartolini et al. 1999, p. 79)*

n	GESTURAL	GRAPHIC	VERBAL	*Wartofsky*	*Edwards / McNeill*	*Radford*
1			push this wheel this way this wheel goes this way	PRIMARY Gesture on a primary artefact to turn the wheel as a whole or pushing a point.	Iconic physical	*Factual generalization*
2			this tooth goes this way		Iconic physical	*Factual generalization*
3			clockwise anticlockwise	Construction / appropriation of secondary artefacts	No gesture	*Contextual generalization*
4			left - right up - down		No gesture	*Contextual generalization*
5			wheels turn in pairs	Gesture to represent a primary artefact (secondary) Construction / appropriation of secondary artefacts	Iconic physical	*Contextual generalization*
5'			this wheel pushes that wheel		Iconic physical	*Contextual generalization*
6			white-black A-B thumb-index this - the other way	*Towards tertiary artefacts* Gesture to represent a mathematical model	metaphorical	*Contextual generalization*
7			one-two-three-four-...		metaphorical	*Symbolic generalization*
8			wheels are paired		metaphorical	*Contextual generalization*

When young pupils (e.g., 2nd grade ones) are asked to represent this experience by drawing, they spontaneously introduce the sign 'arrow' (a semiotic mean of objectification) that seems to objectify on paper the gesture of the hand. Later the sense of the sign changes together with the parallel evolution of drawing and speech. In Table 1 we have related our findings with those of other authors.

When the artefact is a sentence evoking a concrete artefact

In a 4th grade classroom (Bartolini et al. in press), a complex activity about perspective drawing has been started. The first step has been the exploration and the interpretation of an artefact (Dürer's glass) built in wood, metal and Plexiglas, where one observes through the eyehole the perspective drawing of the skeleton of a cube put behind the glass. Some months later, at the beginning of the 5th grade, when the concrete artefact is no longer in the classroom, a very short sentence from L. B. Alberti (De Pictura, 1540) is given to interpret in classroom discussion: "*Thus painting will be nothing more than intersection of the visual pyramid*". Gestures are very important in the interpretation: gestures mime planes and lines and constitute a fundamental support to imagine a pyramid.

Table 2 *"Thus painting will be nothing more than intersection of the visual pyramid"* L.B.Alberti (De Pictura, 1540).	
▸ You have to imagine it. I understood this, if you saw it near the object you obtain a large image; if you saw it near the eye you get a smaller image. [*With gestures, many children saw the visual pyramid].* ▸ If you go down straight, because with our hands we form a kind of plane parallel to the one of the objects [*With his hands he traces two parallel planes in space*]. In this way you certainly obtain a figure which is exactly the same as the base of the pyramid, but smaller. 1a	 1b
2b […] A visual pyramid is a kind of pyramid 'made by you', that is the pyramid helps you to see what you see in different ways, in fact, as I have drawn, it makes you see the sun in several ways. I have drawn that drawing, because it clarifies how a visual pyramid is and also how it must be shaped. I have enjoyed making the sun, bigger and bigger, because it makes one understand much. Anna's eye is open and the other is closed, it is not visible but if you notice there is her arm pointing close to the other side of her face to close the other eye.	2a 

The pupils do not seem troubled by this imaginary context, as the following exchange shows:

Luca: How can you possibly saw the visual pyramid, which is a solid that does not exist?

Alessandro B.: Exactly how you imagine it. If you see it because you imagine it, you can saw it as well. You have to work with the mind.

Three months after this discussion, the pupils are asked to comment individually, in writing (using also drawing if they wish), about the same sentence by Alberti (Maschietto & Bartolini, submitted). In Table 2 some exemplary protocols from the above activities are presented: 1a. The transcript (with comments) of an oral exchange between two pupils in classroom discussion; 1b. The simulation of gesture by means of a dummy; 2a. A drawing produced to explain Alberti's sentence; 2b. An excerpt of the written text, added as a commentary of the sentence and of the drawing.

The right way to produce the gesture ('straight down' i.e. vertically) is verbally explained immediately by the second speaker. This way of cutting an 'imaginary' pyramid in the air becomes a shared action scheme in the classroom, repeatedly used by the pupils and by teacher as well. The gesture works in any position (*contextual generalization*, Radford 2003a). Three months later most pupils prove to have internalized the meaning of the visual pyramid and produce meaningful drawings. In the one reported here there is another instance of *contextual generalization*, which concerns the possibility of tilting any 'imaginary' picture plane in non-vertical position. We know from the history of perspective that this was not a trivial problem.

DISCUSSION

Wartofsky's elaboration of artefacts refers to 'external' objects. He discusses the secondary artefacts as follows:

> Such representations […] are not 'in the mind', as mental entities. They are the products of direct outward action, the transformations of natural materials, or the disposition or arrangement of bodily actions […].

In the classroom pupils construct/appropriate these cultural products by means of social activity carried out together with their peers under the teacher's guidance. We have shown in two cases concerning spatial experience with concrete artefacts how internalization of social activity, is realised by semiotic means of objectification (Radford, 2003a) that are used in parallel and intertwined with each other.

THE ROLE OF GESTURES IN MATHEMATICAL DISCOURSE: REMEMBERING AND PROBLEM SOLVING

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The purpose of this analysis is to examine the role of gestures within the context of a particular setting involving mathematical discourse, specifically, an interview where students were asked to describe how they learned certain mathematical concepts and to explain how they solved problems involving fractions. The overall goal of the study was to examine both the form and function of gestures within a context of mathematical communication and problem solving, and to begin to develop an analytic framework appropriate to understanding gesturing within the domain of mathematics.

Previous research has examined the role of gesture in a number of different mathematical contexts, including learning to count (Alibali & diRusso, 1999; Graham, 1999), classroom communication (Goldin-Meadow, Kim & Singer, 1999), ratio and proportion (Abrahamson, 2003), motion and graphing (Nemirovsky, Tierney & Wright, 1998; Radford, Demers, Guzmán. & Cerulli, 2003, Robutti & Arzarello, 2003), and collaborative problem solving (Reynolds & Reeve, 2002; also see Roth, 2001, for a review of research on gesture in mathematics and science). Gesture is defined as "movements of the arms and hands ... closely synchronized with the flow of speech" (McNeill, 1992, p. 11). In contrast with speech, which is linear, segmented and composed of smaller units, gesture is global and synthetic; it can express meanings as a whole and one gesture can convey a complex of meanings (McNeill, 1992). Gesture can be seen as an important bridge between imagery and speech, and may be seen as a nexus bringing together action, imagery, memory, speech and mathematical problem solving. The investigation of gesture in mathematics takes place within a theoretical context that sees cognition as an embodied phenomenon, and that examines how both evolutionary constraints and individual bodily experience provide a foundation for the distinctive ways that humans think, act, and speak about mathematics (Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999).

The data for the study comprise a set of gestures displayed by twelve adult female students while talking about their memories of learning fractions, and during and after solving problems involving fractions. The participants were prospective elementary school teachers, and the interviews were carried out in pairs. A corpus of more than 80 gestures was collected. The majority of the gestures were displayed in response to questions asking the students to recall how they first learned about fractions.

Figure 1: “I think we did, like, just a stick or a rod…”

These gestures generally fell into four categories, representing an extension of McNeill’s original typography of gestures into iconics and metaphorics:

(1) Iconic gestures referring to physical manipulatives or actions (e.g., “a stick or rod” or “cutting a pie”)

(2) Iconic gestures referring to inscribed representations of physical manipulatives (e.g., “a pie chart”)

(3) Iconic gestures referring to specific written algorithms (Figure 1b)

(4) Metaphoric gestures (referring to an abstract idea or action, e.g. Figure 2)

In Figure 1a, the student describes a manipulative (possibly fraction bars), and goes on to talk about “dividing it again and again,” moving her right hand in a chopping gesture toward the right to indicate the iteration of this division. This chopping motion can also be categorized as an iconic gesture referring to a physical action.

Figure 1b shows an example of a student displaying an “iconic-symbolic” gesture: gestures that refer not to a concrete object but to a remembered written inscription for an algorithm or mathematical symbol; that is, an “algorithm in the air” (Edwards, 2003). The importance of written algorithms for mathematics, and for students memories of learning mathematics, would seem to require this expansion of the typology of gestures that McNeill originally developed to analyze narrative discourse.

Figure 2 shows a part of a gesture made by a student responding to a question about how she would introduce fractions to children. The gesture began with the two hands close together, with whole hands slightly curled and facing each other, and ended with the hands opening out and moving to the right. These somewhat vague metaphorical gestures about generic mathematical operations contrast sharply with the very precise iconic-symbolic gestures used when describing specific arithmetic algorithms with fractions.

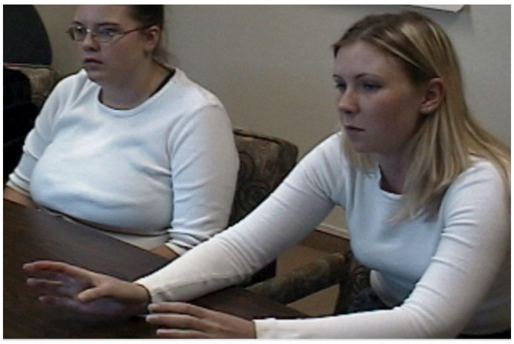

Figure 2: "Like the different formulas"

In addition to gestures displayed in response to the interviewer's questions, one student displayed a complex sequence of gestures associated with a description of how she solved a problem involving comparing two fractions. She and her partner had worked out which was larger, 3/4 or 4/5, and the student was explaining her solution after the fact. The student's spoken words are below (underlining indicating words synchronized with a gesture):

S2: Well, I mean it's like I'm thinking if I had a pie and I had 5 people versus 4 people then,[R: Ah.] you know, we're each kinda getting less of a piece [R: Ah.] because there's a fifth piece we have to like, put out to the other four people.

The four gestures corresponding to the underlined words or phrases consisted of (1) pointing with right index finger to right temple ("thinking"); (2) moving the first two fingers of the right hand from right to left at chest height ("less"); (3) a diagonal chopping motion with the whole right hand at face height ("fifth piece"); which continues into a (4) circular movement of the whole hand in front of and parallel to the face and chest ("put out to the other four"). This use of gesture did not seem to be a static illustration of remembered objects or inscriptions, as some of the other gestures were. Instead, the sequence of gestures was fully synchronized with the description of the problem solution, and may have played a facilitating role in solving the problem. The first gesture would be described as an emblem (a conventionalized gesture for "thinking" by pointing to the temple), but the other four gestures highlighted important aspects of the solution: the relative size of the fractions; i.e., the denominators ("getting less of a piece"), the number of pieces, i.e., the numerators ("a fifth piece") and a sharing operation ("put out to the other four people").

The current study elicited a wide variety of gestures, primarily associated with students' memories of learning fractions, but also occasionally in connection with current problem solving and reasoning. In either context, the gestures were not simple illustrations, but reflected important aspects of the materials and representations present while the students were learning. These findings are similar to those in a study of bodily motion and graphing, in which the authors stated, "The way students describe functions shows deep traces of their actions and interactions with instruments and representations. Such traces are not complementary to the concept but are an essential component of its meaning" (Robutti & Arzarello, 2003, p. 113).

The analysis of the gesturing in mathematical contexts has provoked a re-examination of the categories developed by McNeill for describing gestures elicited in association with narrative descriptions. The initial analysis of the fraction data stimulated a division of McNeill's category of iconic gestures into two sub-categories: iconic-physical and iconic-symbolic. However, the nature of mathematics as a discipline may require an even more refined categorization of gestures. This is because while in everyday life, concrete objects do not "refer" to anything beyond themselves, in mathematics teaching, many concrete objects have been designed to "represent" more abstract mathematical objects. So when a student gestures in a circle when talking about fractions, she may be referring simply to the plastic fraction pieces she remembers from elementary school, or she may be thinking about those pieces in regards to a particular fraction or operation. Furthermore, outside of mathematics, written symbols are not usually manipulated as if they were objects. Thus, descriptions and analyses of gesture in mathematics should take into account these features of mathematical practice and discourse. Furthermore, the analysis of gesture may help to illuminate the relationships and developmental path among physical actions, speech, internalized imagery, written symbols, and mathematical abstractions.

CONNECTING TALK, GESTURE, AND EYE MOTION FOR THE MICROANALYSIS OF MATHEMATICS LEARNING

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TERC, Cambridge, MA (**)

INTRODUCTION AND BACKGROUND

In the last years deep changes have characterised the study of thinking and learning based on ongoing research in neuroscience, psychology, and cognitive science. These changes were supported by the availability of new technologies, which allow for a fine-grained recording of human activity. Different areas of cognition (such as language, vision, motor control, reasoning), which in the past were considered largely autonomous, have started to be studied as integrated and working in unison. This trend entails that research can get a wider and more detailed viewpoint to analyse thinking and learning processes. Examples come from the psychological research on gestures since the '80 (see Kita, 2003) and from vision science (e.g., Tanenhaus *et al.*, 1995). These emerging studies are generating new insights on the nature of thinking in educational research and the study of mathematics learning. For instance, Nemirovsky (2003) argues that "thinking is not a process that takes place "behind" or "underneath" bodily activity, but it is the bodily activities themselves". Within this viewpoint, even "the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending on the circumstances" (ibid.). The integrated study of bodily activity calls for a type of analysis, which is sometimes called "microgenetic analysis"; that is, a detailed examination of the genesis of ideas and approaches by a subject over short periods of time (minutes or seconds), while they are occurring. Microanalytic studies can document variability and actual processes of local change. Furthermore, the advent of digital video and other tools (portable eye trackers are an example), made microanalysis practical and more widespread.

EYE MOTION AND PERCEPTION

Perception and motor control (main constitutive aspects of thinking) are inextricably related in eye motion. Contrary to common belief, the eyes do not take whole snapshots of the surroundings onto our brains. Studies in eye motion provide evidence for Gibson's (2002/1972) thesis that visual perception is not an all-at-once photographic process of image-taking from the retina to the brain but a "process of exploration in time" (p. 84). Since "perception is *not* supposed to occur in the brain but to arise in the retino-neuro-muscular system as an activity of the whole system" (ibid.; p. 79), eye motion is crucial for such a process. Our study focuses on a type of eye motion, the saccadic one, consisting of rapid transitions ("saccades") between "fixations". A fixation is a point in the field of view around which the eyes stay on a relatively long period of time, commonly in the range of tenths of a second. The exploration in time results in some repeated cycles or trajectories formed by the successive fixations, the so-called scanpaths (Norton & Stark, 1971). The scanpaths clearly depend by the circumstances, are idiosyncratic to the individual seeing, and reflect the questions one has in mind. As a consequence, our eyes are constantly and actively traversing the surroundings. They do not record the environment, but they interrogate it, as Yarbus (1967) pointed out in the case of subjects looking at paintings. Other researchers have studied eye motion in context as a means to analyse the strategies different subjects activate when involved in a mathematical activity Some studies (Epelboim and Suppes, 2001) show that eye motion is central not only to seeing what is out there, as it were, but also for imagining things that are not present in the field of view. Therefore given that imagination and visualisation are essential for mathematical understanding, eye motion can be an important tool to reveal thoughts in catching a solution or grasping a meaning.

We will examine the coordination of talk, gesture, and eye motion, moment-by-moment, for a subject interviewed on graphs of motion. In our example, graphs describe a motion story read and interpreted by the subject, who wears an eye tracker recording his eye motions while a second camera films his gestures.

AN EXPLORATIVE EXAMPLE

The example briefly considered is based on an exploratory interview we conducted with a graduate student wearing a state-of-the-art portable eye tracker. The battery-operated eye tracker was carried within a small backpack connected to a head-mounted pair of miniature cameras (for the image of the scene, and for eye motion on the scene: see Fig. 1*a*, where at any time the cross represents the fixation). An external camera recorded gestures and hand motion (Fig. 1*b*). The interview included a "Motion Story" telling the imaginary motion of a person:

Figure 1

> I was quietly walking to the bus stop. I looked back and saw that the bus was fast approaching the stop. Then I ran toward the bus stop. However, the bus went by me and did not stop. I slowed down and kept walking toward the bus stop to wait for the next one. But, I forgot to put a letter in the mailbox, which is placed just a few metres behind where I was. So, I walked quickly toward the mailbox and I posted my letter. As soon as I realized that the next bus was coming, I ran back and I waited for it at the bus stop.

The interviewee (L) was asked to draw on a whiteboard a graph of position vs. time relative to the story and then the corresponding velocity vs. time and acceleration vs. time graphs. The ensuing conversation was about the characteristics of these graphs, maxima and minima, etc. Our analysis strives to trace the process of graph construction over time. For reasons of space we can just sketch the dynamics. At first, L is looking in the story for information to use for drawing the position vs. time graph. His eyes go back and forth from the right side (see Fig. 2*b*) where he has to draw, to the story placed on the left side (Fig. 2*a*). Fixations are located in the written text on places useful to gather important information to be translated in pivotal points of the graph. After L determines the points in time, he draws straight lines connecting them.

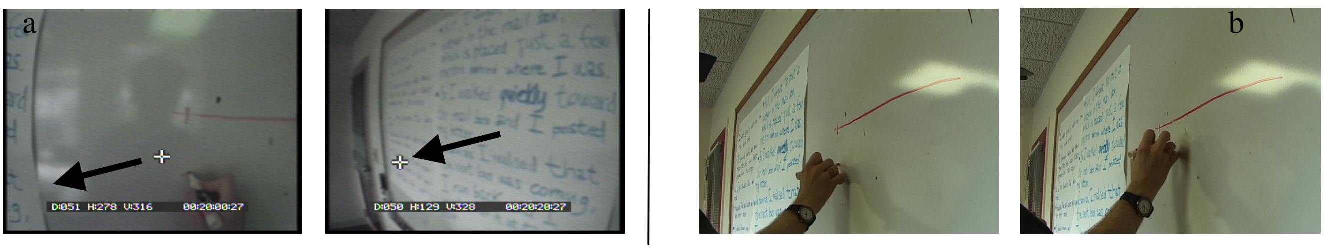

Figure 2

For example at time 3.55.09, L focuses in the story (Fig. 3*a*) on the speed feature (fixation on "quickly") of the piece of the graph he is starting to trace (Fig. 3*b*).

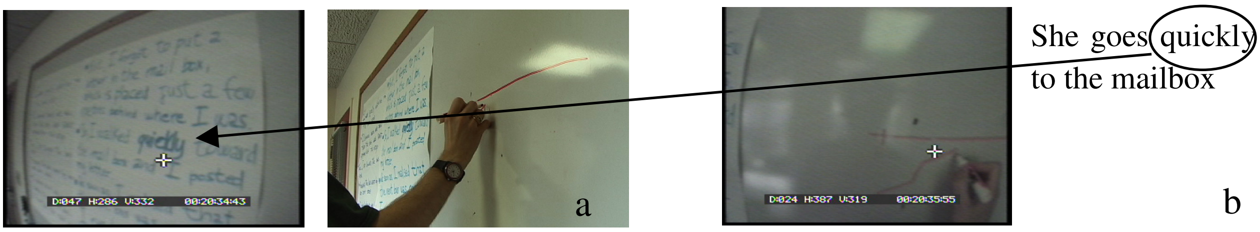

Figure 3

The resulting graph is shown in Fig. 4, where the position of the bus stop is set by L as the zero for the distance axis. Then a second phase started, in which the drawing is checked in relation to the story. The hand is kept still on a graphical element as to not lose the reference in the drawing, while the fixation goes to the text at the corresponding moment (Fig. 5). Then the eye comes back on the graph to traverse, together with the hand (Fig. 6), the motion started at that moment; moreover, L joined this description with his utterance ("She [the character of the story] ran back").

Figure 4

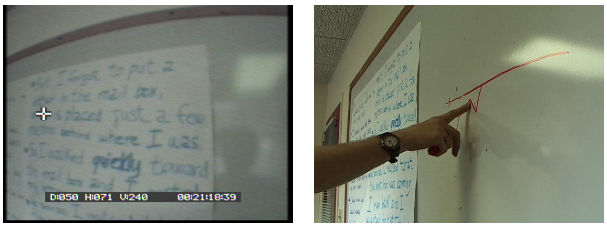

Figure 5

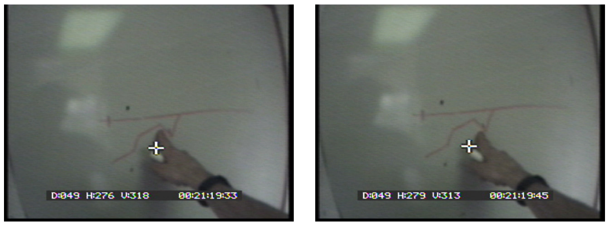

Figure 6: She ran back

In an ensuing phase L gathers from the distance vs. time graph information needed to draw the velocity vs. time graph. L's eyes and hands moved to relate the two graphs, their relations, and the physical quantities related to motion (a sequence of fixations and gestures is shown in Fig. 7).

Figure 7: L draws the velocity vs. time graph

Then a question by the interviewer (F in the following) marks the beginning of a reflection on the shape of the two graphs:

F: So, you suppose that in these three time intervals [hand pointing to the three pieces at the same height on the velocity vs. time graph] she has the same velocity?

To answer L goes back to the story. Then his eyes go from velocity vs. time to the distance vs. time to check the relations between the graphs and the motion described in the motion story; checking leads L to erase and redraw part of the graph (Fig. 8).

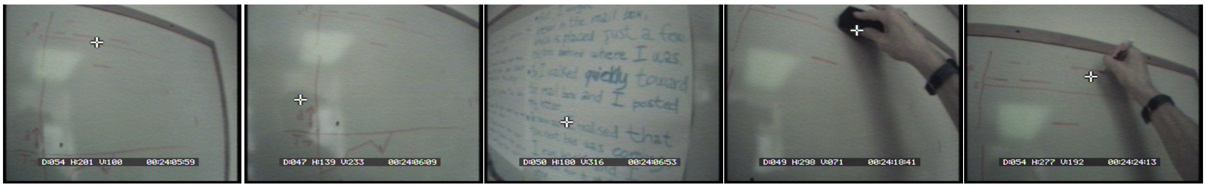

Figure 8: checking relations between graphs and motion

The dialogue between L and F developed further as L justified his changes or choices for the drawing, in trying to assess whether the pieces of distance vs. time indicated by F have the same slope:

L: I mean, I guess, I gave the word quickly the same magnitude basically as the running, so…

F: So, that's the reason because on this graph this part and this part have the same slope [hand pointing to the two pieces on the whiteboard]

L: Yeah.

F: That's the reason. What about these two parts? [hand pointing to the other pieces of the graph with same slope]

L: Those are the same, I think, because… although I guess maybe I'm not so good in drawing. I guess this one [L is pointing to the first segment] could be a little faster than this one… 'cause it says quietly walking [L is pointing to the second segment]… quietly walking versus walking

There seems to be three major functions of L's fixations: *locating*, e.g. when L needs essential information in the story, or when he has to choose where to draw a critical point; *checking*, e.g. when he goes back and forth from one source of information (say, the story) to another (say, the graph) to make sure they cohere; *directing*, e.g. when the eye helps the hand to get the (approximately) correct height of the critical points for the velocity vs. time graph (later for the acceleration vs. time graph). Furthermore, although each completed graph is in some sense a static object, L's eye motion shows that at any given time he is focusing on a very particular aspect, either coordinating with elements of the written story or of another graph. Each visual focusing appears to always have a question motivating it (e.g. should it be steeper? longer? Are these two the same speed?). Each graphical segment has to comply with numerous demands (consistency with the time interval, steeper than another one, etc.) and often his drawing of a segment complies with one or some of them but not with all of them. L goes through an iterative process of repair and re-drawing. As he draws and redraws he also becomes increasingly familiar with the motion story, needing less direct consulting of the text. Examining every single fixation as an effort to address a certain question is significant to a microanalysis of the situation. The sense of the whole for a graph (or a narrative) emerges gradually out of repeated focusing on particular events and shapes. In this sense, knowing how to graph a distance vs.

time graph, or deriving velocity and acceleration from it, entails an intuitive sense of what to look at and how to look at it over time, in order to address ongoing questions.

WHY DO GESTURES MATTER? GESTURES AS SEMIOTIC MEANS OF OBJECTIFICATION

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One of the most intriguing aspects of gestures is that in such varied contexts as face-to-face communication, talking over the phone, and even thinking alone, we all make gestures but we still do not know why. Explanatory models have been proposed by neuro-psychology, information process theories, etc. Our problem here is narrower. We are interested in understanding the role of gestures in the mathematics classroom. However, before going further, we should ask: why do gestures matter? Contemporary forms of knowledge representation are challenging the cognitive primacy with which the written tradition has been endowed since the emergence of printing in the 15th century. The audio and kinesthetic dimensions of oral communication of the pre-print era –dimensions that were replaced by the visual and linear order of the written text– are nowadays viewed with a revived and rejuvenated cognitive interest. Current studies on gestures and perceptual-motor activity belong to this stream.

Now, the way in which each one of us, as mathematics educators, may understand the role of gestures is naturally linked to the theoretical framework underpinning our research. From the semiotic-cultural approach that I have been advocating (Radford, 1998, 2003b), gestures are part of those means that allow the students to objectify knowledge -that is, to become aware of conceptual aspects that, because of their own generality, cannot be fully indicated in the realm of the concrete. In a previous article I have called those means *semiotic means of objectification* (Radford, 2003a). In addition to gestures, they include signs, graphs, formulas, tables, drawings, words, calculators, rules, and so on.

Our answer to the question: "Why do gestures matter?" can then be formulated as follows. Gestures matter because, in learning settings, they fulfill an important function: they are important elements in the students' processes of knowledge objectification. Gestures help the students to make their intentions apparent, to notice abstract mathematical relationships and to become aware of conceptual aspects of mathematical objects.

However, considered in isolation, gestures have -generally speaking- a limited objectifying scope. We have tried again and again the following experiment: we have turned off the volume of many of the hundreds of hours of our video-taped lessons and, even though we *see* the students making gestures and carrying out actions, our understanding of the interaction is very limited. The same can be said of other semiotic systems. Thus, we have also turned off the image and, even though we *hear* the discussion, our understanding of the interaction is again very restricted. We have also stopped both the sound and the image and limited ourselves to *reading* what the students *wrote*, and the result has been as poor as in the previous cases. The reason behind the poor understanding of the students' interaction that results from isolating one or more semiotic systems present in learning is that knowledge objectification is a multi-semiotic mediated activity. It unfolds in a dialectical interplay of diverse semiotic systems. Each semiotic system has a range of possibilities and limitations to express meaning. The conceptuality of mathematical objects cannot be reduced to one of them, not at least in the course of learning, for mathematical meaning is forged out of the interplay of various semiotic systems.

SEMIOTIC NODES

The theoretical construct of *semiotic node* (Radford *et al.* 2003) is an attempt to theorize the interplay of semiotic systems in knowledge objectification. A *semiotic node* is a piece of the students' semiotic activity where action and diverse signs (e.g. gesture, word, formula) work together to achieve knowledge objectification. Since knowledge objectification is a process of becoming aware of certain conceptual states of affairs, semiotic nodes are associated with the progressive course of becoming conscious of something. They are associated with layers of objectification.

Let us illustrate these ideas through a story-problem given to a Grade 10 class. In the story-problem two children, Mireille and Nicolas, walk in opposite directions, as shown in Figure 1. The students were asked to sketch a graph of the relationship between the elapsed time and the remaining distance between the children.

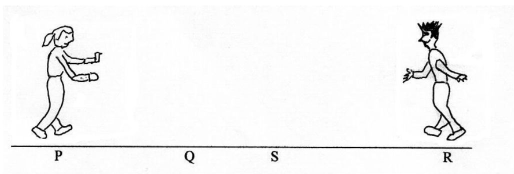

Figure 1. Mireille walks from P to Q. Nicolas walks from R to S

Supported by the students' previous experience, one of the Grade 10 students, Claudine, proposed a compelling -although incorrect- argument: the graph, she suggested, is something like an "S". Ron did not agree, but could not counter Claudine's argument. He claimed that the graph should be something like a decreasing curve, although the details were still unclear for him. In an attempt to better understand the details, he deployed a series of arguments and gestures that were intended not only for his group-mates but for him as well. In Fig. 2 there is an excerpt of the discussion.

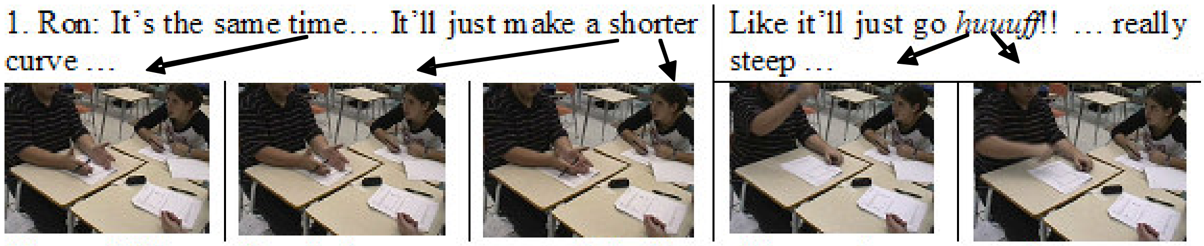

Figure 2. Pictures 1 to 5. Some gestures made by Ron while uttering sentence 1.

To objectify the relationship between distance and time, in the first picture, Ron put his hands one on each one of the students of the story-problem as drawn in the activity sheet. Insofar as the hands stand for something else, they become signs. But in opposition to written signs, which are unavoidably confined to the limits of the paper, hands can move in time and space. Capitalizing on this possibility, to make *apparent* the fact that the distance decreased, Ron moved his hands in opposite directions (pictures 2 and 3). In pictures 4 and 5 he made a vertical gesture sketching the graph time vs. distance, right after have finished the sentence. Three seconds later, remarking that Claudine was not convinced, he started his explanation again. Uttering the first sentence led him to better understand the mathematical relationship, so in the second attempt he was able to produce a more coherent discourse and to better co-ordinate gesture and word. Here, he reached a clearer layer of knowledge objectification.

Pictures 6 to 8 show gestures similar to those in Figure 2, except that now they are made in the air and Ron talks in the first person. In pictures 9 and 10 a familiar situation is invoked (the motion of two trucks). There is, however, another more fundamental aspect that has to be stressed. While in sentence 1, time remained essentially implicit (it was mentioned to emphasize the fact that the children started walking at the same time), in sentence 2, time became an explicit object of reference. Time, however, was not indicated through gestures. It was indicated with words. Even if both are semiotic means of objectification, gestures and words dealt with different aspects of the students' mathematical experience.

In each of the previous cases, the different co-ordination of words and gestures constitutes a distinct semiotic node reflecting different layers of knowledge objectification. One of the research problems that my collaborators and I are currently investigating is related to the theoretical and practical characterization of layers of knowledge objectification. As we saw, gestures play an important role therein. But this role, we suggest, can only be understood if gestures are examined in the larger context of the dialectical interplay of the diverse semiotic systems mobilized by teachers and students in the classroom.

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GESTURES, SIGNS AND MATHEMATISATION

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Where to start: To summarise, criticise, synthesise? The topic of 'gesture' seems so vast, and yet we know (especially with regard to mathematisation) so little. Reading these four papers for the first time, they seem like four ships crossing a huge ocean, moving in different directions, occasionally signalling each other using semaphore!

A SUMMARY: CONTEXT

None of these papers is about gesture alone. All see gesture as part of an integrated communication system with language and, in this case, mathematics. Edwards even defines gesture, after McNeill, in this way, i.e. the gesticulation accompanying speech. Two of these papers are about externalisation in the Vygotskyan sense (Arzarello et al. and Bussi & Maschietto are explicit about this reference) when children are involved in group problem solving. This is also true implicitly of Edwards' students' who gesture as they talk about their previous mathematical work, though her primary reference to theory is in that of Embodied Cognition.

But Ferrara & Nemirovsky's study situates gesture in a more complex setting where seeing (active 'interrogating' with the eye-brain-muscle) is integrated with externalising actions involving gestures, and actually graph-drawing (despite the others' papers' reference to Vygotsky's remark to the effect that gesture gives birth to writing/script, the quote seemed even more apt here!) I highlight the context of gesture, because it influences function and hence categorisation systems.

CLASSIFICATION OF GESTURES/GESTICULATIONS

There is a 2000-year history to the development of classifications of gestures (see Kendon, 2004). Edwards builds her corpus of gestures in the mathematics education context, and this inevitably extends and refines that of McNeill (1992, extended in 2000). Her recognition of context is important: the different functions of gesture in mathematics education imply the need for multiple corpora, each perhaps with its own, albeit related, classification systems.

McNeill's context of interest was mostly that of narrative/narrators, and he was particularly influenced by the significance of 'imagistic' functions of gesture in relation to the emergence of language in an utterance (the so-called growth point, where the gesture precedes the linguistic formulation).

Such an approach has obvious relevance for the emergence of mathematics in children's talk, such as when the child points to figures before articulating (Radford, 2003a, p 46, Episode 1,1, the video clip is not downloadable):

> Josh: It's always the next. Look! [and pointing to the figures with the pencil he says the following] 1 plus 2, 2 plus 3 […]

McNeill's notion that gestures are associated with 'internal', intra-mental images, and their linguistic 'parallels' associated more with the external, inter-mental social/socio-cultural ‘verbal’ representation, is an interesting one for mathematisation (e.g., in Arzarello et al.). The idea here is that the ‘sign’ constituted by a gesture with its linguistic parallel constitutes a unity of internal with external elements, and that conflicts between these elements represent contradictions, and hence opportunities for realignment, or learning. The gesture-and-word unit offers a reflection of Vygotsky’s thought-and-word (or thought-and-utterance) unit of analysis.

So Edwards takes, applies and extends McNeill ‘imagistic’ categories (iconic and metaphoric) to mathematics contexts. This is a good start, and I immediately want to extend this formulation to include McNeill’s non-imagistic gesture categories: I think I see ‘beats’ (Radford speaks of ‘rhythm’) in the gestures used by children to indicate number patterns in ‘factual generalisation’, as in the rhythmic articulation and pointing-beating of the “1 plus 2”, “2+3” etc.

In my own work, I have stressed the significance of deictics in mathematical communication: pointing and waving when associated, or better fused, with models signify mathematics (e.g., Williams & Wake, 2003; Misailidou & Williams, 2003).

In coordination with a model (such as a graph in Roth’s original examples) deictic gestures can signify mathematical objects before they are named, and when the points/segments of a drawing, model or graph have multiple significations, we have an ambiguous moment in communication that can perhaps hold just the right tension in communication.

Beyond gesticulation, there are yet other categories of gesture that mathematics education should consider: ‘Cohesives' and ‘Butterworths' will perhaps emerge or even dominate corpora involving problem solving and proving for instance.

And, to extend further, do the students’ graphing gestures, in Ferrara & Nemirovsky, belong to a different category system, somewhere near the ‘conventional language’ end of the gesturing spectrum (where Kendon and McNeill put sign-languages)?

SEMIOTICS, GENERALISATION AND GESTURE

Arzarello et al. and Bussi & Maschietto inscribe gesture, in part, within Radford’s cultural semiotic theory of ‘semiotic objectification’. Radford’s classification of factual, contextual and symbolic generalisation draws on Peircean categories and conceptions of sign: the index, icon, and symbol, but these are not to be too superficially identified with deictic, iconic, and metaphoric or symbolic gestures.

When a gesture, possibly integrated with parallel action/utterance, is used to denote another object, it constitutes a sign (hence Radford’s term: semiotic objectification). In such a case the gesture can be indexical, iconic, and/or symbolic in Peirce’s (but not McNeill’s) sense. (Peirce, 1955). This now provides a semiotic classification of gestures-in-context that Radford used to analyse significant differences in meanings, such as when the meaning of a formal algebraic expression is indexical for the children but symbolic for the teacher (marking a contradiction between contextual and symbolic generalisation).

I think this difference between McNeill and Peirce/Radford (Wartofsky is another story) explains my concerns with classification systems being equated in Bussi & Maschietto, table 1: a classification system works best if it associated with a particular theoretical scheme. The table thus begs us to examine the relation between the underlying frameworks: Embodied cognition/cognitive linguistics, linguistics, cultural semiotics, that the category systems 'indicate'. (And then there is Wartofsky.)

At this point I would like to consider the disjuncture between the imagistic gestures, or gesticulations in Arzarello et al., Bussi & Maschietto, and Edwards with the gestures and eye foci of the graph-drawing students of Ferrara & Nemirovsky. The gestures of a graph drawer are less strongly bound to the linguistic parallel; but they form a unit of signification with the graph itself, as when the gestures of an operator working a machine form an action because of the mediation of the machine.

In addition, graph drawing has more 'conventional' and 'symbolic' reference rather than iconic, and operate more at the conscious level (in this data anyway, these operations on the graph have not yet descended with practice into the subconscious). In the context of cultural semiotics, this distinction between conscious-unconscious in action-operation suggests an activity theory perspective (Leont'ev, 1981; Williams & Wake, under review) might provide an analytical framework for bringing the two elements together.

It seems there is plenty of empirical and theoretical work to be done still.

BUILDING INTELLECTUAL INFRASTRUCTURE TO EXPOSE AND UNDERSTAND EVER-INCREASING COMPLEXITY

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From the abstract brain-in-a-vat, to the brain neurologically instantiated in a head, to a brain interacting with symbolic tools, to a brain embodied in a walking, talking, gesturing body, to a brain situated in a culture-imbued crowd , … we confront ever increasing complexity in phenomena. Ever more of what was invisible or ignored becomes visible and subject to study, what was excluded becomes included. As so clearly pointed out by Nemirovsky, the subtle new phenomena of gesture, bodily action and perception, eye-movement, and so on, are inevitably and intimately connected with the larger phenomena of thinking, learning, acting and speaking. Indeed, these newly studied phenomena seem, in many cases, to be what the gross phenomena are made of.

With the increasingly complexity comes pressure to expand our repertoire of techniques, conceptual frameworks, and perspectives, our intellectual infrastructure. Each Forum paper reflects a sophisticated response to the new phenomena being exposed, and each reflects the process of building new intellectual infrastructure intending to expose and make sense of these subtle new phenomena. To a significant extent, the value of the papers resides in the intellectual infrastructure that they are making available to the field of Mathematics Education, a contribution that extends well beyond the particulars of the specific studies reported.

DISTINGUISHING FORMS OF GENERALIZATION AND ASSOCIATED SEMIOSIS

Arzarello, Ferrara, Robutti, Paola, & Sabena develop two means of analysis of the processes of semiotically-based objectification, Parallel and Serial, and, most importantly for our purposes, a way of accounting for the *grounded genesis of a new sign*, which in turn includes Radford's notion of contextual generalization. This account is very similar to one developed by Kaput, et al. (in press). However, the latter make a distinction between contextual generalization and the lifting out of repeated actions as the following example illustrates.

Consider a situation where students have been working with open number sentences such as 8+_=13 or perhaps using a literal, $8 + x = 13$. After solving and discussing some number of these kinds of sentences, it is noticed that the answer always seems to be of the form 13 – 8, that is, in verbal terms, "you subtract the left-hand number from the right-hand number to get the answer." The students can be thought of as being in the process of building a rule, a generalization that applies to a parallel set of additive number sentences written in a number-sentence symbol system. This is an example of the grounded genesis of a new sign, where children's intermediate step could be in form of the verbal version of the rule as given. Mathematically, it is a generalization over a subset of the expressions writable in the number sentence system. At some point, as the result of a combination of discussion and perhaps the teacher-led cataloging and recording of cases, the rule gets extended to cover cases where the "unknown" is in the first position, as in "_ + 6 = 15." But now, in order to ensure that the rule covers all such cases and will extend to more cases in the future, the teacher suggests that they think of it as "subtracting the same number from both sides (of the equation)." While it need not be written in what we would recognize as algebraic form, this new verbally described operation on the number-sentence objects is another, and major, contribution to building a new symbol system which consists of expressions of generalizations about *actions on number sentences*. It is a distinct representation of general actions, and as such is part of a new *operative* symbol system being "lifted out" of in order to serve as a new, more general way of thinking about and operating on the number sentence objects.

This is a critically important kind of symbolization in mathematics, but it is a different *kind* of move, I believe, from contextual generalization. Whereas the previously described move involved expressing variation across *statements*, the new one expresses *actions* on the inscription-objects of the initial symbol system. Indeed, the number-sentence statements themselves are likely to be products of such a lifting-out-of-actions. Further, some of the lifted actions based in arithmetic can be represented directly in terms of the structure of the system, such as the distributive law of multiplication over addition in the usual number systems, which allows the substitution of a * (b + c) by a * b + a * c or vice-versa. The action is an equivalence-preserving substitution, which has parallels in the other basic properties of operations as well as substitution actions such as factoring and expanding polynomials that are built directly on them. I expect that the Research Forum will help us unify these different forms of semiosis.

GESTURE, SEMIOSIS AND DELIBERATE GENERALIZATION

I hope that we can jointly address the matter of those acts of communication and sense-making that are driven by deliberate generalization vs. those that are driven by more immediate acts of communication as described in the papers by Arzarello and the paper by Bartolini Bussi & Maschietto. A similar issue can be raised in the study by Ferrara & Nemirovsky, who examine a particular, highly concrete act of representation. Given the essential role of argument and expression in generalization, and the fact that younger learners need to use natural language and other naturally occurring forms of expression, my sense is that we have much to learn about generalization and hence the development of algebraic thinking, from studies of gesture and talk – including intonation.

My sense is that the purposively integrative style embodied in Radford's notion of *semiotic node* holds great promise in deepening our understanding of how speech, gesture and the many different systems of signs interact, particularly if we adopt his perspective that knowledge objectification is almost always, particularly in education, a multi-modal, semiotically mediated phenomenon. His prime example is of particular interest to me because we have used such tasks in a technological context, where the motions of two objects approaching each other, for example, can be created on a computer screen through almost-free-hand drawn graphs produced by students. The interaction between the particular and the general becomes even more pronounced. Indeed, our work also involves activities similar to that used by Ferrara & Nemirovsky, but where the students' graphs can be re-enacted dynamically. Furthermore, these kinds of constructions can be done in a wirelessly connected classroom where different students can systematically contribute different parts of the same graph in the context of a classroom discussion by sending to a shared public display a graph segment produced on their own hand-held device. Or they can import a physical motion that then, as it is relayed (and not merely graphed) interacts in specifiable ways on a public screen with someone else's imported and reenacted motion. In this case, the semiotic acts become highly public and social, and the need for theoretical constructs such as those offered by Radford becomes more acute than ever before.

THE ISSUE OF GENERALITY OF FINDINGS

Edwards' taxonomy of gestures reveals subtleties that any long-term account of gesture in mathematics education would seem to include. Clearly, we need to examine cases of all sorts, from people describing mathematics that they already know, to people learning mathematics, to people teaching mathematics, to people using mathematics in modeling and problem solving, and, most importantly, we need to vary the kinds of mathematics involved, including mathematics centered on generalization vs. mathematics centered on visualization or computation. Taxonomy, of course, helps generate theory, which informs the structuring of the taxonomy. Of particular interest is the use of gesture in the context of technology use, especially because certain actions in a technological environment amount to tracing gestures – as when one drags a hotspot in a dynamic mathematics system, especially a geometric one such as Cabri or Sketchpad. All such actions amount to gestures captured within a mathematically defined system, so the design and use of such systems is an arena for the immediate application of research in gesture.

The eye-tracking microanalytic work by Ferrara and Nemirovsky, pioneering as it is, raises all sorts of questions and tempts all sorts of hypotheses. While more intrusive eye tracking work has been used for many years in areas that involve traditional character-string symbol systems, including arithmetic and algebra, as well as geometry as they cite, the contexts that Ferrara and Nemirovsky investigate are extremely rich, both visually and in mathematical content. In keeping with an underlying theme of the Forum, the authors stress the functional unity of eye motion, kinesthetic experience, and thought. It will be especially interesting to see how differences in eye-tracking patterns relate to prior experience. For example, how would a novice learner of motion-graph interpretation differ from one who is very experienced, or how would the patterns change if the motion were more regular and perhaps algebraically definable? In this case, the graph might, in fact be seen in a more gestalt-like manner.

I will close by briefly offering yet another perspective on the core issues being explored, the perspective of evolutionary psychology, in particular, the highly integrative, culturally oriented approach developed by Merlin Donald (1991, 2001). Donald's analysis of the physical, "mimetic" roots of reference helps explain the intricately intertwined role of physical gesture in thought and communication and, more broadly, the physical-social embodiment of thought and language. Space limitations prevent further exploration of Donald's more recent work on the co-evolution of human consciousness and culture (2001) that helps provide a rationale for Radford's strongly cultural approach that deliberately takes into account layers of objectification that integrate the many forms of symbolic expression and the major modalities (action, speech, writing/drawing) in which they can be instantiated.

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RF03: A PROGRESSION OF EARLY NUMBER CONCEPTS

Kathleen Hart

The purpose of the research forum is to describe the research evidence available concerning a Progression in Early Number Concepts. Children around the world are taught some Arithmetic as soon as they start school. The content matter may be dictated by the school or teacher, a book or very often by the curriculum laid down by the government. The word 'curriculum' may describe the range of experience to which the child is introduced when first attending school. For the purpose of this forum we limit discussion to the Number Syllabus for grades 1 to 4. The speakers involved have some evidence of what appears hard and what easy for young children in different parts of the world. The aim is to consider what gives success for the majority of children not what is possible for a talented few.

Participants are urged to bring a copy of the syllabus [grades 1 to 4] from their own country and evidence from their own research with young children or national surveys carried out on the child population. The intention is not to compare performance among countries but to judge the progression of difficulty of concepts through pupils' success or failure. It is likely that there is a great deal in common.

The allocation of time for the forum is three hours and we want to end with some suggestions of what we know and the identification of areas about which we have little or no information. The following activities are planned:

1. In the first session to study and discuss what is required by the published syllabuses of various countries. In many countries these lists of topics form the base on which the efficiency of schools and teachers are judged. Inspectors and evaluators use the syllabus to judge what is happening in schools. How are these lists drawn up?

The syllabuses we have may have a lot in common. They may make assumptions on the relative difficulty of ideas. Do any of them alert the teacher to a great leap in intellectual demand? Is there an assumption that the great majority of pupils will succeed. Is success measured in terms of mastery of most/all of the content or is a pass mark assigned which admits to success in only 30-40 % of the topics?

2. Talks by invited researchers who have investigated the learning of Number with young children, the steps of increasing difficulty and the pitfalls.

3. Participants are encouraged to add their own evidence.

4. We have planned a debate on the idea of 'achievability' [does everything in the syllabus have to be achievable by the pupils?] with a proposer and opponent, speakers from the floor and a vote. There is however only half an hour available for this activity.

The questions asked in this research forum are:

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 155-169. Melbourne: PME.

- From the accumulated evidence can we suggest a progression of Early Number Concepts that seem to achievable by children in even the most basic of learning circumstances?
- Can we identify from the available evidence parts of Arithmetic which cause problems ?
- Can we provide some help for the teachers concerning these 'bottlenecks'?
- Can we formulate some research questions which could add more evidence?

USING GROWTH POINTS TO DESCRIBE PATHWAYS FOR YOUNG CHILDREN'S NUMBER LEARNING

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One important outcome of the Early Numeracy Research Project was the development of a framework of growth points to describe young children's number learning. This paper provides a brief overview of the development and use of these growth points.

INTRODUCTION

The Early Numeracy Research Project ([ENRP], Clarke, 1999) was a three-year project initiated in 1999 by the then Victorian Department of Education, Employment and Training (DEET). The aim was to enhance the mathematical learning of young children (5-year-olds to 8-year-olds) through increasing the professional knowledge of their teachers. The project was conducted in 35 matched samples of trial and reference schools that were representative of the broader population across the state. It could be expected, therefore, that any underlying dimension of achievement, like most human characteristics, would approximate a normal distribution (Rowley, Horne et al., 2001). This was an underlying assumption of the data analysis undertaken throughout the ENRP.

GROWTH POINTS FOR DESCRIBING MATHEMATICAL LEARNING

A basic premise of the ENRP was that knowledge about children's mathematical understanding and development is needed for teachers to plan effective learning experiences for their students. To increase teacher's knowledge of children's mathematical development, the ENRP research team developed a framework of growth points to:

- describe the development of children's mathematical knowledge and understanding in the first three years of school, through highlighting important ideas in early mathematics understanding in a form and language that was useful for teachers;

- reflect the findings of relevant Australian and international research in mathematics education, building on the work of successful projects such as Count Me in Too (Bobis & Gould, 1999);
- reflect the structure of mathematics;
- form the basis of mathematics curriculum planning and teaching; and
- identify those students who may benefit from additional assistance or intervention.

As the impetus for the ENRP was a desire to improve young children's mathematics learning, in order to document any improvement, it was necessary to develop quantitative measures of children's growth. It was considered that a framework of key growth points in numeracy learning could fulfill this requirement. Further, the framework of growth points enabled the identification and description of any improvements in children's mathematical knowledge and understanding, where it existed, by tracking children's progress through the growth points. Trial school students' growth could then be compared to that of students in the reference schools.

In developing the framework of growth points, the project team studied available research on key "stages" or "levels" in young children's mathematics learning (Bobis, 1996; Boulton-Lewis, 1996; Fuson, 1992b; Mulligan & Mitchelmore, 1996; Pearn & Merrifield, 1998; Wright, 1998) as well as frameworks developed by other authors and groups to describe learning. A major influence on the project design was the New South Wales Department of Education initiative Count Me In Too (Bobis & Gould, 1999; New South Wales Department of Education and Training, 1998) that developed a learning framework in number (Wright, 1998) that was based on prior research and, in particular, on the stages in the construction of the number sequence (Steffe et al., 1988; Steffe et al., 1983). The *Count Me In Too Project* used an interview designed to measure children's learning against the framework of stages. It was decided to use a similar approach for the ENRP, but to expand the content of the interview to include domains in measurement and space, and to extend the range of tasks so that is was possible to measure the mathematical growth of all children in the first three years of school.

Following the review of available research, the ENRP team developed a framework of growth points for Number (incorporating the domains of Counting, Place value, Addition and Subtraction Strategies, and Multiplication and Division Strategies), Measurement (incorporating the domains of Length, Mass and Time), and Space (incorporating the domains of Properties of Shape, and Visualisation and Orientation). Within each mathematical domain, growth points were stated with brief descriptors in each case. There are typically five or six growth points in each domain (see Appendix 1, at the end of the Forum papers), and each growth-point was assigned a numeral so that the growth points reached by each child could be entered into a database and analysed. For example, the six growth points for the Counting domain are:

1. *Rote counting*
   Rote counts the number sequence to at least 20, but is unable to reliably count a collection of that size.
2. *Counting collections*
   Confidently counts a collection of around 20 objects.
3. *Counting by 1s (forward/backward, including variable starting points; before/after)*
   Counts forwards and backwards from various starting points between 1 and 100; knows numbers before and after a given number.
4. *Counting from 0 by 2s, 5s, and 10s*
   Can count from 0 by 2s, 5s, and 10s to a given target.
5. *Counting from x (where x>0) by 2s, 5s, and 10s*
   Can count from x by 2s, 5s, and 10s to a given target beginning at variable starting points.
6. *Extending and Applying*
   Can count from a non-zero starting point by any single digit number, and can apply counting skills in practical tasks.

Each growth point represents substantial expansion in mathematical knowledge, and it is acknowledged that much learning takes place between them. In discussions with teachers, the research team described growth points as key “stepping stones” along paths to mathematical understanding. They provide a kind of conceptual landscape upon which mathematical learning occurs (Rowley, Gervasoni et al., 2001). As with any journey, it is not claimed that every student passes all growth points along the way. Indeed, (Wright, 1998) cautioned that “it is insufficient to think that all children’s early arithmetical knowledge develops along a common developmental path” (p. 702). Also, the growth points should not be regarded as necessarily discrete. As with Wright’s (1998) framework, the extent of the overlap is likely to vary widely across young children. However, the order of the growth points provides a guide to the possible trajectory (Cobb & McClain, 1999) of children’s learning. In a similar way to that described by Owens & Gould (1999) in the *Count Me In Too* project: “the order is more or less the order in which strategies are likely to emerge and be used by children” (p. 4).

So that the stability of the growth point scale could be determined, test-retest correlations over one school year and for a 12 month period were calculated. The correlations for March to November ranged from 0.48 to 0.71 in the trial group and from 0.43 to 0.68 in the reference group (Rowley, Horne et al., 2001). With the addition of the summer break, twelve-month test-retest correlations dropped slightly, as would be expected. Over such a long period of time, when children are developing at a great rate, this represents a high level of stability, in that the relative order amongst the children is preserved quite well, although, as the data showed, considerable growth took place (Rowley, Horne et al., 2001).

The framework of growth points formed the structure for the creation of the assessment items used in the ENRP Assessment Interview. Both the interview and the framework of growth points were refined throughout the first two years of the project in response to data collected from more than 20,000 assessment interviews with children participating in the project. The assessment interviews provided teachers with insights about children's mathematical knowledge that otherwise may not have been forthcoming. Further, teachers were able to use this information to plan instruction that would provide students with the best possible opportunities to extend their mathematical understanding. These themes were also present in responses to a survey asking trial school teachers to explain how their teaching had changed as a result of their involvement in the ENRP (Clarke et al., 2002).

The longitudinal nature of the ENRP and the detailed information collected about individual children's mathematical knowledge meant that the data could be analysed to identify particular issues related to mathematical learning. For example, the complexity of the teaching process was highlighted by the spread of growth points within any particular grade level. For Grade 2 children in 2000, the spread in the Counting domain was from Growth Point 1 to Growth Point 6. It is clear that in providing effective learning experiences for children, teachers needed to cater for a wide range of abilities. This is important knowledge for teachers, and implies that the curriculum in which the children engage needs to be broad enough to cater for the differences. This type of professional knowledge also makes it possible for teachers to transform the curriculum and the mathematics instruction they provide. However, while the aim is for all teachers to be so empowered, the reality is that it is difficult for teachers to cater for all children's learning needs in the classroom. This is why alternative learning opportunities are beneficial for some children.

CONCLUSION

The ENRP framework of growth-points, the professional knowledge gained through the ENRP assessment interview and the professional development program, and the analysis of ENRP data about children's mathematical learning provided teachers with many insights about effective mathematics assessment, learning and teaching. This culminated in teachers being more confident that they were meeting the instructional needs of children, and more assured about the curriculum decisions they made.

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NUMBER ATTAINMENT IN SRI LANKAN PRIMARY SCHOOLS

Kathleen Hart

From 1998 to 2003 the Primary Mathematics Project *was operative in Sri Lanka. Part of the project was a longitudinal survey with a number of cohorts of children. Here only the progress in Number is quoted and only one cohort is considered. Other data are available. For the purpose of the forum the data are used to identify what in the syllabus for Number appears to be available to all the pupils and what concepts cause difficulty.*

Sri Lanka is an island off the southern tip of India having an area of some 66 000 square kilometres. The population is composed of Sinhala,Tamils, and Muslims. About 74% are Sinhala who are predominantly Buddhist , about 18% are Tamil and are predominantly Hindu, the 7% who are Muslim speak mainly Tamil. A civil war has continued for 20 years, waged mainly in the north but with sporadic bombings in the cities and resulting in many refugees in the east of the country.

The country has very nearly universal primary education. There is a school within walking distance of each village and the pupils are provided with school uniforms and learning materials by the government. The literacy rate on the island is one of the highest in Asia [87% in 1986] but repeated surveys have shown that mathematics attainment is low. The *Primary Mathematics Project*, funded by DfID of Great Britain and the Sri Lankan government , from which these data are produced, worked in schools all over the island but had limited access to the north because of the war. Part of the project was the National Basic Mathematics Survey [NBMS] designed to provide information on which reforms could be based. Here we report only those aspects of NBMS which concern mathematics attainment. In 1998, a total of 7400 children in grades 3, 5 and 7 were tested with written papers and a smaller sample from grades 1 and 2 were interviewed. The papers were designed to match the curriculum and to cater for what was emphasised in the school textbooks. A group of 30 teachers studied them , tried the questions in schools and revised items. The papers were produced in Tamil and Sinhala. These teachers became the evaluation team and carried out the testing in the nationwide school sample. The emphasis was on the child completing as much of the test as possible so members of the evaluation team were told to read items to pupils who appeared to have trouble reading them and to allow about an hour for completion. The report of the survey appeared in 1999 (Hart & Yahampath, 1999).

In 1999 a longitudinal study was started, taking three regions of the country and following a sample from schools of the four types found in the state education system, both Tamil and Sinhala speaking and with both boys and girls. Over two hundred children from each of grades 3 and 5, at this time, were tested in consecutive years until 2002. The pupils who were first and second graders in 1999 were tested each year until they were in grade five. The data from these youngest cohorts are reported here. In 1999 we took five children from the first grade and five from the second in each of six schools, in three towns. Tasks which matched contents of the class syllabus and which employed manipulatives and symbols were used. Each child was interviewed by a teacher from another school who had been trained on the tasks. An audio tape of the interview and notes from two observers provided the data.

COHORT ONE

The 87 first grade pupils interviewed in 1999 had only been in school for five months. The syllabus indicated what was considered suitable at this stage and so the tasks were chosen to reflect this. Sorting tasks, the use of vocabulary for 'front', middle' and 'behind' were included but here we will concentrate on Number. A form of the classic Piagetian conservation task was used with questions such as 'Are there the same number?' referring to two piles of objects and then a displacement of one set was made to see if the child changed his/her opinion. Under half the sample responded correctly [47, 42, 40 per cent.] Another task was the recognition of symbols for 1,2,and 3. A card with the symbol was shown and the child asked 'Give me that number of toys'.' Read the card for me'.

Ninety five percent could read '1' and 78% could give the correct amount of toys. For the number '2' this was reduced to 85% and 55% and for '3' the results were 70% and 56%. Given a card with '3' written on it but only two toys with it, 55% could rectify the situation. When asked to count beads [16], 50% could do it correctly, with a further 20% completing part of the count.

We did not interview this group of pupils for another 17 months, towards the end of their second year in school but another group of first graders were interviewed towards the end of their first year in 2000. They were from the same schools. The Piagetian conservation task was more successful, 64, 50 and 58 per cent but it is clear that this task cannot be assumed to be within the grasp of the great majority of the children. However matching groups of objects to the symbols'1', '2', '3', was achieved by 100, 93 and 97% of the children and 90% could correct the number of toys to give '3' In this group 90% could accurately give seven objects, matching the symbol.. The range of objects which could be counted was also extended, so that 88% could count up to 16. However when ,as the syllabus suggested, the children were asked to add 3 and 4 [written on cards] only 67% could do it. Forty five percent counted on their fingers to add these two numbers.

When we tested cohort one towards the end of their second year in school, they were again interviewed on tasks which reflected the class syllabus. By now over 90% of the group [the sample was reduced to 79 from 87] could read number symbols of 1 to 9, say which number was smaller and identify that the cards for 5, 7 and 9 were missing from a sequence of cards. Given a set of dominoes they could total the number of dots on two touching sections, that is provided with objects to count they could provide a total over ten.

In 2000 the interviewers added some questions on subtraction, since it was at the end of the second grade. 'Eight birds were in the tree and three flew away, how many were left?'. Eighty percent had this correct and 95 % when the question was repeated with '8 flew away'.

All the questions given to grades 1 and 2 reported so far were given orally. The syllabus does contain some written computations so the following were given to the pupils, written on paper. The percentage success is shown below in Table 1.

5 +3 ——	7 +8 ——	4 +4 ——	2+4=……	6+6=……	
86%	62%	91%	80%	68%	Success rate
5 -2 ——	9 -2 ——	7 -7 ——	8-4……	3-3……	
72%	61%	58%	58%	54%	Success rate

Table 1. Written Computations Year Two .[2000]

The questions are now too difficult for nearly half the pupils so the syllabus seems to be ahead of the children.

THIRD GRADE. COHORT 1

Towards the end of the third grade the same cohort of children were asked questions pertaining to the syllabus. By now the expectation is that pupils are writing computations in their books and there is a third grade textbook. The tests, given in November, had some questions given orally and a test paper which had printed questions but which the evaluator could read to the child if needed [there were only five in a group]. The oral questions were about Number, Shape and Money and very similar to those asked in Year 2. For Number there was a further question about the number which comes before and after ' 7.' On this latter there was success at the 85% level and on the earlier questions success was at over 95%. The second year work tested here had been consolidated. When it came to the regular third grade questions on the test paper the mean score for the paper was 39%. Failure has arrived.

By the end of third grade the pupils are expected to deal with two and three digit numbers, do addition and subtraction algorithms including decomposition. cope with multiplication of two 2 digit numbers and even shade one half of a diagram. The only question which had a facility of over 85% was completing a sequence of numbers from the five times table. About half the pupils could correctly identify the number of hundreds, tens and ones given a three digit number. The two digit algorithms were adequately completed only if it was single digit work involved, that is no regrouping of tens. This is shown in Table 2 below.

75 -32 _____	81 -25 _____	39 +18 _____	305 +217 _____	
72%	41%	45%	36%	Success rate

Table 2 . Two Digit Algorithms. Grade 3

Cohort One was tested again in grades four and five. Other cohorts were followed and it was obvious that although performance was mixed, those who performed badly or even at a 'middle' level in grade 3 never achieved great success later. Grade 3 seems a very great hurdle. According to the teachers of these children 'place value' is a problem and certainly the algorithms quoted above become not just difficult but very difficult when decomposition is involved.

In the forum we will look at these and other data and try to sequence what is in the syllabus so that the difficulties become more obvious to a teacher. The aim is not to throw out what teachers, certainly in this sample, feel is the mathematics they want or intend to teach but to offer information which might provide a better chance of success. All participants are encouraged to bring data and also the Number Curriculum taught in the first four years of their primary schools.

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MATHEMATICS RECOVERY: FRAMEWORKS TO ASSIST STUDENTS' CONSTRUCTION OF ARITHMETICAL KNOWLEDGE.

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Mathematics Recovery was the outcome of a three-year research and development project at Southern Cross University in northern New South Wales, conducted in 1992-5. The project received major funding from the Australian Research Council and major contributions in the form of teacher time, from regional government and Catholic school systems. Over the 3-year period, the project involved working in 18 schools with 20 teachers and approximately 200 participating first-grade students (Wright, 2000).

MR can be regarded as consisting of two distinct but interrelated components. One component concerns an elaborated body of theory and practice for working with students, that is, teaching early number knowledge (Wright et al., 2000; & Wright et al., 2002). The second component concerns distinctive ways of working with teachers, that is, providing effective, long-term professional development in order to enable teachers to learn about working with students (Wright, 2000, pp.140-4).

The theoretical origins of MR are in the research program of Les Steffe, a professor in mathematics education, at the University of Georgia in the United States. In the 1970s and 1980s, Steffe's research focused almost exclusively on early number learning (e.g. Steffe & Cobb, 1988; Steffe, 1992). The goal of this research is to develop psychological models to explain and predict students' mathematical learning and development. Of particular interest in this approach, is the strategies – for which Steffe uses the Piagetian label of 'schemes', that the student uses in situations that are problematic for the student, and how these schemes develop and are re-organised over the course of an extended teaching cycle, as observed in teaching sessions mainly, but also in pre- and post- interview-based assessments.

Steffe's research and Mathematics Recovery have as their basic orientation, von Glasersfeld's theory of cognitive constructivism – an epistemological theory that has been developed and explicated over the last 30 or more years, (e.g. von Glasersfeld, 1978; 1995). Von Glasersfeld's theory is a theory about knowing – how humans come to know, rather than for example, an approach to teaching.

Assessment in Mathematics Recovery involves a one-on-one interview, in which the student is presented with groups of tasks, where each group relates to a particular aspect of early number learning. The assessment has two broad purposes. First, it should provide a rich, detailed description of the student's current knowledge of early number. Second, the assessment should lead to determination of levels on the relevant tables in the framework of assessment and learning (Wright et al., 2000)

One of the key elements of the MR program is its framework for assessment and learning – usually referred to as the Learning Framework in Number. One important function of the framework is to enable summary profiling of students' current knowledge. The profiling is based on six aspects of number early number knowledge referred to as a model. Each model contains a progression of up to six levels indicating the development of students' knowledge on that particular aspect of early number learning. Taken together, the models can be regarded as laying out a multi-faceted progression of students' knowledge and learning in early number, and in this sense the models are analogous to a framework (Wright et al., 2002, e.g. p. 77).

The view in MR is that models consisting of progressions of levels of student knowledge constitute one important part of a learning framework. A comprehensive learning framework should also contain: (a) descriptions of assessment tasks that relate closely to the levels on each of the models, and thus enable determination of the student's level; (b) descriptions of other assessment tasks which might not relate directly to the models but nevertheless, have the potential to provide important information about early number knowledge; (c) comprehensive descriptions of the likely responses of students to the all assessment tasks; and (d) descriptions of other aspects of early number knowledge considered to be relevant to students' overall learning of early number. A framework as just described can rightly be regarded as a comprehensive framework for assessment and learning.

The Learning Framework in Number (LFIN) is regarded as a rich description of the students' early number knowledge. This includes, but is not limited to, the strategies that student uses to solve what adults might regard as simple number tasks (additive, subtractive). While it is important to document students early arithmetical strategies, it is not sufficient to describe students' knowledge merely in terms of the currently available strategies. As well, there are important aspects of students' knowledge not simply described in terms of strategies used to solve problems. These aspects include for example, facility with spoken and heard number words, and ability to identify (name) numerals.

The six aspects of the framework are described in terms of a progression of levels. These are: (a) strategies for counting and solving simple addition and subtraction tasks; (b) very early place value knowledge, that is, ability to reason in terms of tens and ones; (c) facility with forward number word sequences; (d) facility with backward number word sequences; (e) facility with numeral identification; and (f) early knowledge of multiplication and division. Other aspects of the framework relate to: (a) combining and partitioning small numbers without counting; (b) using five and ten as reference points in numerical reasoning; (c) use of finger patterns in numerical contexts; (d) relating number to spatial patterns; and (e) relating number to temporal sequences. While each aspect can be considered from a distinct perspective, it is also important to focus on the inter-relationships of the aspects.

MR assessment tells the teacher 'where the student is' and the learning framework indicates 'where to take the student', but teachers don't necessarily have the time to design and develop specific instructional procedures. In the period 1999-2000, Wright and colleagues developed an explicit framework for instruction. Thus the instructional settings and activities used in earlier versions of MR were incorporated into an instructional framework (usually referred to as the Instructional Framework for Early Number – IFEN). The instructional framework differs in form from the learning framework because its purpose is different. Nevertheless it is informed by and strongly linked to the learning framework (Wright et al., 2002). The framework sets out a progression of key teaching topics which are organized into three strands as follows:

- Counting — instruction to progressively develop use of counting by ones, to solve arithmetical tasks.
- Grouping — instruction to develop arithmetical strategies other than counting by ones.
- Number words and numerals — instruction to develop facility with FNWSs, BNWSs and a range of aspects related to numerals.

Each of the three strands spans a common set of five phases of instruction. Each key topic contains on average, six instructional procedures. Each instructional procedure includes explicit descriptions of the teachers' words and actions, as well as descriptions of the instructional setting (materials, instructional resources), and notes on purpose, teaching and students' responses. Finally, each instructional procedure typically is linked to a level in one or more of the models (aspects) of the learning framework. Thus the teacher is not only provided with exemplary instructional procedures suited to any particular student but is forearmed with detailed knowledge of ways the student is likely to respond to each instructional procedure.

Recent research (Wright, 1998; 2002), highlights the relative complexities of students' early number knowledge, and the usefulness of close observation and assessment in enabling detailed understanding of students' arithmetical knowledge and strategies. Critical to the efforts of teachers to address students' learning difficulties in mathematics are elaborated exemplars of theory-based practice directed at addressing mathematics learning difficulties.

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Appendix 1
ENRP Number Growth Points (Preparatory – Year 2)

Counting Growth Points

0. Not apparent.
   Not yet able to state the sequence of number names to 20.
1. Rote counting
   Rote counts the number sequence to at least 20, but is not yet able to reliably count a collection of that size.
2. Counting collections
   Confidently counts a collection of around 20 objects.
3. Counting by 1s (forward/backward, including variable starting points; before/after)
   Counts forwards and backwards from various starting points between 1 and 100; knows numbers before and after a given number.
4. Counting from 0 by 2s, 5s, and 10s
   Can count from 0 by 2s, 5s, and 10s to a given target.
5. Counting from x (where x >0) by 2s, 5s, and 10s
   Given a non-zero starting point, can count by 2s, 5s, and 10s to a given target.
6. Extending and applying counting skills
   Can count from a non-zero starting point by any single digit number, and can apply counting skills in practical tasks.

Place Value Growth Points

0. Not apparent
   Not yet able to read, write, interpret and order single digit numbers.
1. Reading, writing, interpreting, and ordering single digit numbers
   Can read, write, interpret and order single digit numbers.
2. Reading, writing, interpreting, and ordering two-digit numbers
   Can read, write, interpret and order two-digit numbers.
3. Reading, writing, interpreting, and ordering three-digit numbers
   Can read, write, interpret and order three-digit numbers.
4. Reading, writing, interpreting, and ordering numbers beyond 1000
   Can read, write, interpret and order numbers beyond 1000.
5. Extending and applying place value knowledge
   Can extend and apply knowledge of place value in solving problems.

Strategies for Addition & Subtraction Growth Points

0. Not apparent
   Not yet able to combine and count two collections of objects.
1. Count all (two collections)
   Counts all to find the total of two collections.
2. Count on
   Counts on from one number to find the total of two collections.
3. Count back/count down to/count up from
   Given a subtraction situation, chooses appropriately from strategies including count back, count down to and count up from.
4. Basic strategies (doubles, commutativity, adding 10, tens facts, other known facts)
   Given an addition or subtraction problem, strategies such as doubles, commutativity, adding 10, tens facts, and other known facts are evident.
5. Derived strategies (near doubles, adding 9, build to next ten, fact families, intuitive strategies)
   Given an addition or subtraction problem, strategies such as near doubles, adding 9, build to next ten, fact families and intuitive strategies are evident.
6. Extending and applying addition and subtraction using basic, derived and intuitive strategies
   Given a range of tasks (including multi-digit numbers), can solve them mentally, using the appropriate strategies and a clear understanding of key concepts.

Strategies for Multiplication & Division Growth Points

0. Not apparent
   Not yet able to create and count the total of several small groups.
1. Counting group items as ones
   To find the total in a multiple group situation, refers to individual items only.
2. Modelling multiplication and division (all objects perceived)
   Models all objects to solve multiplicative and sharing situations.
3. Abstracting multiplication and division
   Solves multiplication and division problems where objects are not all modelled or perceived.
4. Basic, derived and intuitive strategies for multiplication
   Can solve a range of multiplication problems using strategies such as commutativity, skip counting and building up from known facts.
5. Basic, derived and intuitive strategies for division
   Can solve a range of division problems using strategies such as fact families and building up from known facts.
6. Extending and applying multiplication and division
   Can solve a range of multiplication and division problems (including multi-digit numbers) in practical contexts.

RF04: THEORIES OF MATHEMATICS EDUCATION

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The purpose of this Forum is to stimulate critical debate in the area of theory use and theory development, and to consider future directions for the advancement of our discipline. The Forum opens with a discussion of why theories are essential to the work of mathematics educators and addresses possible reasons for why some researchers either ignore or misunderstand/misuse theory in their work. Other issues to be addressed include the social turn in mathematics education, an evolutionary perspective on the nature of human cognition, the use of theory to advance our understanding of student cognitive development, and models and modelling perspectives. The final paper takes a critical survey of European mathematics didactics traditions, particularly those in Germany and compares these to historical trends in other parts of the world.

INTRODUCTION

Our conception and preference for a particular mathematics education theory invariably influences our choice of research questions as well as our theoretical framework in mathematics education research. Although we have made significant advances in mathematics education research, our field has been criticized in recent years for its lack of focus, its diverging theoretical perspectives, and a continued identity crisis (Steen, 1999). At the dawn of this new millennium, the time seems ripe for our community to take stock of the multiple and widely diverging mathematical theories, and chart possible courses for the future. In particular, we need to consider the important role of theory building in mathematics education research.

Issues for consideration include:

1. What is the role of theory in mathematics education research?
2. How does Stokes (1997) model of research in science apply to research in mathematics education?
3. What are the currently accepted and widely used learning theories in mathematics education research? Why have they gained eminence?
4. What is happening with constructivist theories of learning?
5. Embodied cognition has appeared on the scene in recent years. What are the implications for mathematics education research, teaching, and learning?
6. Theories of models and modelling have received considerable attention in the field in recent years. What is the impact of these theories on mathematics research, teaching, and learning?

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7. Is there a relationship between researchers' beliefs about the nature of mathematics and their preference for a particular theory?
8. How do theories used in European mathematics didactics traditions compare with those used in other regions of the world? Do European traditions reveal distinct theoretical trends?

There are several plausible explanations for the presence of multiple theories of mathematical learning, including the diverging, epistemological perspectives about what constitutes mathematical knowledge. Another possible explanation is that mathematics education, unlike "pure" disciplines in the sciences, is heavily influenced by cultural, social, and political forces (e.g., D'Ambrosio, 1999; Secada, 1995; Skovsmose & Valero, 2002). As Lerman indicates in his paper, the switch to research on the social dimensions of mathematical learning towards the end of the 1980s resulted in theories that emphasized a view of mathematics as a social product. Social constructivism, which draws on the seminal work of Vygotsky and Wittgenstein (Ernest, 1994) has been a dominant research paradigm ever since. On the other hand, cognitively oriented theories have emphasized the mental structures that constitute and underlie mathematical learning, how these structures develop, and the extent to which school mathematics curricula should capture the essence of workplace mathematics (e.g., see Stevens, 2000).

Stokes (1997) suggested a new way of thinking about research efforts in science, one that moves away from the linear one-dimensional continuum of "basic, to applied, to applied development, to technology transfer." Although this one-dimensional linear approach has been effective, Stokes argued that it is too narrow and does not effectively describe what happens in scientific research. In Pasteur's Quadrant, Stokes proposed a 2-dimensional model, which he claimed offered a completely different conception of research efforts in science. If one superimposes the Cartesian co-ordinate system on Stokes' model, the Y -axis represents "pure" research (such as the work of theoretical physicists) and the X-axis represents "applied" research" (such as the work of inventors). The area between the two axes is called "Pasteur's Quadrant" because it is a combination (or an amalgam) of the two approaches. If we apply Stokes' model to mathematics education research, we need to clearly delineate what is on the Y-axis of Pasteur's quadrant if we are to call our field a science. Frank Lester elaborates further on this issue in the opening paper of this Forum. Steve Lerman extends the discussion initiated in Lester's contribution on the pivotal, albeit misunderstood role of theories in mathematics education, and presents theoretical frameworks most frequently used in PME papers during the 1990-2001 time period. Lerman's analysis reveals that a wide variety of theories are used by PME researchers with a distinct preference for social theories over cognitive theories. An interesting avenue for discussion is whether the particular social theories used in this time period reveal a distinct geographic distribution, and if so why? Luis Moreno-Armella presents an evolutionary perspective on the nature of human cognition, particularly the evolution of representations, which he aptly terms pre-theory, as it serves as a foundation for the present discussion. John Pegg and David Tall compare neo-Piagetian theories in order to use the similarities and differences among theories to address fundamental questions in learning. Lyn English and Richard Lesh present a models and modeling perspective which innovatively combines the theories of Piaget and Vygotsky to pragmatically address the development and real life use of knowledge via model construction. The Forum concludes with a review by Günter Törner and Bharath Sriraman on European theories of mathematics education, with a focus on German traditions. Eight major tendencies are highlighted in 100 years of mathematics education history in Germany; these tendencies reflect trends that have occurred internationally.

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THE PLACE OF THEORY IN MATHEMATICS EDUCATION RESEARCH

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As most, if not all, of you know, the current emphasis in the United States being placed on so called *scientific research* in education, is driven in large part by political forces. Much of the public and some of the professional conversation has begun with an assumption that the purpose of research is to determine “what works,” and the discourse has focused largely on matters of research design and methods. One consequence has been the rekindling of attention to experimental designs and quantitative methods that had faded from prominence in education research over the past two decades or so. Far less prominent in recent discussions about educational research has been the place of theory.

Scholars in other social science disciplines (e.g., anthropology, psychology, sociology) often justify their research investigations on grounds of developing understanding by building or testing theories. In contrast, the current infatuation in the U.S. with "what works" seems to leave education researchers with less latitude to conduct studies to advance theoretical goals. It is time for a serious examination of the role that theory should play in the formulation of problems, in the design and methods employed, and in the interpretation of findings in education research. In this brief presentation, I speculate about why so many researchers seem to misunderstand or misuse theory and suggest how we might think about the goals of research that might help eliminate this misunderstanding and misuse.

Why is so much of our research atheoretical?

Mathematics education research is an interesting and important area for such an examination. Although math ed research was aptly characterized less than 15 years ago by Kilpatrick (1992) and others as largely *atheoretical*, a perusal of recent articles in major MER journals reveals that theory is alive and well: indeed, Silver and Herbst (2004) have noted that expressions such as "theory-based," "theoretical framework," and "theorizing" are common. In fact, they muse, manuscripts are often rejected for being atheoretical. The same is true of proposals submitted for PME meetings. However, the concerns raised decades ago persist; too often researchers ignore, misunderstand, or misuse theory in their work.

We are our own worst enemies

In my mind there are two basic problems that must be dealt with if we are to expect theory to play a more prominent role in our research. The first has to do with the widespread misunderstanding of what it means to adopt a theoretical stance toward our work. The second is that some researchers, while acknowledging the importance of theory, do not feel qualified to engage in serious theory-based work. I attribute both of these problems to: (a) the failure of our graduate programs to properly equip novice researchers with adequate preparation in theory, and (b) the failure of our research journals to insist that authors of research reports offer serious theory-based explanations of their findings.

Writing about the state of U.S. doctoral programs, Hiebert, Kilpatrick, and Lindquist (2001) suggest that mathematics education is a complex system and that improving the process of preparing doctoral students means improving the entire system, not merely changing individual features of it. They insist that "the absence of system-wide standards for doctoral programs [in mathematics education] is, perhaps, the most serious challenge facing systemic improvement efforts. . . . Indeed, participants in the system have grown accustomed to creating their own standards at each local site" (p. 155). One consequence of the absence of commonly accepted standards is that there is a very wide range of requirements of different programs. At one end of the continuum of requirements are a few programs that focus on the preparation of researchers. At the other end are those programs that require little or no research training beyond taking a research methods course or two. In general, with few exceptions, doctoral programs are replete with courses and experiences in research methodology, but woefully lacking in courses and experiences that provide students with solid theoretical underpinnings for future research. Without solid understanding of the role of theory in conceptualizing and conducting research, there is little chance that the next generation of mathematics education researchers will have a greater appreciation for theory than is currently the case. Put another way, we must do a better job of cultivating a predilection for theory within the mathematics education research community.

During my term as editor of the *Journal for Research in Mathematics Education* in the early1990s, I found the failure of authors of research reports to pay serious attention to explaining the results of their studies one of the most serious shortcomings. A simple example from the expert-novice problem solver research illustrates what I mean. It is not enough simply to report: Experts *do X when they solve problems and novices do Y.* Were the researcher guided by theory, a natural question would be to ask WHY? Having some theoretical perspective guiding the research provides a framework within which to attempt to answer *Why* questions. Without a theoretical orientation, the researcher can speculate at best or offer no explanation at all.

Many mathematics educators hold misconceptions about the role of theory

Time constraints prevent me from providing a detailed discussion of what I see as the most common misconceptions about theory, so I will simply list four and say a few words about them.

1. Theory-based explanation given by "decree" rather than evidence. Some researchers (e.g., Eisenhart, 1991) insist that educational theorists prefer to address and explain the results of their research by "theoretical decree" rather than with solid evidence to support their claims. That is to say, there is a belief among some researchers that theorists make their data fit their theory.

2. Data have to "travel." Sociologist and ethnographer, John Van Maanen (1988), has observed that data collected under the auspices of a theory has to "travel" in the sense that (in his view) data too often must be stripped of context and local meaning in order to serve the theory.

3. Standard for discourse not helpful in day-to-day practice. Related to the previous concern, is the observation that researchers tend to use a theory to set a standard for scholarly discourse that is not functional outside the academic discipline. Conclusions produced by the logic of theoretical discourse too often are not at all helpful in day-to-day practice. Researchers don't speak to practitioners! The theory is irrelevant to the experience of practitioners (cf., Lester & Wiliam, 2002).

4. No triangulation. Sociologist, Norman Denzin (1978) has discussed the importance of theoretical triangulation, by which he means the process of compiling currently relevant theoretical perspectives and practitioner explanations, assessing their strengths, weaknesses, and appropriateness, and using some subset of these perspectives and explanations as the focus of empirical investigation. By using a single theoretical perspective to frame one's research, such triangulation does not happen.

There is no doubt that rigid, uncritical adherence to a theoretical perspective can lead to these sorts of offenses. However, I know of no good researchers who are guilty of such crimes. Instead, more compelling arguments can be marshaled in support of using theory.

Why theory is essential

Again, time constraints for this presentation prevent me from elaborating on the reasons why theory should play an indispensable role in our research. Let me mention a few of the most evident. (In the following brief discussion I borrow heavily from an important paper written about 15 years ago by Andy diSessa [1991])

1. There are no data without theory. We have all heard the claim, "The data speak for themselves!" Dylan Wiliam and I have argued elsewhere that actually data have nothing to say. Whether or not a set of data can count as evidence of something is determined by the researcher's assumptions and beliefs as well as the context in which it was gathered (Lester & Wiliam, 2000). One important aspect of a researcher's beliefs is the theoretical perspective he or she is using; this perspective makes it possible to make sense of a set of data.

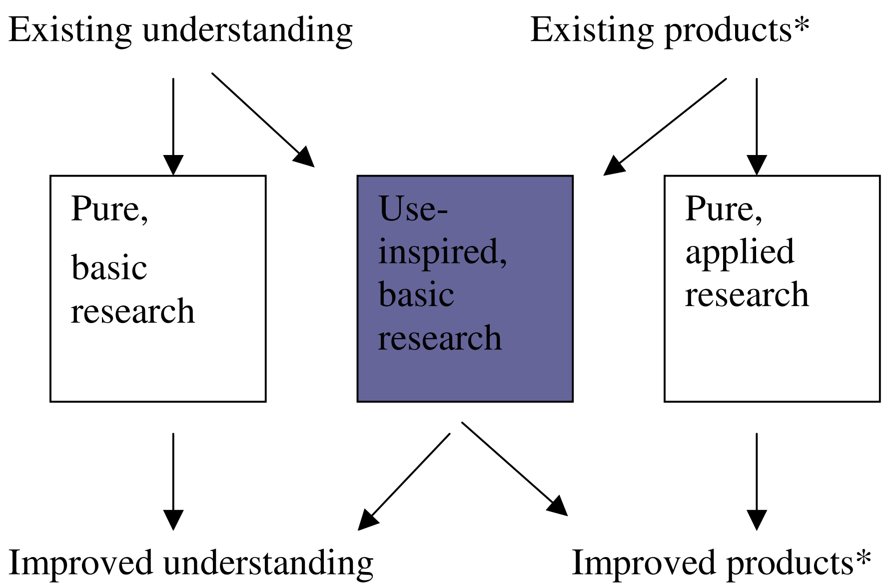
2. Good theory transcends common sense. In the paper mentioned above, diSessa (1991) argues that theoretical advancement is the linchpin in spurring practical progress. He notes that, sure, you don't need theory for many everyday problems—purely empirical approaches often are enough. But often things aren't so easy. Deep understanding that comes from concern for theory building is often essential to deal with truly important problems.

3. Need for deep understanding, not just "for this" understanding. Related to the above, is the need we have to deeply understand some things—the important, big questions (e.g., What does it mean to be intelligent? What does it mean to understand something?)—not simply find solutions to immediate problems and dilemmas. Theory helps us develop deep understanding. (I say more about understanding in the next section.)

A different way to think about the goals of research and the place of theory

In his book, *Pasteur's Quadrant: Basic Science and Technological Innovation,* Donald Stokes (1997) presents a new way to think about scientific and technological research and their purposes. Stokes begins with a detailed discussion of the history of development of the current U.S. policy for supporting advanced scientific study (I suspect similar policies exist in other industrialized countries). He notes that from the beginning of the development of this policy shortly after World War II there has been an inherent tension between the pursuit of *fundamental understanding* and *considerations of use*. This tension is manifest in the, often radical, separation between basic and applied science. He argues that prior to the latter part of the 19th Century, scientific research was conducted largely in pursuit of deep understanding of the world. But, the rise of microbiology in the late 19th Century brought with it a concern for putting scientific understanding to practical use. He illustrates this concern with the work of Louis Pasteur. Of course, Pasteur working in his laboratory wanted to understand the process of disease at the most basic level, but he wanted that understanding to be applicable to dealing with silk worms, anthrax in sheep, cholera in chickens, spoilage in milk, and rabies in people. The work of Pasteur suggests that one could not understand his science without knowing the extent to which he had considerations of use in mind as well as fundamental understanding. Stokes proposed a model for thinking about scientific research that blends the two motives: the *quest for fundamental understanding* and *considerations of use*.

Adapting Stokes's model to educational research in general, and mathematics education research in particular, I have come up with a slightly different model (see Figure 1). In the final section of this short paper, I describe the relationship between my model and the place of theory in mathematics education research.



* "Products" include such things as instructional materials, professional development programs, and district educational policies.

Figure 1. Adaptation of Stokes's model to educational research

A *bricolage* approach to theory in mathematics education research

Even if there is no need to make a case for the importance of theory in our research, there is a need to suggest how researchers, especially novices, can deal with the almost mystifying range of theories and theoretical perspectives that are being used. In a chapter to appear in a forthcoming handbook of research in mathematics education, Cobb (in press) considers how mathematics education researchers might cope with the multiple and frequently conflicting theoretical perspectives that currently exist. He observes:

The theoretical perspectives currently on offer include radical constructivism, sociocultural theory, symbolic interactionism, distributed cognition, information-processing psychology, situated cognition, critical theory, critical race theory, and discourse theory. To add to the mix, experimental psychology has emerged with a renewed vigor in the last few years. . . . In the face of this sometimes bewildering array of theoretical alternatives, the issue . . . is that of how we might make and justify our decision to adopt one theoretical perspective rather than another.[2]

Cobb goes on to question the repeated (mostly unsuccessful) attempts that have been made in mathematics education to derive instructional prescriptions directly from background theoretical perspectives. He insists that it is more productive to compare and contrast various theoretical perspectives in terms of the manner in which they orient and constrain the types of questions that are asked about the learning and teaching of mathematics, the nature of the phenomena that are investigated, and the forms of knowledge that are produced. To his recommendation, I would add that comparing and contrasting various perspectives would have the added benefit of both enhancing our understanding of important phenomena and increasing the usefulness of our investigations (c.f., Lester & Wiliam, 2002).

I propose to view the theoretical perspectives we adopt for our research as sources of ideas that we can appropriate and modify for our purposes as mathematics educators. This process of developing tools for our research is quite similar to that of instructional design as described by Gravemeijer (1994). He suggests that instructional design resembles the thinking process characterized by the French word *bricolage,* a notion borrowed from Claude Levi–Strauss. A *bricoleur* is a handyman who invents pragmatic solutions in practical situations and is adept at using whatever is available. Similarly, I suggest, as do Cobb and Gravemeijer, that rather than adhering to one particular theoretical perspective, we act as *bricoleurs* by adapting ideas from a range of theoretical sources to suit our goals—goals that should aim not only to deepen our fundamental understanding of mathematics learning and teaching, but also to aid us in providing practical wisdom about problems practitioners care about. If we begin to pay serious attention to these goals, the problem of theory is likely to be resolved.

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[2] Because Cobb's paper is currently in draft form and is not yet publicly available, no page numbers are provided.

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THEORIES OF MATHEMATICS EDUCATION: A PROBLEM OF PLURALITY?

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Today, in many countries around the world, constraints on the funding of Universities are leading to restrictions on educational research. In some countries national policy is also placing constraints on the kinds of research that will be funded (e.g. the effects of *No Child Left Behind* policy in the USA). At the same time we see research in mathematics education proliferating, not just in quantity but also, as in the concerns of this Research Forum, in the range of theories that are drawn upon in our research. In my contribution I want to ask: is this surprising, or unusual, and is it necessarily a hindrance to the effectiveness of educational research in mathematics?

In discussing this, I would argue that we need a specific language that enables an analysis of intellectual fields and their growth, a language that will not be provided by mathematics or by psychology. I will draw on some of the later work of the sociologist of education, Basil Bernstein, in particular his 1999 paper on research discourses (Bernstein, 1999). Following that, I will make some remarks about the use of theory.

A Language of Research Fields

Bernstein draws on two notions: hierarchy and verticality. Discourses are described as hierarchical where knowledge in the domain is a process of gradual distancing, or abstraction, from everyday concepts. Hierarchical discourses require an apprenticeship; they position people as initiated or apprenticed. Clearly academic and indeed school mathematics are examples of hierarchical discourses. Research (Cooper & Dunne, 2000) shows that setting mathematics tasks in everyday contexts can mislead some students, namely those from low socio-economic background, into privileging the everyday context and the meanings carried in them over the abstract or esoteric meanings of the discourse of academic mathematics.

His second notion, verticality, describes the extent to which a discourse grows by the progressive integration of previous theories, what he calls a vertical knowledge structure, or by the insertion of a new discourse alongside existing discourses and, to some extent, incommensurable with them. He calls these horizontal knowledge structures. Bernstein offers science as an example of a vertical knowledge structure and, interestingly, both mathematics and education (and sociology) as examples of horizontal knowledge structures. He uses a further distinction that enables us to separate mathematics from education: the former has a strong grammar, the latter a weak grammar, that is, with a conceptual syntax not capable of generating unambiguous empirical descriptions. Both are examples of hierarchical discourses in that one needs to learn the language of linear algebra or string theory just as one needs to learn the language of radical constructivism or embodied cognition. It will be obvious that linear algebra and string theory have much tighter and specific concepts and hierarchies of concepts than radical constructivism or embodied cognition. Adler and Davis (forthcoming) point out that a major obstacle in the development of accepted knowledge in mathematics for teaching may well be the strength of the grammar of the former and the weakness of the latter. Where we can specify accepted knowledge in mathematics, knowledge about teaching is always disputed.

As a horizontal knowledge structure, then, it is typical that mathematics education knowledge will grow both within discourses and by the insertion of new discourses in parallel with existing ones. Thus we can find many examples in the literature of work that elaborates the functioning of the process of reflective abstraction, as an instance of the development of knowledge within a discourse. But the entry of Vygotsky's work into the field in the mid-1980s (Lerman, 2000) with concepts that differed from Piaget's did not lead to the replacement of Piaget's theory (as the proposal of the existence of oxygen replaced the phlogiston theory). Nor did it lead to the incorporation of Piaget's theory into an expanded theory (as in the case of non-Euclidean geometries). Indeed it seems absurd to think that either of these would occur precisely because we are dealing with a social science, that is, we are in the business of interpretation of human behaviour. Whilst all research, including scientific research, is a process of interpretation, in the social sciences, such as education, there is a double hermeneutic (Giddens, 1976) since the 'objects' whose behaviour we are interpreting are themselves trying to make sense of the world.

Education, then, is a social science, not a science. Sociologists of scientific knowledge (Kuhn, Latour) might well argue that science is more of a social science than most of us imagine, but social sciences certainly grow both by hierarchical development but especially by the insertion of new theoretical discourses alongside existing ones. Constructivism grows, and its adherents continue to produce novel and important work; models and modelling may be new to the field but already there are novel and important findings emerging from that orientation.

I referred above to the incommensurability, in principle, of these parallel discourses. Where a constructivist might interpret a classroom transcript in terms of the possible knowledge construction of the individual participants, viewing the researcher's account as itself a construction (Steffe & Thompson, 2000), someone using socio-cultural theory might draw on notions of a zone of proximal development. Constructivists might find that describing learning as an induction into mathematics, as taking on board concepts that are on the intersubjective plane, incoherent in terms of the theory they are using (and a similar description of the reverse can of course be given). In this sense, these parallel discourses are incommensurable.

There is an apparent contradiction between the final sentences of the last two paragraphs. If I am claiming that there is important work emerging in different discourses of mathematics education research, but I also claim that discourses are incommensurable, within which discourse am I positioning myself to write these sentences? Is there a meta-discourse of mathematics education in which we can look across these theories? I will make some remarks about this position in the next section.

Theories in Use in Mathematics Education

First I will make some remarks drawn from a recent research project on the use of theories in mathematics education. Briefly we (Tsatsaroni, Lerman & Xu, 2003) examined a systematic sample of the research publications of the mathematics education research community between 1990 and 2001, using a tool that categorised research in many ways. I will only refer here to our findings concerning how researchers use theories in their work as published in PME Proceedings.

Our analysis showed that just over 85% of all papers in the proceedings had an orientation towards the empirical, with a further 5% moving from the theoretical to the empirical, and this has changed little over the years. A little more than three-quarters are explicit about the theories they are using in the research reported in the article. Again this has not varied across the years. The theories that are used have changed, however. We can notice an expanding range of theories being used and an increase in the use of social theories, based on the explicit references of authors, in some cases by referring to a named authority. These fields or names represent theories used, not the frequency of their occurrence in papers.

Year	Theoretical fields other than educational psychology and/or mathematics
1990	Brousseau
1991	Philosophy of mathematics
1992	Vygotsky
1993	Vygotsky
1994	Brousseau, Chevellard, Poststructuralism
1995	Embodied cognition, Educational research
1996	Vygotsky, Situated cognition, Philosophy of mathematics
1997	Situated cognition, Vygotsky, Philosophy of mathematics
1998	Situated cognition, Vygotsky, Philosophy of mathematics
1999	Socio-historical practice
2000	Chevellard
2001	Semiotics, Bourdieu, Vygotsky, Philosophy

Table 1: Theoretical fields

We might suggest that there is a connection here with creating identities, making a unique space from which to speak in novel ways, but we would need another study to substantiate and instantiate this claim.

We can say that there has been a substantial increase in the number of fields from 1994, although it is too early to say whether this trend will continue, as 1999 and 2000 showed a dropping off. What is clear is that the range of intellectual resources, including sociology, philosophy, semiotics, anthropology, etc., is very broad.

In our analysis of how authors have used theories we have looked at whether, after the research, they have revisited the theory and modified it, expressed dissatisfaction with the theory, or expressed support for the theory as it stands. Alternatively, authors may not revisit the theory at all, content to apply it in their study. We have found that more than three-quarters fall into this last category, just over 10% revisit and support the theory, whilst four percent propose modifications. Two authors in our sample ended by opposing theory. This pattern has not changed over the years. Further findings can be found in Tsatsaroni, Lerman and Xu (2003).

The development and application of an analytical tool in a systematic way, paying attention to the need to make explicit and open to inspection the ways in which decisions on placing articles in one category or another, enables one to make all sorts of evidence-based claims. In particular, I would argue that one can observe and record development within discourses and the development of new parallel discourses because of the adoption of a sociological discourse as a language for describing the internal structure of our intellectual field, mathematics education research.

Conclusion

Finally, I will comment on concerns about the effectiveness of educational research in a time of multiple and sometimes competing paradigms, described here as discourses. ‘Effectiveness’ is a problematic notion, although one that certainly figures highly in current discourses of accountability. It arises because by its nature education is a research field with a face towards theory and a face towards practice. This contrasts with fields such as psychology in which theories and findings can be applied, but practice is not part of the characteristic of research in that field. Research in education, in contrast, draws its problems from practice and expects its outcomes to have applicability or at least significance in practice. Medicine and computing are similar intellectual fields in this respect.

However, what constitutes knowledge is accepted or rejected by the criteria of the social field of mathematics education research. Typically, we might say necessarily, research has to take a step away from practice to be able to say something about it. Taking the results of research into the classroom calls for a process of recontextualisation, a shift from one practice into another in which a selection must take place, allowing the play of ideology. To look for a simple criterion for acceptable research in terms of ‘effectiveness’ is to enter into a complex set of issues.

Indeed 'effectiveness' itself presupposes aims and goals for, in our case, mathematics education. To ignore the complexity is to lose the possibility of critique and hence I am not surprised by the multiplicity of theories in our field and the debates about their relative merits, nor do I see it as a hindrance.

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THE ARTICULATION OF SYMBOL AND MEDIATION IN MATHEMATICS EDUCATION

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I describe some basic elements of a *pre-theory* of Mathematics Education. Our field is at the crossroad of a science, mathematics, and a community of practice, education. The interests of this community include the people whose learning takes place at schools and the corresponding intellectual offer from the institutional sides. But as soon as we enter the space of mathematics, we discover a different discipline from the natural sciences. It is the strictly *symbolic nature* of mathematics that makes a big difference and gives to mathematics education, as a research field, its characteristic features that distinguishe it from similar endeavours with respect to other scientific fields, such as biology for instance. I am not implying, of course, that there is no abstraction or concept development involved in those other fields.

More recently, the presence of computers has introduced a new way of looking at symbols and mathematical cognition and has offered the potentiality to re-shape the goals of our whole research field. The urgency to take care of teaching and learning from the research activities has resulted in practices without corresponding theories. Again, I must make clear I am not dismissing the considerable and important results this community has produced. I simply want to underline that institutional pressures can result more frequently than desirable, in losing track of research goals. Perhaps this is a motive to re-consider the need to enter a more organized level of reflection in our community. There is nothing bad in having the chance to look at educational phenomena from different viewpoints but it is better if we can generate a synergy between those viewpoints that, eventually, has as its output a new and stronger theory. Nevertheless the tension between the local and the global also comes to existence here. Being an interested observer and modest participant in the field, I have come to think that only local explanations are possible in our field. *Local theories* might be the answer to the plethora of explanations we encounter around us. But even if local, a mathematics education theory must be developed from scaffolding that eventually crystallizes in the theory. In our case, part of that scaffolding is constituted by mathematics itself, and by a community of practice, as already mentioned.

What sort of machine is the human brain, that it can give birth to mathematics? – an old question that Stanislas Dehaene has aptly posed anew in his book *The Number Sense* (1997). This is the kind of question that, in the long run, must be answered in order to improve the understanding of our field. Nevertheless, trying to answer it will demand an interdisciplinary and longitudinal effort. At the end of the day, we will need to understand why we are able to create symbolic worlds (mathematics, for instance) and why our minds are essentially incomplete outside the co-development with material and symbolic technologies. Our symbolic and mediated nature comes to the front as soon as we try to characterize our intellectual nature. Evolution and culture have left its traits in our cognition, in particular, in our capacity to duplicate the world at the level of symbols.

Diverging epistemological perspectives about what constitutes mathematical knowledge modulate multiple conceptions of learning and the present theories of what constitutes mathematical education as a research discipline. Today, however, there is substantial evidence that the encounter between the conscious mind and distributed cultural systems has altered human cognition and has changed the tools with which we think. The origins of writing and how writing as a technology changed cognition is key from this perspective (Ong, 1988). These examples suggest the importance of studying the evolution of mathematical systems of representation as a vehicle to develop a proper epistemological perspective for mathematics education.

Human evolution is coextensive with tool development. In a certain sense, human evolution has been an *artificial* process as tools were always designed with the explicit purpose of transforming the environment. And so, since about 1.5 millions years ago, our ancestor Homo Erectus designed the first stone tools and took profit from his/her voluntary memory and gesture capacities (Donald, 2001) to evolve a pervasive technology used to consolidate their early social structures. The increasing complexity of tools demanded optimal coherence in the use of memory and in the transmission, by means of articulate gestures, of the building techniques. We witness here what is perhaps the first example of deliberate teaching. Voluntary memory enabled our ancestors to engender a mental template of their tools. Templates lived in their minds, resulting from activity, granting an objective existence as abstract objects even before they were *extracted* from the stone. Thus, tool production was not only important for plain survival, but also for broadening the mental world of our ancestors –introducing a higher level of objectivity.

The actions of our ancestors were producing a *symbolic* version of the world: A world of intentions and anticipations they could imagine and *crystallize* in their tools. What their tools meant was the same as what they *intended* to do with them. They could *refer* to their tools to *indicate* their *shared* intentions and, after becoming familiar with those tools, they were looked as *crystallized* images of all the activity that was embedded in them.

We suggest that the synchronic analysis of our relationship with technology, no matter how deep, hides profound meanings of this relationship that coheres with the co-evolution of man and his tools. It is then, unavoidable, to revisit our technological past if we want to have an understanding of the present. Let us present a substantial example.

Arithmetic: Ancient Counting Technologies

Evidence of the construction of one–to–one correspondences between arbitrary collections of concrete objects and a *model set* (a template) can already be found between 40000 and 10000 B.C. For instance, hunter-gatherers used bones with marks (tallies). In 1937, a wolf bone dated to about 30000 B.C. was found in Moravia (Flegg, 1983). This reckoning technique (using a one-to-one correspondence) reflects a deeply rooted trait of human cognition. Having a set of stone bits or the marks on a bone as a *modeling set* constitutes, up to our knowledge, the oldest counting technique humans have designed. The modeling set plays, in all cases, an instrumental role for the whole process. In fact, something is crystallized by marking a bone: The *intentional* activity of finding the size of a set of hunted pieces, for instance, or as some authors have argued, the intentional activity of computing time.

The modeling set of marks, plays a role similar to the role played by a stone tool as both mediate an activity, finding the size, and both crystallize that activity. Between 10000 and 8000, B.C. in Mesopotamia, people used sets of pebbles (clay bits) as modeling sets. This technique was inherently limited. If, for instance, we had a collection of twenty pebbles as modeling set then, it would be possible to estimate the size of collections of twenty or less elements. Nevertheless, to deal with larger collections (for instance, of a hundred or more elements), we would need increasingly larger models with evident problems of manipulation and maintenance. And so, the embodiment of the one-to-one technique in the set of pebbles inhibits the extension of it to further realms of experience. It is very plausible that being conscious of these difficulties, humans looked for alternative strategies that led them to the brink of a new technique: the idea that emerged was to replace the elements of the model set with clay pieces of diverse shapes and sizes, *whose numerical value were* *conventional*. Each piece *compacted* the information of a whole former set of simple pebbles —according to its shape and size. The pieces of clay can be seen as embodiments of pre-mathematical symbols. Yet, they lacked rules of transformation that allowed them to constitute a genuine mathematical system.

Much later, the consolidation of the urbanization process (about 4000 B.C.) demanded, accordingly, more complex symbol systems. In fact, the history of complex arithmetic signifiers is almost determined by the occurrence of bullae. These clay envelopes appeared around 3500-3200 B.C. The need to record commercial and astronomic data led to the creation of symbol systems among which mathematical systems seem to be one of the first. The counters that represented different amounts and sorts —according to shape, size, and number— of commodities were put into a bulla which was later sealed. And so, to secure the information contained in a bulla, the shapes of the counters were printed on the bulla outer surface. Along with the merchandise, producers would send a bulla with the counters inside, describing the goods sent. When receiving the shipment, the merchant could verify the integrity of it.

A counter in a bulla *represents* a *contextual* number — for example, the number of sheep in a herd; not an abstract number: there is five of something, but never *just five*. The shape of the counter is impressed in the outer surface of the bulla. The mark on the surface of the bulla *indicates* the counter inside. That is, the mark on the surface keeps an *indexical relation* with the counter inside as its referent. And the counter inside has a *conventional* meaning with respect to amounts and commodities. It must have been evident, after a while, that *counters inside were no longer needed;* impressing them in the outside of the bulla was enough. That decision altered the semiotic status of those external inscriptions. Afterward, instead of impressing the counters against the clay, scribes began using sharp styluses that served *to draw* on the clay the shapes of former counters. From this moment on, the symbolic expression of numerical quantities acquired an infra-structural support that, at its time, led to a new epistemological stage of society. Yet the semiotic contextual constraints, made evident by the simultaneous presence of diverse numerical systems, was an epistemological barrier for the mathematical evolution of the *numerical ideographs*. Eventually, the collection of numerical (and contextual) systems was replaced by one system (Goldstein, La naissance du nombre en Mesopotamie. La Recherche, *L'Univers des Nombres* (hors de serie),1999). That system was the sexagesimal system that also incorporated a new symbolic technique: numerical value according to position. In other words, it was a *positional* system. There is still an obstacle to have a complete numerical system: the presence of zero that is of primordial importance in a positional system to eliminate representational ambiguities. For instance, without zero, how can we distinguish between 12 and 102? We would still need to look for the help of context.

Mathematical objects result from a sequence of crystallization processes that, at a certain level of evolution, has an ostensible social and cultural dimension. As the levels of reference are hierarchical the crystallization process is a kind of recursive process that allows us to state:

Mathematical symbols co-evolve with their mathematical referents and the induced semiotic objectivity makes possible for them to be taken as shared in a community of practice.

In what follows, we should try to articulate some reflections regarding the presence of the computational technologies in mathematical thinking. It is interesting to notice that even if the new technologies are not yet fully integrated within the mathematical universe, their presence will eventually erode the mathematical way of thinking. The blending of mathematical symbol and computers has given way to an *internal mathematical universe* that works as the reference fields to the mathematical signifiers living in the screens of computers. This takes abstraction a large step further.

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USING THEORY TO ADVANCE OUR UNDERSTANDINGS OF STUDENT COGNITIVE DEVELOPMENT

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INTRODUCTION

Over recent years, various theories have arisen to explain and predict cognitive development in mathematics education. Our focus is to raise the debate beyond a simple comparison of detail in different theories to move to use the similarities and differences among theories to address fundamental questions in learning. In particular, a focus of research on fundamental learning cycles provides an empirical basis from which important questions concerning the learning of mathematics can and should be addressed.To assist us with this focus we identify two kinds of theory of cognitive growth:

- **global theories of long-term growth** of the individual, such as the stage-theory of Piaget (e.g., Piaget & Garcia, 1983).
- **local theories of conceptual growth** such as the action-process-object-schema theory of Dubinsky (Czarnocha et al., 1999) or the unistructural-multistructural-relational-extended abstract sequence of SOLO Model (**S**tructure of **O**bserved **L**earning **O**utcomes, Biggs & Collis, 1982, 1991; Pegg, 2003).

Some theories (such as that of Piaget, the SOLO Model, or more broadly, the enactive-iconic-symbolic theory of Bruner, 1966) incorporate both aspects. Others, such as the embodied theory of Lakoff and Nunez (2000) or the situated learning of Lave and Wenger (1990) paint in broader brush-strokes, featuring the underlying biological or social structures involved. A range of global longitudinal theories each begin with physical interaction with the world and, through the use of language and symbols, become increasingly abstract. Table 1 shows four of these theoretical developments.

Piaget Stages	van Hiele Levels (Hoffer,1981)		SOLO Modes	Bruner Modes
Sensori Motor	I	Recognition	Sensori Motor	Enactive
Preoperational	II	Analysis	Ikonic	Iconic
Concrete Operational	III	Ordering	Concrete	Symbolic
Formal Operational	IV	Deduction	Symbolic	
	V	Rigour	Formal	
			Post-formal	

Table 1: Global stages of cognitive development

What stands out from such 'global' perspectives is the gradual biological development of the individual, growing from dependence on sensory perception through physical interaction and on, through the use of language and symbols, to increasingly sophisticated modes of thought. SOLO offers a valuable viewpoint as it explicitly nests each mode within the next, so that an increasing repertoire of more sophisticated modes of operation become available to the learner. At the same time, all modes attained remain available to be used as appropriate. As we go on to discuss fundamental cycles in conceptual learning, we therefore need to take account of the development of modes of thinking available to the individual.

LOCAL CYCLES

Our current focus is on 'local' theories, formulated within a 'global' framework whereby the cycle of learning in a specific conceptual area is related to the overall cognitive structures available to the individual. A recurring theme identified in these theories is a fundamental cycle of growth in the learning of specific concepts, which we frame within broader global theories of individual cognitive growth.

One formulation is found in SOLO. This framework can be considered under the broad descriptor of neo-Piagetian models. It evolved as reaction to observed inadequacies in Piaget's formulations and shares much in common with the ideas of such theorists as Case, Fischer, and Halford.

In particular, SOLO focuses attention upon students' responses rather than their level of thinking or stage of development. It arose, in part, because of the substantial décalage problem associated with Piaget's work when applied to the school learning context, and the identification of a consistency in the structure of responses from large numbers of students across a variety of learning environments in a number of subject and topic areas. While SOLO has its roots in Piaget's epistomelogical tradition, it is based strongly on information-processing theories and the importance of working memory capacity. In addition, familiarity with content and context invariably plays an influential role in determining the response category.

At the 'local' focus SOLO comprises a recurring cycle of three levels referred to as *unistructural, multistructural, and relational* (a UMR cycle). The application of SOLO takes a multiple-cycle form of at least two UMR cycles in each mode where the R level response in one cycle evolves to a new U level response in the next cycle. This not only provides a basis to explore how basic concepts are acquired, but it also provides us with a description of how students react to reality as it presents itself to them. The second cycle then offers the type of development that is most evident and a major focus of primary and secondary education.

Another formulation concerns various theories of process-object encapsulation, in which processes become interiorised and then conceived as mental concepts, which has been variously described as *action*, *process*, *object* (Dubinsky), *interiorization*, *condensation*, *reification* (Sfard) or *procedure*, *process*, *concept* (Gray & Tall).

Theories of 'process-object encapsulation' were formulated at the outset to describe a sequence of cognitive growth. Each of these theories, founded essentially on the ideas of Piaget, saw cognitive growth through actions on existing objects that become interiorized into processes and then encapsulated as mental objects.

Dubinsky described this cycle as part of his APOS theory (action-process-object-schema), although he later asserted that objects could also be formed by encapsulation of schemas as well as encapsulation of processes. Sfard (1991) proposed an operational growth through a cycle she termed interiorization-condensation-reification, which she complemented by a 'structural' growth that focuses on the properties of the reified objects formed in an operational cycle.

Gray and Tall (1994) focused more on the role of *symbols* acting as a pivot, switching from a process (such as addition of two numbers, say 3+4) to a concept (the sum 3+4, which is 7). The entity formed by a symbol and its pivotal link to *pro*cess or con*cept* they named a *procept*. They observed that the growth of procepts occurred often (but not always) through a sequence that they termed procedure-process-procept. In this model a procedure is a sequence of steps carried out by the individual, a process is where a number of procedures (≥0) giving the same input-output are regarded as the same process, and the symbol shared by both becomes process or concept.

The various process-object theories have a spectrum of development from process to object. The process-object theories of Dubinsky and Sfard were mainly based on experiences of students doing more advanced mathematical thinking in late secondary school and at university. For this reason their emphasis is on formal development rather than on earlier acquired forms of thinking such as associated with Piaget's sensori-motor or pre-operational stages. Note too that Sfard's first state is referred to as an 'interiorized process', which is the same name given in Dubinsky's second, however, both see the same main components of the second stage:– that the process is seen as a whole without needing to perform the individual steps.

We now turn to the cycles of development that occur within a range of different theories. These have been developed for differing purposes. The SOLO Model, for instance, is concerned with assessment of performance through observed learning outcomes. Other theories, such as those of Davis (1984), Dubinsky (Czarnocha et al., 1999), Sfard (1991), and Gray and Tall (1994) are concerned with the sequence in which the concepts are constructed by the individual).

SOLO of Biggs & Collis	Davis	APOS of Dubinsky	Gray & Tall
 Unistructural Multistructural Relational Unistructural	 Procedure (VMS) Integrated Process Entity	 Action Process Object Schema	[Base Objects] Procedure Process Procept

Table 2: Local cycles of cognitive development

As can be seen from table 2, there are strong family resemblances between these cycles of development. Note that Davis used the term 'visually moderated sequence' for a step-by-step procedure. Although a deeper analysis of the work of individual authors will reveal discrepancies in detail, there are also insights that arise as a result of comparing one theory with another as assembled in table 3.

<table>
<tr><th>SOLO</th><th>Davis</th><th>APOS</th><th>Gray & Tall</th></tr>
<tr><td rowspan="2">Unistructural</td><td rowspan="3">VMS
Procedure</td><td></td><td>Base Object(s)</td></tr>
<tr><td rowspan="2">Action</td><td rowspan="2">Procedure
[Multi-Procedure]</td></tr>
<tr><td>Multistructural</td></tr>
<tr><td>Relational</td><td>Process</td><td>Process</td><td>Process</td></tr>
<tr><td>Unistructural
(Extended
Abstract)</td><td>Entity</td><td>Object
Schema</td><td>Procept</td></tr>
</table>

Table 3: The fundamental cycle of conceptual construction

CONCLUSION

Our purpose in this brief paper is not so much to attempt to produce a unified theory incorporating these perspectives. Instead, it is to advocate an approach that seeks to understand the meanings implicit in each broad theory and to see where each may shed light on the other, leading to theoretical correspondences and dissonances.

While at first glance there may appear to be irreconcilable differences between the theoretical stances (e.g., van Hiele is concerned with underlying thinking skills and SOLO with observable behaviours), a closer examination can reveal there is much to consider. A synthesis provides a fresh perspective in considering student growth in understanding.

A primary goal of teaching should be to stimulate cognitive development in students. Such development as described by these fundamental learning cycles is not inevitable. Ways to stimulate growth, to assist with the reorganisation of earlier levels need to be explored. Important questions about strategies appropriate for different levels or even if it is true that *all* students pass through all levels in sequence. Research into such questions is sparse. Nevertheless, the notion of fundamental cycles of learning does provide intriguing potential for research.

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TRENDS IN THE EVOLUTION OF MODELS & MODELING PERSPECTIVES ON MATHEMATICAL LEARNING AND PROBLEM SOLVING

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Models and modeling (M&M) research often investigates the nature of understandings and abilities that are needed in order for students to be able to use what they have (presumably) learned in the classroom in “real life” situations beyond school. Nonetheless, M&M perspectives evolved out of research on concept development more than research on problem solving; and, rather than being preoccupied with the kind of word problems emphasized in textbooks and standardized tests, we focus on (simulations of) problem solving “in the wild.” Also, we give special attention to the fact that, in a technology-based *age of information*, significant changes are occurring in the kinds of “mathematical thinking” that is coming to be needed in the everyday lives of ordinary people in the 21st century – as well as in the lives of productive people in future-oriented fields that are heavy users of mathematics, science, and technology.

In modern knowledge economies, systems – ranging from communication systems to economic or accounting systems - are among the most important “things” that impact the lives of ordinary people. Some of these systems occur naturally, while others are created by humans. But, in any case, mathematics is useful for making (or making sense of) such systems precisely because mathematics is the study of structure. That is, it is the study of systemic properties of structurally interesting systems.

In future-oriented fields that range from design sciences to life sciences, industry advisors to university programs consistently emphasize that:

> The kind of people we most want to hire are those who are proficient at (a) making sense of complex systems, (b) working within teams of diverse specialists, (c) adapting rapidly to a variety of rapidly evolving conceptual tools, (d) working on multi-staged projects that require planning and collaboration among many levels and types of participants, and (e) developing sharable and re-useable conceptual tools that usually need to draw on a variety of disciplines – and textbook topic areas.

Both of the preceding trends shift attention beyond *mathematics as computation* toward *mathematics as conceptualization, description, and explanation*. But, they also raise the following kinds of questions that lie at the heart of M&M research in mathematics education.

- What is the nature of the most important classes of problem-solving situations where mathematics, science, and technology are needed for success in real life situations beyond school?
- What mathematical constructs or conceptual systems provide the best foundations for success in these situations?
- What does it mean to "understand" these constructs and conceptual systems?
- How do these understandings develop?
- What kinds of experiences facilitate (or retard) development?
- How can people be identified whose exceptional abilities do not fit the narrow and shallow band of abilities emphasized on standardized tests – or even school work?

Related questions are: (a) Why do students who have histories of getting A's on tests and coursework often do not do well beyond school? (b) What is the relationship between the learning of "basic skills" and a variety of different kinds of deeper or higher-order understandings or abilities? (c) Why do problem solving situations that involve collaborators and conceptual tools tend to create as many conceptual difficulties as they eliminate? (d) In what ways is "mathematical thinking" becoming more multi-media - and more contextualized (in the sense that knowledge and abilities are organized around experience as much as around abstractions, and in the sense that relevant ways of thinking usually need to draw on ways for thinking that seldom fall within the scope of a single discipline or textbook topic area). (e) How can instruction and assessment be changed to reflect the fact that, when you recognize the importance of a broader range of understandings and abilities, a broader range of people often emerge as having exceptional potential?

M&M perspectives assume that such questions should be investigated through research, not simply resolved though political processes - such as those that are emphasized when "blue ribbon" panels of experts develop curriculum standards for teaching or testing. Furthermore, we believe that such questions are not likely to be answered through content-independent investigations about *how people learn* or *how people solve problems*, and they are only indirectly about the nature (and/or the development) of humans - or the functioning of human brains. This is because they are about the nature of mathematical and scientific knowledge, and they are about the ways this knowledge is useful in "real life" situations. So, researchers with broad and deep expertise in mathematics and science should play significant roles in collaborating with experts in the learning and cognitive sciences.

Theoretical perspectives for M&M research trace their lineage to modern descendents of Piaget and Vygotsky - but also (and just as significantly) to *American Pragmatists* such as William James, Charles Sanders Peirce, Oliver Wendell Holmes, George Herbert Mead, and John Dewey. And partly for this reason, M&M perspectives reflect "blue collar" approaches to research. That is, we focus on the development of knowledge (and conceptual tools) to inform "real life" decision-making issues – where (a) the criteria for success are not contained within any preconceived theory, (b) productive ways of thinking usually need to draw on more than a single theory, and (c) useful knowledge usually needs to be expressed in the context of conceptual tools that are powerful (for some specific purpose), sharable (with other people), and re-useable (beyond the context in which they were developed). Thus, M&M research often focuses on model-development rather than proceeding too quickly to theory development and hypothesis testing; and, before rushing ahead to try to teach or test various mathematical concepts, processes, beliefs, habits of mind, or components of a productive problem solving personae, we conduct developmental investigations about the nature of what it means to "understand" them.

One way that mathematics educators have investigated questions about what is needed for success beyond school is by observing people "thinking mathematically" in everyday situations. Sometimes, such studies compare "experts" with "novices" who are working in fields such as engineering, agriculture, medicine, or business management - where "mathematical thinking" often is critical for success. Such ethnographic investigations often have been exceedingly productive and illuminating. Nonetheless, from the perspectives of M&M research, they also tend to have some significant shortcomings. For example, we must be skeptical of observations which depend heavily on preconceived notions about <u>where</u> to observe (in grocery stores? carpentry shops? car dealerships? engineering firms? Internet cafés?), <u>whom</u> to observe (street vendors? shoppers? farmers? cooks? engineers? baseball fans?), <u>when</u> to observe (when they're estimating sizes? calculating with numbers? minimizing routes? describing, explaining, or predicting the behaviors of complex systems?), and <u>what</u> to count as "mathematical thinking" (e.g., planning, monitoring, assessing, explaining, justifying steps during multi-step projects, or deciding what information to collect about specific decision-making issues). Consequently, in simple observational studies, close examinations of underlying assumptions often expose unwarranted prejudices about what it means to "think mathematically" - and about the nature of "real life" situations in which mathematics is useful.

A second way to investigate *what's needed for success beyond school* is to use *multi-tier design experiments* (Lesh, 2002) in which (a) students develop models for making senses of mathematical problem solving situations, (b) teachers develop models for creating (and making sense of) students' modeling activities, and (c) researchers develop models for creating (or making sense of) interactions among students, teachers, and relevant learning environments. We sometimes refer to such studies as *evolving expert studies* (Lesh, Kelly & Yoon, in press) because the final products that are produced tend to represent significant extensions or revisions in the thinking of each of the participants who were involved. Such methodologies respect the opinions of diverse groups of stake holders whose opinions should be considered. On the one hand, nobody is considered to have privileged access to the truth – including, in particular, the researchers. All participants (from students to teachers to researchers) are considered to be in the model development business; and, similar principles are assumed to apply to "scientific inquiry" at all levels. So, everybody's ways of thinking are subjected to examination and possible revision.

For the preceding kind of *three-tiered design experiments*, each tier can be thought of as *a longitudinal development study in a conceptually enriched environment*. That is, a goal is to go beyond studies of <u>typical</u> development in <u>natural</u> environments to also focus on <u>induced</u> development within <u>carefully controlled</u> environments. Finally, because the goal of M&M research is to investigate the nature and development of constructs or conceptual systems (rather than investigating and making claims students per se), we often investigate how understandings evolve in the thinking of "problem solvers" who are in fact teams (or other learning communities) rather than being isolated individuals. So, we often compare individuals with groups in somewhat the same manner that other styles of research might compare experts and novices, or gifted students and average ability students.

Investigations from an M&M perspective have led to the growing realization that, in a technology-based age of information, even the everyday lives of ordinary people are increasingly impacted by systems that are complex, dynamic, and continually adapting; and, this is even more true for people in fields that are heavy users of mathematics and technology. Such fields include design sciences such as engineering or architecture, social sciences such as economics or business management, or life sciences such as new hyphenated fields involving bio-technologies or nano-technologies. In such fields, many of the systems that are most important to understand and explain are dynamic (living), self-organizing, and continually adapting.

M&M research is showing that it is possible for average ability students to develop powerful models for describing complex systems that depend on only new uses of elementary mathematical concepts that are accessible to middle school students. However, when we ask *What kind of mathematical understandings and abilities should students master?* attention should shift beyond asking *What kind of computations can they execute correctly?* to also ask *What kind of situations can they describe productively?* ... This observation is the heart of M&M perspectives on learning and problem solving.

Traditionally, problem solving in mathematics education has been defined as getting from givens to goals when the path is not obvious. But, according to M&M perspectives, goal directed activities only become problematic when the "problem solver" (which may consist of more than an isolated individual) needs to develop a more productive way of thinking about the situation (given, goals, and possible solution processes). So, solutions to non-trivial problems tend to involve a series of modeling cycles in which current ways of thinking are iteratively expressed, tested, and revised; and, each modeling cycle tends to involve somewhat different interpretations of givens, goals, and possible solution steps.

Results from M&M research make it clear that average ability students are indeed capable of developing powerful mathematical models and that the constructs and conceptual systems that underlie these models often are more sophisticated than anything that anybody has tried to teach the relevant students in school.

However, the most significant conceptual developments tend to occur when students are challenged to repeatedly express, test, and revise their own current ways thinking - not because they were guided along a narrow conceptual trajectory toward (idealized versions of) their teachers ways of thinking (Lesh & Yoon, 2004). That is, development looks less like progress along a path; and, it looks more like an inverted genetic inheritance tree - where great grandchildren trace their evolution from multiple lineages which develop simultaneously and interactively.

In general, when knowledge develops through modeling processes, the knowledge and conceptual tools that develop are instances of situated cognition. Models are always molded and shaped by the situations in which they are created or modified; and, the understandings that evolve are organized around experience as much as around abstractions. Yet, the models and underlying conceptual systems that evolve often represent generalizable ways of thinking. That is, they are not simply situation-specific knowledge which does not transfer. This is because models (and other conceptual tools) are seldom worthwhile to develop unless they are intended to by powerful (for a specific purpose in a specific situation), re-useable (in other situations), and sharable (with other people).

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ISSUES AND TENDENCIES IN GERMAN MATHEMATICS-DIDACTICS: AN EPOCHAL PERSPECTIVE

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It is a positive sign that an international discussion on theories of mathematics education is taking place especially in the wake of TIMMS and PISA. It is laudable of PME to take the initiative to closely examine specific geographic trends in mathematics education research in comparison with trends that are concurrently occurring (or occurred) elsewhere (as reported in English et al., 2002; Schoenfeld, 1999, 2002). In doing so we can reflect and hypothesize on why certain trends seem to re-occur, sometimes invariantly across time and geographic location. Numerous reviews about the state of German mathematics didactics are available in German (see [1], Hefendehl et al., 2004; Vollrath et al., 2004). However there are no extant attempts to trace and analyze the last hundred years of "mathematics didactic" trends in Germany in comparison to what is happening internationally. This is our modest attempt to fill this void.

Some preliminary remarks on terminology and history: It has become standard practice for researchers writing in English to use the term "Mathematikdidaktik" when referring to mathematics education in Germany. However, there is no real comprehensive English equivalent for the term "Mathematikdidaktik". Neither "didactics" nor "math-education" describes the full flavor and the historical nuances associated with this German word. Even the adjective "German" is imprecise since educational research approaches in Germany splintered in the aftermath of World War II, with different philosophical schools of thought developing in the former East (GDR) and the west (FRG) on research priorities for university educators, until the reunification which occurred in 1990. Currently the 16 states in Germany reveal a rich heterogeneity in the landscape of mathematics teaching, teacher training and research methods, which manifests itself to insiders who microscopically examine the TIMSS- and PISA-results. However the reasons for this heterogeneity remain a mystery to outsiders. Given the page limits we outline in macroscopic terms the historical reasons for this heterogeneity. In doing so we do not differentiate explicitly between the alignment (or misalignment!) of theories preferred by university educators in comparison to practices of mathematics instruction in schools. The mutual dependencies between the two is certainly an interesting research question which brings into focus the system wide effectiveness (or ineffectiveness) of educational research (see for example Burkhardt & Schoenfeld, 2003).

1. **The Pedagogical tradition of mathematics teaching-Mathematics as Educational Value:** Reflections on the processes of mathematics teaching and learning have been a long-standing tradition in Germany. The early proponents of these theories of teaching and learning are recognizable names even for current

researchers. Chief among these early theorists was Adam Reise "the arithmetician" who stressed hand computation as a foundational learning process in mathematics. This emphasis is found in the pedagogical classics of the 19th century written by Johann Friedrich Herbart (1776-1841), Hugo Gaudig (1860-1923), Georg Kerschensteiner (1854-1932) (see Jahnke, 1990; Führer, 1997; Huster, 1981). The influence of this approach echoed itself until the 1960's in the so-called didactics of mathematics teaching in elementary schools to serve as a learning pre-requisite for mathematics in the secondary schools.

2. **Mathematician-Initiators of traditions in didactics research (20th Century):** In the early part of the previous century, mathematicians like Felix Klein (1849-1925) and Hans Freudenthal (1905-1990) (who was incidentally of German origin) became interested in the complexities of teaching and learning processes for mathematics in schools. The occasionally invoked words "Erlangen program" and "mathematization" are the present day legacy of the contributions of Klein and Freudenthal to mathematics education. Klein characterized geometry (and the teaching of it) by focussing on the related group of symmetries to investigate mathematical objects left invariant under this group. The present day emphasis of using functions (or functional thinking) as the conceptual building block for the teaching and learning of algebra and geometry, is reminiscent of a pre-existing (100 year old) Meraner Program. During this time period one also finds a growing mention in studying the psychological development of school children and its relationship to the principles of arithmetic (Behnke, 1950). This trend was instrumental in the shaping of German mathematics curricula in the 20th century with the goal being to expose students to mathematical analysis at the higher levels. The most notable international development in this time period was the founding of the ICMI in 1908, presided by Felix Klein. One of the founding goals of ICMI was to publish mathematics education books, which were accessible to both teachers and their students. We see this as one of the first attempts to "elementarize" (or simplify) higher level mathematics by basing it on a sound scientific (psychological) foundation. Mathematics educators like Lietzmann (1919) claimed that "didactic" principles were needed in tandem with content to offer methodological support to teachers. This approach mutated over the course of the next 50 years well into the 1970's. The over-arching metaphor for mathematics education researchers during this time period was to be a gardener, one who maintains a small mathematical garden analogous to ongoing research in a particular area of mathematics. The focus of research was on analyzing specific content and using this as a basis to elaborate on instructional design (Reichel 1995, Steiner, 1982). This approach is no longer in vogue and is instrumental in creating a schism between mathematicians and "mathematics-didakters," partly analogous to the math wars in the United States.

3. **"Genetic" Mathematics Instruction: Ineffectual Visionary Bridges (1960 – 1990):** The word "genetic" was used to exemplify an approach to mathematics instruction to prevent the danger of mathematics taught completely via procedures

(Lenné, 1969). Several theorists stressed that mathematics instruction should be focussed on the "genetic" or a natural construction of mathematical objects. This can be viewed as an earlier form of constructivism. This approach to mathematics education did not gather momentum. The word "genetisch" occurs frequently in the didactics research literature until the 1990's.

4. The New Math (1960 – 1975): Parallel to the new math movement occurring in post-Sputnik United States, an analogous reform movement took place in Germany (mostly in the West, but partly adopted by the East, see [1]). A superficial inspection seems to point to a realization of Klein's dream of teaching and learning mathematics by exposing students to its structure. This reform took on the dynamic of polarizing scientists (mathematicians) to work in and with teacher training, the resulting outcome being a lasting influence on mathematics instruction during this time period. Unlike the United States teachers were able to implement a structural approach to mathematics in the classroom. This can be attributed to the fact that during this time period there was no social upheaval in Germany, unlike the U.S where the press for social reform in the classroom (equity and individualized instruction) interfered with this approach to mathematics education. The fact that German "new math" did not survive the tide of time indicates that there was difficulty in implementing it effectively.

5. The birth of didactics as a research discipline (1975): While the new mathematics movement was subject to a host of criticisms, one positive outcome was the founding of the Gesellschaft für Didaktik der Mathematik (German Mathematics Didactics Society), which stresses that mathematics didactics was a science whose concern was to rest the mathematical thinking and learning on a sound theoretical (and empirically verifiable foundation). This was a radical step search for mathematics education research in Germany, one that consciously attempted to move away from the view of a math educator as a part-time mathematician (recall Klein's garden). Needless to say, we could easily write an entire book if we wanted to spell out the ensuing controversy over the definition of this new research discipline in Germany (see Bigalke, 1974; Dress, 1974; Freudenthal, 1974; Griesel, 1974, Laugwitz, 1974; Leuders, 2003; Otte, 1974; Tietz, 1974 Wittmann, 1974; 1992). However, the point to be taken from the founding of this society and a new scientific specialty is that the very debate we have undertaken here, that is, to globally define theories of mathematics education has in fact many localized manifestations such as in Germany.

6. Mathematical Teaching and Learning- A Socialistic and an Individualistic Process (1980 – today): One of the consequences of founding a new discipline of science was the creation of new theories to better explain the phenomenon of mathematical learning. The progress in cognitive science in tandem with interdisciplinary work with social scientists led to the creation of "partial" paradigms about how learning occurs. Bauersfeld's (1988,1995) views of mathematics and mathematical learning as a socio-cultural process within which the individual operates can be viewed as one of the major contributions to theories of mathematics education.

7. An Orientation Crisis - The Conundrums posed by new Technology (1975 – today): Weigand's (1995) work poses the rhetorical question as to whether mathematics instruction is undergoing yet another crisis. The advent of new technologies opened up a new realm of unimagined possibilities for the learner, as well as researchable topics for mathematics educators. The field of mathematics education in Germany oriented itself to address the issues of teaching and learning mathematics with the influx of technology. However the implications of redefining mathematics education, particularly the "*hows*" of mathematics teaching and learning in the face of new technology poses the conundrum of the need to continually re-orient the field, as technology continually evolves (see Noss / Hoyles (1995) for an ongoing global discussion).

8. TIMMS and PISA -The Anti-Climax (1997 – today): The results of TIMMS and PISA brought these seven aforementioned "tendencies" to a collision with mathematics educators and teachers feeling under-appreciated in the wake of the poor results. These assessments also brought mathematicians and politicians back into the debate for framing major policies, which would affect the future of mathematics education in Germany. Mathematics education is now in the midst of new crisis because the results of these assessments painted German educational standing in a poor global light. A detailed statistically sieved inspection of the results indicated that poor scores could be related to factors other than flaws in the mathematics curriculum, and/or its teaching and learning, that is to socioeconomic and cultural variables in a changing modern German society. Thus mathematics education in Germany would now have to adapt to the forces and trends creating havoc in other regions of the globe (see Burton, 2003; Steen, 2001).

Conclusions

Epochal viewpoints: The eight major tendencies that we have highlighted in the 100 years of mathematics education history in Germany reflect trends that have occurred internationally. Each epoch is characterized by an underlying metaphor that shaped the accepted theories of that time period. Felix Klein's view of a mathematics educator was that of a mathematician-gardener tending to all aspects of a specialized domain within mathematics, including its teaching and learning. This shifted to a focus on the structure of modern mathematics itself and partly to the teacher as a "transmitter" of structural mathematics in the 1960's during the New Math period. This was followed by an epoch where the science of mathematics education and the student (finally!) came into focus and brought forth attempts to delineate theories for this new science such as Bauersfeld's socio-cultural theories. New technologies shifted the focus of theories to accommodate how learning occurs in the human-machine interface. Finally TIMMS and PISA brought into focus assessment issues along with societal and political variables that are changing conceptions of mathematics education as we speak. In a sense we have come full circle because we still haven't defined what mathematics didactics is. However, in the search through history for the answer, we have understood the epochal nuances of this interesting term. Perhaps it is time we finally defined it!

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CONCLUDING POINTS

The diversity in the perspectives presented in the six contributions parallel conundrums recently elicited by Tommy Dreyfus at the 4th European Congress in Mathematics Education (Spain, February 2005). In his concluding report about the working group on mathematics education theories, Dreyfus stated that although theories were a vital aspect of mathematics education, they were much too wide of a topic. However the field can take solace from the fact that although contradictions exist, there are also connections and degrees of complementarities among theories. The coordinators of this particular Forum have reached a similar conclusion. Many of the points we make here echo the recommendations of Tommy Dreyfus. Although it is impossible to fully integrate theories, it is certainly possible to bring together researchers from different theoretical backgrounds to consider a given set of data or phenomena and examine the similarities and differences in the ensuing analysis and conclusions. The interaction of different theories can also be studied by applying them to the same empirical study and examining similarities and differences in conclusions. Last but not least, although it is impossible to expect everybody to use the mathematics education "language," a more modest undertaking would be to encourage researchers to understand one or more perspectives different from their own. This will ensure that the discussion continues as well as creates opportunities for researchers to study fruitful interactions of seemingly different theories. We consider such work vital to help move the field forward.